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Global Stability and Plus-Global Stability. An Application to Forward Neural Networks

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"After this paper was ready to be submitted for publication, Prof. Giovanni Di Lena passed away suddenly and prematurely. Basilio and Mario are devastated by this loss and would like to dedicate the paper to the loving memory of Giovanni."

Abstract. *A necessary and sufficient condition for a discrete dynamical system to be globally stable and plus-globally stable are first established in Section 2. The* V-condition *is introduced and Theorems 3.5 and 3.7 are presented in Section 3. The two theorems link the* V-condition *to the most relevant properties of globally stable and plus-globally stable discrete dynamical systems. In Section 4 we provide a simple application to a convergence problem for forward neural networks.*

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1. Introduction

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and consider the discrete dynamical system governed by f. Given $x_0 \in \mathbb{R}$ denote with $O(x_0)$ its orbit, namely the sequence

$$O(x_0) := \{x_n \colon n = 1, 2, \dots\} \tag{1}$$

with

$$x_1 = f(x_0), x_2 = f(x_1) = f(f(x_0)) := f^2(x_0), \dots \tag{2}$$

Notice that

$$x_{n+1} - f(x_n) = 0, \text{ for every } n = 0, 1, \dots \tag{3}$$

Let $\{x_n : n = 0, 1, \dots\}$ be a sequence in $\mathbb{R}$ such that

$$\lim_{n\to\infty} x_{n+1} - f(x_n) = 0. \tag{4}$$

Define

$$P(x_0) = \{x_n : n = 0, 1, \dots\}. \tag{5}$$

Clearly, (3) is a particular case of (4).

The main purpose of this paper is to establish conditions that imply the existence and uniqueness of a fixed point x_s of f, and the convergence of $P(x_0)$ to x_s regardless of the initial position $x_0 \in \mathbb{R}$. A dynamical system with these properties is said to be *plus-globally stable*. The system is simply *globally stable* in the case when f has one and only one fixed point x_s, and $O(x_0)$ converges to x_s for all $x_0 \in \mathbb{R}$.

Obviously, every system that is plus-globally stable is also globally stable. In Section 2 we provide an example of a system that is globally stable, but it is not plus-globally stable.

The results obtained in Sections 2 and 3 are used in Section 4 to establish, in a simple case, the global stability and the plus-global stability of non-scalar systems arising in the study of forward neural networks.

Some historical remarks are in order.

Let $H : \mathbb{R}^q \to \mathbb{R}^q$ and assume that there exists a norm $\|.\|_a$ in $\mathbb{R}^q$, a positive integer p, and a constant $k \in [0, 1)$ such that

$$\|H^p(\mathbf{x}) - H^p(\mathbf{y})\|_a \le k\|\mathbf{x} - \mathbf{y}\|_a, \tag{6}$$

for every pair of points $\mathbf{x}, \mathbf{y}$ of $\mathbb{R}^q$. Inequality (6) implies that there is one and only one point $\mathbf{x}_s$ such that $\mathbf{x}_s = H(\mathbf{x}_s)$. Moreover, every orbit of H, every sequence of iterates

$$O(\mathbf{x}_0) := \{\mathbf{x}_1 = H(\mathbf{x}_0), \mathbf{x}_2 = H(\mathbf{x}_1) = H(H(\mathbf{x}_0)) := H^2(\mathbf{x}_0), \dots\}, \tag{7}$$

where $\mathbf{x}_0$ is any element of $\mathbb{R}^q$, converges to $\mathbf{x}_s$. The existence and uniqueness of the fixed point $\mathbf{x}_s$ and the convergence of $O(\mathbf{x}_0)$ to $\mathbf{x}_s$ for all $\mathbf{x}_0 \in \mathbb{R}^q$ follow, for $p = 1$, from the *Banach Contraction Principle*, established in 1922 by S. Banach for maps defined in complete metric spaces (see [2, 8]). For $p > 1$ both results are derived from an extension, attributed to R. Caccioppoli (see [4, 13]), of the Banach Contraction Principle.

Assume that $H : \mathbb{R}^q \to \mathbb{R}^q$ has a fixed point $\mathbf{x}_s$, and, in addition, the map H is Fréchet differentiable (see, for example, [18]) at every $\mathbf{x} \neq \mathbf{x}_s$. Moreover, let us require the existence of a function $h : \mathbb{R}^q \to \mathbb{R}$ such that $h'(\mathbf{x}) = H(\mathbf{x})$ (namely, H is a gradient). Then, for every $\mathbf{x}_0 \in \mathbb{R}^q$, the orbit $O(\mathbf{x}_0)$ converges to $\mathbf{x}_s$, provided that, for all $\mathbf{x} \neq \mathbf{x}_s$, the spectral radius of the Fréchet derivative $H'_F(\mathbf{x})$ is smaller than 1 (see [13]). Since the Fréchet derivative of F at every $\mathbf{x} \neq \mathbf{x}_s$ is a symmetric matrix, one can use the Euclidean norm to prove the uniqueness of the fixed point $\mathbf{x}_s$ and the convergence of $O(\mathbf{x}_0)$ to $\mathbf{x}_s$ for every $\mathbf{x}_0 \in \mathbb{R}^q$.

