University of Trieste

thesis submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Robust Nonlinear Receding Horizon Control with Constraint Tightening

Off-line Approximation and Application to Networked Control Systems

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a Tania,
alla sua disponibilità e comprensione

Abstract

Nonlinear Receding Horizon (RH) control, also known as moving horizon control or nonlinear Model Predictive Control (MPC), refers to a class of algorithms that make explicit use of a nonlinear process model to optimize the plant behavior, by computing a sequence of future manipulated variable adjustments. Usually the optimal control sequence is obtained by minimizing a multi-stage cost functional on the basis of open-loop predictions. The presence of uncertainty in the model used for the optimization raises the question of robustness, i.e., the maintenance of certain properties such as stability and performance in the presence of uncertainty.

The need for guaranteeing the closed-loop stability in presence of uncertainties motivates the conception of robust nonlinear MPC, in which the perturbations are explicitly taken in account in the design of the controller. When the nature of the uncertainty is know, and it is assumed to be bounded in some compact set, the robust RH control can be determined, in a natural way, by solving a min–max optimal control problem, that is, the performance objective is optimized for the worst-case scenario. However, the use of min-max techniques is limited by the high computational burden required to solve the optimization problem. In the case of constrained system, a possibility to ensure the robust constraint satisfaction and the closed-loop stability without resorting to min-max optimization consists in imposing restricted (tightened) constraints on the the predicted trajectories during the optimization.

In this framework, an MPC scheme with constraint tightening for discrete-time nonlinear systems affected by state-dependent and norm bounded uncertainties is proposed and discussed. A novel method to tighten the constraints relying on the nominal state prediction is described, leading to less conservative set contractions than in the existing approaches. Moreover, by imposing a stabilizing state constraint at the end of the control horizon (in place of the usual terminal one placed at the end of the prediction horizon), less stringent assumptions can be posed

on the terminal region, while improving the robust stability properties of the MPC closed-loop system.

The robust nonlinear MPC formulation with tightened constraints is then used to design off-line approximate feedback laws able to guarantee the practical stability of the closed-loop system. By using off-line approximations, the computational burden due to the on-line optimization is removed, thus allowing for the application of the MPC to systems with fast dynamics. In this framework, we will also address the problem of approximating possibly discontinuous feedback functions, thus overcoming the limitation of existent approximation scheme which assume the continuity of the RH control law (whereas this condition is not always verified in practice, due to both nonlinearities and constraints).

Finally, the problem of stabilizing constrained systems with networked unreliable (and delayed) feedback and command channels is also considered. In order to satisfy the control objectives for this class of systems, also referenced to as Networked Control Systems (NCS's), a control scheme based on the combined use of constraint tightening MPC with a delay compensation strategy will be proposed and analyzed.

The stability properties of all the aforementioned MPC schemes are characterized by using the regional Input-to-State Stability (ISS) tool. The ISS approach allows to analyze the dependence of state trajectories of nonlinear systems on the magnitude of inputs, which can represent control variables or disturbances. Typically, in MPC the ISS property is characterized in terms of Lyapunov functions, both for historical and practical reasons, since the optimal finite horizon cost of the optimization problem can be easily used for this task. Note that, in order to study the ISS property of MPC closed-loop systems, global results are in general not useful because, due to the presence of state and input constraints, it is impossible to establish global bounds for the multi-stage cost used as Lyapunov function. On the other hand local results do not allow to analyze the properties of the predictive control law in terms of its region of attraction. Therefore, regional ISS results have to employed for MPC controlled systems. Moreover, in the case of NCS, the resulting control strategy yields to a time-varying closed-loop system, whose stability properties can be analyzed using a novel regional ISS characterization in terms of time-varying Lyapunov functions.

Acknowledgements

Almost all the research pertaining to this thesis was done in the Control Laboratory at the University of Trieste. The thesis has been completed after three years of development and deployment of some of the academic ideas expounded herein. It is the product of my interaction with a large number of people, with whom I have had the pleasure to discuss a wide range of topics in control, engineering, mathematics and physics.

I'd first like to thank my supervisor, Prof. Thomas Parisini, for his guidance and support. Throughout my studies he has been a source of inspiration and advices, giving me the latitude to be an independent researcher. I am deeply indebted to him for being a great mentor and, most important, a caring and trusted friend.

I would also like to thank Prof. Lalo Magni, who introduced me to the world of nonlinear MPC, and dr. Davide Raimondo, without whom this work would have never seen the light.

In addition, I would like to thank Prof. Franco Blanchini, for the many interesting and productive discussions we have had.

The Control Lab in Trieste has been my home for the last three years and everybody who has passed through has contributed to my understanding of control systems in some way or another. In particular I would like to thank Marco Filippo, Gianfranco Fenu, Felice Andrea Pellegrino, Daniele Casagrande, Andrea Petronio, Riccardo Ferrari and all the other colleagues who overlapped my PHD experience in Trieste.

I would also like to express my thanks to the administrative people at the Department of Electric, Electronic and Computer Engineering, for their support in the process of my research. In particular, I gratefully acknowledge Piero Riosa, Giovanni Lucci and Germana Trebbi, whose mastery and efficiency of administrative matters took lots of worries from my head.

During the last three years I had the opportunity to extend my network of mentors and friends, by attending several workshops, schools and conferences all over the world. I will never forget all the people I met in Bertinoro, Seattle, London, Pavia and Cancun. I reserve special thanks to Claudio Vecchio for the time spent together in Bertinoro, Pavia and Pula.

I would also like to take this opportunity to thank Sertubi S.p.A. - Duferco Group - that have provided me with financial support during the course of my research. Particular thanks go to Daniele Deana and Dario Majovsky, who patiently explained to me what a PLC is and why control systems need to be simpler than just simple for practical deployment. Working in industry before finishing my thesis helped me in grounding my ideas in reality. Thanks also to the members of the R & D Dept. at Danieli Automation S.p.A. for the time spent together and the support they provided. Particular thanks go to Lorenzo Ciani, Luciano Olivo and Alessandro Ardesi.

Finally, I would like to acknowledge the guests of Collegio Marianum in Opicina: Giovanni, Robert, Federico, Claudio and Manuel, for making my stay in Trieste unforgettable.

I would also like to thank all my friends in San Vito, for the pleasant moments spent together during these years.

Last but not least, my deep gratitude also goes to all the members of my family, in particular to my mom Alda and my father Nadir, who instilled in me their strong work ethic, and to Tania, which has always been my major source of inspiration. Without their loving support, the whole thesis would have been impossible.

Contents

Chapter 1

Introduction

Model Predictive Control (MPC) refers to a class of algorithms which make explicit use of a process model to optimize the plant behavior, by computing a sequence of future manipulated variable adjustments.

Originally developed to meet the specialized control needs of power plants and chemical plants, MPC technology can now be found in a wide variety of application areas including food processing, automotive, aerospace and medical applications, [97]. MPC has gained increasing popularity in industry, mainly due to the ease with which constraints can be included in the controller formulation.

It is worth to note that this control technique has achieved increasing attention among control practitioners, since the 1980s, in spite of the original lack of theoretical results concerning some crucial points such as stability and robustness.

In fact, a solid theoretical basis for this technique started to emerge more than 15 years after it appeared in industry. Several recent publications provide a good introduction to theoretical and practical issues associated with MPC technology (see e.g. the books [19, 37, 71, 104], and the survey papers [31, 77, 81, 97, 103]).

Figure 1.1 depicts the basic principle of Model Predictive Control, which usually relies on the following two ideas, [19]:

1) *Model-based optimization*: Relying on measurements obtained at time t (let us assume, at this point, that the whole state vector $x_t \in \mathbb{R}^n$ is measured), the controller predicts the future dynamic behavior of the system over a prediction horizon $N_p \in \mathbb{N}$ and determines

(over a control horizon $N_c \leq N_p$) an input sequence such that a predetermined open-loop performance objective functional is optimized. Optionally, also constraints on input variables ($u_t \in U$) and on state trajectories ($x_t \in X$) are imposed. If there were no disturbances and no model-plant mismatch, and if the optimization problem could be solved for infinite horizons, then one could apply the computed input sequence for all times from t to $t + N_c - 1$ in open-loop. However, this is not possible in general. Indeed, due to external perturbations and model uncertainty, the true system behavior is different from the predicted one;

2) *Receding Horizon (RH) paradigm*: In order to incorporate some feedback mechanism, the open-loop input sequence obtained by the optimization will be implemented only until the next state measurement becomes available. The time difference between the recalculation/measurements can vary, however often it is assumed to be fixed (typically, a state measurement is available at each recalculation instant, such that only the first control move of the computed sequence is applied to the plant). Using the new state measurement x_{t+1} at time $t + 1$, the whole procedure (comprising prediction and optimization) is repeated to find a new input sequence with control and prediction horizons moved forward.

Since the Receding Horizon strategy and the model-based optimization are intrinsically connected and represent the basic ingredients of the method, MPC is also called, with slight abuse of terminology, RH control or moving horizon control.

Remarkably, the described underlying procedure applies both in linear and nonlinear MPC formulations. However, apart from those basic common features, linear and nonlinear MPC are usually approached separately in literature, mainly due to the implementation issues posed by the nonlinear optimization and to the different theoretical tools needed to prove the closed-loop stability in the two frameworks.

Linear MPC refers to a family of MPC schemes in which linear models are used to design the controller. Linear MPC approaches have found successful applications, especially in the process industries [97]. By now, linear MPC is fairly mature (see [81] and the reference therein) from the theoretical point of view.

Many systems are, however, in general inherently nonlinear. In addition, tighter environmental regulations and demanding economical considerations in the process industry require to operate systems closer to the boundary of the admissible operating region. In these cases, linear models are often inadequate to describe the process dynamics and the nonlinearities have to be taken in account. Moreover, in practical applications, the assumption that the system behavior is identical to the model used for prediction is unrealistic. In fact, model/plant mismatch or unknown disturbances are always present. The introduction of uncertainty in the system description raises the question of robustness, i.e., the maintenance of certain properties such as stability and performance in the presence of uncertainty. These needs motivate the conception of robust nonlinear MPC schemes (see e.g., [6, 77, 103]), that stem from the consideration of the uncertainties directly in the synthesis of the controller. The incorporation of uncertainties in the control formulation adds complexity to the MPC design, in particular in the constrained case, because the satisfaction of the constraints must be ensured for any possible realization of uncertainty.

In the remainder of the present chapter, we will describe the different solutions proposed in the current literature to cope with the presence of state and input constraints, as well as the robust formulations aimed to cope with uncertainties, due, for instance, to the presence of external disturbances of poor knowledge of the process dynamics. Finally, we will introduce the original contributions presented in the thesis in the framework of robust nonlinear MPC.

1.1 Overview on Robust Nonlinear MPC

This section aims to describe the fundamental results raised in the last few years in the framework of Model Predictive Control of nonlinear discrete–time systems. Before reviewing the main contributions related to robust nonlinear MPC, let us introduce its unconstrained *nominal* formulation, which does not explicitly account for uncertainty and constraints in the problem setup.

1.1.1 MPC formulation for nominal nonlinear systems

Although the problem of designing MPC schemes for unconstrained and unperturbed systems appears simple at first sight, many different different formulation have been proposed to achieve closed-loop stability in the nonlinear setup. Nonetheless, all the existent implementable MPC formulations for discrete-time system rely on the solution, at each sampling instant, of a Finite Horizon Optimal Control Problem (FHOCP), which is introduced, in its simplest form, in Definition 1.1.1 above.

Consider the nonlinear discrete-time dynamic system

$$x_{t+1} = f(x_t, u_t, \upsilon_t),\ \ t \in \mathbb{Z}_{\geq 0},\ \ x_0 = \overline{x}_0\,, \tag{1.1}$$

where $x_t \in \mathbb{R}^n$ denotes the state vector, $u_t \in \mathbb{R}^m$ the control vector and $\upsilon_t \in \Upsilon$ is an uncertain exogenous input vector, with $\Upsilon \subset \mathbb{R}^r$ compact and $\{0\} \subset \Upsilon$. Assume that state and control variables are subject to the following constraints

$$x_t \in X,\ \ t \in \mathbb{Z}_{\geq 0}\,, \tag{1.2}$$

$$u_t \in U,\ \ t \in \mathbb{Z}_{\geq 0}\,, \tag{1.3}$$

where X and U are compact subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, containing the origin as an interior point.

Given the system (1.1), let $\hat{f}(x,u)$, with $\hat{f}(0,0) = 0$, denote the *nominal* model used for control design purposes. Moreover, when it will be necessary to point out the dependence of a nominal trajectory on the initial condition x_t with a specific input sequence $\mathbf{u}_{t,t+i-1}$, we will also use the notation $\hat{x}(i, x_t, \mathbf{u}_{t,t+i-1}) = \hat{x}_{t+i|t}$.

The complete list of notations used in the sequel and some basic definitions are given in the Appendix A.

Definition 1.1.1 (FHOCP)**.** *Given a state measurement x_t at time t, two positive integers $N_c, N_p \in \mathbb{Z}_{>0}$, an auxiliary state-feedback control law $\kappa_f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, a stage cost function $h(\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a terminal penalty function $h_f(\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and a compact set $X_f \subset \mathbb{R}^n$, the Finite Horizon Optimal Control Problem (FHOCP) consists in minimizing, with respect to a sequence of control moves $\mathbf{u}_{t,t+N_c-1}$ the performance index*

$$J_{FH}(x_t, \mathbf{u}_{t,t+N_c-1}, N_c, N_p) = \sum_{l=t}^{t+N_c-1} h(\hat{x}_{l|t}, u_l) + \sum_{l=t+N_c}^{t+N_p-1} h(\hat{x}_{l|t}, \kappa_f(\hat{x}_{l|t})) + h_f(\hat{x}_{t+N_p|t}) \quad (1.4)$$

subject to:

1) *the nominal state dynamics initialized with* $\hat{x}_{t|t} = x_t$*;*

2) *(optionally) the control variable and state constraints;* $u_{j-1} \in U$, $\hat{x}_{t+j|t} \in X$, $j \in \{1, \ldots, N_c\}$*;*

3) *the terminal state constraints* $\hat{x}_{t+N_p|t} \in X_f$.

□

Then the RH strategy consists in applying to the plant the input $u_t = \kappa_{RH}(x_t) = u^{\circ}_{t|t}$, where $u^{\circ}_{t|t}$ is the first element of the optimal sequence $\mathbf{u}^{\circ}_{t,t+N_c-1|t}$ (implicitly dependent on x_t) which gives the minimum value of the multi-stage cost functions, that is

$$J^{\circ}_{FH}(x_t) = J_{FH}(x_t, \mathbf{u}^{\circ}_{t,t+N_c-1}, N_c, N_p) = \min_{\mathbf{u}_{t,t+N_c-1}} J_{FH}(x_t, \mathbf{u}_{t,t+N_c-1}, N_c, N_p) \quad (1.5)$$

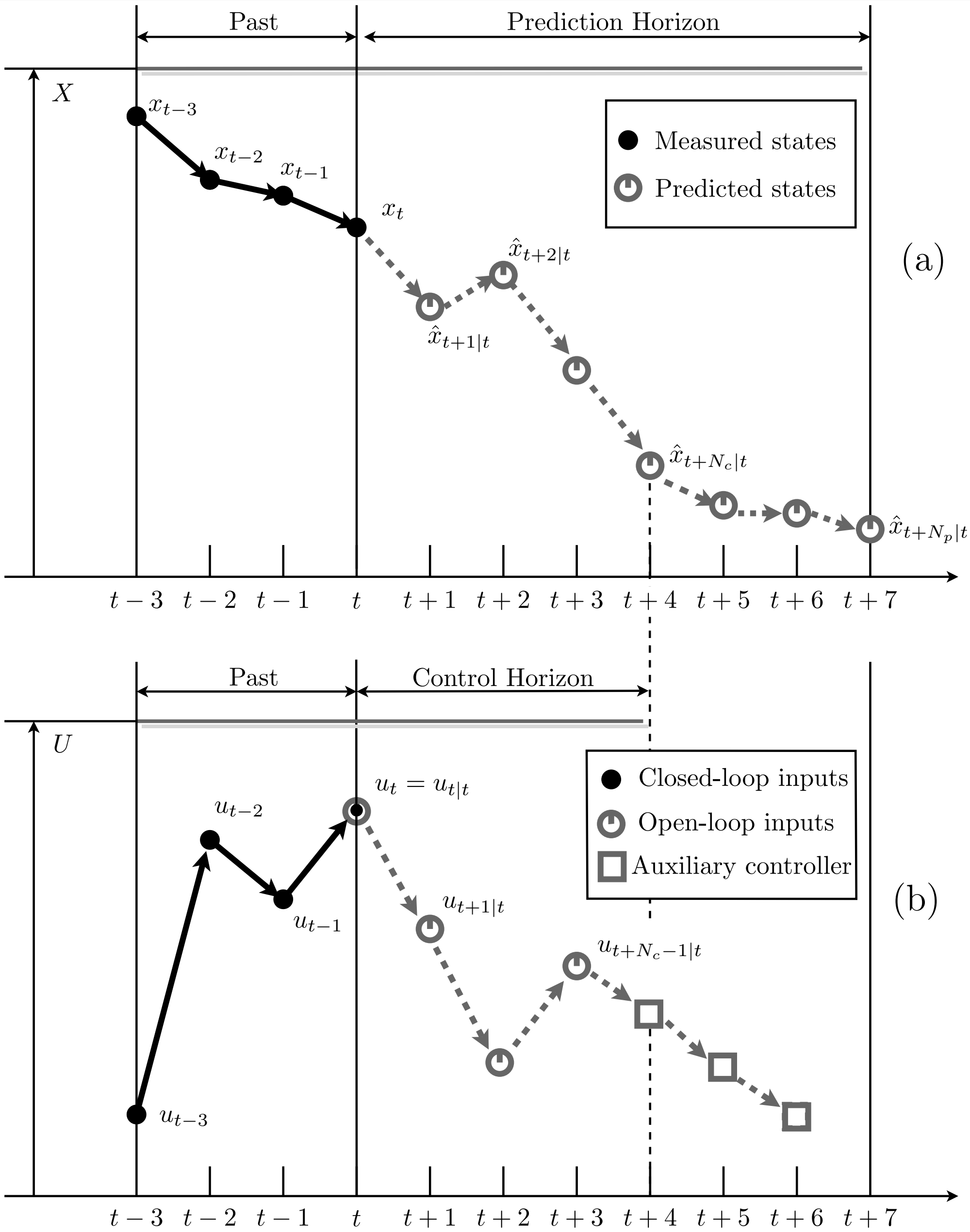
subject to the specified constraints.

Apart from the length of control and prediction horizon, N_c and N_p, what distinguishes the various proposals are the different design criteria for the stage cost h, the terminal cost h_f and/or the terminal constraint X_f in the FHOCP.

First, let us consider the case in which the dynamics of the system are perfectly known and there are no exogenous perturbations (i.e., $\upsilon_t = 0$, $\forall t \in \mathbb{Z}_{\geq 0}$ and $\hat{f}(x, u) = f(x, u, 0)$, $\forall (x, u) \in X \times U$).

The simplest condition that can be posed in the FHOCP to guarantee the nominal stability of the closed-loop system consists in a terminal equality constraint. In this version of MPC, the following constraint is introduced $\hat{x}_{t+N_c|t} \in X_f = \{0\}$ (i.e., the state at the end of the control horizon is forced to reach the origin). The terminal equality constrained MPC can be regarded as the earliest and, conceptually, the simplest tool to guarantee the stability of the controlled system, whenever feasibility is satisfied. The first proposal of this form of MPC for nonlinear, discrete–time systems was made in [54]. This paper is particularly important, since it provides a definitive stability analysis for this version of discrete-time receding horizon control

Figure 1.1 Scheme of the underlying principle of Model Predictive Control, based (a) on the prediction of system trajectories over an horizon of N_p steps and (b) on the computation of open-loop sequences over a control horizon of $N_c \leq N_p$ steps.



and shows that the optimal value function $J^{\circ}_{FH}(x_t, \mathbf{u}_{t,t+N_c-1}, N_c)$ associated with the finite horizon optimal control problem approaches that of the infinite horizon problem as the horizon approaches infinity. Other important papers on terminal constrained MPC are [24, 76] and [30]. Due to the simplicity and limitedness of this formulation, a detailed description of this approach is omitted. Although this approach appears quite simple, it is capable to stabilize systems that cannot be stabilized by feedback control laws continuous in the state [78].

One of the earliest proposals to overcome the feasibility issue of terminal equality constrained MPC, thus enlarging the domain of attraction of the resulting closed-loop system, consists in the use of a terminal cost function h_f. In this version of model predictive control no terminal constraints are introduced, so that $X_f = \mathbb{R}^n$. If the system under analysis is globally stabilizable, the terminal cost can be constructed as a global control Lyapunov function (i.e., a Lyapunov function associated to the system in closed loop with an auxiliary nominally stabilizing controller [89]). Related works are [3] and [72], in which a control horizon $N_c \in \mathbb{Z}_{>0}$ and a cost horizon (also referred as prediction horizon) $N_p \in \mathbb{Z}_{>N_c}$ are employed to show that closed-loop stability ensues if the chosen N_p is large enough. Hence, in the latter approaches, the terminal cost function $h_f(\cdot)$ is not given analytically, but is evaluated by extending the prediction horizon.

In the more recent work [42], the explicit characterization of a control Lyapunov function for the system is not strictly required as in [72], since a generic terminal cost is employed, which can be constructed assuming only the existence of a global value function for the system bounded by a linear $\mathcal{K}$–function, with argument the distance from the target set (the origin in the zero–regulation problem). On the other side, similarly to the result in [72], global stability can be ensured only if the given cost horizon is long enough.

Finally, a further approach to design stabilizing RH schemes consists in imposing terminal inequality or terminal set constraints. In this version of MPC, the terminal constraint set $X_f \subset \mathbb{R}^n$ is chosen as a neighborhood of the origin and no terminal cost is used to penalize the finite horizon value function. The purpose of this formulation is to steer the state to X_f in finite time, in nominal conditions. Inside X_f, a local stabilizing controller κ_f is employed; this form of model predictive control is therefore sometimes referred to as *Dual Mode*, and was first proposed, in the context of constrained, continuous-time, nonlinear systems, in the seminal paper [80].

As far we have discussed about stabilizing MPC schemes for unconstrained nonlinear system. In many applications the state of the system and/or the input variables are subject to constraints, hence, it is of great interest to develop control strategies capable to keep the state and the input within the prescribed bounds.

Now, assume that state and control variables are subjected to the constraints (1.2) and (1.3). In the context of nonlinear MPC, a natural way to incorporate these requirements consists in directly imposing constraints (1.2) on the predicted state and input sequences. In this way the constrained MPC leads to the formulation of a constrained optimization problem at each time step. Different solutions have been proposed to provide stability results in presence state and input constraints, but basically all these approaches are based on the imposition of both a terminal cost function and a terminal constraint.

In the context of receding horizon control of constrained continuous time systems, the most used MPC formulation consists in the position of both terminal cost and terminal constraints. Such a formulation was firstly proposed in [25] where the terminal constraint X_f, has been chosen as a positively invariant set for the nonlinear system, satisfying the conditions $X_f \subset X$ and $\kappa_f(x) \in U, \forall x \in X_f$, where κ_f is an auxiliary locally stabilizing control law. The terminal cost is chosen as the local Lyapunov–function h_f associated to the linear optimal static state feedback law for the linearized system at the origin. This approach is referred by the authors as *quasi–infinite horizon* predictive control because the finite horizon optimal control problem approximates the full infinite–horizon constrained one.

For the case when the system is discrete–time and there are state and control constraints, in [89] and [88] a generic locally stabilizing control law κ_f has been used as auxiliary controller, while the terminal cost function h_f has been chosen as a local Lyapunov function for the stabilized system. Finally, it has been suggested to choose the terminal constraint set X_f as a positively invariant sub-level set of h_f under the closed–loop system with κ_f.

1.1.2 Robust RH control of nonlinear systems with constraints

The introduction of uncertainty in the system description raises the question of robustness, i.e. the maintenance of certain properties such as stability and performance in presence of uncertainty. As studied in [41], a nominal stabilizing MPC may exhibit zero-robustness.

Earliest studies on the robustness of RH controlled systems do not consider the presence of constraints, establishing that if a global Lyapunov function for the nominal closed–loop system maintains its descent property if the disturbance (uncertainty) is sufficiently small, then perturbed (uncertain) closed-loop system preserves stability. In this respect, the inherent robustness of RH controllers for unconstrained nonlinear discrete–time systems has been investigated in [30] and [107]. By inherent robustness we mean robustness of the closed-loop system using model predictive control designed ignoring uncertainty.

However, when constraints on states and controls are present, it is necessary to ensure, in addition, that the constraints are satisfied also in presence of uncertainties. This adds an extra level of complexity.

When the nature of the uncertainty is known, and it is assumed to be bounded in some compact set, the robust MPC can be determined, in a natural way, by solving a min–max optimal control problem, as proposed in the seminal paper [79]. It consists basically in imposing that the state constraints, as well as the terminal set constraint, are satisfied for all the possible realization of uncertainties by a feasible sequence of controls. The complexity of this problem increases exponentially with horizon length. A defect of the classical formulation of MPC for uncertain systems relies on the open-loop nature of the optimal control problem; in order to overcome this limitation, recent papers propose to optimize over a parametrized family of control feedback strategies rather then over a sequence of control moves, [38, 74, 106]. In this approach, a vector of feedback control policies is considered in the minimization of the cost, for the worst case perturbations. This closed-loop method allows to take into account the reaction to the effect of the uncertainty in the predictions at expense of a practically intractable optimization problem. In this context robust stability issues have been recently studied and some novel contributions on this topic have appeared in the literature [32, 44, 45, 73, 57, 64, 68]. Although the solid underlying theoretical basis, the high computational burden required to solve min-max optimizations has limited the application of min-max nonlinear MPC to small dimensional problems or very slow plant. The implementation issue still remains an open problem in the min-max literature. Therefore, other approaches have been tackled to ensure robust closed-loop stability in nonlinear MPC.

In order to alleviate the implementation issues of min-max MPC, open-loop formulations with restricted constraints have been conceived (see for instance [26], for the linear case and [43, 66] for the nonlinear one). This method for the design of robust MPC consists in minimizing a nominal performance index, while imposing the fulfillment of tightened constraints on the trajectories of the nominal system. In this way, the nominal constraint are satisfied by the perturbed (uncertain) system when the optimal sequence is applied to the plant. The main drawback of this open-loop strategy is the large spread of trajectories along the optimization horizon due to the effect of the disturbances and leads to very conservative solutions or even to unfeasible problems. Indeed, the dramatic reduction of computational effort at the optimization stage can be obtained at the cost of an increase of conservativeness: in order to enforce the robust constraint satisfaction, restricted set constraints are imposed to the predicted state as well as a restricted terminal state constraint. Due to the aforementioned limitations of existent schemes, the development of more efficient and less conservative constraint-tightening algorithms is a very active area of research [43, 62].

When uncertainties affect the system dynamics, the stability analysis of the closed-loop systems given by both min-max or constraint-tightening MPC schemes is usually carried out in the framework of Input-to-State Stability (ISS). The concept of ISS was first introduced in [109, 110] and then further exploited by many authors in view of its equivalent characterization in terms of robust stability, dissipativity and input-output stability (see e.g. [50, 51, 49, 61, 85, 86]). The ISS approach allows to analyze the dependence of state trajectories of nonlinear systems on the magnitude of inputs, which can represent control variables or disturbances.

Now, several variants of the ISS property have been conceived and applied in different contexts (see [34, 51, 111, 112]). Typically, in MPC the ISS property is characterized in terms of Lyapunov functions, both for historical and practical reasons. Indeed, since the first theoretical results on the stability MPC, [54, 76], the optimal value function of the FHOCP was employed as a Lyapunov function for establishing the stabilizing properties of RH control schemes applied to time-varying, constrained, nonlinear, discrete-time systems. Nowadays, the value function is universally employed as a Lyapunov function for studying the ISS property of nonlinear MPC (see for example [43, 62, 63, 64, 66, 67, 68]). Note that, in order to study the ISS property of MPC closed-loop systems, global results are in general not useful because, due to the presence of state and input constraints, it is impossible to establish global bounds for the finite horizon cost used as Lyapunov function. On the other hand local results (see e.g., [50, 51]) do not allow to analyze the properties of the predictive control law in terms of its region of attraction. Therefore, regional ISS results have been recently introduced to apply the ISS theory to MPC closed-loop systems.

In this work, we will extensively use the ISS concept to analyze the stability properties of several novel MPC schemes, based on constraint tightening, in presence of different classes of uncertainties.

In particular, we will study the closed-loop behaviour of MPC-controlled systems affected by state-dependent and bounded additive uncertainties, delays in the feedback and control information paths, and perturbations due, e.g., by the off-line approximation of the exact RH control law. In the sequel, we will describe the structure of the thesis and the content of each chapter, evidencing the original contributions in the field of robust MPC for nonlinear discrete-time systems.

1.2 Contents and Structure of The Thesis

The present thesis is mainly concerned with the use of the Input-to-State Stability (ISS) as a tool to assess the robust stability properties of a class of MPC schemes based on constraint tightening. This technique will be studied and analyzed in detail, and several improvements will be proposed to reduce the inherent conservatism of the method. Indeed, the conception of methodologies to alleviate this drawback represents a key point toward the possibility to use this technique as an alternative to min-max MPC for uncertain nonlinear systems with fast dynamics. Indeed, due to ease of implementation and to the reduced computational burden required by the constraint tightening method, this class of algorithms is more attractive than min-max formulations for practical deployment. In the same direction, we will also consider the possibility to completely remove the need for the on-line optimization by approximating off-line the control law. In this respect, we will establish a set of conditions under which the closed-loop system with the approximate controller would preserve the Input-to-State practical stability property. Finally, the constraint tightening MPC, together with a delay compensation strategy, will be used to stabilize networked systems. In this case a novel characterization of ISS in terms of time-varying Lyapunov functions will be proposed to analyze the closed-loop behavior of the devised scheme.

The Thesis is organized as follows.

In Chapter 2 we will introduce the notion of regional ISS, together with its characterization in terms of Lyapunov functions. Moreover, the regional ISS property will be characterized by means of time-varying Lyapunov functions, allowing to extend the ISS analysis to time-varying systems.

The regional ISS concept will be used in Chapter 3 to prove the robust stability of an MPC scheme with constraint tightening for systems affected by state-dependent and norm bounded uncertainties. In this setup, a novel method to tighten the constraints relying on the nominal state prediction will be proposed, leading to less conservative set contractions than in the existing approaches. Moreover, by imposing a stabilizing state constraint at the end of the control horizon (in place of the usual terminal one placed at the end of the prediction horizon), it will be shown that less stringent assumptions can be posed on the terminal region, while improving the robust stability properties of the MPC closed-loop system.

In Chapter 4 the robust nonlinear MPC formulation with tightened constraints will be used to design off-line approximate feedback laws able to guarantee the practical stability of the closed-loop system. In this framework, we will also address the problem of approximating possibly discontinuous control laws, thus overcoming the limitation of existent approximation scheme which assume the continuity of the RH controller (whereas this condition is not always verified in practice).

Finally, the problem of stabilizing constrained systems with networked unreliable (and delayed) feedback and command channels will be addressed in Chapter 5. In order to satisfy the control objectives for this class of systems, also referenced to as Networked Control Systems (NCS's), a control scheme based on the combined use of MPC with a delay compensation strategy will be proposed and analyzed. Notably, the resulting control strategy yields to a time-varying closed-loop system, whose stability properties can be analyzed using the regional ISS characterization in terms of time-varying Lyapunov functions described in Chapter 2.

The results discussed in the thesis are based on the following original contributions by the author:

PIN, G., RAIMONDO, D., MAGNI, L., AND PARISINI, T. Robust model predictive control of nonlinear systems with bounded and state-dependent uncertainties. *IEEE Trans. on Automatic Control, to appear* (2009)

PIN, G., AND PARISINI, T. Networked predictive control of uncertain constrained nonlinear systems: recursive feasibility and input-to-state stability analysis. *Submitted for publication on IEEE Trans. on Automatic Control* (2009)

PIN, G., PARISINI, T., MAGNI, L., AND RAIMONDO, D. Robust receding-horizon control of nonlinear systems with state dependent uncertainties: an input-to-state stability approach. In *Proc. American Control Conference* (2008), pp. 1667 – 1672

PIN, G., FILIPPO, M., PELLEGRINO, F. A., AND PARISINI, T. Approximate off-line receding horizon control of constrained nonlinear discrete-time systems. In *Submitted to the European Control Conference* (Budapest, 2009)

PIN, G., AND PARISINI, T. Stabilization of networked control systems by nonlinear model predictive control: a set invariance approach. In *Proc. of International Workshop on Assessment and Future Directions of* NMPC (Pavia, 2008)

PIN, G., AND PARISINI, T. Set invariance under controlled nonlinear dynamics with application to robust RH control. In *Proc. of the IEEE Conf. on Decision and Control* (2008), pp. 4073 – 4078

Chapter 2

Regional Input-to-State Stability for NMPC

Input-to-state stability (ISS) is one of the most important tools to study the dependence of state trajectories of nonlinear continuous and discrete time systems on the magnitude of inputs (which can represent control variables or disturbances) and on the initial conditions (see e.g., [51, 87, 110, 112]). Due to the possibility to characterize the ISS in terms of Lyapunov functions, the ISS has been widely used to analyze stabilizing properties of closed-loop systems in that the controller is designed accordingly to Lyapunov-based techniques, or for which a control Lyapunov function can be constructed with ease. In the framework of MPC controllers, it is well known that the optimal value function of the FHOCP can serve as a Lyapunov function to study the stability of the closed-loop system.

However, in order to analyze the ISS properties of a system controlled by an MPC policy, global results are in general not useful in view of the presence of state and input constraints. On the other hand, local results do not allow to characterize the region of attraction of the predictive control law. Then, in this chapter, the notion of regional-ISS is introduced (see also [75]), and the equivalence between the ISS property and the existence of a suitable Lyapunov function is established. Notably, this Lyapunov function is not required to be continuous nor to be upper bounded in the whole region of attraction. An estimation of the region where the state of the system converges asymptotically is also given.

The ISS results presented in this chapter will be successively used in the dissertation to characterize the stability properties of a specific class of robust MPC algorithms for constrained discrete-time nonlinear systems, based on constraint tightening and open-loop optimization (i.e., the decision variables consist in a sequence of control actions over a finite time horizon, while the prediction is performed on the basis of a nominal model of the controlled system).

Furthermore, we will use the ISS tool to develop an off-line approximate MPC control law capable to guarantee the input-to-state practical stability of the closed-loop system toward an equilibrium which may be not stabilizable by continuous static state-feedback laws.

Finally, a novel characterization of the regional-ISS property in terms of time-varying Lyapunov functions will be here introduced, and then used in Chapter 5 to study the closed-loop stability of Networked Control Systems (i.e., systems in which the informations are exchanged between sensors, controller, and actuators over an unreliable packet-based communication network with delays), in which an MPC controller is used in combination with a Network Delay Compensation (NDC) strategy to mitigate the perturbing effect of communication delays.

2.1 Problem Statement and Definitions

Consider the discrete-time autonomous perturbed nonlinear dynamic system described by

$$x_{t+1} = g(x_t, \upsilon_t), \;\; x_0 = \overline{x}_0, \, t \in \mathbb{Z}_{\geq 0},$$

where $g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ is a nonlinear function, while $x_t \in \mathbb{R}^n$ and $\upsilon_t \in \mathbb{R}^q$ denote respectively the state vector and an exogenous (unmeasurable) disturbance term. In order to point out the effect of the disturbance term on the state evolution, given an initial condition $x_0 = \overline{x}_0$ and a disturbance sequence $\boldsymbol{\upsilon}_{0,t-1}$ from time 0 to $t-1$, we will denote the state vector at time t as $x_t = x(t, \overline{x}_0, \boldsymbol{\upsilon}_{0,t-1})$. The transition function g and the disturbance are supposed to fulfill the following assumption.

Assumption 2.1.1.

1) The origin is an equilibrium point (i.e., $g(0,0) = 0$*);*

2) The disturbance υ_t *is such that* $\upsilon_t \in \Upsilon$, $\forall t \in \mathbb{Z}_{\geq 0}$*, where* $\Upsilon \subseteq \mathcal{B}^q(\overline{\upsilon})$*, where* $\overline{\upsilon} \in (0, \infty)$ *is a finite scalar; moreover* Υ *contains the origin as interior point.* □

The following regularity assumption will be needed.

Assumption 2.1.2. *For every $t \in \mathbb{R}_{>0}$, the state trajectories $x(t, \overline{x}_0, \boldsymbol{\upsilon}_{0,t-1})$ of the system (2.1) are continuous in $\overline{x}_0 = 0$ and $\upsilon = 0$ with respect to the initial condition $\overline{x}_0$ and the disturbance sequence $\boldsymbol{\upsilon}_{0,t-1}$.* □

Let us introduce the following definitions. For more details about the notation and the acronyms used in the following, the reader is referred to Section A.3 of the Appendix. In particular, the notion of Robust Positively Invariant (RPI) set given in Definition A.3.1 will be used.

Definition 2.1.1 (UAG in Ξ)**.** *Given a compact set $\Xi \in \mathbb{R}^n$ including the origin as interior point, the system (2.1), with $\boldsymbol{\upsilon} \in \mathcal{M}_\Upsilon$, satisfies the Uniform Asymptotic Gain (UAG) property in Ξ, if Ξ is a RPI set for system (2.1) and if there exists a $\mathcal{K}$-funtion γ such that for any arbitrary $\epsilon \in \mathbb{R}_{>0}$ and $\forall\, \overline{x}_0 \in \Xi$, $\exists T^{\epsilon}_{\overline{x}_0}$ finite such that*

$$|x(t, \overline{x}_0, \boldsymbol{\upsilon})| \leq \gamma(||\upsilon||) + \epsilon,$$

for all $t \geq T^{\epsilon}_{\overline{x}_0}$. □

Definition 2.1.2 (LS)**.** *System (2.1), with $\boldsymbol{\upsilon} \in \mathcal{M}_\Upsilon$, satisfies the Local Stability (LS) property if for any arbitrary $\epsilon \in \mathbb{R}_{>0}$, $\exists \delta \in \mathbb{R}_{>0}$ such that*

$$|x(t, \overline{x}_0, \boldsymbol{\upsilon})| \leq \epsilon, \ \ \forall t \in \mathbb{Z}_{\geq 0},$$

for all $|\overline{x}_0| \leq \delta$ and all $\Upsilon \subseteq \mathcal{B}^r(\delta)$. □

Definition 2.1.3 (ISS in Ξ)**.** *Given a compact set $\Xi \subset \mathbb{R}^n$ including the origin as interior point, the system (2.1), with $\boldsymbol{\upsilon} \in \mathcal{M}_\Upsilon$, is said to be Input-to-State Stable (ISS) in Ξ if Ξ is a RPI set for system (2.1) and if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that*

$$|x(t, \overline{x}_0, \boldsymbol{\upsilon})| \leq \beta(|\overline{x}_0|, t) + \gamma(||\boldsymbol{\upsilon}||), \ \ \forall \overline{x}_0 \in \Xi, \forall t \in \mathbb{Z}_{\geq 0}. \tag{2.1}$$

□

Note that, by causality, the same definitions of ISS in Ξ would result if (2.1) is replaced by

$$|x(t, \overline{x}_0, \boldsymbol{\upsilon})| \leq \beta(|\overline{x}_0|, t) + \gamma(||\boldsymbol{\upsilon}_{[t-1]}||), \ \ \forall \overline{x}_0 \in \Xi, \forall t \in \mathbb{Z}_{\geq 0},$$

where $\boldsymbol{\upsilon}_{[t-1]}$ denotes a truncation of $\boldsymbol{\upsilon}$ at the time instant $k-1$.

It can be proven that if a system satisfies both the UAG in Ξ and the LS properties, then it is ISS in Ξ (see [34]). This result, originally developed under the assumption of continuity of $g(\cdot,\cdot)$, can be applied also to discontinuous systems if a bound on the trajectories can be established. In particular, the trajectories are bounded if the set Ξ is RPI under g for all the possible realizations of uncertainties. Hence, the following result can be stated.

Lemma 2.1.1. *Suppose that Assumption 2.1.1 holds. System (2.1) is ISS in Ξ if and only if the properties UAG in Ξ and LS hold.*

The proof of this theorem can be found in [34] for discrete-time systems. We point out that if also Assumption 2.1.2, then the LS property is redundant. In fact, the following proposition holds.

Proposition 2.1.1. *Under Assumptions 2.1.1 and 2.1.2, if the system (2.1) is UAG in Ξ, then it verifies the LS property.*

Conversely, Assumption 2.1.2 is necessary in order to have ISS. In fact, in view of (2.1), if the solution of (2.1) is not continuous in $(x, \upsilon) = (0, 0)$, then the ISS property does not hold.

2.2 Regional ISS Characterization in Terms of Lyapunov Functions

The regional-ISS stability property will now be associated to the existence of a suitable regional ISS-Lyapunov function (in general, a-priori non smooth) defined as follows.

Definition 2.2.1 (ISS-Lyapunov Function). *Given the system (2.1) and a pair of compact sets* $\Xi \subset \mathbb{R}^n$ *and* $\Omega \subseteq \Xi$*, with* $\{0\} \subset \Omega$*, a function* $V(\cdot): \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *is called a (Regional) ISS-Lyapunov function in* Ξ*, if there exist some* $\mathcal{K}_\infty$*-functions* α_1*,* α_2*,* α_3*, and two* $\mathcal{K}$*-function* σ_1 *and* σ_2 *such that*

1) Ξ *is a compact RPI set including the origin as an interior point;*

2) the following inequalities hold $\forall \upsilon \in \Upsilon$

$$V(x) \geq \alpha_1(|x|), \qquad \forall x \in \Xi, \tag{2.2}$$

$$V(x) \leq \alpha_2(|x|) + \sigma_1(|\upsilon|), \qquad \forall x \in \Omega, \tag{2.3}$$

$$V(g(x, \upsilon)) - V(x) \leq -\alpha_3(|x|) + \sigma_2(|\upsilon|), \ \forall x \in \Xi, ; \tag{2.4}$$

3) there exist some suitable $\mathcal{K}_\infty$*-functions* ϵ *and* ρ *(with* ρ *such that* $(id - \rho)$ *is a* $\mathcal{K}_\infty$*-function, too) such that the following compact set*

$$\Theta \triangleq \{x : V(x) \leq b(\overline{\upsilon})\}, \tag{2.5}$$

verifies the inclusion $\Theta \subset \Omega \backsim \mathcal{B}^n(c)$*, for some suitable constant* $c \in \mathbb{R}_{>0}$*, where* $b(s) \triangleq \alpha_4^{-1} \circ \rho^{-1} \circ \sigma_4(s)$*,* $\alpha_4 \triangleq \underline{\alpha}_3 \circ \overline{\alpha}_2^{-1}$*,* $\underline{\alpha}_3(s) \triangleq \min(\alpha_3(s/2), \epsilon(s/2)), \overline{\alpha}_2(s) \triangleq \alpha_2(s) + \sigma_1(s)$*,* $\sigma_4 = \epsilon(s) + \sigma_2(s)$ *and* $\overline{\upsilon} \triangleq \max_{\upsilon \in \Upsilon}\{|\upsilon|\}$*.* □

Notably, the ISS-Lyapunov inequalities (2.2),(2.3) and (2.4) differ from those posed in the original Regional-ISS formulation [75], since an input-dependent upper bound is admitted in (2.3) (thus allowing for a more general characterization).

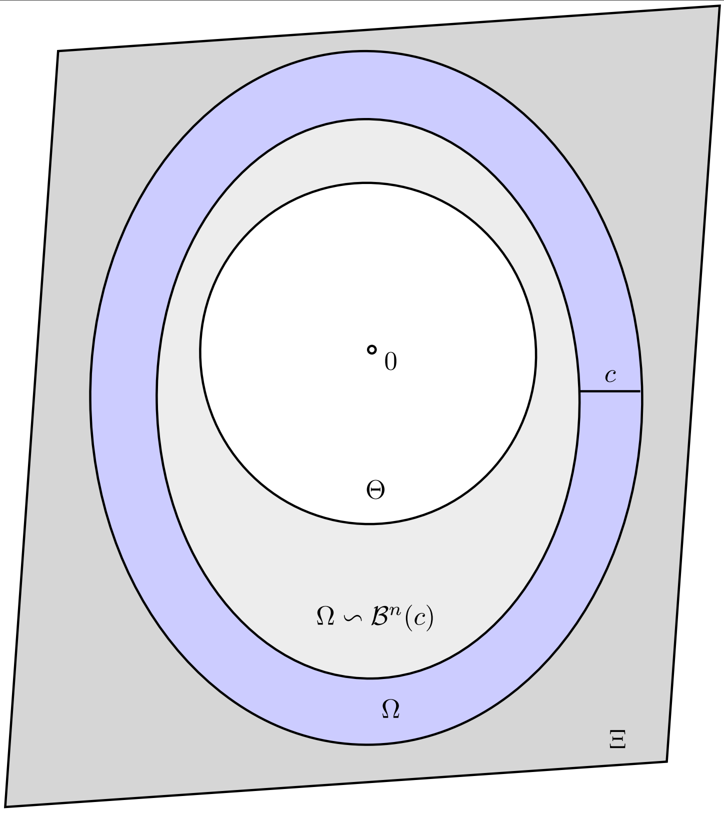
A scheme of the sets introduced in Definition 2.2.1 is depicted in Figure 2.1.

