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Asymptotic behavior for the elasticity system with a nonlinear dissipative term

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Abstract. *We study the asymptotic behavior of an elasticity problem with a nonlinear dissipative term in a bidimensional thin domain* Ω^{ε}. *We prove some convergence results when the thickness tends to zero. The specific Reynolds limit equation and the limit of Tresca free boundary conditions are obtained.*

Keywords: A priori estimates, dissipative term, elasticity system, Tresca Law, Reynolds equation, variational formulation.
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1. Introduction and mathematical model

The topic dealing with propagation of elastic waves with dissipative term is a subject of considerable interest due to its industrial applications such as the dynamics of rubbers, silicones, and gels. Furthermore, in quantum mechanics the dissipation term determines the phenomenon according to which a dynamic system (wave, oscillation...) loses energy with time, where this energy turns into heat. Heat production occurs usually when there is friction between two bodies, and is mathematically modeled adding to the equation of motion a term dependent on the velocity. From a theoretical point of view, the mathematics and mechanics of wave phenomena with dissipation is a classical yet still active subject of research, where many studies have been published in this field. We cite among these the article [15], where Lions studied theoretically the problem for the wave equation with Dirichlet boundary conditions and a nonlinear dissipative term $\left|\frac{\partial u}{\partial t}\right|^{p}\frac{\partial u}{\partial t}$, in which the author proved the existence and the uniqueness of the solution. In [11] Georgiev and Todorova studied the nonlinear wave equation involving the nonlinear damping term $\left|\frac{\partial u}{\partial t}\right|^{m-1}\frac{\partial u}{\partial t}$ and a source term of type $|u|^{p-1}\ u$, from large initial data, they proved a global existence theorem for $1 < p \leq m$ and a blow-up result for $1 < m < p$. In [3], Benaissa and Messaoudi studied the stability of solutions to the nonlinear wave equation with the nonlinear dissipative term $\alpha\left(1+\left|\frac{\partial u}{\partial t}\right|^{m-2}\right)\frac{\partial u^{\varepsilon}}{\partial t}$ and proved for his solution that energy decays exponentially. Lagnese [13] , proved some

uniform stability results of elasticity systems with linear dissipative term.
In our paper, we study the asymptotic behavior of the hyperbolic equation governed by a thin, isotropic and homogeneous elastic membrane in the dynamic regime with a dissipative term $\left(\alpha^{\varepsilon}+\left|\frac{\partial u^{\varepsilon}}{\partial t}\right|\right)\frac{\partial u^{\varepsilon}}{\partial t}$ in a two dimensional thin domain Ω^{ε}. It is worth noting that the boundary conditions for our problem consist of two conditions: The first is Dirichlet boundary condition on the top and lateral parts, the other condition is Tresca's friction law over lower part of the border. This friction law has a threshold of friction (coefficient of friction) k^{ε}, when the elastic membrane and the foundation are in contact, the foundation exerts on the elastic membrane a tangential effort which does not exceed the threshold k^{ε}. As long as the tangential stress has not reached the threshold k^{ε}, the elastic membrane can not move relative to the foundation and there is blockage. When this threshold is reached, the elastic membrane can move tangentially relative to the foundation and then there is a slip. Some research for initial and boundary value problems involving Tresca friction law can be found in [10, 17].
In the literature, the asymptotic behavior of partial differential equations in a thin domain, particularly those governed by elastic systems has been widely studied. Ciarlet and Destuynder [9] studied equilibrium states of a thin plate $\Omega\times(-\varepsilon,+\varepsilon)$ under external forces where Ω is a smooth domain in $\mathbb{R}^2$ and ε is a small parameter, to justify the two-dimensional model of the plates. In the paper [16] Paumier studied the asymptotic modeling of a thin elastic plate in unilateral contact with friction against a rigid obstacle (Signorini problem with friction) where he proved that any family of solutions of the three-dimensional problem of Signorini with friction strongly converges towards an unique solution of a two-dimensional problem of plate of the type Signorini without friction. Léger and Miara in [14] justified of a mechanical model for an elastic shallow shell in frictionless unilateral contact with an obstacle using the asymptotic analysis. In [5, 6] Benseridi and Dilmi studied the asymptotic analysis of linear elasticity with the nonlinear terms $|u^{\varepsilon}|^{p-2}u^{\varepsilon}$ in the stationary case, in [4] they analyzed the asymptotic behavior of a dynamical problem of isothermal elasticity with nonlinear friction of Tresca's type but without including the nonlinear dissipative term. Bayada and Lhalouani [2] investigated the asymptotic and numerical analysis for a unilateral contact problem with Coulomb's friction between an elastic body and a thin elastic soft layer. The reader can also review some articles that are interested in studying the asymptotic analysis of some fluid mechanics problems in a thin domain for the stationary case [1, 7, 8].
Our paper is structured as follows. In Section 1 we present the form of the domain Ω^{ε}, then we give the basic equations. In Section 2 we derive the variational formulation of the problem and give the theorem of existence and uniqueness of the weak solution. In Section 3 by a scale change we carry out the asymptotic analysis, in which the small parameter (thickness) of the domain tends to

zero. Using Gronwall's lemma and Korn's inequality we establish some parameter independent estimates for the displacement and velocity fields. Finally in Section 4 we go to the limit when the thickness tends to zero, we derive the convergence theorem and find the limiting problem, for which we study the solution.

Let Ω^ε be a bounded domain of $\mathbb{R}^2$, where ε is a small parameter which ultimately will tend to zero, the boundary of Ω^ε will be denoted by $\Gamma^\varepsilon = \bar{\Gamma}_1^\varepsilon \cup \bar{\Gamma}_L^\varepsilon \cup \bar{\omega}$, where Γ_1^ε is the upper boundary of equation $y = \varepsilon h(x)$, $\Gamma_L^\varepsilon = \{x = 0\} \cup \{x = l\}$ is the lateral boundary and $\omega =]0, l[$ is a bounded interval, which constitutes the bottom of the domain Ω^ε. For all $x' = (x, y) \in \mathbb{R}^2$, the domain Ω^ε is given by

$$\Omega^\varepsilon = \{x' \in \mathbb{R}^2 : 0 < x < l,\ 0 < y < \varepsilon h(x)\},$$

where $h(.)$ is a function of class C^1 defined on $[0, l]$ such that

$$0 < \underline{h} = h_{min} \leq h(x) \leq h_{max} = \overline{h},\ \forall x \in [0, l]\,.$$

Let $u^\varepsilon(x', t)$ be the displacement field, then the law of elastic behavior is given by

$$\sigma_{ij}^\varepsilon(u^\varepsilon) = 2\mu d_{ij}(u^\varepsilon) + \lambda \sum_{k=1}^{2} d_{kk}(u^\varepsilon)\delta_{ij},$$
$$1 \leq i, j \leq 2;\ d_{ij}(u^\varepsilon) = \frac{1}{2}\left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i}\right),$$

where δ_{ij} is the Krönecker symbol, λ, μ are the Lamé constants and $d_{ij}(.)$ the strain tensor.

The equation which governs the deformations of an isotropic elastic homogeneous body with a nonlinear dissipative term in dynamic regime is the following

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} - \operatorname{div}\left(\sigma^\varepsilon\left(u^\varepsilon\right)\right) + \left(\alpha^\varepsilon + \left|\frac{\partial u^\varepsilon}{\partial t}\right|\right)\frac{\partial u^\varepsilon}{\partial t} = f^\varepsilon, \text{ in } \Omega^\varepsilon \times]0, T[\,, \tag{1}$$

where $|.|$ denotes the Euclidean norm of $\mathbb{R}^2$, f^ε represents a force density and $\alpha^\varepsilon \in \mathbb{R}_+$.

To describe the boundary conditions we use the usual notation

$$u_n^\varepsilon = u^\varepsilon.n,\ \ u_\tau^\varepsilon = u^\varepsilon - u_n^\varepsilon.n,\ \ \sigma_n^\varepsilon = \left(\sigma^\varepsilon.n\right).n,\ \ \sigma_\tau^\varepsilon = \sigma^\varepsilon.n - \left(\sigma_n^\varepsilon\right).n,$$

where $n = (n_1, n_2)$ is the unit outward normal to Γ^ε.

