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Period Two implies Chaos for a Class of ODEs: a Dynamical System Approach

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Abstract. *The aim of this note is to set in the field of dynamical systems a recent theorem by Obersnel and Omari in [19] about the presence of subharmonic solutions of all orders for a class of scalar time-periodic first order differential equations without uniqueness, provided a subharmonic solution (for instance, of order two) does exist. Indeed, making use of the Bebutov flow, we try to clarify in what sense the term "chaos" has to be understood and which dynamical features can be inferred for the system under analysis.*

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1. Introduction and Motivation

In the recent papers [17, 19] Obersnel and Omari and in [8] De Coster, Obersnel and Omari, using upper and lower solutions techniques, provide a complete description of the structure of the set of solutions of the scalar time-periodic first order differential equation

$$\dot{x} = f(t, x), \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is 1-periodic in the time-variable, that is, $f(t+1, x) = f(t, x)$, $\forall (t, x) \in \mathbb{R} \times \mathbb{R}$, and satisfies the L^1-Carathéodory conditions. In particular, the authors show that the periodic solutions are assembled in mutually ordered connected components and the existence of subharmonic solutions of all orders for (1) is achieved, under the hypothesis that a subharmonic solution does exist. In [18] the case of differential inclusions is studied as well, by direct techniques.

Subsequently, the result on the existence of subharmonic solutions of all orders for (1) has been reconsidered in [2, 20], employing different approaches.

In fact, in [2] Andres, Fürst and Pastor give a proof in terms of multivalued maps, under the additional assumption of global existence for the solutions of (1), while in [20] Sędziwy exploits direct arguments, similar to those used in [18].

The precise statement of the result proven in [19] reads as follows:

THEOREM 1.1. *Let* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be* 1*-periodic in the first variable and satisfy the* L^1*-Carathéodory conditions on* $[0, 1] \times \mathbb{R}$. *If equation* (1) *admits a subharmonic solution of order* $m > 1$, *then, for every integer* $n \geq 1$, *there exists a subharmonic solution of* (1) *of order* n.

We recall that in [19] it was also shown that the set of all the subharmonic solutions of (1) of order n has dimension at least n as a subset of $L^\infty(\mathbb{R})$. The treatment of such topic is however out of the scope of the present paper, as it doesn't fall within our dynamical approach. Of course, in the statement above the case of a map f which is T-periodic in the time-variable, for some $T > 0$, could be considered as well. For the sake of simplicity, we confine ourselves to the setting considered in [19], presenting an elementary verification of Theorem 1.1 based on connectivity. The arguments we employ bear resemblance to [18, 20]: our proof has however been obtained independently and we present it in full details because it is along the course of such proof that we lay the foundations for the study of the system generated by the solutions of (1).

Our contribution is indeed twofold. On the one hand, we propose an alternative dynamical approach to the study of the system under consideration. Namely, since the uniqueness of the solutions is missing, instead of dealing with the multivalued Poincaré operator as in [2], we introduce the Bebutov flow, defined on a function space. This allows to study the case of differential inclusions as well, without the additional hypothesis of global existence for the solutions of (1). On the other hand, we try to explain which are the chaotic features that can be inferred for the system generated by the solutions of (1). For instance, we are able to show the positivity of the topological entropy and the presence of chaos in the sense of Li-Yorke and Devaney.

The paper is organized as follows. In Section 2 the term "chaos" appearing in the title is better specified and the dynamical features of the solutions of equation (1) are more deeply analyzed. In particular we introduce the tools from the theory of dynamical systems we need along the proof of our main result, Theorem 2.2, where we show that the Bebutov flow restricted to a suitable invariant set generated by a subharmonic solution and the Bernoulli shift are conjugate. According to [14], this fact has several consequences for the dynamical system generated by the solutions of (1), since it turns out to be Li-Yorke chaotic, sensitive with respect to initial data, topologically transitive, the set of the periodic solutions is dense therein and the topological entropy is positive. The definition of such concepts can be found in Section 2, too.

