



# ANALYSING THE CAPACITY OF A TRANSPORTATION NETWORK. A GENERAL THEORETICAL APPROACH

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## Abstract

Estimation of transportation network capacity is important in analysing network performance. In the existing literature, the capacity of a network is defined in conditional terms as a theoretical construct called “reserve capacity”. This may be limited because, when considering local land-use development, the hypothetical uniform increase in O-D flows is not always realistic. This paper, introducing the concept of “capacity function”, proposes a generalized concept of road network capacity which does not require information on either current O-D demand or the corresponding growth trend. Attention focuses on deterministic and stationary situations in which only one path is available for each O-D pair. Some examples regarding simple study cases demonstrate the capacity of this approach to solve problems and, consequently, to contribute to the analysis of network performance.

*Keywords:* network capacity, traffic flow, optimization, reserve capacity

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Information on the capacity of a transportation network, and on the extent to which this depends on the characteristics of its constituent elements, is important both for managing traffic, i.e., traffic calming measures to reduce existing congestion problems, exploiting reserve capacity, road pricing, etc., and for managing demand such as activity scheduling. It is also important when undertaking land-use planning and development for modification of the existing network (Wang, Meng and Yang, 2013) or planning and management of actions on transportation system to address extraordinary events in post-emergency situations (Rossi et al., 2012; Carturan et al., 2013).

*In the context of network speed-flow theory*, the problem of determining capacity has been stated in the hypothesis of homogeneous origin-destination (O-D) flows, considering capacity constraints on each link; the procedure is based on the maximization of the sum of O-D flows without violating any constraint. Accordingly, this is a constrained maximization problem which can clearly be identified in mathematical terms and easily solved (Ahuja et al., 1993). In general, maximizing the sum of O-D flows does not appear to be acceptable, in view of

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their typically heterogeneous nature, e.g., traffic flows with different origins and/or destinations.

In the case of heterogeneous flows, the procedure reported in the literature to determine the capacity of a road network is a “conditional maximization process”: starting from an O-D matrix ( $\tilde{d}_{ij}$ ) and hypothesizing the invariance of its structure, each single O-D flow is expanded by applying a common factor  $\mu$  to the point of saturating a link ( $\mu = \mu_{max}$ ). This configuration is assumed to be an indicator of capacity, and the expansion coefficient defines the “reserve capacity” in terms of percentages.

An example is provided by the situation shown in Figure 1 which deals with a symbolic two-dimensional general case. The concept of reserve capacity for a given network, with an initial O-D flow matrix represented by point A, is linked to transformation AB (where B represents the maximum O-D flow configuration corresponding to  $\mu_{max}$ ). It is important to establish the initial status of the problem, i.e., to determine the initial matrix of O-D movements and the relative growth trend: the capacity afforded by a transportation network depends on these initial conditions.

In addition, the hypothesis of a uniform increase in O-D flows is barely realistic in situations in which land is subject to local development. It is thus important to take into account other possible situations, such as transformations  $AB^I$  or  $AB^{II}$ , characterized by the variation of only one flow, or transformation  $AB^{III}$ . In this way, we introduce a generalization of the concept of reserve capacity in terms.

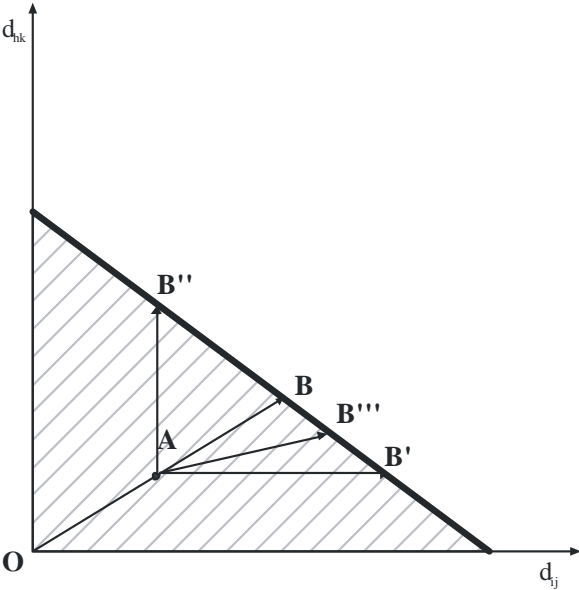


Figure 1 – Generalization of concept of reserve capacity in a two-dimensional case.  $d_{ij}$  and  $d_{hk}$ : two O-D flows with differing origins and destinations.

In this sense, this paper proposes a different definition of network capacity (in particular, relating to a road network) which does not rely on knowledge of initial conditions: a systems approach is adopted, and the concept of network capacity function is introduced.

Let us assume that the state of the system under analysis is defined by the values of a set of state variables,  $x_1 \dots x_i \dots x_n$ ; in general, there is no system configuration in which all state variables are at maximum. Rather, as a consequence of link capacity constraints, there is a set of non-dominated configurations, each of which does not allow any increase in the value of

one state variable without reducing that of another. Capacity may thus be expressed by the set of vectors, defined by the values of the state variables, which coincides with these non-dominated configurations. In this sense, it is more correct to speak in terms of capacity function.

