

On a Quasilinear Parabolic System Modelling the Diffusion of Radioactive Isotopes

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Dedicated to the memory of Fabio Rossi, with deep affection

SUMMARY. - *We consider a model for the diffusion of N species of isotopes of the same element in a medium, consisting in a parabolic quasilinear system, with Dirichlet boundary condition, in the general hypothesis that the diffusion coefficients possibly are all different. We prove existence and uniqueness of classical solution in the physically relevant assumption that the total concentration of the element is positive and bounded.*

1. Introduction

We will consider a model proposed in [8] for the diffusion of N species of isotopes of the same element in a medium and based on the assumption that the flux of the i component J_i is given by

$$J_i = - \left(\tilde{D}_i \nabla c_i + D_i \frac{c_i}{c} \nabla c \right), \quad i = 1, \dots, N \quad (1)$$

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with

$$c = \sum_{i=1}^N c_i. \quad (2)$$

In the above constitutive law, the coefficients \tilde{D}_i are the usual parabolic diffusion coefficients, which account for the interaction of the i -component with the surrounding medium, while the coefficients D_i are related to the interaction among the isotopes. We assume that the coefficients \tilde{D}_i and D_i are positive constants.

The main idea is that each component feels the gradient of the total element concentration in a relative percentage $\frac{c_i}{c}$, because the isotopes are chemically indistinguishable. For details of the original physical model see [8]. A similar model is presented in [3].

In the case of radioactive isotopes, we have to take into account the radioactive decay law, which for spacially homogeneous distributions is a linear ODE system

$$\frac{d\underline{C}}{dt} = \underline{\Lambda}\underline{C}, \quad \underline{C} \in \mathbf{R}^N, \quad (3)$$

with $\underline{\Lambda}$ a suitable $N \times N$ constant matrix.

An important example is the couple $U_{238}(c_1), U_{234}(c_2)$ for which the decay law is

$$\begin{cases} \frac{dc_1}{dt} = -\lambda_1 c_1, \\ \frac{dc_2}{dt} = \lambda_1 c_1 - \lambda_2 c_2, \end{cases} \quad (4)$$

with $0 < \lambda_1 \ll \lambda_2$.

The model is relevant in various physical applications, among which let us mention the distribution of radionuclides in the ground water around a deep repository for used nuclear fuel, whose study is an essential requirement for future safety analysis, see e.g. [6].

In the paper [4] we have studied some qualitative properties of the solution in the physically relevant assumption that the diffusion coefficients \tilde{D}_i are much smaller than the D_i , thus showing the appearance of a ‘‘hyperbolic’’ behaviour for the c_i , quite interesting for the applications. Here we will consider the general case of positive diffusion coefficients, possibly all different.

We will prove the existence and uniqueness of classical solution of the resulting system with Dirichlet boundary conditions in the physically relevant assumption that

$$K \geq c_i \geq k > 0, \quad i = 1, \dots, N, \tag{5}$$

k, K constant.

2. Statement of the problem

Let Ω be a bounded domain of \mathbf{R}^n with regular boundary $\partial\Omega$, the model problem is

$$\begin{cases} \frac{\partial c_i}{\partial t} = -\operatorname{div} J_i + \sum_{j=1}^N \lambda_{ij} c_j, & \text{in } \Omega \times (0, T), \\ c_i|_{\partial\Omega} = f_i, & \text{in } \partial\Omega \times (0, T), \\ c_i(x, 0) = c_{i0}(x) & \text{in } \bar{\Omega}, \end{cases} \tag{6}$$

for $i = 1, \dots, N$, with J_i given by (1), (2).

The assumptions on the boundary-initial data are the physical one (5) plus smoothness, that is, denoting by Γ the parabolic boundary $\partial\Omega \times [0, T] \cup \Omega \times \{t = 0\}$

Hp A) $c_i|_{\Gamma}$ smooth, $c_i|_{\Gamma}$ satisfies (5) for $i = 1, \dots, N$.