- The displacement is known on $\Gamma_1^\varepsilon \times]0, T[$ and on $\Gamma_L^\varepsilon \times]0, T[$

$$\begin{aligned} &u^\varepsilon\left(x, h\left(x\right), t\right) = 0 \ \text{ on } \ \Gamma_1^\varepsilon \times]0, T[, \\ &u^\varepsilon\left(0, y, t\right) = u^\varepsilon\left(l, y, t\right) = 0 \ \text{ on } \ \Gamma_L^\varepsilon \times]0, T[. \end{aligned} \tag{2}$$

• On ω the velocity is assumed unknown and satisfies the following condition

$$\frac{\partial u^{\varepsilon}}{\partial t}.n=0 \quad \text{on} \quad]0,l[\times]0,T[. \tag{3}$$

• There exists friction on ω, this friction is modeled by the nonlinear Tresca's law (see [10])

$$\left.\begin{array}{l} |\sigma_{\tau}^{\varepsilon}|<k^{\varepsilon}\Rightarrow\left(\frac{\partial u^{\varepsilon}}{\partial t}\right)_{\tau}=0, \\ |\sigma_{\tau}^{\varepsilon}|=k^{\varepsilon}\Rightarrow\exists\beta>0 \text{ such that } \left(\frac{\partial u^{\varepsilon}}{\partial t}\right)_{\tau}=-\beta\sigma_{\tau}^{\varepsilon} \end{array}\right\} \text{ on }]0,l[\times]0,T[, \tag{4}$$

where $k^{\varepsilon}\in C_0^{\infty}(]0,l[)$, $k^{\varepsilon}>0$ does not depend of t.

The problem consists in finding u^{ε} satisfying (1)-(4) and the following initial conditions

$$u^{\varepsilon}(x',0)=\vartheta_0(x'), \quad \frac{\partial u^{\varepsilon}}{\partial t}(x',0)=\vartheta_1(x'), \, \forall x'\in\Omega^{\varepsilon}. \tag{5}$$

2. Weak formulation

Let $L^p(\Omega)$ be the space of real scalar or real vector functions on Ω whose p^{th} power is absolutely integrable with respect to Lebesgue measure dx'. This is a Banach space with the norm

$$\|u\|_{L^p(\Omega)}=\left(\int_{\Omega}|u|^p\,dx'\right)^{\frac{1}{p}}, \, 1\leq p<\infty.$$

The Sobolev space $H^1(\Omega)$ is the space of functions in $L^2(\Omega)$ with first order distributional derivatives also in $L^2(\Omega)$. The norm of this space is

$$\|u\|_{H^1(\Omega)}=\left(\|u\|_{L^2(\Omega)}^2+\|\nabla u\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}}.$$

To find the weak formulation, we recall that Tresca's boundary condition (4) is equivalent to

$$\left(\frac{\partial u^{\varepsilon}}{\partial t}\right)_{\tau}.\sigma_{\tau}^{\varepsilon}+k^{\varepsilon}\left|\left(\frac{\partial u^{\varepsilon}}{\partial t}\right)_{\tau}\right|=0. \tag{6}$$

Multiplying (1) by $\left(\varphi - \frac{\partial u^\varepsilon}{\partial t}\right)$ where φ is test-function, then integrating over Ω^ε and using the Green's formula, we obtain

$$\begin{aligned}
&\int_{\Omega^\varepsilon} \frac{\partial^2 u^\varepsilon}{\partial t^2}\left(\varphi - \frac{\partial u^\varepsilon}{\partial t}\right) dx' + \int_{\Omega^\varepsilon} \sigma^\varepsilon.\nabla\left(\varphi - \frac{\partial u^\varepsilon}{\partial t}\right) dx' \\
&\quad - \int_{\Gamma^\varepsilon} \sigma^\varepsilon.n\left(\varphi - \frac{\partial u^\varepsilon}{\partial t}\right) dx' + \int_{\Omega^\varepsilon}\left(\alpha^\varepsilon + \left|\frac{\partial u^\varepsilon}{\partial t}\right|\right)\frac{\partial u^\varepsilon}{\partial t}\left(\varphi - \frac{\partial u^\varepsilon}{\partial t}\right) dx' \\
&\qquad = \int_{\Omega^\varepsilon} f^\varepsilon\left(\varphi - \frac{\partial u^\varepsilon}{\partial t}\right) dx', \quad (7)
\end{aligned}$$

on the other hand, the boundary condition (2)-(3) implies that

$$\int_{\Gamma^\varepsilon} \sigma^\varepsilon.n\left(\varphi - \frac{\partial u^\varepsilon}{\partial t}\right) dx' = \int_0^l \sigma_\tau^\varepsilon\left(\varphi_\tau - \left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau\right) dx,$$

going back to (7), we get

$$\begin{aligned}
&\int_{\Omega^\varepsilon} \frac{\partial^2 u^\varepsilon}{\partial t^2}\left(\varphi - \frac{\partial u^\varepsilon}{\partial t}\right) dx' + \int_{\Omega^\varepsilon} \sigma^\varepsilon.\nabla\left(\varphi - \frac{\partial u^\varepsilon}{\partial t}\right) dx' \\
&+ \int_{\Omega^\varepsilon}\left(\alpha^\varepsilon + \left|\frac{\partial u^\varepsilon}{\partial t}\right|\right)\frac{\partial u^\varepsilon}{\partial t}\left(\varphi - \frac{\partial u^\varepsilon}{\partial t}\right) dx' + \int_0^l k^\varepsilon\left(|\varphi_\tau| - \left|\left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau\right|\right) dx \\
&\quad = \int_{\Omega^\varepsilon} f^\varepsilon\left(\varphi - \frac{\partial u^\varepsilon}{\partial t}\right) dx' + \int_0^l \sigma_\tau^\varepsilon\left(\varphi_\tau - \left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau\right) dx \\
&\qquad + \int_0^l k^\varepsilon\left(|\varphi_\tau| - \left|\left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau\right|\right) dx.
\end{aligned}$$

Using (6) and the fact that

$$\int_0^l \sigma_\tau^\varepsilon\left(\varphi_\tau - \left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau\right) dx + \int_0^l k^\varepsilon\left(|\varphi_\tau| - \left|\left(\frac{\partial u^\varepsilon}{\partial t}\right)_\tau\right|\right) dx \geq 0\,,$$

we get the following variational formulation

$$\left\{\begin{aligned}
&\text{Find } u^\varepsilon, \text{ with } u^\varepsilon(\cdot,t), \frac{\partial u^\varepsilon}{\partial t}(\cdot,t) \in K^\varepsilon \text{ for all } t \in [0,T]\,, \text{ such that} \\
&\left(\frac{\partial^2 u^\varepsilon}{\partial t^2}, \varphi - \frac{\partial u^\varepsilon}{\partial t}\right) + a\left(u^\varepsilon, \varphi - \frac{\partial u^\varepsilon}{\partial t}\right) + \alpha^\varepsilon\left(\frac{\partial u^\varepsilon}{\partial t}, \varphi - \frac{\partial u^\varepsilon}{\partial t}\right) \\
&\qquad + \left(\left|\frac{\partial u^\varepsilon}{\partial t}\right|\frac{\partial u^\varepsilon}{\partial t}, \varphi - \frac{\partial u^\varepsilon}{\partial t}\right) + j^\varepsilon(\varphi) - j^\varepsilon\left(\frac{\partial u^\varepsilon}{\partial t}\right) \\
&\qquad\qquad \geq \left(f^\varepsilon, \varphi - \frac{\partial u^\varepsilon}{\partial t}\right),\ \forall \varphi \in K^\varepsilon, \\
&u^\varepsilon(x',0) = \vartheta_0(x'), \qquad \frac{\partial u^\varepsilon}{\partial t}(x',0) = \vartheta_1(x'),
\end{aligned}\right. \quad (8)$$

where

$$K^{\varepsilon} = \left\{ v \in H^1(\Omega^{\varepsilon})^2 : v = 0 \text{ on } \Gamma_1^{\varepsilon} \cup \Gamma_L^{\varepsilon},\ v.n = 0 \text{ on } \omega \right\},$$

$$j^{\varepsilon}(v) = \int_0^l k^{\varepsilon} \, |v| \, dx, \ \forall v \in H^1(\Omega^{\varepsilon})^2,$$