The concept of transportation system capacity, as defined by the capacity function, is affected by one or more constrained maximization problems.

If the capacity function of a network is determined, the performance of the network can be evaluated analysing the sensitivity of the network to variations in travel demand and the geometrical parameters can be modified, yielding the greatest increase in terms of capacity.

The paper is organized as follows: section 2 provides a brief overview of the techniques reported in the literature to determine network capacity. Section 3 introduces the concept of capacity function and describes the method proposed to identify it. Section 4 presents examples of how capacity function is determined for simple networks (motorway corridors, roundabouts), with the aim of demonstrating the usefulness and simplicity of the evaluation process. Lastly, concluding remarks and future research are presented.

Studies on the capacity of transportation networks in the current literature can be divided into two categories, according to whether or not the route choice behavior of users is taken into account in formulating the problem.

Asakura (1992), Akamatsu and Miyawaki (1995), Akamatsu and Heydecker (2003) proposed models and algorithms for solving the problem of road network capacity by adopting approaches which simulate static or dynamic equilibrium assignments. Wong and Yang (1997) extended the concept of reserve capacity to a fully signal-controlled network: this concept had previously been used to assess the performance of isolated signal-controlled intersections (Allsop, 1972; Heydecker, 1983) and priority intersections and roundabouts (Wong, 1996). Chen, Chootinan and Wong (Chen et al., 2006) used a capacity-reliability index (Chootinan et al., 2005) to define reserve capacity models in the context of signal-controlled road networks, and Yang and Bell (Yang et al., 1998) used the concept of network capacity in their procedure for designing a road network. Wang and Deng (Wang et al., 2013) proposed an integrated method to maximize the reserve capacity of multi-phase signalized road network with stochastic user equation. Yang, Bell and Meng (Yang et al., 2000) analysed the problem of determining the reserve capacity of a transportation network, proposing a formulation in which the assumption of a common multiplier of O-D flows to determine network reserve capacity was relaxed, in a combined distribution and assignment model which may be used to obtain a maximum O-D flow solution differentiating the multipliers of the O-D pairs. In another work (Chen et al., 2002), a method of assessment, integrating capacity reliability and uncertainty analysis, and network equilibrium models, was developed to evaluate the changing performance of a road network.

One limiting factor common to these studies is the assumption of a predetermined initial O-D matrix and in general the identification of a single direction of matrix growth, and these assumptions do influence the maximum O-D flow configurations obtainable. Choice of the initial O-D matrix and/or the growth direction of its components should reflect future developments in terms of land-use. This means that choosing an invariant matrix structure (all O-D flows increasing proportionally) is unsuitable. However, differentiation of the growth trends for the various O-D pairs (Gao et al., 2002) may lead to predictions and assessments which are not robust, since they are strongly conditioned by the uncertainty of land-use growth predictions. The methods listed above assume that the quantities in play are fixed and predetermined; in reality, there are various sources of uncertainty which affect the assessment of transportation network capacity (Zhu and Zhang, 2008; Chen and Panatda, 2011).

### 3.1 The concept of system capacity: the capacity function

The concept of capacity, as mentioned above, may be formulated in general terms with a systems approach: a transportation network is generally configured as a complex system of many elements interacting with one another.

At application level, the concept of “analysis system” is introduced, composed only of the elements and their interactions (subsystems) which are important in relation to the phenomenon under analysis or the problem to which a solution is required. External elements affecting the analysis system (environment) are expressed through inputs and outputs, without considering their mutual interactions. In the case of road network capacity assessment, state variables such as the physical, geometrical and control characteristics of the network elements, travel demand, etc. are taken into consideration.

The state variables of a generic system are subject to certain constraints (equalities and/or inequalities) which identify feasible configurations.

The concept of *capacity* becomes significant when these variables are upper-bounded: consequently, capacity is generally associated with constrained maximization problems. Constraints may directly involve all or some of the variables for which capacity is defined: however, constraints can directly influence only variables not included in the capacity function, but which nonetheless affect the values of the variables in that function (induced constraints).

In the case of a road network, two components are important: infrastructure and circulation. If the infrastructural component, i.e., the topological, technical and control characteristics of the network, is fixed, then capacity refers to the circulation component. However, the variables characterizing infrastructure, e.g., speed limits, numbers of lanes, etc., may also take on different values, in which case they must also be included in the capacity function.

With  $\tilde{x}$  denoting all state variables and  $\tilde{z}$  the subset of variables for which capacity is defined (these will appear explicitly in the capacity function) - with reference to a road network - state variables  $\tilde{x}$  may be the values of the infrastructure characteristics, such as network topology link characteristics, etc., and the speed-flow relationship, such as O-D flows, density, speed, and flows on links.  $\tilde{z}$  variables, as will be explained, may be O-D flows.