As for the matrix Λ we have to assume that either it is zero, (for stable isotopes such as the couple (Cl_{37}, Cl_{35})), or it models a radioactive decay, (such as the couple (U_{238}, U_{234}) we mentioned in the introduction).

Therefore it is natural to require a positive property for the solution of the ODE initial value problem

$$\begin{cases} \frac{d\underline{C}(t)}{dt} = \Lambda \underline{C}(t), \quad \underline{C} = (c_1, \dots, c_N) \\ \underline{C}(0) = \underline{C}_0, \quad \underline{C}_0 = (c_{10}, \dots, c_{N0}), \end{cases} \tag{7}$$

namely

Hp B) Λ is a constant matrix such that $\forall \underline{C}_0 \in \mathbb{R}^N$, there exists a unique bounded solution $\underline{C}(t; \underline{C}_0)$ for $t \in (0, +\infty)$ and if

- (i) if $c_{i0} \geq 0$ then $c_i(t) \geq 0$, for $i = 1, \dots, N$;
- (ii) if for some i , $c_{i0} > 0$ then $c_i(t) > 0$, for any $t > 0$.

Assumption B) implies that, if $\underline{C}_0 \neq \underline{0}$ and (i) holds, then

$$c(t) = \sum_{i=1}^N c_i(t) > 0, \quad t > 0.$$

REMARK 2.1.

Actually for radioactive isotopes there often exists a so called “secular equilibrium”, i.e. an asymptotically stable positive equilibrium for the ratios $\frac{c_i}{c_j}$. For the couple U_{238}, U_{234} it is $\frac{U_{238}}{U_{234}} \cong 18\,000$. In this situation we should add to assumption B) the following

Hp **B’)** Λ has N real negative eigenvalues, the eigenvector of the largest one has all components of the same sign.

Since it is relevant to consider the total concentration c , we will consider the problem obtained from (6) substituting to the N -th equation the sum of all the equations, setting

$$c_N = c - \sum_{i=1}^{N-1} c_i.$$

Then denoting by

$$\mathcal{C} = (c_1, \dots, c_{N-1}, c) \in \mathbb{R}^N, \tag{8}$$

we can write the problem in the following way, using the notations of [2]:

$$\left\{ \begin{array}{ll} \frac{\partial \mathcal{C}}{\partial t} = \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left(\mathcal{A}_{jk}(\mathcal{C}) \frac{\partial \mathcal{C}}{\partial x_k} \right) + \tilde{\Lambda} \mathcal{C}, & \text{in } \Omega \times (0, T) \\ \mathcal{C}|_{\partial\Omega} = f, & \text{in } \partial\Omega \times (0, T) \\ \mathcal{C}(x, 0) = \mathcal{C}_0(x) & \text{in } \Omega, \end{array} \right. \tag{9}$$

where \mathcal{A}_{jk} , $j, k = 1, \dots, n$ are a family of real matrices $N \times N$ given by

$$\mathcal{A}_{jk}(\mathcal{C}) = A(\mathcal{C})\delta_{jk}, \quad j, k = 1, \dots, n \tag{10}$$

δ_{jk} is the Kronecker symbol, and $A(\mathcal{C})$ is the $N \times N$ real matrix

$$\begin{pmatrix} \tilde{D}_1 & 0 & \dots & 0 & D_1 \frac{c_1}{c} \\ 0 & \tilde{D}_2 & \dots & 0 & D_2 \frac{c_2}{c} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \tilde{D}_{N-1} & D_{N-1} \frac{c_{N-1}}{c} \\ \tilde{D}_1 - \tilde{D}_N & \tilde{D}_2 - \tilde{D}_N & \dots & \tilde{D}_{N-1} - \tilde{D}_N & \tilde{D}_N + D_N + \sum_{i=1}^{N-1} (D_i - D_N) \frac{c_i}{c} \end{pmatrix} \quad (11)$$

