Rend. Istit. Mat. Univ. Trieste
Vol. XXX, 81–90 (1998)

On a Family of Topological Invariants Similar to Homotopy Groups

A. P. Caetano and R. F. Picken (*)

Summary. - *The intimacy relation between smooth loops, which is a strong homotopy relation, is generalized to smooth maps defined on the n-cube, leading to a family of groups similar to the classical homotopy groups. The formal resemblance between the two families of groups is explored. Special attention is devoted to the role of these groups as topological invariants for manifolds and as tools for describing geometrical structures defined on manifolds such as bundles and connections.*

1. Introduction

In [5] the intimacy relation between smooth based loops on a manifold was introduced. For two loops to be related in this way it is required that they may be linked by a smooth homotopy whose differential has rank ≤ 1 throughout its domain. The chief property of this relation is that it preserves parallel transport whatever the connection that is present. As a side product this relation produces a new topological invariant for differentiable manifolds which is similar to the fundamental group. In this article we carry out a full comparison between these two groups and generalize intimacy to smooth maps defined on the n-cube along the lines of higher homotopy groups. It turns out that we still get topological invariants

(*) Authors' addresses: A. P. Caetano, Centro de Matemática e Aplicações Fundamentais, Universidade de Lisboa, Av. Prof. Gama Pinto, 2, 1699 Lisboa Codex, Portugal
R. F. Picken, Departamento de Matemática & Centro de Matemática Aplicada, Instituto Superior Técnico, Av. Rovisco Pais, 1096 Lisboa Codex, Portugal

similar to the classical homotopy groups but having additional properties. A few words are devoted to the usefulness of the first intimacy group in bundle and connection theory in particular through the *kernel theorem.* Several routes for further developments are outlined in the last section

2. Smooth singular cubes and boundaries

Let $I^n = [0,1]^n = \{(t_1, ..., t_n) : 0 \leq t_i \leq 1 \wedge 1 \leq i \leq n\}$ be the *standard* n*-dimensional cube* and let $I^0 = \{0\}$ to suit the operations defined later on. For $n = 1, 2, ...$ an $(n-1)$-*face* of I^n is either $\{(t_1, ..., t_n) \in I^n : t_i = 0\}$ or $\{(t_1, ..., t_n) \in I^n : t_i = 1\}$. The *initial*-$(n-1)$-*face* is chosen to be $\{(t_1, ..., t_n) \in I^n : t_n = 0\}$ and will be identified with I^{n-1} in the obvious way. The union of all $(n-1)$-*faces* of the n-*cube*, numbering $2n$, is just the *boundary* ∂I^n. J^{n-1} is defined to be the union of all $(n-1)$-*faces* with the single exception of the *initial*-$(n-1)$-*face*. Some obvious relations which follow are $\partial I^n = I^{n-1} \bigcup J^{n-1}$, $\partial I^{n-1} = I^{n-1} \bigcap J^{n-1}$ and $\partial I^{n-1} \subseteq J^{n-1}$. These concepts and this notation are standard in homotopy theory and a nice summary can be found in [10]. For our purposes we need to introduce the neighbourhood $K_\epsilon^{n-1} = \{(t_1, ..., t_n) \in I^n : 0 \leq t_1 < \epsilon \vee 1 - \epsilon < t_1 \leq 1\}$ where $1 > \epsilon > 0$.

Let M be a smooth paracompact and connected manifold, $A \subseteq M$ a connected topological subspace and $* \in A \subseteq M$ a fixed point. For $n = 0, 1, 2, ...$ $F^n(M, A, *)$ shall denote the space of smooth functions $f : I^n \to M$ for which there is some $1 > \epsilon > 0$ such that $f(K_\epsilon^{n-1}) \subseteq \{*\}$, $f(J^{n-1}) \subseteq \{*\}$ and $f(I^{n-1}) \subseteq A$, conditions which are taken for granted when $n = 0$. Notice that these functions are smooth versions of the usual n-*singular cubes* appearing in the construction of higher homotopy groups $\pi_n(M, A, *)$ as carried out in [10], except for the requirement for them to "stop for a while" in a neighbourhood of the faces '$t_1 = 0$' and '$t_1 = 1$'. This technicality allows one to glue smooth singular cubes to get cubes that are still smooth, which opens up the possibility of defining the usual binary operation leading to the group strucure of $\pi_n(M, A, *)$, without leaving the smooth category. Non-degenerate examples of such cubes may be built using the same smooth functions that appear in the construction of *partitions of unity* [7][5][4][3]. $F^0(M,A,*)$ can be identified with the set M. It follows easily from the definition that $F^1(M,A,*)$ does not depend on $A \subseteq M$ and $F^1(M,A,*) = F^1(M,\{*\},*) = \Omega^\infty(M)$ is the space of smooth loops based at $* \in M$ that was used in [5][4][3].