As already mentioned, the concept of capacity is linked to one or more constrained maximization problems: if maximization involves only one variable  $z$  - for example, the flow of homogeneous vehicles on a road section - the concept of capacity is reduced to the maximization of this same variable. Then, in a deterministic and stationary situation, capacity assumes a clearly defined value. When there are two or more  $\tilde{z}$  variables - for example, heterogeneous O-D flows on a road network - vector maximization is used. As already stated, in general there is no system configuration in which all state variables are simultaneously at maximum, but rather a set of non-dominated configurations: for each of these, the value of one variable cannot be increased without reducing that of at least one other, in accordance with the constraints applied. As capacity may be expressed through the set of vectors defined by the values of the variables when they coincide with the non-dominated configurations, it is therefore correct to describe capacity function  $C(z_1 \dots z_n) = 0$  as being satisfied by non-dominated configurations alone.

### 3.2 Capacity function of a network used by homogeneous vehicles, with one path only for each O-D pair

This section formalizes the procedure for determining the capacity function of a network used by homogeneous vehicles, with only one path available (or in use) for each O-D pair: such a situation arises, for example, in the case of an intersection or a roundabout

(unsignalized or signalized), or a segment of motorway, and is typical of the Italian motorway network.

In this case, the procedure for determining capacity function is relatively simple, at least from the logical standpoint, and is also a topic of interest, both because it affects some significant specific cases and because it allows us to introduce the general case of a network with several paths used by at least one O-D pair. Note that, in this instance, the problem of determining system capacity becomes significantly more complex: analysts must consider choice behavior (in this case, route selection), which is also conditioned by the performance of the transportation network itself (congestion factors).

If topological and technical features are defined and vehicles are homogeneous, the variables considered to characterize capacity are the O-D flows which, being differentiated by spatial collocation, cannot be summed. Referring in particular to path flows, which in this case coincide with demand flows (O-D matrix), feasible flow configurations on the network are defined by link capacity constraints and non-negativity conditions.

Given:

$A$  set of network links

$a$  a generic link of the network

$c_a$  capacity of link  $a$

$N_A$  number of network links

$d_{ij}^a$  flow on link  $a$  relative to O-D pair  $(ij)$  (equal to 0 or to total demand  $d_{ij}$ )

$\delta_{a,ij}$  generic element of link-path incidence matrix (0 or 1)

$f_a = \sum_{ij} d_{ij}^a = \sum_{ij} \delta_{a,ij} d_{ij}$  overall flow of link  $a$ ,

feasible flow configurations are the solutions of the system of constraints:

$$\left\{ \begin{array}{l} \sum_{ij} \delta_{a,ij} d_{ij} \leq c_a \quad (ij) \in OD, \quad a = 1 \dots N_A \\ d_{ij} \geq 0 \quad (ij) \in OD \end{array} \right.$$

The domain of the feasible O-D flow configurations,  $\tilde{d}_{ij}$ , is represented by a convex hyperpolyhedron in  $\mathbb{R}^n$ , where  $n$  is the number of O-D pairs, defined inferiorly by non-negativity constraints and superiorly by “ $\leq$ ” constraints, considered as equalities; each constraint, considered as an equality, is represented by a hyperplane with  $n-1$  dimensions in  $\mathbb{R}^n$ .

If a “ $\leq$ ” constraint is “active”, i.e., it affects the determination of feasible flows and therefore contributes to defining the frontier, the part of the corresponding hyperplane which also satisfies all the other constraints constitutes one face of the upper frontier of the domain; this therefore coincides with the set of hyperpolyhedron faces. Note that the “upper frontier” is the set of points on the frontier for which it is not possible to increase all state variables simultaneously.

As mentioned, network capacity is expressed by a capacity function,  $C(\tilde{d}_{ij}) = 0$ , which is satisfied only by non-dominated configurations (Pareto optimal), in which:

- one variable (in this instance, an O-D flow) cannot be increased without reducing at least one other variable;
- no increase in variable values is admitted.

In the first case, the network capacity function is represented by the union of:

- the strictly decreasing faces of the upper frontier, corresponding to the constraints singly containing all the variables, considered as equalities. The  $\delta_{a,ij}$  coefficients of the corresponding active “ $\leq$ ” constraints are all positive;
- the intersections among some other faces of the upper frontier: no variables can be increased without reducing at least one other, so that the solution is non-dominated.

In the second case, the network capacity function is reduced to a single point.

Accordingly, the capacity function may coincide with the upper frontier or with part of it.

In general, in the case of a transportation network, none of the link capacity constraints contains all the variables: the capacity function therefore corresponds to a strict subset of the upper frontier.

By way of example, let us consider the network in Figure 2, consisting of four one-way links.

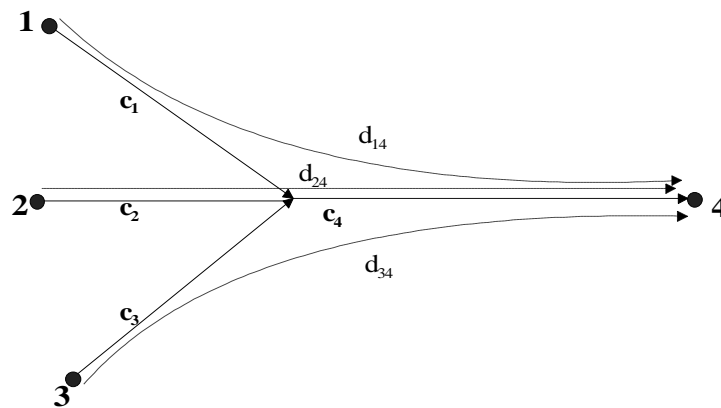


Figure 2: Example 1- network of four one-way links.