The proof of Theorem 2.2 is however postponed to Section 3, where we first prove Theorem 1.1, by splitting its verification into the Cancellation Lemma 3.1 and Lemma 3.2. More precisely, Lemma 3.1 states that, whenever equation (1) admits a subharmonic solution of order $m \geq 3$, then it also has a subharmonic of order 2. In Lemma 3.2 we show instead that, whenever a subharmonic solution of period two exists for (1), then the presence of periodic solutions of all periods follows. As mentioned above, in the proof of Theorem 1.1 the language and the notation for the dynamical analysis of the system generated by the solutions of (1) are introduced. Such notation is in fact used along the proof of Theorem 2.2, which concludes the work.

2. Chaotic Dynamics

Before stating our main result on chaotic dynamics, i.e., Theorem 2.2, we recall the fundamental tools from the theory of dynamical systems we are going to use along its proof. In particular, at first we introduce the Bebutov flow [4, 21] and then we collect some general definitions and facts about chaotic dynamics.

We denote by $\mathcal{C}$ the set of the continuous functions $f : \mathbb{R} \to \mathbb{R}$, that is,

$$\mathcal{C} = \mathcal{C}(\mathbb{R}). \tag{2}$$

On this space we define a metric ρ as follows: given an integer $m \geq 1$ and $I_m := [-m, m]$, for $f, g \in \mathcal{C}$ we set

$$\begin{aligned} \vartheta_m(f,g) &:= \max\{|f(t) - g(t)| : t \in I_m\}, \\ \rho_m(f,g) &:= \frac{\vartheta_m(f,g)}{1 + \vartheta_m(f,g)} \end{aligned}$$

and

$$\rho(f,g) := \sum_{m=1}^{\infty} \frac{\rho_m(f,g)}{2^m}. \tag{3}$$

One may verify that ρ is indeed a metric on $\mathcal{C}$ and that with this choice $\mathcal{C}$ is complete. Moreover, the convergence induced by ρ on $\mathcal{C}$ is the uniform convergence on compact sets, that is, if $(f_n)_n$ is a sequence in $\mathcal{C}$, then $\rho(f_n, f) \to 0$ as $n \to \infty$ if and only if $f_n(t) \to f(t)$ uniformly on compact subsets of $\mathbb{R}$ [4].

On the metric space $(\mathcal{C}, \rho)$, we define the *Bebutov dynamical system* (or *shift dynamical system* [21]) $\pi : \mathcal{C} \times \mathbb{R} \to \mathcal{C}$ as

$$\pi(f,t) = g,$$

where

$$g(s) = f(t+s), \ \forall s \in \mathbb{R}.$$

The verification that π is a dynamical system can be found in [4]. When s is fixed, it is also possible to define the continuous function

$$\psi_s : \mathcal{C} \to \mathcal{C}, \quad f(\cdot) \mapsto f(\,\cdot + s). \tag{4}$$

This is the map we will consider, for $s = 1$ and restricted to a suitable compact set, along Theorem 2.2.

The interested reader can find applications of the Bebutov flow to the theory of control in [7, 11] and to the study of differential equations in [5, 13], just to quote a few contributions in such directions.

Given an integer $m \geq 2$, we denote by $\Sigma_m := \{0, \dots, m-1\}^{\mathbb{Z}}$ the set of two-sided sequences of m symbols. On Σ_m we introduce the distance

$$\hat{d}(\mathbf{s}', \mathbf{s}'') := \sum_{i \in \mathbb{Z}} \frac{|s_i' - s_i''|}{m^{|i|+1}}\,, \qquad \text{for } \mathbf{s}' = (s_i')_{i\in\mathbb{Z}}\,,\ \mathbf{s}'' = (s_i'')_{i\in\mathbb{Z}} \in \Sigma_m, \tag{5}$$

that makes it a compact metric space. Here we define the *two-sided Bernoulli shift* on m symbols $\sigma : \Sigma_m \to \Sigma_m$ as $\sigma((s_i)_i) := (s_{i+1})_i$, $\forall i \in \mathbb{Z}$. Observe that σ is a bijection and, by the choice of the metric, it is also continuous (and hence a homeomorphism).

