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Dirichlet Boundary Value Problems for Second Order p-Laplacian Difference Equations[1]

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Abstract. *In this paper, the solutions to second order Dirichlet boundary value problems of p-Laplacian difference equations are investigated. By using critical point theory, existence and multiplicity results are obtained. The proof is based on the Mountain Pass Lemma in combination with variational techniques.*

Keywords: Discrete Dirichlet Boundary Value Problem, Mountain Pass Lemma, p-Laplacian, Critical point theory.
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1. Introduction

Let $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For a, $b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a,b) = \{a, a+1, \cdots, b\}$ when $a \leq b$. Let k be a positive integer and * denote the transpose of a vector.

In this paper, we consider the second order difference equation

$$\Delta(\varphi_p(\Delta u_{n-1})) + f(n, u_{n+1}, u_n, u_{n-1}) = 0, \ n \in \mathbf{Z}(1,k) \tag{1}$$

with Dirichlet boundary value conditions

$$u_0 = u_{k+1} = 0, \tag{2}$$

where Δ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, $\varphi_p(s)$ is the p-Laplacian operator, that is, $\varphi_p(s) = |s|^{p-2}s (1 < p < \infty)$, $f \in C(\mathbf{Z} \times \mathbf{R}^3, \mathbf{R})$.

We may think of Eq. (1) as being a discrete analogue of the second order functional differential equation

$$[\varphi_p(u')]' + f(t, u(t+1), u(t), u(t-1)) = 0, \ t \in \mathbf{R} \tag{3}$$

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which includes the following equation

$$c^2u''(t) = V'(u(t+1) - u(t)) - V'(u(t) - u(t-1)),\ t \in \mathbf{R} \tag{4}$$

where $f \in C(\mathbf{R}^4, \mathbf{R})$. Eq. (3) has been studied extensively by many scholars. For example, Smets and Willem have obtained the existence of solitary waves of lattice differential equations, see [30] and the references cited therein.

The theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. Since the last decade, there has been much literature on qualitative properties of difference equations, those studies over many of the branches of difference equations, such as [1, 3, 10, 12, 13, 18–21, 23–25, 27, 28].

In recent years, the study of boundary value problems for differential equations has received much attention. By using various methods and techniques, such as Schauder fixed point theorem, the cone theoretic fixed point theorem, the method of upper and lower solutions, and coincidence degree theory, a series of results showing the existence of nontrivial solutions for differential equations have been obtained in the literatures, we refer to [2, 4–8, 17, 32, 36]. It is well known that critical point theory is an powerful tool to deal with differential equations [22, 26, 36]. Starting in 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. In particular, Yu, Guo, Chen, Shi, Tian and their collaborators considered second order nonlinear difference equations [9,14-16,29,31,33-35]. However, to our best knowledge, the results on the boundary value problem (BVP) of p-Laplacian difference equations are very scarce in the literature.

Our aim in this paper is to use the critical point theory to give some sufficient conditions for the existence and multiplicity of the BVP (1) with (2). The main idea in this paper is to transfer the existence of the BVP (1) with (2) into the existence of the critical points of some functional.

Our main results are as follows.

THEOREM 1.1. *Assume that the following hypotheses are satisfied:*

(F_1) *there exists a functional* $F(n, v_1, v_2) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ *with* $F(0, \cdot) = 0$ *such that for any* $n \in \boldsymbol{Z}(1, k)$,

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3);$$

(F_2) *there exist constants* $\delta > 0$, $\alpha \in \left(0, \frac{1}{2^{p/2}p}\left(\frac{c_1}{c_2}\right)^p \lambda_{\min}^{\frac{p}{2}}\right)$ *such that*

$$F(n, v_1, v_2) \le \alpha \left(\sqrt{v_1^2 + v_2^2}\right)^p, \quad \text{for} \quad n \in \boldsymbol{Z}(1, k) \quad \text{and} \quad v_1^2 + v_2^2 \le \delta^2\,;$$

(F_3) *there exist constants* $\rho > 0$, $\gamma > 0$, $\beta \in \left(\frac{1}{p}\left(\frac{c_2}{c_1}\right)^p \lambda_{\max}^{\frac{p}{2}}, +\infty\right)$ *such that*

$$F(n, v_1, v_2) \geq \beta \left(\sqrt{v_1^2+v_2^2}\right)^p - \gamma, \quad \text{for} \quad n \in \boldsymbol{Z}(1,k) \quad \text{and} \quad v_1^2 + v_2^2 \geq \rho^2 ,$$

where c_1, c_2 *are constants which can be referred to (8), and* $\lambda_{\min}$, $\lambda_{\max}$ *are constants which can be referred to (10).*

Then the BVP (1) with (2) possesses at least two nontrivial solutions.