Symmetricity of $H'_F(\mathbf{x})$ is no longer available when H is lower triangular, i.e.

$$H(\mathbf{x}) = H(x_1, \dots, x_q) = (k_1(x_1), k_2(x_1, x_2), \dots, k_q(x_1, \dots, x_q)),$$

unless

$$H(\mathbf{x}) = H(x_1, \dots, x_q) = (h_1(x_1), h_2(x_2), \dots, h_q(x_q)),$$

namely H is a diagonal map. However, one can still obtain that every orbit $O(\mathbf{x}_0)$ converges to $\mathbf{x}_s$ by placing suitable restrictions on the diagonal elements of the matrix $H'_F(\mathbf{x})$, for all $\mathbf{x} \neq \mathbf{x}_s$. The convergence is established using the Euclidean norm (see [1, 7, 14]).

Assume that $H(\mathbf{0}) = \mathbf{0}$. It has been recently proved (see [9]) that $O(\mathbf{x}_0)$ converges to $\mathbf{0}$ for every $\mathbf{x}_0 \in \mathbb{R}^q$ provided that H is continuous, its Gateaux derivative $H'_G(\mathbf{x})$ (see, for example, [18]) exists except possibly on a linearly countable set $S \subset \mathbb{R}^q$, and the spectral radius of the product of $H'_G(\mathbf{x})$ and of its transpose is smaller than 1 at every point $\mathbf{x} \notin S$. Recall that S is linearly countable if for every $\mathbf{z} \in \mathbb{R}^q$ the set $[\mathbf{0}, \mathbf{z}] \cap S$ is at most countable, where the symbol $[\mathbf{0}, \mathbf{z}]$ denotes the line segment $\{\mathbf{y} = t\mathbf{z} \colon t \in [0, 1]\}$. The convergence is established using the Euclidean norm.

Notice that with the exception of the case when H or one of its iterates is a contraction (see inequality (6)), all remaining cases mentioned above use some form of differentiability. Moreover, the plus-global stability is never considered. No differentiability assumptions are made in this paper. Moreover, we investigate the theory and an application of plus-globally stable systems.

The paper is divided into four sections. After this introduction (Section 1), we define, in Section 2, *globally stable* and *plus-globally stable* systems and we establish necessary and sufficient conditions for global stability and for plus-global stability. In Section 3 the *V-condition* is introduced and its implications for dynamical systems are analyzed. We explore the interplay between the V-condition and global stability (see Theorem 3.5). We also analyze the relations between the V-condition and plus-global stability (see Theorem 3.7). Section 4 presents a simple application of the theory developed in Sections 2 and 3 to a convergence problem that arises in the study of forward neural networks.

2. Notations, Definitions, and Preliminary Results

Consider the discrete dynamical system governed by a function $f \colon \mathbb{R} \to \mathbb{R}$. Recall that the symbol $O(x_0)$ denotes the orbit of x_0 (see (1) and (2)).

Definition 2.1. We write

$$x_0 < O(x_0) \tag{8}$$

to indicate that $x_0 < x_n := f^n(x_0)$, for every $n = 1, 2, \dots$ Obviously $O(x_0) < x_0$ denotes that $f^n(x) := x_n < x_0$, for every $n = 1, 2, \dots$

Definition 2.2. We say that x_0 is a *periodic point of period* 2 of f if $x_0 = x_2 \neq f(x_0) = x_1$. Notice that $x_3 = x_1$. The definition of a *periodic point of period* q, $3 \leq q$, is analogous.

Definition 2.3. The dynamical system governed by f is said to be *globally stable* (or, simply, f is globally stable) if f has a unique fixed point x_s, and, for every $x_0 \in \mathbb{R}$, the orbit $O(x_0)$ converges to x_s (see, for example, [17]).

Definition 2.4. The dynamical system governed by f is *plus-globally stable* (or, simply, f is plus-globally stable) if f has a unique fixed point x_s, and every sequence $\{x_n,\ n = 0, 1, \dots\}$ (not necessarily an orbit) such that

$$\lim_{n\to\infty} x_{n+1} - f(x_n) = 0, \tag{9}$$

converges to x_s.

Globally stable and *plus-globally stable* systems (or maps) in $\mathbb{R}^q$ are defined (and denoted) in a similar manner.

Remark 2.5. Notice that, as mentioned in Section 1, a plus-globally stable system is globally stable. In fact, along every orbit of the dynamical system governed by f we have

$$x_{n+1} - f(x_n) = 0, \text{ for every } n = 0, 1, \dots \text{ (see (3))}. \tag{10}$$

As pointed out by P. Cull (see [17]), *local stability* is easier to check than *global stability* (see [12]) for population models either continuous or discrete and governed by functions that are at least C^1.