A sufficient condition to establish the regional-ISS of system (2.1) can now be stated.

Theorem 2.2.1 (Lyapunov characterization of ISS). *Suppose that Assumption 2.1.2 holds. If the system (2.1) admits an ISS-Lyapunov function in* Ξ*, then it is ISS in* Ξ *with respect to* υ *and for all* $\overline{x}_0 \in \Xi$ *it holds that* $\lim_{t \to \infty} \mathrm{d}(x(t, \overline{x}_0, \boldsymbol{\upsilon}_{0,t-1}), \Theta) = 0$, $\forall \boldsymbol{\upsilon} \in \mathcal{M}_\Upsilon$. □

Proof Let $\bar{x} \in \Xi$. The proof will be carried out in three steps:

Figure 2.1 Scheme of the sets introduced in Definition 2.2.1



1) First, we are going to show that the set Θ defined in (2.5) is RPI for the system. From the definition of $\overline{\alpha}_2(s)$ it follows that $\alpha_2(|x|)+\sigma_1(|\upsilon|) \leq \overline{\alpha}_2(|x|+|\upsilon|)$. Therefore $V(x) \leq \overline{\alpha}_2(|x|+|\upsilon|)$ and hence $|x|+|\upsilon| \geq \overline{\alpha}_2^{-1}(V(x))$. Moreover, thanks to Point 3) of Definition 2.2.1, there exists a $\mathcal{K}_\infty$-function ϵ such that

$$\alpha_3(|x|) + \epsilon(|\upsilon|) \geq \underline{\alpha}_3(|x| + |\upsilon|) \geq \alpha_4(V(x)).$$

Then, considering the perturbed state transition from x_t to x_{t+1}, we have

$$\begin{aligned} V(g(x_t, \upsilon_t)) - V(x_t) &\leq -\alpha_4(V(x_t)) + \epsilon(|\upsilon_t|) + \sigma_2(|\upsilon_t|) \\ &\leq -\alpha_4(V(x_t)) + \sigma_4(|\upsilon_t|), \quad \forall x \in \Omega, \forall \upsilon \in \Upsilon, \forall t \in \mathbb{R}_{\geq 0}. \end{aligned} \tag{2.6}$$

Let us assume now that $x_t \in \Theta$. Then $V(x_t, \upsilon_t) \leq b(\overline{\upsilon})$; this implies $\rho \circ \alpha_4(V(x_t, \upsilon_t)) \leq \sigma_4(\overline{\upsilon})$. Without loss of generality, assume that $(id - \alpha_4)$ is a $\mathcal{K}_\infty$-function, otherwise pick a bigger α_2

so that $\underline{\alpha}_3 < \overline{\alpha}_2$. Then

$$\begin{aligned} V(g(x_t, \upsilon_t)) &\leq (id - \alpha_4)(V(x_t)) + \sigma_4(\overline{\upsilon}) \\ &\leq (id - \alpha_4)(b(\overline{\upsilon})) + \sigma_4(\overline{\upsilon}) \\ &= -(id - \rho) \circ \alpha_4(b(\overline{\upsilon})) + b(\overline{\upsilon}) - \rho \circ \alpha_4(b(\overline{\upsilon})) + \sigma_4(\overline{\upsilon}). \end{aligned}$$

From the definition of b, it follows that $\rho \circ \alpha_4(b(\overline{\upsilon})) = \sigma_4(\overline{\upsilon})$ and, owing to the fact that $(id - \rho)$ is a $\mathcal{K}_\infty$-function, we obtain

$$V(g(x_t, \upsilon_t)) \leq (id - \rho) \circ \alpha_4(b(\overline{\upsilon})) + b(\overline{\upsilon}) \leq b(\overline{\upsilon})$$

By induction one can show that $V(g(x_{t+j}, \upsilon_{t+j})) \leq b(\overline{\upsilon})$ for all $j \in \mathbb{Z}_{\geq 0}$, that is $x_t \in \Theta, \forall j \in \mathbb{Z}_{\geq 0}$. Hence Θ is RPI for system (2.1).

2) Next, we are going to show that the state, starting from $\Xi \backslash \Theta$, tends asymptotically to Θ. Firstly, if $x \in \Omega \backslash \Theta$, then

$$\rho \circ \alpha_4(V(x_t)) \geq \sigma_4(\overline{\upsilon}).$$

From the inequality $\alpha_3(|x_t|) + \epsilon(|\upsilon_t|) \geq \alpha_4(V(x_t))$, we have that

$$\rho(\alpha_3(|x_t|) + \epsilon(|\upsilon_t|)) > \sigma_4(\overline{\upsilon}).$$

Being $(id - \rho)$ a $\mathcal{K}_\infty$-function, it holds that $id(s) > \rho(s), \ \forall s \in \mathbb{R}_{>0}$, then

$$\begin{aligned} \alpha_3(|x_t|) + \epsilon(\overline{\upsilon}) &> \alpha_3(|x_t|) + \epsilon(|\upsilon_t|) > \rho(\alpha_3(|x_t|) + \epsilon(|\upsilon_t|)) \\ &> \sigma_4(\overline{\upsilon}) = \epsilon(\overline{\upsilon}) + \sigma_2(\upsilon), \ \forall x_t \in \Omega \backslash \Theta, \forall \upsilon_t \in \Upsilon, \end{aligned}$$

which in turn implies that

$$\begin{aligned} V(g(x_t, \upsilon_t)) - V(x_t) &\leq -\alpha_3(|x_t|) + \sigma_2(\overline{\upsilon}) + \sigma_3(\overline{\upsilon}) \\ &< 0, \quad \forall x_t \in \Omega \backslash \Theta, \forall \upsilon_t \in \Upsilon. \end{aligned} \tag{2.7}$$

Moreover, in view of (2.5), $\exists \overline{c} \in \mathbb{R}_{>0}$ such that for all $x^{'} \in \Xi \backslash \Theta$ there exists $x^{''} \in \Omega \backslash D$ such that $\alpha_3(|x^{''}|) \leq \alpha_3(|x^{'}|) - \overline{c}$. Then, from (2.7) it follows that

$$-\alpha_3(|x^{'}|) + \overline{c} \leq -\alpha_3(|x^{''}|) < -\sigma_2(\overline{\upsilon}) - \sigma_3(\overline{\upsilon}), \ \forall x^{'} \in \Xi \backslash \Omega, \ \forall x^{''} \in \Omega \backslash \Theta.$$

Then,

$$\begin{aligned} V(g(x_t, \upsilon_t)) - V(x_t) &\leq -\alpha_3(|x_t|) + \sigma_2(\overline{\upsilon}) + \sigma_3(\overline{\upsilon}) \\ &< -\overline{c}, \quad \forall x_t \in \Xi \backslash \Omega, \forall \upsilon_t \in \Upsilon. \end{aligned}$$

Hence, for any $\overline{x}_0 \in \Xi$, there exists $T^{\Omega}_{\overline{x}_0} \in \mathbb{Z}_{\geq 0}$ finite such that $x_{T^{\Omega}_{\overline{x}_0}} = x(T^{\Omega}_{\overline{x}_0}, \overline{x}_0, \boldsymbol{\upsilon}) \in \Omega$, that is, starting from Ξ, the region Ω will be reached in finite time.

Now, we will prove that starting from Ω, the state trajectories will tend asymptotically to the set Θ. Since Θ is RPI, it holds that $\lim_{j\to\infty} \mathrm{d}\left(x(T^{\Omega}_{\overline{x}_0} + j, \overline{x}_{T^{\Omega}_{\overline{x}_0}}, \boldsymbol{\upsilon}), \Theta\right) = 0$. Otherwise, posing $t = T^{\Omega}_{\overline{x}_0}$, if $x_t \notin \Theta$, then we have that $\rho \circ \alpha_4(V(x_t)) > \sigma_4(\overline{\upsilon})$; moreover, from (2.7) it follows that

$$\begin{aligned} V(g(x_t, \upsilon_t)) - V(x_t) &\leq -\alpha_4(V(x_t)) + \sigma_4(\overline{\upsilon}) \\ &= -(id - \rho) \circ \alpha_4(V(x_t)) - \rho \circ \alpha_4(V(x_t)) + \sigma_4(\overline{\upsilon}) \\ &\leq -(id - \rho) \circ \alpha_4(V(x_t)) \\ &\leq -(id - \rho) \circ \alpha_4 \circ \alpha_1(|x_t|), \forall x_t \in \Omega \backslash \Theta, \forall \upsilon \in \Upsilon \end{aligned}$$

Then, we can conclude that $\forall \epsilon' \in \mathbb{R}_{>0}$, $\exists T^{\Theta}_{\overline{x}_0} \geq T^{\Omega}_{\overline{x}_0}$ such that

$$V(x_t) \leq \epsilon' + b(\overline{\upsilon}).$$

Therefore, starting from Ξ, the state will arrive arbitrarily close to Θ in finite time and the state trajectories will tend to Θ asymptotically. Hence $\lim_{t\to\infty} \mathrm{d}(x(t, \overline{x}_0, \boldsymbol{\upsilon}), \Theta) = 0$, $\forall \overline{x}_0 \in \Xi, \forall \boldsymbol{\upsilon} \in \mathcal{M}_{\Upsilon}$.

3) Finally, we will show that system (2.1) is regionally ISS in Ξ. Given $e \in \mathbb{R}_{\geq 0}$, let us define the sub-level set $\mathcal{N}_{[V,e]} \triangleq \{x \in \mathbb{R}^n : V(x) \leq e, \forall \upsilon \in \Upsilon\}$. Let $\overline{e} \triangleq \max\{e \in \mathbb{R}_{>0} : \mathcal{N}_{[V,e]} \in \Omega\}$ and consider $\mathcal{N}_{[V,\overline{e}]}$. Note that $\overline{e} > b(\overline{\upsilon})$ and $\Theta \subset \mathcal{N}_{[V,\overline{e}]}$. Since the region Θ is reached asymptotically from Ξ, the state will arrive in $\mathcal{N}_{[V,\overline{e}]}$ in finite time, that is, given $\overline{x}_0 \in \Xi$ there exists $T^{\mathcal{N}_{[V,\overline{e}]}}_{\overline{x}_0}$ such that

$$V\left(x_{T^{\mathcal{N}_{[V,\overline{e}]}}_{\overline{x}_0} + j}\right) \leq \overline{e}, \ \ \forall j \in \mathbb{Z}_{\geq 0}$$

Hence, the region $\mathcal{N}_{[V,\overline{e}]}$ is RPI. Now, proceeding as in the Proof of Lemma 3.5 in [50], for any $\overline{x}_0 \in \mathcal{N}_{[V,\overline{e}]}$, there exist a $\mathcal{KL}$-function $\hat{\beta}$ and a $\mathcal{K}$-function $\hat{\gamma}$ such that

$$V(x_t) \leq \max \hat{\beta}\left(V(\overline{x}_0), t\right), \hat{\gamma}(||\upsilon_{[t]}||), \ \ \forall t \in \mathbb{Z}_{\geq 0}, \forall \boldsymbol{\upsilon} \in \mathcal{M}_{\Upsilon}$$

, with $x_t \in \mathcal{N}_{[V,\bar{e}]}$ and where $\hat{\gamma}$ can be chosen as $\hat{\gamma} = \alpha_4^{-1} \circ \rho^{-1} \circ \sigma_4$. Hence, considering that $\hat{\beta}(r+s,t) \le \hat{\beta}(2r,t) + \hat{\beta}(2s,t), \forall (s,t) \in \mathbb{R}^2_{\ge 0}$ (see [68]), it follows that

$$\begin{aligned}
\alpha_1(|x_t|) &\le \max \hat{\beta}\left(\alpha_2(|\overline{x}_0|) + \sigma_1(|\upsilon_0|), t\right), \hat{\gamma}(\|\upsilon_{[t]}\|) \\
&\le \max \hat{\beta}\left(2\alpha_2(|\overline{x}_0|), t\right) + \hat{\beta}\left(2\sigma_1(|\upsilon_0|), t\right), \hat{\gamma}(\|\upsilon_{[t]}\|), \ \forall t \in \mathbb{Z}_{\ge 0}, \forall \overline{x}_0 \in \mathcal{N}_{[V,\bar{e}]}, \forall \upsilon \in \mathcal{M}_\Upsilon.
\end{aligned}$$

Now, let us define the $\mathcal{KL}$-functions $\tilde{\beta}(s,t) \triangleq \alpha_1^{-1} \circ \hat{\beta}(2s,t)$, $\beta(s,t) \triangleq \tilde{\beta}(\alpha_2(s),t)$, and the $\mathcal{K}$-functions $\tilde{\gamma}(s) \triangleq \alpha_1^{-1} \circ \hat{\gamma}(s)$ and $\gamma(s) \triangleq \tilde{\beta}(\sigma_1(s),0) + \tilde{\gamma(s)}$, we have that

$$\begin{aligned}
|x_t| &\le \max \tilde{\beta}\left(\alpha_2(|\overline{x}_0|), t\right) + \tilde{\beta}\left(\sigma_1(|\upsilon_0|), t\right), \tilde{\gamma}(\|\upsilon_{[t]}\|) \\
&\le \tilde{\beta}\left(\alpha_2(|\overline{x}_0|), t\right) + \tilde{\beta}\left(\sigma_1(|\upsilon_0|), t\right) + \tilde{\gamma}(\|\upsilon_{[t]}\|) \\
&\le \tilde{\beta}\left(\alpha_2(|\overline{x}_0|), t\right) + \tilde{\beta}\big(\sigma_1(\|\upsilon_[t]\|), 0\big) + \tilde{\gamma}(\|\upsilon_{[t]}\|) \\
&\le \beta\left(|\overline{x}_0|, t\right) + \gamma(\|\upsilon_{[t]}\|), \ \forall t \in \mathbb{Z}_{\ge 0}, \forall \overline{x}_0 \in \mathcal{N}_{[V,\bar{e}]}, \forall \upsilon \in \mathcal{M}_\Upsilon.
\end{aligned} \tag{2.8}$$

Hence, by (2.8), the system (2.1) is ISS in $\mathcal{N}_{[V,\bar{e}]}$ with ISS-asymptotic gain γ. Considering that starting from Ξ the set $\mathcal{N}_{[V,\bar{e}]}$ is reached in finite time, the UAG in $\mathcal{N}_{[V,\bar{e}]}$ implies the UAG in Ξ.

Now, thanks to Lemma 2.1.1 Assumption 2.1.2, together with the UAG in Ξ, implies the LS and UAG, as well, in Ξ, and hence the ISS property in Ξ. ■

For systems in which the asymptotic stability cannot be proved even in absence of perturbations, a property slightly different than ISS, namely the Input-to-State practical Stability (ISpS), can be used to characterize the region of attraction (see [111]). In the next section, we will introduce the ISpS, establishing its connections with the ISS.

2.3 Regional Input-to-State Practical Stability

In this section, the ISpS tool for the stability analysis of discrete-time autonomous perturbed nonlinear systems is presented. The ISpS allows to address systems for which, even in absence of perturbations, the asymptotic convergence of the trajectories toward the origin cannot be proven (the reader is referred to [111] for a deeper insight into this topic). The results that are going to be discussed will be employed in Chapter 4 to study the behavior of nonlinear system in closed-loop with approximate MPC control laws. Indeed, in order to take in account the effect of non-vanishing perturbations on the controlled system, due to the use of an approximate controller, the ISpS has been regarded as one of the most appropriate method of analysis.

The definition of the ISpS for perturbed discrete-time dynamic systems is given below.

Definition 2.3.1 (ISpS in Ξ)**.** *Given a compact set $\Xi \subset \mathbb{R}^n$ including the origin as interior point, the system (2.1), with $\boldsymbol{\upsilon} \in \mathcal{M}_\Upsilon$, is said to be Input-to-State practically Stable (ISpS) in Ξ if Ξ is a RPI set for system (2.1) and if there exist a $\mathcal{KL}$-function β, a $\mathcal{K}$-function γ and a constant $c \in \mathbb{R}_{\geq 0}$ such that*

$$|x(t, \overline{x}_0, \boldsymbol{\upsilon})| \leq \beta(|\overline{x}_0|, t) + \gamma(||\boldsymbol{\upsilon}||) + c, \ \ \forall \overline{x}_0 \in \Xi, \forall t \in \mathbb{Z}_{\geq 0}. \tag{2.9}$$

□

Note that, by causality, the same definitions of ISpS in Ξ would result if the inequality (2.9) is replaced by

$$|x(t, \overline{x}_0, \boldsymbol{\upsilon})| \leq \beta(|\overline{x}_0|, t) + \gamma(||\boldsymbol{\upsilon}_{[t-1]}||), \ \ \forall \overline{x}_0 \in \Xi, \forall t \in \mathbb{Z}_{\geq 0} + c,$$

where $\boldsymbol{\upsilon}_{[t-1]}$ denotes a truncation of $\boldsymbol{\upsilon}$ at the time instant $k-1$.

Moreover it is worth to notice that, if the inequality (2.1) holds with $c = 0$, then the definition of ISS in Ξ follows.

Analogously to the ISS property, regional results are need in the framework of of MPC controlled system in order to use the ISpS for assessing the stability properties. Moreover, also the ISpS can be associated to the existence of a suitable Lyapunov function (in general, a priori non-smooth) with respect to υ. Sufficient conditions for characterizing the ISpS property through Lyapunov functions have been introduced in [99], where the ISS result of [75] has been extended to the ISpS case.

In order to briefly recall the basic result on the Lyapunov characterization of the regional ISpS property, let us introducing the following definition.

Definition 2.3.2 (ISpS-Lyapunov Function)**.** *Given the system (2.1) and a pair of compact sets $\Xi \subset \mathbb{R}^n$ and $\Omega \subseteq \Xi$, with $\{0\} \subset \Omega$, a function $V(\cdot)\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a (Regional) ISpS-Lyapunov* *function in* Ξ*, if there exist some* $\mathcal{K}_\infty$*-functions* α_1*,* α_2*,* α_3*, two* $\mathcal{K}$*-function* σ_1 *and* σ_2 *and two non-negative scalars* c_1 *and* $c_2 \in \mathbb{R}_{\geq 0}$ *such that*

1) Ξ *is a compact RPI set including the origin as an interior point;*

2) the following inequalities hold $\forall \upsilon \in \Upsilon$

$$V(x) \geq \alpha_1(|x|), \qquad \forall x \in \Xi, \tag{2.10}$$

$$V(x) \leq \alpha_2(|x|) + c_1, \qquad \forall x \in \Omega, \tag{2.11}$$

$$V(g(x,\upsilon)) - V(x) \leq -\alpha_3(|x|) + \sigma_2(|\upsilon|) + c_2, \ \forall x \in \Xi, ; \tag{2.12}$$

3) there exist some suitable $\mathcal{K}_\infty$*-functions* ϵ *and* ρ *(with* ρ *such that* $(id - \rho)$ *is a* $\mathcal{K}_\infty$*-function, too) such that the following compact set*

$$\Theta_c \triangleq \{x : V(x) \leq b_c\left(\sigma_2(\overline{\upsilon}) + c_3\right)\}, \tag{2.13}$$

verifies the inclusion $\Theta_c \subset \Omega \backsim \mathcal{B}^n(c)$*, for some suitable constant* $c \in \mathbb{R}_{>0}$*, where* $b_c(s) \triangleq \alpha_4^{-1} \circ \rho^{-1}$*,* $\alpha_4 \triangleq \underline{\alpha}_3 \circ \overline{\alpha}_2^{-1}$*,* $\underline{\alpha}_3(s) \triangleq \min(\alpha_3(s/2), \epsilon(s/2)), \overline{\alpha}_2(s) \triangleq \alpha_2(s) + s$*,* $c_3 \triangleq c_2 + \epsilon(c_1)$ *and* $\overline{\upsilon} \triangleq \max_{\upsilon \in \Upsilon}\{|\upsilon|\}$*.* □

By using the same arguments exploited in in the proof of Theorem 2.2.1, the following results can be proven.

Theorem 2.3.1 (Lyapunov Characterization of regional ISpS)**.** *Suppose that Assumption 2.1.2 holds. If system (2.1) admists a regional ISpS Lyapunov function in* Ξ *with respect to* υ*, then it is regional ISpS in* Ξ *with respect to* υ *and, for all* $\overline{x}_0 \in \Xi$*, it holds that* $\lim_{t\to\infty} \mathrm{d}(x(t, \overline{x}_0, \boldsymbol{\upsilon}_{0,t-1}), \Theta_c) = 0$, $\forall \boldsymbol{\upsilon} \in \mathcal{M}_\Upsilon$.

The reader is referred to [99], for a complete proof of Theorem 2.3.1.

2.4 Regional ISS in terms of Time-varying Lyapunov Functions

In the present section, the notion of regional ISS will be extended to time-varying systems. In this case, as mentioned in the case of the regional ISS result for time-invariant systems, global results are not suited for NMPC-controlled constrained dynamics, due to impossibility to obtain global bounds for the optimal multi-stage cost function (see [21] and [86] for the global ISS characterizations in the case of time-varying systems).

In the following, the Regional Input-to-State Stability property will be characterized for a class time-varying systems, which admit a possibly time-varying Lyapunov function satisfying suitable time-invariant comparison inequalities.

Let us consider the time-varying discrete-time dynamic system

$$x_{t+1} = g(t, x_t, \upsilon_t),\ t \in \mathbb{Z}_{\geq 0},\ x_0 = \overline{x}\,, \tag{2.14}$$

with $g(t, 0, 0) = 0$, $\forall t \geq \overline{T}$ with $\overline{T} \in \mathbb{Z}_{\geq 0}$, and where $x_t \in \mathbb{R}^n$ and $\upsilon_t \in \Upsilon \subset \mathbb{R}^r$ denote the state and the bounded input of the system, respectively. The discrete-time state trajectory of the system (2.14), with initial state $x_0 = \overline{x}$ and input sequence $\boldsymbol{\upsilon} \in \mathcal{M}_\Upsilon$, is denoted by $x(t, \overline{x}, \boldsymbol{\upsilon}_{0,t}),\ t \in \mathbb{Z}_{\geq 0}$.

In the case of time-varying controlled transition maps $g(t, x_0, \upsilon)$, the following definition of RPI set will be used (see the Appendix for an analogous definition in the time-invariant case).

Definition 2.4.1 (RPI set)**.** *A set $\Xi \subset \mathbb{R}^n$ is a Robust Positively Invariant (RPI) set for system (2.14) if, for all $t \in \mathbb{Z}_{\geq 0}$, it holds that $g(t, x, \upsilon) \in \Xi$, $\forall x \in \Xi$ and $\forall \upsilon \in \Upsilon$.* □

Moreover, the regional ISS property for time-varying discrete-time nonlinear systems of the form (2.14) is given below.

Definition 2.4.2 (Time-varying regional ISS)**.** *Given a compact set $\Xi \subset \mathbb{R}^n$, if Ξ is RPI for (2.14) and if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that*

$$|x(t,\overline{x}_0,\boldsymbol{v}_{0,t-1})|\leq\max\{\beta(|\overline{x}_0|,t),\gamma(\|\boldsymbol{v}_{[t-1]}\|)\},\forall t\in\mathbb{Z}_{\geq 0},\forall\overline{x}_0\in\Xi, \tag{2.15}$$

then the system (2.14), with $\boldsymbol{v}\in\mathcal{M}_\Upsilon$, is said to be Input-to-State Stable (ISS) with respect to v for initial conditions in Ξ. □

In the literature there exist some recent results concerning the characterization of the ISS property in terms of time-varying Lyapunov functions for perturbed (uncertain) discrete-time system [51, 59, 60]; on the other hand those results guarantee the Input-to-State Stability property in a semi-global sense, and cannot be trivially used in the MPC setup due to the impossibility to obtain global bounds for the candidate ISS Lyapunov function. Indeed, for systems controlled by predictive control schemes the stability analysis needs to be carried out by using non smooth ISS-Lyapunov functions with an upper bound guaranteed only in a sub-region of the domain of attraction [75]. Therefore, a novel regional ISS result for a family of time-varying Lyapunov functions has been derived to assess the stability properties of MPC-based NCS's.

To this end, let us first consider the following definition.

Definition 2.4.3 (ISS-Lyapunov Function)**.** *Given a pair of compact sets $\Xi\subset\mathbb{R}^n$ and $\Omega\subseteq\Xi$, with Ξ RPI for system (2.14) and $\{0\}\subset\Omega$, a function $V(\cdot,\cdot)$: $\mathbb{Z}_{\geq 0}\times\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ is called a (Regional) ISS-Lyapunov function in Ξ, if there exist $\mathcal{K}_\infty$-functions α_1, α_2, α_3, and $\mathcal{K}$-function σ_1 and σ_2, such that*

1) the following inequalities hold $\forall v\in\Upsilon$ and $\forall t\in\mathbb{Z}_{\geq 0}$

$$V(t,x)\geq\alpha_1(|x|), \qquad \forall x\in\Xi, \tag{2.16}$$

$$V(t,x)\leq\alpha_2(|x|)+\sigma_1(|v|), \qquad \forall x\in\Omega, \tag{2.17}$$

$$V(t+1,g(t,x,v))-V(t,x)\leq-\alpha_3(|x|)+\sigma_2(|v|),\ \forall x\in\Xi, \tag{2.18}$$

2) there exist some suitable $\mathcal{K}_\infty$-functions ϵ and ρ (with ρ such that $(id-\rho)$ is a $\mathcal{K}_\infty$-function, too) and a positive scalar $c\in\mathbb{R}_{>0}$ such that the set

$$\Theta\triangleq\{x:V(t,x)\leq b(\overline{v}),\forall t\in\mathbb{Z}_{\geq 0}\}, \tag{2.19}$$

verifies the inclusion

$$\Theta \subseteq \Omega \backsim \mathcal{B}^n(c), \tag{2.20}$$

with $\{0\} \in \Theta$ *and where* $b(s) \triangleq \alpha_4^{-1} \circ \rho^{-1} \circ \sigma_4(s)$, $\alpha_4 \triangleq \underline{\alpha}_3 \circ \overline{\alpha}_2^{-1}$, $\underline{\alpha}_3(s) \triangleq \min(\alpha_3(s/2), \epsilon(s/2)), \overline{\alpha}(s) \triangleq \alpha_2(s) + \sigma_1(s)$, $\sigma_4 = \epsilon(s) + \sigma_2(s)$ *and* $\overline{\upsilon} \triangleq \max_{\upsilon \in \Upsilon}\{|\upsilon|\}$.

□

The following remark will provide some insight into the meaning of Condition 2) in Definition 2.4.3 above.

Remark 2.4.1. *Due the fact that, in Definition 2.4.3, the set* Ξ *has been assumed to be compact, there always exists a set* Θ *satisfying condition (2.20) for a suitably small uncertainty bound* $\overline{\upsilon} \in \mathbb{R}_{>0}$ *(and hence for a suitable non empty uncertainty set* Υ*). Indeed, by setting* $\overline{\xi} \triangleq \inf_{\xi \in \mathbb{R}^n \setminus \Xi}\{|\xi|\}$, *and noting that* $\overline{\xi}$ *is strict positive, a sufficient condition for (2.20) to hold is that*

$$\overline{\upsilon} \leq b^{-1}\left(\alpha_1(\overline{\xi} - c_\upsilon)\right), \tag{2.21}$$

for some $c_\upsilon \in \mathbb{R}_{>0}$, *with* $c_\upsilon < \overline{\xi}$. *Indeed from (2.21) it follows that* $b(\overline{\upsilon}) \leq \alpha_1(\overline{\xi} - c_\upsilon)$. *Then* $\forall \xi : |\xi| > \overline{\xi} - c_\upsilon$ *it holds that* $V(t, \xi) \geq \alpha_1(\xi) > b(\overline{\upsilon})$, *which implies* $\Theta \subseteq \mathcal{B}^n(\overline{\xi} - c_\upsilon) \subseteq \Xi \backsim \mathcal{B}^n(c_\upsilon)$.

Due to the inherent conservativeness of the comparison function approach, in practice it turns out that the uncertainty bound given by (2.21) is in general smaller than that for which the invariance of Ξ *can be guaranteed. However, this observation is nonetheless important, since it permits to guarantee the convergence towards the origin in presence of small uncertainty, while the robust constraint satisfaction (related to the concept of set invariance rather then to comparison inequalities) can be enforced for larger uncertainties.* □

Notably, the ISS-Lyapunov inequalities (2.16),(2.17) and (2.18) differ from those posed in the original regional ISS formulation [75], since an input-dependent upper bound is admitted in (2.17) (thus allowing for a more general characterization). Moreover, with regard to the regional ISS result presented in [33], the ISS-Lyapunov function $V(t, x)$ is allowed to belong a family of time-varying functions. Remarkably, the possibility to incorporate an input-dependent upper bound in (2.17) and to admit a time-varying characterization will be instrumental for characterizing the ISS property for NCS's, as it will clearly emerge in Section 5.4.

Now, under Assumption 2.1.2, the characterization of the regional ISS property in terms of Lyapunov functions can be stated.

Theorem 2.4.1 (Lyapunov characterization of regional ISS)**.** *Suppose that Assumption 2.1.2 holds. If the system (2.14) admits an ISS-Lyapunov function in* Ξ*, then it is ISS in* Ξ *with respect to* υ *and*

$$\lim_{t\to\infty} \mathrm{d}(x(t,\overline{x}_0,\boldsymbol{\upsilon}_{0,t-1}),\Theta)=0\,,\ \ \forall \overline{x}_0\in\Xi.$$

□

Proof [Theorem 2.4.1] Let $x\in\Xi$. The proof will be carried out in three steps

1) First, we are going to show that the set Θ defined in (2.19) is RPI for the system. From the definition of $\overline{\alpha}_2(s)$ it follows that $\alpha_2(|x|)+\sigma_1(|\upsilon|)\le\overline{\alpha}_2(|x|+|\upsilon|)$. Therefore $V(t,x)\le\overline{\alpha}_2(|x|+|\upsilon|)$ and hence $|x|+|\upsilon|\ge\overline{\alpha}_2^{-1}(V(t,x))$. Moreover, thanks to Point 2) of Definition 2.4.3, there exists a $\mathcal{K}_\infty$-function ϵ such that

$$\alpha_3(|x|)+\epsilon(|\upsilon|)\ge\underline{\alpha}_3(|x|+|\upsilon|)\ge\alpha_4(V(t,x)).$$

Then, considering the transition from (t,x) to $(t+1,g(t,x,\upsilon))$, we have

$$\begin{aligned} V(t+1,g(t,x,\upsilon))-V(t,x) &\le -\alpha_4(V(t,x))+\epsilon(|\upsilon|)+\sigma_2(|\upsilon|)\\ &\le -\alpha_4(V(t,x))+\sigma_4(|\upsilon|),\quad \forall x\in\Omega,\forall\upsilon\in\Upsilon,\forall t\in\mathbb{R}_{\ge 0}. \end{aligned} \tag{2.22}$$

Let us assume now that $x\in\Theta$. Then $V(t,x)\le b(\overline{\upsilon})$; this implies $\rho\circ\alpha_4(V(t,x))\le\sigma_4(\overline{\upsilon})$. Without loss of generality, assume that $(id-\alpha_4)$ is a $\mathcal{K}_\infty$-function, otherwise pick a bigger α_2 so that $\underline{\alpha}_3<\overline{\alpha}_2$. Then

$$\begin{aligned} V(t+1,g(t,x,\upsilon)) &\le (id-\alpha_4)\,(V(t,x))+\sigma_4(\overline{\upsilon})\\ &\le (id-\alpha_4)\,(b(\overline{\upsilon}))+\sigma_4(\overline{\upsilon})\\ &= -(id-\rho)\circ\alpha_4\,(b(\overline{\upsilon}))+b(\overline{\upsilon})-\rho\circ\alpha_4\,(b(\overline{\upsilon}))+\sigma_4(\overline{\upsilon}). \end{aligned}$$

From the definition of b, it follows that $\rho\circ\alpha_4\,(b(\overline{\upsilon}))=\sigma_4(\overline{\upsilon})$ and, owing to the fact that $(id-\rho)$

is a $\mathcal{K}_\infty$-function, we obtain

$$V(t+1, g(t,x,\upsilon)) \le (id-\rho)\circ\alpha_4\,(b(\overline{\upsilon})) + b(\overline{\upsilon}) \le b(\overline{\upsilon}).$$

By induction it is possible to show that, $V(t, x(t,\bar{x}_0, \boldsymbol{\upsilon}_{0,t-1})) \le b(\overline{\upsilon})$, $\forall \bar{x}_0 \in \Theta$, $\forall t \in \mathbb{Z}_{\ge 0}$, that is $x_t \in \Theta, \forall t \in \mathbb{Z}_{\ge 0}$. Hence Θ is RPI for system (2.14).

2) Next, we are going to show that the state, starting from $\Xi\backslash\Theta$, tends asymptotically to Θ. Firstly, if $x \in \Omega\backslash\Theta$, then

$$\rho\circ\alpha_4\,(V(t,x)) \ge \sigma_4(\overline{\upsilon}).$$

From the inequality $\alpha_3(|x|) + \epsilon(|\upsilon|) \ge \alpha_4\,(V(t,x))$, we have that

$$\rho\,(\alpha_3(|x|) + \epsilon(|\upsilon|)) > \sigma_4(\overline{\upsilon}).$$

Being $(id-\rho)$ a $\mathcal{K}_\infty$-function, it holds that $id(s) > \rho(s),\ \forall s \in \mathbb{R}_{>0}$, then

$$\begin{aligned}\alpha_3(|x|) + \epsilon(\overline{\upsilon}) &> \alpha_3(|x|) + \epsilon(|\upsilon|) > \rho(\alpha_3(|x|) + \epsilon(|\upsilon|)) \\ &> \sigma_4(\overline{\upsilon}) = \epsilon(\overline{\upsilon}) + \sigma_2(\upsilon),\ \ \forall x \in \Omega\backslash\Theta, \forall \upsilon \in \Upsilon,\end{aligned}$$

which in turn implies that

$$\begin{aligned}V(t+1, g(t,x,\upsilon)) - V(t,x) &\le -\alpha_3(|x|) + \sigma_2(\overline{\upsilon}) + \sigma_3(\overline{\upsilon}) \\ &< 0, \quad \forall x \in \Omega\backslash\Theta, \forall \upsilon \in \Upsilon.\end{aligned} \tag{2.23}$$

Moreover, in view of (2.19), $\exists \bar{c} \in \mathbb{R}_{>0}$ such that for all $x' \in \Xi\backslash\Theta$ there exists $x'' \in \Omega\backslash D$ such that $\alpha_3(|x''|) \le \alpha_3(|x'|) - \bar{c}$. Then, from (2.23) it follows that

$$-\alpha_3(|x'|) + \bar{c} \le -\alpha_3(|x''|) < -\sigma_2(\overline{\upsilon}) - \sigma_3(\overline{\upsilon}),\ \ \forall x' \in \Xi\backslash\Omega,\ \ \forall x'' \in \Omega\backslash\Theta.$$

Then,

$$\begin{aligned}V(t+1, g(t,x,\upsilon)) - V(t,x) &\le -\alpha_3(|x|) + \sigma_2(\overline{\upsilon}) + \sigma_3(\overline{\upsilon}) \\ &< -\bar{c}, \quad \forall x \in \Xi\backslash\Omega, \forall \upsilon \in \Upsilon.\end{aligned}$$

Hence, for any $\overline{x}_0 \in \Xi$, there exists $T^{\Omega}_{\overline{x}_0} \in \mathbb{Z}_{\ge 0}$ finite such that $x_{T^{\Omega}_{\overline{x}_0}} = x(T^{\Omega}_{\overline{x}_0}, \overline{x}_0, \boldsymbol{\upsilon}) \in \Omega$, that is, starting from Ξ, the region Ω will be reached in finite time.

Now, we will prove that starting from Ω, the state trajectories will tend asymptotically to the set Θ. Since Θ is RPI, it holds that $\lim_{j\to\infty} \mathrm{d}\left(x(T^{\Omega}_{\overline{x}_0}+j, \overline{x}_{T^{\Omega}_{\overline{x}_0}}, \boldsymbol{\upsilon}), \Theta\right) = 0$. Otherwise, posing $t = T^{\Omega}_{\overline{x}_0}$, if $x_t \notin \Theta$, then we have that $\rho \circ \alpha_4(V(t,x)) > \sigma_4(\overline{\upsilon})$; moreover, from (2.23) it follows that

$$\begin{aligned} V(t+1, g(t,x,\upsilon)) - V(t,x) &\leq -\alpha_4(V(t,x)) + \sigma_4(\overline{\upsilon}) \\ &= -(id-\rho)\circ\alpha_4(V(t,x)) - \rho\circ\alpha_4(V(t,x)) + \sigma_4(\overline{\upsilon}) \\ &\leq -(id-\rho)\circ\alpha_4(V(t,x)) \\ &\leq -(id-\rho)\circ\alpha_4\circ\alpha_1(|x|), \forall x \in \Omega\backslash\Theta, \forall \upsilon \in \Upsilon \end{aligned}$$

Then, we can conclude that $\forall \epsilon' \in \mathbb{R}_{>0}$, $\exists T^{\Theta}_{\overline{x}_0} \geq T^{\Omega}_{\overline{x}_0}$ such that

$$V(T^{\Theta}_{\overline{x}_0}+j, x_{T^{\Theta}_{\overline{x}_0}+j}) \leq \epsilon' + b(\overline{\upsilon}), \ \ \forall j \in \mathbb{Z}_{\geq 0}.$$

Therefore, starting from Ξ, the state will arrive arbitrarily close to Θ in finite time and the state trajectories will tend to Θ asymptotically. Hence $\lim_{t\to\infty} \mathrm{d}(x(t, \overline{x}_0, \boldsymbol{\upsilon}_{0,t-1}), \Theta) = 0$, $\ \forall \overline{x}_0 \in \Xi, \forall \boldsymbol{\upsilon} \in \mathcal{M}_\Upsilon$.

3) The present part of the proof is intended to show that system (2.14) is regionally ISS in the sub-level set $\mathcal{N}_{[V,\overline{e}]}$, where $\overline{e} \triangleq \max\{e \in \mathbb{R}_{>0} : \mathcal{N}_{[V,e]} \in \Omega\}$, having denoted with $\mathcal{N}_{[V,e]} \triangleq \{x \in \mathbb{R}^n : V(t,x) \leq e, \forall \upsilon \in \Upsilon, \forall t \in \mathbb{Z}_{\geq 0}\}$ a sub-level set of V for a specified $e \in \mathbb{R}_{\geq 0}$. Let and consider . Note that $\overline{e} > b(\overline{\upsilon})$ and $\Theta \subset \mathcal{N}_{[V,\overline{e}]}$. Since the region Θ is reached asymptotically from Ξ, the state will arrive in $\mathcal{N}_{[V,\overline{e}]}$ in finite time, that is, given $\overline{x}_0 \in \Xi$ there exists $T^{\mathcal{N}_{[V,\overline{e}]}}_{\overline{x}_0}$ such that

$$V\left(T^{\mathcal{N}_{[V,\overline{e}]}}_{\overline{x}_0}+j \ , \ x_{T^{\mathcal{N}_{[V,\overline{e}]}}_{\overline{x}_0}+j}\right) \leq \overline{e}, \ \ \forall j \in \mathbb{Z}_{\geq 0}$$

Hence, the region $\mathcal{N}_{[V,\overline{e}]}$ is RPI. Now, proceeding as in the Proof of Lemma 3.5 in [50], for any $\overline{x}_0 \in \mathcal{N}_{[V,\overline{e}]}$, there exist a $\mathcal{KL}$-function $\hat{\beta}$ and a $\mathcal{K}$-function $\hat{\gamma}$ such that

$$V(t, x_t) \leq \max \hat{\beta}\left(V(0, \overline{x}_0), t\right), \ \hat{\gamma}(||\boldsymbol{\upsilon}_{[t-1]}||), \ \ \forall t \in \mathbb{Z}_{\geq 0}, \forall \boldsymbol{\upsilon} \in \mathcal{M}_\Upsilon,$$

with $x_t \in \mathcal{N}_{[V,\overline{e}]}$ and where $\hat{\gamma}$ can be chosen as $\hat{\gamma} = \alpha_4^{-1} \circ \rho^{-1} \circ \sigma_4$. Hence, considering that

$\hat{\beta}(r+s,t) \leq \hat{\beta}(2r,t)+\hat{\beta}(2s,t), \forall(s,t) \in \mathbb{R}^2_{\geq 0}$ (see [68]), it follows that

$$\begin{aligned} \alpha_1(|x_t|) &\leq \max \hat{\beta}\left(\alpha_2(|\overline{x}_0|)+\sigma_1(|\upsilon_0|),t\right), \hat{\gamma}(||\boldsymbol{\upsilon}_{[t-1]}||) \\ &\leq \max \hat{\beta}\left(2\alpha_2(|\overline{x}_0|),t\right)+\hat{\beta}\left(2\sigma_1(|\upsilon_0|),t\right), \hat{\gamma}(||\boldsymbol{\upsilon}_{[t-1]}||), \ \forall t \in \mathbb{Z}_{\geq 0}, \forall \overline{x}_0 \in \mathcal{N}_{[V,\overline{e}]}, \forall \upsilon \in \mathcal{M}_{\Upsilon}. \end{aligned}$$

Now, let us define the $\mathcal{KL}$-functions $\tilde{\beta}(s,t) \triangleq \alpha_1^{-1} \circ \hat{\beta}(2s,t)$, $\beta(s,t) \triangleq \tilde{\beta}(\alpha_2(s),t)$, and the $\mathcal{K}$-functions $\tilde{\gamma}(s) \triangleq \alpha_1^{-1} \circ \hat{\gamma}(s)$ and $\gamma(s) \triangleq \tilde{\beta}(\sigma_1(s),0)+\tilde{\gamma(s)}$, we have that

$$\begin{aligned} |x_t| &\leq \max \tilde{\beta}\left(\alpha_2(|\overline{x}_0|),t\right)+\tilde{\beta}\left(\sigma_1(|\upsilon_0|),t\right), \tilde{\gamma}(||\boldsymbol{\upsilon}_{[t-1]}||) \\ &\leq \tilde{\beta}\left(\alpha_2(|\overline{x}_0|),t\right)+\tilde{\beta}\left(\sigma_1(|\upsilon_0|),t\right)+\tilde{\gamma}(||\boldsymbol{\upsilon}_{[t-1]}||) \\ &\leq \tilde{\beta}\left(\alpha_2(|\overline{x}_0|),t\right)+\tilde{\beta}\left(\sigma_1(||\boldsymbol{\upsilon}_{[t-1]}||),0\right)+\tilde{\gamma}(||\boldsymbol{\upsilon}_{[t-1]}||) \\ &\leq \beta\left(|\overline{x}_0|,t\right)+\gamma(||\boldsymbol{\upsilon}_{[t-1]}||), \ \forall t \in \mathbb{Z}_{\geq 0}, \forall \overline{x}_0 \in \mathcal{N}_{[V,\overline{e}]}, \forall \upsilon \in \mathcal{M}_{\Upsilon}. \end{aligned} \tag{2.24}$$

Hence, by (2.24), the system (2.14) is ISS in $\mathcal{N}_{[V,\overline{e}]}$ with ISS-asymptotic gain γ. Considering that starting from Ξ the set $\mathcal{N}_{[V,\overline{e}]}$ is reached in finite time, the ISS in $\mathcal{N}_{[V,\overline{e}]}$ implies the UAG in Ξ.

Now, thanks to Lemma 2.1.1 Assumption 2.1.2, the UAG in Ξ implies the LS, as well, in Ξ, and hence the regional ISS property in Ξ. ■

2.5 Concluding Remarks

In this chapter, the notion of regional Input-to-State Stability for discrete-time nonlinear constrained systems has been recalled. In particular, an equivalent characterization of the regional ISS property in terms of (non-necessarily continuous) time-invariant Lyapunov functions has been discussed (see [75]). This result will be used, in the sequel, to study the robustness of MPC algorithms derived according to open-loop formulations.