We have chosen to present this definition in the generic case of $m \geq 2$ symbols because of the discussion along the proof of Lemma 3.2. However, in view of Theorem 2.2, from now on we will mainly confine ourselves to the special framework of two symbols. In particular, this holds true for the definition of chaos in the coin-tossing sense below, that we directly give in the less general version, but that could as well be formulated for an arbitrary number of symbols (greater or equal than 2).

Given two continuous self-maps $f : Y \to Y$ and $g : Z \to Z$ of the metric spaces Y and Z, we say that f and g are *topologically conjugate* if there exists a homeomorphism $\phi : Y \to Z$ that makes the diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y \\ \phi\downarrow & & \downarrow\phi \\ Z & \xrightarrow[g]{} & Z \end{array}$$

commute, i.e., such that $\phi \circ f = g \circ \phi$. Any such map ϕ is named *conjugacy.*

A precious tool for the detection of complex dynamics is the *topological entropy* and indeed its positivity is generally considered as one of the trademarks of chaos. Such object can be introduced for any continuous self-map f of a compact metric space X and we indicate it with the symbol $h_{\rm top}(f)$. Its original definition due to Adler, Konheim and McAndrew [1] is based on the open coverings. More precisely, for an open cover α of X, we define the *entropy of* α

as $H(\alpha) := \log N(\alpha)$, where $N(\alpha)$ is the minimal number of elements in a finite subcover of α. Given two open covers α and β of X, we define their *join* $\alpha \vee \beta$ as the open cover of X made by all sets of the form $A \cap B$, with $A \in \alpha$ and $B \in \beta$. Similarly one can define the join $\vee_{i=1}^{n} \alpha_i$ of any finite collection $\alpha_1, \dots, \alpha_n$ of open covers of X. If α is an open cover of X and $f : X \to X$ a continuous map, we denote by $f^{-1}\alpha$ the open cover consisting of all sets $f^{-1}(A)$, with $A \in \alpha$. By $\vee_{i=0}^{n-1} f^{-i}\alpha$ we mean $\alpha \vee f^{-1}\alpha \vee \cdots \vee f^{-n+1}\alpha$. Finally, we have:

$$h_{\mathrm{top}}(f) := \sup_{\alpha} \left(\lim_{n\to\infty} \frac{1}{n} \left(H \left(\vee_{i=0}^{n-1} f^{-i}\alpha \right) \right) \right),$$

where α ranges over all open covers of X.

Among the several properties of the topological entropy, we recall just the ones that are useful in view of Theorem 2.2. For instance, in regard to the Bernoulli shift σ on m symbols, it holds that $h_{\mathrm{top}}(\sigma) = \log(m)$. On the other hand, given two topologically conjugate continuous self-maps $f : X \to X$ and $g : Y \to Y$ of the compact metric spaces X and Y, respectively, we have $h_{\mathrm{top}}(f) = h_{\mathrm{top}}(g)$. Hence, when a continuous self-map f of a compact metric space X is conjugate to the Bernoulli shift σ on $m \geq 2$ symbols, then

$$h_{\mathrm{top}}(f) = h_{\mathrm{top}}(\sigma) = \log(m), \tag{6}$$

and the topological entropy of f is positive.

We stress that the concepts of topological conjugacy and of topological entropy could be defined in the more general setting of topological spaces, as well. However, in order to make the presentation more uniform, we have decided to introduce all the notions in the context of metric spaces. For instance, this remark applies to the definition of coin-tossing chaoticity, too. For further features of h_{top} and additional details, see [1, 12, 15, 22]. With reference to the case of compact metric spaces, alternative definitions of entropy can be found in [6, 10].