The demand flows on the paths are the variables  $d_{14}, d_{24}, d_{34}$  and the capacities of the links are  $c_1, c_2, c_3, c_4$ . There is only one path for each O-D pair; demand flows may thus be referred to the paths, thereby automatically satisfying the law of flow conservation at the nodes. Feasible demand flows can therefore be defined solely by the “active” capacity constraints on the links and by non-negativity conditions:

$$\left\{ \begin{array}{l} d_{14} \leq c_1 \text{ constraint 1} \\ d_{24} \leq c_2 \text{ constraint 2} \\ d_{34} \leq c_3 \text{ constraint 3} \\ d_{14} + d_{24} + d_{34} \leq c_4 \text{ constraint 4} \\ d_{14}, d_{24}, d_{34} \geq 0 \text{ non-negativity constraints} \end{array} \right.$$

The domain of feasible demand configurations  $(d_{14}, d_{24}, d_{34})$ , the upper frontier and the capacity function  $C(d_{14}, d_{24}, d_{34}) = 0$  naturally depend on the capacity values of the links. An example is shown in Figure 3: “ $\leq$ ” constraints 2 and 4 and the non-negativity constraints are “active”, whereas constraints 1 and 3 are not active. The domain of feasible configurations is represented by polyhedron  $OABCDE$  and bounded by face  $ABDE$  (constraint 4), and by face  $DCB$  (constraint 2, in which  $d_{14}$  and  $d_{34}$  are missing and the face is therefore parallel with a coordinate plane). As a consequence, the upper frontier is defined by the union of faces  $DCB$  and  $ABDE$ , and the capacity function is represented by the subset of the upper frontier coinciding with polygon  $ABDE$ , the strictly decreasing face defined by constraint 4.

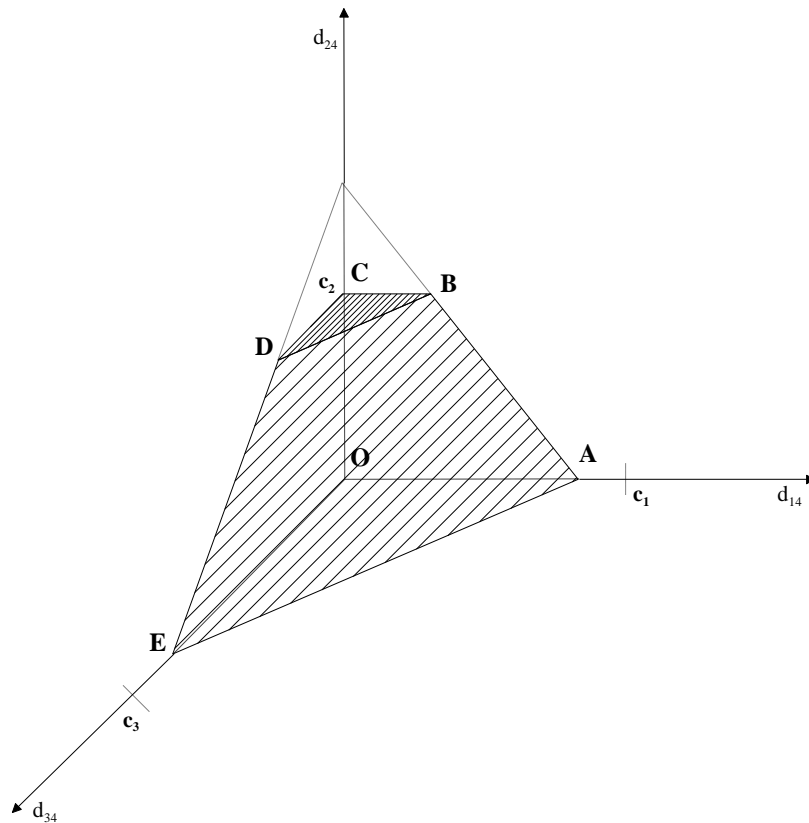


Figure 3: Example 1 - representation of capacity function (face ABDE).

### 3.3 Analytical identification of capacity function

In general, O-D flow configurations belonging to the capacity function may be carried out in the following ways.

#### First method

Each O-D flow variable is maximized, according to the constraints defining the domain, parameterizing all the other variables and then considering the solutions common to each optimization process. Each optimization process is of this type:

$$\max d_{(ij)^*}$$

with constraints

$$\left\{ \begin{array}{l} \dots \dots \dots \dots \dots \\ \sum_{ij \neq (ij)^*} \delta_{a,ij} \bar{d}_{ij} + \delta_{a,(ij)^*} d_{(ij)^*} \leq c_a \quad a = 1 \dots N_A \\ \dots \dots \dots \dots \dots \\ d_{(ij)^*} \geq 0 \\ \bar{d}_{ij} \geq 0 \end{array} \right.$$

where:

$d_{(ij)^*}$  is the demand relative to O-D pair  $(ij)^*$

$\bar{d}_{ij}$  are demand variables parameterized with  $(ij) \neq (ij)^*$

$\bar{d}_{ij}$  is the set of  $\bar{d}_{ij}$ .