Notice that Cull's investigation addresses the study of population models. Consequently, the variable x is necessarily *positive.* In this paper we consider functions $f\colon \mathbb{R} \to \mathbb{R}$, we only assume continuity and we study globally stable as well as plus-globally stable systems.

The following examples and theorems regard dynamical systems that are globally stable and the ones that are plus-globally stable.

Example 2.6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x(1 - e^{-|x|}). \tag{11}$$

Notice that $x_s = 0$ is the only fixed point of f. Moreover, it can be easily verified that $O(x_0)$ converges to 0 for every $x_0 \in \mathbb{R}$. Hence, the dynamical system is globally stable. The sequence

$$\{x_n = \ln(n+2),\ n = 0, 1, \dots\} \tag{12}$$

has the property

$$\lim_{n\to+\infty} x_{n+1} - f(x_n) = 0.$$

Therefore, f is not plus-globally stable.

EXAMPLE 2.7. Let $a \in (-1,1)$ and define $f(x) = ax$. Clearly, $x_s = 0$ is the only fixed point of f and $O(x_0)$ converges to 0 for every $x_0 \in \mathbb{R}$. It is not difficult to verify that this globally stable dynamical system is also plus-globally stable.

Theorem 2.16 gives a necessary and sufficient condition for global stability. Its proof is based on the following results and remarks.

We begin with a theorem of A. N. Sharkovsky [19].

THEOREM 2.8. *Let $I \subset \mathbb{R}$ be an interval and $f\colon I \to I$ be continuous. Assume that f has a periodic orbit of period p. Then f has a periodic orbit of period q for every q that follows p in the arrangement* (Sharkovsky′s ordering)

$$3 \prec 5 \prec 7 \prec \ldots \prec 6 \prec 10 \prec 14 \prec \ldots \prec 12 \prec 20 \prec 28 \prec \ldots \prec 2^3 \prec 2^2 \prec 2 \prec 2^0 = 1.$$

Notice that the first group contains, in increasing order, all odd numbers starting with 3. The second group consists of the entries of the first group multiplied by 2 and with the increasing order preserved. The elements of the third group are obtained from the ones of the second group by multiplying its entries by 2 and preserving the increasing order ... Each group is infinite on the right, but the last one that is infinite on the left, and it consists of all powers of 2 in reverse order. The string ends with $2^0 = 1$.

The following remark is an easy consequence of Sharkovsky's result.

REMARK 2.9. Theorem 2.8 (see also [16]) implies that the following statements are equivalent:
a. there exists $x_s \in \mathbb{R}$ such that $f(x_s) = x_s$ is the only periodic point of f;
b. f has one and only one fixed point x_s, and it does not have any periodic point of period 2 (see Definition 2.2).

REMARK 2.10. It is easy to show that every bounded orbit of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ that satisfies condition *a.* (or *b.*) of Remark 2.9 must converge to x_s.

We also need the following result of L. S. Block and W. A. Coppel [3].

THEOREM 2.11. *Let I be a compact interval and $J \subset I$ be a subinterval which contains no periodic points of a continuous function $f\colon I \to I$. Then, for every $x_0 \in I$, the points of the orbit $O(x_0)$ which lie in J form a strictly monotonic (finite or infinite) sequence.*

Finally, we shall use a result established by A. Crannell and M. Martelli [6].

Theorem 2.12. *Suppose that $I \subset \mathbb{R}$ is a (possibly infinite) interval and $f\colon I \to I$ is continuous. Suppose, moreover, that there is $x_0 \in I$ such that $O(x_0)$ has infinitely many limit points. Then for every number $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$, $q > p$, and $y_0 \in I$ such that y_0 is a periodic point of period q of f.*

We now present some additional remarks and a result (see Proposition 2.14) that will be helpful in the proof of Theorem 2.16.

Remark 2.13. The reasoning used in [6] shows that when $O(x_0)$ has only $m \in \mathbb{N},\ m \geq 1$, distinct limit points, say

$$\{z_1,\ z_2,\ \ldots, z_m\},$$

then each $z_k,\ k = 1, \ldots, m$ is a periodic point of period m of f.

As mentioned above, the proof of Theorem 2.16 is based on the following

Proposition 2.14. *Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, and for every $x \in \mathbb{R}$, $x \neq 0$, whenever one of the ratios*

$$\frac{f(x)}{x},\ \frac{f(f(x))}{x} \tag{13}$$

is positive, it belongs to $(0,1)$. Then f does not have any fixed point different from 0, or any periodic orbit of period 2.

Proof. The existence of a fixed point $x_s \neq 0$ would imply that the first quotient is 1. The existence of a periodic orbit of period 2 (see Definition 2.2) would imply

$$z = f(f(z)).$$

for some $z \neq 0$. □

Remark 2.15. In what follows we shall always assume that $0 \notin O(x_0)$, and it is easy to see that any orbit of this type belongs to one and only one of the following disjoint sets:

- j. the set A of orbits with infinitely many states on one side of 0, and with only finitely many states on the other side of 0;
- jj. the set B of orbits with infinitely many states on both sides of 0, and with the additional property that we can find, on each side, at least one pair of entries belonging to three successive states of the orbit;
- jjj. the set C of orbits with infinitely many states on both sides of 0, and such that we cannot find, on one side (say the left side), a pair of entries belonging to any three successive states of the orbit.