A self-map $f : X \to X$ of the metric space X is called *chaotic in the sense of coin-tossing* [14] if there exist two nonempty disjoint compact sets

$$\mathcal{K}_0, \mathcal{K}_1 \subseteq X,$$

such that, for each two-sided sequence $(s_i)_{i\in\mathbb{Z}} \in \Sigma_2$, there exists a corresponding sequence $(x_i)_{i\in\mathbb{Z}} \in X^{\mathbb{Z}}$ such that

$$x_i \in \mathcal{K}_{s_i} \quad \text{and} \quad x_{i+1} = f(x_i), \;\; \forall\, i \in \mathbb{Z}.$$

According to [14], we say that a self-map $f : X \to X$ of the metric space (X, d) is *chaotic in the sense of Li-Yorke* if there exists an uncountable invariant set $S \subseteq X$ and $\delta > 0$ such that, for any $x, y \in S$, with $x \neq y$, it holds that

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq \delta. \tag{7}$$

Notice that the above definition of Li-Yorke chaos differs from the standard one in [16], where the set S need not be invariant and (7) is replaced by the weaker condition that for any $x, y \in S$, with $x \neq y$,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

However, in order to employ a result from [14] (see Lemma 2.1 below), we follow the terminology introduced therein. We also recall that the Bernoulli shift is Li-Yorke chaotic according to the former definition [14]. This fact will be used in Theorem 2.2.

A self-map $f : X \to X$ of the infinite metric space (X, d) is called *chaotic in the sense of Devaney* if

- f is *topologically transitive*, i.e., for any couple of nonempty open subsets U, V of X there exists an integer $n \geq 1$ such that $U \cap f^n(V) \neq \emptyset$;
- the set of the periodic points for f is dense in X.

In the original definition of Devaney chaos [9], it was also required the map f to be *sensitive with respect to initial data on* X, i.e., there exists $\delta > 0$ such that for any $x \in X$ there is a sequence $(x_i)_{i\in\mathbb{N}}$ in X such that $x_i \to x$ when $i \to \infty$ and for each $i \in \mathbb{N}$ there exists a positive integer m_i with $d(f^{m_i}(x_i), f^{m_i}(x)) \geq \delta$. This third condition has however been proved in [3] to be redundant for any continuous self-map of an infinite metric space, as it is implied by the previous two, and hence it is usually omitted.

In Theorem 2.2 below we are going to show that the map ψ_1 in (4) (restricted to a suitable compact set W of solutions of (1)) and the Bernoulli shift σ are conjugate. From such fact, many chaotic features of the Bernoulli system can be directly transferred to the Bebutov flow by using the next result from [14], that we recall for the reader's convenience, rewritten in conformity with our notation and limited to what is indeed needed in the course of the proof of Theorem 2.2.

LEMMA 2.1 (Kirchgraber and Stoffer [14, Proposition 1]). *Let $f : X \to X$ and $g : Y \to Y$ be homeomorphisms of the complete metric spaces X and Y, respectively. Assume that f and g are topologically conjugate and that X is compact. Then, if f is chaotic in the sense of coin-tossing, or in the sense of Devaney, or in the sense of Li-Yorke, so is g.*

We are now in position to state the following:

THEOREM 2.2. *In the hypotheses of Theorem 1.1, if equation* (1) *admits a subharmonic solution $x(t)$ of order $m > 1$, then there exists a compact set $W(= W(x)) \subset \mathcal{C}$ such that the map $\psi := \psi_1|_W$, with ψ_1 defined in* (4)*, is a homeomorphism which is topologically conjugate to the two-sided Bernoulli shift σ on two symbols. As a consequence, ψ displays the following chaotic features:*

(*i*) $h_{\mathrm{top}}(\psi) = h_{\mathrm{top}}(\sigma) = \log(2)$;

(*ii*) *the map ψ is chaotic in the sense of coin-tossing, Li-Yorke and Devaney;*

(*iii*) *as regards the coin-tossing chaoticity, the map ψ actually displays the stronger property that the periodic sequences of symbols in Σ_2 get realized by periodic orbits of ψ* [1].