In order to use the ISS tools to study the stability properties of discrete-time-varying nonlinear constrained system, such as those arising from the application of the MPC to networked system (see Chapter 5 for further details), the regional ISS property has been characterized in terms of (possibly discontinuous) time-varying Lyapunov functions satisfying suitable time-invariant comparison inequalities.

This result will be instrumental to study the region of attraction for MPC-controlled system in which the loop is closed through unreliable and delayed communication channels. It is also believed that this contribution can be used to improve the stability analysis of existing MPC algorithms as well as to develop new design methods with enhanced robustness properties.

Chapter 3

Robust NMPC based on Constraint Tightening

The idea of using restricted constraints in the formulation of the MPC, to provide the desired degree of robustness to the closed-loop system, was first introduced in [26] for linear systems and then extended to nonlinear systems in [66] and [100]. The main drawback of MPC with tightened constraints is represented by the conservative set-restrictions introduced to account for the disturbances, which consider a large spread of trajectories along the optimization horizon.

In order to overcome this limitation, the use of a closed-loop policy was suggested in [102], where the concept of uncertainty tube, (an envelope of all the possible trajectories introduced in [16] for uncertain linear system) was extended to some classes of nonlinear system.

It must be remarked that all existent constraint tightening approaches rely on an additive description of uncertainties, that is, they do not consider the possibility to reduce the conservativeness by exploiting some knowledge on the structure of the perturbation.

If the system is affected by state-dependent disturbances, and the state is limited in a compact set, it is always possible to bound the state-dependent perturbation with a worst-case value and to apply the algorithms described in [66, 100, 102].

However, if the particular state-dependent structure of the disturbance is considered, significant advantages can be clearly obtained.

In the following, we are going to propose a modification the nonlinear constraint tightening algorithms presented in [43, 62] and [66] in order to handle state-dependent disturbances more efficiently.

In this setup, the restricted sets are computed on-line iteratively by exploiting the state sequence obtained by the open-loop optimization, thus accounting for a possible reduction of the state dependent component of the uncertainty due to the control action. In this regard, it is possible to show that the devised technique yields to an enlarged feasible region compared to the one obtainable if just an additive disturbance is considered. Moreover, with respect to the previous scheme, the proposed algorithm uses a control horizon shorter than the prediction one. The terminal stabilizing constraint is imposed at the end of the control horizon, and not of the prediction horizon as in [72], in order to reduce the propagation of the uncertainty. The use of a long prediction horizon, along which an auxiliary controller is employed, is suggested to better approximate the performance of the so-called infinite horizon control law (see e.g., [72]).

A graphical representation of the underlying principle of Model Predictive Control with tightened constraints is given in Figure 3.1.

In order to analyze the stability properties of the closed-loop system in the presence of bounded persistent disturbances and state-dependent uncertainties, the regional characterization of Input-to-State Stability (ISS) in terms of Lyapunov functions is used (see Section 2.2 of Chapter 2).

The robustness with respect to state-dependent disturbances is analyzed using the nonlinear stability margin concept.

3.1 Problem Formulation

Consider the nonlinear discrete-time dynamic system

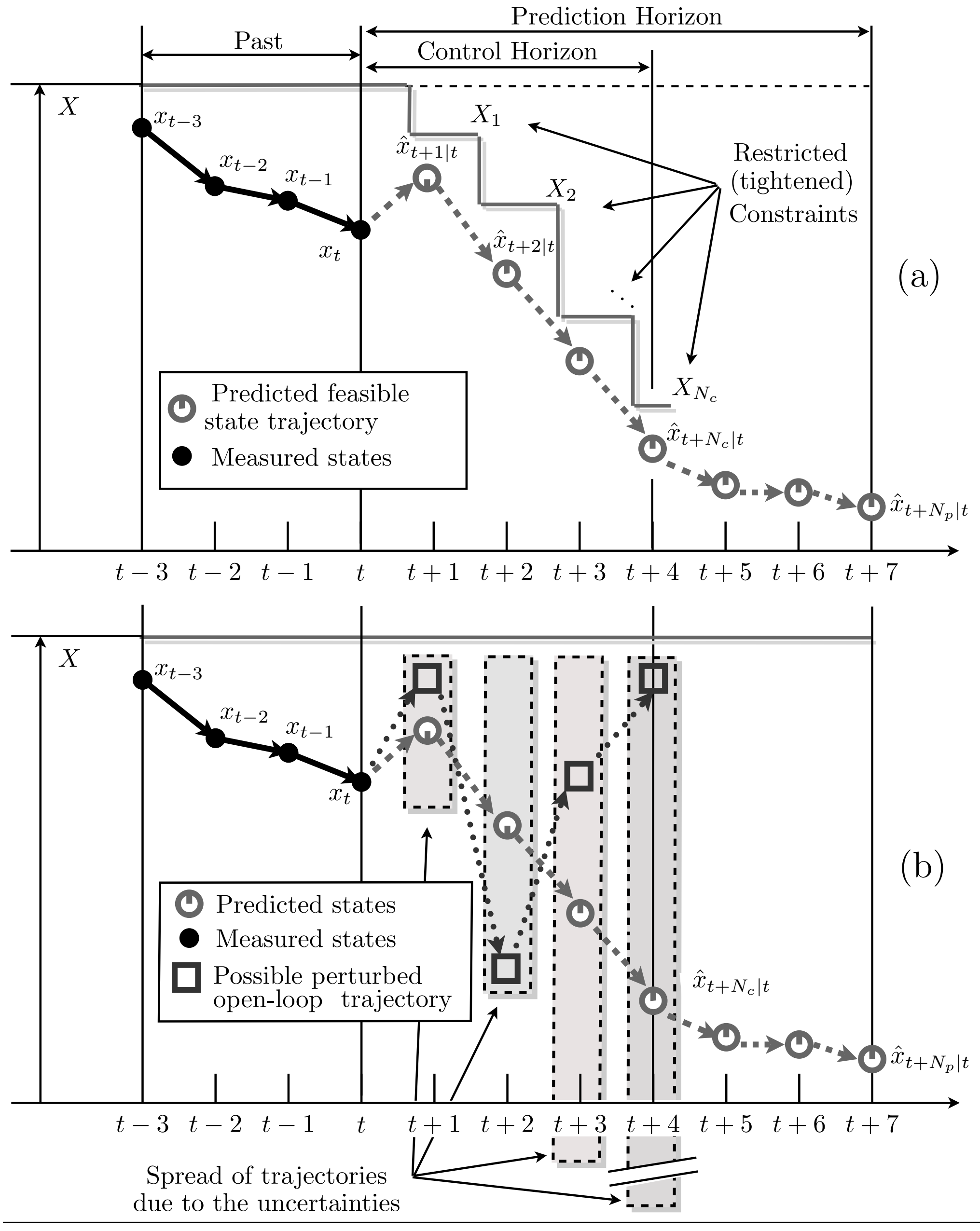
$$x_{t+1} = f(x_t, u_t, \upsilon_t),\ t \in \mathbb{Z}_{\geq 0}, \tag{3.1}$$

where $x_t \in \mathbb{R}^n$ denotes the system state, $u_t \in \mathbb{R}^m$ the control vector and $\upsilon_t \in \mathbb{R}^r$ an exogenous input which models the disturbance. The state and control variables are subject to the following constraints

$$x \in X, \tag{3.2}$$

$$u \in U, \tag{3.3}$$

Figure 3.1 Scheme of the underlying principle of MPC with Constraint Tightening: the restricted constraints set ($X_i \subset X,\ \forall i \in \{1, \dots, N_c\}$) are applied along the control horizon to the nominal system trajectories in (a). In this way, all the possible perturbed trajectories obtained with a feasible control sequence will respect the nominal constraint X, (b).



where X and U are compact subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, containing the origin as an interior point. Given the system (3.1), let $\hat{f}(x_t, u_t)$, with $\hat{f}(0,0) = 0$, denote the *nominal* model used for control design purposes, such that

$$x_{t+1} = \hat{f}(x_t, u_t) + d_t, \; t \in \mathbb{Z}_{\geq 0}, \tag{3.4}$$

where $d_t = d_t(x_t, u_t, \upsilon_t) \triangleq f(x_t, u_t, \upsilon_t) - \hat{f}(x_t, u_t) \in \mathbb{R}^n$ denotes the discrete-time state transition uncertainty. In the sequel, for the sake of brevity, we will not point out the functional dependence of $d_t(x_t, u_t, \upsilon_t)$ on its arguments except where strictly needed. The following assumptions will be used throughout the note.

Assumption 3.1.1. *$\hat{f}$ is Lipschitz with respect to x for all $x \in X$, with constant $L_{f_x} \in \mathbb{R}_{>0}$.* □

Assumption 3.1.2 (Uncertainties)**.** *The additive transition uncertainty d_t is limited in a time varying compact ball D_t, that is $d_t(x_t, u_t, \upsilon_t) \in D_t \triangleq \mathcal{B}^n(\delta(|x_t|) + \mu(\Upsilon^{sup}))$, $\forall x_t \in X$, $\forall u_t \in U$, $\forall \upsilon_t \in \Upsilon$, where δ and μ are two $\mathcal{K}$-functions. The $\mathcal{K}$-function δ is such that $L_\delta \triangleq \min\{L \in \mathbb{R}_{>0} : \delta(|x|) \leq L|x|, \forall x \in X\}$ exists finite. It follows that d_t is bounded by the sum of two contributions: a state-dependent component and a non-state-dependent one.* □

The control objective consists in designing a state-feedback control law capable to achieve ISS closed-loop stability and to satisfy state and control constraints in presence of state-dependent uncertainties and persistent disturbances.

On the basis of the previous assumptions, let us formulate the control problem. To this end, a suitable FHOCP (see Definition 1.1.1 for the standard formulation) should be introduced. At any time $t \in \mathbb{Z}_{\geq 0}$, let $\mathbf{u}_{t,t+N_p-1|t} \triangleq \text{col}[u_{t|t}, u_{t+1|t}, \ldots, u_{t+N_c-1|t}, u_{t+N_c|t}, \ldots, u_{t+N_p-1|t}]$ denote a sequence of input variables over the time-horizon N_p. Moreover, given x_t and $\mathbf{u}_{t,t+N_p-1|t}$, let $\hat{x}_{t+j|t}$ denote the state "predicted" at time $t+j$, $j \in \{1, \cdots, N_p\}$ by means of the nominal model.

Definition 3.1.1 (FHOCP)**.** *Consider system (3.4). Given two positive integers $N_c \in \mathbb{Z}_{\geq 0}$ and $N_p \in \mathbb{Z}_{\geq 0}$, with $N_p \geq N_c$ respectively representing the control and the prediction horizons, a transition cost function h, an auxiliary control law κ_f, a terminal penalty function h_f, a terminal set X_{N_c} and a sequence of constraint sets $\hat{X}_{t+j|t} \subseteq X$, $j \in \{1, \ldots, N_c - 1\}$ (to be described later* *on), the FHOCP consists in minimizing, with respect to* $\mathbf{u}_{t,t+N_c-1|t}$*, the performance index*

$$J_{FH}(x_t, \mathbf{u}_{t,t+N_c-1|t}, N_c, N_p) \triangleq \sum_{l=t}^{t+N_c-1} h(\hat{x}_{l|t}, u_{l|t}) + \sum_{l=t+N_c}^{t+N_p-1} h(\hat{x}_{l|t}, \kappa_f(x_{l|t})) + h_f(\hat{x}_{t+N_p|t}) \quad (3.5)$$

subject to

1) the nominal state dynamics $\hat{x}_{t+j|t} = \hat{f}(\hat{x}_{t+j-1|t}, u_{t+j-1|t})$*, with* $\hat{x}_{t|t} = x_t$*;*

2) the control and the state constraints $u_{t+j|t} \in U$*,* $\hat{x}_{t+j|t} \in \hat{X}_{t+j|t}, \forall j \in \{0, \ldots, N_c - 1\}$*;*

3) the terminal state constraints $\hat{x}_{t+N_c|t} \in X_{N_c}$*;*

4) the auxiliary control law $u_{t+j|t} = \kappa_f(\hat{x}_{t+j|t}), \forall j \in \{N_c, \ldots, N_p - 1\}$*.* □

Assumption 3.1.3. *The transition cost function* h *is such that* $\underline{h}(|x|) \leq h(x,u), \forall x \in X, \forall u \in U$ *where* $\underline{h}$ *is a* $\mathcal{K}_\infty$*-function. Moreover,* h *is Lipschitz with respect to* x *and* u *in* $X \times U$*, with Lipschitz constants* $L_h \in \mathbb{R}_{\geq 0}$ *and* $L_{h_u} \in \mathbb{R}_{\geq 0}$ *respectively.* □

The usual RH control technique can now be stated as follows: given a time instant $t \in \mathbb{Z}_{\geq 0}$, let $\hat{x}_{t|t} = x_t$, and find the optimal control sequence $\mathbf{u}^{\circ}_{t,t+N_c-1|t}$ by solving the FHOCP. Then, according to the RH strategy, apply

$$u_t = \kappa_{MPC}(x_t)\,, \quad (3.6)$$

where $\kappa_{MPC}(x_t) \triangleq u^{\circ}_{t,t}$ and $u^{\circ}_{t,t}$ is the first element of the optimal control sequence $\mathbf{u}^{\circ}_{t,t+N_c-1|t}$ (implicitly dependent on x_t).

With particular reference to the underlined definition of the FHOCP, note that, with respect to the usual formulation, in this case the constraint sets are defined only within the control horizon and the terminal constraint is stated at the end of the control horizon. Another peculiarity is the use of a state constraint that changes along the horizon. In the following, it will be shown how to choose accurately the stage cost h, the terminal cost function h_f, the control and prediction horizon N_c and N_p, the constraint sets $\hat{X}_{t+j|t}$, $j \in \{1, \ldots, N_c - 1\}$, the terminal constraint X_{N_c} and the auxiliary control law κ_f in order to guarantee closed-loop ISS. In particular the set X_{N_c} will be chosen such that, starting from any $x \in X_{N_c}$ in $N_p - N_c$ steps the auxiliary control law can steer the state of the nominal system into a set X_f which satisfies the assumption asked for the terminal set of standard stabilizing MPC algorithm [77]. In the following, X_{MPC} will denote the set containing all the state vectors for which a feasible control sequence exists, i.e. a control sequence $\mathbf{u}_{t,t+N_c-1|t}$ satisfying all the constraints of the FHOCP.

3.2 Robust MPC Strategy

In order to formulate the robust MPC algorithm, let us introduce the following further assumptions.

Assumption 3.2.1. *A terminal cost function h_f, an auxiliary control law κ_f, and a set X_f are given such that*

1) $X_f \subset X$, X_f closed, $0 \in X_f$;

2) $\kappa_f(x) \in U$, $\forall x \in X_f$; $\kappa_f(x)$ is Lipschitz in X_f, with constant $L_{\kappa_f} \in \mathbb{R}_{>0}$;

3) the closed loop map $\hat{f}(x, \kappa_f(x))$, is Lipschitz in X_f, with constant $L_{f_c} \in \mathbb{R}_{>0}$;

4) $\hat{f}(x, \kappa_f(x)) \in X_f$, $\forall x \in X_f$;

5) $h_f(x)$ is Lipschitz in X_f, with constant $L_{h_f} \in \mathbb{R}_{>0}$;

6) $h_f(\hat{f}(x, \kappa_f(x))) - h_f(x) \leq -h(x, \kappa_f(x))$, $\forall x \in X_f$;

7) $\tilde{\mathbf{u}}_{t,t+N_p-1|t} \triangleq \text{col}[\kappa_f(\hat{x}_{t|t}),\ \kappa_f(\hat{x}_{t+1|t}), \ldots,\ \kappa_f(\hat{x}_{t+N_p-1|t})]$, with $\hat{x}_{t|t} = x_t$, is a feasible control sequence for the FHOCP, $\forall x \in X_f$. □

Assumption 3.2.2 (Robust constraint X_{N_c})**.** *The robust terminal constraint set of the FHOCP, X_{N_c}, is chosen such that*

1) for all $x \in X_{N_c}$ the state can be steered to X_f in $N_p - N_c$ steps under the nominal dynamics in closed-loop with the auxiliary control law κ_f;

2) there exists a positive scalar $\epsilon \in \mathbb{R}_{>0}$ such that $\hat{f}(x_t, \kappa_f(x_t)) \in X_{N_c} \backsim \mathcal{B}^n(\epsilon)$, $\forall x_t \in X_{N_c}$. □

3.2.1 Shrunk State Constraints

In order to show the difference among the existing constraint tightening formulations and the proposed scheme, let us recall the definition of restricted constraint sets under the usual assumption of norm-bounded uncertainties, neglecting the possible state-dependent structure (see e.g., [66]).

Definition 3.2.1 (Tightened Constraints)**.** *Under Assumptions 3.1.1 and 4.2.2, suppose* [1]*, without loss of generality,* $L_{f_x} \neq 1$*. The tightened constraints are defined as*

$$X_i(\bar{d}) \triangleq X \backsim \mathcal{B}^n\left(\frac{L_{f_x}^i - 1}{L_{f_x} - 1}\bar{d}\right), \ \ \forall i \in \mathbb{Z}_{>0}. \tag{3.7}$$

□

In the following, we will exploit the particular state-dependent nature of the uncertainty to reduce the conservativeness of the set contraction relying on the nominal prediction of state trajectories along the control (optimization) horizon.

Under Assumption 3.1.2, given x_t, a norm-bound on the state prediction error will be derived. Subsequently, it is shown that the satisfaction of the original state constraints is ensured, for any admissible disturbance sequence, by imposing suitably restricted constraints to the predicted open-loop trajectories.

Throughout this section, the following notation will be used: given an optimal sequence $\mathbf{u}^\circ_{t,t+N_c-1|t}$ of control actions obtained by solving the FHOCP at time t, let us define the sequence $\bar{\mathbf{u}}_{t+1,t+N_c|t+1} \triangleq \text{col}[u^\circ_{t+1|t}, \ldots, u^\circ_{t+N_c-1|t}, \bar{u}]$, where $\bar{u} \in U$ is a suitably defined feasible control action implicitly depending on $\hat{x}_{t+N_c|t+1}$. The following result will be instrumental for the subsequent analysis.

Lemma 3.2.1 (Constraints tightening)**.** *Under Assumptions 3.1.1 and 3.1.2, given the state vector* x_t *at time* t*, let a control sequence,* $\bar{\mathbf{u}}_{t,t+N_c-1|t}$*, be feasible with respect to the restricted state constraints of the FHOCP,* $\hat{X}_{t+j|t}$*, computed as follows*

$$\hat{X}_{t+j|t} \triangleq X \ \backsim \ \mathcal{B}^n(\hat{\rho}_{t+j|t}), \tag{3.8}$$

where

$$\begin{cases} \hat{\rho}_{t+1|t} & \triangleq \bar{\mu} + L_\delta |x_t|, \\ \hat{\rho}_{t+j|t} & = (L_\delta + L_{f_x})\hat{\rho}_{t+j-1|t} + \bar{\mu} + L_\delta |\hat{x}_{t+j-1|t}|, \ \ \forall j \in \{2, \ldots, N_c\} \end{cases} \tag{3.9}$$

with $\bar{\mu} \triangleq \mu(\Upsilon^{\sup})$*. Then, the sequence* $\bar{\mathbf{u}}_{t,t+N_c-1|t}$*, applied to the perturbed system (3.1), guar-*

[1] The very special case $L_{f_x} = 1$ can be trivially addressed by a few suitable modifications to the proof of Lemma 3.2.1.

antees $x_{t+j} \in X,\ \forall j \in \{1, \ldots, N_c\}, \forall x_t \in X_{MPC}, \forall \upsilon \in \mathcal{M}_\Upsilon$. □

Proof. Given x_t, consider the state x_{t+j} obtained applying the first j elements of a feasible control sequence $\bar{\mathbf{u}}_{t,t+j-1|t}$ to the uncertain system (3.1). Then, the prediction error $\hat{e}_{t+j|t} \triangleq x_{t+j} - \hat{x}_{t+j|t}$, with $j \in \{1, \ldots, N_c\}$, is upper bounded by

$$\begin{aligned} |\hat{e}_{t+j|t}| &= |\hat{f}(x_{t+j-1}, u_{t+j-1|t}) + d_{t+j-1} - \hat{f}(\hat{x}_{t+j-1|t}, u_{t+j-1|t})| \\ &\leq L_{f_x}|\hat{e}_{t+j-1|t}| + |d_{t+j-1}| \leq L_{f_x}|\hat{e}_{t+j-1|t}| + \bar{\mu} + L_\delta |x_{t+j-1}| \\ &\leq (L_{f_x} + L_\delta)|\hat{e}_{t+j-1|t}| + \bar{\mu} + L_\delta|\hat{x}_{t+j-1|t}|. \end{aligned} \tag{3.10}$$

Finally, comparing (3.10) with (3.9), it follows that $|\hat{e}_{t+j|t}| \leq \hat{\rho}_{t+j|t}$, which in turn proves the statement. □

Remark 3.2.1. *The constraint tightening (3.8), compared to previous approaches [66, 100], may lead to less conservative computations. In fact, rather then using only the state information x_t at time t, it relies on the whole predicted state sequence $\hat{x}_{t+j|t}, j \in \{1, \ldots, N_c\}$, thus accounting for a possible reduction of the state-dependent component of the uncertainty along the horizon. The effectiveness of the proposed approach in enlarging the feasible region of the FHOCP will be shown in Section 3.3 by a simulation example.*

3.2.2 Feasibility

In order to show the robust positive invariance of the feasible region, X_{MPC}, under the closed loop dynamics given by (3.1) and (3.6), an upper norm bound for the maximal admissible uncertainty will be stated in Assumption 3.2.3, motivated by Lemma 3.2.2 [METTERE PROOF].

Lemma 3.2.2 (Technical)**.** *Given a set X_{N_c} for which Assumption 3.2.2 holds, let us define $\bar{d}_{\kappa_f} \triangleq \epsilon / L_{f_x}$ and $\bar{d} \triangleq \mathrm{dist}(\mathbb{R}^n \backslash \mathcal{C}_1(X_{N_c}), X_{N_c})$. Under Assumption 3.1.1, it holds that*

1) $X_{N_c} \subset X_{N_c} \oplus \mathcal{B}^n(\bar{d}_{\kappa_f}) \subseteq \mathcal{C}_1(X_{N_c})$*;*

2) $\bar{d} \geq \bar{d}_{\kappa_f}$. □

Proof Notice that, given a vector $x \in X_{N_c} \oplus \mathcal{B}(\bar{d}_{\kappa_f})$, there exist at least one vector $x^{'} \in X_{N_c}$ such that $|x - x^{'}| \leq \epsilon / L_{f_x}$. Since $\hat{f}(x^{'}, \kappa_f(x^{'})) \in X_{N_c} \backsim \mathcal{B}(\epsilon)$, with $\kappa_f(x^{'}) \in U$, then, by Assumption 3.1.1, it follows that $\hat{f}(x, \kappa_f(x^{'})) \in \mathcal{B}(\hat{f}(x^{'}, \kappa_f(x^{'})), \epsilon) \subseteq X_{N_c}$, and hence $x \in \mathcal{C}_1(X_{N_c}), \forall x \in X_{N_c} \oplus \mathcal{B}(\bar{d}_{\kappa_f})$, thus proving the statement. ■

It must be remarked that, for nonlinear systems, the numerical computation of $\mathcal{C}_1(X_{N_c})$ is a very difficult task, although the underlying theory is well established and many different methods have been proposed since the seminal paper [16]. In this regard, for some classes of nonlinear systems, there exist efficient numerical procedures for the computation of pre-images and predecessor sets (see [18, 56, 101]). In addition, a novel algorithm to compute inner approximations of controllability sets will be given in Section 3.4.

Assumption 3.2.3 (Bound on uncertainties)**.** *The $\mathcal{K}$-functions δ and μ are such that the following inequality holds*

$$\delta(|x_t|) + \mu(\Upsilon^{\sup}) \leq L_{f_x}^{1-N_c} \bar{d}, \ \ \forall x_t \in X. \tag{3.11}$$

□

The robust positively invariance of the feasible region, X_{MPC}, under the closed-loop dynamics, can now be stated and proved.

Theorem 3.2.1 (Feasibility)**.** *Let a system be described by equation (3.1) and subject to (3.2) and (3.3). Under Assumptions 3.1.1-3.2.3, the set in which the FHOCP is feasible, X_{MPC}, is also RPI for the closed-loop system under the action of the control law given by (3.6).* □

Proof. It will be shown that the region X_{MPC} is RPI for the closed-loop system, proving that, for all $x_t \in X_{MPC}$, there exists a feasible solution of the FHOCP at time instant $t+1$, based on the optimal solution in t, $\mathbf{u}^{\circ}_{t,t+N_c-1|t}$. In particular, a possible feasible control sequence is given by $\bar{\mathbf{u}}_{t+1,t+N_c|t+1} = \mathrm{col}\big[u^{\circ}_{t+1|t}, \ldots, u^{\circ}_{t+N_c-1|t}, \bar{u}\big]$, where $\bar{u} = \bar{u}(\hat{x}_{t+N_c|t+1}) \in U$ is a feasible control action, suitably chosen to satisfy the robust constraint $\hat{x}_{t+N_c+1|t+1} \in X_{N_c}$.

First, let us introduce the following technical lemma (the proof is given in Appendix).

Lemma 3.2.3 (Technical)**.** *Given x_t and $x_{t+1} = \hat{f}(x_t, \kappa_{MPC}(x_t)) + d_t$, with $d_t \in D_t$, consider the predictions $\hat{x}_{t+N_c|t}$ and $\hat{x}_{t+N_c+1|t+1}$, obtained respectively using the input sequences $\mathbf{u}^{\circ}_{t,t+N_c-1|t}$ and $\bar{\mathbf{u}}_{t+1,t+N_c|t+1}$, and initialized with $\hat{x}_{t|t} = x_t$ and $\hat{x}_{t+1|t+1} = x_{t+1}$. Given X_{N_c} as described in Assumption 3.2.2, suppose that $\hat{x}_{t+N_c|t} \in X_{N_c}$. In view of Assumption 3.2.3, if $\delta(|x_t|) + \bar{\mu} \leq L_{f_x}^{1-N_c}\bar{d}$, then $\hat{x}_{t+N_c|t+1} \in \mathcal{C}_1(X_{N_c})$. Moreover, if $\delta(|x_t|) + \bar{\mu} \leq L_{f_x}^{1-N_c}\bar{d}_{\kappa_f}$, then $\hat{x}_{t+N_c|t+1} \in X_{N_c} \oplus \mathcal{B}^n(\bar{d}_{\kappa_f})$.* □

Now, the proof will be divided in two steps.

1) $\hat{x}_{t+j|t+1} \in \hat{X}_{t+j|t+1}$: First, in view of Assumptions 3.1.1, 3.1.2 and (3.9), it follows that

$$\begin{cases} \hat{\rho}_{t+1|t} - \hat{\rho}_{t+1|t+1} = L_\delta |x_t| + \bar{\mu}, \\ \hat{\rho}_{t+j|t} - \hat{\rho}_{t+j|t+1} = (L_{f_x} + L_\delta)\left(\hat{\rho}_{t+j-1|t} - \hat{\rho}_{t+j-1|t+1}\right) + L_\delta(|\hat{x}_{t+j|t}| - |\hat{x}_{t+j|t+1}|) \\ \qquad \geq (L_{f_x} + L_\delta)\left(\hat{\rho}_{t+j-1|t} - \hat{\rho}_{t+j-1|t+1}\right) - L_\delta L_{f_x}^{j-2}(L_\delta|x_t| + \bar{\mu}), \\ \qquad\qquad \forall j \in \{2, \dots, N_c\}. \end{cases}$$

Proceeding by induction, it follows that, for all $j \in \{2, \dots, N_c\}$

$$\hat{\rho}_{t+j|t} - \hat{\rho}_{t+j|t+1} \geq \left[(L_{f_x} + L_\delta)^{j-1} - L_\delta\,(L_{f_x} + L_\delta)^{j-2} \sum_{k=0}^{j-2}\left(\frac{L_{f_x}}{L_{f_x}+L_\delta}\right)^k\right](L_\delta|x_t| + \bar{\mu})$$

which yields

$$\hat{\rho}_{t+j|t} - \hat{\rho}_{t+j|t+1} \geq L_{f_x}^{j-1}(L_\delta|x_t| + \bar{\mu}), \;\; \forall j \in \{1, \dots, N_c\}. \tag{3.12}$$

Now, consider the predictions $\hat{x}_{t+j|t}$ and $\hat{x}_{t+j|t+1}$, with $j \in \{1, \dots, N_c\}$, made respectively using the input sequences $\mathbf{u}^{\circ}_{t,t+N_c-1|t}$ and $\bar{\mathbf{u}}_{t+1,t+N_c-1|t+1}$, and initialized with $\hat{x}_{t|t} = x_t$ and $\hat{x}_{t+1|t+1} = \hat{f}(x_t, \kappa_{MPC}(x_t))$. Assuming that $\hat{x}_{t+j|t} \in \hat{X}_{t+j|t} \triangleq X \backsim \mathcal{B}^n(\hat{\rho}_{t+j|t})$, with $\hat{\rho}_{t+j|t}$ given by (3.9), let us introduce $\eta \in \mathcal{B}^n(\hat{\rho}_{t+j|t+1})$. Furthermore, let $\xi \triangleq \hat{x}_{t+j|t+1} - \hat{x}_{t+j|t} + \eta$. Then, under Assumption 3.1.1, it follows that $|\xi| \leq |\hat{x}_{t+j|t+1} - \hat{x}_{t+j|t}| + \hat{\rho}_{t+j|t+1} \leq L_{f_x}^{j-1}(L_\delta|x_t| + \bar{\mu}) + \hat{\rho}_{t+j|t+1}$. In view of (3.12), it turns out that $|\xi| \leq \hat{\rho}_{t+1|t}$, and hence, $\xi \in \mathcal{B}^n(\hat{\rho}_{t+j|t})$. Since $\hat{x}_{t+j|t} \in \hat{X}_{t+j|t}$, it follows that $\hat{x}_{t+j|t} + \xi = \hat{x}_{t+j|t+1} + \eta \in X$, $\forall \eta \in \mathcal{B}^n(\hat{\rho}_{t+j|t+1})$, which finally yields $\hat{x}_{t+j|t+1} \in \hat{X}_{t+j|t+1}$.

2) $\hat{x}_{t+N_c+1|t+1} \in X_{N_c}$: if $L_{f_x}^{N_c-1}(\delta(|x_t|) + \bar{\mu}) \leq \bar{d}_{\kappa_f}$, in view of Lemma 3.2.2 there exists a feasible control action such that the statement holds. If $\bar{d}_{\kappa_f} < L_{f_x}^{N_c-1}(\delta(|x_t|) + \bar{\mu}) \leq \bar{d}$, thanks to Lemma 3.2.3, it follows that $\hat{x}_{t+N_c|t+1} \in \mathcal{C}_1(X_{N_c})$. Hence, there exists a feasible control action, namely

$\bar{u} \in U$, such that $\hat{x}_{t+N_c+1|t+1} = \hat{f}(\hat{x}_{t+N_c|t+1}, \bar{u}) \in X_{N_c}$, thus ending the proof. □

Remark 3.2.2. *With respect to previous literature [66, 100], the use of X_{N_c} instead of X_f as stabilizing constraint set and the possibility to compute $\bar{d}$ relying on $\mathcal{C}_1(X_{N_c})$, together allow to enlarge the bound on admissible uncertainties which the controller can cope with. In fact, considering that the restricted constraints are based on Lipschitz constants, which lead to conservative computations, the limitation of their use to a shorter horizon (and not to the whole prediction one) may enlarge the feasible set of the FHOCP.* □

3.2.3 Regional Input-to-State Stability

In the following, the stability properties of system (3.1) in closed-loop with (3.6) are analyzed. Let denote $\alpha_1(s) = \underline{h}(s)$, $\alpha_2(s) = L_{h_f} s$ and $c_1 = 0$. In order to state the main theorem concerning the stability properties of the devised control scheme, the following alternate assumptions are introduced.

Assumption 3.2.4. *With reference to Definition 2.1.3, given*

1) $\alpha_3(s) = \underline{h}(s) - \Big[L_h \dfrac{L_{f_x}^{N_c} - 1}{L_{f_x} - 1} + L_{h_f} L_{f_c}^{N_p-(N_c+1)} L_{f_x}^{N_c} + (L_h + L_{h_u} L_{\kappa_f}) \dfrac{L_{f_c}^{N_p-(N_c+1)} - 1}{L_{f_c} - 1} L_{f_x}^{N_c}\Big] \delta(s)$;

2) $\sigma(s) = \Big[L_h \dfrac{L_{f_x}^{N_c} - 1}{L_{f_x} - 1} + (L_h + L_{h_u} L_{\kappa_f}) \dfrac{L_{f_c}^{N_p-N_c-1} - 1}{L_{f_c} - 1} L_{f_x}^{N_c} + L_{h_f} L_{f_c}^{N_p-(N_c+1)} L_{f_x}^{N_c}\Big] \mu(s)$;

3) $c_2 = 0$;

let

i) $\delta(|x_t|) + \mu(|\upsilon_t|) \leq L_{f_x}^{1-N_c} \bar{d}_{\kappa_f}$, $\forall x_t \in X_{MPC}$, $\forall \upsilon_t \in \Upsilon$*;*

ii) α_3 *be a* $\mathcal{K}_\infty$*-function* $\forall s \leq \sup_{x \in X_{MPC}} \{|x|\}$ *;*

iii) Υ *be such that* Θ *defined in (2.5) is contained in* $\Omega \backsim \mathcal{B}^n(c)$*, for some* $c \in \mathbb{R}_{>0}$ *and for all* $\upsilon \in \Upsilon$*.* □

Assumption 3.2.5. *With reference to Definition 2.1.3, given*

1) $\alpha_3(s) = \underline{h}(s) - L_h \dfrac{L_{f_c}^{N_c} - 1}{L_{f_c} - 1} \delta(s)$;

2) $\sigma(s) = L_h \dfrac{L_{f_c}^{N_c} - 1}{L_{f_c} - 1}\mu(s)$*;*

3) $c_2 = \Big[(L_h{+}L_{h_u}L_{\kappa_f})\dfrac{L_{f_c}^{N_p-(N_c+1)}-1}{L_{f_c}-1}{+}L_{h_f}L_{f_c}^{N_p-(N_c+1)}\Big]\max\big\{|x-\xi|, (x,\xi){\in}X_{N_c}{\times}(X_{N_c}{\oplus}\mathcal{B}^n(\bar{d}))\big\}{+}$ $L_{h_f}L_{f_c}^{N_p-(N_c+1)}L_{h_u}\max\{|u-w|, (u,w){\in}U{\times}U\}$ *;*

let

i) $L_{f_x}^{1-N_c}\bar{d}_{\kappa_f} < \delta(|x_t|) + \mu(|\upsilon_t|) \le L_{f_x}^{1-N_c}\bar{d}$, $\forall x_t \in X_{MPC}$, $\forall \upsilon_t \in \Upsilon$*;*

ii) α_3 *be a* $\mathcal{K}_\infty$*-function* $\forall s \le \sup_{x\in X_{MPC}}\{|x|\}$ *;*

iii) Υ *be such that* Θ *defined in (2.5) is contained in* $\Omega \backsim \mathcal{B}^n(c)$*, for some* $c \in \mathbb{R}_{>0}$ *for all* $\upsilon \in \Upsilon$. □

Theorem 3.2.2 (Regional Input-to-State Stability)**.** *Let a system be described by equation (3.1) and subject to (3.2) and (3.3). Under Assumptions 3.1.1-3.2.3,*

1) if Assumption 3.2.4 holds, then the closed-loop system (3.1), (3.6) is regional ISS in $\Xi = X_{MPC}$ *with respect to* $\upsilon_t \in \Upsilon$*;*

2) if Assumption 3.2.5 holds, then the closed-loop system (3.1), (3.6) is regional ISpS in $\Xi = X_{MPC}$ *with respect to* $\upsilon \in \Upsilon$*.* □

Proof. In view of Assumptions 3.1.1-3.2.3 it follows from Theorem 3.2.1 that X_{MPC} is a RPI set for system (3.1) under the action of the control law (3.6). So, the proof consists in showing that $V(x_t) = J_{FH}(x_t, \mathbf{u}^{\circ}_{t,t+N_c-1|t}, N_c, N_p)$ is an ISS-Lyapunov function in X_{MPC}. First, by Assumption 3.2.2, the set X_{MPC} is not empty. In fact, for any $x_t \in X_f$, a feasible control sequence for FHOCP is given by $\tilde{\mathbf{u}}_{t,t+N_p-1|t} \triangleq \text{col}\big[\kappa_f(\hat{x}_{t|t}),\ \kappa_f(\hat{x}_{t+1|t}),\ \ldots,\ \kappa_f(\hat{x}_{t+N_c-1|t})\big]$. Then $X_{MPC} \supseteq X_f$. Then, in view of Point 5) of Assumption 3.2.1 and Assumptions 3.1.3-3.2.2 it holds that

$$\begin{aligned} V(x_t) \le J_{FH}(x_t, \tilde{\mathbf{u}}_{t,t+N_c-1|t}, N_c, N_p) &= \sum_{l=t}^{t+N_p-1} h(\hat{x}_{l|t}, \kappa_f(\hat{x}_{l|t})) + h_f(\hat{x}_{t+N_p|t}) \\ \le \sum_{l=t}^{t+N_p-1}\Big[h_f(\hat{x}_{l|t}) - h_f(\hat{x}_{l+1|t})\Big] + h_f(\hat{x}_{t+N_p|t}) & \\ \le \quad h_f(\hat{x}_{t|t}) \le L_{h_f}|x_t|, \qquad & \forall x_t \in X_f. \end{aligned}$$

Hence, there exists a $\mathcal{K}$-function $\alpha_2(|x_t|)$ such that

$$V(x_t) \leq \alpha_2(|x_t|), \ \ \forall x_t \in X_f. \tag{3.13}$$

The lower bound on $V(x_t)$ can be easily obtained using Assumption 3.1.3:

$$V(x_t) \geq \underline{h}(|x_t|), \ \forall x_t \in X_{MPC}. \tag{3.14}$$

Inequalities (2.2) and (2.3) hold respectively with $\Xi = X_{MPC}$ and $\Omega = X_f$. Suppose[2] that $L_{f_c} \neq 1$. Now, in view of Assumption 3.2.2 and Theorem 3.2.1, given the optimal control sequence at time t, $\mathbf{u}^{\circ}_{t,t+N_c-1|t+1}$, the sequence $\bar{\mathbf{u}}_{t+1,t+N_c|t} \triangleq \mathrm{col}\big[u^{\circ}_{t+1|t}, \dots, u^{\circ}_{t+N_c-1|t}, \bar{u}\big]$ with

$$\bar{u} = \begin{cases} \kappa_f(\hat{x}_{t+N_c|t}), & \text{if } \delta(|x_t|) + \mu(|\upsilon_t|) \leq L_{f_x}^{1-N_c}\bar{d}_{\kappa_f} \\ \bar{u} \in U : \hat{f}(\hat{x}_{t+N_c|t+1}, \bar{u}) \in X_{N_c}, & \text{if } L_{f_x}^{1-N_c}\bar{d}_{\kappa_f} < \delta(|x_t|) + \mu(|\upsilon_t|) \leq L_{f_x}^{1-N_c}\bar{d} \end{cases}$$

is a feasible (in general, suboptimal) control sequence for the FHOCP initiated with $x_{t+1} = f(x_t, \kappa_{MPC}(x_t), \upsilon_t)$ at time $t+1$, with cost

$$\begin{aligned} & J_{FH}(x_{t+1}, \bar{\mathbf{u}}_{t+1,t+N_c|t+1}, N_c, N_p) = \\ & \quad V(x_t) - h(x_t, u^{\circ}_{t,t}) + \sum_{l=t+1}^{t+N_c-1} \Big[h(\hat{x}_{l|t+1}, u^{\circ}_{l|t}) - h(\hat{x}_{l|t}, u^{\circ}_{l|t}) \Big] + h(\hat{x}_{t+N_c|t+1}, \bar{u}) \\ & \quad - h(\hat{x}_{t+N_c|t}, \kappa_f(\hat{x}_{t+N_c|t})) + \sum_{l=t+(N_c+1)}^{t+N_p-1} \Big[h(\hat{x}_{l|t+1}, \kappa_f(\hat{x}_{l|t+1})) - h(\hat{x}_{l|t}, \kappa_f(\hat{x}_{l|t})) \Big] \\ & \quad + h(\hat{x}_{t+N_p|t+1}, \kappa_f(\hat{x}_{t+N_p|t+1})) + h_f(\hat{f}(\hat{x}_{t+N_p|t+1}, \kappa_f(\hat{x}_{t+N_p|t+1}))) - h_f(\hat{x}_{t+N_p|t}). \end{aligned}$$

Using Assumptions 3.1.1 and 3.1.3, it follows that

$$\Big| h(\hat{x}_{t+j|t+1}, u^{\circ}_{t+j|t}) - h(\hat{x}_{t+j|t}, u^{\circ}_{t+j|t}) \Big| \leq L_h L_{f_x}^{j-1} (\delta(|x_t|) + \mu(|\upsilon_t|)), \tag{3.15}$$

for all $j \in \{1, \dots, N_c - 1\}$. Moreover, for $j = N_c$, we have

$$\begin{aligned} & \Big| h(\hat{x}_{t+N_c|t+1}, \bar{u}) - h(\hat{x}_{t+N_c|t}, \kappa_f(\hat{x}_{t+N_c|t})) \Big| \leq \\ & \qquad L_h L_{f_x}^{N_c-1} (\delta(|x_t|) + \mu(|\upsilon_t|)) + L_{h_u} \Delta_u (\delta(|x_t|) + \mu(|\upsilon_t|)), \end{aligned} \tag{3.16}$$

[2]The very special case $L_{f_c} = 1$ can be trivially addressed by a few suitable modifications to the proof of Theorem 3.2.2.

where

$$\Delta_u(s) \triangleq \begin{cases} 0, & \text{if } s \leq L_{f_x}^{1-N_c} \bar{d}_{\kappa_f} \\ \max\{|u-w|, (u,w) \in U \times U\}, & \text{if } L_{f_x}^{1-N_c} \bar{d}_{\kappa_f} < s \leq L_{f_x}^{1-N_c} \bar{d}. \end{cases} \tag{3.17}$$

Finally, under Assumptions 3.1.1, 3.1.3 and 3.2.2, for all $j \in \{N_c+1, \ldots, N_p-1\}$, the following intermediate results hold

$$\begin{aligned} &\left| h(\hat{x}_{t+j|t+1}, \kappa_f(\hat{x}_{t+j|t+1})) - h(\hat{x}_{t+j|t}, \kappa_f(\hat{x}_{t+j|t})) \right| \leq \\ &\qquad (L_h + L_{h_u} L_{\kappa_f}) L_{f_c}^{j-(N_c+1)} \Big[\Delta_x(\delta(|x_t|) + \mu(|\upsilon_t|)) + L_{f_x}^{N_c}(\delta(|x_t|) + \mu(|\upsilon_t|)) \Big], \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} &\left| h_f(\hat{x}_{t+N_p|t+1}) - h_f(\hat{x}_{t+N_p|t}) \right| \\ &\qquad \leq L_{h_f} L_{f_c}^{N_p-(N_c+1)} \Big[\Delta_x(\delta(|x_t|) + \mu(|\upsilon_t|)) + L_{f_x}^{N_c}(\delta(|x_t|) + \mu(|\upsilon_t|)) \Big], \end{aligned} \tag{3.19}$$

where

$$\Delta_x(s) \triangleq \begin{cases} 0, & \text{if } s \leq L_{f_x}^{1-N_c} \bar{d}_{\kappa_f}, \\ \max\{|x-\xi|, (x,\xi) \in X_{N_c} \times (X_{N_c} \oplus \mathcal{B}^n(\bar{d}))\} - L_{f_x}^{N_c} s, & \text{if } L_{f_x}^{1-N_c} \bar{d}_{\kappa_f} < s \leq L_{f_x}^{1-N_c} \bar{d}. \end{cases} \tag{3.20}$$

Consider now the case $\delta(|x_t|) + \mu(|\upsilon_t|) \leq L_{f_x}^{1-N_c} \bar{d}_{\kappa_f}, \forall x_t \in X, \forall \upsilon_t \in \Upsilon$. Then, in view of Points 5) and 6) of Assumption 3.2.1, Assumption 3.2.2 and by using (3.15)-(3.20), the following inequalities hold

$$\begin{aligned} &J_{FH}(x_{t+1}, \bar{\mathbf{u}}_{t+1,t+N_c|t}, N_c, N_p) \\ &\leq V(x_t) - h(x_t, u^{\circ}_{t,t}) + \sum_{j=1}^{N_c} L_h L_{f_x}^{j-1}(\delta(|x_t|) + \mu(|\upsilon_t|)) + \sum_{j=N_c+1}^{N_p-1} (L_h + L_{h_u} L_{\kappa_f}) L_{f_c}^{j-(N_c+1)} \\ &\quad \times L_{f_x}^{N_c}(\delta(|x_t|) + \mu(|\upsilon_t|)) + h(\hat{x}_{t+Np|t+1}, \kappa_f(\hat{x}_{t+Np|t+1})) + h_f(\hat{x}_{t+N_p+1|t+1}) - h_f(\hat{x}_{t+N_p|t+1}) \\ &\quad + L_{h_f} L_{f_c}^{N_p-(N_c+1)} L_{f_x}^{N_c}(\delta(|x_t|) + \mu(|\upsilon_t|)). \end{aligned}$$

Now, from inequality $V(x_{t+1}) \leq J_{FH}(x_{t+1}, \bar{\mathbf{u}}_{t+1,t+N_c|t}, N_c, N_p)$ it follows that

$$V\left(f(x_t, \kappa_{MPC}(x_t), \upsilon_t)\right) - V(x_t) \leq -\alpha_3(|x_t|) + \sigma(|\upsilon_t|), \ \forall x_t \in X_{MPC}, \forall \upsilon_t \in \Upsilon \tag{3.21}$$

with $\alpha_3(s)$ and $\sigma(s)$ defined as is Assumption 3.2.4. If $\delta(s)$ is such that $\alpha_3(s)$ is a $\mathcal{K}_\infty$-function $\forall s \le \sup_{x\in X_{MPC}}\{|x|\}$, then the closed-loop system has a stability margin for [50]. Therefore, by (3.13), (3.14) and (3.21), if $\delta(|x_t|) + \mu(|\upsilon_t|) \le L_{f_x}^{1-N_c}\bar{d}_{\kappa_f}$, $\forall x_t \in X_{MPC}$, $\forall \upsilon_t \in \Upsilon$, then the optimal cost $J_{FH}(x_t,\mathbf{u}^\circ_{t,t+N_c-1|t},N_c,N_p)$ is an ISS-Lyapunov function for the closed-loop system in X_{MPC}. Hence, it is possible to conclude that the closed-loop system is regional ISS in X_{MPC} with respect to $\boldsymbol{\upsilon} \in \mathcal{M}_\Upsilon$.