3. Intimacy

Given two cubes $\alpha, \beta \in F^n(M,A,*)$, $n = 0,1,2,...$, we say that they are *intimate*, and write $\alpha \overset{\rho}{\sim} \beta$, if there exist $1 > \epsilon > 0$ and a *homotopy* $H : [0,1] \times I^n \longrightarrow M$ such that:

1. $H([0,1] \times K_\epsilon^{n-1}) \subseteq \{*\}$
2. $H([0,1] \times J^{n-1}) \subseteq \{*\}$
3. $H([0,1] \times I^{n-1}) \subseteq A$
4. $H(s,t_1,...,t_n) = \alpha(t_1,...,t_n),\ \forall s \in [0,\epsilon[$
5. $H(s,t_1,...,t_n) = \beta(t_1,...,t_n),\ \forall s \in]1-\epsilon,1]$
6. H is smooth throughout $[0,1] \times I^n$
7. $rank(DH_{(s,t_1,...,t_n)}) \leq n,\ \forall (s,t_1,...,t_n) \in [0,1] \times I^n$
8. $rank(DH_{|_{[0,1]\times I^{n-1}}(s,t_1,...,t_{n-1},0)}) \leq n-1,\ \forall (s,t_1,...,t_{n-1},0) \in [0,1] \times I^{n-1}$

For $n = 0$ conditions 1,2,3 are taken for granted. Such a homotopy is called a *rank-n-homotopy*. Reparametrizing and glueing rank-n-homotopies one can easily check that *intimacy* is an equivalence relation in $F^n(M,A,*)$. Rank-one-homotopies along with intimacy for loops evolved from the concept of *thin homotopy* introduced by Barrett in [1] and have been used in [5][4][3][11].

4. The intimacy group

The usual operations with cubes $\alpha, \beta \in F^n(M,A,*)$, $n \geq 1$, leading to higher homotopy groups consist of a composition $\alpha\beta(t_1,...,t_n) =$

$\alpha(2t_1, ..., t_n)$ for $0 \leq t_1 \leq \frac{1}{2}$, $\alpha\beta(t_1, ..., t_n) = \beta(2t_1 - 1, ..., t_n)$ for $\frac{1}{2} < t_1 \leq 1$ and an inversion $\alpha^{-1}(t_1, ..., t_n) = \alpha(1 - t_1, ..., t_n)$. Both operations go over to the quotient by intimacy $F^n(M, A, *)/ \overset{\rho}{\sim}$ to produce an *intimacy group* which will be denoted by $\pi_n^n(M, A, *)$ (where the upper index serves as a reminder of the upper bound imposed on the ranks of the homotopies). The proof of this statement is merely a repetition of the proof given for $\pi_1^1(M, A, *) = \pi_1^1(M, \{*\}, *)$ in [5] where it was denoted by $\mathcal{GL}^\infty(M)$. If we drop conditions 7 and 8 in the definition of rank-n-homotopy the whole group construction that follows still works and the group emerging will be the classical homotopy group $\pi_n(M, A, *)$. To check this we just have to go through some technical results about approximations of continuous cubes and homotopies by means of smooth ones. Since rank-n-homotopy is stronger than usual homotopy, a surjective group morphism $\mathcal{C} : \pi_n^n(M, A, *) \longrightarrow \pi_n(M, A, *)$ arises in a canonical way. Therefore when the manifold is n-dimensional $M = {}^nM$ we get $\pi_k^k({}^nM, A, *) = \pi_k({}^nM, A, *)$ for $k \geq n+1$ and $\pi_k^k({}^nM, \{*\}, *) = \pi_k({}^nM)$ for $k \geq n$. Apart from the trivial cases $\pi_0^0(M, A, *) \simeq M$ (which is not a group), $\pi_1^1(\mathbf{R}, A, *) = \mathcal{GL}^\infty(\mathbf{R}) = \{1\}$ and $\pi_1^1(\mathbf{S^1}, A, *) = \mathcal{GL}^\infty(\mathbf{S^1}) = \mathbf{Z}$, a few concrete examples are $\pi_n^n(\mathbf{S^n}, \{*\}, *) = \pi_n(\mathbf{S^n}) = \mathbf{Z}$ for $n \geq 1$, $\pi_3^3(\mathbf{S^2}, A, *) = \pi_3(\mathbf{S^2}) = \mathbf{Z}$ (Hopf), $\pi_4^4(\mathbf{S^3}, A, *) = \pi_4(\mathbf{S^3}) = \mathbf{Z}_2$. For an n-dimensional manifold $M = {}^nM$ and $0 \leq k < n$ the intimacy group $\pi_k^k({}^nM, A, *)$ is not countable.