$$a(u,v) = 2\mu \int_{\Omega^{\varepsilon}} d\,(u)\, d\,(v)\, dxdy + \lambda \int_{\Omega^{\varepsilon}} \operatorname{div}(u) \operatorname{div}(v)\, dxdy,$$

with

$$d\,(u)\, d\,(v) = \sum_{i,j=1}^{2} d_{ij}\,(u)\,.d_{ij}\,(v)\,.$$

THEOREM 2.1. *Under the assumptions*

$$f^{\varepsilon}, \frac{\partial f^{\varepsilon}}{\partial t}, \frac{\partial^2 f^{\varepsilon}}{\partial t^2} \in L^2\left(0, T, L^2\left(\Omega^{\varepsilon}\right)^2\right),$$

$$\vartheta_0 \in H^1(\Omega^{\varepsilon})^2, \ \ \vartheta_1 \in H^1(\Omega^{\varepsilon})^2, \ \ (\vartheta_1)_{\tau} = 0, \tag{9}$$

there exists a unique solution u^{ε} of (8) such that

$$u^{\varepsilon}, \frac{\partial u^{\varepsilon}}{\partial t} \in L^{\infty}\left(0, T, H^1(\Omega^{\varepsilon})^2\right),$$

$$\frac{\partial^2 u^{\varepsilon}}{\partial t^2} \in L^{\infty}\left(0, T, L^2\left(\Omega^{\varepsilon}\right)^2\right).$$

The proof of this theorem proceeds in a similar fashion as in Lions [10, 15].

3. Change of the domain and some estimates

In this section, we use the technique of scaling $z = y/\varepsilon$ for studying the asymptotic analysis of the problem (8). This method consists in transposing the initial problem posed in the domain Ω^{ε} to an equivalent problem posed in a fixed domain Ω independent of ε :

$$\Omega = \{(x,z) \in \mathbb{R}^2 : 0 < x < l,\ 0 < z < h(x)\},$$

and $\Gamma = \Gamma_1 \cup \Gamma_L \cup \omega$ its boundary. We define on Ω the new unknowns and the data

$$\begin{cases} \hat{u}_1^{\varepsilon}\,(x,z,t) = u_1^{\varepsilon}\,(x,y,t)\,, \\ \hat{u}_2^{\varepsilon}\,(x,z,t) = \varepsilon^{-1} u_2^{\varepsilon}\,(x,y,t)\,, \end{cases}$$

$$\begin{cases} \hat{f}_i\,(x,z,t) = \varepsilon^2 f_i^{\varepsilon}\,(x,y,t)\,,\ i = 1,2, \\ \hat{k} = \varepsilon k^{\varepsilon},\ \hat{\alpha} = \varepsilon^2 \alpha^{\varepsilon}, \end{cases}$$

where $\hat{f}_i$, $i = 1,2$, $\hat{k}$ and $\hat{\alpha}$ do not depend on ε.

Moreover, we define some function spaces on Ω

$$K=\left\{\varphi\in H^{1}(\Omega)^{2}:\varphi=0 \text{ on } \Gamma_1\cup\Gamma_L \text{ and } \varphi.n=0 \text{ on } \omega\right\},$$

$$\Pi(K)=\left\{\varphi\in H^{1}(\Omega):\varphi=0 \text{ on } \Gamma_1\cup\Gamma_L\right\},$$

$$V_z=\left\{v\in L^{2}(\Omega):\frac{\partial v}{\partial z}\in L^{2}(\Omega),\ v=0 \text{ on } \Gamma_1\right\},$$

V_z is a Banach space for the norm

$$\|v\|_{V_z}=\left(\|v\|_{L^2(\Omega)}^{2}+\left\|\frac{\partial v}{\partial z}\right\|_{L^2(\Omega)}^{2}\right)^{\frac{1}{2}}.$$

Multiplying (8) by ε then we inject the new variables and the new data, we obtain the following variational formulation on the fixed domain Ω.

$$\begin{cases}\text{Find } \hat{u}^{\varepsilon}, \text{ with } \hat{u}^{\varepsilon}(\cdot,t), \dfrac{\partial \hat{u}^{\varepsilon}}{\partial t}(\cdot,t)\in K \text{ for all } t\in[0,T], \text{ such that}\\ \varepsilon^{2}\left(\dfrac{\partial^{2}\hat{u}_1^{\varepsilon}}{\partial t^{2}},\hat{\varphi}_1-\dfrac{\partial \hat{u}_1^{\varepsilon}}{\partial t}\right)+\varepsilon^{4}\left(\dfrac{\partial^{2}\hat{u}_2^{\varepsilon}}{\partial t^{2}},\hat{\varphi}_2-\dfrac{\partial \hat{u}_2^{\varepsilon}}{\partial t}\right)+\hat{a}\left(\hat{u}^{\varepsilon},\hat{\varphi}-\dfrac{\partial \hat{u}^{\varepsilon}}{\partial t}\right)\\ \quad+\hat{\alpha}\left(\dfrac{\partial \hat{u}_1^{\varepsilon}}{\partial t},\hat{\varphi}_1-\dfrac{\partial \hat{u}_1^{\varepsilon}}{\partial t}\right)+\hat{\alpha}\varepsilon^{2}\left(\dfrac{\partial \hat{u}_2^{\varepsilon}}{\partial t},\hat{\varphi}_2-\dfrac{\partial \hat{u}_2^{\varepsilon}}{\partial t}\right)\\ \quad\quad+\varepsilon^{2}\left(\left[\left(\dfrac{\partial \hat{u}_1^{\varepsilon}}{\partial t}\right)^{2}+\left(\varepsilon\dfrac{\partial \hat{u}_2^{\varepsilon}}{\partial t}\right)^{2}\right]^{\frac{1}{2}}\dfrac{\partial \hat{u}_1^{\varepsilon}}{\partial t},\hat{\varphi}_1-\dfrac{\partial \hat{u}_1^{\varepsilon}}{\partial t}\right)\\ \quad\quad\quad+\varepsilon^{4}\left(\left[\left(\dfrac{\partial \hat{u}_1^{\varepsilon}}{\partial t}\right)^{2}+\left(\varepsilon\dfrac{\partial \hat{u}_2^{\varepsilon}}{\partial t}\right)^{2}\right]^{\frac{1}{2}}\dfrac{\partial \hat{u}_2^{\varepsilon}}{\partial t},\hat{\varphi}_2-\dfrac{\partial \hat{u}_2^{\varepsilon}}{\partial t}\right)\\ \quad\quad\quad\quad+\hat{J}(\hat{\varphi})-\hat{J}\left(\dfrac{\partial \hat{u}^{\varepsilon}}{\partial t}\right)\\ \quad\quad\quad\quad\quad\geq\left(\hat{f}_1,\hat{\varphi}_1-\dfrac{\partial \hat{u}_1^{\varepsilon}}{\partial t}\right)+\varepsilon\left(\hat{f}_2,\hat{\varphi}_2-\dfrac{\partial \hat{u}_2^{\varepsilon}}{\partial t}\right),\ \forall\hat{\varphi}\in K,\\ \hat{u}^{\varepsilon}(0)=\hat{\vartheta}_0,\quad \dfrac{\partial \hat{u}^{\varepsilon}}{\partial t}(0)=\hat{\vartheta}_1,\end{cases}\tag{10}$$

where

$$\hat{J}(\hat{\varphi})=\int_0^{l}\hat{k}\,|\hat{\varphi}|\,dx,$$

and

$$\hat{a}\left(\hat{u}^{\varepsilon},\hat{\varphi}-\frac{\partial\hat{u}^{\varepsilon}}{\partial t}\right)=2\mu\varepsilon^{2}\int_{\Omega}\left(\frac{\partial\hat{u}_{1}^{\varepsilon}}{\partial x}\right)\frac{\partial}{\partial x}\left(\hat{\varphi}_{1}-\frac{\partial\hat{u}_{1}^{\varepsilon}}{\partial t}\right)dxdz$$
$$+\mu\int_{\Omega}\left(\frac{\partial\hat{u}_{1}^{\varepsilon}}{\partial z}+\varepsilon^{2}\frac{\partial\hat{u}_{2}^{\varepsilon}}{\partial x}\right)\left[\frac{\partial}{\partial z}\left(\hat{\varphi}_{1}-\frac{\partial\hat{u}_{1}^{\varepsilon}}{\partial t}\right)+\varepsilon^{2}\frac{\partial}{\partial x}\left(\hat{\varphi}_{2}-\frac{\partial\hat{u}_{2}^{\varepsilon}}{\partial t}\right)\right]dxdz$$
$$+2\mu\varepsilon^{2}\int_{\Omega}\frac{\partial\hat{u}_{2}^{\varepsilon}}{\partial z}\cdot\frac{\partial}{\partial z}\left(\hat{\varphi}_{2}-\frac{\partial\hat{u}_{2}^{\varepsilon}}{\partial t}\right)dxdz$$
$$+\lambda\varepsilon^{2}\int_{\Omega}\operatorname{div}\left(\hat{u}^{\varepsilon}\right)\cdot\operatorname{div}\left(\hat{\varphi}-\frac{\partial\hat{u}^{\varepsilon}}{\partial t}\right)dxdz.$$

For the rest of this paper, we will denote by c possibly different positive constants and we establish some estimates for the displacement field $\hat{u}^{\varepsilon}$ in the domain Ω.