Conversely, if $L_{f_x}^{1-N_c}\bar{d}_{\kappa_f} < \delta(|x_t|)+\mu(|\upsilon_t|) \le L_{f_x}^{1-N_c}\bar{d}$, the following inequality can be straightforwardly obtained

$$V\left(f(x_t,\kappa_{MPC}(x_t),\upsilon_t)\right) - V(x_t) \le -\alpha_3(|x_t|) + \sigma(|\upsilon_t|) + c_2,\ \forall x_t \in X_{MPC}, \forall \upsilon_t \in \Upsilon \tag{3.22}$$

with $\alpha_3(s)$, $\sigma(s)$ and c_2 defined as is Assumption 3.2.5. Hence, in the latter case, provided that α_3 is a $\mathcal{K}_\infty$-function, only ISpS can be guaranteed, although the invariance of X_{MPC} and the fulfillment of constraints are preserved thanks to Theorem 3.2.1. □

The characterization of the ISS property for the controlled system in terms of Lyapunov function, as well as the characterization of the maximal admissible uncertainty under which such a property is guaranteed, allows to design effective robust MPC schemes with ease. In the following section, we will show how the basic set-invariance theoretic tools and the regional ISS can be used for the synthesis and the analysis of the stabilizing property of the controller and the estimation of its domain of attraction.

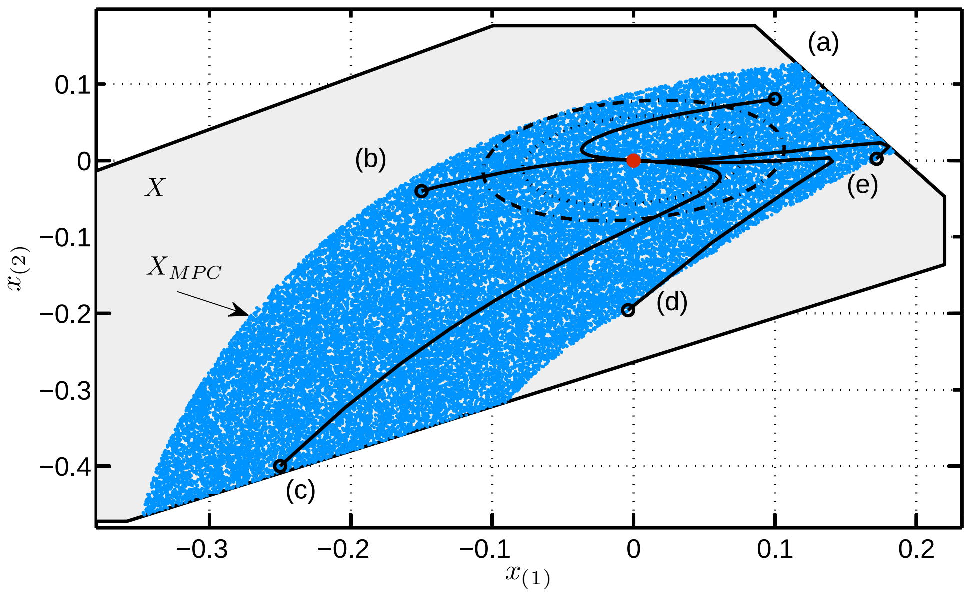
3.3 Simulation Results

Consider the following discrete-time model of an undamped nonlinear oscillator

$$\begin{cases} x_{(1)_{t+1}}=x_{(1)_t}+0.05\left[-x_{(2)_t}+0.5\left(1+x_{(1)_t}\right)u_t\right]+d_{(1)_t} \\ x_{(2)_{t+1}}=x_{(2)_t}+0.05\left[x_{(1)_t}+0.5\left(1-4x_{(2)_t}\right)u_t\right]+d_{(2)_t} \end{cases} \tag{3.23}$$

where the subscript (i) denotes the i-th component of a vector. The uncertainty vector is given by $d_t=1\cdot10^{-3}x_t+\upsilon_t$, with $|\upsilon_t| \le 1\cdot10^{-4}$. System (3.23) is subject to state and input constraints (3.2) and (3.3), where the set X is depicted in Figure 3.2, while $U \triangleq \{u \in \mathbb{R} : |u| \le 2\}$. The Lipschitz constant of the system is L_{f_x}=1.1390.

Figure 3.2 Perturbed closed-loop trajectories with initial points: (a)=$(0.1, 0.08)^T$, (b)=$(-0.15, -0.04)^T$, (c)=$(-0.25, -0.40)^T$, (d)=$(0.00, -0.20)^T$, (e)=$(-0.17, 0.00)^T$. The robust constraint set X_{N_c} (dash-dotted) and the set X_f (dotted) are emphasized.



Since affordable algorithms exist for the numerical computation of the Pontryagin difference set of polytopes, for implementation purposes the balls to be "subtracted" (in the Potryagin sense) from the constraint set X to obtain $\hat{X}_{t+j|t}$, $\forall j \in \{1, \ldots, N_c\}$ are outer approximated by convex parallelotopes.

A linear state feedback control law $u_t = \kappa_f(x_t) = k^T x_t$, with $k \in \mathbb{R}^2$, stabilizing (3.23) in a neighborhood of the origin, can be designed as described in [88]. Choosing $k = [0.5955 \;\; 0.9764]^T$ and $N_c = 5$, the following ellipsoidal sets, X_f and X_{N_c}, satisfy Assumption 3.2.1 and 3.2.2 respectively

$$X_f \triangleq \left\{ x_t \in \mathbb{R}^n \colon x_t^T \begin{bmatrix} 167.21 & -43.12 \\ -43.12 & 305.50 \end{bmatrix} x_t \leq 1 \right\}, X_{N_c} \triangleq \left\{ x_t \in \mathbb{R}^n \colon x_t^T \begin{bmatrix} 114.21 & -29.45 \\ -29.45 & 208.67 \end{bmatrix} x_t \leq 1 \right\},$$

with $L_{\kappa_f} = 1.1437$, $L_{f_c} = 1.0504$ and $N_p = 12$. Let the stage cost h be given by $h(x, u) \triangleq x^T \mathrm{Q} x + u^T \mathrm{R} u$, and the final cost h_f by $h_f(x) \triangleq x^T P x$, with

$$Q = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \text{R} = 1, \text{P} = \begin{bmatrix} 91.56 & -23.61 \\ -23.61 & 167.28 \end{bmatrix},$$

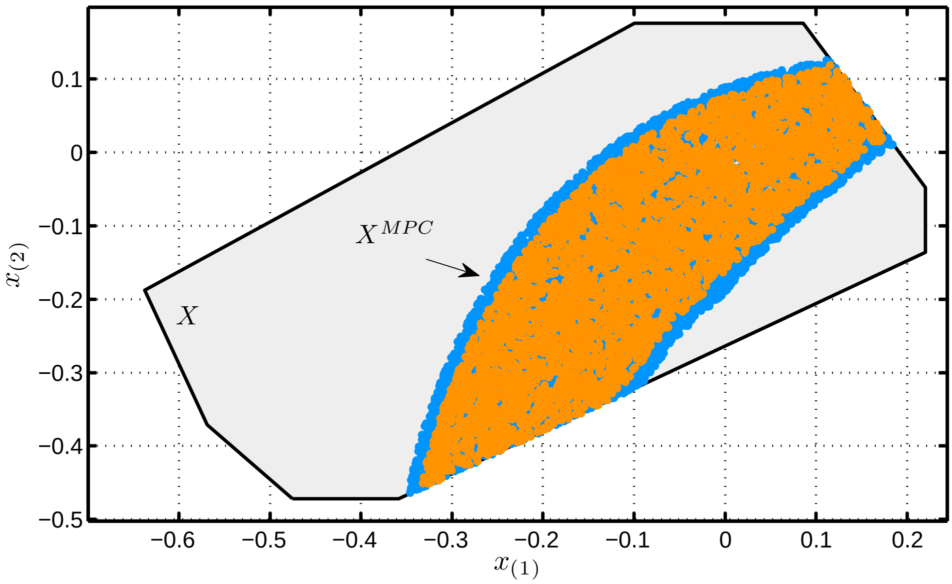
then h_f satisfies Points 2) and 6) of Assumption 3.2.1 in $X_f \subseteq X$. Moreover $\kappa_f(x)$ satisfies point 7) of Assumption 3.2.1 in X_f. The following values for $\bar{d}_{\kappa_f}$ and $\bar{d}$ can be computed: $\bar{d}_{\kappa_f}$=$2.3311 \cdot 10^{-4}$ and $\bar{d}$=$1.2554 \cdot 10^{-3}$. It follows that the admissible uncertainties, for which the feasibility set X_{MPC} is RPI under the closed-loop dynamics, are bounded by

$$\delta(|x_t|) + \mu(\Upsilon^{\sup}) \leq 7.4591 \cdot 10^{-4}, \ \ \forall x_t \in X.$$

Sample perturbed closed-loop trajectories obtained by simulation are depicted in Figure 3.2.

Figure 3.3 compares the feasible set X_{MPC} obtained for the developed MPC policy (cyan) with the feasible region of the controller designed according to [66] (orange), using the same control horizon length with terminal constraint set X_f. It is enhanced the improving of the new algorithm in terms of domain of attraction.

Figure 3.3 Comparison between the feasible region of the controller designed accordingly to [66] with terminal set X_f and the enlarged feasible set obtained by using the robust constraint set X_{N_c}.



In this section, we used some basic notions of set-invariance theory, such as controllability sets, to characterize the robustness of the resulting MPC scheme. In order not to be reduced to a vague concept, in the sequel of the chapter we are going to provide a numerical tool to compute inner approximations of controllability sets, which can be used as conservative estimates of the true ones in order to assess the robustness of the devised schemes. The algorithm proposed has been used in the last given example to provide an effective estimate of the maximal admissible uncertainty tolerable by the controller.

On the other hand, not to oversell the devised numerical procedure, it must be remarked that, making use of gridding and computationally expensive set iterations, it can apply only to small dimensional systems. However, by using some important computability results for predecessor operators based on set-valued function theory, it has been possible to prove some interesting properties of the proposed method; among them, the convergence and the monotonicity results are recalled, since they guarantee that the set-estimates are improved at each iteration step, while the convergence to the a desired distance metric is ensured for infinite iterations.

3.4 Computing Convex Inner Approximations of Controllability Sets

In the previous sections, a design methodology for RH control schemes has been presented, and the ISS properties of the resulting closed-loop scheme have been established, provided that the uncertainties/perturbations are suitably bounded.

In particular, the bound on additive transition uncertainty, under which the invariance of the feasible set for the MPC is guaranteed, has been shown to depend on the invariance properties of the terminal set constraint X_f (see inequality (3.11)).

Indeed, the distance metric $\text{dist}(\mathbb{R}^n \backslash \mathcal{C}_1(X_f), X_f)$, on which the uncertainty bound relies on, can be viewed as a measure of inherent contractivity of X_f under the controlled transition map $\hat{f}(x, u)$ with the input constraint $u \in U$, and is therefore independent on the particular auxiliary terminal controller used in the stability proof.

At a first glance, the exact evaluation of $\text{dist}(\mathbb{R}^n \backslash \mathcal{C}_1(X_f), X_f)$ can be carried out by directly computing $\mathcal{C}_1(X_f)$. However, in most situations, only an inner approximation $\hat{\mathcal{C}}_1(X_f)$ can be obtained numerically.

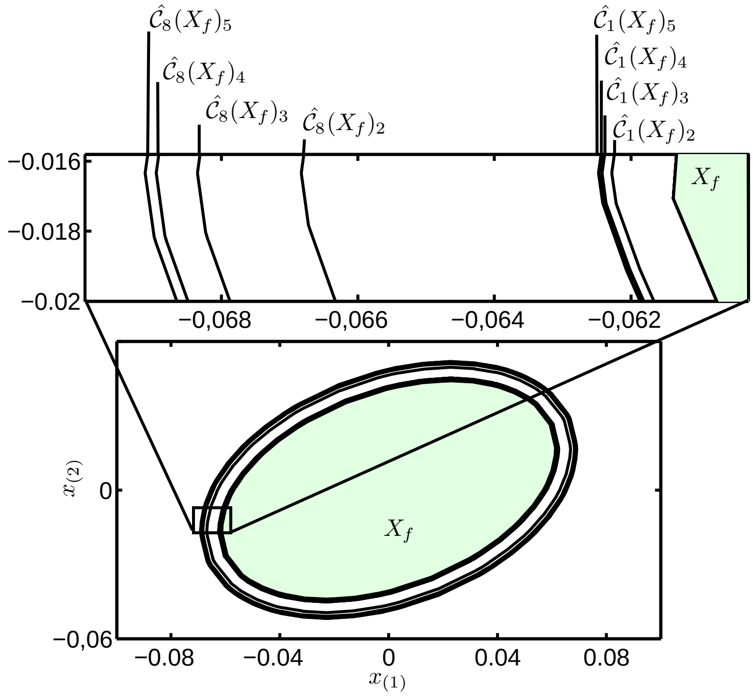
In this respect, given a compact invariant set $X_f \subset \mathbb{R}^n$ an iterative procedure will be described to compute convex inner approximations of controllability sets $\mathcal{C}_i(X_f)$, $i \in \mathbb{Z}_{>0}$. In turn, the procedure will allow to compute lower approximations of the metric $\text{dist}(\mathbb{R}^n \backslash \mathcal{C}_i(X_f), X_f)$, $i \in \mathbb{Z}_{>0}$, which is a multi-step generalization of the metric $\text{dist}(\mathbb{R}^n \backslash \mathcal{C}_1(X_f), X_f)$ used to bound the admissible uncertainty for RH schemes. The multi-step result will be used later on in the framework of Networked Control System to characterize the stability property of a control strategy, based on MPC, devised for this particular class of systems (see Chapter 5).

The proposed algorithm algorithm is based on the following recursion

$$\begin{cases} \hat{\mathcal{C}}_i(\Xi)_1 = X_f, \\ \hat{\mathcal{C}}_i(\Xi)_{j+1} = \vartheta(\hat{\mathcal{C}}_i(\Xi)_j, X_f), \;\; j \in \mathbb{Z}_{\geq 1}, \end{cases} \tag{3.24}$$

where ϑ is a suitable set-valued function and $\hat{\mathcal{C}}_i(\Xi)_j$ is the convex inner approximation of $\mathcal{C}_i(X_f)$ computed at the j-th iteration. Referring to a numerical example reported in Section 3.3, Figure 3.4 shows a graphical representation of two sequences of sets generated by such a recursion to approximate $\mathcal{C}_1(X_f)$ and $\mathcal{C}_8(X_f)$.

Figure 3.4 Sequence of sets generated by the iterative procedure (3.24) for the terminal set X_f used in the example reported in Section 3.3.



The main objective of the following analysis consists in designing the above set-valued operator ϑ, such as to guarantee the convergence the algorithm toward the desired euclidean metric $\text{dist}(\mathbb{R}^n \backslash \mathcal{C}_i(X_f), X_f)$. In order to determine a function ϑ capable to satisfy the above requirements, we need to address the issue of numerical computability of set-valued operators, which poses indeed some constraints on the structure of the approximation algorithm. A detailed analysis of computability of set-valued operators for nonlinear discrete-time autonomous and controlled systems is given in [8] and [29]. In order to present a key result on the computability of controllability sets, some notions of set-valued analysis [10] are needed (see Appendix A.3). In particular, the notion of Robust Controllability Set of $\Xi \subset \mathbb{R}^n$ in $X \subset \mathbb{R}^n$, denoted as $\mathcal{RC}_i(X, \Xi)$ (see A.3.5), will be used extensively in the sequel. In the following, the shorthand $\mathcal{RC}_i(\Xi)$ will be used to denote $\mathcal{RC}_i(\mathbb{R}^n, \Xi)$.

The possibility to obtain an arbitrary accurate numerical approximation of the robust predecessor set is guaranteed by the following approximate computability result.

Theorem 3.4.1 ([29]). *Given a set $\Xi \subseteq X$, if the map $\hat{F}(x)$ is LSC in x, $\forall x \in X$, then $\mathcal{C}_1(\Xi)$ is open whenever Ξ is open. Hence, the operator $\Xi \mapsto \mathcal{RC}_1(\Xi)$ is always lower semicomputable, i.e., it can be approximated arbitrarily well by a sequence of compact sets $\{\hat{\mathcal{C}}_1(\Xi)_j\}$, $j \in \mathbb{Z}_{\geq 1}$, with $\hat{\mathcal{C}}_1(\Xi)_j \supset \hat{\mathcal{C}}_1(\Xi)_{j-1}$, given an initial lower approximation $\hat{\mathcal{C}}_1(\Xi)_1 \subset \mathcal{RP}(\Xi) \subseteq \mathcal{C}_1(\Xi)$.* □

Noting that a set-valued map defined as in (A.6) is LSC under Assumption 3.1.1, Theorem 3.4.1 can be readily extended to characterize the computability of the i-step robust controllability set $\mathcal{RC}_i(\Xi)$. In this regard, let us introduce the following problem.

Problem 3.4.1. *Given a finite integer $i \in \mathbb{Z}_{>0}$ and an invariant compact set $X_f \subset \mathbb{R}^n$, we look for a numerical set-iterative procedure, in the form of (3.24), capable to generate a sequence $\{\hat{\mathcal{C}}_i(X_f)_j,\ j \in \mathbb{Z}_{\geq 1}\}$ of compact sets lower approximating $\mathcal{RC}_i(X_f)$, such that*

1) $\hat{\mathcal{C}}(X_f)_j \subset \mathcal{C}_i(X_f),\ \forall j \in \mathbb{Z}_{>0}$;

2)if $\mathrm{dist}(\mathbb{R}^n \backslash \hat{\mathcal{C}}_i(X_f)_j, X_f) < \mathrm{dist}(\mathbb{R}^n \backslash \mathcal{RC}_i(X_f), X_f)$

$\Rightarrow \mathrm{dist}(\mathbb{R}^n \backslash \hat{\mathcal{C}}_i(X_f)_{j+1}, X_f) > \mathrm{dist}(\mathbb{R}^n \backslash \hat{\mathcal{C}}_i(X_f)_j, X_f), \forall j \in \mathbb{Z}_{>0}$;

3) $\mathrm{dist}(\mathbb{R}^n \backslash \mathcal{RC}_i(X_f), X_f) \leq \lim_{j \to \infty} \mathrm{dist}(\mathbb{R}^n \backslash \hat{\mathcal{C}}_i(X_f)_j, X_f) \leq \mathrm{dist}(\mathbb{R}^n \backslash \mathcal{C}_i(X_f), X_f)$.

□

Now, we are going to introduce a numerical framework to address the issues raised by Problem 3.4.1. In particular, it will be shown that the function ϑ in (3.24) can be designed such that all requirements are satisfied.

To this end, let us introduce the *Finite Horizon Distance Optimal Control Problem* (FHDOCP).

Problem 3.4.2 (FHDOCP). *Given system (3.4), an invariant compact set $X_f \subset \mathbb{R}^n$ and a compact set $X \subseteq \mathcal{RC}_i(X_f)$ (first guess inner approximation), consider a vector $x_0 \in \partial\Xi$. The (i-steps) Finite Horizon Distance Optimal Control Problem (FHDOCP) consists in finding the sequence of control moves $\mathbf{u}_{0,i-1} = \mathrm{col}[u_0, \ldots, u_{i-1}]$, subject to (1.3), such that the following value function, $J_{FHD}(x_0, \mathbf{u}_{0,i-1}, X_f)$, is maximized:*

$$J_{FHD}(x_0,\mathbf{u}_{0,i-1}(x_0),X_f) \triangleq \Phi\left(\hat{x}(i,x_0,\mathbf{u}_{0,i-1}),X_f\right).$$

□

In the following, we will denote the optimal value function as

$$J^{\circ}_{FHD}(x_0) \triangleq= \max_{\mathbf{u}_{0,i-1}\in U^i}\{\Phi\left(\hat{x}(i,x_0,\mathbf{u}_{0,i-1}),X_f\right)\},$$

and the optimal input sequence as $\mathbf{u}^{\circ}_{0,i-1}(x_0)$. An effective optimization-based algorithm, which satisfies the requirements raised by Problem 3.4.1, is now presented (for the sake of notational simplicity, the dependency of J°_{FHD} on X_f will be omitted).

Theorem 3.4.2 ($\hat{\mathcal{C}}_i(X_f)$)**.** *Given a positive integer $i\in\mathbb{Z}_{>0}$ and a compact invariant set $X_f\subset\mathbb{R}^n$, consider the recursion (3.24) with ϑ defined as follows*

$$\vartheta(\hat{\mathcal{C}}_i(X_f)_j,X_f) \triangleq \hat{\mathcal{C}}_i(X_f)_j \oplus \mathcal{B}\left(L_{f_x}^{-i}\min_{x\in\partial\hat{\mathcal{C}}_i(X_f)_j}\{J^{\circ}_{FHD}(x)\}\right). \tag{3.25}$$

Then the sequence of sets $\{\hat{\mathcal{C}}_i(X_f)_j, j\in\mathbb{Z}_{\geq 1}\}$ satisfies Points 1)-3) of Problem 3.4.1. □

Proof First, it is worth to note that, in a numerical framework, the geometric condition for invariance of a compact set $\Xi\subset\mathbb{R}^n$ stated in Theorem A.3.1 must be replaced by its robust counterpart, i.e., by the condition $\Xi\subseteq\mathcal{RC}_1(\Xi)$. Now, it is possible to prove Theorem 3.4.2. Points 1), 2) and 3) of Problem 3.4.1 are addressed separately in the following.

1) $\hat{\mathcal{C}}_i(X_f)_j\subset\mathcal{C}_i(X_f)\Rightarrow\hat{\mathcal{C}}_i(X_f)_{j+1}\subset\mathcal{C}_i(X_f), \forall j\in\mathbb{Z}_{\geq 1}$: Given a vector $x'\in\hat{\mathcal{C}}_i(X_f)_j\oplus\mathcal{B}(L_{f_x}^{-i}\underline{J}_j)$, with $\underline{J}_j \triangleq \min_{x\in\partial\hat{\mathcal{C}}_i(X_f)_j} J^{\circ}_{FHD}(x)$, then $\exists x''\in\partial\hat{\mathcal{C}}_i(X_f)_j$ $(\subset\mathcal{C}_i(X_f))$ such that $|x'-x''|\leq L_{f_x}^{-i}\underline{J}_j$. Hence, there exists a feasible sequence of controls $\bar{\mathbf{u}}_{0,i-1}$ which yields to $\hat{x}(i,x'',\bar{\mathbf{u}}_{0,i-1})\in X_f \backsim \mathcal{B}(\underline{J}_j)$, with $\underline{J}_j\in\mathbb{R}_{\geq 0}$. Then, under Assumption 3.1.1, the inequality $|\hat{x}(i,x',\bar{\mathbf{u}}_{0,i-1})-\hat{x}(i,x'',\bar{\mathbf{u}}_{0,i-1})|\leq L_{f_x}^{i}|x'-x''|$, yields to $\hat{x}(i,x',\mathbf{u}_{0,i-1})\in X_f$, and hence $x'\in\mathcal{C}_i(X_f)$. These arguments also prove the second inequality at Point 3) of Problem 3.4.1.

2) $\mathrm{dist}(\mathbb{R}^n\backslash\hat{\mathcal{C}}_i(X_f)_{j+1},X_f) < \mathrm{dist}(\mathbb{R}^n\backslash\mathcal{RC}_i(X_f),X_f) \Rightarrow \hat{\mathcal{C}}_i(X_f)_{j+1}\supset\hat{\mathcal{C}}_i(X_f)_j$ $(\supset X_f), \forall j\in\mathbb{Z}_{\geq 1}$: Under the stated assumption, being $\underline{J}_j\in\mathbb{R}_{>0}$, the subsequent inclusion follows from the properties of Minkowski addition. If $\underline{J}_j=0$ for some $j\in\mathbb{Z}_{\geq 1}$, then $\hat{\mathcal{C}}_i(X_f)_{j+1}=\hat{\mathcal{C}}_i(X_f)_j, \forall j\in$

$\mathbb{Z}_{\geq j}$, and hence the limit is finitely determined.

3) $\text{dist}(\mathbb{R}^n \backslash \mathcal{RC}_i(X_f), X_f) \leq \lim_{j\to\infty} \text{dist}(\mathbb{R}^n \backslash \hat{\mathcal{C}}_i(X_f)_j, X_f)$: The proof can be carried out by contradiction. First, by Point 2), notice that $\vartheta(\hat{\mathcal{C}}_i(X_f)_\infty) = \hat{\mathcal{C}}_i(X_f)_\infty$. Assume there exists $\epsilon^{'} \in \mathbb{R}_{>0}$ such that $\lim_{j\to\infty}\text{dist}(\mathbb{R}^n \backslash \hat{\mathcal{C}}_i(X_f)_j, X_f) = \text{dist}(\mathbb{R}^n \backslash \mathcal{RC}_i(X_f), X_f) - \epsilon^{'}$. Then, the set $\partial\hat{\mathcal{C}}_i(X_f)_\infty \subset \mathcal{C}_i(\text{int}(X_f))$. Since there exists $\epsilon^{''} \in \mathbb{R}_{>0}$ such that $\min_{x\in\partial\hat{\mathcal{C}}_i(X_f)_\infty} J^\circ_{FHD}(x) \geq \epsilon^{''}$, then there exist a set $\vartheta(\hat{\mathcal{C}}_i(X_f)_\infty) \supset \hat{\mathcal{C}}_i(X_f)_\infty$, which invalidates the original assumption. ■

Remark 3.4.1. *Notice that Theorem 3.4.2 assumes that the optimal value $J^\circ_{FHD}(x)$ for each $x \in \partial\hat{\mathcal{C}}_i(X_f)_j$ as well as the global minimum $\min_{x\in\partial\hat{\mathcal{C}}_i(X_f)_j}\{J^\circ_{FHD}(x)\}$ can be actually obtained. For a generic nonlinear system this is not always the case, therefore a numerical method to approximate the set-valued function ϑ is described. Notably, in this case the constraints imposed by Points 1)-3) of Problem 3.4.1 cannot be strictly fulfilled, but can be violated with an arbitrarily small tolerance specified by the designer, as described in the following section.* □

3.4.1 Numerical implementation of the set-iterative scheme

In order to derive a numerically affordable implementation of the set iterations (3.24)-(3.25), some properties of the optimal value function $J^\circ_{FHD}(\cdot)$ are going to be analyzed. In particular, in a neighborhood of a point $x_0^{'}$ for which the optimal value function yields to $J^\circ_{FHD}(x_0^{'}) = J_{FHD}(x_0^{'}, \mathbf{u}^\circ_{0,i}(x_0^{'}))$, a conic lower bound can be established.

Lemma 3.4.1. *Under Assumptions 3.1.1, given a vector $x_0^{'} \in \partial\hat{\mathcal{C}}_i(X_f)_j$, the optimal cost $J^\circ_{FHD}(x_0^{'}, \mathbf{u}^\circ_{0,i}(x_0^{'}))$, the optimal control sequence $\mathbf{u}^\circ_{0,i}(x_0^{'})$, and the optimal state prediction $\hat{x}(x_0^{'}, \mathbf{u}^\circ_{0,i}(x_0^{'}), i)$, then the optimal value of the function $J^\circ_{FHD}(x_0^{''}, \mathbf{u}^\circ_{0,i}(x_0^{''}))$ is lower bounded by*

$$J^\circ_{FHD}(x_0^{''}) \geq J_{FHD}(x_0^{''}, \mathbf{u}^\circ_{0,i}(x_0^{'})) \geq J^\circ_{FHD}(x_0^{'}) - \alpha L_{f_x}^{\ i} \tag{3.26}$$

for any vector $x_0^{''} \in \mathbb{R}^n : |x_0^{'} - x_0^{''}| \leq \alpha$, with $\alpha \in \mathbb{R}_{>0}$. □

Figure 3.5 shows a pictorial representation of the lower bound result stated by Lemma 3.4.1, which is proved below.

Proof In view of Assumption 3.1.1, it follows that

$$|\hat{x}(i, x_0^{'}, \mathbf{u}_{0,i}^{\circ}(x_0^{'})) - \hat{x}(i, x_0^{''}, \mathbf{u}_{0,i}^{\circ}(x_0^{'}))| \leq L_{f_x}^{\ i}\alpha. \tag{3.27}$$

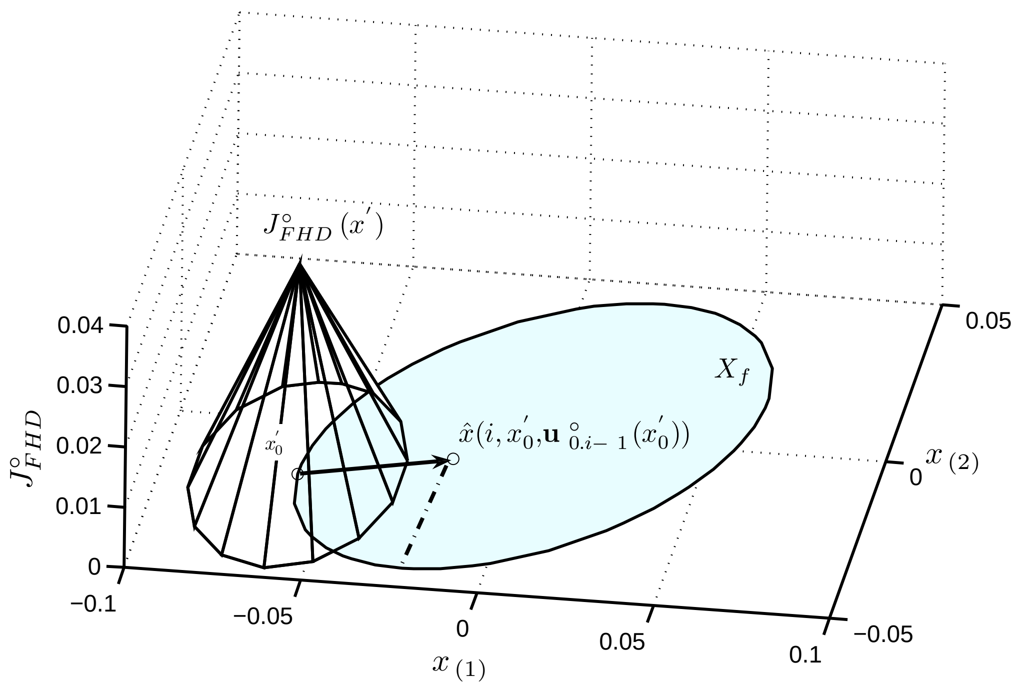
Let $\eta \in \mathbb{R}_{>0}$ be such that $\alpha = \eta L_{f_x}^{-i} J_{FHD}^{\circ}(x_0^{'})$, then it follows that the state vector $\hat{x}(i, x_0^{''}, \mathbf{u}_{0,i}^{\circ}(x_0^{'})) \in \mathcal{B}^n\left(\hat{x}(i, x_0^{'}, \mathbf{u}_{0,i}^{\circ}(x_0^{'})), \eta J_{FHD}^{\circ}(x_0^{'})\right)$. Considering that $J_{FHD}^{\circ}(x_0^{''}) \geq J_{FHD}(x_0^{''}, \mathbf{u}_{0,i}^{\circ}(x_0^{'}))$, then

$$J_{FHD}^{\circ}(x_0^{''}) \geq (1-\eta) J_{FHD}^{\circ}(x_0^{'}).$$

Finally, substituting the expression for η, the statement of the lemma trivially follows. ■

In the sequel, an algorithm for numerically approximating the set-valued function ϑ in (3.25) is discussed.

Figure 3.5 Graphical representation of the conic lower bound on $J_{FHD}^{\circ}(x_0^{''})$, for $|x_0^{''} - x_0^{'}| < L_{f_x}^{-i} J_{FHD}^{\circ}(x^{'})$.



Procedure 3.4.1 (Numerical recipe for $\vartheta(\hat{\mathcal{C}}_i(X_f)_j, X_f)$)**.** *First, notice that, given a lower bound*

$$\underline{J}_j \leq \min_{x \in \partial\hat{\mathcal{C}}_i(x_f)_j} \{J^\circ_{FHD}(x)\}, \;\; \underline{J}_j \in \mathbb{R}_{>0}$$

then the following inequality holds

$$\mathcal{C}_i(x_f)_j \subseteq \left(\hat{\mathcal{C}}_i(x_f)_j \oplus (L_{f_x}^{-1}\underline{J}_j)\right) \subseteq \vartheta(\hat{\mathcal{C}}_i(X_f)_j, X_f)$$

Thanks to Lemma 3.4.1, $\underline{J}_j$ can be obtained by performing a series of FHDOCP's in suitably chosen vectors belonging to $\partial\hat{\mathcal{C}}_i(X_f)_j$. In order to ensure the termination of the procedure in a finite number of steps, let us fix an arbitrary tolerance $\delta \in \mathbb{R}_{>0}$, whose significance will be cleared later on. At this point, let us consider a grid-like subset $X_H \subset \partial\hat{\mathcal{C}}_i(X_f)_j$ such that $\mathrm{d}(x, X_H) \leq \delta$, $\forall x \in \partial\hat{\mathcal{C}}_i(X_f)_j$ and $\exists \epsilon \in \mathbb{R}_{>0}$: $\mathrm{d}(x, X_H \backslash \{x\}) \geq \epsilon$, $\forall x \in X_H$. Being $\hat{\mathcal{C}}_i(X_f)_j$ compact, X_H is numerable. Then, performing a finite number of FHDOCP's on all the vectors of X_H, we can compute

$$\underline{J}_j = (1 - 0.5 L_{f_x}\delta) \min_{x \in X_H} \{J^\circ_{FHD}(x)\}$$

If $\underline{J}_j < 0$, the recursion is terminated. Notice that δ can be reduced to allow for a finer gridding, which permits to compute a possibly tighter bound, at the cost of an increase in the computational load. Conversely, if $\underline{J}_j > 0$, the set $\hat{\mathcal{C}}_i(X_f)_{j+1} = \hat{\mathcal{C}}_i(x_f)_j \oplus (L_{f_x}^{-1}\underline{J}_j)$ is computed and the recursion is continued.

For implementation purposes, it is convenient that X_f is given as a polytope [4], such that the sets $\hat{\mathcal{C}}_i(X_f)_{j+1}$, $j \in \{1, \ldots, N_c\}$ can be obtained by performing the Minkowski addition between polytopes, having previously inner approximated the ball addendum in (3.25) by a parallelotope.

□

3.5 Concluding Remarks

In this chapter, a robust model predictive control design method for constrained discrete-time nonlinear systems with state-dependent uncertainty and persistent disturbances has been presented.

Under suitable assumptions, by employing the devised technique, based on tightening the original state-constraints, the robust constraint satisfaction and the recursive feasibility of the scheme can be guaranteed.

Furthermore, the closed-loop system under the action of the MPC law has been shown to be Input-to-State Stable with respect to additive bounded perturbations.

Remarkably, the method proposed to tighten the constraints, which uses the nominal state predictions to compute the set restrictions, allows for less conservative results and yields to enlarged feasible regions with respect to previous approaches.

In the belief of the author, the conception of methodologies to reduce the inherent conservativeness of constraint-tightening MPC represents a key point toward the possibility to use this technique as an alternative to min-max MPC for uncertain nonlinear systems with fast dynamics. Indeed, due to ease of implementation and to the reduced computational burden required by the open-loop optimization of constraint-tightening MPC, it is more attractive than min-max formulations for practical deployment.

Chapter 4

Off-line Approximated Nonlinear MPC

In the last few years, the problem of reducing the computational complexity of MPC has attracted increasing interest among the control engineers. Indeed, nonlinear plants with fast dynamics, which require the computation of the control action with small sampling periods that do not allow to solve the optimization problem on-line, call for the design of explicit RH control laws. In practice, for a generic nonlinear system, only an approximation of the true RH control law can be obtained by off-line computations. However the approximated control law is still required to enforce the robust constraint satisfaction and to guarantee some stability property.

The problem of obtaining explicit RH controllers with quadratic costs and linear constraints for linear systems can be solved by using parametric quadratic programming techniques [1, 15]. The multi-parametric optimization approach has also been used to obtain robust explicit feedback laws for uncertain linear systems [14, 27, 83].

On the other side, in the context of nonlinear systems, an exact explicit solution cannot in general be obtained. In this framework, the idea of approximating MPC control laws with general function approximators, such as artificial neural networks, has been proposed in several works [23, 89, 88], and more recently in [2] . Although the literature concerning the application of approximate nonlinear RH controllers to real plants is rich [40, 82], there is a need for a further investigation toward the effect of approximation errors on the robustness of the closed-loop system, in particular when the dynamics are driven by strong nonlinearities and when state and input variables are subjected to hard constraints.

Recently, several approaches have been proposed to obtain explicit solutions of MPC problems for special classes of constrained uncertain nonlinear systems. An efficient off-line formulation of the robust MPC problem has been developed in [119] for constrained linear time-varying systems affected by polytopic uncertainty. Explicit MPC controllers for constrained piecewise affine systems with bounded disturbances have been designed in [58] and [84].

An approximate RH control technique that can be applied to a generic nonlinear system with state and input constraints has been proposed in [39], where the control law has been obtained off-line by recursively partitioning the state space with a binary search tree. The major drawback of the state space partitioning approach [52, 117] is that the structure of the approximator is fixed by the algorithm which solves the multi-parametric optimization associated to the RH controller. Moreover, a large number of regions may be necessary to guarantee the stability of the closed-loop system. A method to reduce the complexity of the controller in case of nonlinear input-affine systems has been proposed in [11].

With the aim of decoupling the computation of the optimal control law from the function approximation stage, a detailed stability analysis has been carried out in [20] for a generic nonlinear system driven by a set-membership approximate RH controller. In the aforementioned work, the stability properties of the closed-loop system are studied under the assumption that the true RH control law is Lipschitz continuous with respect to the state variables, and that the Lipschitz constant (or a suitable upper bound) is known at the approximation stage. In this respect, it must be remarked that this assumption is not always verified in practice, since the optimal MPC law may be discontinuous, as deeply discussed in [78, 107].

In [78] it was shown that MPC could generate discontinuous feedback control law with respect to the state variables even if the dynamic system is continuous. This is due to the fact that the feedback law comes from the solution of a constrained optimization problem (when constraints, as for example the terminal constraint, are considered). Only when the plant is unconstrained and the terminal constraint is not active, [48], or when only constraints on the inputs are present, [69], discontinuity of the control law is avoided. When the transition function of the system is discontinuous (as for instance the case of hybrid systems) or the system is constrained, this problem remains open, [63].

In this connection, we will show that it is possible to analyze the stability of a system driven by a generic approximate static state-feedback without formulating any “a priori” assumption on the continuity of the resulting control law. To this end, the effect of the approximation error will be decomposed in two bounded perturbation terms, one acting on the state measurements and the other perturbing directly the control input. Finally, we will show that the Nearest Neighbor (NN) interpolation (see [9]) can be effectively used to approximate (possibly discontinuous) MPC laws.

To summarize, the main features of the proposed approximate RH control design are

1) it removes any "a priori" assumption on the continuity of the RH control law, thus permitting to apply the method to systems which are not asymptotically stabilizable by continuous state-feedback;

2) hard constraints on state and input variables can be robustly enforced;

3) it allows to compute a conservative bound on the quantization of the input command values (due to the numerical implementation of the approximate control law).

4.1 Motivating example

In this section, we provide an example of a non-autonomous system which can not be locally asymptotically stabilized to the 0-equilibrium by any static state-feedback control law continuous in the state variables. We will prove that a discontinuous control law can effectively achieve the asymptotic stabilization, and that a smooth Lyapunov function can be found for the closed-loop system as described in [53]. This stabilizing discontinuous controller will be used, in section 4.4, as an auxiliary feedback to design a RH control scheme capable to optimize a given performance index and to enforce the satisfaction of state and and input constraint. In addition, assuming that the discrete dynamics evolves with a small time-step that is not compatible with real-time optimization, our goal is to obtain, by off-line optimization, a suitable approximation of the RH (discontinuous) control law, preserving some stability property to be specified. Consider the system

$$\begin{cases} {x_{(1)}}_{t+1} &= {x_{(1)}}_t\left[2+({x_{(1)}}_t+0.95)u_t\right]+0.1{x_{(2)}}_t u_t \\ {x_{(2)}}_{t+1} &= e^{-1}\left[0.5-0.5\dfrac{{x_{(1)}}_t+0.95}{2}u_t\right]{x_{(2)}}_t \\ x_0 = ({x_{(1)}}_0, {x_{(2)}}_0) = \bar{x}, \end{cases} \quad , t \in \mathbb{Z}_{\geq 0} \tag{4.1}$$

where the subscript (i), $i \in \{1, 2\}$, denotes the i-th component of x_t.

Proposition 4.1.1. *For system (4.1) there does not exist a continuous time-invariant feedback control law capable to asymptotically stabilize the* 0*-equilibrium.* □

Proof The proof will be carried out by contradiction. Assume that there exists a (bounded) continuous state feedback control law $u_t=\kappa(x_t)$, $\kappa : \mathbb{R}^2 \to \mathbb{R}$, $\kappa(x)\leq R$, $\forall x \in \mathbb{R}^2$, with $R \in \mathbb{R}_{>0}$, capable to render locally asymptotically stable the closed-loop system toward the origin. For local asymptotic stability to hold, for all $\epsilon>0$ there must exist $\delta>0$ such that $|x_t| < \epsilon$, $\forall t\geq 0$, whenever $|x_0| < \delta$. Let us fix $\epsilon=(2 - r - 0.95)/(R+1)$, with $r > 1$, and consider an initial condition $x_0=\bar{x}=(\bar{x}_{(1)}, 0)$, with $|x_0| < \delta\leq\epsilon$. Notice that, with the given initial condition, the solution verifies ${x_{(2)}}_t \equiv 0$, $\forall t\geq 0$. Moreover, it holds that $|({x_{(1)}}_t + 0.95)\kappa({x_{(1)}}_t, {x_{(2)}}_t)|\leq 2 - r$, $\forall t\geq 0$, which implies $|{x_{(1)}}_{t+1}|\geq r|{x_{(1)}}_t|\forall t\geq 0$, and in turn that $|x_t|\geq r^t|x_0|$. From the assumption $r > 1$, it follows that the local uniform stability property $|x_t| < \epsilon$, $\forall t\geq 0$ cannot be verified for any initial condition satisfying $|x_0| > 0$. ■

On the other hand, the following discontinuous feedback law is able to asymptotically stabilize the closed-loop system

$$\kappa_d(x_{(1)}, x_{(2)})=\begin{cases} 0, & |x_{(1)}|\leq\sqrt{|x_{(2)}|} \\ -2\frac{1}{x_{(1)}+0.95} & |x_{(1)}| > \sqrt{|x_{(2)}|} \end{cases} \tag{4.2}$$

The function κ_d is bounded by $|\kappa_d(x)|\leq 2.1053$,$\forall x \in \mathbb{R}^2$. Moreover the closed-loop system admits the following Lyapunov function $W : \mathbb{R}^2 \to \mathbb{R}_{>0}$

$$W(x) = |x_{(1)}| + \frac{2e^{0.5}}{e^{0.25} - 1}\sqrt{|x_{(2)}|}, \tag{4.3}$$

which satisfies the following inequality

$$W(\hat{f}(x, \kappa_d(x))) - W(x)\leq -\left(1 - e^{-0.25}\right) W(x) \tag{4.4}$$

where $\hat{f}(x, u)$ denotes the state transition map given in (4.1).