The group $\pi_n^n(M, \{*\}, *)$ is *abelian* for $n \geq 2$. A proof of this statement consists of an exact repetition of the proof for the same statement for the usual higher homotopy groups [2] since all homotopies involved are in fact rank-n-homotopies.

By construction, the intimacy group $\pi_n^n(M, A, *)$ is an *invariant* with respect to *diffeomorphisms*. Invariants are expected to provide some help in distinguishing between "topologically different spaces", in this case non-diffeomorphic ones. An example of the usefulness of intimacy groups in this sense is the following: while the fundamental group doesn't separate the cilinder from the circle (despite their having different dimensions), $\pi_1(\mathbf{S^1} \times [0,1]) = \mathbf{Z} = \pi_1(\mathbf{S^1})$, with the intimacy group we get $\pi_1^1(\mathbf{S^1}, \{*\}, *) = \mathbf{Z}$ which is clearly different from the non-countable $\pi_1^1(\mathbf{S^1} \times [0,1], \{*\}, *)$. This example also points out that a major characteristic of homotopy groups is lost by intimacy groups: homotopy groups behave well with respect to cartesian products while intimacy groups do not.

Another role for these new groups will emerge in the description of *differential geometric structures on M* (such as bundles and connections) and this will help to reinforce their value as topological invariants (see the results in section 6).

5. The long intimacy sequence

The *boundary operator* between usual homotopy groups $\partial_n : \pi_n(M,A,*) \longrightarrow \pi_{n-1}(A,\{*\},*)$, as found in [10], works with the intimacy groups in precisely the same way: for $n = 2,3,...$

$$\partial_n : \pi_n^n(M,A,*) \longrightarrow \pi_{n-1}^{n-1}(A,\{*\},*)$$
$$[\alpha] \longmapsto [\alpha_{|_{I^{n-1}}}]$$

For $[\alpha] \in \pi_n^n(M,A,*)$, $\alpha \in F^n(M,A,*)$ and since $\alpha_{|_{I^{n-1}}}(I^{n-2}) \subseteq \alpha(J^{n-1}) \subseteq \{*\}$, $\alpha_{|_{I^{n-1}}}(J^{n-2}) \subseteq \alpha(J^{n-1}) \subseteq \{*\}$ and $\alpha_{|_{I^{n-1}}}(K_\epsilon^{n-2}) \subseteq \alpha(K_\epsilon^{n-1}) \subseteq \{*\}$ it follows that $\alpha_{|_{I^{n-1}}} \in F^{n-1}(A,\{*\},*)$. If $[\alpha] = [\beta]$ then α and β are linked by a rank-n-homotopy H and the restriction $H_{|_{[0,1]\times I^{n-1}}}$ satisfies $rank(DH_{|_{[0,1]\times I^{n-1}}(s,t_1,...,t_{n-1},0)}) \leq n-1$ and $rank(DH_{|_{[0,1]\times I^{n-2}}(s,t_1,...,t_{n-2},0,0)}) = 0 \leq n-2$ thus being the homotopy which ensures that $[\alpha_{|_{I^{n-1}}}] = [\beta_{|_{I^{n-1}}}]$. We may conclude that ∂_n is well defined and a quick look at the way the group operations are defined shows that this boundary operator is a group morphism: $\partial_n([\alpha][\beta]) = \partial_n([\alpha])\partial_n([\beta])$. There are also two natural inclusions coming from higher homotopy group theory [10]:

$$\pi_n^n(A,\{*\},*) \xrightarrow{i_n} \pi_n^n(M,\{*\},*) \xrightarrow{j_n} \pi_n^n(M,A,*)$$