THEOREM 3.1. *Under the hypotheses of Theorem 2.1, there exists a constant c independent of ε such that*

$$\left\|\frac{\partial\hat{u}_{1}^{\varepsilon}}{\partial z}\right\|_{L^{2}(\Omega)}^{2}+\left\|\varepsilon\frac{\partial\hat{u}_{1}^{\varepsilon}}{\partial t}\right\|_{L^{2}(\Omega)}^{2}+\left\|\varepsilon^{2}\frac{\partial\hat{u}_{2}^{\varepsilon}}{\partial x}\right\|_{L^{2}(\Omega)}^{2}+\left\|\varepsilon\frac{\partial\hat{u}_{1}^{\varepsilon}}{\partial x}\right\|_{L^{2}(\Omega)}^{2}+\left\|\varepsilon\frac{\partial\hat{u}_{2}^{\varepsilon}}{\partial z}\right\|_{L^{2}(\Omega)}^{2}+\left\|\varepsilon^{2}\frac{\partial\hat{u}_{2}^{\varepsilon}}{\partial t}\right\|_{L^{2}(\Omega)}^{2}$$
$$+\left\|\varepsilon^{\frac{2}{3}}\frac{\partial\hat{u}_{1}^{\varepsilon}}{\partial t}\right\|_{L^{3}(0,T,L^{3}(\Omega))}^{3}+\left\|\varepsilon^{\frac{5}{3}}\frac{\partial\hat{u}_{2}^{\varepsilon}}{\partial t}\right\|_{L^{3}(0,T,L^{3}(\Omega))}^{3}\leq c,\quad(11)$$

$$\left\|\frac{\partial^{2}\hat{u}_{1}^{\varepsilon}}{\partial z\partial t}\right\|_{L^{2}(\Omega)}^{2}+\left\|\varepsilon\frac{\partial^{2}\hat{u}_{1}^{\varepsilon}}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}+\left\|\varepsilon^{2}\frac{\partial^{2}\hat{u}_{2}^{\varepsilon}}{\partial x\partial t}\right\|_{L^{2}(\Omega)}^{2}+\left\|\varepsilon\frac{\partial^{2}\hat{u}_{1}^{\varepsilon}}{\partial x\partial t}\right\|_{L^{2}(\Omega)}^{2}+\left\|\varepsilon\frac{\partial\hat{u}_{2}^{\varepsilon}}{\partial z\partial t}\right\|_{L^{2}(\Omega)}^{2}+\left\|\varepsilon^{2}\frac{\partial^{2}\hat{u}_{2}^{\varepsilon}}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\leq c.\quad(12)$$

Proof. First, we recall some inequalities

- Poincaré's inequality

$$\left\|u^{\varepsilon}\right\|_{L^{2}(\Omega^{\varepsilon})}\leq\varepsilon\bar{h}\left\|\nabla u^{\varepsilon}\right\|_{L^{2}(\Omega^{\varepsilon})}.$$

- Young's inequality

$$ab\leq\eta^{2}\frac{a^{2}}{2}+\eta^{-2}\frac{b^{2}}{2},\ \forall\left(a,b\right)\in\mathbb{R}^{2},\forall\eta>0.$$

- Korn's inequality [12]

$$\|d(u^\varepsilon)\|^2_{L^2(\Omega^\varepsilon)} \geq C_K \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)},$$

where $\bar{h}$ and C_K are constants independent of ε.
Let u^ε be a solution of the problem (8), we take $\varphi = 0$, then

$$\left(\frac{\partial^2 u^\varepsilon}{\partial t^2}, \frac{\partial u^\varepsilon}{\partial t}\right) + a\left(u^\varepsilon, \frac{\partial u^\varepsilon}{\partial t}\right) + \left(\left(\alpha^\varepsilon + \left|\frac{\partial u^\varepsilon}{\partial t}\right|\right)\frac{\partial u^\varepsilon}{\partial t}, \frac{\partial u^\varepsilon}{\partial t}\right) + j^\varepsilon\left(\frac{\partial u^\varepsilon}{\partial t}\right) \leq \left(f^\varepsilon, \frac{\partial u^\varepsilon}{\partial t}\right),$$

whence

$$\frac{1}{2}\frac{d}{dt}\left[\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)} + a(u^\varepsilon, u^\varepsilon)\right] + \alpha^\varepsilon \left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)} + \left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^3_{L^3(\Omega^\varepsilon)} \leq \left(f^\varepsilon, \frac{\partial u^\varepsilon}{\partial t}\right).$$

For $s \in]0, t[$ by integration we get

$$\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)} + a(u^\varepsilon, u^\varepsilon) + 2\int_0^t \left\|\frac{\partial u^\varepsilon(s)}{\partial t}\right\|^3_{L^3(\Omega^\varepsilon)} ds \leq \|\vartheta_1\|^2_{L^2(\Omega^\varepsilon)} + (2\mu + 3\lambda)\|\nabla \vartheta_0\|^2_{L^2(\Omega^\varepsilon)} + 2\int_0^t \left(f^\varepsilon(s), \frac{\partial u^\varepsilon(s)}{\partial t}\right) ds. \quad (13)$$

On the other hand, we have

$$2\int_0^t \left(f^\varepsilon(s), \frac{\partial u^\varepsilon(s)}{\partial t}\right) ds = 2(f^\varepsilon, u^\varepsilon) - 2(f^\varepsilon(0), \vartheta_0) - 2\int_0^t \left(\frac{\partial f^\varepsilon(s)}{\partial t}, u^\varepsilon(s)\right) ds,$$

using Poincaré's and Young's inequalities, we obtain

$$\left|2\int_0^t \left(f^\varepsilon(s), \frac{\partial u^\varepsilon(s)}{\partial t}\right) ds\right| \leq \mu C_K \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + \frac{4\varepsilon^2\bar{h}^2}{\mu C_K}\|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + 4\varepsilon^2\bar{h}^2\|f^\varepsilon(0)\|^2_{L^2(\Omega^\varepsilon)} + \|\nabla\vartheta_0\|^2_{L^2(\Omega^\varepsilon)} + \mu C_K \int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)} ds + \frac{4(\varepsilon\bar{h})^2}{\mu C_K}\int_0^t \left\|\frac{\partial f^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)} ds. \quad (14)$$

By inserting (14) in (13), and using Korn's inequality, we find

$$\begin{aligned}
&\left[\left\|\frac{\partial u^{\varepsilon}}{\partial t}\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}+\mu C_{K}\left\|\nabla u^{\varepsilon}\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}\right]+2\int_{0}^{t}\left\|\frac{\partial u^{\varepsilon}(s)}{\partial t}\right\|^{3}_{L^{3}(\Omega^{\varepsilon})}ds\\
&\quad\leq\left\|\vartheta_{1}\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}+(1+2\mu+3\lambda)\left\|\nabla\vartheta_{0}\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}+\frac{4\varepsilon^{2}\bar{h}^{2}}{\mu C_{K}}\left\|f^{\varepsilon}\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}\\
&\qquad+4\varepsilon^{2}\bar{h}^{2}\left\|f^{\varepsilon}(0)\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}+\frac{4\left(\varepsilon\bar{h}\right)^{2}}{\mu C_{K}}\int_{0}^{t}\left\|\frac{\partial f^{\varepsilon}(s)}{\partial t}\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}ds\\
&\qquad\quad+\int_{0}^{t}\left[\left\|\frac{\partial u^{\varepsilon}(s)}{\partial t}\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}+\mu C_{K}\left\|\nabla u^{\varepsilon}(s)\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}\right]ds. \qquad (15)
\end{aligned}$$