Next, we are going to address the problem of designing robust MPC schemes for systems that are non smoothly asymptotically stabilizable, as the one proposed in this section, and that are subjected to hard constraint on state and input variables. Since we will prove that MPC can asymptotically stabilize this class of systems, then it is possible to conclude that also RH policies with terminal set constraint and terminal penalty (besides the formulation with terminal equality constraint already analyzed in [78]), can give rise to discontinuous control laws. Finally, we will determine the conditions under which those systems can be effectively controlled by approximate feedback laws obtained by off-line computations.

4.2 Regional ISS Result for Discontinuous MPC Feedback Laws

Consider the nonlinear discrete-time perturbed dynamic system

$$x_{t+1} \;=\; f(x_t, u_t, \varsigma_t),\; t \in \mathbb{Z}_{\geq 0},\; x_0 = \bar{x}\,, \tag{4.5}$$

where $x_t \in \mathbb{R}^n$ denotes the system state, $u_t \in \mathbb{R}^m$ the control vector and $\varsigma_t \in \mathbb{R}^r$ an exogenous disturbance input. The state and control variables are subject to constraints (1.2) and (1.3). Given the system (4.5), let $\hat{f}(x_t, u_t)$, with $\hat{f}(0,0)=0$, denote the *nominal* model used for control design purposes, such that

$$x_{t+1} = \hat{f}(x_t, u_t) + d_t,\; t \in \mathbb{Z}_{\geq 0},\; x_0 = \bar{x}\,, \tag{4.6}$$

where $d_t \triangleq f(x_t, u_t, \varsigma_t) - \hat{f}(x_t, u_t) \in \mathbb{R}^n$ denotes the discrete-time state transition uncertainty.

In the sequel the following assumptions will be needed.

Assumption 4.2.1. *The function $\hat{f} : X \times U \to X$ is Lipschitz (L.) continuous with respect to $x \in X$, with L. constant $L_{f_x} \in \mathbb{R}_{>0}$, uniformly in $u \in U$ (i.e., for any fixed $u \in U$, it holds that $|\hat{f}(x,u) - \hat{f}(x', u)| \leq L_{f_x} |x - x'|$ for all $(x, x') \in X^2$).*

Furthermore, the function $\hat{f}$ is uniformly continuous in u: there exists a $\mathcal{K}$-function η_u such that $|\hat{f}(x,u) - \hat{f}(x, u')| \leq \eta_u(|u - u'|)$ for all $x \in X$ and for all $(u, u') \in U^2$. □

Assumption 4.2.2 (Uncertainties)**.** *The additive transition uncertainty verifies $d_t \leq \mu(|\varsigma_t|)$, $\forall t \in \mathbb{Z}_{\geq 0}$ where μ is a $\mathcal{K}$-function. Moreover, d_t is bounded in a compact ball D, that is $d_t \in D \triangleq \mathcal{B}(\bar{d})$, $\forall t \in \mathbb{Z}_{\geq 0}$, with $\bar{d} \in \mathbb{R}_{\geq 0}$ finite.* □

Assumption 4.2.3 (Input-to-state stabilizing controller)**.** *There exist a compact set $\tilde{\Xi} \in X$, with $\{0\} \in \tilde{\Xi}$, and a state-feedback control law (possibly non-smooth)*

$$u_t = \kappa(x_t), \ \ \kappa(x_t) : \tilde{\Xi} \to U, \tag{4.7}$$

such that the following system, given by (4.6) in closed-loop with (4.7)

$$x_{t+1} = \hat{f}(x_t, \kappa(x_t)) + d_t, \ \ t \in \mathbb{Z}_{\geq 0}, \ \ x_0 = \bar{x}, \tag{4.8}$$

enjoies the properties:

1) *it is ISS in $\tilde{\Xi}$ with respect to additive disturbances $d_t \in D$. In particular, there exists a ISS-Lyapunov function for which Equation 2) and 3) of Definition 2.2.1 hold.*

2) *the set $\tilde{\Xi}$ is RPI for system (4.6) with additive disturbances $d_t \in D$.*

□

Notice that a control law κ satisfying Assumption 4.2.3 can be designed by using some control techniques recently developed in the framework of RH control. In this regard, the methodologies described in [66, 94] guarantee the input-to-state stability of the closed-loop system with respect to bounded additive disturbances and allow to evaluate the bounds on additive uncertainties under which the feasible set of the optimization problem associated to the RH control can be rendered robust positively invariant. In the following section, we will extend the aforementioned RH control design procedures to systems that are not asymptotically stabilizable by continuous static state feedback.

4.2.1 Formulation and Stability Properties of the Exact RH Control Law

Given a perturbed nonlinear system (4.5), and a nominal model of the form (4.6) , the control objective consists in designing a state-feedback control law (possibly discontinuous), capable to meet the requirements posed by Assumption 4.2.3 and to satisfy state and input constraints in the presence of additive uncertainties. On the basis of Assumptions 4.2.1 and 4.2.2, let us formulate the MPC policy. To this end, a suitable Finite-Horizon Optimal Control Problem (FHOCP) should be introduced.

Definition 4.2.1 (FHOCP)**.** *Given a positive integer $N_c \in \mathbb{Z}_{\geq 0}$, at any time $t \in \mathbb{Z}_{\geq 0}$, let $\mathbf{u}_{t,t+N_c-1|t} \triangleq \mathrm{col}[u_{t|t}, \, u_{t+1|t}, \, \ldots, \, u_{t+N_c-1|t}]$ denote a sequence of input variables over the control* *horizon N_c. Moreover, given x_t and $\mathbf{u}_{t,t+N_c-1|t}$, let $\hat{x}_{t+j|t}$ denote the state "predicted" by means of the nominal model, such that*

$$\hat{x}_{t+j|t}=\hat{f}(\hat{x}_{t+j-1|t}, u_{t+j-1|t}), \hat{x}_{t|t}=x_t, \forall j\in\{1,\ldots,N_c\}\,. \tag{4.9}$$

Then, given a transition cost function h, an auxiliary control law κ_f, a terminal cost function h_f, a terminal set X_f and a sequence of constraint sets $\hat{X}_{t+j|t}\subseteq X$, $j\in\{1,\ldots,N_c-1\}$, to be described later on, the FHOCP consists in minimizing, with respect to $\mathbf{u}_{t,t+N_c-1|t}$, the cost function

$$J_{FH}(x_t, \mathbf{u}_{t,t+N_c-1|t}, N_c) \triangleq \sum_{l=t}^{t+N_c-1} h(\hat{x}_{l|t}, u_{l|t}) + h_f(\hat{x}_{t+N_c|t}) \tag{4.10}$$

subject to

1) the nominal dynamics (4.9), with $\hat{x}_{t|t}=x_t$;

2) the control and the state constraints $u_{t+j|t} \in U$, $\hat{x}_{t+j|t}\in\hat{X}_{t+j|t}, \forall j\in\{0,\ldots,N_c-1\}$;

3) the terminal state constraint $\hat{x}_{t+N_c|t} \in X_f$. □

The usual RH control technique can now be stated as follows: given a time instant $t \in \mathbb{Z}_{\geq 0}$, let $\hat{x}_{t|t}=x_t$, and find the optimal control sequence $\mathbf{u}^{\circ}_{t,t+N_c-1|t}$ by solving the FHOCP. Then, according to the RH strategy, apply

$$u_t=\kappa_{MPC}(x_t)\,, \tag{4.11}$$

where $\kappa_{MPC}(x_t)\triangleq u^{\circ}_{t|t}$ and $u^{\circ}_{t|t}$ is the first element of the optimal control sequence $\mathbf{u}^{\circ}_{t,t+N_c-1|t}$ (implicitly dependent on x_t).

It can be shown that the satisfaction of the original state constraints is ensured, for any admissible disturbance sequence, by imposing restricted constraints to the predicted open-loop trajectories. The tightened constraints can be computed as prescribed by Definition 3.2.1, under the assumption of norm-bounded uncertainties[1].

In order to prove the ISS property for the closed-loop system, let us introduce the following

[1] In the current chapter we neglect the presence of state-dependent uncertainties in order to enhance the clarity in the presentation. However, the state dependent contraction of constraints described in Lemma 3.2.1 can be used without invalidating the stability results.

assumptions.

Assumption 4.2.4. *The transition cost function* h *is such that* $\underline{h}(|x|) \leq h(x,u), \forall x \in X, , \forall u \in U$, *where* $\underline{h}$ *is a* $\mathcal{K}_\infty$*-function. Moreover,* h *is Lipschitz with respect to* x*, uniformly in* u*, with L. constant* $L_h > 0$. □

Assumption 4.2.5. *A terminal cost function* h_f*, an auxiliary control law* κ_f*, and a set* X_f *are given such that*

1) $X_f \subset X$, X_f *closed,* $0 \in X_f$*;*

2) $\exists \delta > 0 : \kappa_f(x) \in U, \forall x \in X_f \oplus \mathcal{B}^n(\delta)$*;*

3) $\hat{f}(x, \kappa_f(x)) \in X_f, \ \forall x \in X_f \oplus \mathcal{B}^n(\delta)$*;*

4) $h_f(x)$ *is Lipschitz in* X*, with L. constant* $L_{h_f} > 0$*;*

5) $h_f(\hat{f}(x, \kappa_f(x))) - h_f(x) \leq -h(x, \kappa_f(x)), \ \forall x \in X_f \oplus \mathcal{B}^n(\delta)$*;* □

With respect to previous works [66, 75, 94] concerning the design of input-to-state stabilizing MPC controllers, in order to cope with possibly discontinuous auxiliary control laws (consider, for instance, the motivating example and the proposed controller (4.2)), here we do not require neither $\kappa_f(x)$ nor the closed-loop map $\hat{f}(x, \kappa_f(x))$ to be Lipschitz continuous with respect to $x \in X_f$. In addition, in order to establish the ISS property for the closed-loop system, we require the following assumptions to be verified together with Assumption 4.2.5.

Assumption 4.2.6. *Let* X_f *be a sub-level set of* h_f *(i.e.* $X_f = \{x \in \mathbb{R}^n : h_f(x) \leq \overline{h}_f\}$ *); the transition cost function* h *and the terminal cost* h_f *satisfy the condition*

$$\min_{u \in U} \left\{ \inf_{x \in \mathcal{C}_1(X_f) \setminus (X_f \oplus \mathcal{B}^n(\delta))} \{h_f(x) - h(x,u)\} \right\} > \overline{h}_f. \tag{4.12}$$

where $\delta > 0$ *is a positive scalar for which Points 3) and 5) of Assumption 4.2.5 hold.* □

Assumption 4.2.7 (X_{κ_f})**.** *Suppose that there exists a compact set* $X_{\kappa_f} \supseteq X_f$ *for which* $\tilde{\mathbf{u}}_{t,t+N_c-1|t} \triangleq \text{col}[\kappa_f(\hat{x}_{t|t}), \ \kappa_f(\hat{x}_{t+1|t}), \ldots, \ \kappa_f(\hat{x}_{t+N_c-1|t})]$ *is a feasible control sequence for the FHOCP and for which Points 1), 2) and 5) of Assumption 4.2.5 are satisfied.* □

Remark 4.2.1. *Notice that Assumption 4.2.6, which is an extension of Point 5) of Assumption 4.2.5 to the set* $\mathcal{C}_1(X_f)$*, can be easily verified if* h_f *is Lipschitz continuous in* $\mathcal{C}_1(X_f)$ *with L. constant* L_{h_f} *and if* h_f *admits an exponential decay along the trajectories of the closed-loop system under the auxiliary control law. In order to design a MPC controller for the system described in the example given in Section 4.1, the Lyapunov function* W *can serve as a terminal cost for the FHOCP, while* X_f *can be chosen as a sub-level set* $X_f = \{x \in X : W(x) \leq \overline{h}_f\}$*. Indeed, thanks to (4.4), it is possible to pick* $\delta = (e^{0.5/2} - 1)\overline{h}_f / L_{h_f}$*, and then choose* h *such that Point 5) of Assumption 4.2.5 holds and*

$$\sup_{(x,u)\in[C_1(X_f)\setminus(X_f\oplus\mathcal{B}^n(\delta))]\times U} \{h(x,u)\} < (e^{0.5/2}-1)\overline{h}_f, \tag{4.13}$$

in order to meet inequality (4.12). □

Under the stated assumptions, the following theorem characterizes the ISS property of the closed-loop system with respect to bounded additive uncertainties. Moreover, in order to guarantee the recursive feasibility of the FHOCP, an upper bound on the admissible uncertainty is introduced, which is shown to depend on the invariance properties of X_f. This theorem represents the extension of the ISS result presented in [94] to the case of systems which are not asymptotically stabilizable by smooth feedback.

Theorem 4.2.1 (Regional ISS)**.** *Let us denote as* $X_{MPC} \subset \mathbb{R}^n$ *the set of state vectors for which the FHOCP is feasible. Under Assumptions 4.2.1,4.2.2, 4.2.5-4.2.7, the system (4.5), driven by the MPC control law (4.11), is regional ISS in* X_{MPC} *with respect to additive perturbations* $d_t \in D$*, with* $D \subseteq \mathcal{B}^n(\overline{d})$ *and*

$$\overline{d} \leq L_{f_x}^{1-N_c} \mathrm{dist}(\mathbb{R}^n \setminus \mathcal{C}_1(X_f), X_f). \tag{4.14}$$

□

Proof The proof will be carried out in two steps. The first step is aimed to prove the recursive feasibility of the scheme under the prescribed bound on uncertainties, thus establishing the robust positive invariance of the feasible set X_{MPC} with respect to $d_t \in D$. The second step consists in showing that $V(x_t) = J_{FH}(x_t, \mathbf{u}^{\circ}_{t,t+N_c-1|t}, N_c)$ is an ISS-Lyapunov function for the closed-loop system in X_{MPC}.

1) First, by Assumption 4.2.7, the set X_{MPC} is not empty. In fact, for any $x_t \in X_{\kappa_f}$, a feasible control sequence for FHOCP is given by $\tilde{\mathbf{u}}_{t,t+N_c-1|t} \triangleq \mathrm{col}\big[\kappa_f(\hat{x}_{t|t}),\ \kappa_f(\hat{x}_{t+1|t}),\ \ldots,\ \kappa_f(\hat{x}_{t+N_c-1|t})\big]$. Then $X_{MPC} \supseteq X_{\kappa_f} \supseteq X_f$. Moreover, since $d_{t+j} \in D, \forall j \in \mathbb{Z}_{\geq 0}$, with $D \subseteq \mathcal{B}^n(\overline{d})$ and $\overline{d} \leq \mathrm{dist}(\mathbb{R}^n \backslash \mathcal{C}_1(X_f), X_f)$, and by using standard arguments [66, 94], it is also possible to show that, if the FHOCP at time t is feasible, then the recursive feasibility of the scheme is guaranteed with respect to the restricted constraints. Furthermore, it possible to show that, under the stated assumption on $\overline{d}$, also the recursive feasibility with respect to the terminal constraint set can be guaranteed. Indeed, from the assumption $x_t \in X_{MPC}$, it follows that the predicted state $\hat{x}_{t+N_c|t}$, obtained with the optimal sequence $\mathbf{u}^{\circ}_{t,t+N_c-1|t}$, verifies $\hat{x}_{t+N_c|t} \in X_f$. Now we claim that at time $t+1$, given $x_{t+1} = \hat{f}(x_t, u^{\circ}_{t|t}) + d_t$, there exists a feasible input sequence $\bar{\mathbf{u}}_{t+1,t+N_c|t+1}$, based on the optimal sequence $\mathbf{u}^{\circ}_{t,t+N_c-1|t}$ at time t, such that $\hat{x}_{t+N_c+1|t+1} \in X_f$. Indeed, let us pick $\bar{\mathbf{u}}_{t+1,t+N_c|t+1} = \mathrm{col}[u^{\circ}_{t+1|t}, u^{\circ}_{t+2|t}, \ldots, u^{\circ}_{t+N_c-1|t}, \bar{u}]$, where $\bar{u} \in U$ is a feasible control action to be specified later on. From the Lipschitz continuity of $\hat{f}(x,u)$ with respect to x, it follows that $|\hat{x}_{t+j|t} - \hat{x}_{t+j|t+1}| \leq L_{f_x}^{j-1} \bar{d}$, $\forall j \in \{1, \ldots, N_c\}$. Then, in view of (4.14), it holds that $\hat{x}_{t+N_c|t+1} \in \mathcal{C}_1(X_f)$, which implies the existence of a feasible $\bar{u} \in U$ such that $\hat{x}_{t+N_c+1|t+1} = \hat{f}(\hat{x}_{t+N_c|t+1}, \bar{u}) \in X_f$. Thus, we can conclude that X_{MPC} is RPI with respect to $d_t \in D$.

2) Suppose[2] that $L_{f_x} \neq 1$; then, in view of Point 5) of Assumption 4.2.5, for all $x_t \in X_{\kappa_f}$ it holds

$$\begin{aligned} V(x_t) \leq J_{FH}(x_t, \tilde{\mathbf{u}}_{t,t+N_c-1|t}, N_c) &= \sum_{l=t}^{t+N_c-1} h(\hat{x}_{l|t}, \kappa_f(\hat{x}_{l|t})) + h_f(\hat{x}_{t+N_c|t}) \\ &\leq \sum_{l=t}^{t+N_c-1} [h_f(\hat{x}_{l|t}) - h_f(\hat{x}_{l+1|t})] + h_f(\hat{x}_{t+N_c|t}) \leq h_f(|x_t|). \end{aligned}$$

Hence, there exists a $\mathcal{K}$-function $\alpha_2(s) = h_f(s)$ such that

$$V(x_t) \leq \alpha_2(|x_t|),\ \ \forall x_t \in X_{\kappa_f}. \tag{4.15}$$

The lower bound on $V(x_t)$ can be easily obtained by using Assumption 4.2.4

$$V(x_t) \geq \underline{h}(|x_t|),\ \forall x_t \in X_{MPC}. \tag{4.16}$$

[2] The very special case $L_{f_x} = 1$ can be trivially addressed by a few suitable modifications to the proof of Theorem 4.2.1.

Then, inequalities (2.2) and (2.3) hold respectively with $\Xi = X_{MPC}$ and $\Omega = X_{\kappa_f}$.

Given the optimal control sequence at time t, $\mathbf{u}^{\circ}_{t,t+N_c-1|t+1}$, consider now the sequence $\bar{\mathbf{u}}_{t+1,t+N_c|t+1} \triangleq \text{col}\big[u^{\circ}_{t+1|t}, \ldots, u^{\circ}_{t+N_c-1|t}, \bar{u}_f(\hat{x}_{t+N_c|t+1})\big]$, with $\bar{u}_f : \mathcal{C}_1(X_f) \to U$ defined as

$$\bar{u}_f(x) \triangleq \arg \min_{u \in U: \hat{f}(x,u) \in X_f} \{|u - \kappa_f(x)|\} .$$

Clearly, $\bar{\mathbf{u}}_{t+1,t+N_c|t+1}$ is a feasible (in general, suboptimal) control sequence for the FHOCP at time $t+1$, with cost

$$\begin{aligned} J_{FH}(x_{t+1}, \bar{\mathbf{u}}_{t+1,t+N_c|t+1}, N_c) = \\ V(x_t) - h(x_t, u^{\circ}_{t|t}) + \sum_{l=t+1}^{t+N_c-1} [h(\hat{x}_{l|t+1}, u^{\circ}_{l|t}) - h(\hat{x}_{l|t}, u^{\circ}_{l|t})] \\ + h(\hat{x}_{t+N_c|t+1}, \bar{u}_f(\hat{x}_{t+N_c|t+1})) + h_f\Big(\hat{f}(\hat{x}_{t+N_c|t+1}, \bar{u}_f(\hat{x}_{t+N_c|t+1}))\Big) - h_f(\hat{x}_{t+N_c|t}). \end{aligned} \tag{4.17}$$

In view of Assumptions 4.2.5 and 4.2.6, and considering that $\hat{x}_{t+N_c|t+1} \in \mathcal{C}_1(X_f)$, the following inequalities hold

$$\begin{aligned} & h(\hat{x}_{t+N_c|t+1}, \bar{u}_f(\hat{x}_{t+N_c|t+1})) + h_f\Big(\hat{f}(\hat{x}_{t+N_c|t+1}, \bar{u}_f(\hat{x}_{t+N_c|t+1}))\Big) - h_f(\hat{x}_{t+N_c|t}) \\ & \quad \le h(\hat{x}_{t+N_c|t+1}, \bar{u}_f(\hat{x}_{t+N_c|t+1})) + h_f\Big(\hat{f}(\hat{x}_{t+N_c|t+1}, \bar{u}_f(\hat{x}_{t+N_c|t+1}))\Big) - h_f(\hat{x}_{t+N_c|t+1}) \\ & \qquad + |h_f(\hat{x}_{t+N_c|t+1}) - h_f(\hat{x}_{t+N_c|t})| \\ & \quad \le L_{h_f} L_{f_x}^{N_c-1} \mu(|\upsilon_t|). \end{aligned} \tag{4.18}$$

Moreover, Assumption 4.2.4 implies that

$$|h(\hat{x}_{t+j|t+1}, u^{\circ}_{t+j|t}) - h(\hat{x}_{t+j|t}, u^{\circ}_{t+j|t})| \le L_h L_{f_x}^{j-1} |d_t|, \ \ \forall j \in \{1, \ldots, N_c - 1\} \tag{4.19}$$

Now, in view of (4.17), (4.18), (4.19) and Assumption 4.2.4, it is possible to conclude that the optimal cost $V(x_{t+1})$ satisfies

$$\begin{aligned} V(x_{t+1}) & \le J_{FH}(x_{t+1}, \bar{\mathbf{u}}_{t+1,t+N_c|t+1}, N_c) \\ & \le V(x_t) - \underline{h}(|x|) + \left(L_h \frac{L_{f_x}^{N_c} - 1}{L_{f_x} - 1} + L_{h_f} L_{f_x}^{N_c-1} \right) |d_t|. \end{aligned} \tag{4.20}$$

Finally, inequality (4.20) implies the existence of two $\mathcal{K}$-functions $\alpha_3(s) = \underline{h}(s)$ and $\sigma(s) = [L_h L_{f_x}^{N_c} - 1)/(L_{f_x} - 1) + L_{h_f} L_{f_x}^{N_c-1}]s$, such that

$$V(x_{t+1}) - V(x_t) \leq -\alpha_3(|x_t|) + \sigma(|d_t|) \tag{4.21}$$

■

In this section we have shown how to design an input-to-state stabilizing exact MPC control law κ_{MPC} for system (4.6), which renders RPI the set $X_{MPC} \subseteq X$ with respect to additive disturbances $d_t \in D$. Therefore, Assumption 4.2.3 is verified by the MPC controller with $\kappa = \kappa_{MPC}$ and $\tilde{\Xi} = X_{MPC}$.

Next, we are going to infer, from the stabilizing properties of κ, the stability properties of the closed-loop system driven by an approximate control law κ^*, satisfying suitable requirements to be specified later on.

4.3 Approximation of the NMPC Control Law: Sufficient Conditions for Practical Stabilization

Consider the following dynamic system:

$$x_{t+1} = \hat{f}(x_t, \kappa^*(x_t)) + w_t, \ \ t \in \mathbb{Z}_{\geq 0}, \ \ x_0 = \bar{x}. \tag{4.22}$$

where $w_t \in W \triangleq \mathcal{B}^n(\overline{d}_w)$ is a disturbance input and the function $\kappa^* : \mathbb{R}^n \to \mathbb{R}^m$ is an approximation of the given ISS stabilizing κ satisfying Assumption 4.2.3. We will show that the stability properties of (4.22) can be inferred from those of (4.8) provided that κ^* satisfies the following additional requirements.

Assumption 4.3.1. *Let us define the $\mathcal{K}_\infty$-function $\eta_x(s) = L_{f_x} s + s$ for $s \geq 0$ and let $\overline{d}_q \in \mathbb{R}_{\geq 0}$ and $\overline{d}_v \in \mathbb{R}_{\geq 0}$ be two positive scalars satisfying the following inequality*

$$\overline{d}_q + \overline{d}_v + \overline{d}_w \leq \overline{d}. \tag{4.23}$$

Posing $\overline{q} \triangleq \eta_x^{-1}(\overline{d}_q)$, assume that $\forall \xi \in \operatorname{dom}(\kappa)$, $\exists \zeta_\xi \in \mathcal{B}^n(\xi, \overline{q}) \cap \operatorname{dom}(\kappa)$ such that

$$\eta_u(|\kappa^*(\xi) - \kappa(\zeta_\xi)|) + \eta_x(|\zeta_\xi - \xi|) + \overline{d}_w \leq \overline{d}(\zeta_\xi) \tag{4.24}$$

where $\overline{d}(\zeta_\xi)$ *is the local uncertainty bound under which the state can be driven from* ζ_ξ *in* Ξ *by the control law* κ*, i.e.,*

$$\overline{d}(\zeta_\xi) \triangleq \inf\{c \in \mathbb{R}_{>0} | \exists d \in \mathcal{B}^n(c) : \hat{f}(\zeta_\xi, \kappa(\zeta_\xi)) + d \notin \Xi\}.$$

□

In this connection, it is worth to notice that for the global uncertainty bound d, it holds that $\overline{d} \leq \overline{d}(\zeta_\xi)$. Thus, the conditions posed on the approximating function are less restrictive than those formulated in [90], in which a global bound had been used.

Then, the stability properties of the closed-loop system driven by the approximate control law κ^* can be established.

Theorem 4.3.1. *Suppose that Assumptions 4.2.1-4.3.1 hold and let* $\Xi \triangleq \tilde{\Xi} \backsim Q$*, with* $Q \triangleq \mathcal{B}^n(\overline{q})$*. Then, the following statements hold:*
1) The set $\Xi \subset X$ *is RPI for the closed-loop system (4.22) with* $w_t \in W$*, with* $W \triangleq \mathcal{B}^n(\overline{d}_w)$*;*
2) The closed-loop system (4.22) is ISS in Ξ*.* □

Proof Points 1) and 2) of Theorem 4.3.1 will be addressed separately in the following

1) Let $x \in \tilde{\Xi}, q \in Q$ and $w \in W$. Now we will prove that Ξ is RPI for (4.22).

First, under Assumption 4.3.3, $\forall x \in \tilde{\Xi}$ there exist $\zeta_x = q_x + x$, with $q_x \in \mathcal{B}^n(\overline{q})$ such that (4.24) holds. Then, let us consider that

$$\hat{f}(x, \kappa^*(x)) + w + q_x = \hat{f}(x_t, \kappa^*(\zeta_x)) - \hat{f}(x, \kappa(\zeta_x)) + \hat{f}(x, \kappa(\zeta_x)) + \hat{f}(\zeta_x, \kappa(\zeta_x)) - \hat{f}(\zeta_x, \kappa(\zeta_x)) + w + q_x,$$

which can be written in compact form as follows

$$\hat{f}(x, \kappa^*(x)) + w + q_x = \hat{f}(\zeta_x, \kappa(\zeta_x)) + d_{x,w} \tag{4.25}$$

where

$$d_{x,w} \triangleq \hat{f}(x, \kappa^*(x)) - \hat{f}(x, \kappa(\zeta_x)) + \hat{f}(x, \kappa(\zeta_x)) - \hat{f}(\zeta_x, \kappa(\zeta_x)) + w + q_x. \tag{4.26}$$

Using Assumption 4.2.1, we note that, for all $w \in W$ and for all $x \in \tilde{\Xi}$, the following inequalities hold:

$$\begin{aligned}|d_{x,w}| &= |\hat{f}(x,\kappa^*(x)) - \hat{f}(x,\kappa(\zeta_x)) + \hat{f}(x,\kappa(\zeta_x)) - \hat{f}(\zeta_x,\kappa(\zeta_x)) + w + q_x| \\ &\leq |\hat{f}(x,\kappa^*(x)) - \hat{f}(x,\kappa(\zeta_x))| + \left|\hat{f}(x,\kappa(\zeta_x)) - \hat{f}(\zeta_x,\kappa(\zeta_x))\right| + |w| + |q_x| \\ &\leq \eta_u\left(|\kappa^*(x) - \kappa(\zeta_x)|\right) + L_{f_x}|q_x| + |w| + |q_x| \\ &\leq \eta_u\left(|\kappa^*(x) - \kappa(\zeta_x)|\right) + \eta_x(|\zeta_x - x|) + |w|.\end{aligned} \tag{4.27}$$

In view of Assumption 4.3.1 it follows that

$$|d_{x,w}| \leq \overline{d}(\zeta_x) \tag{4.28}$$

Since $\hat{f}(\zeta_x,\kappa(\zeta_x)) + d \in \tilde{\Xi},\ \ \forall d \in \mathcal{B}^n(\overline{d}(\zeta_x))$, then (4.25) and (4.28) together imply that

$$\hat{f}(x,\kappa^*(x)) + w + q_x \in \tilde{\Xi},\ \forall q_x \in Q,\ \ \forall w \in W. \tag{4.29}$$

We can conclude that, under Assumptions 4.2.1 and 4.3.3, for any $x \in \Xi$, $\hat{f}(x,\kappa^*(x)) + w \in \Xi$, $\forall w \in W$.

2) The ISS property for the closed-loop system can be straightforwardly proven considering that, in view of Theorem 4.2.1 and taking in account inequalities (4.21) and (4.27), the optimal finite horizon cost function satisfies the condition

$$\begin{aligned}V(\hat{f}(x,\kappa^*(x)) + w) - V(x) &\leq -\alpha_3(|x|) + \sigma(d_{x,w}) \\ &\leq -\alpha_3(|x|) + \sigma(\eta_u\left(|\kappa^*(x) - \kappa(\zeta_x)|\right) + \eta_x(|\zeta_x - x|) + |w|)\end{aligned} \tag{4.30}$$

Now, posing $v_x \triangleq \kappa^*(x) - \kappa(\zeta_x)$, in view of (4.28) and being Ξ compact, it holds that $|v_x| \leq \overline{v},\ \forall x \in \Xi$ for some $\overline{v} \in \mathbb{R}_{>0}$. Then we can conclude that

$$\begin{aligned}V(\hat{f}(x,\kappa^*(x)) + w) - V(x) &\leq -\alpha_3(|x|) + \sigma(3\eta_u(|v_x|)) + \sigma(3\eta_x(|q_x|)) + \sigma(3|w|) \\ &= -\alpha_3(|x|) + \sigma_v(|v_x|) + \sigma_q(|q_x|) + \sigma_w(|w|),\end{aligned} \tag{4.31}$$

where $\sigma_v(s) \triangleq \sigma(3\eta_u(s))$, $\sigma_q(s) \triangleq \sigma(3\eta_x(s))$, $\sigma_w(|w_t|) \triangleq \sigma(3s)$, $s \in \mathbb{R}_{\geq 0}$.

Hence, in view of Theorem 2.2.1, the closed-loop system is regional-ISS in Ξ with respect to the bounded approximation-induced perturbations $v \in V \triangleq \mathcal{B}^n(\overline{v})$ $q_x \in Q$ and $w \in W$. ■

In the following sections, we will show that the NN approximator (see [9]) can be used to approximate possibly discontinuous MPC control laws with ISS guarantees.

4.3.1 Approximate MPC control law by off-line NN approximation

The NN has been chosen, among many available function approximators, for the possibility to easily satisfy the requirements specified in Assumption 4.3.3, thus permitting to achieve the closed-loop stability, thanks to Theorem 4.3.1, even in presence of discontinuous control laws such as the ones possibly arising from MPC schemes.

At this point, we are going to design the NN approximator in such a way that Assumption 4.3.1 can be verified in practice.

First, assuming that a bound on the additive transition uncertainty is given (i.e., $|w_t| \leq \overline{d}_w$) the designer must assign arbitrary values to the scalars $\overline{d}_v$ and $\overline{d}_q$ such that inequality (4.23) holds true. These parameters are used to specify how close the approximate control law κ^* will be to κ_{MPC}, at the cost of increasing the complexity of κ^*. Then, the off-line procedure starts with the construction of a suitable training data set by evaluating the MPC control law in a finite number of points (knots) belonging to a (possibly non uniform) grid X_G which covers the whole region X. Such a grid must fulfill the following requirement.

Assumption 4.3.2 (X_G)**.** *Given the set X and $\overline{d}_q \in \mathbb{R}_{>0}$ satisfying (4.23), the set X_G verifies*

1) $\forall \xi \in X, \ \exists \zeta_\xi \in X_G : |\xi - \zeta| \leq \overline{q}_{NN} < \eta_x^{-1}(\overline{d}_q)$;

2) $\exists \psi_{NN} \in \mathbb{R}_{>0} : \ |\zeta' - \zeta''| \geq \underline{\psi}_{NN}, \forall (\zeta', \zeta'') \in {X_G}^2$,

where $\overline{q}_{NN}$ and $\underline{\psi}_{NN}$ are referred respectively as knot density and knot separation parameters, (see [9] and the references therein). □

Notice that, since X is compact, then, by Point 2), X_G is made up of a finite number of knots. Moreover, the cardinality of the training set grows with the decrease of $\overline{d}_q$. However, considering that a lower limit on $\overline{d}_q$ is imposed by (4.23), there exists a finite upper bound on the knot density $\overline{q}_{NN}$. Once the quantization (spatial sampling) of X, operated by X_G, has been performed, the control law must be evaluated at each point of X_G. Noting that X_{MPC}=dom(κ_{MPC}), the NN data are given by the pair $(\mathcal{X}, \mathcal{Y})$, with

$$\mathcal{X}{=}X_G \cap X_{MPC}{\subset}\mathrm{dom}\,(\kappa_{MPC})\,,\;\; \mathcal{Y}{\triangleq}\bigcup_{\zeta\in\mathcal{X}}\tilde{\kappa}_{MPC}(\zeta), \tag{4.32}$$

where $\tilde{\kappa}_{MPC}(\zeta) = y\,(\kappa_{MPC}\,(\zeta))$ and $y\colon U \to \mathcal{U} \subset U$ is a quantizer in the command input space which models the error that can be due to the coding of input command values with a finite alphabet. This problem always affects numerical approximation schemes.

Now, it is straightforward to show that if an approximating function $\kappa^*(\cdot)$ verifies the following conditions, then Assumption 4.3.1 is satisfied. Assumption 4.3.3 below, although more restrictive than Assumption 4.3.1, is going to be introduced because it can be verified with ease by NN approximation schemes.

Assumption 4.3.3. *Let $\overline{d}_q \in \mathbb{R}_{\geq 0}$ and $\overline{d}_v \in \mathbb{R}_{\geq 0}$ be two positive scalars satisfying the inequality 4.23. Defining $\overline{v} \triangleq \eta_u^{-1}(\overline{d}_v)$ and $\overline{q} \triangleq \eta_x^{-1}(\overline{d}_q)$, there exists $\lambda \in (0,1) : \forall \xi \in \mathrm{dom}(\kappa)$, $\exists \zeta_\xi \in \mathcal{B}^n(\xi, \overline{q}) \cap \mathrm{dom}(\kappa)$ such that*
1) $|\kappa(\zeta_\xi) - \kappa^*(\zeta_\xi)| \leq (1-\lambda)\overline{v}$;
2) $|\kappa^*(\xi) - \kappa^*(\zeta_\xi)| \leq \lambda\overline{v}$.
Moreover, let us assume that $\kappa^*(\xi) \in U, \forall \xi \in \mathrm{dom}(\kappa)$. □

For the NN approximator, in order to meet the requirement posed by Point 1) of Assumption 4.3.3, the input space quantizer is required to satisfy the following condition in correspondence of points belonging to the training set

$$|y(\kappa_{MPC}(\zeta)) - \kappa_{MPC}(\zeta)| \leq (1-\lambda)\overline{v} \quad \left(= (1-\lambda)\eta_u^{-1}(\overline{d}_v)\right) \;,\;\; \forall \zeta \in \mathcal{X}. \tag{4.33}$$

Hence, a local error on the sampling points may be tolerated. Conversely, if the output map is exact on the grid points, then Point 2) of Assumption 4.3.3 is satisfied with $|\kappa^*(\xi) - \kappa^*(\zeta_\xi)| \equiv 0$ by this approximation scheme.

Then, given a state measurement $x_t \in X_{MPC} \backsim \mathcal{B}^n(\overline{q}_{NN})$ at time t, the NN control law is given by

$$u_t{=}\kappa_{NN}(x_t){=}\tilde{\kappa}_{MPC}\left(\mathcal{N}_{\mathcal{X}}(x_t)\right), \tag{4.34}$$

where $\mathcal{N}_{\mathcal{X}}\colon \mathbb{R}^m {\to} \mathbb{R}^n$ denotes a single-valued NN search in the data-set $\mathcal{X}$. Notice that the NN approximator intrinsically verifies Point 2) of Assumption 4.3.3, since, $\forall \xi {\in} X_{MPC}$, $\exists\, \zeta_\xi {\in} \mathcal{N}_{\mathcal{X}}(\xi) {\subseteq} \mathcal{X}\colon |\xi - \zeta_\xi| {\leq} \overline{q}_{NN} {<} \overline{q} \;\; ({=}\eta_x^{-1}(\overline{d}_q)\,)$. Moreover it holds that $\kappa^*(\xi) {\in} \kappa^*(\mathcal{N}_{\mathcal{X}}(\xi)) {=} \tilde{\kappa}_{MPC}(\mathcal{N}_{\mathcal{X}}(x_t))$.

Finally, in view of Theorem 4.3.1, it is possible to conclude that the closed-loop system (4.22), driven by the approximate MPC control law (4.34), is regional-ISS in $X_{NN} \triangleq X_{MPC} \backsim \mathcal{B}^n(\overline{q}_{NN})$ with respect to the approximation-induced perturbations and the model uncertainty.

Remark 4.3.1 (ISpS)**.** *In order to complete the analysis on the stability properties of the closed-loop system under the approximate control law $\kappa^*{=}\kappa_{NN}$, it is necessary to take in explicit consideration the fact that the perturbations due to the finite knot density $\overline{q}_{NN}$ and to the input quantization $\overline{v}_{NN}$ do not vanish along the system trajectories.*

In order to assess the stability properties of the closed-loop system under non-vanishing perturbations, the ISpS tool, introduced in Section 2.3 of Chapter 2, will be used.

In view of (4.28) and the first inequality in (4.30), it follows that

$$V(x_{t+1}){-}V(x_t){\leq}{-}\alpha_3(|x_t|){+}\sigma\Big(2\eta_u(\overline{v}_{NN}){+}2\eta_x(\overline{q}_{NN})\Big)\;{+}\sigma(2|w_t|)$$

Hence, by Theorem 2.3.1, the closed-loop system driven by the approximate NN-MPC control law κ_{NN} is ISpS in $X_{MPC}{\backsim}\mathcal{B}^n(\overline{q}_{NN})$ with respect to the model uncertainty $w_t{\in}W$. This result implies that the closed-loop trajectories, driven by the approximate controller, cannot be guaranteed to asymptotically converge to the origin. □

4.3.2 Smooth approximation of the control law

In the previous section, we have shown that a suitably designed NN approximator can fulfill the requirements posed by Assumption 4.3.1, and therefore the resulting closed-loop system guarantees the invariance of the region Ξ. However, the conditions posed by Assumption 4.3.1 can be satisfied even by other types of approximators, such as Neural Networks, with smooth basis functions and smooth output function. With lack of formalism, the "smoothness" of a Neural Network approximator with a given structure (number of layers, interconnections, number of neurons) depends upon the shape of the activation functions and on the parameters (weights) of the network $\mathrm{w} \in \mathbb{R}^{n_{\mathrm{w}}}$. At this point, assuming that the kind of the basis functions (shape) is fixed "a priori" by the designer, and that the degrees of freedom of the output function with respect to the parameters is sufficiently high[3], then the approximation procedure consists in finding a set of parameters which guarantees the fulfillment of Assumption 4.3.1.

[3]In general, the complexity of a Neural Network depends on the number of neurons. Therefore, we assume the network complexity is such that a set of parameters satisfying Assumption 4.3.1 exists.

In the following, given a grid of reference points X_G satisfying Assumption 4.3.2 and set of network parameters, w, we will show how the fulfillment of inequality (4.24) can be checked.