These three types of group morphism originate the *long intimacy sequence* which is shown below when $A = M$:

$$\begin{aligned} \ldots \xrightarrow{j_4} \pi_4^4(M,M,*) \xrightarrow{\partial_4} \pi_3^3(M,\{*\},*) \xrightarrow{j_3} \pi_3^3(M,M,*) \\ \xrightarrow{\partial_3} \pi_2^2(M,\{*\},*) \xrightarrow{j_2} \pi_2^2(M,M,*) \xrightarrow{\partial_2} \pi_1^1(M,\{*\},*) \\ \xrightarrow{j_1\equiv id.} \pi_1^1(M,M,*) \xrightarrow{\partial_1\equiv *} M \xrightarrow{j_0\equiv id.} M \end{aligned}$$

The long intimacy sequence is exact. A proof of this statement consists of an exact repetition of the proof for the same statement for the usual higher homotopy groups [10] since all homotopies involved are in fact rank-n-homotopies.

6. Intimacy, bundles, connections and topological invariants

THEOREM 6.1. *Given two n-cubes* $\alpha, \beta \in F^n(M, A, *)$ *and an* $\mathbf{R}$ *-valued n-form over M* $\omega \in \bigwedge^n(M)$*, if* α, β *are intimate* $\alpha \overset{\ell}{\sim} \beta$*, then*

$$\int_\alpha \omega = \int_\beta \omega$$

Proof. Suppose $\alpha, \beta \in F^n(M, A, *)$ are linked by a rank-n-homotopy $H : [0,1] \times I^n \longrightarrow M$, as described in section 3. Then:

$$\int_\beta \omega - \int_\alpha \omega = \int_{I^n} \beta^* \omega - \int_{I^n} \alpha^* \omega =$$

$$= \int_{\partial([0,1]\times I^n)} H^*\omega - \int_{[0,1]\times J^{n-1}} H^*\omega - (-1)^{n+1} \int_{[0,1]\times I^{n-1}} H^*\omega =$$

$$= \int_{\partial([0,1]\times I^n)} H^*\omega = \int_{[0,1]\times I^n} d(H^*\omega) = \int_{[0,1]\times I^n} H^*(d\omega) = 0$$

where $\int_{[0,1]\times J^{n-1}} H^*\omega = 0$ because $H([0,1] \times J^{n-1}) \subseteq \{*\}$ and $\int_{[0,1]\times I^{n-1}} H^*\omega = 0$ because $rank(DH_{|_{[0,1]\times I^{n-1}}}) \leq n-1$ so that this integral involves the evaluation of ω over n *linearly dependent* vector fields. In a similar way $\int_{[0,1]\times I^n} H^*(d\omega) = 0$ since it involves the evaluation of $d\omega$ over $n+1$ *linearly dependent* vector fields due to $rank(DH) \leq n$ $\square$

This theorem may be applied to (finite dimensional) *Lie-algebra valued forms* $\omega \in \bigwedge^n(M) \otimes \mathcal{G}$. When $\mathcal{G}$ is associated with an *abelian Lie group* G, the theorem means that the *holonomy* of *abelian connections* may be expressed as a function defined on $\pi_1^1(M, \{*\}, *)$

since the formula $\mathcal{H}_\omega(l) = exp \int_l \omega$ which describes the *parallel transport* along a *loop* $l \in F^1(M, \{*\}, *)$, is shown not to depend on the loop itself but only on its intimacy class. The properties of the abelian exponential $exp : \mathcal{G} \to G$ indicate that

$$\pi_1^1(M, \{*\}, *) \xrightarrow{\mathcal{H}_\omega} G$$
$$[l] \longmapsto exp \int_l \omega$$

is a group morphism.