The matrix $\tilde{\Lambda}$ is derived in obvious way from Λ :

$$\begin{pmatrix} \lambda_{11} - \lambda_{1N} & \dots & \lambda_{1N} \\ \lambda_{21} - \lambda_{2N} & \dots & \lambda_{2N} \\ \vdots & \ddots & \vdots \\ \sum_{m=1}^N (\lambda_{m1} - \lambda_{mN}) & \dots & \sum_{m=1}^N \lambda_{mN} \end{pmatrix} \quad (12)$$

and

$$\begin{cases} f = (f_1, \dots, f_{N-1}, \sum_{i=1}^N f_i), \\ \mathcal{C}_0 = (c_{10}, \dots, c_{(N-1)0}, \sum_{i=1}^N c_{i0}). \end{cases} \quad (13)$$

Since we are interested in the physical assumptions, the boundary value cannot be zero, so we reduce to the homogeneous Dirichlet problem in the usual way, see also [2], Sect.11, that is we extend smoothly f in the interior, let us denote the extension by \tilde{f} , and arrive for $u = \mathcal{C} - \tilde{f}$ to the following problem for u

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left(\tilde{\mathcal{A}}_{jk} \frac{\partial u}{\partial x_k} \right) + F(u, \partial u, x, t), & \text{in } \Omega \times (0, T) \\ u|_{\partial\Omega} = 0, & \text{in } \partial\Omega \times (0, T) \\ u(x, 0) = \mathcal{C}_0 - \tilde{f}(x, 0) & \text{in } \Omega, \end{cases} \quad (14)$$

where

$$\begin{cases} \tilde{\mathcal{A}}_{jk} = A(u + \tilde{f}) \delta_{ik}, \\ F = \tilde{\Lambda}u + \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left(\tilde{\mathcal{A}}_{jk} \frac{\partial \tilde{f}}{\partial x_k} \right) + \tilde{\Lambda} \tilde{f} - \frac{\partial \tilde{f}}{\partial t}. \end{cases} \quad (15)$$

Since F is affine in ∂u , the principal part of the operator, see the first of (15), is in “separated divergence form” in the sense of [2] and

we have Dirichlet boundary conditions, we can apply the results of [1], [2], provided we prove a “normal ellipticity condition” (see [2] p.151 and [1] p.219, 220) which in this case (see Theorem 4.4 of [2]) reduces to prove that

$$\text{“all the eigenvalues of the matrix } A \text{ have positive real part”}. \quad (16)$$

Let us remark that for our problem the “normal ellipticity condition” corresponds to the original definition of parabolic system by Petrovskii (see [9]) (see Def. 1,2 Cap. VII, Sect.8 of [7]), and that in space dimension one a simpler proof of Theorem 4.4 of [2] can be found in [7], p.624.

We will prove in the next Section 3 the “normal ellipticity condition” on A , however we want to stress here that our operator (15) is not “strongly elliptic”, that is it does not satisfy the condition (see [2], (10) of Sect.1, [7], Def.7 Cap. VII, Sect.8)

$$\sum_{i,j=1}^N A_{ij} \xi_i \xi_j > 0, \quad \forall \underline{\xi} \in \mathbb{R}^N \setminus \{0\}, \quad (17)$$

in all the domain of interest (see (5)).

In order to see this, it is sufficient to consider the special, but physically relevant, case in which

$$\tilde{D}_i = \tilde{D}, \quad D_i = D, \quad i = 1, \dots, N. \quad (18)$$

In fact in this case the matrix A is upper triangular and has $N-1$ eigenvalues equal to \tilde{D} and one eigenvalue equal to $\tilde{D} + D$, so it is “normal elliptic”, but a straightforward calculation shows that (17) does not hold for all the interesting range of \tilde{D} , D and for c_i satisfying (5). In particular, one can see that for $N = 2$ the condition (17) does not hold for any \tilde{D} positive, but only for sufficiently large \tilde{D} (remark that the matrix A is not symmetric).