Remark 1.2. *By* (F_3) *it is easy to see that there exists a constant* $\gamma' > 0$ *such that*

(F_3') $F(n, v_1, v_2) \geq \beta \left(\sqrt{v_1^2 + v_2^2}\right)^p - \gamma'$, $\forall (n, v_1, v_2) \in \boldsymbol{Z}(1,k) \times \boldsymbol{R}^2$.

As a matter of fact, let

$$\gamma_1 = \max\left\{ \left| F(n, v_1, v_2) - \beta \left(\sqrt{v_1^2 + v_2^2}\right)^p + \gamma \right| : n \in \mathbf{Z}(1,k) v_1^2 + v_2^2 \leq \rho^2 \right\},$$
$$\gamma' = \gamma + \gamma_1 ,$$

we can easily get the desired result.

The rest of the paper is organized as follows. In Sect. 2 we shall establish the variational framework for the BVP (1) with (2) in order to apply the critical point method and give some useful lemmas. In Sect. 3 we shall complete the proof of the main results and give an example to illustrate the result.

2. Variational Structure and some Lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for the BVP (1) with (2) and give some basic notation and useful lemmas.

Let $\mathbf{R}^k$ be the real Euclidean space with dimension k. Define the inner product on $\mathbf{R}^k$ as follows:

$$\langle u, v \rangle = \sum_{j=1}^{k} u_j v_j, \ \forall u, v \in \mathbf{R}^k, \tag{5}$$

by which the norm $\| \cdot \|$ can be induced by

$$\|u\| = \left(\sum_{j=1}^{k} u_j^2\right)^{\frac{1}{2}}, \ \forall u \in \mathbf{R}^k. \tag{6}$$

On the other hand, we define the norm $\|\cdot\|_r$ on $\mathbf{R}^k$ as follows:

$$\|u\|_r = \left(\sum_{j=1}^{k} |u_j|^r\right)^{\frac{1}{r}}, \tag{7}$$

for all $u \in \mathbf{R}^k$ and $r > 1$.

Since $\|u\|_r$ and $\|u\|_2$ are equivalent, there exist constants $c_1,\ c_2$ such that $c_2 \geq c_1 > 0$, and

$$c_1\|u\|_2 \leq \|u\|_r \leq c_2\|u\|_2,\ \forall u \in \mathbf{R}^k. \tag{8}$$

Clearly, $\|u\| = \|u\|_2$. For the BVP (1) with (2), consider the functional J on $\mathbf{R}^k$ as follows:

$$J(u) = \sum_{n=0}^{k} \left[\frac{1}{p}|\Delta u_n|^p - F(n, u_{n+1}, u_n)\right], \tag{9}$$

$\forall u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k,\ u_0 = u_{k+1} = 0$ and F is as in assumption (F_1).

Clearly, $J \in C^1(\mathbf{R}^k, \mathbf{R})$ and for any $u = \{u_n\}_{n\in\mathbf{Z}(1,k)} \in \mathbf{R}^k$, by using $u_0 = u_{k+1} = 0$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_n} = -\left[\Delta(\varphi_p(\Delta u_{n-1})) + f(n, u_{n+1}, u_n, u_{n-1})\right],\ n \in \mathbf{Z}(1,k).$$

Thus, u is a critical point of J on $\mathbf{R}^k$ if and only if

$$\Delta(\varphi_p(\Delta u_{n-1})) + f(n, u_{n+1}, u_n, u_{n-1}) = 0,\ \forall n \in \mathbf{Z}(1,k).$$

We reduce the existence of the BVP (1) with (2) to the existence of critical points of J on $\mathbf{R}^k$. That is, the functional J is just the variational framework of the BVP (1) with (2).