In [5] it is shown that the holonomy $\mathcal{H}_\nabla$ of every *G-connection* ∇ (abelian or otherwise) defined on some *principal G-bundle* $\pi : P \longrightarrow M$ may be presented as a group morphism:

$$\pi_1^1(M, \{*\}, *) \xrightarrow{\mathcal{H}_\nabla} G$$

If we require that a group morphism $\mathcal{H} : \pi_1^1(M, \{*\}, *) \longrightarrow G$ be smooth in the sense that every *smooth family of loops* $\psi : [0,1] \longrightarrow F^1(M, \{*\}, *)$ (meaning that $\psi'(s,t) = \psi(s)(t)$ is smooth throughout $[0,1] \times [0,1]$) be transformed by $\mathcal{H}$ into a smooth curve $\mathcal{H}([\psi(s)])$ in G, then from every such object it is possible to retrieve a *G-bundle* together with a connection ∇ whose holonomy is $\mathcal{H}_\nabla = \mathcal{H}$ and these are unique up to isomorphism (see [5] for a proof).

The strongest result about this relationship between holonomies and bundles equipped with connections is the *kernel theorem* which first appeared in [8] (see also [4]). This theorem states that whenever $\mathcal{H}_{\nabla_1} : \pi_1^1(M, \{*\}, *) \longrightarrow G_1$ and $\mathcal{H}_{\nabla_2} : \pi_1^1(M, \{*\}, *) \longrightarrow G_2$ are surjective holonomies which share the same kernel then the Lie groups G_1 and G_2 are isomorphic and the connections ∇_1 and ∇_2 are also isomorphic (there is a principal bundle isomorphism which relates one to the other).

As promised in section 4, we now show that these considerations concerning holonomies bring about a reinforcement of the role of intimacy groups as topological invariants:

THEOREM 6.2. *If M and N are smooth, paracompact and connected manifolds and $\Phi : \pi_1^1(M, \{*\}, *) \longrightarrow \pi_1^1(N, \{*\}, *)$ is a group morphism then Φ carries homotopic (intimacy classes of) loops into homotopic (intimacy classes of) loops.*

Proof. Given a group morphism $\Phi : \pi_1^1(M, \{*\}, *) \longrightarrow \pi_1^1(N, \{*\}, *)$, let $\mathcal{C} : \pi_1^1(N, \{*\}, *) \longrightarrow \pi_1(N)$ be the canonical morphism (see section 4). Note that $\pi_1(N)$ is *countable* since the topology of N is second countable and therefore $\pi_1(N)$ is a zero-dimensional Lie group. It follows that the group morphism $\mathcal{C} \circ \Phi : \pi_1^1(M, \{*\}, *) \longrightarrow \pi_1(N)$ is smooth in the sense defined above and therefore it is possible to build a $\pi_1(N)$-*bundle* $\pi : C \to M$ together with a connection ∇ whose holonomy is $\mathcal{H}_\nabla = \mathcal{C} \circ \Phi$. Since the structure group is zero dimensional ∇ is the unique connection available in this bundle and it is *flat.* Given $[l], [k] \in \pi_1^1(M, \{*\}, *)$ such that $l, k \in F^1(M, \{*\}, *)$ are homotopic it follows that $\mathcal{C}(\Phi([l])) = \mathcal{H}_\nabla([l]) = \mathcal{H}_\nabla([k]) = \mathcal{C}(\Phi([k]))$ because ∇ is flat, therefore $\mathcal{C}(\Phi([lk^{-1}])) = 1$, so that $\Phi([lk^{-1}]) \in Ker\mathcal{C}$ and hence $\Phi(l)$ and $\Phi(k)$ are homotopic. $\square$

Theorem 6.3. *If M and N are smooth, paracompact and connected manifolds such that $\pi_1^1(M, \{*\}, *)$ and $\pi_1^1(N, \{*\}, *)$ are isomorphic groups, then $\pi_1(M)$ and $\pi_1(N)$ are also isomorphic.*