Inequality (4.24) can be rewritten as

$$\eta_u\left(|\kappa^*(\zeta_\xi)-\kappa^*(\xi)|+|\kappa^*(\zeta_\xi)-\kappa(\zeta_\xi)|\right)+\eta_x\left(|\zeta_\xi-\xi|\right)\leq d(\zeta_\xi)-\overline{d}_w \tag{4.35}$$

Let the approximation function κ^* be locally Lipschitz in dom(κ). In particular, there exists a function $L_{\kappa^*}(x):\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ such that

$$|\kappa^*(x)-\kappa(x^{'})|\leq L_{\kappa^*}(x)|x-x^{'}|,\ \ \forall x^{'}\in\mathcal{B}^n(x,\overline{q})$$

Then, from (4.35), we have that, for all $\zeta_\xi\in\mathcal{B}^n(\xi,\overline{q})$

$$\begin{aligned}
&\eta_u\left(|\kappa^*(\zeta_\xi)-\kappa^*(\xi)|+|\kappa^*(\zeta_\xi)-\kappa(\zeta_\xi)|\right)+\eta_x\left(|\zeta_\xi-\xi|\right)\\
&\quad\leq \eta_u\left(L_{\kappa^*}(\zeta_\xi)\left|\zeta_\xi-\xi\right|+|\kappa^*(\zeta_\xi)-\kappa(\zeta_\xi)|\right)+\eta_x\left(|\zeta_\xi-\xi|\right)\\
&\quad\leq \eta_u\left(L_{\kappa^*}(\zeta_\xi)\overline{q}+|\kappa^*(\zeta_\xi)-\kappa(\zeta_\xi)|\right)+\eta_x\left(\overline{q}\right)\\
&\quad\leq \eta_u\left(L_{\kappa^*}(\zeta_\xi)\overline{q}+|\kappa^*(\zeta_\xi)-\kappa(\zeta_\xi)|\right)+L_{f_x}\overline{q}+\overline{q}
\end{aligned}$$

Therefore, the following implication holds

$$\begin{aligned}
&\eta_u\left(L_{\kappa^*}(\zeta_\xi)\overline{q}+|\kappa^*(\zeta_\xi)-\kappa(\zeta_\xi)|\right)+(L_{f_x}+1)\overline{q}\leq\overline{d}(\zeta_\xi)-\overline{d}_w\\
&\Rightarrow\eta_u\left(|\kappa^*(\xi)-\kappa(\zeta_\xi)|\right)+\eta_x\left(|\zeta_\xi-\xi|\right)\leq\overline{d}(\zeta_\xi)-\overline{d}_w
\end{aligned}$$

Then, by posing $\epsilon(\zeta_\xi)=\overline{d}(\zeta_\xi)-\bar{d}_w-(L_{f_x}+1)\overline{q}$, let us consider the following inequality

$$\eta_u\left(L_{\kappa^*}(\zeta_\xi)\overline{q}+|\kappa^*(\zeta_\xi)-\kappa(\zeta_\xi)|\right)\leq\epsilon(\zeta_\xi)$$

In order to point out the dependence of the approximating function and of the local Lipschitz bound on the networks parameters[4], we will use the notations $\kappa^*(\zeta_\xi|\mathrm{w})$ and $L_{\kappa^*}(\zeta_\xi|\mathrm{w})$. Then

[4]For the case of a two-layer Neural Network with Lipschitz continuous activation functions at the first layer and linear output layer, denoting as $\mathrm{w}\in\mathbb{R}^{n_\mathrm{w}}$ the overall network parameters, it holds that the approximating function is locally Lipschitz for any value of w. However, the function $L_{\kappa^*}(\xi)$ (which locally bonds the Lipschitz constant) is a function of the parameters.

we have that

$$\eta_u \left(L_{\kappa^*}(\zeta_\xi|\mathrm{w})\overline{q} + |\kappa^*(\zeta_\xi|\mathrm{w}) - \kappa(\zeta_\xi)| \right) \leq \epsilon(\zeta_\xi)$$

and, finally,

$$L_{\kappa^*}(\zeta_\xi|\mathrm{w})\overline{q} + |\kappa^*(\zeta_\xi|\mathrm{w}) - \kappa(\zeta_\xi)| \leq \eta_u^{-1} \left(\epsilon(\zeta_\xi) \right).$$

From a practical point of view, one can first sample the domain with a grid X_G with density parameter $\overline{q}$, then, by posing $\epsilon^{'}(\zeta) \triangleq \eta_u^{-1}\left(\epsilon(\zeta)\right),\ \forall \zeta \in X_G$, one can evaluate the map $\epsilon^{'}(\cdot)$ on the grid points. Then, the approximating function must verify the condition

$$L_{\kappa^*}(\zeta|\mathrm{w})\overline{q} + |\kappa^*(\zeta|\mathrm{w}) - \kappa(\zeta)| \leq \epsilon^{'}(\zeta),\ \ \forall \zeta \in X_G.$$

Notice that, in order to compute $\epsilon^{'}(\zeta)$, the $\mathcal{K}$-function $\eta_u(\cdot)$ must be known. For a general nonlinear transition map $\hat{f}$, we can compute a local linear bound on $\eta_u(\cdot)$ with ease. That is, the problem of finding a global $\mathcal{K}$-function $\eta_u(\cdot)$ is simplified in that of computing a local Lipschitz bound $L_{f_u}(\zeta)$ such that

$$\left|\hat{f}(\xi, u) - \hat{f}(\xi, u^{'})\right| \leq L_{f_u}(\zeta) \left|u - u^{'}\right|,\ \ \forall \xi \in \mathcal{B}^n(\zeta, \overline{q}),\ \forall (u, u^{'}) \in U^2.$$

A conservative bound on L_{f_u} can be evaluated as

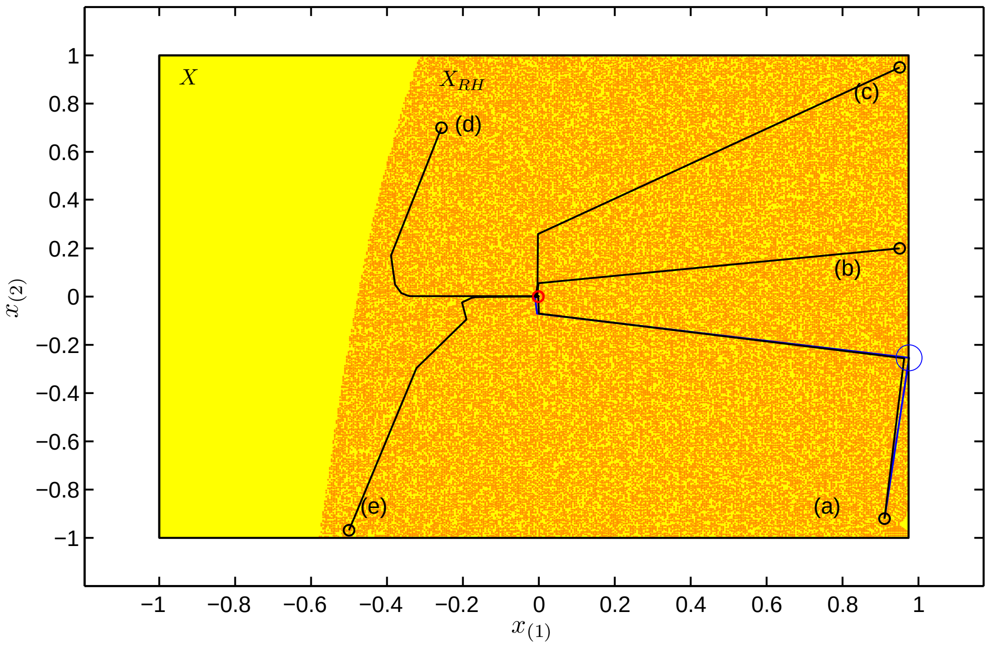
$$L_{f_u}(\zeta) = \max_{(\xi,u)\in\mathcal{B}(\zeta,\overline{q})\times U} \sum_{j=1}^{m}\sum_{i=1}^{n} \left| \left.\frac{\partial \hat{f}_{(i)}}{\partial u_{(j)}}\right|_{\zeta,u} \right|$$

where $f_{(i)}$ and $u_{(j)}$ are the i-th and the j-th components of $\hat{f}$ and u respectively.

4.4 Simulation Results

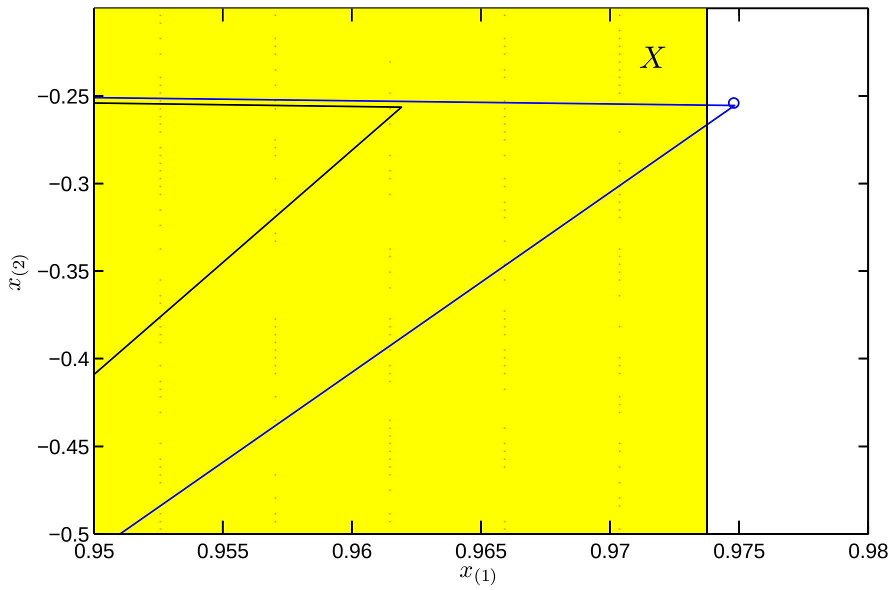
In order to show the effectiveness of the proposed approach, the behavior of the system proposed in Section 4.1, under the action of the devised approximate NN control law κ_{NN}, has been simulated by choosing different starting points inside the feasible domain $X_{MPC} \subset X$ (see Figure 4.1). Relying on the proposed control design methodology, the following value of $\bar{d}$=$6.4 \cdot 10^{-3}$ can

Figure 4.1 Closed loop trajectories of the system under the action of the approximate MPC law κ_{NN}, with starting points: (a)=[0.91,-0.921]; (b)=[0.95,0.2]; (c)=[0.95,0.95]; (d)=[-0.257,0.7]; (e)=[-0.5,-0.97]. The feasible area X_{MPC} and the constraint set X are put in evidence.



be computed considering the length of the control horizon N_c=5, the Lipschitz constant of the system L_{f_x}=2.05, the input constraint set U=$[-2.1053, 0.0263]$, and the invariance properties of the terminal set used to set-up the optimization problem X_f=$\{x{\in}X{:}\; h_f(x){\leq}\bar{h}_f{=}0.8\}$, where the function h_f, used as terminal cost, has been chosen as the Lyapunov function W for the closed-loop system under the action of the discontinuous stabilizing auxiliary controller given in Section 4.1. The black trajectories in Figure 4.1 show that the system has been effectively steered toward $\{0\}$ by κ_{NN}, while the state has been kept inside the constraint set X. Notably, if the constraints are not tightened in the computation of the approximation control law, it may happen that, due to the approximation-induced perturbations, the approximate controller fails to preserve the state within X. Indeed, the blue trajectory in Figure 4.1, generated by an approximate controller without tightening, violates the constraint in correspondence of the blue circled area, whose magnification is depicted in Figure 4.2. Finally, the approximate κ_{NN} law obtained by off-line computations over a uniform grid with knot density parameter $\bar{q}_{NN}$=$1.9{\cdot}10^{-3}$ is depicted in Figure 4.3, where the discontinuous nature of the MPC control law is enhanced.

Figure 4.2 Magnification of the circled area in Fig 4.1. The approximate control law with constraint tightening (left) keeps the trajectories inside X, while the approximate controller without tightening (right) fails in achieving the constraint satisfaction.

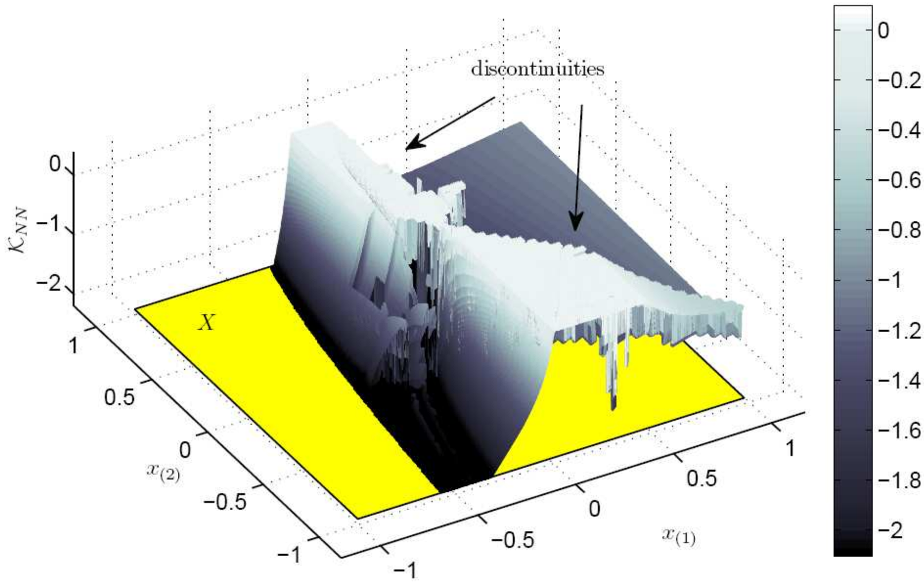


4.5 Concluding Remarks

With the aim to reduce the on-line computational complexity of nonlinear constrained Receding Horizon controllers, a method to obtain approximated off-line MPC laws has bee proposed. The possibility to obtain an approximate RH control law by performing off-line optimization, leads to a dramatic reduction of the real-time computational complexity with respect to on-line algorithms, and allows the application of the developed control technique to plants with fast dynamics, that require small sampling periods.

The robust stability properties of the off-line approximated RH control law have been analyzed by using the regional ISS tool. In this regard, the closed-loop system driven by the approximated controller is shown to be Input-to-State Stable with respect to approximation-induced perturbations in a subset of the feasible region of the exact RH controller.

Figure 4.3 The approximate MPC control law κ_{NN} in the domain $X_{MPC} \subset X$.



Finally, a Nearest Neighbor-based implementation of the approximate control law has been proposed, which guarantees the satisfaction of hard constraints and allows efficient on-line computations.

The distinctive feature of the proposed approximation scheme consists in the possibility to cope with possibly discontinuous state-feedback control laws, such as those arising from nonlinear constrained optimization, while guaranteeing the fulfillment of hard constraints on state and input variable despite the perturbations due to the use of an approximate controller.

It is believed that the proposed method can be further improved in terms of memory consumption (for the storage of the off-line approximated control law) and speed of the on-line computations by considering that, at least locally, some smooth approximation scheme (such as Neural Networks) can be used to fulfill the robust stability requirements.

Chapter 5

Networked Predictive Control of Uncertain Constrained Nonlinear Systems

5.1 Motivations

In the past few years, control applications in which sensor data and actuator commands are sent through a shared communication network have attracted increasing attention in control engineering, since network technologies provide a convenient way to remotely control large distributed plants [7, 46, 123]. Major advantages of these systems, usually referenced to as Networked Control Systems (NCS's), include low cost, reduced weight and power requirements and simple installation and maintenance. Conversely, NCS's are affected by the dynamics introduced by both the physical link and the communication protocol, that, in general, need to be taken in account in the design of the control schemes.

As many applications converge in sharing computing and communication resources, issues of scheduling, network delay and data loss will need to be dealt with systematically. In particular, the random nature of transmission delays in shared networks makes it difficult to analyze stability and performance of the closed-loop systems. Remarkably, random delays are inherently related with the problem of data losses in NCS's. Indeed the stringent bounds imposed on time-delays by closed-loop stability requirements lead to the necessity to discard those packets arriving later than a maximum tolerable delay threshold. In addition, when the design of feedback control systems concerns wireless sensor networks, the implicit assumption of data availability no longer holds, as data packets are randomly dropped and delayed.

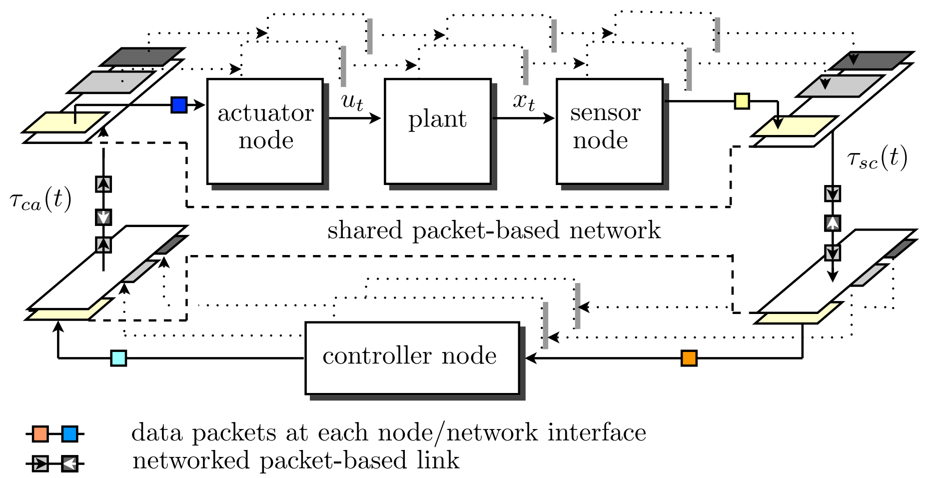
While classical control theories provide solid analytical results to design the various components of the control system, they critically rely on the assumption that the underlying communication technology is ideal. In the networked communication setting, with possibly shared resources, neglecting network-induced perturbations such as delays and packet losses can eventually compromise the stability of the closed-loop system, if no proper provision are adopted.

Various control strategies have been presented in the past literature to design effective NCS's for linear time-invariant systems [36, 70, 105, 116] in presence of lossy or delayed communications. In particular, many recent results are focused on characterizing the stability properties of the closed-loop NCS's in a stochastic framework when static state-feedback control laws or Linear Quadratic Gaussian (LQG) policies are adopted in presence of random transmission delays and packet dropouts [22, 35, 47, 108].

Besides the development of inherently stable controllers for these systems, another important aspect in the deployment of an effective NCS is the choice of the communication protocol to be used. In this regard, the packet structure of most transmission networks has important implications from the control point of view [118]. As it is well known, the performances of digital control systems increases with the sampling rate; nonetheless, when shared resources are used, it is not possible to increase arbitrarily the data transfer rate, due to the subsequent increase of network congestion, delays and packet dropouts. An effective way to overcome this limitation consists in using protocols which allow to transmit fewer but more informative packets [5, 36]. Thus, large data packets can be used to collect multiple sensors data and send predictions on future control inputs, without significantly increasing the network load [96, 114]. The basic layout of an NCS with multiple loops sharing a packet-based communication network is depicted in Figure 5.1, where, in order to distinguish the time delays in the sensor-to-controller and controller-to-actuator links, the network has been partitioned in two segments affected respectively by delays $\tau_{sc}(t)$ and $\tau_{ca}(t)$.

When strict bounds on data delays and losses can be assumed and large data packets are allowed, model predictive control strategies have been proposed to cope with the design of a stabilizing NCS [22, 115], due to their intrinsic features of generating a future input sequence

Figure 5.1 Scheme of a NCS with multiple loops closed through a shared packet-based network with delayed data transmission.



that can be transmitted within a single data packet.

While the aforementioned control design/scheduling techniques rely on linear process models, if the system to be controlled is subjected to constraints and nonlinearities, the formulation of an effective networked control strategy becomes a really hard task [98]. In this framework, the present chapter provides theoretical results that motivate, under suitable assumptions, the combined use of nonlinear Model Predictive Control (MPC) with a Network Delay Compensation (NDC) strategy [13, 96], in order to cope with the simultaneous presence of model uncertainties, time-varying transmission delays and data-packet losses. In the current literature, for the specific class of MPC schemes which impose a fixed terminal constraint set, X_f, as a stabilizing condition, the robustness of the overall c-l system, in absence of transmission delays, has been shown to depend on the invariance properties of X_f, [66, 94]. In this regard, by resorting to invariant set theoretic arguments [17, 55], we will show that the proposed NCS can robustly stabilize a nonlinear constrained system even in presence of data transmission delays and model uncertainty. In particular, the tool recursive feasibility in constrained networked nonlinear MPC, first addressed in [92], will be exploited to prove the Input-to-State Stability (ISS) of the scheme w.r.t. additive perturbations. Indeed, by using the novel regional characterization of ISS in terms of time-varying Lyapunov functions given in Chapter 2 (the regional ISS for time invariant case has been introduced in [75], while semi-global results for time-varying discrete-time systems are given in [51, 60]), the ISS property of the closed-loop system will be established also in presence of unreliable networked communications.

5.2 Problem Formulation

Consider the nonlinear discrete-time dynamic system

$$x_{t+1} = f(x_t, u_t, \upsilon_t), \ \ t \in \mathbb{Z}_{\geq 0}, \ x_0 = \overline{x}\,, \tag{5.1}$$

where $x_t \in \mathbb{R}^n$ denotes the state vector, $u_t \in \mathbb{R}^m$ the control vector and $\upsilon_t \in \Upsilon$ is an uncertain exogenous input vector, with $\Upsilon \subset \mathbb{R}^r$ compact and $\{0\} \subset \Upsilon$. Assume that state and control variables are subject to the state and input constraints (1.2) and (1.3).

The transition function is supposed to be Lipschitz continuous in the state (see Assumption 3.1.1). Moreover we assume that the additive transition uncertainty is bounded, as stated by Assumption 4.2.2, such that the system (5.1) can be posed in the form.

$$x_{t+1} = \hat{f}(x_t, u_t) + d_t \tag{5.2}$$

with $d_t \in \mathcal{B}^n(\overline{d})$ and $\overline{d} \in \mathbb{R}_{\geq 0}$ finite.

Under the posed assumptions, the control objective consists in guaranteeing the ISS property for the closed-loop system with respect to the prescribed class of uncertainties, while enforcing the fulfillment of constraints in presence of packet dropouts, bounded transmission delays and bounded disturbances.

With regard to the network dynamics and communication protocol, it is assumed that a set of data (packet) can be sent, at a given time instant, through the network by a node, while both the sensor-to-controller and the controller-to-actuator links are supposed to be affected by delays and dropouts due to the unreliable nature of networked communications. In order to cope with network delays, the data packets sent by the sensor node are Time-Stamped (TS) [114], that is, they contain the information on when the transmitted state measurement had been collected. Analogously, the controller node is required to attach to each data packet the time stamp of the state measurement which the computed control action relies on. The advantage of using a time-stamping policy in NCS's is well documented [12, 123], however it requires, in general, that all the nodes of the network have access to a common system's clock, or that a proper clock synchronization service is provided by the network protocol. In our setup, we will assume that perfect clock synchronization is maintained between sensors, actuators and controller. This task can be achieved in different ways (see [113, 122, 124] and the references therein), however we will abstract from the particular method used to maintain synchronization, since we are mainly focused on the control design issues rather than on the transmission protocol and the network scheduling policy.

The next section will describe how the TS mechanism can be used to compensate for transmission delays.

5.2.1 Network dynamics and delay compensation

In the sequel, $\tau_{ca}(t)$ and $\tau_{sc}(t)$ will denote the delays occurred respectively in the controller-to-actuator and in the sensor-to-controller links, while $\tau_a(t)$ will represent the "age" (in discrete time instant) of the control sequence used by the actuator to compute the current input and $\tau_c(t)$ the age of the state measurement which had been used by the controller at time t to compute the control actions to be sent to the actuator. Finally, $\tau_{rt}(t) \triangleq \tau_a(t) + \tau_c(t - \tau_a(t))$ is the so called *round trip time*, i.e., the age of the state measurement used to compute the input applied at time t.

The NDC strategy adopted in the present work, which relies on the one devised in [96] (originally developed for unconstrained systems nominally stabilized by a generic nonlinear controller), is based on exploiting the time stamps of the data packets in order to retain only the most recent informations at each destination node: when a novel packet is received, if it carries a more recent time-stamp than the one already in the buffer, then it takes the place of the older one and, in the TCP-like case, an acknowledgment of successful packet receipt is sent to the node which transmitted the packet. The TS-based packet arrival management implies $\tau_a(t) \leq \tau_{ca}(t)$ and $\tau_c(t) \leq \tau_{sc}(t)$. Moreover, the NDC strategy comprises a Future Input Buffering (FIB) mechanism (also known as "play-back buffer", see [65] for details), which requires that the controller node send a packeted sequence of N_c control actions (with N_c sufficiently large) to the actuator node (in order to compensate for future packet dropouts or delays). In turn the actuator, at the arrival of each new packet, first stores the entire sequence in its internal buffer, then, at each time instant t, selects a time-consistent control action to be applied to the plant, by setting $u_t = u_t^b$, where u_t^b is the $\tau_a(t)$-th element of the buffered sequence $\mathbf{u}^b_{t-\tau_a(t),t-\tau_a(t)+N_c-1}$, which, in turn, is given by

$$\mathbf{u}^b_{t-\tau_a(t),t-\tau_a(t)+N_c-1} = \mathrm{col}[u^b_{t-\tau_a(t)}, \ldots, u^b_t, \ldots, u^b_{t-\tau_a(t)+N_c-1}] = \mathbf{u}^c_{t-\tau_a(t),t-\tau_a(t)+N_c-1|t-\tau_{rt}(t)}.$$

where the sequence $\mathbf{u}^c_{t-\tau_a(t),t+N_c-1|t-\tau_{rt}(t)}$ had been computed at time $t-\tau_a(t)$ by the controller on the basis of the state measurement collected at time $t-\tau_{rt}(t) = t-\tau_a(t)-\tau_c(t-\tau_a(t))$. In this framework, our control-theoretical approach will exploit the input buffering as a service provided transparently by an underlying communication channel, therefore we will not focus on the implementation details. Due to the capability of performing synchronization, buffering operations and management of time stamped packets, the actuation device will be addressed to as "smart"actuator. For a deeper insight on such a mechanism, the reader is referred to [5] and [65].

In most situations, it is natural to assume that the age of the data-packets available at the controller and actuator nodes subsume an upper bound [96], as specified by the following assumption.

Assumption 5.2.1 (Network reliability)**.** *The quantities* $\tau_c(t)$ *and* $\tau_a(t)$ *verify* $\tau_c(t) \leq \overline{\tau}_c$ *and* $\tau_a(t) \leq \overline{\tau}_a$, $\forall t \in \mathbb{Z}_{>0}$, *with* $\overline{\tau}_c \in \mathbb{Z}_{\geq 0}$ *and* $\overline{\tau}_a \in \mathbb{Z}_{\geq 0}$ *finite.* □

Notably, we don't impose bounds on $\tau_{sc}(t)$ and $\tau_{ca}(t)$, allowing the presence of packet losses (infinite delay). In this way, an actuator buffer with finite length can be used.

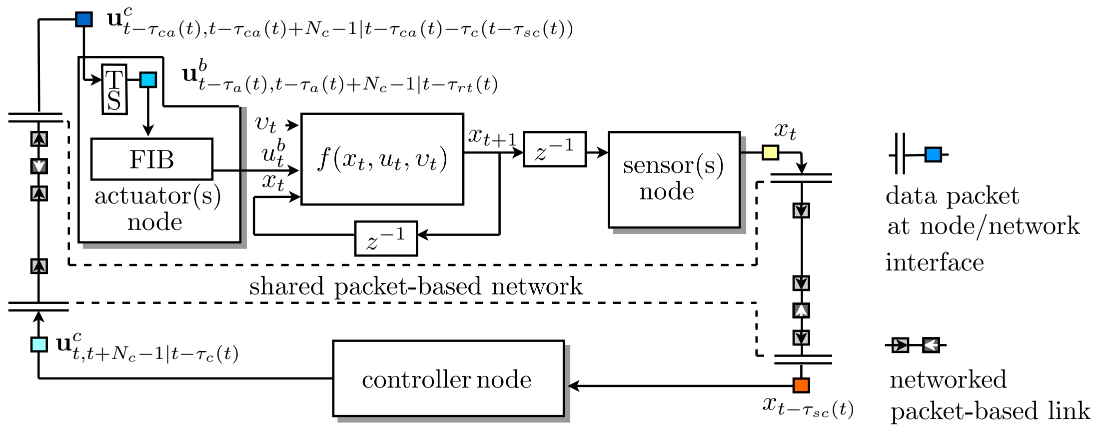
Assumption 5.2.2 (Buffer length)**.** *The actuator buffer length, which is equal to the length of the input sequence sent by the controller to actuator, verifies* $N_c \geq \overline{\tau}_a + 1$.

Under Assumptions 5.2.1 and 5.2.2 , the round trip time verifies $\tau_{rt}(t) \leq \overline{\tau}_{rt} = \overline{\tau}_c + \overline{\tau}_a \leq \overline{\tau}_c + N_c - 1$, $\forall t \in \mathbb{Z}_{>0}$.

We will first consider the case of networks with acknowledged communication protocols, also known as *TCP-like* [47], in which the destination node sends an acknowledgment packet (ACK) of successful packet receipt to the source node, and then the results will be extended to non-acknowledged protocols, which are usually referred to as *UDP–like* [47]. In a TCP-like scenario, the acknowledgment messages are assumed to have the highest priority among all the routed packets, such that, after each successful packet receipt, the source node receives a deterministic notification within a single time-interval.

In this regard, the presence of ACKs in TCP-like networks can be exploited by the controller (which is acknowledged of successful packet receipt by the actuator) to internally reconstruct the true sequence of controls which has been applied to the plant [96] from time instant $t-\tau_c(t)$ to $t-1$, in order to get an estimation $\hat{x}_{t|t-\tau_c(t)}$ of the current state x_t, on the basis of the most recent available state measurement $x_{t-\tau_c(t)}$. A graphical representation of the overall NCS layout is depicted in Figure 5.2.

Figure 5.2 Scheme of the NDC strategy. In evidence the Time-Stamping packet arrival management (TS) and the Future Input Buffering (FIB) mechanism at the actuator node.



5.2.2 State reconstruction in TCP–like networks

At time t, the computation of the control sequence to be sent to the actuator must rely on a state measurement $x_{t-\tau_c(t)}$ performed at time $t-\tau_c(t)$. In order to recover the standard MPC formulation, the current (possibly unavailable) state x_t has to be reconstructed by means of the nominal model (4.9) and of the true input sequence $\mathbf{u}_{t-\tau_c(t),t-1} \triangleq \text{col}[u_{t-\tau_c(t)}, \ldots, u_{t-1}]$ applied by the smart actuator to the plant from time $t-\tau_c(t)$ to $t-1$. In this regard, the benefits due to the use of a state predictor in NCS's are deeply discussed in [96, 120, 121] and [114, 115]. The sequence $\mathbf{u}_{t-\tau_c(t),t-1}$ can be internally reconstructed by the controller thanks to an acknowledgment-based protocol.

Moreover, in presence of delays in the controller-to-actuator link, we must consider that the computed control sequence may not be applied entirely to the plant.

Indeed, as shown in Figures 5.3 5.4 and 5.5, the presence of delays may lead to infeasibility and even destabilize the plant, due to the fact that the sequence of commands applied to the plant may consists of control actions belonging to spurious sequences (computed in different time instants and with different informations) and for which the satisfaction of nominal constraints is not ensured.

In order to ensure that the sequence used for prediction would coincide with the one that will be applied to the plant, we can retain, at time t, some of the elements of the control sequence computed at time $t-1$ (i.e., the subsequence $\mathbf{u}^b_{t,t+\overline{\tau}_a-1|t-1-\tau_c(t-1)}$), and optimize only over the remaining elements (i.e., computing a sequence $\mathbf{u}^\circ_{t+\overline{\tau}_a,t+N_c-1}$), initiating a Reduced Horizon Optimal Control Problem (RHOCP) with the state prediction $\hat{x}_{t+\overline{\tau}_a}$.

In the next section we will describe in detail how RHOCP can be formulated.

5.2.3 Reduced horizon optimization

In the following, we will describe the mechanism used by the controller to compute the sequence of control actions to be forwarded to the smart actuator.

The class of algorithms which the considered controller belongs to is that of Model Predictive Control (MPC), in which a Finite Horizon Optimal Control Problem, based on the current state measurement, is solved at each time step to obtain a control action to be applied to the plant, thus implicitly obtaining a closed-loop scheme. With regard to the aforementioned class of controllers, in which the length of the horizon is usually kept fixed and equals the number of decision variables of the optimization, the proposed method relies on the solution, at each time instant t, of a Reduced Horizon Optimal Control Problem, that is, the number of decision variables will be, in general, reduced by reusing some elements of previous optimizations. This feature will allow to address the problem of delayed communication in the controller-to-actuator path.

Moreover, a constraint tightening technique [66] will be used to robustly enforce the constraints.

Figure 5.3 Example of a delayed command sequence delivery to the actuator node. At time $t = 0$, the FIB is filled with a 3-steps feasible sequence $\mathbf{u}^b$, (a). The controller, after having generated a control sequence $\mathbf{u}^c$, forms a packet and sends it through the network, (b). Due to network delays, the transmitted sequence cannot be applied by the smart actuator, which picks a control action from the sequence already stored in the FIB, (c).

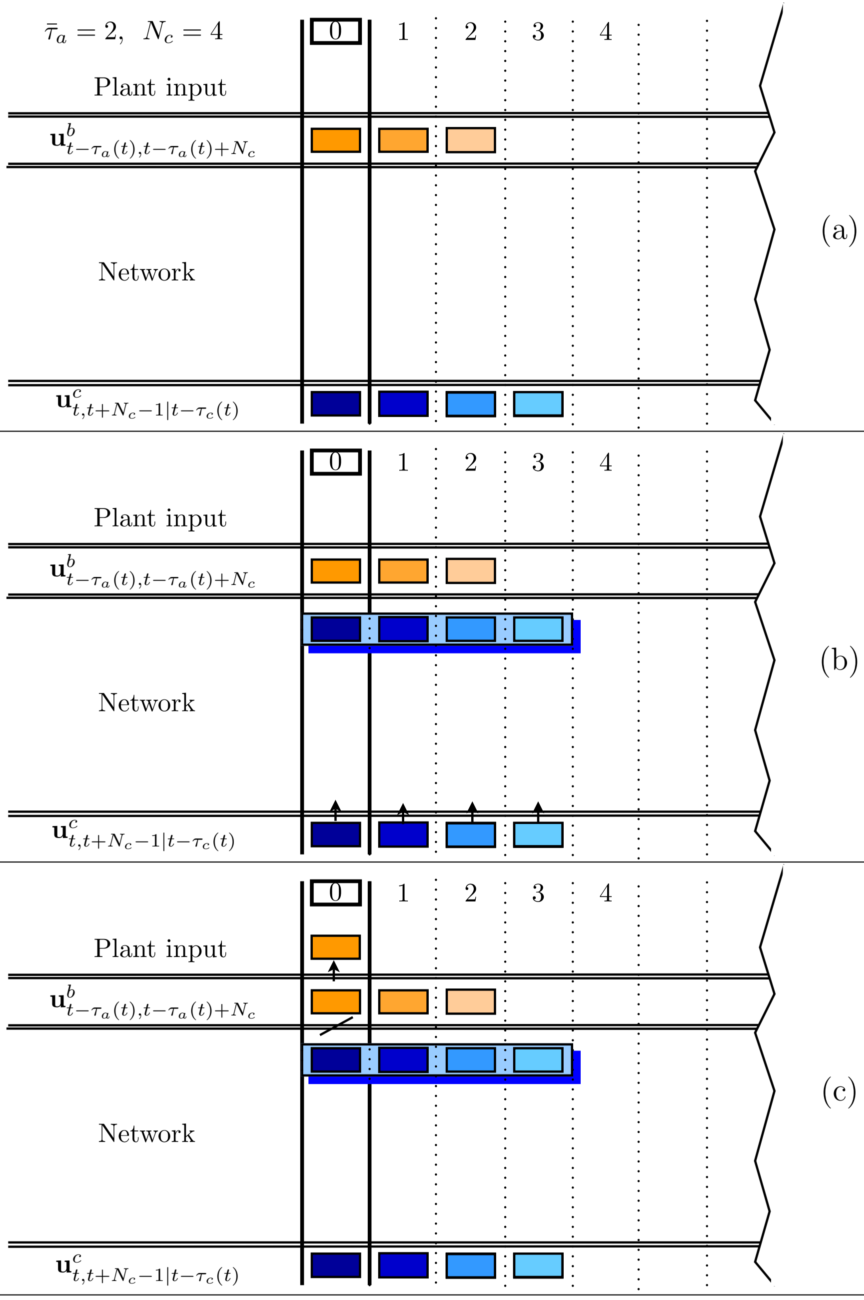


Figure 5.4 At time $t = 1$, the controller generates a new input sequence and sends a new data packet through the network, but the actuator still does not receive any new packets; thus, the FIB provides the required control action, (d). At time $t = 2$ the controller computes a new sequence and sends a packet (e), but the actuator node receives the sequence computed at time $t = 0$, with $\tau_a(t) = 2$, (f). The sequence in the FIB is replaced by the newest one, and the control action to be applied is taken from the last received sequence.

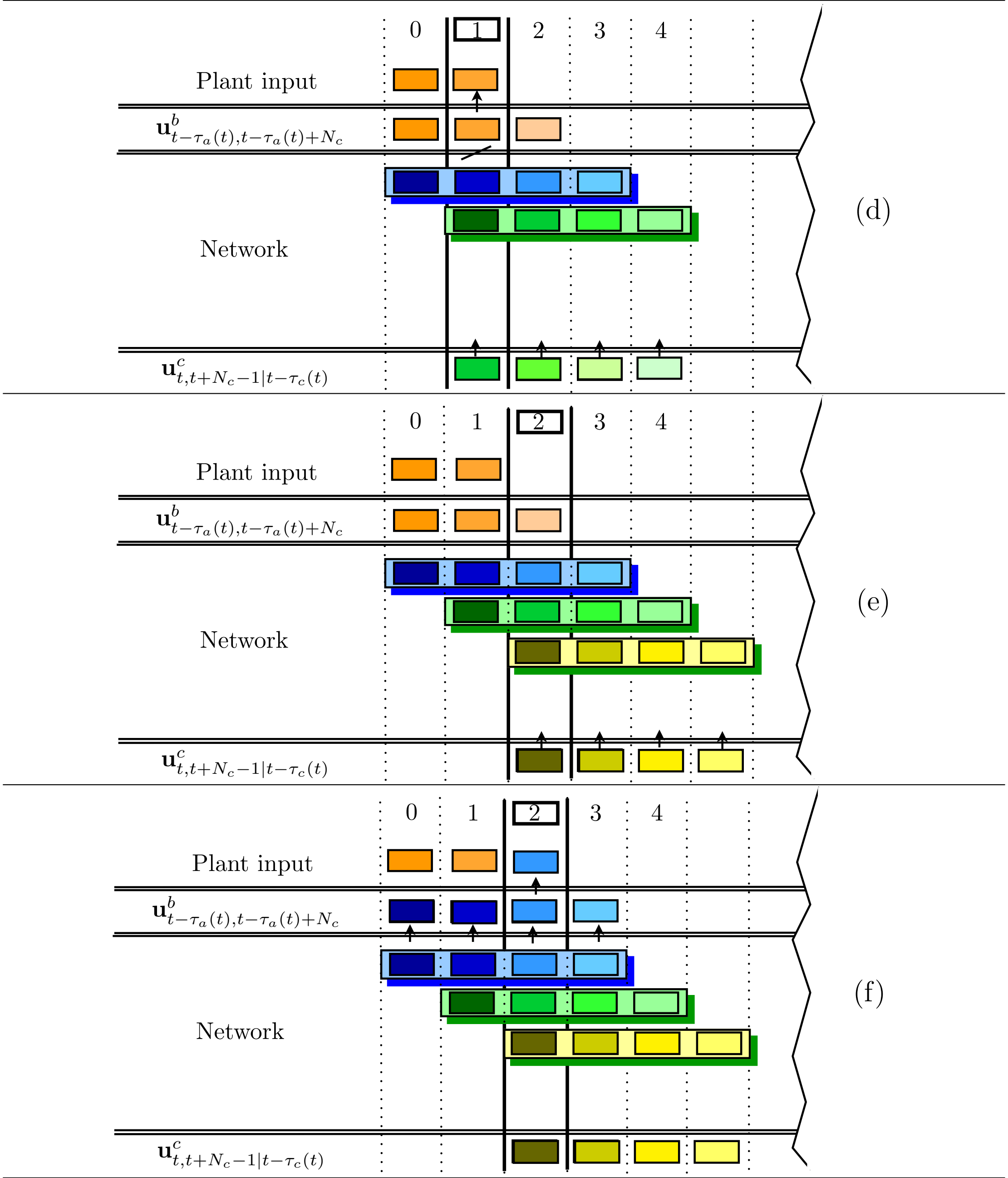
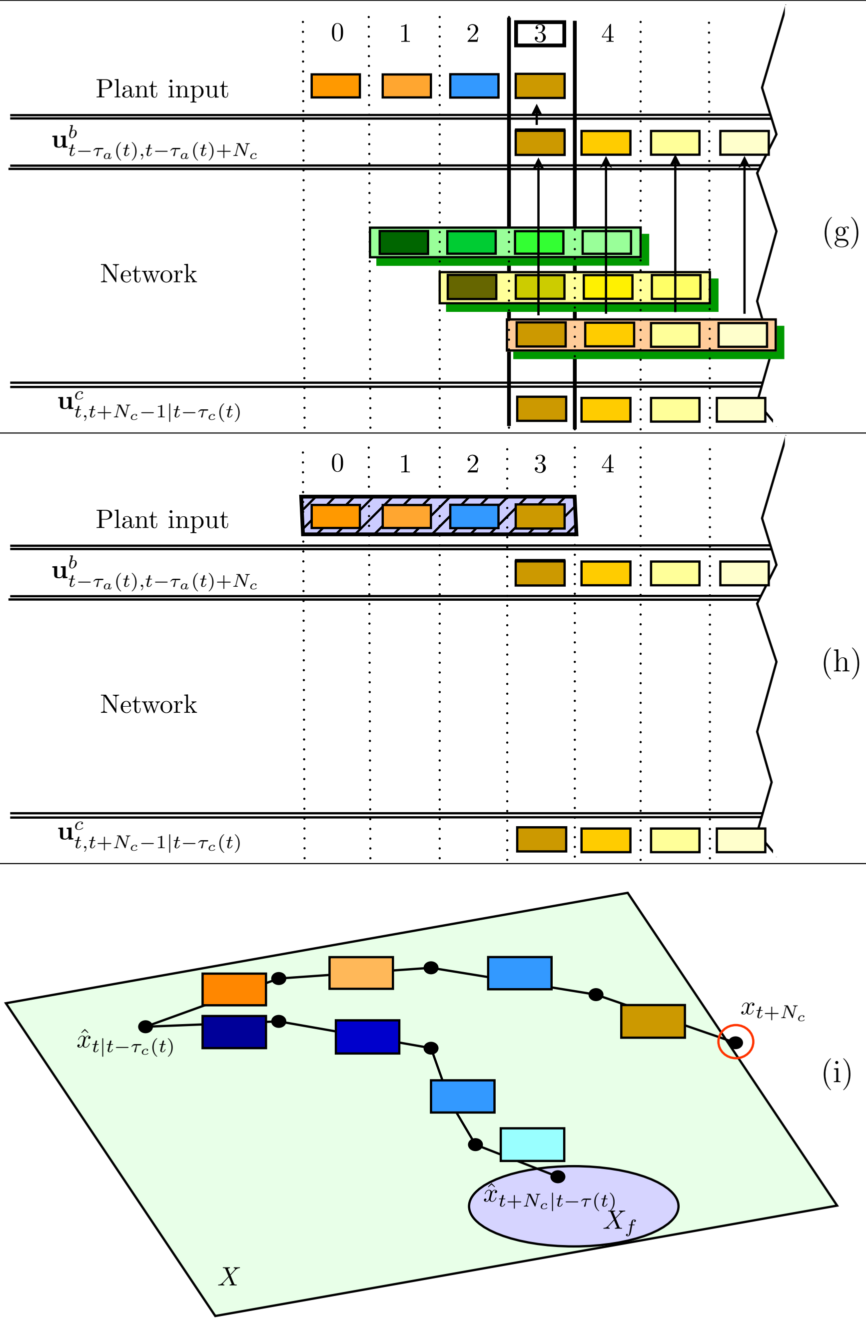


Figure 5.5 Finally, at time $t = 3$, the sequence computed by the controller reaches the actuator node with no delay, (g). It follows that the true sequence applied to the plant in the interval $\{0, \ldots, 3\}$ is a combination of sequences computed in different instants, (h), which is not guaranteed (in general) to be feasible for the perturbed system, (i). Therefore, network induced delays may lead to infeasibility if no proper provisions are adopted.



First, let us introduce the following sets, obtained by restricting the nominal constraint X.

Definition 5.2.1 ($X_i(\overline{d})$). *Under Assumptions 3.1.1 and 4.2.2, suppose*[1]*, without loss of generality,* $L_{f_x}\neq 1$. *The tightened sets* $X_i(\overline{d})$, *are defined as*

$$X_i(\overline{d})\triangleq X\backsim\mathcal{B}^n\left(\frac{L_{f_x}^i-1}{L_{f_x}-1}\overline{d}\right),\ \forall i\in\mathbb{Z}_{>0}\ . \tag{5.3}$$

□

Problem 5.2.1 (RHOCP). *Given a positive integer* $N_c\in\mathbb{Z}_{\geq 0}$, *at any time* $t\in\mathbb{Z}_{\geq 0}$, *let* $\hat{x}_{t|t-\tau_c(t)}$ *be the estimate of the current state,* x_t, *obtained from the last available state measurement* $x_{t-\tau_c(t)}$ *with the controls* $\mathbf{u}_{t-\tau_c(t),t-1}$ *already applied to the plant. Moreover let* $\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}$ *be the state computed from* $\hat{x}_{t|t-\tau_c(t)}$ *by extending the prediction using the input sequence computed at time* $t-1$, $\mathbf{u}^c_{t,t+\overline{\tau}_a-1}$. *Then, given a stage-cost function* h, *the constraint sets* $X_i(\overline{d})\subseteq X, i\in\{\tau_c(t)+\overline{\tau}_a+1,\ldots,\tau_c(t)+N_c\}$, *a terminal cost function* h_f *and a terminal set* X_f, *the Reduced Horizon Optimal Control Problem (RHOCP) consists in solving, with respect to a* $(N_c-\overline{\tau}_a)$*-steps input sequence,* $\mathbf{u}_{t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)}\triangleq \mathrm{col}[u_{t+\overline{\tau}_a|t-\tau_c(t)},\ldots,u_{t+N_c-1|t-\tau_c(t)}]$, *the following minimization problem*

$$J^\circ_{FH}(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)},\mathbf{u}^\circ_{t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)},N_c-\overline{\tau}_a)$$
$$\triangleq \min_{\mathbf{u}_{t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)}}\left\{\sum_{l=t+\overline{\tau}_a}^{t+N_c-1}h(\hat{x}_{l|t-\tau_c(t)},u_{l|t-\tau_c(t)})+h_f(\hat{x}_{t+N_c|t-\tau_c(t)})\right\}$$

subject to the

i) nominal dynamics (4.9);

ii) input constraints $u_{t-\tau_c(t)+i|t-\tau_c(t)}\in U$, *with* $i\in\{\tau_c(t)+\overline{\tau}_a,\ldots,\tau_c(t)+N_c-1\}$;

iii) restricted state constraints $\hat{x}_{t-\tau_c(t)+i|t-\tau_c(t)}\in X_i(\overline{d})$, *with* $i\in\{\tau_c(t)+\overline{\tau}_a+1,\ldots,\tau_c(t)+N_c\}$;

iv) terminal state constraint $\hat{x}_{t+N_c|t-\tau_c(t)}\in X_f$.