3. Normal ellipticity condition

In view of assumption (5) and of the definition of A , let us define

$$r = (r_1, \dots, r_{N-1}) \in \mathbb{R}^{N-1}, \quad r_i = \frac{c_i}{c}. \quad (19)$$

We will consider the coefficients of the matrix A defined by (10), as functions of r , affine in r , in the open domain

$$G = \left\{ r \in \mathbb{R}^{N-1} : r_i > 0, \quad i = 1, \dots, N-1, \quad \sum_{j=1}^{N-1} r_j < 1 \right\} \subset \mathbb{R}^{N-1}. \tag{20}$$

We will assume, without loss of generality, the order

$$\tilde{D}_N \leq \tilde{D}_{N-1} \leq \dots \leq \tilde{D}_1. \tag{21}$$

Let us list first some simple cases, however of interest in the applications.

case 1. $D_i = D, \quad \tilde{D}_i = \tilde{D}, \quad i = 1, \dots, N$

We have already seen in Sect.2 that we have an eigenvalue \tilde{D} with algebraic multiplicity $N-1$ and an eigenvalue $\tilde{D} + D$.

case 2. $\tilde{D}_i = \tilde{D}, \quad i = 1, \dots, N$

Again A is upper triangular, \tilde{D} is an eigenvalue with multiplicity $N-1$ and the last one is $A_{NN}(r)$

case 3. $D_i = D, \quad i = 1, \dots, N$

Then in the matrix A the term $A_{NN} = \tilde{D} + D$ is constant.

Let us remark that for A_{NN} we have

$$\tilde{D}_N + \min_i D_i \leq A_{NN}(r) \leq \tilde{D}_N + \max_i D_i, \quad r \in \bar{G}. \tag{22}$$

Since A_{NN} is affine in r and (22) holds in \bar{G} , we have that

$$\bar{G} \subset \tilde{G} = \{r \in \mathbb{R}^{N-1} : A_{NN} > 0\}, \tag{23}$$

actually \bar{G} is bounded away from $\partial\tilde{G}$.

We will now prove the following

LEMMA 3.1. *The matrix A has, in G , N real positive bounded eigenvalues, say $\lambda_i, \quad i = 1, \dots, N$ and*

$$0 < \tilde{D}_N \leq \lambda_1 \leq \tilde{D}_{N-1} \leq \dots \leq \lambda_{N-1} \leq \tilde{D}_1 \leq \lambda_N \leq \tilde{D}_1 + \max_i D_i.$$

Proof. Let us consider first the case of \tilde{D}_i all different.

Then the eigenvalues of A are $\neq \tilde{D}_i$, $i = 1, \dots, N-1$. In fact, assume the contrary, i.e. one eigenvalue is \tilde{D}_i for some i , and look at the eigenvector equation $(A - \tilde{D}_i Id) \underline{v} = \underline{0}$, that is:

$$\begin{cases} (\tilde{D}_j - \tilde{D}_i)v_j + D_j r_j v_N = 0, & j \neq i, N, \\ D_i r_i v_N = 0, \\ \sum_{j=1}^{N-1} (\tilde{D}_j - \tilde{D}_N)v_j + (A_{NN} - \tilde{D}_i)v_N = 0. \end{cases} \quad (24)$$

Since $r \in G$, $r_i > 0$ hence $v_N = 0$, therefore $v_j = 0$ for $j \neq i$, and from the last row $v_i = 0$ (recall that we are assuming that the \tilde{D}_i are all different). So $\underline{v} = \underline{0}$ and we get a contradiction.

Multiplying the i -th row for $-\frac{(\tilde{D}_i - \tilde{D}_N)}{\tilde{D}_i - \lambda}$ and summing it to the N -th row, for $i = 1, \dots, N-1$, we get

$$\begin{aligned} \det(A - \lambda Id) &= \det \begin{pmatrix} \tilde{D}_1 - \lambda & 0 & \dots & 0 & D_1 r_1 \\ 0 & \tilde{D}_2 - \lambda & \dots & 0 & D_2 r_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \tilde{D}_{N-1} - \lambda & D_{N-1} r_{N-1} \\ 0 & 0 & \dots & 0 & -g(\lambda; r) \end{pmatrix} \\ &= -(\tilde{D}_1 - \lambda) \dots (\tilde{D}_{N-1} - \lambda) g(\lambda; r) \end{aligned} \quad (25)$$

where $g(\lambda; r)$ is given by

$$g(\lambda; r) = \sum_{i=1}^{N-1} \frac{(\tilde{D}_i - \tilde{D}_N) D_i r_i}{\tilde{D}_i - \lambda} + \lambda - A_{NN}(r). \quad (26)$$

Since $\lambda \neq \tilde{D}_i$, $i \neq N$, the eigenvalues are the zeros of $g(\lambda; r)$.