With the results and remarks mentioned above we are in a position to prove

THEOREM 2.16. *Assume that f is continuous and $f(0) = 0$. Then $O(x_0)$ converges to 0 for every $x_0 \in \mathbb{R}$ if and only if f satisfies the condition that whenever one of the ratios (13) is positive, it belongs to $(0, 1)$.*

Proof. We first show that the convergence of $O(x_0)$ to 0 follows from the property that *whenever one of the ratios* (13) *is positive, it belongs to* $(0, 1)$. Recall that, by Proposition 2.14, f does not have any fixed point $x_s \neq 0$ nor any periodic orbit of period 2. Consequently, by Theorem 2.8, the function does not admit any periodic orbit of period $p \geq 2$. Moreover, Remark 2.15 implies that $O(x_0)$ belongs to one and only one of the sets A, B, and C.

Let us assume that $O(x_0) \in A$. Since only finitely many states of the orbit are on one side of 0, there exists a positive integer k such that the states $\{x_k = f^k(x_0),\ x_{k+1} = f^{k+1}(x_0), \dots\}$ belong to the other side of 0. We can assume, without loss of generality, that *the other side* is the half-line $(0, +\infty)$. Since the ratios $\{\frac{x_{p+1}}{x_p}, : p \geq k\}$ are positive, they belong to $(0, 1)$. Hence, the sequence $\{x_{k+q} : q = 0, 1, \dots\}$ is strictly monotone decreasing and it converges to the fixed point 0 (see also Remark 2.10).

Let us now assume that $O(x_0) \in B$ and denote by $R(x_0)$ and $L(x_0)$ the subsequences of $O(x_0)$ contained in $(0, +\infty)$ and $(-\infty, 0)$ respectively. Since there are no periodic orbits of f in $(0, +\infty)$ we can use Proposition 2.14 and Theorem 2.11 to derive that $R(x_0)$ is stricly decreasing. In an analogous manner we obtain that $L(x_0)$ is strictly increasing. Therefore, $R(x_0)$ converges to $z_R \geq 0$ and $L(x_0)$ converges to $z_L \leq 0$. It is easy to prove that $z_R = z_L = 0$. It remains to show that the result holds when $O(x_0) \in C$. Without loss of generality we can assume that the subset $L(x_0)$ does not have any pair of entries belonging to a group of three successive states of the orbit. Moreover, Proposition 2.14, Theorem 2.11 and the same strategy outlined in the case $O(x_0) \in B$, require $R(x_0)$ to be strictly decreasing. Hence, it converges to a point $z_R \geq 0$. We cannot have $z_R > 0$, since either from Theorem 2.12 or from Remark 2.13, we would obtain the existence of a periodic point $x_p \neq 0$ of period $p \geq 1$ of f, contradicting Proposition 2.14 and Theorem 2.8. Hence, $R(x_0)$ converges to 0. The continuity of f implies that the entire orbit converges to 0.

We now prove that when the discrete dynamical system governed by f has the property that every orbit $O(x_0)$ converges to 0, then if one of the ratios (13) is positive, it belongs to $(0, 1)$. Clearly, the system does not have a fixed point $x_s \neq 0$ nor any periodic orbit of period $p \geq 2$. In addition, for every $x_0 \in \mathbb{R}$, $0 \notin O(x_0)$, there exists a positive constant $M(x_0) = \max\{|x_n| : n = 1,\ 2, \dots\}$. Theorem 2.11 implies that the elements of $O(x_0)$ belonging to either one of the two intervals

$$I_1 = (0, M(x_0)),\ I_2 = (-M(x_0), 0)$$

are arranged in a strictly monotonic manner. More precisely, the convergence of $O(x_0)$ to 0 requires the ones of I_1 to be strictly decreasing and the ones of I_2 to be strictly increasing. Hence, the ratio of any two states of $O(x_0)$ on the same side of 0 is smaller than 1 whenever the subscript of the one in the numerator is larger than the subscript of the one in the denominator. Hence, in particular, the ratios (13) that are positive must be smaller than 1. □

REMARK 2.17. As Example 2.6 shows the necessary and sufficient condition of Theorem 2.16 is not enough to guarantee that a globally stable system is plus-globally stable. We will provide, at the end of this section, a sufficient condition and a separate necessary condition for plus-global stability.

REMARK 2.18. The property $f(0) = 0$ can obviously be replaced by $f(x_s) = x_s$, provided that the ratios (13) are changed accordingly. This observation holds also for Theorem 2.19 (see below).

We now establish a sufficient condition for the global stability of dynamical systems governed by functions belonging to a special family.

THEOREM 2.19. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and such that*

a) f(0)=0 is the only periodic point of f;

b) there exists $r > 0$ such that $|f(x)| < r$ when $|x| < r$.

Then the dynamical system governed by f is globally stable.