Finally, the sequence of controls forwarded by the controller to the actuator is constructed as $\mathbf{u}^c_{t,t+N_c-1|t-\tau_c(t)}\triangleq\mathrm{col}[\mathbf{u}^c_{t,t+\overline{\tau}_a-1|t-1-\tau_c(t-1)},\mathbf{u}^\circ_{t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)}]$ *(i.e., it is obtained by appending the solution of the RHOCP to the control sequence computed at time* $t-1$*).* □

[1] The very special case L_{f_x}=1 can be trivially addressed by a few suitable modifications to the Definition 5.2.1

The following definitions will be used in the rest of the chapter.

Definition 5.2.2 ($X_{MPC}(\tau)$)**.** *Given a non-negative integer $\tau \in \mathbb{Z}_{\geq 0}$, the set containing all the vectors $\overline{x}_0 \in \mathbb{R}^n$ for which there exists a sequence of N_c control moves which satisfies all the constraints specified below is said feasible set with τ-delay restriction, and it is denoted with $X_{MPC}(\tau)$.*

$$X_{MPC}(\tau) \triangleq \left\{ \overline{x}_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\, \overline{\mathbf{u}}_{0,N_c-1} \in U^{N_c} : \\ \hat{x}(i, \overline{x}_0, \overline{\mathbf{u}}_{0,i-1}) \in X_{\tau+i}(\overline{d}), \forall i \in \{1, \ldots, N_c\} \\ \text{and } \hat{x}(N_c, \overline{x}_0, \overline{\mathbf{u}}_{0,N_c-1}) \in X_f \end{array} \right\} \tag{5.4}$$

□

For the sake of brevity, the set $X_{MPC}(0)$ will be denoted as X_{MPC}.

Definition 5.2.3 (Feasible sequence at time t)**.** *Given a delayed state measurement $x_{t-\tau_c(t)}$, available at at time t to the controller, let us consider the prediction $\hat{x}_{t|t-\tau_c(t)}$ of the actual state x_t obtained with the nominal model and with the actual control sequence applied from time $t - \tau_c(t)$ to $t-1$, $\overline{\mathbf{u}}_{t-\tau_c(t),t-1}$, which is known to the controller. Moreover consider a sequence of N_c control moves $\overline{\mathbf{u}}^c_{t,t+N_c-1}$ and its two subsequences $\overline{\mathbf{u}}^c_{t,t+\overline{\tau}_a-1}$ and $\overline{\mathbf{u}}^c_{t+\overline{\tau}_a,t+N_c-1}$ such that $\overline{\mathbf{u}}^c_{t,t+N_c-1} = \mathrm{col}[\overline{\mathbf{u}}^c_{t,t+\overline{\tau}_a-1}, \overline{\mathbf{u}}^c_{t+\overline{\tau}_a,t+N_c-1}]$.*

The input sequence $\overline{\mathbf{u}}^c = \overline{\mathbf{u}}^c_{t,t+N_c-1}$ is said feasible at time t if the subsequence $\overline{\mathbf{u}}^c_{t,t+\overline{\tau}_a-1}$ yields to $\hat{x}_{t-\tau_c(t)+i|t-\tau_c(t)} \in X_i(D)$, $\forall i \in \{\tau_c(t) + 1, \ldots, \tau_c(t) + \overline{\tau}_a\}$ and if the second subsequence satisfies all the constraints of the RHOCP initiated with $\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)} = \hat{x}(\overline{\tau}_a, x_{t-\tau_c(t)}, \overline{\mathbf{u}}^*_{t-\tau_c(t),t+\overline{\tau}_a-1})$.

□

Remark 5.2.1. *Note that what we call feasible sequence in t is not just an input sequence which satisfies the constraints of the RHOCP (specified in the horizon $[t + \overline{\tau}_a + 1, \ldots, t + N_c]$), but it is required to keep the nominal trajectories inside the restricted constraints for an horizon of N_c steps from $t + 1$ to $t + N_c$, that is larger than the one considered by the optimization.*

Now, by accurately choosing the stage cost h, the constraints $X_i(\overline{d})$, the terminal cost function h_f, and by imposing a terminal constraint X_f at the end of the control horizon, it is possible to show that the recursive feasibility of the scheme can be guaranteed for $t \in \mathbb{Z}_{>0}$, also in presence of norm-bounded additive transition uncertainties and network delays. Moreover, the devised control scheme will be proven to be Input-to-State stabilizing if the following assumptions are verified.

Assumption 5.2.3. *The transition cost function* $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ *is such that* $\underline{h}(|x|) \leq h(x,u), \forall x \in X, , \forall u \in U$, *where* $\underline{h}$ *is a* $\mathcal{K}_\infty$*-function. Moreover,* h *is Lipschitz w.r.t.* x*, uniformly in* u*, with L. constant* $L_h \in \mathbb{R}_{>0}$. □

Assumption 5.2.4 (κ_f, h_f, X_f)**.** *There exist an auxiliary control law* $\kappa_f(x) : X \to U$*, a function* $h_f(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$*, a positive constant* $L_{h_f} \in \mathbb{R}_{>0}$*, a level set of* h_f*,* $X_f \subset X$ *and a positive constant* $\delta \in \mathbb{R}_{>0}$ *such that the following properties hold:*

i) $X_f \subset X$*,* X_f *closed,* $\{0\} \in X_f$*;*

ii) $\kappa_f(x) \in U$*,* $\forall x \in X_f \oplus \mathcal{B}^n(\delta)$*;*

iii) $\hat{f}(x, \kappa_f(x)) \in X_f$*,* $\forall x \in X_f \oplus \mathcal{B}^n(\delta)$*;*

iv) $h_f(x)$ *Lipschitz in* X_f*, with L. constant* $L_{h_f} \in \mathbb{R}_{>0}$*;*

v) $h_f(\hat{f}(x, \kappa_f(x))) - h_f(x) \leq -h(x, \kappa_f(x))$*,* $\forall x \in (X_f \oplus \mathcal{B}^n(\delta)) \backslash 0$*.*

□

In addition, we require the following assumptions to be verified together with 5.2.3 and 5.2.4.

Assumption 5.2.5. *Let* X_f *be a sub-level set of* h_f *(i.e.* $X_f = \{x \in \mathbb{R}^n : h_f(x) \leq \overline{h}_f\}$ *); then we assume that the transition cost function* h *and the terminal cost* h_f *satisfy the condition*

$$\min_{u \in U} \left\{ \inf_{x \in \mathcal{C}_1(X_f) \backslash (X_f \oplus \mathcal{B}^n(\delta))} \{h_f(x) - h(x,u)\} \right\} > \overline{h}_f. \tag{5.5}$$

where $\delta \in \mathbb{R}_{>0}$ *is a positive scalar for which Points iii) and v) of Assumption 5.2.4 hold.* □

Remark 5.2.2. *With regard to the choice of the terminal set, a sufficient procedure for obtaining a set* X_f *satisfying Assumption 5.2.4 has been proposed in [66]. First, notice that, given a locally stabilizing auxiliary state-feedback controller* $\kappa_f(x)$*, a control Lyapunov function* $h_f(x)$ *for* $\hat{f}(x, \kappa_f(x))$ *and a sub-level set* Ω_f *RPI under* $\hat{f}(x, \kappa_f(x))$ *(i.e.,* $\Omega_f \triangleq \{x \in \mathbb{R}^n : h_f(x) \leq$

$\overline{h}_f$, $\overline{h}_f \in \mathbb{R}_{>0}\}$ such that $\hat{f}(x, \kappa_f(x)) \in \Omega_f \backsim \mathcal{B}(\delta), \forall x \in \Omega_f$ for some $\delta \in \mathbb{R}_{>0}$) it is always possible to find a positive definite functions $h(x, u)$ such that Point v) of Assumption 5.2.4 holds. Then, it has been suggested to choose $X_f = \Omega_f \backsim \mathcal{B}(\delta)$, imposing a bound on the maximal admissible uncertainties which depends on δ.

On the other hand the constructive assumption posed in [66] is somewhat conservative. In this regard, Assumption 5.2.5 allows to decouple the uncertainty bound which ensures the recursive feasibility of the scheme from the particular choice of κ_f. In this way, the robustness of the scheme depends only on the invariant properties of X_f through $\mathcal{C}_1(X_f)$. □

Now, the following Lemma ensures that the original state constraints can be satisfied by imposing to the nominal trajectories in the RHOCP the restricted constraints introduced in Definition 5.2.1.

Lemma 5.2.1 (Robust Constraint Satisfaction)**.** *Any feasible control sequence $\overline{\mathbf{u}}^c_{t,t+N_c-1|t-\tau_c(t)}$, applied in open-loop to the perturbed system from time t to $t + N_c - 1$, guarantees that the true (networked/perturbed) state will satisfy $x_{t+j} \in X$, $\forall j \in \{1, \ldots, N_c\}$.* □

Proof [Lemma 5.2.1] Given the state measurement $x_{t-\tau_c(t)}$, available at time t at the controller node, let us consider the combined sequence of controls $\mathbf{u}^*$ formed by: i) the subsequence used for estimating $\hat{x}_{t|t-\tau_c(t)}$ (i.e., the true control sequence $\mathbf{u}_{t-\tau_c(t),t-1}$applied by the NDC to the plant from $t - \tau_c(t)$ to $t - 1$) and ii) a feasible control sequence $\overline{\mathbf{u}}^c_{t,t+N_c-1|t-\tau_c(t)}$, that is

$$\mathbf{u}^*_{t-\tau_c(t),t+N_c-1|t-\tau_c(t)} \triangleq \text{col}[\mathbf{u}_{t-\tau_c(t),t-1}, \overline{\mathbf{u}}^c_{t,t+N_c-1|t-\tau_c(t)}]. \tag{5.6}$$

Then, the prediction error $\hat{e}_{t-\tau_c(t)+i|t-\tau_c(t)} \triangleq x_{t-\tau_c(t)+i} - \hat{x}_{t-\tau_c(t)+i|t-\tau_c(t)}$, with $i \in \{1, \ldots, N_c + \tau_c(t)\}$ and $x_{t-\tau_c(t)+i}$ obtained by applying $\mathbf{u}^*_{t-\tau_c(t),t+N_c-1|t-\tau_c(t)}$ in open loop to the uncertain system (5.1), is upper bounded by

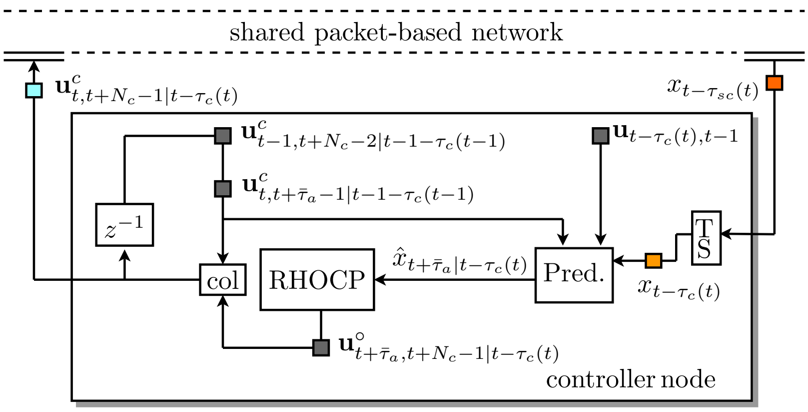
$$|\hat{e}_{t-\tau_c(t)+i|t-\tau_c(t)}| \leq \frac{L^i_{f_x} - 1}{L_{f_x} - 1}\overline{d}, \;\; \forall i \in \{1, \ldots, N_c + \tau_c(t)\}$$

where $\overline{d}$ is defined as in Assumption 4.2.2. Being $\overline{\mathbf{u}}^c_{t,t+N_c-1|t-\tau_c(t)}$ feasible, it holds that $\hat{x}_{t-\tau_c(t)+i|t-\tau_c(t)} \in X_i(\overline{d}), \forall i \in \{\tau_c(t) + 1, \ldots, N_c + \tau_c(t)\}$, then it follows immediately that $x_{t-\tau_c(t)+i} = \hat{x}_{t-\tau_c(t)+i|t-\tau_c(t)} + \hat{e}_{t-\tau_c(t)+i|t-\tau_c(t)} \in X$. ■

The proposed control scheme, which uses the MPC technique to compute the control sequences and a NDC strategy to compensate for network delays and dropouts, will be address as MPC–NDC scheme.

A functional scheme of the proposed controller is depicted in Figure 5.6.

Figure 5.6 Scheme of the mechanism used to compute the control sequence, based on prediction (Pred.) and reduced horizon optimization (RHOCP). In evidence the input sequences used to perform the prediction, $\mathbf{u}^c_{t,t+\overline{\tau}_a-1|t-1-\tau_c(t-1)}$ and $\mathbf{u}_{t-\tau_c(t),t-1}$, and the control sequence computed by the reduced horizon optimization, $\mathbf{u}^{\circ}_{t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)}$.



5.3 Formalization of the MPC–NDC Scheme for TCP-like Networks

The overall control scheme for NCS that has been presented in previous sections can be formally described by the Procedure 5.3.1 below, which gives the sequence of operations that have to be performed by the NCS components [2].

The sensor node, the controller and the smart actuator are in charge of processing informations and forming suitably structured data packets, by using some internal storage buffers and computational resources. In this regard, we will neglect the issue of quantization raised by the numerical implementation of the procedure.

[2]The low-level TCP–like communication protocol, in charge for packet routing and synchronization, is considered as a service provided by the network transparently to the components of the NCS

In the sequel, we will denote as $\mathbf{P}_{sc}$ and $\mathbf{P}_{ca}$ the data packets sent by the sensor to the controller and by the controller to the actuator respectively, while $\mathbf{P}_{ack}$ will represent the acknowledgment (which is, in turn, a data packet) transmitted by the actuator to the controller. For the sake of clarity, all the packets will be addressed to as data structures of the form $\mathbf{P} = \{\ \mathbf{P}.data,\quad \mathbf{P}.time\ \}$, containing a data field and a time stamp field.

Moreover, denoting as $\mathbf{M}_a$ the overall storage memory of the smart actuator, we assume that $\mathbf{M}_a$ is structured in buffers: *i)* $\mathbf{M}_a.\mathbf{u} \in \mathbb{R}^m \times N_c$, which is used to store a sequence of N_c future control actions and *ii)* $\mathbf{M}_a.T \in \mathbb{Z}_{\geq 0}$, which contains the time stamp of the information stored in $\mathbf{M}_a.\mathbf{u}$.

The storage memory of controller node, $\mathbf{M}_c$, in turn, is structured in buffers: *i)* $\mathbf{M}_c.\mathbf{U} \in (\mathbb{R}^m \times N_c) \times \overline{\tau}_a$, which is a First-In-First-Out (FIFO) buffer used to store the control sequences computed in the past $\overline{\tau}_a$ time instants (each element is a sequence); *ii)* $\mathbf{M}_c.\mathbf{u} \in \mathbb{R}^m \times \overline{\tau}_c$, which is used to store the inputs applied to the plant from time $t - \overline{\tau}_c$ to $t - 1$ (each element is a control move); *iii)* $\mathbf{M}_c.x \in \mathbb{R}^n$, which stores the last available state measurement; *iv)* $\mathbf{M}_c.T \in \mathbb{Z}_{\geq 0}$, which contains the time stamp relative to $\mathbf{M}_c.x$ and *v)* two counters $\mathbf{M}_c.i_{seq} \in \mathbb{Z}_{\geq 0}$ and $\mathbf{M}_c.i_u \in \mathbb{Z}_{\geq 0}$.

Let us denote as $\leftarrow$ a data assignment operation. Given a buffer (array) $\mathbf{B}$ containing N elements, let us denote as $\mathbf{B}(i)$ the i-th element of the array, with $i \in \{1, \ldots, N\}$. Given a buffer $\mathbf{B}$ containing M sequences of N elements each, let us denote as $\mathbf{B}(i, j)$ the j-th element of the i-th sequence, with $i \in \{1, \ldots, M\}$ and $j \in \{1, \ldots, N\}$. Then, the following procedure can be outlined.

Procedure 5.3.1 (MPC–NDC scheme for TCP–like networks)**.** *Assume that, starting from time instant* $t = 0$*, the initial condition* x_0 *is known.*

Initialization

1 *Given* x_0*, let* $\mathbf{M}_c.x \leftarrow x_0$*;*

2 $\mathbf{M}_a.\mathbf{u} = \mathbf{M}_c.\mathbf{u} = \mathbf{M}_c.\mathbf{U}(1) \leftarrow \overline{\mathbf{u}}_{0,N_c-1}$*, with* $\overline{\mathbf{u}}_{0,N_c-1}$ *feasible for* x_0*;*

3 $\mathbf{M}_a.T = \mathbf{M}_c.T \leftarrow 0$*;*

4 $\mathbf{M}_c.i_{seq} = \mathbf{M}_c.i_u \leftarrow 0$.

Sensor node

1 *for* $t \in \mathbb{Z}_{\geq 0}$

2 *form the packet* $\begin{cases} \mathbf{P}_{sc}.x \leftarrow x_t \\ \mathbf{P}_{sc}.T \leftarrow t \end{cases}$;

3 *send* $\mathbf{P}_{sc}$.

Controller node

1 *for* $t \in \mathbb{Z}_{\geq 0}$

2 *if a packet* $\mathbf{P}_{sc}$ *arrived*

3 *if* $\mathbf{P}_{sc}.T > \mathbf{M}_c.T$

4 $\mathbf{M}_c.x \leftarrow \mathbf{P}_{sc}.x; \quad (= x_{t-\tau_c(t)})$

5 $\mathbf{M}_c.T \leftarrow \mathbf{P}_{sc}.T; \quad (= t - \tau_c(t))$

6 *if the acknowledgment* $\mathbf{P}_{ack}$ *arrived*

7 $\mathbf{M}_c.i_{seq} \leftarrow t - \mathbf{P}_{ack}.T + 1$;

8 $\mathbf{M}_c.i_u \leftarrow t - \mathbf{M}_c.T_{ack} + 1$;

9 *else*

10 $\mathbf{M}_c.i_{seq} \leftarrow \mathbf{M}_c.i_{seq} + 1$;

11 $\mathbf{M}_c.i_u \leftarrow \mathbf{M}_c.i_u + 1$;

12 $\mathbf{M}_c.\mathbf{u} \leftarrow \mathrm{col}[\mathbf{M}_c.\mathbf{u}(2), \ldots, \mathbf{M}_c.\mathbf{u}(\overline{\tau}_c), \mathbf{M}_c.\mathbf{U}(\mathbf{M}_c.i_{seq}, \mathbf{M}_c.i_u)]$;

13 *considering that* $\mathbf{M}_c.x = x_{t-\tau_c(t)}$, *compute the prediction* $\hat{x}_{t|t-\tau_c(t)}$ *by using (4.9)*
and the input sequence $\mathbf{u}_{t-\tau_c(t),t} = \mathrm{col}[\mathbf{M}_c.\mathbf{u}(\overline{\tau}_c - \tau_c(t) + 1), \ldots, \mathbf{M}_c.\mathbf{u}(\overline{\tau}_c)]$
where $\tau_c(t) = t - \mathbf{M}_c.T$ *(see 5) ;*

14 *starting from* $\hat{x}_{t|t-\tau_c(t)}$, *compute the prediction* $\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}$ *by using (4.9)*
and the input sequence $\mathbf{u}^c_{t,t+\overline{\tau}_a-1|t-1-\tau_c(t-1)}$, *which can be retrieved*
from the $\mathbf{M}_c.\mathbf{U}(1)$ *(see line 17);*

15 *solve the RHOCP initiated with* $\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}$, *obtaining* $\mathbf{u}^\circ_{t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)}$;

16 *form* $\mathbf{u}^c_{t,t+N_c-1|t-\tau_c(t)} = \mathrm{col}[\mathbf{u}^c_{t,t+\overline{\tau}_a-1|t-1-\tau_c(t-1)}, \mathbf{u}^\circ_{t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)}]$;

17 *shift by one position the sequences in the register* $\mathbf{M}_c.\mathbf{U}$
and store $\mathbf{M}_c.\mathbf{U}(1) \leftarrow \mathbf{u}^c_{t,t+N_c-1|t-\tau_c(t)}$;

form the packet $\begin{cases} \mathbf{P}_{ca}.\mathbf{u} \leftarrow \mathbf{u}^c_{t,t+N_c-1|t-\tau_c(t)} \\ \mathbf{P}_{ca}.T \leftarrow t \end{cases}$;

send $\mathbf{P}_{ca}$.

Actuator node

for $t \in \mathbb{Z}_{\geq 0}$

  if a packet $\mathbf{P}_{ca}$ *arrived*

    if $\mathbf{P}_{ca}.T > \mathbf{M}_a.T$

      $\mathbf{M}_a.\mathbf{u} \leftarrow \mathbf{P}_{ca}.\mathbf{u}$; $\quad (= \mathbf{u}^c_{t-\tau_a(t),t-\tau_a(t)+N_c-1|t-\tau_{rt}(t)})$

      $\mathbf{M}_a.T \leftarrow \mathbf{P}_{ca}.T$; $\quad (= t - \tau_a(t))$

      form the packet $\mathbf{P}_{ack}.T \leftarrow \mathbf{M}_a.T$;

      send $\mathbf{P}_{ack}$;

  apply the control action $u_t = \mathbf{M}_a.\mathbf{u}(t - \mathbf{M}_a.T + 1)$. $\quad (= u^c_{t|t-\tau_{rt}(t)})$

□

In the next section, the robust stability properties of the described control scheme will be analyzed in presence of transmission delays and model uncertainty.

5.4 Recursive Feasibility and Regional Input-to-State Stability

The following Theorem states the recursive feasibility of the combined MPC–NDC scheme.

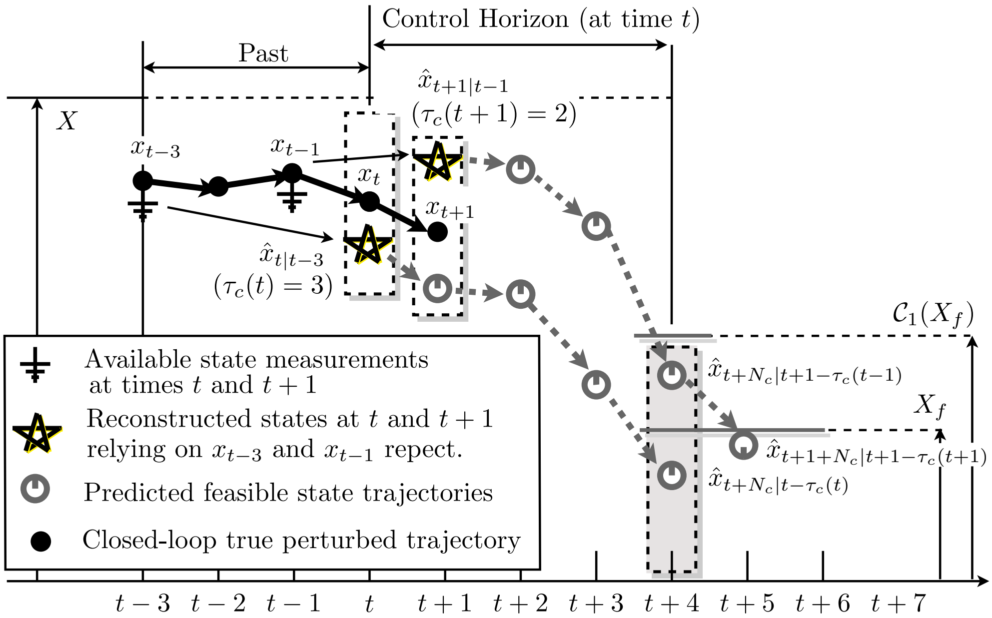
Theorem 5.4.1 (Invariance of the feasible set)**.** *Under Assumption 3.1.1, suppose that* $L_{f_x} > 1$[3]*. Assume that at time instant t the control sequence computed by the controller,* $\overline{\mathbf{u}}^c_{t,t+N_c-1|t-\tau_c(t)}$*, is feasible. Then, in view of Assumptions 3.1.1-5.2.4, if the norm bound on the uncertainty satisfies*

$$\overline{d} \leq \min_{k \in \{0,\overline{\tau}_c\}} \left\{ \min\left(\frac{L_{f_x}-1}{L_{f_x}^{N_c+k} - L_{f_x}^{N_c-1}} \mathrm{dist}(\mathbb{R}^n \backslash \mathcal{C}_1(X_f), X_f),\ \frac{L_{f_x}-1}{L_{f_x}^{N_c+k} - 1} \mathrm{dist}\big(\mathbb{R}^n \backslash X_{k+N_c}(\overline{d}), X_f\big) \right) \right\}, \tag{5.7}$$

[3]The very special case $L_{f_x} = 1$ can be trivially addressed by a few suitable modifications to the proof of Lemma 5.2.1. In general, if the true Lipschitz constant of the system is lower than 1, considering $L_{f_x} > 1$ only leads to more conservative results.

then, the recursive feasibility of the scheme in ensured for every time instant $t+i, \forall i \in \mathbb{Z}_{>0}$, while the closed-loop trajectories are confined into X. Hence, the feasible set X_{MPC} is RPI under the c-l networked dynamics w.r.t. bounded uncertainties. □

Figure 5.7 Sketch of the recursive feasibility result for the RHOCP with respect to the terminal state constraint X_f. Given a feasible solution of the RHOCP at time t, initiated with the reconstructed state $\hat{x}_{t|t-\tau_c(t)}$ (relying on a state measurement performed at time $t-\tau_c(t)$, with $\tau_c(t) = 3$ in the example), if the uncertainty verifies the condition (5.7), then at time $t+1$, given the reconstructed state $\hat{x}_{t+1|t+1-\tau_c(t+1)}$ (relying on the state measurement $x_{t+1-\tau_c(t+1)}$, with $\tau_c(t+1) = 2$ in the example), there exists a sequence which brings the nominal predicted trajectory $\hat{x}_{t+N_c|t+1-\tau_c(t+1)}$ in $\mathcal{C}_1(X_f)$. Hence there exists a feasible control move which can steer the state in X_f.



Proof The proof consists in showing that if, at time t, the input sequence computed by the controller $\overline{\mathbf{u}}^c_{t,t+N_c-1|t-\tau_c(t)}$ is feasible in the sense of Definition 5.2.3, then, if the perturbed system evolves under the action of the MPC–NDC scheme, there will exist a feasible control sequence at time instant $t+1$. Finally, the recursive feasibility follows by induction. First, notice that Points ii) and iii) of Assumption 5.2.4 together imply that $\text{dist}(\mathbb{R}^n \backslash \mathcal{C}_1(X_f), X_f) \geq \delta > 0$. Now, the proof will be carried out in three steps.

i) $\hat{x}_{t+N_c|t-\tau_c(t)} \in X_f \Rightarrow \hat{x}_{t+N_c+1|t+1-\tau_c(t+1)} \in X_f$: Let us consider the sequence $\mathbf{u}^*_{t-\tau_c(t),t+N_c-1|t-\tau_c(t)}$ defined in (5.6). It is straightforward to prove that the norm difference between the predictions $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)}$ and $\hat{x}_{t-\tau_c(t)+j|t+1-\tau_c(t+1)}$ (initiated by $x_{t-\tau_c(t)}$ and $x_{t+1-\tau_c(t+1)}$), respectively obtained by applying to the nominal model the sequence $\mathbf{u}^*_{t-\tau_c(t),t-\tau_c(t)+j-1|t-\tau_c(t)}$ and its subsequence $\mathbf{u}^*_{t+1-\tau_c(t+1),t-\tau_c(t)+j-1|t-\tau_c(t)}$, can be upper bounded by

$$|\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i} - \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)}| \leq \frac{1}{L_{f_x}} \sum_{l=1}^{i} L_{f_x}^{j-l+1} \overline{d} = \frac{L_{f_x}^{j} - L_{f_x}^{j-i}}{L_{f_x} - 1} \overline{d} \tag{5.8}$$

where we have posed $i=\tau_c(t)-\tau_c(t+1)+1$ and with $j \in \{i, \ldots, N_c+\tau_c(t)\}$. Considering now the case $j=N_c+\tau_c(t)$, then (5.8) yields to $|\hat{x}_{t+N_c|t-\tau_c(t)+i} - \hat{x}_{t+N_c|t-\tau_c(t)}| = |\hat{x}_{t+N_c|t+1-\tau_c(t+1)} - \hat{x}_{t+N_c|t-\tau_c(t)}| \leq (L_{f_x}^{N_c+\tau_c(t)} - L_{f_x}^{N_c+\tau_c(t)-i})/(L_{f_x}-1)\overline{d}$. If the following inequality holds $\forall k \in \{1, \ldots, \overline{\tau}_c\}$

$$\overline{d} \leq \frac{L_{f_x} - 1}{L_{f_x}^{N_c+k} - L_{f_x}^{N_c-1}} \operatorname{dist}(\mathbb{R}^n \backslash \mathcal{C}_1(X_f), X_f),$$

then $\hat{x}_{t+N_c|t+1-\tau_c(t+1)} \in \mathcal{C}_1(X_f)$, whatever be the values of $\tau_c(t)$ and $\tau_c(t+1)$. Hence, there exists a control move $\overline{u}_{t+N_c|t+1-\tau_c(t+1)} = \overline{u}_f(\hat{x}_{t+N_c|t+1-\tau_c(t+1)}) \in U$, with $\overline{u}_f : \mathcal{C}_1(X_f) \to U$ defined as

$$\overline{u}_f(x) \triangleq \arg \min_{u \in U: \hat{f}(x,u) \in X_f} \{|u - \kappa_f(x)|\}, \tag{5.9}$$

which can steer the state vector from $\hat{x}_{t+N_c|t+1-\tau_c(t+1)}$ to $\hat{x}_{t+N_c+1|t+1-\tau_c(t+1)} \in X_f$. A pictorial representation of the recursive feasibility result is given in Figure 5.7.

ii) $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X_j(\overline{d}) \Rightarrow \hat{x}_{t-\tau_c(t)+j|t+1-\tau_c(t+1)} \in X_{j-i}(\overline{d})$, with $i = \tau_c(t) - \tau_c(t+1) + 1$ and $\forall j \in \{\tau_c(t)+1, \ldots, N_c + \tau_c(t)\}$: Consider the predictions $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)}$ and $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i}$(initiated respectively by $x_{t-\tau_c(t)}$ and $x_{t-\tau_c(t)+i}$), respectively obtained with the sequence $\mathbf{u}^*_{t-\tau_c(t),t-\tau_c(t)+j-1|t-\tau_c(t)}$ and its subsequence $\mathbf{u}^*_{t-\tau_c(t)+i,t-\tau_c(t)+j-1|t-\tau_c(t)}$. Assuming that $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} \in X \backsim \mathcal{B}^n((L_{f_x}^{j} - 1)/(L_{f_x} - 1)\overline{d})$, let us introduce $\eta \in \mathcal{B}^n((L_{f_x}^{j-i} - 1)/(L_{f_x} - 1)\overline{d})$. Let $\xi \triangleq \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i} - \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} + \eta$, then, in view of Assumption 3.1.1 and thanks to (5.8), it follows that

$$|\xi| \leq |\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i} - \hat{x}_{t-\tau_{rt}(t)+j|t-\tau_c(t)}| + |\eta| \leq \frac{L_{f_x}^{j} - 1}{L_{f_x} - 1} \overline{d}, \tag{5.10}$$

and hence, $\xi \in \mathcal{B}^n((L_{f_x}^j - 1)/(L_{f_x} - 1)\overline{d})$. Since $\hat{x}_{t-\tau_c(t)+j|t} \in X_j(\overline{d})$, it follows that $\hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)} + \xi = \hat{x}_{t-\tau_c(t)+j|t-\tau_c(t)+i} + \eta \in X, \ \forall \eta \in \mathcal{B}^n((L_{f_x}^{j-i} - 1)/(L_{f_x} - 1)\overline{d})$, yielding to $\hat{x}_{t-\tau_c(t)+j|t+1-\tau_c(t+1)} \in X_{j-\tau_c(t)+\tau_c(t+1)-1}(\overline{d})$.

iii) $\hat{x}_{t+N_c|t-\tau_c(t)} \in X_f \Rightarrow \hat{x}_{t+N_c+1|t+1-\tau_c(t+1)} \in X_{N_c+\tau_c(t+1)}(\overline{d})$; Thanks to Point i), there exists a feasible control sequence at time $t+1$ which yields to $\hat{x}_{t+1+N_c|t+1-\tau_c(t+1)} \in X_f$. If $\overline{d}$ satisfies

$$\overline{d} \leq \min_{j \in \{N_c, \ldots, N_c+\overline{\tau}_c\}} \left\{ \frac{L_{f_x} - 1}{L_{f_x}^j - 1} \text{dist}(\mathbb{R}^n \backslash X_j(\overline{d}), X_f) \right\},$$

it follows that $\hat{x}_{t+1+N_c|t+1-\tau_c(t+1)} \in X_{N_c+\tau_c(t+1)}$, whatever be the value of $\tau_c(t+1)$.

Then, under the assumptions posed in the statement of Theorem 5.4.1, given $x_0 \in X_{MPC}$, and being $\tau_c(0) = 0$ (i.e. at the first time instant the actuator buffer is initiated with a feasible sequence) in view of Points i)–iii) it holds that at any time $t \in \mathbb{Z}_{>0}$ a feasible control sequence exists and can be chosen as $\overline{\mathbf{u}}^c_{t+1,t+N_c+1|t+1-\tau_c(t+1)} = \text{col}[\overline{\mathbf{u}}^c_{t+1,t+N_c-1|t-\tau_c(t)}, \overline{u}_{t+N_c|t+1-\tau_c(t+1)}]$. Therefore the recursive feasibility of the scheme is ensured. ■

Remark 5.4.1 (Recursive feasibility and invariance of X_{MPC}). *Given a delayed state measurement $x_{t-\tau_c(t)}$, if there exists a feasible sequence at time t, $\overline{\mathbf{u}}_{t,t+N_c-1}$, we have that $\hat{x}_{t|t-\tau_c(t)} = \hat{x}(t, \overline{x}_{t|t-\tau_c(t)}, \overline{\mathbf{u}}_{t-\tau_c(t),t-1})$, verifies the inclusion $\hat{x}_{t|t-\tau_c(t)} \in X_{MPC}(\tau_c(t))$, since $\overline{\mathbf{u}}_{t,t+N_c-1}$ satisfies all the constraints specified in (5.4) with $i = \tau_c(t)$. Thus, proving that the scheme is recursively feasible (that is, given a feasible sequence at time t, there exists a feasible sequence at time $t+1$), would prove that $\hat{x}_{t+1|t+1-\tau_c(t+1)}$, will belong to $X_{MPC}(\tau_c(t+1))$, whatever be the value of $\tau_c(t+1)$ in the set $\{0, \ldots, \overline{\tau}_c\}$. Without loss of generality, assume that $\tau_c(t+1) = 0$, then it holds that $x_{t+1} = \hat{x}_{t+1|t+1} \in X_{MPC}$.*

Now, assuming that the initial condition $\overline{x}_0$, at time $t = 0$, is known to the controller (i.e.,$\tau_c(0) = 0$) and that the sequence stored in the actuator buffer is feasible, by induction it follows that

$$x_t \in X_{MPC}, \ \forall t \in \mathbb{Z}_{\geq 0}. \tag{5.11}$$

We can conclude that X_{MPC} is RPI for the NCS driven by the MPC-NDC scheme. □

Considering that the transmission delay is bounded as well as the sequence of control action forwarded by the controller, then the control input applied to the plant can be viewed as the output of a time-varying control law $\kappa_{MPC-NDC}(t)$. Notably, the closed-loop perturbed system becomes time-varying, i.e.,

$$x_{t+1} = g(t, x_t, d_t), \quad x_0 = \overline{x}_0, t \in \mathbb{Z}_{\geq 0} \tag{5.12}$$

with $g(t, x_t, d_t) \triangleq \hat{f}(x_t, \kappa_{MPC-NDC}(t)) + d_t$.

Now, the following important stability result can be stated.

Theorem 5.4.2 (Regional Input-to-State Stability). *Under Assumptions 3.1.1-5.2.4, if the bound on uncertainties verifies (5.7), then the closed-loop system system (5.12), controlled by the proposed MPC–NDC strategy $\kappa_{MPC-NDC}(t, \mathbf{x}_{t-\overline{\tau}_{rt},t})$, is regional ISS in X_{MPC} with respect to additive perturbations $d_t \in \mathcal{B}^n(\overline{d})$.* □

Proof [Theorem 5.4.2] Recalling that we have posed the assumption that, at time $t = 0$, the FIB contains a feasible control sequence, then, in a worst case situation, the system will be driven in open-loop for $\overline{\tau}_a$ time instants (see Procedure 5.3.1). With regard to the ISS property, this observation implies that the bound on the trajectories after $\overline{\tau}_a$ should depend on $x_{\overline{\tau}_a}$ and the regional ISS inequality (2.15) has to be modified as follows

$$|x(t + \overline{\tau}_a, \overline{x}_{\overline{\tau}_a}, \boldsymbol{\upsilon})| \leq \max\left\{\beta(|\overline{x}_{\overline{\tau}_a}|, t), \gamma(\|\boldsymbol{\upsilon}_{[t+\overline{\tau}_a-1]}\|)\right\}, \forall t \in \mathbb{Z}_{\geq 0}, \forall \overline{x}_{\overline{\tau}_a} \in \Xi, \tag{5.13}$$

where $\overline{x}_{\overline{\tau}_a}$ is the state at time $\overline{\tau}_a$ after the system has been driven for $\overline{\tau}_a$ steps by the open-loop policy stored in the buffer at time $t = 0$. In view of previous consideration, the proof consists in showing that there exist a ISS-Lyapunov function $V(t + \overline{\tau}_a, x_{t+\overline{\tau}_a})$ for the closed-loop system. To this end, let us define the following positive-definite function $V^\circ : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$

$$V^\circ(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}) \triangleq J^\circ_{FH}\Big(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}, \mathbf{u}^\circ_{t+\overline{\tau}_a, t+N_c-1|t-\tau_c(t)}, N_c - \overline{\tau}_a\Big) \tag{5.14}$$

where $\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)} = \hat{x}(t+\tau_a, x_{t-\tau_c(t)}, \mathbf{u}_{t-\tau_c(t), t+\overline{\tau}_a-1})$ is a prediction obtained with the nominal model initiated with $x_{t-\tau_c(t)}$. Notice that V° corresponds to the optimal cost subsequent to the reduced horizon optimization. Now, consider the following candidate ISS-Lyapunov function

$V : \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$

$$
\begin{aligned}
V(t+\overline{\tau}_a, x_{t+\overline{\tau}_a}) &\triangleq J_{FH}\Big(x_{t+\overline{\tau}_a}, \mathbf{u}^{\circ}_{t+\overline{\tau}_a, t+N_c-1|t-\tau_c(t)}, N_c-\overline{\tau}_a\Big) \\
&= \sum_{l=t+\overline{\tau}_a}^{t+N_c-1} h\Big(\hat{x}_{l|t+\overline{\tau}_a}, u^{\circ}_{l|t-\tau_c(t)}\Big) + h_f(\hat{x}_{t+N_c|t+\overline{\tau}_a})
\end{aligned}
\tag{5.15}
$$

where $\hat{x}_{t+\overline{\tau}_a+j|t+\overline{\tau}_a}$, $j \in \{1, \dots, N_c - \overline{\tau}_a\}$ are obtained using the nominal model initialized with $\hat{x}_{t+\overline{\tau}_a|t+\overline{\tau}_a} = x_{t+\overline{\tau}_a}$ and the sequence $\mathbf{u}^{\circ}_{t+\overline{\tau}_a, t+N_c-1|t-\tau_c(t)}$ (which is optimal for $\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}$ and not for $x_{t+\overline{\tau}_a}$). Notice that, since $\mathbf{u}^{\circ}_{t+\overline{\tau}_a, t+N_c-1|t-\tau_c(t)}$ is not computed in correspondence of $x_{t+\tau_a}$, but exploiting a past state information $x_{t-\tau_c(t)}$, V becomes a time-varying function of the state. We will show in the following that $V(t+\overline{\tau}_a, x_{t+\overline{\tau}_a})$ verifies the ISS inequalities with time-invariant bounds.

Suppose, without loss of generality[4], that $L_{f_x} \neq 1$. Now, let us point out that, in view of (5.8), the inclusion $x_{t+\overline{\tau}_a} \in \Omega \triangleq X_f \backsim \mathcal{B}^n((L_{f_x}^{\overline{\tau}_c+\overline{\tau}_a} - 1)/(L_{f_x} - 1)\overline{d})$ implies $\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)} \in X_f$ whatever be the value of $\tau_c(t)$. Then, by Assumption 5.2.4, the control sequence $\tilde{\mathbf{u}}_{t+\overline{\tau}_a, t+N_c-1|t-\tau_c(t)} \triangleq \text{col}[\kappa_f(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}), \kappa_f(\hat{x}_{t+\overline{\tau}_a+1|t-\tau_c(t)}), \dots, \kappa_f(\hat{x}_{t+N_c-1|t-\tau_c(t)})]$ is feasible for the RHOCP, hence the set X_{MPC} is not empty.