The geometric interpretation of each optimization process (in symbolic terms, as we are generally operating in the hyperspace of the  $d_{ij}$  variables representing the O-D flows) may be traced to the determination of the intersection point between the line, parallel with the axis corresponding to the objective variable, identified by the values of the  $(n-1)$  parameterized variables and the upper frontier of the domain defined by the inequality equations.

Lastly, indicating:

$$d_{(ij)_{max}} = \psi_{(ij)^*}(\tilde{d}_{ij}) \quad \text{with } (ij) \neq (ij)^*$$

as the function corresponding to the solutions of the generic optimum process, the capacity function is given by the solutions of the system of functions previously mentioned:

$$\left\{ \begin{array}{c} \dots \dots \dots \dots \\ d_{(ij)_{max}} = \psi_{(ij)^*}(\tilde{d}_{ij}) \\ \dots \dots \dots \dots \end{array} \right.$$

and may be expressed as  $C(\tilde{d}_{ij}) = 0$ .

By way of example, let us go back to the network of Figure 2, in the case illustrated in Figure 3. Since the upper frontier is not strictly decreasing, the capacity function is obtained by maximizing each of the three demand flows, parameterizing the others, and then taking into account the solutions common to the three partial optimization processes. The analytical procedure is illustrated below:

$$\begin{array}{ll} d_{14}, d_{24}, d_{34} & \text{status variables} \\ \left\{ \begin{array}{l} d_{24} \leq c_2 \\ d_{14} + d_{24} + d_{34} \leq c_4 \\ d_{14}, d_{24}, d_{34} \geq 0 \end{array} \right. & \text{active constraints defining domain of feasible solutions} \end{array}$$

$$\left\{ \begin{array}{l} \max d_{14} \text{ (parameters } \bar{d}_{24} \text{ and } \bar{d}_{34}) \\ \left\{ \begin{array}{l} \bar{d}_{24} \leq c_2 \\ d_{14} + \bar{d}_{24} + \bar{d}_{34} \leq c_4 \\ d_{14}, \bar{d}_{24}, \bar{d}_{34} \geq 0 \end{array} \right. \end{array} \right. \rightarrow d_{14_{max}} = \Psi_{14}(\bar{d}_{24}, \bar{d}_{34}) \text{ face of frontier ABDE}$$

$$\left\{ \begin{array}{l} \max d_{24} \text{ (parameters } \bar{d}_{14} \text{ and } \bar{d}_{34}) \\ \left\{ \begin{array}{l} \bar{d}_{14} + d_{24} + \bar{d}_{34} \leq c_4 \\ d_{24} \leq c_2 \\ \bar{d}_{14}, d_{24}, \bar{d}_{34} \geq 0 \end{array} \right. \end{array} \right. \rightarrow d_{24_{max}} = \Psi_{24}(\bar{d}_{14}, \bar{d}_{34}) \text{ faces of frontier ABDE and BCD}$$

$$\left\{ \begin{array}{l} \max d_{34} \text{ (parameters } \bar{d}_{14} \text{ and } \bar{d}_{24}) \\ \left\{ \begin{array}{l} \bar{d}_{14} + \bar{d}_{24} + d_{34} \leq c_4 \\ \bar{d}_{24} \leq c_2 \\ \bar{d}_{14}, \bar{d}_{24}, d_{34} \geq 0 \end{array} \right. \end{array} \right. \rightarrow d_{34_{max}} = \Psi_{34}(\bar{d}_{14}, \bar{d}_{24}) \text{ face of frontier ABDE}$$



The intersection among the three sets of solutions is:

$$\left( \begin{array}{c} ABDE \\ ABDE \cup BCD \\ ABDE \end{array} \right) \rightarrow ABDE$$

Second method

The procedure described below for identifying the capacity function involves direct analysis of both the upper frontier faces and their intersections.

*Case b1*): each “ $\leq$ ” constraint contains all the variables. The single constraints are resolved in terms of equalities, keeping only the solutions which also satisfy the other constraints.

*Case b2*): only some of the constraints contain all the variables.

If the number of constraints,  $m$ , is greater than the number of variables,  $n$ , the capacity function in  $R^n$  may be obtained by a procedure consisting of the following steps:

- Step 1 For constraints including all the variables, the situation of case *b1* applies. These constraints will not be considered in subsequent steps to form combinations of constraints. The solutions to these combinations would in fact be subsets of the faces identified in this step;
- Step 2 Examination of all possible pairs formed with the remaining constraints considered as equalities, and analysis only of the set of pairs in which, taken together, all the variables are present. The solution to each system of paired constraints which satisfies all the other constraints forms part of the capacity function. These pairs are not considered in subsequent steps, since analysis would produce subsets of solutions already identified by the same pairs;
- Step 3 Examination of triplets of constraints, formed taking into account previous considerations, and similar progression to the second step;
- ...
- Step  $n$  Examination of the corresponding combinations formed of systems of  $n$  constraints in  $n$  variables, considered as equalities which do not contain combinations already examined according to the procedure described above. A system formed of linearly independent equations identifies a point in  $R^n$  of the capacity function. Any system with linearly dependent equations identifies solutions already determined by previous steps;
- Step  $(n+1)$  The usual procedure leads to systems of  $n+1$  equations in  $n$  variables. If the equations of a system are linearly independent, the combination has no solution; if they are dependent, there will be no new solutions with respect to the previous steps.