We have:

$$\frac{dg}{d\lambda} = \sum_{i=1}^{N-1} \frac{(\tilde{D}_i - \tilde{D}_N) D_i r_i}{(\tilde{D}_i - \lambda)^2} + 1 > 1, \quad (27)$$

for all r in G .

For any $i \neq N$ we have $\forall r_i > 0$, $r \in \mathbb{R}^{N-1}$

$$\begin{cases} \lim_{\lambda \rightarrow \tilde{D}_i^-} g(\lambda; r) = +\infty, \\ \lim_{\lambda \rightarrow \tilde{D}_i^+} g(\lambda; r) = -\infty, \\ \lim_{\lambda \rightarrow +\infty} g(\lambda; r) = +\infty. \end{cases} \quad (28)$$

Therefore there exists a unique zero in each interval $(\tilde{D}_{i+1}, \tilde{D}_i)$, $i = 1, N - 2$, and $(\tilde{D}_1, +\infty)$.

As for the smallest eigenvalue, consider for $r \in G$

$$\begin{aligned} g(\tilde{D}_N; r) &= \sum_{i=1}^{N-1} D_i r_i + \tilde{D}_N - \left(\tilde{D}_N + D_N + \sum_{i=1}^{N-1} (D_i - D_N) r_i \right) \\ &= -D_N \left(1 - \sum_{i=1}^{N-1} r_i \right) < 0. \end{aligned} \tag{29}$$

Therefore, the smallest eigenvalue is $> \tilde{D}_N$, since (28) holds for $i = N - 1$ and g is increasing.

To prove the upper bound for the eigenvalues we simply show that

$$g(\tilde{D}_1 + \max_i D_i; r) \geq 0 \quad \forall r \in \overline{G}. \tag{30}$$

Therefore we have proved the Lemma in G with the strict inequalities in the case of \tilde{D}_i all different.

As we already remarked if all the \tilde{D}_i are equal the Lemma is true in \overline{G} (see (22)), so we are left with two situations

- (i) there exists one index i , $1 < i \leq N - 1$, such that $\tilde{D}_i = \tilde{D}_N$,
- (ii) there exist two indices i, j , $1 \leq i < j \leq N - 1$, such that $\tilde{D}_i = \tilde{D}_j$,

In case (i) all the columns with index i are zero but for the i -th element, therefore we have $N - i$ eigenvalues equal to \tilde{D}_N and we can reduce to a matrix of order $i \times i$ of the same form of A (suppressing rows and columns of index $i, \dots, N - 1$) to which we can apply again the Lemma.

In case (ii) a direct examination either of the characteristic equation or of the system $(A - \lambda Id)v = 0$ shows that A has $j - i$ eigenvalues \tilde{D}_i and we can reduce to a matrix $(N - (j - i)) \times (N - (j - i))$ of the same form of A and repeat the initial argument, thus concluding the proof. □

REMARK 3.2. *In the realistic assumption on the diffusion coefficients, we have that*

$$\tilde{D}_N + \min_i D_i > \tilde{D}_1.$$

Then we have that

$$g(\tilde{D}_N + \min_i D_i; r) \leq 0 \quad \forall r \in \overline{G}.$$

Therefore we have in this case that the largest eigenvalue λ_N is such that in \overline{G}

$$\tilde{D}_1 < \tilde{D}_N + \min_i D_i \leq \lambda_N \leq \tilde{D}_N + \max_i D_i.$$

In other words the first $N-1$ eigenvalues are of the order of \tilde{D}_i , while the last one is of the order $\tilde{D}_i + D_i$, i.e. larger than the previous ones.