3. Proofs

As explained in the Introduction, before proving Theorem 2.2, we give a proof of Theorem 1.1, which depends on two steps. The first one is the Cancellation Lemma 3.1, which states that whenever equation (1) admits a subharmonic solution of order $m \geq 3$, then it also admits a subharmonic of order 2. The second step consists instead in the verification of Lemma 3.2 below, which asserts that, whenever a subharmonic solution of order two exists for (1), then the presence of subharmonic solutions of all orders follows. This is the main part of the proof of Theorem 1.1 and it is here that the language and notation then used in the verification of Theorem 2.2 are introduced.

Lemma 3.1 and Lemma 3.2, as well as the proof of Theorem 2.2, are presented hereinafter.

LEMMA 3.1 (Cancellation Lemma). *If equation* (1) *admits a subharmonic solution of order m, with $m \geq 3$, then it also admits a subharmonic of order two.*

Proof. Let $u(t)$ be a solution of (1) defined for $t \geq t_0$, for a certain $t_0 \in \mathbb{R}$, such that there exists $m \geq 3$ with $u(t+m) = u(t)$, $\forall t \geq t_0$, and $u(t^*+1) \neq u(t^*)$, for some $t^* \geq t_0$. Then we claim that we can suppose the set $\mathcal{U} := \{u(t^*+1), \ldots, u(t^*+m-1)\}$ to be composed by pairwise distinct terms. Indeed, if this were not the case, we could join two of the coinciding elements $u(t^*+j)$ and $u(t^*+k)$, for some $j < k \in \{1, \ldots, m-1\}$, in order to obtain an $m-(k-j)$-periodic solution $\widetilde{u}$ of (1) defined as

$$\widetilde{u}(t) := \begin{cases} u(t) & t \leq t^*+j \\ u(t+k-j) & t \geq t^*+j\,. \end{cases} \tag{8}$$

Since this can be done for any couple of coinciding elements, the claim is true for the solution $\widetilde{\widetilde{u}}$ of (1) so obtained, that for simplicity we still denote by u. With a similar argument, we can also suppose that $u(t^*) \notin \mathcal{U}$.

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[1]In symbols, this means that whenever $(s_i)_{i\in\mathbb{Z}} \in \Sigma_2$ is a k-periodic sequence (that is, $s_{i+k} = s_i\,, \forall i \in \mathbb{Z}$) for some $k \geq 1$, then there exists a corresponding k-periodic sequence $(w^{(i)})_{i\in\mathbb{Z}} \in W^{\mathbb{Z}}$ satisfying

$$w^{(i)} \in \mathcal{K}_{s_i} \quad \text{and} \quad w^{(i+1)} = \psi(w^{(i)}), \ \ \forall\, i \in \mathbb{Z},$$

where $\mathcal{K}_0$ and $\mathcal{K}_1$ are compact disjoint subsets of W to be defined in the course of the proof.

If u has period two, the lemma is proved. Otherwise, following an argument similar to [18], let us call $\bar{t}$ the element among $t^*, t^*+1, \ldots, t^*+m-1$ such that $u(\bar{t}) = \min\{u(t^*+i) : 0 \le i \le m-1\}$, so that, setting $v(t) := u(t+m-1)$, we find $v(\bar{t}) = u(\bar{t}+m-1) > u(\bar{t})$. On the other hand, $v(\bar{t}+1) = u(\bar{t}) < u(\bar{t}+1)$. Hence, by Bolzano theorem there exists $\tilde{t} \in (\bar{t}, \bar{t}+1)$ with $u(\tilde{t}) = v(\tilde{t}) = u(\tilde{t}+m-1)$. Therefore, calling $s := \tilde{t}+m-1$ we get $u(s+1) = u(\tilde{t}+m) = u(\tilde{t}) = u(\tilde{t}+m-1) = u(s)$. Thus we can obtain an $(m-1)$-periodic solution of (1) with the same procedure as in (8).