The procedure thus ends at Step  $n$ , and the capacity function is represented by joining the identified solutions. It may end before step  $n$ .

If  $m \leq n$ , the procedure adopted for  $m > n$  may be followed up to Step  $m$ . Again, the procedure may end before step  $m$ . An example is given in section 4.2.

*Case b3*): no single constraint contains all the variables (as occurs most frequently in transportation networks). The procedure described in case *b2*) is followed, starting from step 2.

Third method

The capacity function may also be determined by the procedure used to identify “reserve capacity”. Given an initial situation characterized by null travel demand, a growth direction of the demand defined by a segment starting from the origin in the positive orthant, and indicating by  $\tilde{\gamma}_{ij}$  the matrix of unit vector components (which defines the segment direction), the following constrained maximization problem, for each  $\tilde{\gamma}_{ij}$ , will have to be solved:

$$\max \mu$$

with constraints

$$\left\{ \begin{array}{l} \sum_{ij} \delta_{a,ij} \cdot \mu \cdot \gamma_{ij} \leq c_a \\ \tilde{\gamma}_{ij} \geq 0 \end{array} \right.$$

where:

$$\begin{aligned} \mu \cdot \gamma_{ij} &= O/D \text{ flow}(ij) \\ \delta_{a,ij} &= \begin{cases} 1 & \text{if arc } a \text{ belongs to pth used by } O/D \text{ flow}(ij) \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

In this way, the maximum O-D flow configuration obtained is  $\mu_{max} \cdot \tilde{\gamma}_{ij}$ , which belongs to the upper frontier of the domain, but it belongs to the capacity function only if the upper frontier is strictly decreasing at that point. The values of the elements of initial matrix  $\tilde{\gamma}_{ij}$  may be varied to obtain all the configurations of the upper frontier. They belong to the capacity function if they satisfy the non-dominance conditions.

Figure 4a and Figure 4b show some examples of a two-dimensional case (but in symbolic terms they may represent the general case), where  $A$  represents the maximum O-D flow configuration corresponding to  $\mu_{max}$ .

In the case shown in Figure 4a, as the upper frontier is strictly decreasing, varying the elements of initial matrix  $\tilde{\gamma}_{ij}$ , all the configurations of the capacity function are obtained (coinciding with those of the upper frontier).

In the case shown in Figure 4b, which is more frequent in road networks, the upper frontier of the domain is not strictly decreasing and the capacity function is represented by segment  $CD$ . The optimization procedure mentioned above may lead to point  $A$ , which does not belong to the capacity function (in effect, flow  $d_{ik}$  can be increased as far as point  $D$  without reducing the other flow).

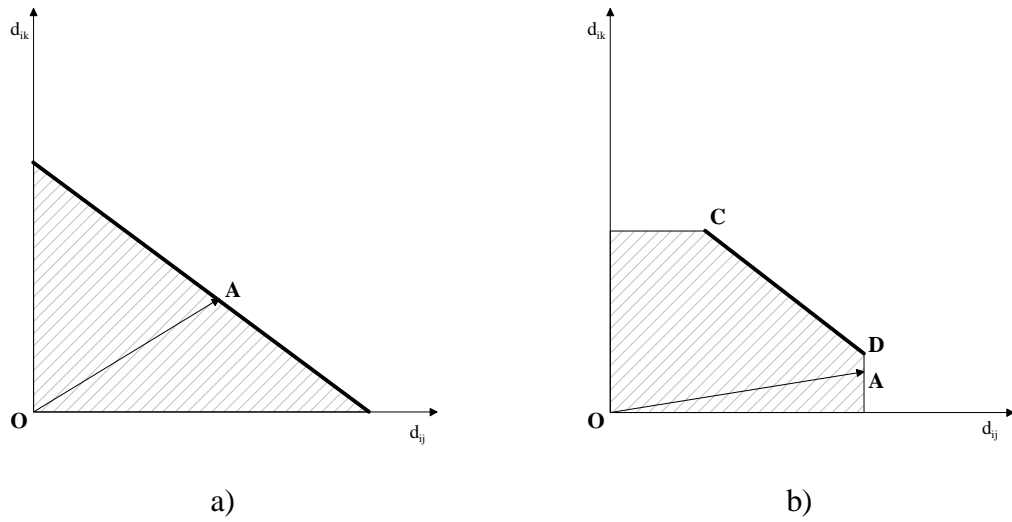


Figure 4: Representation of capacity function: the two-dimensional case.

## 4.1 Motorway segment between two toll plazas

Let us examine the simple case of a motorway segment which includes two one-way links having capacity  $c$  and therefore three end-sections (Figure 5).