Consider now the boundary of G , we have the following:

LEMMA 3.3. *On the boundary ∂G A still has N real positive eigenvalues and:*

$$\tilde{D}_N \leq \min_i \lambda_i < \max_i \lambda_i \leq \tilde{D}_1 + \max_i D_i, \quad \forall r \in \partial G. \quad (31)$$

Let us remark in particular that, looking at (29), we have

$$\min_i \lambda_i = \tilde{D}_N \text{ if } r : \sum_{i=1}^{N-1} r_i = 1. \quad (32)$$

Hence also if the \tilde{D}_i are different, if the smallest one is reduced to zero, the problem degenerates.

Proof. Let us consider only the case of all \tilde{D}_i different, since, as we have seen previously, we can always reduce to such a case in a smaller dimension than N .

If $r \in \{r \in \mathbb{R}^{N-1} : r_i > 0, \sum_{i=1}^{N-1} r_i = 1\}$, then the same proof of Lemma 3.1 holds and hence its assertion with the strict inequalities but for the first one, i.e. $\lambda_1 = \tilde{D}_N$ (see (29)).

If $r \in \Gamma_i = \{r \in \mathbb{R}^{N-1} : r_i = 0, r_j \geq 0, \sum_{j=1}^{N-1} r_j \leq 1\}$, then, looking at matrix $(A - \lambda Id)$, we have that $\lambda = \tilde{D}_i$ is an eigenvalue and we can reduce the problem to the minor i, i of A , say $A^{i,i}$.

If all the $r_j, j \neq i$ are positive, we can apply directly Lemma 3.1 to $A^{i,i}$. If some other $r_j = 0$, then we have that the corresponding \tilde{D}_j are eigenvalues and reduce again the problem, repeating the present argument.

Let us remark that in the origin $r = 0$ the matrix A is lower triangular and the eigenvalues are $\tilde{D}_{N-1}, \dots, \tilde{D}_1, \tilde{D}_N + D_N$, the last one is the largest one in the “physical” assumption on \tilde{D}_i, D_i . Let us also remark that there are points in ∂G in which the eigenvalues have algebraic multiplicity 2. \square

REMARK 3.4. *Since the elements of A are affine in $r \in \mathbb{R}^{N-1}$, we can write $A(r)$, given any $r_0 \in \mathbb{R}^{N-1}$:*

$$A(r) = A(r_0) + T,$$

where the matrix $T, N \times N$, is:

$$\begin{pmatrix} 0 & \dots & 0 & D_1(r_1 - r_{10}) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & D_{(N-1)}(r_{(N-1)} - r_{(N-1)0}) \\ 0 & \dots & 0 & \sum_{i=1}^{N-1} (D_i - D_N)(r_i - r_{i0}) \end{pmatrix}. \tag{33}$$

Then we have:

$$\|T\| \leq \max_i D_i \|r - r_0\|. \tag{34}$$

Since the spectrum of A is continuous, uniformly for $\|r - r_0\| < 1$, (see [5], Thm. 5.14, Cap.II, Sect. 1.2, 1.7, 5.7, Cap.II) and Lemmas 3.1, 3.3 hold, we have that, strictly: $\overline{G} \subset G_{ne} = \{r \in \mathbb{R}^{N-1} : \text{eigenvalues of } A \text{ have positive real part}\}$.

Let us also remark that when $N = 2$ one can easily determine G_{ne} , since

$$G_{ne} = \{r \in \mathbb{R} : \text{Tr}A > 0, \det A > 0\},$$

and $\text{Tr}A, \det A$ are affine in r . Moreover we have:

$$\begin{cases} \text{Tr}A \geq \tilde{D}_1 + \tilde{D}_2 + \min_i D_i, \\ \det A \geq \tilde{D}_1 \tilde{D}_2 + \min(\tilde{D}_1 D_2, \tilde{D}_2 D_1), \end{cases} \tag{35}$$

for $r \in [0, 1] = \overline{G}$.