As

$$\varepsilon^{2}\left\|f^{\varepsilon}\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}=\varepsilon^{-1}\left\|\hat{f}\right\|^{2}_{L^{2}(\Omega)},$$

multiplying (15) by ε we deduce that

$$\begin{aligned}
&\varepsilon\left[\left\|\frac{\partial u^{\varepsilon}}{\partial t}\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}+\mu C_{K}\left\|\nabla u^{\varepsilon}\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}\right]+2\varepsilon\int_{0}^{t}\left\|\frac{\partial u^{\varepsilon}(s)}{\partial t}\right\|^{3}_{L^{3}(\Omega^{\varepsilon})}ds\\
&\qquad\leq\int_{0}^{t}\varepsilon\left[\left\|\frac{\partial u^{\varepsilon}(s)}{\partial t}\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}+\mu C_{K}\left\|\nabla u^{\varepsilon}(s)\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}\right]ds+A,
\end{aligned}$$

where A is a constant does not depend of ε with

$$\begin{aligned}
A=&\left\|\hat{\vartheta}_{1}\right\|^{2}_{L^{2}(\Omega)}+(1+2\mu+3\lambda)\left\|\nabla\hat{\vartheta}_{0}\right\|^{2}_{L^{2}(\Omega)}+4\bar{h}^{2}\left\|\hat{f}(0)\right\|^{2}_{L^{2}(\Omega)}\\
&+\frac{4\bar{h}^{2}}{\mu C_{K}}\left\|\hat{f}\right\|^{2}_{L^{\infty}\left(0,T,L^{2}(\Omega)^{2}\right)}+\frac{4\bar{h}^{2}}{\mu C_{K}}\left\|\frac{\partial\hat{f}}{\partial t}\right\|^{2}_{L^{2}\left(0,T,L^{2}(\Omega)^{2}\right)}.
\end{aligned}$$

Now using Gronwall's lemma, we have

$$\varepsilon\left\|\frac{\partial u^{\varepsilon}}{\partial t}\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}+\varepsilon\left\|\nabla u^{\varepsilon}\right\|^{2}_{L^{2}(\Omega^{\varepsilon})}\leq c,$$

from which (11) follows.

The functional $j^{\varepsilon}(\cdot)$ is convex but nondifferentiable. To overcome this difficulty, we shall use the following approach. Let $j^{\varepsilon}_{\zeta}(\cdot)$ be a functional defined by

$$j^{\varepsilon}_{\zeta}(v)=\int_{0}^{l}k^{\varepsilon}(x)\,\phi_{\zeta}\left(\left|v_{\tau}\right|^{2}\right)dx,\text{ where }\phi_{\zeta}(\lambda)=\frac{1}{1+\zeta}\left|\lambda\right|^{(1+\zeta)},\ \zeta>0.$$

To show (12) we consider the approximate equation as in [15]

$$\left(\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2},\varphi\right)+a(u_\zeta^\varepsilon,\varphi)+\left(\left(\alpha^\varepsilon+\left|\frac{\partial u_\zeta^\varepsilon}{\partial t}\right|\right)\frac{\partial u_\zeta^\varepsilon}{\partial t},\varphi\right)$$
$$+\left(\left(j_\zeta^\varepsilon\right)'\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}\right),\varphi\right)=\left(f^\varepsilon,\varphi\right),\forall\varphi\in K^\varepsilon,\qquad(16)$$

$$u_\zeta^\varepsilon(x',0)=\vartheta_0(x'),\quad \frac{\partial u_\zeta^\varepsilon\left(x',0\right)}{\partial t}=\vartheta_1(x').$$

We differentiate (16), in t and we take $\varphi=\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}$, we get

$$\left(\frac{\partial^3 u_\zeta^\varepsilon}{\partial t^3},\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right)+a\left(\frac{\partial u_\zeta^\varepsilon}{\partial t},\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right)+\alpha^\varepsilon\left(\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2},\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right)$$
$$+\left(\left|\frac{\partial u_\zeta^\varepsilon}{\partial t}\right|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2},\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right)+\left(\frac{\partial}{\partial t}\left(j_\zeta^\varepsilon\right)'\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}\right),\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right)=\left(\frac{\partial f^\varepsilon}{\partial t},\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right),$$

as $\left(\frac{\partial}{\partial t}\left(j_\zeta^\varepsilon\right)'\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}\right),\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right)\geq 0$, we have

$$\frac{1}{2}\frac{d}{dt}\left[\left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right\|_{L^2(\Omega^\varepsilon)}^2+a\left(\frac{\partial u_\zeta^\varepsilon}{\partial t},\frac{\partial u_\zeta^\varepsilon}{\partial t}\right)\right]\leq\left(\frac{\partial f^\varepsilon}{\partial t},\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right).$$

Integrating this inequality over $(0,t)$ and use Korn's inequality, we get

$$\left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right\|_{L^2(\Omega^\varepsilon)}^2+2\mu C_K\left\|\nabla\frac{\partial u_\zeta^\varepsilon}{\partial t}\right\|_{L^2(\Omega^\varepsilon)}^2$$
$$\leq\left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0)\right\|_{L^2(\Omega^\varepsilon)}^2+(2\mu+3\lambda)\left\|\nabla\vartheta_1\right\|_{L^2(\Omega^\varepsilon)}^2+2\left(\frac{\partial f^\varepsilon}{\partial t},\frac{\partial u_\zeta^\varepsilon}{\partial t}\right)$$
$$-2\left(\frac{\partial f^\varepsilon}{\partial t}(0),\vartheta_1\right)-2\int_0^t\left(\frac{\partial^2 f^\varepsilon(s)}{\partial t^2},\frac{\partial u_\zeta^\varepsilon(s)}{\partial t}\right)ds.$$

On the other hand, using Cauchy-Schwarz's, Poincaré's and Young's inequalities, we obtain

$$\begin{aligned} 2\left(\frac{\partial f^{\varepsilon}}{\partial t},\frac{\partial u_{\zeta}^{\varepsilon}}{\partial t}\right) &\leq 2\left\|\frac{\partial f^{\varepsilon}}{\partial t}\right\|_{L^2(\Omega^{\varepsilon})}\left\|\frac{\partial u_{\zeta}^{\varepsilon}}{\partial t}\right\|_{L^2(\Omega^{\varepsilon})} \\ &\leq 2\varepsilon\bar{h}\left\|\frac{\partial f^{\varepsilon}}{\partial t}\right\|_{L^2(\Omega^{\varepsilon})}\left\|\nabla\frac{\partial u_{\zeta}^{\varepsilon}}{\partial t}\right\|_{L^2(\Omega^{\varepsilon})} \\ &\leq \frac{4\left(\varepsilon\bar{h}\right)^2}{\mu C_K}\left\|\frac{\partial f^{\varepsilon}}{\partial t}\right\|^2_{L^2(\Omega^{\varepsilon})}+\mu C_K\left\|\nabla\frac{\partial u_{\zeta}^{\varepsilon}}{\partial t}\right\|^2_{L^2(\Omega^{\varepsilon})}. \end{aligned}$$