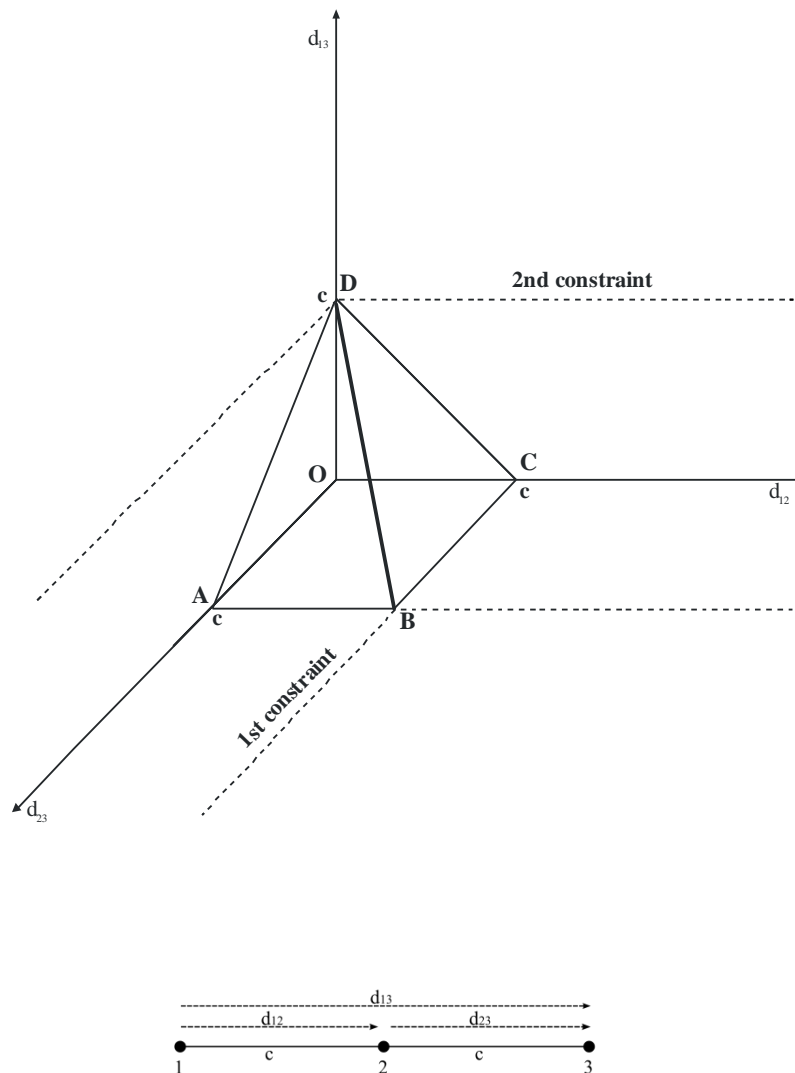


Figure 5: Example 2 – capacity function of a motorway segment including two one-way links of capacity  $c$ .

Feasible flow configurations must satisfy the system of constraints:

$$\begin{cases} d_{12} + d_{13} \leq c \\ d_{13} + d_{23} \leq c \\ d_{12}, d_{13}, d_{23} \geq 0 \end{cases}$$

Each constraint, considered as an equality, is shown by a plane (in  $\mathbb{R}^3$ ) parallel to the coordinate axis, corresponding to the variable not present in the constraint, and strictly decreasing with respect to the other variables.

The domain is represented by polyhedron OABED in  $\mathbb{R}^3$ , defined by two planes identified by the “ $\leq$ ” constraints, considered as equalities, and by the coordinate planes. More

precisely, each face of the polyhedron corresponds to part of one of the planes, i.e., the one which also satisfies the other “ $\leq$ ” constraints; the upper frontier is composed of faces ABD and BED.

Using the second method to determine which flow configurations belong to the capacity function we note that only the system consisting of the two constraints contains all the variables. The capacity function therefore consists of the solutions of the system of two constraints considered as equalities, which of course also satisfy non-negativity conditions. The capacity function is therefore represented by segment BD in  $\mathbb{R}^3$  and may also be obtained by maximizing each variable (demand flow) and parameterizing the others according to the system of inequality constraints (domain), and then considering the solutions common to all the maximization processes. In particular, flow  $d_{13}$  at point D is equal to  $c$  and saturates both links, whereas the other two flows are null. Reducing  $d_{13}$ , flows  $d_{12}$  and  $d_{23}$  increase by the same amount; at point B, flow  $d_{13}$  is null, whereas flows  $d_{12}$  and  $d_{23}$  both assume value  $c$ , saturating each of the two links.