Applying repeatedly the previous argument, in at most $m-2$ steps we get the desired solution of period two. The thesis is so achieved. $\square$

LEMMA 3.2. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and* 1*-periodic in the time-variable. If equation* (1) *admits a subharmonic solution of order* 2*, then, for every $n \ge 1$, there exists a subharmonic solution of* (1) *of order n.*

Proof. Let $x(t)$ be a solution of (1) of minimum period two defined for $t \ge t_0$, for some $t_0 \in \mathbb{R}$. Then $x(t+2) = x(t)$, $\forall t \ge t_0$ and there exists $t_1 \ge t_0$ such that $x(t_1+1) \ne x(t_1)$. Without loss of generality we can assume $x(t_1+1) > x(t_1)$, since otherwise it would be sufficient to consider t_1+1 in place of t_1. Defining $y(t) := x(t+1)$, we find $y(t_1) = x(t_1+1) > x(t_1)$ and $y(t_1+1) = x(t_1+2) = x(t_1) < x(t_1+1)$. Thus, by Bolzano theorem there exists $\xi \in (t_1, t_1+1)$ such that $y(\xi) = x(\xi)$. Hence, $y(\xi+1) = x(\xi+2) = x(\xi) = y(\xi) = x(\xi+1)$ and, more generally, $y(\xi+n) = x(\xi+n)$, $\forall n \in \mathbb{N}$. Just to fix ideas, we start by supposing that ξ is the only instant in (t_1, t_1+1) where $x(\cdot)$ and $y(\cdot)$ coincide. By such assumption, on each interval of the form $(\xi+n, \xi+n+1)$, with $n \in \mathbb{N}$, it holds that $x(t) > y(t)$ or $x(t) < y(t)$ and the situation gets inverted when moving from $(\xi+n, \xi+n+1)$ to $(\xi+n+1, \xi+n+2)$ (more precisely, $x(t) > y(t)$ on the intervals of the kind $(\xi+2m, \xi+2m+1)$ and $y(t) > x(t)$ on $(\xi+2n+1, \xi+2n+2)$, for $m, n \in \mathbb{N}$). Thus, in correspondence to every interval of the form $(\xi+n, \xi+n+1)$, we can choose between staying "up" or "down" by suitably selecting $x(t)$ or $y(t)$. In particular we associate to any such interval the label "0" when we stay "up" and the label "1" when we stay "down". This procedure can obviously be adopted also on the intervals $(\xi+i, \xi+i+1)$ with i negative integer, by extending $x(\cdot)$ and $y(\cdot)$ to the whole real line by 2-periodicity, thanks to the fact that f is 1-periodic in the time-variable. In such way we are led to work with the two-sided sequences on two symbols $\eta = (\eta_i)_{i\in\mathbb{Z}}$, with $\eta_i \in \{0,1\}$, $\forall i \in \mathbb{Z}$. For any $t \in \mathbb{R} \setminus \{\xi+i : i \in \mathbb{Z}\}$, we call $x_0(t)$ the one between $x(t)$ and $y(t)$ that stays "up" and $x_1(t)$ the one that stays "down". Hence, to any sequence $\eta = (\eta_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ it is possible to associate the continuous function

$$w_\eta : \mathbb{R} \to \mathbb{R}, \quad w_\eta(t) := x_{\eta_i}(t), \text{ for } \xi+i < t < \xi+i+1. \tag{9}$$

It is easy to check that, setting $p := x(t_1)$, $q := y(t_1)$, $s_1 := t_1+1$ and recalling that $p < q$, then $w(\cdot) = w_\eta(\cdot)$ is a solution of (1) which satisfies $w(s_1+i) = q$

if $\eta_i = 0$ and $w(s_1 + i) = p$ if $\eta_i = 1$. Moreover, it is immediate to see that if $\eta = (\eta_i)_i$ is a periodic sequence of some period $l \geq 1$, that is, $\eta_{i+l} = \eta_i$, $\forall i \in \mathbb{Z}$, then the corresponding solution $w_\eta(t) = (x_{\eta_i}(t))_{i\in\mathbb{Z}}$ is periodic of the same period, i.e., $x_{\eta_i}(t+l) = x_{\eta_i}(t)$, $\forall t \in \mathbb{R}$. In this way, we have proved the existence of subharmonic solutions of each period for (1). The thesis is achieved.