We use the same techniques for the other terms, so we will have the following inequality

$$\begin{aligned} &\left\|\frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}\right\|^2_{L^2(\Omega^{\varepsilon})}+2\mu C_K\left\|\nabla\frac{\partial u_{\zeta}^{\varepsilon}}{\partial t}\right\|^2_{L^2(\Omega^{\varepsilon})} \\ &\quad\leq\left\|\frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}(0)\right\|^2_{L^2(\Omega^{\varepsilon})}+(2\mu+3\lambda)\left\|\nabla\vartheta_1\right\|^2_{L^2(\Omega^{\varepsilon})} \\ &\quad+\mu C_K\left\|\nabla\frac{\partial u_{\zeta}^{\varepsilon}}{\partial t}\right\|^2_{L^2(\Omega^{\varepsilon})}+\frac{4\left(\varepsilon\bar{h}\right)^2}{\mu C_K}\left\|\frac{\partial f^{\varepsilon}}{\partial t}(0)\right\|^2_{L^2(\Omega^{\varepsilon})} \\ &\quad+\frac{4\left(\varepsilon\bar{h}\right)^2}{\mu C_K}\int_0^t\left\|\frac{\partial^2 f^{\varepsilon}(s)}{\partial t^2}\right\|^2_{L^2(\Omega^{\varepsilon})}ds+\mu C_K\left\|\nabla\vartheta_1\right\|^2_{L^2(\Omega^{\varepsilon})} \\ &\quad+\frac{4\left(\varepsilon\bar{h}\right)^2}{\mu C_K}\left\|\frac{\partial f^{\varepsilon}}{\partial t}\right\|^2_{L^2(\Omega^{\varepsilon})}+\mu C_K\int_0^t\left\|\nabla\frac{\partial u_{\zeta}^{\varepsilon}(s)}{\partial t}\right\|^2_{L^2(\Omega^{\varepsilon})}ds. \end{aligned} \tag{17}$$

Now let us estimate $\frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}(0)$. From (16) and (9) we deduce

$$\left(\frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}(0),\varphi\right)=\left(f^{\varepsilon}(0),\varphi\right)-a(\vartheta_0,\varphi)-\alpha^{\varepsilon}\left(\vartheta_1,\varphi\right)-\left(\left|\vartheta_1\right|\vartheta_1,\varphi\right),\forall\varphi\in K^{\varepsilon}.$$

Therefore

$$\begin{aligned}
&\left|\left(\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0),\varphi\right)\right| \\
&\quad\leq \varepsilon\bar{h}\left\|f^\varepsilon(0)\right\|_{L^2(\Omega^\varepsilon)}\left\|\nabla\varphi\right\|_{L^2(\Omega^\varepsilon)}+(2\mu+3\lambda)\left\|\vartheta_0\right\|_{H^1(\Omega^\varepsilon)}\left\|\varphi\right\|_{H^1(\Omega^\varepsilon)} \\
&\qquad+\alpha^\varepsilon\left\|\vartheta_1\right\|_{L^2(\Omega^\varepsilon)}\left\|\varphi\right\|_{L^2(\Omega^\varepsilon)}+\left(\int_{\Omega^\varepsilon}\left|\vartheta_1\right|^4dxdy\right)^{\frac{1}{2}}\left\|\varphi\right\|_{L^2(\Omega^\varepsilon)} \\
&\qquad\leq\left(\varepsilon\bar{h}\left\|f^\varepsilon(0)\right\|_{L^2(\Omega^\varepsilon)}+(2\mu+3\lambda)\left\|\vartheta_0\right\|_{H^1(\Omega^\varepsilon)}\right)\left\|\varphi\right\|_{H^1(\Omega^\varepsilon)} \\
&\qquad\quad+\left(\hat{\alpha}\bar{h}^2\left\|\nabla\vartheta_1\right\|_{L^2(\Omega^\varepsilon)}+\varepsilon\bar{h}\left(\int_{\Omega^\varepsilon}\left|\vartheta_1\right|^4dxdy\right)^{\frac{1}{2}}\right)\left\|\varphi\right\|_{H^1(\Omega^\varepsilon)}.
\end{aligned}$$

As $\varepsilon^{\frac{3}{2}}\left\|f^\varepsilon(0)\right\|_{L^2(\Omega^\varepsilon)}=\left\|\hat{f}(0)\right\|_{L^2(\Omega)}$, we multiply this last inequality by $\sqrt{\varepsilon}$. Then using Sobolev embedding $\|v\|_{L^4(\Omega)}\leq c_s\|v\|_{H^1(\Omega)}$, we get

$$\sqrt{\varepsilon}\left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0)\right\|_{L^2(\Omega^\varepsilon)}\leq C',$$

where

$$C'=\bar{h}\left\|\hat{f}(0)\right\|_{L^2(\Omega)}+(2\mu+3\lambda)\left\|\hat{\vartheta}_0\right\|_{H^1(\Omega)}+\hat{\alpha}\bar{h}^2\left\|\hat{\vartheta}_1\right\|_{H^1(\Omega)}+\bar{h}c_s\left\|\hat{\vartheta}_1\right\|_{H^1(\Omega)}^2$$

is independent of ε. Passing to the limit in (17) when ζ tends to zero, we find

$$\begin{aligned}
&\left[\left\|\frac{\partial^2 u^\varepsilon}{\partial t^2}\right\|_{L^2(\Omega^\varepsilon)}^2+\mu C_K\left\|\nabla\frac{\partial u^\varepsilon}{\partial t}\right\|_{L^2(\Omega^\varepsilon)}^2\right]\leq\left\|\frac{\partial^2 u^\varepsilon}{\partial t^2}(0)\right\|_{L^2(\Omega^\varepsilon)}^2 \\
&\quad+(2\mu+3\lambda+\mu C_K)\left\|\nabla\vartheta_1\right\|_{L^2(\Omega^\varepsilon)}^2+\frac{4\left(\varepsilon\bar{h}\right)^2}{\mu C_K}\left\|\frac{\partial f^\varepsilon}{\partial t}(0)\right\|_{L^2(\Omega^\varepsilon)}^2 \\
&\quad+\frac{4\left(\varepsilon\bar{h}\right)^2}{\mu C_K}\int_0^t\left\|\frac{\partial^2 f^\varepsilon(s)}{\partial t^2}\right\|_{L^2(\Omega^\varepsilon)}^2ds+\frac{4\left(\varepsilon\bar{h}\right)^2}{\mu C_K}\left\|\frac{\partial f^\varepsilon}{\partial t}\right\|_{L^2(\Omega^\varepsilon)}^2 \\
&\quad+\int_0^t\left[\left\|\frac{\partial^2 u^\varepsilon(s)}{\partial t^2}\right\|_{L^2(\Omega^\varepsilon)}^2+\mu C_K\left\|\nabla\frac{\partial u^\varepsilon(s)}{\partial t}\right\|_{L^2(\Omega^\varepsilon)}^2\right]ds. \quad (18)
\end{aligned}$$

Multiplying now (18) by ε, we obtain

$$\varepsilon\left[\left\|\frac{\partial^2 u^\varepsilon}{\partial t^2}\right\|^2_{L^2(\Omega^\varepsilon)}+\mu C_K\left\|\nabla\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)}\right]$$
$$\leq\int_0^t\varepsilon\left[\left\|\frac{\partial^2 u^\varepsilon(s)}{\partial t^2}\right\|^2_{L^2(\Omega^\varepsilon)}+\mu C_K\left\|\nabla\frac{\partial u^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)}\right]ds+B,$$

where B is a constant does not depend of ε

$$B=(2\mu+3\lambda+\mu C_K)\left\|\nabla\hat{\vartheta}_1\right\|^2_{L^2(\Omega)}+(C')^2+\frac{4\bar{h}^2}{\mu C_K}\left\|\frac{\partial\hat{f}}{\partial t}(0)\right\|^2_{L^2(\Omega)}$$
$$+\frac{4\bar{h}^2}{\mu C_K}\left\|\frac{\partial\hat{f}}{\partial t}\right\|^2_{L^\infty\left(0,T,L^2(\Omega)^2\right)}+\frac{4\bar{h}^2}{\mu C_K}\left\|\frac{\partial^2\hat{f}}{\partial t^2}\right\|^2_{L^2\left(0,T,L^2(\Omega)^2\right)}.$$