Now, our objective consists in finding a suitable comparison function to upper bound the candidate ISS-Lyapunov function $V(t+\overline{\tau}_a, x_{t+\overline{\tau}_a})$. By adding and subtracting $V^{\circ}(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)})$ to the right-hand side of (5.14), we obtain

$$
\begin{aligned}
V(t+\overline{\tau}_a, x_{t+\overline{\tau}_a}) \;\leq\; & \sum_{l=t+\overline{\tau}_a}^{t+N_c-1} h\Big(\hat{x}_{l|t+\overline{\tau}_a}, u^{\circ}_{l|t-\tau_c(t)}\Big) - h\left(\hat{x}_{l|t-\tau_c(t)}, u^{\circ}_{l|t-\tau_c(t)}\right) \\
& + h_f(\hat{x}_{t+N_c|t+\overline{\tau}_a}) - h_f(\hat{x}_{t+N_c|t-\tau_c(t)}) \\
& + J^{\circ}_{FH}\Big(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}, \mathbf{u}^{\circ}_{t+\overline{\tau}_a, t+N_c-1|t-\tau_c(t)}, N_c-\overline{\tau}_a\Big),
\end{aligned}
\tag{5.16}
$$

In view of Assumptions 3.1.1,5.2.3 and thanks to (5.8), the following inequalities holds

$$
\begin{aligned}
& \sum_{l=t+\overline{\tau}_a}^{t+N_c-1} \Big| h\Big(\hat{x}_{l|t-\tau_c(t)}, u^{\circ}_{l|t-\tau_c(t)}\Big) - h\left(\hat{x}_{l|t+\overline{\tau}_a}, u^{\circ}_{l|t-\tau_c(t)}\right) \Big| \\
& \leq L_h \frac{L_{f_x}^{\tau_c(t)+\overline{\tau}_a} - 1}{L_{f_x} - 1} \sum_{j=0}^{N_c-\overline{\tau}_a-1} L_{f_x}^j ||\boldsymbol{d}_{[t+\overline{\tau}_a-1]}|| \\
& \leq L_h \frac{L_{f_x}^{\overline{\tau}_{rt}} - 1}{L_{f_x} - 1} \frac{L_{f_x}^{N_c-\overline{\tau}_a} - 1}{L_{f_x} - 1} ||\boldsymbol{d}_{[t+\overline{\tau}_a-1]}||,
\end{aligned}
\tag{5.17}
$$

[4]The case L_{f_x}=1 can be trivially addressed with a few suitable modification to the proof of theorem 5.4.2

moreover

$$|h_f(\hat{x}_{t+N_c|t+\overline{\tau}_a}) - h_f(\hat{x}_{t+N_c|t-\tau_c(t)})| \leq L_{h_f}\frac{L_{f_x}^{\overline{\tau}_{rt}}-1}{L_{f_x}-1}L_{f_x}^{N_c-\overline{\tau}_a-1}||\boldsymbol{d}_{[t+\overline{\tau}_a-1]}||, \tag{5.18}$$

and

$$\begin{aligned} &J_{FH}\Big(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)},\mathbf{u}^{\circ}_{t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)},N_c-\overline{\tau}_a\Big)\\ &\quad\leq J_{FH}\Big(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)},\tilde{\mathbf{u}}_{t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)},N_c-\overline{\tau}_a\Big)\\ &\quad=\sum_{l=t+\overline{\tau}_a}^{t+N_c-1}h\left(\tilde{x}_{l|t-\tau_c(t)},\tilde{u}_{l|t-\tau_c(t)}\right)+h_f(\tilde{x}_{t+N_c|t-\tau_c(t)}).\end{aligned} \tag{5.19}$$

where, given $\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)} \in X_f$, we have posed

$$\tilde{x}_{t+\overline{\tau}_a+j|t-\tau_c(t)} = \hat{f}(\tilde{x}_{t+\overline{\tau}_a+j-1|t-\tau_c(t)}, \kappa_f(\tilde{x}_{t+\overline{\tau}_a+j-1|t-\tau_c(t)})) \in X_f,\ \ \forall j \in \{1,\ldots,N_c-\overline{\tau}_a\}.$$

Considering that $\sum_{l=t+\overline{\tau}_a}^{t+N_c-1} h\left(\tilde{x}_{l|t-\tau_c(t)},\tilde{u}_{l|t-\tau_c(t)}\right)+h_f(\tilde{x}_{t+N_c|t-\tau_c(t)} \leq h_f(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)})$, then the following bound can be established

$$\begin{aligned} J_{FH}\Big(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)},\mathbf{u}^{\circ}_{t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)},N_c-\overline{\tau}_a\Big) &\leq h_f(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}) - h_f(x_{t+\overline{\tau}_a}) + h_f(x_{t+\overline{\tau}_a})\\ &\leq L_{h_f}\frac{L_{f_x}^{\overline{\tau}_{rt}}-1}{L_{f_x}-1}||\boldsymbol{d}_{[t+\overline{\tau}_a-1]}|| + h_f(x_{t+\overline{\tau}_a}).\end{aligned} \tag{5.20}$$

Finally, in view of (5.17) (5.18) and (5.20) the following inequalities hold

$$\begin{aligned} V(t+\overline{\tau}_a,x_{t+\overline{\tau}_a}) \leq&\ \frac{L_{f_x}^{\overline{\tau}_{rt}}-1}{L_{f_x}-1}\left[L_h\frac{L_{f_x}^{N_c-\overline{\tau}_a}-1}{L_{f_x}-1}+L_{h_f}L_{f_x}^{N_c-\overline{\tau}_a-1}+L_{h_f}\right]||\boldsymbol{d}_{[t+\overline{\tau}_a-1]}||+h_f(x_{t+\overline{\tau}_a})\\ \leq&\ \alpha_1(|x_{t+\overline{\tau}_a}|)+\sigma_1(||\boldsymbol{d}_{[t+\overline{\tau}_a-1]}||),\ \forall x_{t+\overline{\tau}_a}\in X_f, \forall \mathbf{d}\in\mathcal{M}_{\mathcal{B}^n(\overline{d})}\end{aligned} \tag{5.21}$$

where

$$\begin{aligned}\alpha_1(s)&\triangleq L_{h_f}|s|\\ \sigma_1(s)&\triangleq\frac{L_{f_x}^{\overline{\tau}_{rt}}-1}{L_{f_x}-1}\left[L_h\frac{L_{f_x}^{N_c-\overline{\tau}_a}-1}{L_{f_x}-1}+L_{h_f}L_{f_x}^{N_c-\overline{\tau}_a-1}+L_{h_f}\right]s\end{aligned}$$

The lower bound on $V(t+\overline{\tau}_a, x_{t+\overline{\tau}_a})$ can be easily obtained using Assumption 5.2.3

$$V(t+\overline{\tau}_a, x_{t+\overline{\tau}_a}) \geq \underline{h}(x_{t+\overline{\tau}_a}),\ \ \forall x_{t+\tau_a} \in X_{MPC} \tag{5.22}$$

Then, in view (5.21) of (5.22), the ISS inequalities (2.2) and (2.3) hold respectively with

$\Xi = X_{MPC}$ and $\Omega = X_f \backsim \mathcal{B}^n((L_{f_x}^{\overline{\tau}_{rt}} - 1)/(L_{f_x} - 1)\overline{d})$. Moreover, in view of Point i) in the proof of Theorem 5.4.1, given the (feasible) control sequence computed at time t, $\mathbf{u}^c_{t,t+N_c-1|t-\tau_c(t)} = \text{col}[\mathbf{u}^c_{t,t+\overline{\tau}_a-1|t-1-\tau_c(t-1)}, \text{u}^\circ_{t+\overline{\tau}_a,t+N_c-1}]$, the sequence $\overline{\mathbf{u}}^c_{t+1,t+N_c|t+1-\tau_c(t+1)} = \text{col}[\mathbf{u}^c_{t+1,t+N_c-1|t-\tau_c(t)}, \overline{u}_f(\hat{x}_{t+N_c|t+1-\tau_c(t+1)})]$, with $\overline{u}_f(\cdot)$ defined as in (5.9), is a feasible sequence (in general, suboptimal) at time $t+1$. The subsequence $\mathbf{u}^c_{t+\overline{\tau}_a+1,t+N_c|t-\tau_c(t)}$ along the reduced horizon gives rise to a cost which verifies the inequality

$$\begin{aligned}
J_{FH}&\Big(\hat{x}_{t+\overline{\tau}_a+1|t+1-\tau_c(t+1)}, \mathbf{u}^c_{t+\overline{\tau}_a+1,t+N_c|t-\tau_c(t)}, N_c - \overline{\tau}_a\Big) \\
&\leq J^\circ_{FH}\Big(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}, \mathbf{u}^\circ_{t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)}, N_c - \overline{\tau}_a\Big) - h\Big(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}, u^\circ_{t|t-\tau_c(t)}\Big) \\
&+ \sum_{l=t+\overline{\tau}_a+1}^{t+N_c-1} h\Big(\hat{x}_{l|t+1-\tau_c(t+1)}, u^\circ_{l|t-\tau_c(t)}\Big) - h\Big(\hat{x}_{l|t-\tau_c(t)}, u^\circ_{l|t-\tau_c(t)}\Big) \\
&+ h\big(\hat{x}_{t+N_c|t+1-\tau_c(t+1)}, \overline{u}_f(\hat{x}_{t+N_c|t+1-\tau_c(t+1)})\big) \\
&+ h_f\big(\hat{x}_{t+N_c+1|t+1-\tau_c(t+1)}\big) - h_f\big(\hat{x}_{t+N_c|t-\tau_c(t)}\big)
\end{aligned} \tag{5.23}$$

Now, by (5.16), (5.17) and (5.18) we have that

$$\begin{aligned}
V(t+\overline{\tau}_a+1, x_{t+\overline{\tau}_a+1}) \leq \; & J_{FH}\Big(\hat{x}_{t+\overline{\tau}_a+1|t+1-\tau_c(t+1)}, \mathbf{u}^\circ_{t+\overline{\tau}_a+1,t+N_c|t-\tau_c(t)}, N_c - \overline{\tau}_a\Big) \\
& + \frac{L_{f_x}^{\overline{\tau}_{rt}} - 1}{L_{f_x} - 1}\left[L_h \frac{L_{f_x}^{N_c-\overline{\tau}_a} - 1}{L_{f_x} - 1} + L_{h_f} L_{f_x}^{N_c-\overline{\tau}_a-1}\right] ||\boldsymbol{d}_{[t+\overline{\tau}_a]}||,
\end{aligned} \tag{5.24}$$

and that

$$\begin{aligned}
J_{FH}&\Big(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}, \mathbf{u}^\circ_{t+\overline{\tau}_a,t+N_c-1|t-\tau_c(t)}, N_c - \overline{\tau}_a\Big) \leq \\
& V(t+\overline{\tau}_a, x_{t+\overline{\tau}_a}) + \frac{L_{f_x}^{\overline{\tau}_{rt}} - 1}{L_{f_x} - 1}\left[L_h \frac{L_{f_x}^{N_c-\overline{\tau}_a} - 1}{L_{f_x} - 1} + L_{h_f} L_{f_x}^{N_c-\overline{\tau}_a-1}\right] ||\boldsymbol{d}_{[t+\overline{\tau}_a-1]}||.
\end{aligned} \tag{5.25}$$

Moreover, in view of Point v) of Assumption 5.2.4 and thanks to Assumption 5.2.5, it follows that

$$\begin{aligned}
h\big(\hat{x}_{t+N_c|t+\overline{\tau}_a+1}, \overline{u}_f(\hat{x}_{t+N_c|t+\overline{\tau}_a+1})\big) &+ h_f\big(\hat{x}_{t+N_c+1|t+\overline{\tau}_a+1}\big) - h_f\big(\hat{x}_{t+N_c|t+\overline{\tau}_a}\big) \\
&\leq h_f\big(\hat{x}_{t+N_c|t+\overline{\tau}_a+1}\big) - h_f\big(\hat{x}_{t+N_c|t+\overline{\tau}_a}\big) \\
&\leq L_{h_f} L_{f_x}^{N_c-\overline{\tau}_a-1} ||\boldsymbol{d}_{[t+\overline{\tau}_a]}||
\end{aligned} \tag{5.26}$$

Then, we have that the following inequality follows from (5.23) by using (5.24), (5.25), (5.26)

$$
\begin{aligned}
&V(t+\overline{\tau}_a+1, x_{t+\overline{\tau}_a+1}) - V(t+\overline{\tau}_a, x_{t+\overline{\tau}_a}) \\
&\qquad \leq -h\Big(\hat{x}_{t+\overline{\tau}_a|t-\tau_c(t)}, u^{\circ}_{t|t-\tau_c(t)}\Big) \\
&\qquad + \sum_{l=t+\overline{\tau}_a+1}^{t+N_c-1} h\Big(\hat{x}_{l|t+1-\tau_c(t+1)}, u^{\circ}_{l|t-\tau_c(t)}\Big) - h\Big(\hat{x}_{l|t-\tau_c(t)}, u^{\circ}_{l|t-\tau_c(t)}\Big) \\
&\qquad + \left[L_{h_f} L_{f_x}^{N_c-\overline{\tau}_a-1} + 2\frac{L_{f_x}^{\overline{\tau}_{rt}}-1}{L_{f_x}-1}\left(L_h \frac{L_{f_x}^{N_c-\overline{\tau}_a}-1}{L_{f_x}-1} + L_{h_f} L_{f_x}^{N_c-\overline{\tau}_a-1}\right)\right] ||\boldsymbol{d}_{[t+\overline{\tau}_a]}|| \\
&\qquad \leq -h\Big(x_{t+\overline{\tau}_a}, u^{\circ}_{t|t-\tau_c(t)}\Big) \\
&\qquad + L_h \frac{L_{f_x}^{\overline{\tau}_{rt}}-1}{L_{f_x}-1} ||\boldsymbol{d}_{[t]}|| + L_h \frac{L_{f_x}^{\overline{\tau}_c}-1}{L_{f_x}-1} \frac{L_{f_x}^{N_c-\overline{\tau}_a}-1}{L_{f_x}-1} ||\boldsymbol{d}_{[t]}|| \\
&\qquad + \left[L_{h_f} L_{f_x}^{N_c-\overline{\tau}_a-1} + 2\frac{L_{f_x}^{\overline{\tau}_{rt}}-1}{L_{f_x}-1}\left(L_h \frac{L_{f_x}^{N_c-\overline{\tau}_a}-1}{L_{f_x}-1} + L_{h_f} L_{f_x}^{N_c-\overline{\tau}_a-1}\right)\right] ||\boldsymbol{d}_{[t+\overline{\tau}_a]}||
\end{aligned}
$$

Finally, by using Point iv) of Assumption 5.2.4, the third ISS inequality can be obtained

$$
\begin{aligned}
&V(t+\overline{\tau}_a+1, x_{t+\overline{\tau}_a+1}) - V(t+\overline{\tau}_a, x_{t+\overline{\tau}_a}) \\
&\quad \leq -\underline{h}(|x_{t+\overline{\tau}_a}|) + \left[L_h \frac{L_{f_x}^{\overline{\tau}_{rt}}-1}{L_{f_x}-1} + L_h \frac{L_{f_x}^{\overline{\tau}_c}-1}{L_{f_x}-1} \frac{L_{f_x}^{N_c-\overline{\tau}_a}-1}{L_{f_x}-1} + L_{h_f} L_{f_x}^{N_c-\overline{\tau}_a-1}\right. \\
&\qquad\qquad \left. +2\frac{L_{f_x}^{\overline{\tau}_{rt}}-1}{L_{f_x}-1}\left(L_h \frac{L_{f_x}^{N_c-\overline{\tau}_a}-1}{L_{f_x}-1} + L_{h_f} L_{f_x}^{N_c-\overline{\tau}_a-1}\right)\right] ||\boldsymbol{d}_{[t+\overline{\tau}_a]}|| \\
&\quad \leq -\alpha_2(|x_{t+\overline{\tau}_a}|) + \sigma_2(||\boldsymbol{d}_{[t+\overline{\tau}_a]}||), \ \ \forall x_{t+\overline{\tau}_a} \in X_{MPC}, \forall \boldsymbol{d} \in \mathcal{M}_{\mathcal{B}^n(\overline{d})}
\end{aligned} \tag{5.27}
$$

where

$$
\begin{aligned}
&\alpha_2(s) \triangleq \underline{h}(s) \\
&\sigma_2(s) \triangleq \left[L_h \frac{L_{f_x}^{\overline{\tau}_{rt}}-1}{L_{f_x}-1} + L_h \frac{L_{f_x}^{\overline{\tau}_c}-1}{L_{f_x}-1} \frac{L_{f_x}^{N_c-\overline{\tau}_a}-1}{L_{f_x}-1} + L_{h_f} L_{f_x}^{N_c-\overline{\tau}_a-1}\right. \\
&\qquad\qquad \left. +2\frac{L_{f_x}^{\overline{\tau}_{rt}}-1}{L_{f_x}-1}\left(L_h \frac{L_{f_x}^{N_c-\overline{\tau}_a}-1}{L_{f_x}-1} + L_{h_f} L_{f_x}^{N_c-\overline{\tau}_a-1}\right)\right].
\end{aligned}
$$

Finally, in view of (5.21), (5.22) and (5.27), it is possible to conclude that the closed-loop system is regional ISS in X_{MPC} with respect to $d \in \mathcal{B}^n(\overline{d})$. ■

5.5 Formalization of the NDC–MPC Scheme for UDP–like Networks

In the case of UDP–like networks, no ACKs are sent by the actuator node to the controller. In this scenario, the problem of delayed arrival of packeted input sequences to the actuator (which may lead to wrong open-loop predictions at the controller side, due to the fact that the truly applied input sequence is not known to the controller if a specific strategy is not adopted to compensate for the lack of ACK's), could represent a major source of uncertainty. Thus, with the aim to recast the formulation in a deterministic framework, such that the sequence used by the predictor to obtain $\hat{x}_t$ would coincide with the true input sequence applied by the actuator to the plant from time $t - \tau_c(t)$ to $t - 1$, a possible solution consists enlarging the buffer length N_c, as specified in the following assumption.

Assumption 5.5.1 (Buffer length in UDP–like networks)**.** *The buffer length of the smart actuator, which is equal to the length of the input sequence sent by the controller to actuator, verifies* $N_c \geq \overline{\tau}_c + \overline{\tau}_a + 1$.

In this set up, being N_c also the length of the sequence computed by the controller to be forwarded to the actuator, we have that $N_c \geq \overline{\tau}_{rt} + 1$. The optimization, in the UDP–like case, has to be performed over an input sequence, namely $\mathbf{u}_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)}$, consisting of $N_c - \overline{\tau}_{rt}$ control actions. Moreover the RHOCP has to be initiated with the predicted state $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}$. The input sequence used to obtain $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}$ is

$$\mathbf{u}^*_{t-\tau_c(t),t+\overline{\tau}_{rt}-1} = \mathrm{col}[\mathbf{u}^*_{t-\tau_c(t),t-2}, \mathbf{u}^c_{t-1,t+\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}] \tag{5.28}$$

where $\mathbf{u}^*_{t-\tau_c(t),t-2}$ and $\mathbf{u}^c_{t,t+\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}$ are respectively a subsequence of $\mathbf{u}^*_{t-1-\tau_c(t-1),t+N_c-2}$ (which can be retrieved recursively from a buffer in which the sequences $\mathbf{u}^*$ are stored at each computational step) and a subsequence of the control sequence $\mathbf{u}^c_{t-1,t+N_c-2|t-1-\tau_c(t-1)}$ computed at time $t-1$. At this point, noting that the first $\overline{\tau}_{rt}$ elements of $\mathbf{u}^c_{t-1,t+N_c-2|t-1-\tau_c(t-1)}$ coincide with the subsequence $\mathbf{u}^*_{t-1,t+\overline{\tau}_{rt}-2}$, then (5.28) can be rearranged as

$$\mathbf{u}^*_{t-\tau_c(t),t+\overline{\tau}_{rt}-1} = \mathrm{col}[\mathbf{u}^*_{t-\tau_c(t),t+\overline{\tau}_{rt}-2}, u^c_{t-\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}] \tag{5.29}$$

where the control action $u^c_{t-\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}$ is the element of the optimal subsequence $\mathbf{u}^\circ_{t-\overline{\tau}_{rt}-1,t+N_c-\overline{\tau}_{rt}-2|t-1-\tau_c(t-1)}$, obtained by solving the RHOCP at time $t-1$ (i.e., $u^c_{t-\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)} = u^\circ_{t-\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}$). By this position, with suitable few modifications to the proof Theorem 5.4.1, it is possible to show that the proposed scheme is robustly recursively feasible in the UDP–like framework and that the closed-loop system is regionally ISS stable.

Remarkably, given a buffer sequence length N_c, the further shortening of the optimization horizon may reduce the degree of optimality of the control action with regard to the TCP–like formulation.

The MPC–NDC control scheme for UDP–like networks[5] can be formally described by the Procedure 5.5.1 listed below. In this case, the buffers $\mathbf{M}_c.\mathbf{U}$, $\mathbf{M}_c.i_{seq}$ and $\mathbf{M}_c.i_u$, introduced in Procedure 5.3.1, are not used, while the buffer $\mathbf{M}_c.\mathbf{u}$ must be enlarged to store $\overline{\tau}_{rt} + \overline{\tau}_c$ control actions.

Procedure 5.5.1 (MPC–NDC scheme for UDP–like networks)**.** *Assume that, starting from time instant $t = 0$, the initial condition x_0 is known.*

Initialization

Given x_0, let $\mathbf{M}_c.x \leftarrow x_0$;

$\mathbf{M}_a.\mathbf{u} = \mathbf{M}_c.\mathbf{u} \leftarrow \overline{\mathbf{u}}_{0,N_c-1}$, *with $\overline{\mathbf{u}}_{0,N_c-1}$ feasible for x_0;*

$\mathbf{M}_a.T = \mathbf{M}_c.T \leftarrow 0.$

Sensor node

for $t \in \mathbb{Z}_{\geq 0}$

form the packet $\begin{cases} \mathbf{P}_{sc}.x \leftarrow x_t \\ \mathbf{P}_{sc}.T \leftarrow t \end{cases}$;

send $\mathbf{P}_{sc}$.

Controller node

for $t \in \mathbb{Z}_{\geq 0}$

if a packet $\mathbf{P}_{sc}$ arrived

if $\mathbf{P}_{sc}.T > \mathbf{M}_c.T$

$\mathbf{M}_c.x \leftarrow \mathbf{P}_{sc}.x; \quad (= x_{t-\tau_c(t)})$

$\mathbf{M}_c.T \leftarrow \mathbf{P}_{sc}.T; \quad (= t - \tau_c(t)\,)$

considering that $\mathbf{M}_c.x = x_{t-\tau_c(t)}$, compute the prediction $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}$ by using (4.9) and the input sequence $\mathbf{u}^*_{t-\tau_c(t),t+\overline{\tau}_{rt}-1}$, which can be retrieved from $\mathbf{M}_c.\mathbf{u}$

[5] The low-level UDP–like communication protocol, in charge for packet routing and synchronization, is considered as a service provided by the network transparently to the components of the NCS

(see (5.29) and Line 9);

7 *solve the RHOCP initiated with* $\hat{x}_{t+\overline{\tau}_{rt}|t-\tau_c(t)}$, *obtaining* $\mathbf{u}^{\circ}_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)}$;

8 *form* $\mathbf{u}^c_{t,t+N_c-1|t-\tau_c(t)} = \mathrm{col}[\mathbf{u}^c_{t,t+\overline{\tau}_{rt}-1|t-1-\tau_c(t-1)}, \mathbf{u}^{\circ}_{t+\overline{\tau}_{rt},t+N_c-1|t-\tau_c(t)}]$;

9 *store* $\mathbf{M}_c.\mathbf{u} \leftarrow \mathrm{col}[\mathbf{M}_c.\mathbf{u}(2), \ldots, \mathbf{M}_c.\mathbf{u}(\overline{\tau}_{rt}),\, u^c_{t+\overline{\tau}_{rt}|t-\tau_c(t)}]$;

10 *form the packet* $\begin{cases} \mathbf{P}_{ca}.\mathbf{u} \leftarrow \mathbf{u}^c_{t,t+N_c-1|t-\tau_c(t)} \\ \mathbf{P}_{ca}.T \leftarrow t \end{cases}$;

11 *send* $\mathbf{P}_{ca}$.

Actuator node

for $t \in \mathbb{Z}_{\geq 0}$

if a packet $\mathbf{P}_{ca}$ *arrived*

if $\mathbf{P}_{ca}.T > \mathbf{M}_a.T$

$\mathbf{M}_a.\mathbf{u} \leftarrow \mathbf{P}_{ca}.\mathbf{u}$; $\quad (= \mathbf{u}^c_{t-\tau_a(t),t-\tau_a(t)+N_c-1|t-\tau_{rt}(t)})$

$\mathbf{M}_a.T \leftarrow \mathbf{P}_{ca}.T$; $\quad (= t - \tau_a(t))$

apply the control action $u_t = \mathbf{M}_a.\mathbf{u}(t - \mathbf{M}_a.T + 1)$. $\quad (= u^c_{t|t-\tau_{rt}(t)})$

□

5.6 Example

In order to show the effectiveness of the devised control scheme, the closed-loop behavior of the following nonlinear system (forward-Euler discretized version of an undamped single-link flexible-joint pendulum) is simulated first in nominal conditions and then under the simultaneous presence of model uncertainty and unreliable communications between sensors, controller, and actuators

$$\begin{cases} x_{(1)_{t+1}} = x_{(1)_t} + T_s x_{(2)_t} \\ x_{(2)_{t+1}} = x_{(2)_t} - \dfrac{T_s}{I}\left[MgL\sin(x_{(1)_t}) + k\big(x_{(1)_t} - x_{(3)_t}\big)\right] \\ x_{(3)_{t+1}} = x_{(3)_t} + T_s x_{(4)_t} \\ x_{(4)_{t+1}} = x_{(4)_t} + \dfrac{T_s}{J}\left[k\left(x_{(1)_t} - x_{(3)_t}\right) + u\right] \end{cases}$$
$$x_0 = \overline{x},\ t \in \mathbb{Z}_{\geq 0} \tag{5.30}$$

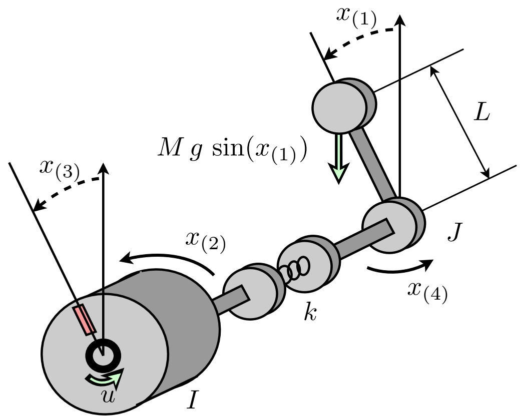
where $x_{(i)_t}, i \in \{1, \ldots, 4\}$ denotes the i-th component of the vector x_t, $T_s = 0.05\,s$ is the sampling interval, $I = 0.25\,kg \cdot m^2$ the inertia of the arm, $J = 2\,kg \cdot m^2$ the rotor inertia, $g = 9.8\,ms^2$ the gravitational acceleration, $M = 1\,kg$ the mass of the link, $L = 0.5\,m$ the distance between the rotational axis and the center of gravity of the pendulum-arm, $k = 20\,N \cdot m/rad$ the stiffness coefficient of the link. The control objective consists in stabilizing the system toward the (open-loop unstable) 0-state equilibrium, while keeping in the trajectories within prescribed bounds depicted in Figure 5.10 (green).

The following auxiliary linear controller is used $\kappa_f(x) = [-55.92\ \ -7.46\ 124.01\ 19.22] \cdot x$, with $X_f = \{x \in \mathbb{R}^4 : x^T \cdot P_f \cdot x \leq 1\}$, $h_f(x) = 10^3(x^T \cdot P_f \cdot x)$ and

$$P_f = 10^3 \begin{bmatrix} 1.3789 & -0.0629 & -1.7904 & -0.1508 \\ -0.0629 & 0.0186 & 0.1404 & 0.0074 \\ -1.7904 & 0.1404 & 3.1580 & 0.2216 \\ -0.1508 & 0.0074 & 0.2216 & 0.0292 \end{bmatrix}$$

The predictive controller has been set up with control sequence length N_c=12, and quadratic stage cost $h(x)$=$x^T \cdot Q \cdot x + Ru^2$, where Q=diag$(10, 0.1, 0.1, 0.1)$ and R=10^{-3}.

Figure 5.8 Scheme of the single-link flexible-joint pendulum used in the example.

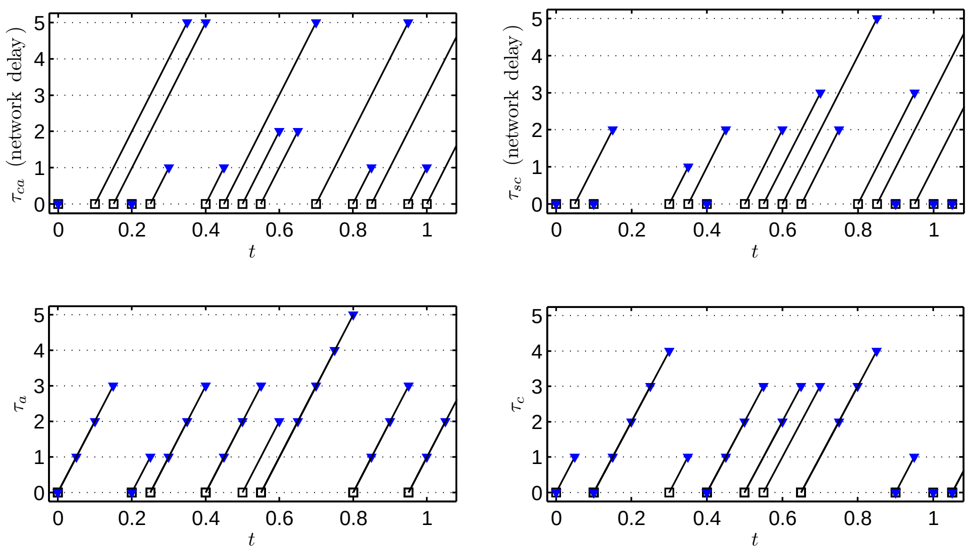


In the uncertain/unreliable networked scenario, a TCP–like protocol has been simulated, with delay bounds $\overline{\tau}_c$=$\overline{\tau}_a$=5, while the nominal model is subjected to the following parametric uncertainty M_{nom}=1.05M.

The timing diagrams of the simulated networked packet-based communication links are given in Figure 5.9. Notice that, due to the use of a TS strategy, the networks delays τ_{ca} and τ_{sc} have been decoupled from the age of information used in the nodes τ_a and τ_c, retaining only the packets which carry the most recent information.

Finally, Figure 5.10 shows the trajectories of the state variables in the nominal case (black) and in the uncertain/delayed conditions (blue). Notably, the constraints are fulfilled and the recursive feasibility of the scheme is guaranteed even in the networked case. At the opposite, if a network delay compensation strategy is not used, then system (5.30), controlled by a nominal MPC, becomes unstable even for small delays $\overline{\tau}_c = \overline{\tau}_a = 2$, as shown in Figure 5.11.

Figure 5.9 Timing diagrams of feedback and control communication links. Each slanted segment in τ_{ca} and τ_{sc} diagrams represents a successfully delivered data packet form the sending time (square) to the arrival time (triangle). The length of each segment represent the age of the packet at the receipt instant. In τ_c and τ_a diagrams the triangles represent the age of the information retained in each node thanks to the TS strategy.



5.7 Concluding Remarks

In this chapter a networked control system, based on the combined use of a constraint tightening based model predictive controller with a network delay compensation strategy, has been designed with the aim to stabilize toward an equilibrium a constrained nonlinear discrete-time system, affected by unknown bounded perturbations and subjected to delayed packet-based communications in both sensor-to-controller and controller-to-actuator links.

The characterization of the robust stability properties of the devised scheme represents a

Figure 5.10 Trajectories of the state variables of system (5.30) controlled by the combined strategy MPC+NDC over an unreliable network with uncertainty (blue : $\overline{\tau}_c = \overline{\tau}_a = 5$) and in nominal conditions (black : $\overline{\tau}_c = \overline{\tau}_a = 0$).

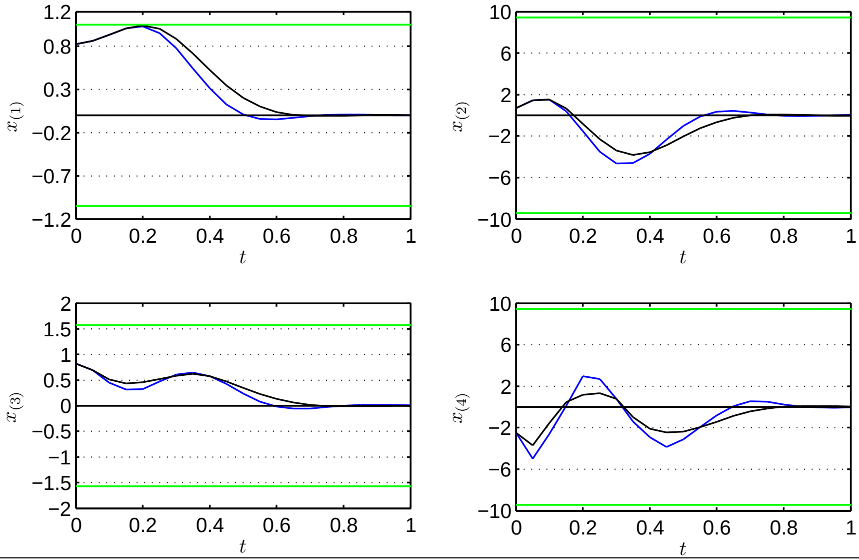
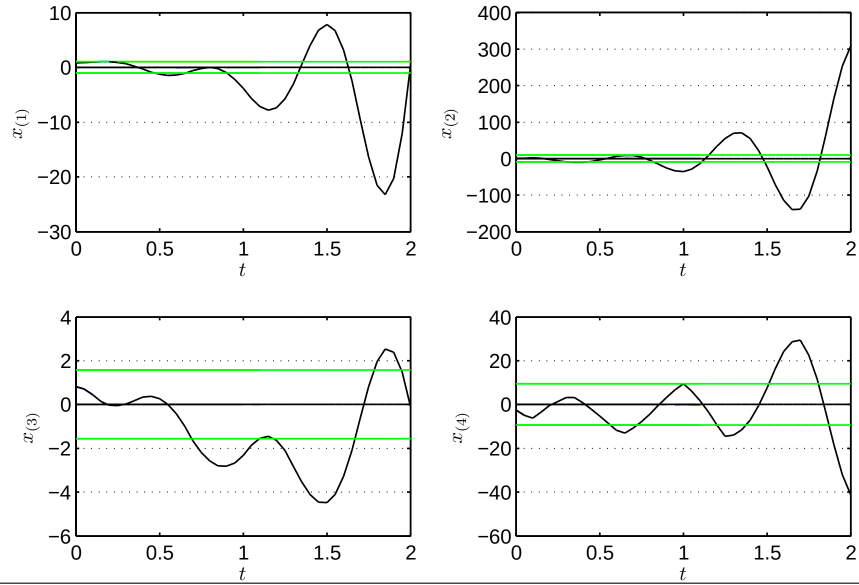


Figure 5.11 Trajectories of the state variables for system (5.30) controlled by a nominal NMPC, without delay compensation ($\overline{\tau}_c$=$\overline{\tau}_a$=2). Feasibility gets lost and instability occurs.



significant contribution in the context of nonlinear networked control systems, since it establishes the possibility to enforce the robust satisfaction of constraints under unreliable networked communications in the feedback and command channels, even in presence of model uncertainty.

Moreover, the problem of guaranteeing the recursive feasibility of the scheme has been addressed.

By exploiting a novel characterization of the regional Input-to-State Stability in terms of time-varying Lyapunov functions, the networked closed-loop system has been proven to be Input-to-State Stable with respect to bounded perturbations.

In the belief of the author, the strategy presented in this chapter can be further used to deploy constrained advanced-step MPC schemes (in which the control actions computed by the controller are applied to the plant after a fixed time interval) with ISS guarantees. Indeed, the advanced-step MPC, in which a constant delay affect only the controller-to-actuator path, turns out to be a particular case of the delayed networked configuration considered in the present work.

Appendix A

A.1 Main Notations and Basic Definitions

Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, and $\mathbb{Z}_{\geq 0}$ denote the real, the non-negative real, the integer, and the non-negative integer sets of numbers, respectively. The Euclidean norm is denoted as $|\cdot|$.

For any discrete-time sequence $\boldsymbol{\upsilon} : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}^m$, $\|\boldsymbol{\upsilon}\| \triangleq \sup_{k\geq 0}\{|\upsilon_k|\}$ and $\|\boldsymbol{\upsilon}_{[\tau]}\| \triangleq \sup_{0\leq k\leq \tau}\{|\upsilon_k|\}$, where ϕ_k denotes the value that the sequence $\boldsymbol{\upsilon}$ takes on in correspondence with the index k. The set of discrete-time sequences of $\boldsymbol{\upsilon}$ taking values in some subset $\Upsilon \subset \mathbb{R}^m$ is denoted by $\mathcal{M}_\Upsilon$. Given a sequence $\boldsymbol{\upsilon} \in \mathcal{M}_\Upsilon$ and two non-negative integers $k \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{Z}_{\geq 0}$, we will denote as $\boldsymbol{\upsilon}_{k,t}$ the subsequence formed by elements indexed from k to t (i.e., $\boldsymbol{\upsilon}_{k,t} \triangleq \{\upsilon_k, \upsilon_{k+1}, \ldots, \upsilon_{t-1}, \upsilon_t\}$).

Given a compact set $A \subseteq \mathbb{R}^n$, let ∂A denote the boundary of A. Given a vector $x \in \mathbb{R}^n$, $\mathrm{d}(x, A) \triangleq \inf\{|\xi - x|, \xi \in A\}$ is the point-to-set distance from $x \in \mathbb{R}^n$ to A. Given two sets $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$, $\mathrm{dist}(A, B) \triangleq \inf\{\mathrm{d}(\zeta, A), \zeta \in B\}$ is the minimal set-to-set distance. The difference between two given sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$, with $B \subseteq A$, is denoted as $A \backslash B \triangleq \{x : x \in A, x \notin B\}$. Given two sets $A \in \mathbb{R}^n$, $B \in \mathbb{R}^n$, the Pontryagin difference set C is defined as $C = A \backsim B \triangleq \{x \in \mathbb{R}^n : x + \xi \in A, \forall \xi \in B\}$. Given a vector $\eta \in \mathbb{R}^n$ and a scalar $\rho \in \mathbb{R}_{>0}$, the closed ball in $\mathbb{R}^n$ centered in η of radius ρ is denoted as $\mathcal{B}^n(\eta, \rho) \triangleq \{\xi \in \mathbb{R}^n : |\xi - \eta| \leq \rho\}$. The shorthand $\mathcal{B}^n(\rho)$ is used when $\eta = 0$.

A.2 Comparison Functions

The notions of functions of class $\mathcal{K}$, class $\mathcal{K}_\infty$, and class $\mathcal{KL}$ are used to characterize stability properties of the control schemes presented in the dissertation.

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing.

It belongs to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and is unbounded.

A function $\beta:\mathbb{R}_{\geq 0}\times\mathbb{Z}_{\geq 0}\to\mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if it is nondecreasing in its first argument, nonincreasing in its second argument, and $\lim_{s\to 0}\beta(s,t)=\lim_{t\to\infty}\beta(s,t)=0$.

In the sequel, a collection of some well-known properties of comparison functions is presented.

Property A.2.1 (Comparison functions)**.** *Let $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ be both $\mathcal{K}$-functions;$\alpha_3(\cdot)$ and $\alpha_4(\cdot)$ be $\mathcal{K}$-functions and let $\beta(\cdot,\cdot)$ be a $\mathcal{KL}$-function. Finally, let $s, s_1, s_2 \in \mathbb{R}_{\geq 0}$ be positive scalars. Then*

1) $\alpha_1^{-1}(\cdot)$ *is a $\mathcal{K}$-function;*

2) $\alpha_1 \circ \alpha_2(\cdot)$ *is a $\mathcal{K}$-function;*

3) $\alpha_1 \circ \beta(\cdot,\cdot)$ *is a $\mathcal{KL}$-function;*

4) $\max(\alpha_1(s), \alpha_2(s))$ *and* $\min(\alpha_1(s), \alpha_2(s))$ *are both $\mathcal{K}$-functions;*

5) $\alpha_1(s_1 + s_2) \leq \alpha_1(2s_1)) + \alpha_1(2s_2)$, $\forall s_1, s_2 \in \mathbb{R}_{\geq 0}$*;*

6) $\alpha_1(s_1) + \alpha_2(s_2) \leq \alpha_1(s_1 + s_2) + \alpha_2(s_1 + s_2)$, $\forall s_1, s_2 \in \mathbb{R}_{\geq 0}$*;*

7) $\alpha_1(s_1) + \alpha_2(s_2) \leq \min(\alpha_1(0.5(s_1 + s_2)) + \alpha_2(0.5(s_1 + s_2)))$, $\forall s_1, s_2 \in \mathbb{R}_{\geq 0}$*;*

8) there exists a $\mathcal{K}_\infty$-function $\alpha_5(\cdot)$ such that $\alpha_5(s) \leq \alpha_3(s)$, $\forall s \in \mathbb{R}_{\geq 0}$ and $s - \alpha_5(s)$ is a κ_∞-function.

A.3 Brief Introduction to Set-Invariance Theory

Let us consider the discrete-time dynamic system

$$x_{t+1} = g(x_t, \upsilon_t),\ t \in \mathbb{Z}_{\geq 0}, x_0 = \overline{x} \tag{A.1}$$

with $g(0,0) = 0$ and where $x_t \in \mathbb{R}^n$ and $\upsilon_t \in \Upsilon \subset \mathbb{R}^r$ denote the state of the system and a bounded disturbance (exogenous input), respectively. The discrete-time state trajectory of the system (A.1), with initial state $\bar{x}$ and disturbance sequence $\boldsymbol{\upsilon} \in \mathcal{M}_{\Upsilon}$, is denoted by $x(t, \bar{x}, \boldsymbol{\upsilon}),\ t \in \mathbb{Z}_{\geq 0}$.

Definition A.3.1 (RPI set). *A set $\Xi \subset \mathbb{R}^n$ is a Robust Positively Invariant (RPI) set for system (A.1) if $g(x_t, \upsilon_t) \in \Xi,\ \forall x_t \in \Xi$ and $\forall \upsilon_t \in \Upsilon$.* □

Consider the nonlinear discrete-time dynamic system

$$x_{t+1} = f(x_t, u_t, \upsilon_t),\ \ t \in \mathbb{Z}_{\geq 0},\ x_0 = \overline{x}_0\,, \tag{A.2}$$

where $x_t \in \mathbb{R}^n$ denotes the state vector, $u_t \in \mathbb{R}^m$ the control vector and $\upsilon_t \in \Upsilon$ is an uncertain exogenous input vector, with $\Upsilon \subset \mathbb{R}^r$ compact and $\{0\} \subset \Upsilon$. Assume that state and control variables are subject to the following constraints

$$x_t \in X,\ \ t \in \mathbb{Z}_{\geq 0}\,, \tag{A.3}$$

$$u_t \in U,\ \ t \in \mathbb{Z}_{\geq 0}\,, \tag{A.4}$$

where X and U are compact subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, containing the origin as an interior point.

Given the system (A.2), let $\hat{f}(x_t, u_t)$, with $\hat{f}(0,0) = 0$, denote the *nominal* model used for control design purposes.

Let $\hat{x}_{t+i|t}$, $i \in \mathbb{Z}_{>0}$ denote the state prediction generated by the nominal model on the basis of the state informations at time t under the control sequence $\mathbf{u}_{t,t+i-1} = \text{col}[u_t, \ldots, u_{t+i-1}]$, that is

$$\hat{x}_{t+i|t} = \hat{f}(\hat{x}_{t+i-1|t}, u_{t+i-1}),\ \ \hat{x}_{t|t} = x_t,\ \ t \in \mathbb{Z}_{\geq 0}, i \in \mathbb{Z}_{>0}. \tag{A.5}$$

Moreover, when it will be necessary to point out the dependence of a nominal trajectory on the initial condition x_t with a specific input sequence $\mathbf{u}_{t,t+i-1}$, we will also use the notation $\hat{x}(i, x_t, \mathbf{u}_{t,t+i-1}) = \hat{x}_{t+i|t}$.

Having introduced the nominal transition map $\hat{f}(x,u)$, the following definition can now be posed.

Definition A.3.2 ($\mathcal{C}_i(X,\Xi)$)**.** *Given a set $\Xi \subseteq X$, the i-step Controllability Set to Ξ, $\mathcal{C}_i(X,\Xi)$, is the set of states which can be steered to Ξ by a control sequence of length i, $\mathbf{u}_{0,i-1}$, under the nominal map $\hat{f}(x,u)$, subject to constraints (A.3) and (A.4), i.e.*

$$\mathcal{C}_i(X,\Xi) \triangleq \left\{ \begin{array}{ll} x_0 \in X: & \exists \mathbf{u}_{0,i-1} \in U \times \ldots \times U \quad \textit{such that} \\ & \hat{x}(t, x_0, \mathbf{u}_{0,i-1}) \in X,\ \forall t \in \{1,\ldots,i-1\}, \\ & \text{and } \ \hat{x}(i, x_0, \mathbf{u}_{0,i-1}) \in \Xi. \end{array} \right\}$$

□

The shorthand $\mathcal{C}_1(\Xi)$ will be used in place of $\mathcal{C}_1(\mathbb{R}^n,\Xi)$ to denote the one-step controllability set to Ξ, [17].

In order to give a complete our overview on basic concepts of set-invariance theory, also some notions of set-valued analysis (see [10]) must be introduced. In this regard, let us introduce the following definition.

Definition A.3.3 ($\hat{F}(x)$)**.** *Given the nominal transition function $\hat{f}(x,u)$, the set-valued map $\hat{F}: X \mapsto Y, Y \subseteq \mathbb{R}^n$ is defined as*

$$\hat{F}(x) \triangleq \bigcup_{u \in U} \hat{f}(x,u). \tag{A.6}$$

□

Definition A.3.4 (LSC)**.** *A set-valued map $\hat{F}: X \mapsto Y$ is called Lower Semi-Continuous (LSC) in X if $\forall x^{'} \in X$, given $\epsilon \in \mathbb{R}_{>0}$, $\exists \delta \in \mathbb{R}_{>0}$ such that inequality $|x^{'} - x^{''}| < \delta$ implies*

$$\hat{F}(x^{'}) \subseteq \hat{F}(x^{''}) \oplus \mathcal{B}(\epsilon).$$

□

Noting that the predecessor operator generates the natural *weak set-valued preimage* [28] of a given set $\Xi \subseteq Y$ under $\hat{F}$, $\mathcal{C}_1(\Xi) = \hat{F}^{-1}(\Xi) \triangleq \{x \in X : \hat{F}(x) \cap \Xi \neq \emptyset\}$, a computability result for $\hat{F}^{-1}(\Xi)$ would imply the computability of $\mathcal{C}_1(\Xi)$.

To this end, let us introduce the notion of robust robust controllability set, which plays a key role in the computability theory for set-valued operators, since it represents the best computable approximation of the true predecessor set [29].

Definition A.3.5 ($\mathcal{RC}_i(X, \Xi)$)**.** *Given a set* $\Xi \subseteq X$ *and the nominal map* $\hat{f}(x, u)$*, the* i*-step Robust Controllability set to* Ξ *is defined as* $\mathcal{RC}_i(X, \Xi) \triangleq \mathcal{C}_i(X, \text{int}(\Xi))$. □

The following important theorem can be stated.

Theorem A.3.1 (Geometric Condition for Invariance,[17])**.** *A set* $\Xi \in \mathbb{R}^n$ *is a control/positively invariant set if and only if* $\Xi \subseteq \mathcal{C}_1(\Xi)$. □

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