In the more general case in which $x(\cdot)$ and $y(\cdot)$ meet several times in (t_1, t_1+1), let ξ be the first instant in (t_1, t_1+1) such that $x(\xi) = y(\xi)$. Then the same proof presented above still works, with the only difference that the label to assign to the generic interval $(\xi + i, \xi + i + 1)$ is now decided by looking at the value that the maps $x(\cdot)$ and $y(\cdot)$ assume in $t_1 + i + 1$. Indeed, $t_1 + i + 1 \in (\xi + i, \xi + i + 1)$, for any $i \in \mathbb{Z}$, and it holds that $x(t_1 + i + 1) > y(t_1 + i + 1)$ when i is even, while $y(t_1 + i + 1) > x(t_1 + i + 1)$ when i is odd. Moreover, for any $t \in \mathbb{R} \setminus \{\xi + k : k \in \mathbb{Z}\}$, we have that $t \in (\xi + i, \xi + i + 1)$, for a unique $i \in \mathbb{Z}$. Then we call $x_0(t)$ the one between $x(t)$ and $y(t)$ that stays "up" in $t_1 + i + 1$ and $x_1(t)$ the one that stays "down" in $t_1 + i + 1$. More precisely, for any $\bar{t} \in (\xi + i, \xi + i + 1)$, we set $x_0(\bar{t}) = x(\bar{t})$, if $x(t_1 + i + 1) > y(t_1 + i + 1)$, otherwise $x_0(\bar{t}) = y(\bar{t})$, and we set $x_1(\bar{t}) = x(\bar{t})$, if $x(t_1 + i + 1) < y(t_1 + i + 1)$, otherwise $x_1(\bar{t}) = y(\bar{t})$. In this way, to any sequence $\eta = (\eta_i)_{i\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ it is possible to associate the function w_η as in (9) and conclude as before.

We stress that the case in which $x(\cdot)$ and $y(\cdot)$ meet several times in (t_1, t_1+1) could be more deeply analyzed. Indeed, when there are $m > 1$ intersections $\xi_1, \ldots, \xi_m$ between $x(\cdot)$ and $y(\cdot)$ in $(t_1, t_1 + 1)$, it is possible to work with the sequences on a higher number of symbols, that can be assigned as follows: since on any interval of the form (ξ_k, ξ_{k+1}), with $k = 1, \ldots, m$, (where we identify ξ_{m+1} with $\xi_1 + 1$) we can choose between staying "up" or "down" by suitably selecting $x(t)$ or $y(t)$, we associate to (ξ_k, ξ_{k+1}) the label "0" when we stay "up" and the label "1" when we stay "down". Hence, to describe our m choices on the interval $(\xi_1, \xi_1 + 1)$, we use a string of m symbols, in which any element can be 0 or 1. Such 2^m strings can be ordered lexicographically and each of them may be identified with the integer between 0 and $2^m - 1$ that denotes its position in this order decreased by one. In such way we are led to work with the two-sided sequences on 2^m symbols. This allows to infer stronger consequences from a dynamical point of view, as it permits to prove the conjugacy between the space generated by the solutions of (1) and the Bernoulli shift on 2^m symbols, instead of considering the shift on just two symbols. Actually, since in [17, 19] the existence of non-degenerate continua of periodic solutions is proven, the presence of chaos on infinite symbols could be shown as well. However, since all the relevant chaotic features are present even with a finite number of symbols, we confine ourselves to the easier framework. □