By the Gronwall's lemma, there exists a constant c that does not depend of ε such that

$$\varepsilon\left\|\frac{\partial^2 u^\varepsilon}{\partial t^2}\right\|^2_{L^2(\Omega^\varepsilon)}+\varepsilon\left\|\nabla\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)}\leq c,$$

we conclude (12). □

4. Convergence theorem and limiting problem

THEOREM 4.1. *Under the hypotheses of Theorem 3.1, there exists* $u_1^*\in L^2(0,T,V_z)\cap L^\infty(0,T,V_z)$, *such that*

$$\left.\begin{array}{l}\hat{u}_1^\varepsilon\rightharpoonup u_1^*\\ \frac{\partial\hat{u}_1^\varepsilon}{\partial t}\rightharpoonup\frac{\partial u_1^*}{\partial t}\end{array}\right\}\begin{array}{c}\text{weakly in } L^2(0,T,V_z)\\ \text{and weakly } *\text{ in } L^\infty(0,T,V_z),\end{array}\tag{19}$$

$$\left.\begin{array}{l}\varepsilon\frac{\partial\hat{u}_1^\varepsilon}{\partial x}\rightharpoonup 0\\ \varepsilon\frac{\partial^2\hat{u}_1^\varepsilon}{\partial x\partial t}\rightharpoonup 0\end{array}\right\}\begin{array}{c}\text{weakly in } L^2\left(0,T,L^2(\Omega)\right)\\ \text{and weakly } *\text{ in } L^\infty\left(0,T,L^2(\Omega)\right),\end{array}\tag{20}$$

$$\left.\begin{array}{l}\varepsilon\frac{\partial\hat{u}_1^\varepsilon}{\partial t}\rightharpoonup 0\\ \varepsilon\frac{\partial^2\hat{u}_1^\varepsilon}{\partial t^2}\rightharpoonup 0\end{array}\right\}\begin{array}{c}\text{weakly in } L^2\left(0,T,L^2(\Omega)\right)\\ \text{and weakly } *\text{ in } L^\infty\left(0,T,L^2(\Omega)\right),\end{array}\tag{21}$$

$$\left.\begin{array}{l}\varepsilon^2\frac{\partial\hat{u}_2^\varepsilon}{\partial x}\rightharpoonup 0\\ \varepsilon^2\frac{\partial^2\hat{u}_2^\varepsilon}{\partial x\partial t}\rightharpoonup 0\end{array}\right\}\begin{array}{c}\text{weakly in } L^2\left(0,T,L^2(\Omega)\right)\\ \text{and weakly } *\text{ in } L^\infty\left(0,T,L^2(\Omega)\right),\end{array}\tag{22}$$

$$\left.\begin{array}{l}\varepsilon^2 \frac{\partial \hat{u}_2^\varepsilon}{\partial t} \rightharpoonup 0 \\ \varepsilon^2 \frac{\partial^2 \hat{u}_2^\varepsilon}{\partial z \partial t} \rightharpoonup 0 \\ \varepsilon^2 \frac{\partial^2 \hat{u}_2^\varepsilon}{\partial t^2} \rightharpoonup 0\end{array}\right\} \begin{array}{c}\textit{weakly in } L^2\left(0, T, L^2(\Omega)\right) \\ \textit{and weakly } * \textit{ in } L^\infty\left(0, T, L^2(\Omega)\right),\end{array} \tag{23}$$

$$\left.\begin{array}{l}\varepsilon^{\frac{2}{3}} \frac{\partial \hat{u}_1^\varepsilon}{\partial t} \rightharpoonup 0 \\ \varepsilon^{\frac{5}{3}} \frac{\partial \hat{u}_2^\varepsilon}{\partial t} \rightharpoonup 0\end{array}\right\} \textit{weakly in } L^3\left(0, T, L^3(\Omega)\right). \tag{24}$$

Proof. According to Theorem 3.1, there exists a constant c independent of ε such that

$$\left\| \frac{\partial \hat{u}_1^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \leq c.$$

Using this estimate with the Poincaré inequality in the domain Ω, we get

$$\|\hat{u}_1^\varepsilon\|_{L^2(\Omega)}^2 \leq \bar{h} \left\| \frac{\partial \hat{u}_1^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \leq c.$$

So $(\hat{u}_1^\varepsilon)_\varepsilon$ is bounded in $L^2(0, T, V_z) \cap L^\infty(0, T, V_z)$, which implies the existence of an element u_1^* in $L^2(0, T, V_z) \cap L^\infty(0, T, V_z)$ such that $(\hat{u}_1^\varepsilon)_\varepsilon$ converges weakly to u_1^* in $L^2(0, T, V_z) \cap L^\infty(0, T, V_z)$, the same for $\left(\frac{\partial \hat{u}_1^\varepsilon}{\partial t}\right)_\varepsilon$, thus we obtain (19). For (20)-(24), according to (11), (12) and (19). □

THEOREM 4.2. *Under the hypotheses of Theorem 4.1, the limit u_1^* satisfies the following variational inequality*

$$\mu \int_\Omega \frac{\partial u_1^*}{\partial z} \cdot \frac{\partial}{\partial z}\left(\hat{\varphi}_1 - \frac{\partial u_1^*}{\partial t}\right) dxdz + \hat{\alpha} \int_\Omega \frac{\partial u_1^*}{\partial t} \cdot \left(\hat{\varphi}_1 - \frac{\partial u_1^*}{\partial t}\right) dxdz$$
$$+ \hat{J}(\hat{\varphi}) - \hat{J}\left(\frac{\partial u_1^*}{\partial t}\right) \geq \left(\hat{f}_1, \hat{\varphi}_1 - \frac{\partial u_1^*}{\partial t}\right), \qquad \forall \hat{\varphi} \in \Pi(K), \tag{25}$$

and the parabolic problem

$$\left\{\begin{array}{c} -\mu \frac{\partial^2 u_1^*}{\partial z^2}(t) + \hat{\alpha} \frac{\partial u_1^*}{\partial t}(t) = \hat{f}_1(t), \quad \textit{in } L^2(\Omega), \\ u_1^*(x, z, 0) = \hat{\vartheta}_{0,1}. \end{array}\right. \tag{26}$$

Proof. As $\hat{J}(\cdot)$ is convex and lower semicontinuous i.e

$$\lim_{\varepsilon \to 0}\left(\inf \int_0^l \hat{k} \left|\frac{\partial \hat{u}_1^\varepsilon}{\partial t}\right| dx\right) \geq \int_0^l \hat{k} \left|\frac{\partial u_1^*}{\partial t}\right| dx,$$

we pass to the limit when ε tends to zero in (10) and using the convergence results of the Theorem 4.1, we find the following limit inequality

$$\mu \int_\Omega \frac{\partial u_1^*}{\partial z} \cdot \frac{\partial}{\partial z}\left(\hat{\varphi}_1 - \frac{\partial u_1^*}{\partial t}\right) dxdz + \hat{\alpha} \int_\Omega \frac{\partial u_1^*}{\partial t} \cdot \left(\hat{\varphi}_1 - \frac{\partial u_1^*}{\partial t}\right) dxdz + \hat{J}(\hat{\varphi}) - \hat{J}\left(\frac{\partial u_1^*}{\partial t}\right) \geq \int_\Omega \hat{f}_1 \left(\hat{\varphi}_1 - \frac{\partial u_1^*}{\partial t}\right) dxdz. \quad (27)$$

We now choose in the variational inequality (27)

$$\hat{\varphi}_1 = \frac{\partial u_1^*}{\partial t} \pm \psi, \qquad \psi \in H_0^1(\Omega),$$

we find

$$\mu \int_\Omega \frac{\partial u_1^*}{\partial z} \frac{\partial \psi}{\partial z} dxdz + \hat{\alpha} \int_\Omega \frac{\partial u_1^*}{\partial t} \cdot \psi dxdz = \int_\Omega \hat{f}_1 \psi dxdz.$$

According to Green's formula, we obtain

$$- \int_\Omega \mu \frac{\partial}{\partial z}\left(\frac{\partial u_1^*}{\partial z}\right) \psi dxdz + \int_\Omega \hat{\alpha} \frac{\partial u_1^*}{\partial t} \cdot \psi dxdy = \int_\Omega \hat{f}_1 \psi dxdz, \quad \forall \psi \in H_0^1(\Omega).$$