#### 4.2 Four-arm roundabout

This section illustrates the case of a four-arm roundabout, to which the capacity formula proposed by the Swiss standard (Bovy, 1991) was applied. There are twelve variables in play (assuming  $d_{ii}=0$ ) and the system of constraints is:

$$\begin{cases} d_{12} + d_{13} + d_{14} \leq 1500 - 8/9[\beta(d_{32} + d_{42} + d_{43}) + \alpha(d_{21} + d_{31} + d_{41})] \\ d_{23} + d_{24} + d_{21} \leq 1500 - 8/9[\beta(d_{43} + d_{13} + d_{14}) + \alpha(d_{32} + d_{42} + d_{12})] \\ d_{34} + d_{31} + d_{32} \leq 1500 - 8/9[\beta(d_{14} + d_{24} + d_{21}) + \alpha(d_{43} + d_{13} + d_{23})] \\ d_{41} + d_{42} + d_{43} \leq 1500 - 8/9[\beta(d_{21} + d_{31} + d_{32}) + \alpha(d_{14} + d_{24} + d_{34})] \\ d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{21}, d_{34}, d_{31}, d_{32}, d_{41}, d_{42}, d_{43} \geq 0 \end{cases}$$

⇓

$$\begin{cases} d_{12} + d_{13} + d_{14} + 8/9\alpha d_{21} + 8/9\alpha d_{31} + 8/9\beta d_{32} + 8/9\alpha d_{41} + 8/9\beta d_{42} + 8/9\beta d_{43} \leq 1500 \\ 8/9\alpha d_{12} + 8/9\beta d_{13} + 8/9\beta d_{14} + d_{21} + d_{23} + d_{24} + 8/9\alpha d_{32} + 8/9\alpha d_{42} + 8/9\beta d_{43} \leq 1500 \\ 8/9\alpha d_{13} + 8/9\beta d_{14} + 8/9\beta d_{21} + 8/9\alpha d_{23} + 8/9\beta d_{24} + d_{31} + d_{32} + d_{34} + 8/9\alpha d_{43} \leq 1500 \\ 8/9\alpha d_{14} + 8/9\beta d_{21} + 8/9\alpha d_{24} + 8/9\beta d_{31} + 8/9\beta d_{32} + 8/9\alpha d_{34} + d_{41} + d_{42} + d_{43} \leq 1500 \\ d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{21}, d_{34}, d_{31}, d_{32}, d_{41}, d_{42}, d_{43} \geq 0 \end{cases}$$

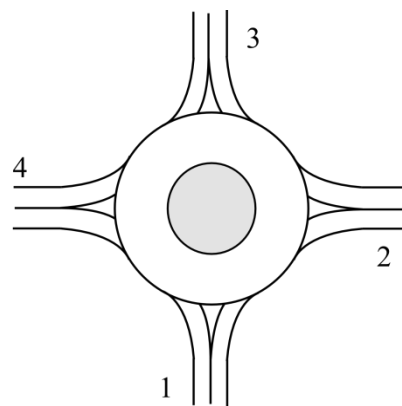


Figure 6: Example 3: four-arm roundabout.

The domain of feasible configurations is a hyperpolyhedron in  $\mathbb{R}^{12}$  (12 variables); the upper frontier consists of four hyperplanes with 11 dimensions, each of which defined by the solutions of a constraint considered as an equality, which also satisfies all the other constraints; each hyperplane is parallel to three coordinate axes: the corresponding face thus does not entirely belong to the capacity function, which is defined by faces (intersections of upper frontier faces) with dimensions of less than 11.

Using the second method (see §3.3), in the case in point:

- the system consisting of the first two constraints (considered as equalities) does not contain variable  $d_{34}$  and therefore does not belong to the capacity function;
- the intersection between constraints 1 and 3 belongs to the capacity function (all variables are present);
- variable  $d_{23}$  is missing at the intersection between constraints 1 and 4;
- variable  $d_{41}$  is missing at the intersection between constraints 2 and 3;
- all variables are present at the intersection between constraints 2 and 4: this intersection belongs to the capacity function;
- variable  $d_{12}$  is missing (and does not belong to the capacity function) at the intersection between constraints 3 and 4;
- in the case of triplets of constraints, it is sufficient to consider combinations not containing pairs of constraints which already incorporate all the variables in play (constraints 1 and 3; constraints 2 and 4): there are no triplets of this type;
- the same is true of the intersection of the four constraints.

Lastly, the intersections belonging to the capacity function are those between constraints 1 and 3 and between constraints 2 and 4, i.e., two hyperplanes with 10 dimensions.

A more detailed analysis of this subject is reported in Gastaldi et al. (2009).

This paper describes the capacity of a transportation network and addresses the problem of determining capacity, with special reference to a road network. The authors first briefly review previous studies, and indicate some of their limitations.

In particular, they highlight the importance of referring to a *network capacity function*, rather than to a single value. When predicting the performance of a network, this approach allows analysts to measure the sensitivity of the system to variations in travel demand and therefore to consider the uncertainty which always characterizes land-use predictions. Methods for determining capacity function are also proposed. Attention focuses on deterministic and stationary situations in which only one path is available for each O-D pair. Some examples regarding simple study cases demonstrate the capacity of the approach to solve the problem and, consequently, to contribute to the analysis of network performance.

Analysis of the capacity function of a given road network may be extremely useful to engineers working in the transportation field, especially during study of transportation/land-use interactions (analyses assessing the effects of pursuing different directions in land use and transportation system modifications). In this context, the method offers a useful decision-making tool, which does not rely either on knowledge of current travel demand or the predetermined, predicted growth trend of that demand. Once the capacity function is known, it is relatively easy to assess what increases in demand can be absorbed by the system and, consequently, what modifications of the road network may be necessary.

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