Proof. Let us show that the existence of an unbounded orbit of f would imply the existence of a point $z = f^p(z),\ z \neq 0,\ p \in \mathbb{N},\ p \geq 1$, contradicting the property that $0 = f(0)$ is the only periodic point of f.

From assumptions a), b) and Remark 2.10 we derive that

$$\lim_{n\to\infty} f^n(x) = 0 \tag{14}$$

for every $x \in I = (-r, r)$. Therefore, given $c \in (-r, r),\ c \neq 0$, there exists $k \in \mathbb{N}, k \geq 1$, such that for every $q \in \mathbb{N},\ q \geq k$, we have $f^q(c) < c$ when $c > 0$ and $c < f^q(c)$ when $c < 0$. Assume that f has an unbounded orbit $O(x_0)$. Without loss of generality we may assume that $x_0 > c > 0$ and $f^p(x_0) > x_0$ for some $p \geq q$. Hence

$$(c, x_0) \subset (f^p(c), f^p(x_0)). \tag{15}$$

By the Intermediate Value Theorem f^p has a fixed point $z \in [c, x_0]$. This conclusion contradicts property a) of f. Hence every orbit of f is bounded. The result now follows from Remark 2.10. □

REMARK 2.20. Notice that property b) of Theorem 2.19 is sufficient but not necessary for global stability.

We are now interested in establishing a sufficient condition and a separate necessary condition for a discrete dynamical system to be plus-globally stable.

THEOREM 2.21. *Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is globally stable with $f(x_s) = x_s$. Then f is plus-globally stable if there exist positive constants K and T such that*

$$T < f(x) - x, \text{ whenever } x < -K, \tag{16}$$

and

$$T < x - f(x), \text{ whenever } x > K. \tag{17}$$

Proof. We shall do the proof only in the case when $x > 0$. The case $x < 0$ can be analyzed in a similar manner. Let us show that the plus-global stability of the system can be derived from inequality (17).

Let $\{z_n\colon n = 1, 2, \dots\}$ be such that

$$\lim_{n\to\infty} z_{n+1} - f(z_n) = 0. \tag{18}$$

We need to prove that the sequence converges to x_s. Assume that the sequence is bounded and let L be the set of its limit points. Although $\{z_n\colon n = 0, 1, \dots\}$ may not be an orbit, we obtain (see for example [7]) $f(L) = L$. We can now use either Theorem 2.12 or Remark 2.13 to conclude that f has periodic orbits. Since the only periodic point of f is x_s we conclude that the sequence must converge to x_s.

It remains to show that the sequence is bounded. From (18) we derive the existence of positive integer n_0 such that for all $n \geq n_0$ we have

$$z_{n+1} \in \left(f(z_n) - \frac{T}{2}, f(z_n) + \frac{T}{2} \right). \tag{19}$$

Moreover, (17) implies that

$$f(x_q) + T < x_q \tag{20}$$

whenever $x_q > K$. Let $p \geq n_0$ be such that $x_p > K$. Then

$$f(x_p) + T < x_p \tag{21}$$

and

$$x_{p+1} < f(x_p) + \frac{T}{2} < x_p. \tag{22}$$

Consequently, the sequence is bounded. □

We now provide a necessary condition for plus-global stability.

THEOREM 2.22. *Let $f: \mathbb{R} \to \mathbb{R}$ be globally stable and such that $f(x_s) = x_s$. Assume that at least one of the two limits*

$$\lim_{x\to+\infty} x - f(x), \quad \lim_{x\to-\infty} x - f(x) \tag{23}$$

exists and it is 0. Then the dynamical system governed by f is not plus-globally stable.

Proof. We shall prove the result when the first limit is 0. The remaining case can be analyzed in a similar manner.

From (23) we derive that every monotone sequence $\{x_n : n = 1, 2, \dots\}$ such that $x_1 = 1$ and

$$\lim_{n\to\infty} x_n = +\infty \tag{24}$$

has the property

$$\lim_{n\to\infty} x_n - f(x_n) = 0. \tag{25}$$

Choose $\{x_n : n = 1, 2, \dots\}$ so that

$$x_{n+1} = x_n + \frac{1}{n}. \tag{26}$$

Clearly, the sequence has the property required by (24). Then, from (25) and (26), we obtain

$$\lim_{n\to\infty} x_{n+1} - f(x_n) = 0. \tag{27}$$

Therefore, the system is not plus-globally stable. □

REMARK 2.23. From Theorem 2.22 we derive that a necessary condition for plus-global stability is that both limits (23) should be different from 0. However, it is not difficult to verify that this requirement does not guarantee plus-global stability.

3. The V-Condition

We first would like to characterize those functions $f: \mathbb{R} \to \mathbb{R}$ such that for every $r \in \mathbb{R}$

1. the point x_r is the one and only one solution of the equation $f(x)+r = x$;
2. the dynamical system governed by $g_r: \mathbb{R} \to \mathbb{R}$, $g_r(x) = f(x)+r$ is globally stable.