Therefore

$$-\mu \frac{\partial^2 u_1^*(t)}{\partial z^2} + \hat{\alpha} \frac{\partial u_1^*(t)}{\partial t} = \hat{f}_1(t), \quad in\ H^{-1}(\Omega), \forall t \in]0,T[, \quad (28)$$

and, as $\hat{f}_1 \in L^2(\Omega)$, then (28) is valid in $L^2(\Omega)$. □

THEOREM 4.3. *Under the same assumptions of Theorem 4.1, the traces*

$$\tau^*(x,t) = \frac{\partial u_1^*}{\partial z}(x,0,t) \ \text{ and } s^*(x,t) = u_1^*(x,0,t)$$

satisfy the following inequality

$$\int_0^l \hat{k}\left(\left|\psi + \frac{\partial s^*}{\partial t}\right| - \left|\frac{\partial s^*}{\partial t}\right|\right) dx - \int_0^l \mu \tau^* \psi dx \geq 0, \quad \forall \psi \in L^2(]0,l[), \quad (29)$$

and the following limit form of the Tresca boundary conditions

$$\left.\begin{array}{l} \mu|\tau^*| < \hat{k} \Rightarrow \frac{\partial s^*}{\partial t} = 0, \\ \mu|\tau^*| = \hat{k} \Rightarrow \exists \beta > 0 \text{ such that } \frac{\partial s^*}{\partial t} = -\beta \tau^*, \end{array}\right\} \quad a.e \text{ on }]0,l[\times]0,T[. \quad (30)$$

Moreover u_1^* and s^* satisfies the following weak form of the Reynolds equation

$$\int_0^l \left(\tilde{F} - \frac{h}{2}s^* + \int_0^h u_1^*(x,z,t)\,dz + \tilde{U}_1\right) \psi'(x)\,dx = 0, \quad \forall \psi \in H^1(]0,l[), \quad (31)$$

where

$$\tilde{F}(x,h,t) = \frac{1}{\mu}\int_0^h F(x,z,t)\,dz - \frac{h}{2\mu}F(x,h,t),$$
$$\tilde{U}_1(x,h,t) = -\frac{\hat{\alpha}}{\mu}\int_0^h U_1(x,z,t)\,dz + \frac{\hat{\alpha}h}{2\mu}U_1(x,h,t),$$
$$F(x,z,t) = \int_0^z\int_0^\zeta \hat{f}_1(x,\eta,t)\,d\eta d\zeta,$$
$$U_1(x,z,t) = \int_0^z\int_0^\zeta \frac{\partial u_1^*}{\partial t}(x,\eta,t)\,d\eta d\zeta.$$

Proof. For the proof of (29), (30), we follow the same steps as in [1]. To prove (31) by integrating (26) from 0 to z, we see that

$$-\mu\frac{\partial u_1^*}{\partial z}(x,z,t) + \mu\frac{\partial u_1^*}{\partial z}(x,0,t) + \hat{\alpha}\int_0^z \frac{\partial u_1^*}{\partial t}(x,\eta,t)\,d\eta = \int_0^z \hat{f}_1(x,\eta,t)\,d\eta.$$

Integrating again between 0 to z, we obtain

$$u_1^*(x,z,t) = s^* + z\tau^* + \frac{\hat{\alpha}}{\mu}\int_0^z\int_0^\zeta \frac{\partial u_1^*}{\partial t}(x,\eta,t)\,d\eta d\zeta - \frac{1}{\mu}\int_0^z\int_0^\zeta \hat{f}_1(x,\eta,t)\,d\eta d\zeta, \quad (32)$$

in particular for $z = h(x)$ we get

$$s^* + h\tau^* = -\frac{\hat{\alpha}}{\mu}\int_0^h\int_0^\zeta \frac{\partial u_1^*}{\partial t}(x,\eta,t)\,d\eta d\zeta + \frac{1}{\mu}\int_0^h\int_0^\zeta f_1(x,\eta,t)\,d\eta d\zeta. \quad (33)$$

Integrating (32) from 0 to h, we obtain

$$\int_0^h u_1^*(x,z,t)\,dz = hs^* + \frac{1}{2}h^2\tau^* + \frac{\hat{\alpha}}{\mu}\int_0^h\int_0^z\int_0^\zeta \frac{\partial u_1^*}{\partial t}(x,\eta,t)\,d\eta d\zeta dz - \frac{1}{\mu}\int_0^h\int_0^z\int_0^\zeta \hat{f}_1(x,\eta,t)\,d\eta d\zeta dz. \quad (34)$$

From (33) and (34), we deduce that

$$\int_0^h u_1^*(x,z,t)\,dz - \frac{h}{2}s^* + \tilde{F} + \tilde{U}_1 = 0,$$

with

$$\tilde{F}(x,h,t) = \frac{1}{\mu}\int_0^h F(x,z,t)\,dz - \frac{h}{2\mu}F(x,h,t),$$

$$\tilde{U}_1(x,h,t) = -\frac{\hat{\alpha}}{\mu}\int_0^h U_1(x,z,t)\,dz + \frac{\hat{\alpha}h}{2\mu}U_1(x,h,t),$$

$$F(x,z,t) = \int_0^z \int_0^\zeta \hat{f}_1(x,\eta,t)\,d\eta d\zeta,$$

$$U_1(x,z,t) = \int_0^z \int_0^\zeta \frac{\partial u_1^*}{\partial t}(x,\eta,t)\,d\eta d\zeta.$$

Therefore

$$\int_0^l \left(\int_0^h u_1^*(x,z,t)\,dz - \frac{h}{2}s^* + \tilde{F} + \tilde{U}_1\right)\psi^{'}(x)\,dx = 0.$$

□

THEOREM 4.4. *The solution* u_1^* *of the limiting problem* (25), (26) *is unique in* $L^2(0,T,V_z) \cap L^\infty(0,T,V_z)$.

Proof. Suppose that there exist two solutions u_1^* and u_1^{**} of the variational inequality (25), we have

$$\mu\int_\Omega \frac{\partial u_1^*}{\partial z}\cdot\frac{\partial}{\partial z}\left(\hat{\varphi} - \frac{\partial u_1^*}{\partial t}\right)dxdz + \hat{\alpha}\int_\Omega \frac{\partial u_1^*}{\partial t}\cdot\left(\hat{\varphi} - \frac{\partial u_1^*}{\partial t}\right)dxdz + \hat{J}(\hat{\varphi}) - \hat{J}\left(\frac{\partial u_1^*}{\partial t}\right) \geq \left(\hat{f}_1, \hat{\varphi} - \frac{\partial u_1^*}{\partial t}\right), \quad (35)$$

and

$$\mu\int_\Omega \frac{\partial u_1^{**}}{\partial z}\cdot\frac{\partial}{\partial z}\left(\hat{\varphi} - \frac{\partial u_1^{**}}{\partial t}\right)dxdz + \hat{\alpha}\int_\Omega \frac{\partial u_1^{**}}{\partial t}\cdot\left(\hat{\varphi} - \frac{\partial u_1^{**}}{\partial t}\right)dxdz + \hat{J}(\hat{\varphi}) - \hat{J}\left(\frac{\partial u_1^{**}}{\partial t}\right) \geq \left(\hat{f}_1, \hat{\varphi} - \frac{\partial u_1^{**}}{\partial t}\right). \quad (36)$$

We take $\hat{\varphi} = \frac{\partial u_1^{**}}{\partial t}$ in (35), then $\hat{\varphi} = \frac{\partial u_1^*}{\partial t}$ in (36), and by summing the two inequalities, we obtain

$$\mu\int_\Omega \frac{\partial}{\partial z}(u_1^* - u_1^{**})\cdot\frac{\partial}{\partial z}\left(\frac{\partial u_1^*}{\partial t} - \frac{\partial u_1^{**}}{\partial t}\right)dxdz + \hat{\alpha}\int_\Omega \left(\frac{\partial u_1^*}{\partial t} - \frac{\partial u_1^{**}}{\partial t}\right)\cdot\left(\frac{\partial u_1^*}{\partial t} - \frac{\partial u_1^{**}}{\partial t}\right)dxdz \leq 0.$$

If we put $\bar{W}(t) = u_1^*(t) - u_1^{**}(t)$, this implies

$$\mu \frac{d}{dt} \left\| \frac{\partial \bar{W}}{\partial z} \right\|_{L^2(\Omega)}^2 + \hat{\alpha} \left\| \frac{\partial \bar{W}}{\partial t} \right\|_{L^2(\Omega)}^2 \leq 0.$$

We have $\bar{W}(0) = 0$, then we find

$$\left\| \frac{\partial \bar{W}}{\partial z} \right\|_{L^2(\Omega)}^2 \leq 0.$$

Using Poincaré's inequality, we conclude

$$\|\bar{W}\|_{L^2(0,T,V_z)} = \|\bar{W}\|_{L^\infty(0,T,V_z)} = 0.$$

□

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