Let P is a $k \times k$ matrix defined by

$$P = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Clearly, P is positive definite. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the eigenvalues of P. Applying matrix theory, we know $\lambda_j > 0,\ j = 1, 2, \cdots, k$.

Denote

$$0<\lambda_{\min}=\min\left\{\lambda_1,\lambda_2,\cdots,\lambda_k\right\},\ \lambda_{\max}=\max\left\{\lambda_1,\lambda_2,\cdots,\lambda_k\right\}, \tag{10}$$

Let E be a real Banach space, $J\in C^1(E,\mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition[11] for short) if any sequence $\left\{u^{(k)}\right\}\subset E$ for which $\left\{J\left(u^{(k)}\right)\right\}$ is bounded and $J'\left(u^{(k)}\right)\to 0$ as $k\to\infty$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1. *(Mountain Pass Lemma [26]). Let E be a real Banach space and suppose $J\in C^1(E,\boldsymbol{R})$ satisfies the P.S. condition. If $J(0)=0$ and*
(J_1) there exist constants $\rho,\ a>0$ such that $J|_{\partial B_\rho}\geq a$, and
(J_2) there exists $e\in E\setminus B_\rho$ such that $J(e)\leq 0$.
Then J possesses a critical value $c\geq a$ given by

$$c=\inf_{g\in\Gamma}\max_{s\in[0,1]}J(g(s)), \tag{11}$$

where

$$\Gamma=\{g\in C([0,1],E)|g(0)=0,\ g(1)=e\}. \tag{12}$$

Lemma 2.2. *Suppose that $(F_1)-(F_3)$ are satisfied. Then the functional J satisfies the P.S. condition.*

Proof. Let $u^{(l)}\in\mathbf{R}^k,\ l\in\mathbf{Z}(1)$ be such that $\left\{J\left(u^{(l)}\right)\right\}$ is bounded. Then there exists a positive constant M_2 such that

$$-M_2\leq J\left(u^{(l)}\right)\leq M_2,\ \forall l\in\mathbf{N}.$$

By (F_3'), we have

$$\begin{aligned}
-M_2\leq J\left(u^{(l)}\right)&=\frac{1}{p}\sum_{n=0}^{k}\left|\Delta u_n^{(l)}\right|^p-\sum_{n=1}^{k}F\left(n,u_{n+1}^{(l)},u_n^{(l)}\right)\\
&=\frac{1}{p}\left[\left(\sum_{n=0}^{k}\left|\Delta u_n^{(l)}\right|^p\right)^{\frac{1}{p}}\right]^p-\sum_{n=1}^{k}F\left(n,u_{n+1}^{(l)},u_n^{(l)}\right)\\
&\leq\frac{1}{p}\left[c_2\left(\sum_{n=0}^{k}\left|\Delta u_n^{(l)}\right|^2\right)^{\frac{1}{2}}\right]^p-\sum_{n=1}^{k}\left[\beta\left(\sqrt{\left(u_{n+1}^{(l)}\right)^2+\left(u_n^{(l)}\right)^2}\right)^p-\gamma'\right]\\
&=\frac{1}{p}c_2^p\left(\left(u^{(l)}\right)^*Pu^{(l)}\right)^{\frac{p}{2}}-\beta\left[\left(\sum_{n=1}^{k}\left(\sqrt{\left(u_{n+1}^{(l)}\right)^2+\left(u_n^{(l)}\right)^2}\right)^p\right)^{\frac{1}{p}}\right]^p+k\gamma'
\end{aligned}$$

$$\leq \frac{1}{p}c_2^p\lambda_{\max}^{\frac{p}{2}}\left\|u^{(l)}\right\|_2^p - \beta c_1^p\Bigg[\sum_{n=1}^{k}\left(\left(u_{n+1}^{(l)}\right)^2+\left(u_n^{(l)}\right)^2\right)\Bigg]^{\frac{p}{2}} + k\gamma'$$
$$\leq \frac{1}{p}c_2^p\lambda_{\max}^{\frac{p}{2}}\left\|u^{(l)}\right\|_2^p - \beta c_1^p\left\|u^{(l)}\right\|_2^p + k\gamma'$$
$$= \left(\frac{1}{p}c_2^p\lambda_{\max}^{\frac{p}{2}} - \beta c_1^p\right)\left\|u^{(l)}\right\|_2^p + k\gamma'.$$