To achieve the stated goals we introduce a class of maps with the property (28) listed below.

Definition 3.1. A function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the **V-condition** whenever $a < b$ does not imply that

$$f(b) \leq a < b \leq f(a). \tag{28}$$

It can be shown that when f is continuous and the orbits of f are bounded, inequality (28) is equivalent to the statement that f does not have periodic points of period 2 (see [16]).

The following result holds.

Proposition 3.2. *Assume that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the V-condition. Then*

i) $x_0 < f(x_0) = x_1$ *implies* $x_0 < O(x_0)$,

and

ii) $x_1 = f(x_0) < x_0$ *implies* $O(x_0) < x_0$.

Proof. We shall only prove *i*), namely that $x_0 < x_1$ implies $x_0 < O(x_0)$. The proof of *ii*) is similar.

Assume that $x_0 < O(x_0)$ is not verified for some $x_0 \in \mathbb{R}$. Let $p \geq 2$ be the smallest integer such that

$$x_p \leq x_0 < f(x_0) = x_1. \tag{29}$$

In the case when $p = 2$ we conclude that f does not satisfy the V-condition with $a = x_0$ and $b = x_1$.

Hence, assume that $p \geq 3$ and notice that we must have $x_0 < x_{p-1}$ for, otherwise, x_p could not be the first element to the left of x_0. The inequality $x_{p-1} \leq x_1$ implies that f does not satisfy the V-condition with $a = x_0$ and $b = x_{p-1}$. Hence, we can suppose that $x_p \leq x_0 < x_1 < x_{p-1}$. Let $1 < k < p-1$ be the largest integer such that $x_k < x_{p-1} \leq x_{k+1}$. Then, once again, f does not satisfy the V-condition with $a = x_k$ and $b = x_{p-1}$. □

We now present two propositions that give us the information we need to establish Theorems 3.5 and 3.7.

Define

$$h_1\colon \mathbb{R} \to \mathbb{R} \text{ by } h_1(x) := f(x) - x\,;$$

$$h_2\colon \mathbb{R} \to \mathbb{R} \text{ by } h_2(x) := f(x) + x\,.$$

We do not include a proof of Proposition 3.3 since the result is evident.

Proposition 3.3. *The following properties are equivalent* :

j) the function h_1 is bijective;

jj) the equation $g_r(x) = x$ *has one and only one solution for every* $r \in \mathbb{R}$.

Proposition 3.4 establishes the equivalence of three different statements linking the V-condition, the strictly increasing character of the map h_2, and a lower bound for the ratio

$$\frac{f(b) - f(a)}{b - a}$$

with $a \neq b$.

PROPOSITION 3.4. *The following properties are equivalent* :

1. $\forall r \in \mathbb{R}$ *the function* g_r *verifies the V-condition;*

2. *for every pair of real numbers* a, b *such that* $a < b$ *we have*

$$\frac{f(b) - f(a)}{b - a} > -1;$$

3. *the function* h_2 *is strictly increasing.*

Proof. Assume the existence of $a < b$ such that

$$\frac{f(b) - f(a)}{b - a} \leq -1. \tag{30}$$

Then, we can find $r \in \mathbb{R}$ satisfying the following inequality

$$f(b) - a \leq -r \leq f(a) - b. \tag{31}$$

Rearrange (31) to obtain

$$f(b) + r \leq a < b \leq f(a) + r \tag{32}$$

which implies that g_r violates the V-condition. Therefore, Property 1 implies Property 2.

Property 2 clearly implies Property 3. It remains to show that Property 3 implies Property 1.

Assume that there exists r such that g_r does not satisfy the V-condition. Then we can find $a < b$ such that

$$f(b) + r \leq a < b \leq f(a) + r. \tag{33}$$

From (33) we derive

$$b - a \leq f(a) - f(b).$$

against the assumption that h_2 is strictly increasing. □

We are now ready to prove an important result regarding global stability.

THEOREM 3.5. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that h_1 is bijective and h_2 is strictly increasing and let x_r be the only fixed point of g_r* (see Proposition 3.3). *Then, the following two properties are equivalent*:

1. *the dynamical system governed by g_r is globally stable;*
2. *for every $x \neq x_r$, we have*

$$|g_r(x) - x_r| < |x - x_r|.$$

Proof. Since the global stability of the system governed by g_r can easily be derived from the inequality

$$|g_r(x) - x_r| < |x - x_r|, \tag{34}$$

we only need to show that the global stability of g_r implies (34).

We prove the result only in the case when $x_r < x$. A similar reasoning can be used for the case $x < x_r$.

We first show that the following inequality holds

$$2x_r - x < g_r(x). \tag{35}$$

In fact, assume that

$$g_r(x) - x_r \leq x_r - x. \tag{36}$$

Inequality (36) implies

$$\frac{f(x) - f(x_r)}{x - x_r} \leq -1. \tag{37}$$

Since (37) violates the stricly increasing character of h_2, we obtain that (35) holds.