Proof. Given a group isomorphism $\Phi : \pi_1^1(M, \{*\}, *) \longrightarrow \pi_1^1(N, \{*\}, *)$, let $\mathcal{C}_M : \pi_1^1(M, \{*\}, *) \longrightarrow \pi_1(M)$ and $\mathcal{C}_N : \pi_1^1(N, \{*\}, *) \longrightarrow \pi_1(N)$ be the canonical morphisms associated to each manifold (see section 4). Applying the above theorem to Φ and Φ^{-1} one concludes that $Ker\mathcal{C}_M = Ker\mathcal{C}_N$. Quotienting by these kernels we get isomorphisms $\widetilde{\mathcal{C}}_M : \frac{\pi_1^1(M,\{*\},*)}{Ker\mathcal{C}_M} \longrightarrow \pi_1(M)$, $\widetilde{\mathcal{C}}_N : \frac{\pi_1^1(N,\{*\},*)}{Ker\mathcal{C}_N} \longrightarrow \pi_1(N)$ and $\widetilde{\Phi} : \frac{\pi_1^1(M,\{*\},*)}{\mathcal{C}_M} \longrightarrow \frac{\pi_1^1(N,\{*\},*)}{\mathcal{C}_N}$. Then $\phi = \widetilde{\mathcal{C}}_N \circ \widetilde{\Phi} \circ \widetilde{\mathcal{C}}_M^{-1}$ is an isomorphism between $\pi_1(M)$ and $\pi_1(N)$. $\square$

In this proof, the *kernel theorem* could have been used at a certain stage to show the existence of a isomorphism between the fundamental groups. Notice that this last theorem settles the issue as far as the topological abilities of $\pi_1^1(M, \{*\}, *)$ are concerned: it is a more powerful invariant than $\pi_1(M)$ (see section 4 for a counterexample for the reciprocal of theorem 3). One more remark: part of the conclusions of the above theorems may be achieved reasoning within *covering spaces theory* [9] since each connected component of the $\pi_1(N)$-bundle $\pi : C \to M$ associated with $\mathcal{C}_N \circ \Phi$ is a covering space of M and the $\pi_1(M)$-bundle associated with $\mathcal{C}_M$ is the *universal covering space* of M.

7. Final remarks

The comparison between the fundamental group $\pi_1(M)$ and the first intimacy group $\pi_1^1(M,\{*\},*)$ achieved in theorem 3 (see also section 4) answers the question concerning the topological abilities of $\pi_1^1(M,\{*\},*)$. Future work could be devoted to seeking similar results for intimacy groups $\pi_n^n(M,\{*\},*)$ of higher order $n \geq 2$. A more 'structural quest' would be to look for the *geometrical structure* 'hidden' in a group morphism $\Phi : \pi_n^n(M,\{*\},*) \to G$ with values in a topological group and whether this can be related to gauge theory, since for $n = 1$ there is a nice start for the whole idea (see section 6). Also these groups might prove useful within theoretical physics whenever topological considerations are an issue as in the case of monopoles in gauge theories or in loop space formulations of physical theories [6][3][11].

8. Acknowledgements

This work has benefitted from several remarks made by Owen Brison and João Pedro Boto as well as a partial oral presentation at the second annual meeting of Física-Matemática praxis/2/2.1/mat/19 /94 which took place in Madeira in February 97 and financial support from fct, praxis xxi, feder, cmaf praxis/2/2.1/mat/125 /94, jnict.

References

[1] J. W. Barrett, *Holonomy and path structures in general relativity and Yang-Mills theory*, International J. Theoret. Phys. **30** (1991), 1171–1215.

[2] R. Bott and L. W. Tu, *Differential forms in algebraic topology*, Springer Verlag, 1982.

[3] A.P. Caetano, *On the Abelian projection of a connection*, Acta Cosmologica **Fasciculus XXI-2** (1995), 181–193.

[4] A.P. Caetano, *On the kernel of holonomy*, Publicacions Matemàtiques **40** (1996), 373–381.

[5] A.P. Caetano and R.F. Picken, *An axiomatic definition of holonomy*, Intern. Jour. Math. **5** (1994), no. 6, 835–848.

[6] CHAN HONG-MO AND TSOU SHEUNG TSUN, *Some elementary gauge theory concepts*, World Scientific Lecture Notes in Physics (1993).

[7] S. KOBAYASHI AND K. NOMIZU, *Foundations of differential geometry*, Interscience Publishers, 1963.

[8] J. LEWANDOWSKI, *The group of loops, holonomy maps, path bundle and path connection*, Classical and Quantum Gravity **10** (1993), 879–904.

[9] W. S. MASSEY, *A basic course in algebraic topology*, Springer Verlag, 1991.

[10] N. STEENROD, *The topology of fibre bundles*, Princeton University Press, 1951.

[11] E. E. WOOD, *Reconstruction theorem for groupoids and principal fiber bundles*, International J. Theoret. Phys. **36** (1997).

Received December 10, 1997.