That is,

$$\left(\beta c_1^p - \frac{1}{p}c_2^p\lambda_{\max}^{\frac{p}{2}}\right)\left\|u^{(l)}\right\|_2^p \leq M_2 + k\gamma'.$$

Since $\beta > \frac{1}{p}\left(\frac{c_2}{c_1}\right)^p\lambda_{\max}^{\frac{p}{2}}$, there exists a constant $M_3 > 0$ such that

$$\left\|u^{(l)}\right\|_2 \leq M_3,\ \forall l \in \mathbf{N}.$$

Therefore, $\{u^{(l)}\}$ is bounded on $\mathbf{R}^k$. As a consequence, $\{u^{(l)}\}$ possesses a convergence subsequence in $\mathbf{R}^k$. And thus the P.S. condition is verified. □

3. Proof of the Main Results

In this section, we shall complete the proof of Theorem 1.1.

3.1. Proof of Theorem 1.1

Proof. For any $n \in \mathbf{Z}(1,k)$, by (F_2) and (8),

$$J(u) = \sum_{n=0}^{k}\left[\frac{1}{p}|\Delta u_n|^p - F(n,u_{n+1},u_n)\right]$$
$$= \frac{1}{p}\left[\left(\sum_{n=0}^{k}|\Delta u_n|^p\right)^{\frac{1}{p}}\right]^p - \sum_{n=1}^{k}F(n,u_{n+1},u_n)$$
$$\geq \frac{1}{p}c_1^p\left[\left(\sum_{n=0}^{k}|\Delta u_n|^2\right)^{\frac{1}{2}}\right]^p - \sum_{n=1}^{k}F(n,u_{n+1},u_n)$$
$$= \frac{1}{p}c_1^p\left(u^*Pu\right)^{\frac{p}{2}} - \sum_{n=1}^{k}F(n,u_{n+1},u_n)$$
$$\geq \frac{1}{p}c_1^p\lambda_{\min}^{\frac{p}{2}}\|u\|_2^p - \alpha\sum_{n=1}^{k}\left(\sqrt{u_{n+1}^2+u_n^2}\right)^p$$

$$
\begin{aligned}
&= \frac{1}{p} c_1^p \lambda_{\min}^{\frac{p}{2}} \|u\|_2^p - \alpha \left[\left(\sum_{n=1}^{k} \left(\sqrt{u_{n+1}^2 + u_n^2} \right)^p \right)^{\frac{1}{p}} \right]^p \\
&\geq \frac{1}{p} c_1^p \lambda_{\min}^{\frac{p}{2}} \|u\|_2^p - \alpha c_2^p \left[\sum_{n=1}^{k} \left(u_{n+1}^2 + u_n^2 \right) \right]^{\frac{p}{2}} \\
&\geq \frac{1}{p} c_1^p \lambda_{\min}^{\frac{p}{2}} \|u\|_2^p - \alpha c_2^p \left(2\|u\|_2^2 \right)^{\frac{p}{2}} \\
&\geq \left(\frac{1}{p} c_1^p \lambda_{\min}^{\frac{p}{2}} - 2^{\frac{p}{2}} c_2^p \alpha \right) \|u\|_2^p.
\end{aligned}
$$

Take $a \triangleq \left(\frac{1}{p} c_1^p \lambda_{\min}^{\frac{p}{2}} - 2^{\frac{p}{2}} c_2^p \alpha \right) \delta^p$. Therefore,

$$J(u) \geq a > 0, \ \forall u \in \partial B_\rho.$$

At the same time, we have also proved that there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho} \geq a$. That is to say, J satisfies the condition (J_1) of the Mountain Pass Lemma.

For our setting, clearly $J(0) = 0$. In order to exploit the Mountain Pass Lemma in critical point theory, we need to verify other conditions of the Mountain Pass Lemma. By Lemma 2.2, J satisfies the P.S. condition. So it suffices to verify the condition (J_2).

From the proof of the P.S. condition, we know

$$J(u) \leq \left(\frac{1}{p} c_2^p \lambda_{\max}^{\frac{p}{2}} - \beta c_1^p \right) \|u\|_2^p + k\gamma'.$$

Since $\beta > \frac{1}{p} \left(\frac{c_2}{c_1} \right)^p \lambda_{\max}^{\frac{p}{2}}$, we can choose $\bar{u}$ large enough to ensure that $J(\bar{u}) < 0$.