We now show that $g_r(x) < x$. In fact, if this is not the case, we can find x_0 such that $x_r < x_0 \leq g_r(x_0) = x_1$. Propositions 3.2 and 3.4 imply that $x_r < x_0 \leq O(x_0)$. This result, however, is unacceptable since the orbit $O(x_0)$ must converge to x_r. Hence

$$2x_r - x < g_r(x) < x. \tag{38}$$

Subtracting x_r from each term of (38) we obtain

$$x_r - x < g_r(x) - x_r < x - x_r. \tag{39}$$

Notice that (34) is equivalent to (39). □

Theorem 3.5 implies

Corollary 3.6. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that h_1 is bijective, h_2 is strictly increasing, and the dynamical system governed by g_r is globally stable for every $r \in \mathbb{R}$. Then f is continuous.*

We now establish an important result regarding plus-global stability.

Theorem 3.7. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and assume that the following conditions are verified :*

j) h_1 *is bijective;*

jj) $\displaystyle\inf_{x \neq y} \frac{f(x) - f(y)}{x - y} > -1;$

jjj) $\displaystyle\limsup_{|x| \to \infty} \left| \frac{f(x)}{x} \right| < 1.$

Then, for every $r \in \mathbb{R}$, the dynamical system governed by the function g_r is plus-globally stable.

Proof. The plus-global stability will be established when $r = 0$ ($g_0 = f$). In a similar manner we can analyze the situation $r \neq 0$.

From assumption j) we derive, using Proposition 3.3, that the function f has one and only one fixed point α. Let $\{y_n\colon n = 0, 1, \dots\}$ be such that

$$\lim_{n \to \infty} y_{n+1} - f(y_n) = 0. \tag{40}$$

We need to show that

$$\lim_{n \to \infty} y_n = \alpha. \tag{41}$$

We first prove that the sequence $\{y_n\colon n = 0, 1, \dots\}$ is bounded. From jjj) and from the continuity of f we derive that there are constants $P \geq 0$ and $k \in [0, 1)$ such that

$$|f(x)| \leq P + k|x| \tag{42}$$

for all $x \in \mathbb{R}$. Since

$$\lim_{n \to \infty} y_{n+1} - f(y_n) = 0,$$

inequality (42) implies the existence of a non-negative constant Q and $n_0 \in \mathbb{N}$ such that

$$|y_{n+1}| \leq Q + k|y_n| \tag{43}$$

for all $n \geq n_0$. Hence, the sequence is bounded.

Let L be the set of its limit points. Notice that the sequence $\{y_n\colon n = 0, 1, \dots\}$ may not be an orbit. However, it is known (see, for example, [7]) that L is compact and $f(L) = L$. Define

$$\beta_1 = \liminf_{n \to \infty} y_n\ ,\ \ \beta_2 = \limsup_{n \to \infty} y_n.$$

It is easy to verify that $\beta_1 \leq \alpha \leq \beta_2$. Our goal is to prove that $\beta_1 = \beta_2 = \alpha$. Let us show that

$$\beta_1 < \beta_2, \tag{44}$$

leads to a contradiction. Observe first that we cannot have

$$\beta_1 = \alpha < \beta_2 \tag{45}$$

since, by the Intermediate Value Theorem, we could find a fixed point of f different from α. Analogously, we cannot have

$$\beta_1 < \alpha = \beta_2. \tag{46}$$

It remains to show that we cannot have $\beta_1 < \alpha < \beta_2$. In fact, using once more the Intermediate Value Theorem, we can prove that f has a fixed point different from α either in the case when there exists $a \in (\beta_1, \alpha)$ such that $f(a) = \beta_1$, or in the case when we can find $b \in (\alpha, \beta_2)$ such that $f(b) = \beta_2$ (recall that $f(L) = L$). Hence, the only possibility is to have $a \in (\beta_1, \alpha)$ and $b \in (\alpha, \beta_2)$ such that $f(a) = \beta_2$ and $f(b) = \beta_1$. This however, contradicts assumption jj) of the theorem. Hence, $\beta_1 = \beta_2 = \alpha$. It follows that (41) holds and the dynamical system governed by $f = g_0$ is plus-globally stable. □

4. An Application to Foreward Neural Networks

We now present an application of the theory established in the previous sections to dynamical systems governed by lower triangular maps. This type of systems are used to model the evolution of forward neural networks (see [5, 10, 11]). We would like to point out that the purpose of this section is not to investigate the most general situation regarding global convergence of lower (or upper) triangular maps. We just want to underline how the theory presented in the previous sections can be used to study this class of problems.