Proof of Theorem 2.2. Recalling the definition of $\mathcal{C}$ in (2) and of w_η in (9), let

$$W := \{w_\eta : \eta = (\eta_i)_{i\in\mathbb{Z}} \in \Sigma_2\} \subset \mathcal{C} .$$

In order to show that there exists a conjugacy φ between σ and ψ, let us first check that $\psi(W) \subseteq W$. Applying ψ to $w_\eta(\cdot) \in W$, we get $\psi(w_\eta(t)) = w_\eta(t+1) = w_{\sigma(\eta)}(t)$ and this is an element of W, as $\sigma(\eta) \in \Sigma_2$. Let us now define $\varphi : \Sigma_2 \to W$ in the natural way, i.e., as $\varphi(\eta) = w_\eta(\cdot)$. Then the diagram

$$\begin{tikzcd} \Sigma_2 \arrow[r, "\sigma"] \arrow[d, "\varphi"'] & \Sigma_2 \arrow[d, "\varphi"] \\ W \arrow[r, "\psi"'] & W \end{tikzcd}$$

commutes, since $\varphi(\sigma(\eta)) = w_{\sigma(\eta)}(\cdot) = w_\eta(\cdot + 1) = \psi(w_\eta(\cdot)) = \psi(\varphi(\eta))$. The fact that φ is a bijection directly follows from its definition. The (uniform) continuity of φ comes from the choice of the distances $\hat{d}$ on Σ_2 and ρ on W according to (5) and (3), respectively. Indeed, given an arbitrary $\varepsilon > 0$ we have to find a $\delta > 0$ such that, for any $\eta = (\eta_i)_i, \nu = (\nu_i)_i \in \Sigma_2$ with $\hat{d}(\eta, \nu) < \delta$, then $\rho(w_\eta, w_\nu) < \varepsilon$. To such aim, let us fix an integer $m >> 0$ so that $1/2^m < \varepsilon$ and observe that, in order to have $\rho(w_\eta, w_\nu) < \varepsilon$, it is sufficient to prove that $\vartheta_m = 0$, i.e., $w_\eta \equiv w_\nu$ on $I_m = [-m, m]$. Namely, if this is the case, $\rho(w_\eta, w_\nu) < 1/2^m < \varepsilon$. Let m' be a positive integer such that $[\xi - m', \xi + m'] \supseteq [-m, m]$. Choosing $\delta = 1/2^{m'+1}$, we have that $\hat{d}(\eta, \nu) < \delta$ implies $\eta_i = \nu_i$, $\forall |i| \leq m'$. Hence, $w_\eta \equiv w_\nu$ holds on $[\xi - m', \xi + m' + 1] \supseteq [-m, m]$ and thus $\rho(w_\eta, w_\nu) < \varepsilon$. The continuity of φ^{-1} comes from the fact that φ is a continuous bijection between the compact set Σ_2 and the Hausdorff space W. Notice that, by the continuity of φ and the compactness of Σ_2, the set $W = \varphi(\Sigma_2)$ is compact, too, and thus complete.

Recalling (6), conclusion (i) immediately follows.

As regards (ii), according to [14], the Bernoulli system is chaotic in the sense of coin-tossing, Devaney and Li-Yorke. In order to apply Lemma 2.1, we only have to check that $\psi : W \to W$ is a homeomorphism. By its definition and the choice of W, it is straightforward to see that ψ is a bijection on W with inverse $\psi^{-1}(w_\eta) = w_{\sigma^{-1}(\eta)}$ and again the continuity of ψ^{-1} comes from the fact that ψ is a continuous bijection between the compact set W and the Hausdorff space W. Thus Lemma 2.1 allows to reach conclusion (ii). In particular, we stress that the compact disjoint subsets of W in the definition of coin-tossing chaos are $\mathcal{K}_i := \{w_\eta \in W : \eta_0 = i\}$, $i = 0, 1$.

Finally, (iii) is a direct consequence of the definition of φ, which indeed maps periodic sequences of symbols into periodic orbits of ψ.

The proof is complete. □

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