By the Mountain Pass Lemma, J possesses a critical value $c \geq a > 0$, where

$$c = \inf_{h \in \Gamma} \sup_{s \in [0,1]} J(h(s)),$$

and

$$\Gamma = \{ h \in C([0,1], \mathbf{R}^k) \mid h(0) = 0, \ h(1) = \bar{u} \}.$$

Let $\tilde{u} \in \mathbf{R}^k$ be a critical point associated to the critical value c of J, i.e., $J(\tilde{u}) = c$. Similar to the proof of the P.S. condition, there exists $u \in \mathbf{R}^k$ such that $J(u) \geq a$. Hence, there exists $\hat{u} \in \mathbf{R}^k$ such that $J(\hat{u}) = c_{\max} = \max_{u \in \mathbf{R}^k} J(u) \geq a > 0$.

Clearly, $\hat{u} \neq 0$. If $\tilde{u} \neq \hat{u}$, then the conclusion of Theorem 1.1 holds. Otherwise, $\tilde{u} = \hat{u}$. Then $c = J(\tilde{u}) = J(\hat{u}) = c_{\max}$. That is,

$$\sup_{u \in \mathbf{R}^k} J(u) = \inf_{h \in \Gamma} \sup_{s \in [0,1]} J(h(s)).$$

Therefore,

$$c_{\max} = \max_{s\in[0,1]} J(h(s)),\ \forall h \in \Gamma.$$

By the continuity of $J(h(s))$ with respect to s, $J(0) = 0$ and $J(\bar{u}) < 0$ imply that there exists $s_0 \in (0, 1)$ such that

$$J\left(h\left(s_0\right)\right) = c_{\max}.$$

Choose $h_1,\ h_2 \in \Gamma$ such that $\{h_1(s) \mid s \in (0,1)\} \cap \{h_2(s) \mid s \in (0,1)\}$ is empty, then there exists $s_1,\ s_2 \in (0, 1)$ such that

$$J\left(h_1\left(s_1\right)\right) = J\left(h_2\left(s_2\right)\right) = c_{\max}.$$

Thus, we get two different critical points of J on $\mathbf{R}^k$ denoted by

$$u^1 = h_1\left(s_1\right),\ u^2 = h_2\left(s_2\right).$$

The above argument implies that the BVP (1) with (2) possesses at least two nontrivial solutions $\tilde{u}$, $\hat{u}$ or $\tilde{u}$, u^1, u^2. The proof of Theorem 1.1 is finished. □

REMARK 3.1. *As an application of Theorem 1.1, finally, we give an example to illustrate our result.*

For $n \in \mathbf{Z}(1, k)$, assume that

$$\Delta(\varphi_p(\Delta u_{n-1}))+\mu u_n\left[(e^n-1)\left(u_{n+1}^2+u_n^2\right)^{\frac{\mu}{2}-1}+\left(e^{n-1}-1\right)\left(u_n^2+u_{n-1}^2\right)^{\frac{\mu}{2}-1}\right] = 0, \tag{13}$$

with boundary value conditions

$$u_0 = u_{k+1} = 0, \tag{14}$$

where $1 < p < +\infty$, $\mu > p$.

We have

$$f(n, v_1, v_2, v_3) = \mu v_2 \left[(e^n - 1)\left(v_1^2 + v_2^2\right)^{\frac{\mu}{2}-1} + \left(e^{n-1} - 1\right)\left(v_2^2 + v_3^2\right)^{\frac{\mu}{2}-1}\right]$$

and

$$F(n, v_1, v_2) = (e^n - 1)\left(v_1^2 + v_2^2\right)^{\frac{\mu}{2}}.$$

Then

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = \mu v_2 \left[(e^n - 1)\left(v_1^2 + v_2^2\right)^{\frac{\mu}{2}-1} + \left(e^{n-1} - 1\right)\left(v_2^2 + v_3^2\right)^{\frac{\mu}{2}-1}\right].$$

It is easy to verify all the assumptions of Theorem 1.1 are satisfied and then the BVP (13) with (14) possesses at least two nontrivial solutions.

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