Let $F\colon \mathbb{R}^q \to \mathbb{R}^q$ be of the form

$$F(\mathbf{x}) = (f_1(x_1), f_2(x_1, x_2), \ldots, f_q(x_1, \ldots, x_q)) + \mathbf{x}_I \tag{47}$$

where

$$\mathbf{x}_I = (x_{I,1}, \ldots, x_{I,q}). \tag{48}$$

Assume that $g_1(t) = f_1(t) + x_{I,1},\ t \in \mathbb{R}$, satisfies the assumptions of Theorem 3.7 and let α_1 be its unique fixed point. Next, consider $g_2(t) = f_2(\alpha_1, t) + x_{I,2},\ t \in \mathbb{R}$, and assume that also g_2 satisfies the assumptions of Theorem 3.7. Denote by α_2 its unique fixed point. Continuing in the manner just described we arrive at $g_q(t) = f_q(\alpha_1, \ldots, \alpha_{q-1}, t) + x_{I,q},\ t \in \mathbb{R}$. We assume that g_q

satisfies the assumptions of Theorem 3.7 and we denote by α_q its unique fixed point. Notice that

$$F(\alpha_1, \dots, \alpha_q) = (\alpha_1, \dots, \alpha_q) := \boldsymbol{\alpha} \tag{49}$$

i.e. $\boldsymbol{\alpha}$ is a fixed point of F. The dynamical system governed by $F\colon \mathbb{R}^q \to \mathbb{R}^q$ is said to be *globally stable* (or, simply, F is globally stable) if for every $\mathbf{x}_0 \in \mathbb{R}^q$, the orbit $O(\mathbf{x}_0)$ converges to $\boldsymbol{\alpha}$.

The following result holds.

THEOREM 4.1. *Assume that every $f_i,\ i \in \{2, \dots, q\}$, is uniformly continuous, $\mathbf{x}_I = (x_{I,1}, \dots, x_{I,q})$ is a given point of $\mathbb{R}^q$ and the functions $g_i,\ i \in \{1, \dots, q\}$, satisfy the assumptions of Theorem 3.7 with fixed points $\alpha_1, \dots, \alpha_q$, respectively. Let $F\colon \mathbb{R}^q \to \mathbb{R}^q$ be defined as in* (47). *Then the dynamical system governed by F is globally stable.*

Proof. Let $\mathbf{x}_0$ be a point of $\mathbb{R}^q$ and consider the sequence of iterates of (47) starting from $\mathbf{x}_0 = (x_{1,0}, \dots, x_{q,0})$. We have

$$\begin{aligned} x_{1,n+1} &= f_1(x_{1,n}) + x_{I,1}\ , \\ x_{2,n+1} &= f_2(x_{1,n}, x_{2,n}) + x_{I,2}\ , \\ & \dots \\ x_{q,n+1} &= f_q(x_{1,n}, \dots, x_{q,n}) + x_{I,q}. \end{aligned}$$

Since g_1 is plus-globally stable we obtain that

$$\lim_{n\to\infty} x_{1,n+1} = \alpha_1. \tag{50}$$

Moreover,

$$x_{2,n+1} = g_2(\alpha_1, x_{2,n}) + \delta_{2,n} \tag{51}$$

where

$$\delta_{2,n} = f_2(x_{1,n}, x_{2,n}) - f_2(\alpha_1, x_{2,n}).$$

The uniform continuity of f_2 implies that

$$\lim_{n\to\infty} \delta_{2,n} = 0. \tag{52}$$

Since g_2 is plus-globally stable we obtain

$$\lim_{n\to\infty} x_{2,n+1} = \alpha_2.$$

Continuing in the same manner we conclude that

$$\lim_{n\to\infty} x_{q,n+1} = \alpha_q. \tag{53}$$

Therefore,

$$\lim_{n\to\infty} \mathbf{x}_{n+1} = \boldsymbol{\alpha}, \tag{54}$$

and Theorem 4.1 is fully established. □

Remark 4.2. Notice that a sequence $\{\mathbf{z}_n,\ n=1,2,\dots\}$ such that

$$\lim_{n\to+\infty} \mathbf{z}_{n+1} - F(\mathbf{z}_n) = \mathbf{0} \tag{55}$$

need not be bounded. In the case when all such sequences are bounded, for example when $\limsup_{\|\mathbf{z}\|\to+\infty} \frac{\| F(\mathbf{z}) \|}{\| \mathbf{z} \|} < 1$, and with the same assumptions of Theorem 4.1, we obtain that the dynamical system governed by F is plus-globally stable.

Remark 4.3. The neural networks studied in [5, 10, 11] are of the form

$$H(\mathbf{x}) = (h_{11}(x_1), h_{21}(x_1) + h_{22}(x_2), \dots, h_{q1}(x_1) + \cdots + h_{qq}(x_q)) + \mathbf{x}_I \tag{56}$$

where $H\colon \mathbb{R}^q \to \mathbb{R}^q$ is continuous and h_{kk} is a contraction for every $k = 1, \dots, q$. It can be easily proved that the functions

$$g_k(t) = h_{k1}(\alpha_1) + \cdots + h_{k,k-1}(\alpha_{k-1}) + h_{kk}(t) + x_{I,k},\ \ k = 1, \dots, q$$

are plus-globally stable. Moreover, in this particular case, the uniform continuity of the functions $f_i,\ i \in \{2, \dots, q\}$, is no longer needed and it can be replaced by the simple continuity. The proof of Theorem 4.1 is modified accordingly. For more general situations one can consult [1, 14].

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