4. Existence and uniqueness

The result of Sect.3, Lemmas 3.1, 3.3, Remark 3.4, allows us to apply the existence and uniqueness theorem of [2], see Corollary 9.3, p.57, which gives a classical solution in a given interval $[0, T]$.

More regularity on the data, $f \in C^\infty$, gives a C^∞ solution, see Corollary 9.4 of [2], but one can consider less regular initial datum, see Theorem 9.2 of [2].

We can then repeat the argument in [2], p.18, to show that \underline{C} is bounded away from the boundary of

$$\tilde{G} = \{\underline{C} \in \mathbb{R}^N : c_i > 0, \quad i = 1, \dots, N, \} \subset \mathbb{R}^N$$

and we can use then Theorem 2 of [1] to get global existence.

In fact once we have a classical solution, we can rewrite the equation for c_i as a linear parabolic equation, namely

$$\frac{\partial c_i}{\partial t} = \tilde{D}_i \Delta c_i + b \cdot \nabla c_i + a c_i, \quad i = 1, \dots, N \quad (36)$$

with $b \in \mathbb{R}^N$, a a function of x, t .

Then by the maximum principle we have that in the assumption A), $c_i > 0$ in $\overline{\Omega} \times [0, T]$ for any $T > 0$, and bounded.

We have thus proved the following

THEOREM 4.1. *In assumptions A) and B) there exists a unique classical solution of problem (6) for any $T > 0$.*

Actually, from the results of Sect.3, see Remark 3.4, we can weaken assumption A), assuming instead the following:

Hp **A')** $c_i|_\Gamma$ smooth, $K \geq c_i|_\Gamma \geq 0$ for $i = 1, \dots, N$ and such that $\sum_{i=1}^N c_i|_\Gamma \geq k > 0$.

REMARK 4.2.

With the same method one can deal with the Neumann boundary value problem, that is

$$J_i \cdot \nu = g_i, \quad \text{in } \partial\Omega \times (0, T), \quad i = 1, \dots, N, \quad (37)$$

where ν is the exterior normal to $\partial\Omega$, J_i given by (1). Moreover one can also treat the case of mixed Dirichlet and Neumann boundary condition as in [2], p.14, introducing, with the notations of Sect.2, the boundary operator

$$B(\mathcal{C})\mathcal{C} = \delta \left(\sum_{j,k=1}^n \mathcal{A}_{jk} \nu_j \frac{\partial \mathcal{C}}{\partial x_k} \right) + (1 - \delta)\mathcal{C}, \tag{38}$$

with either $\delta = 0$ (Dirichlet condition) or $\delta = 1$ (Neumann condition). In fact Theorem 4.2 of [2], p.32, holds also in this situation.

Let us make some comments on the special cases 1. and 2. of Sect.3, because of their relevance in the applications.

case 1. ($D_i = D$, $\tilde{D}_i = \tilde{D}$, $i = 1, \dots, N$)

If $\Lambda = 0$ (stable isotopes) the system (9) is actually decoupled, since one can solve first the equation for c_j which is a heat equation with diffusion coefficient equal to $(\tilde{D} + D)$. Then the equations for each c_i are linear parabolic equations (see (6)) with a much less diffusion coefficient \tilde{D} . If $\Lambda \neq 0$ (radioactive isotopes) the system (9) is still coupled. However, looking at the equation for the total concentration c , we can see that c has a purely diffusive behaviour, while the single specie's concentrations have a lower diffusion coefficient and a possibly strong gradient term, thus exhibiting a behaviour close to a "hyperbolic one (see also [4]), in agreement with physical observations, see [8].

case 2. $\tilde{D}_i = \tilde{D}$, $i = 1, \dots, N$

Also if $\Lambda = 0$ the system (9) is not decoupled since the diffusion coefficient for c is A_{NN} which depends on c_1, \dots, c_{N-1}, c (see (10)). However the equation for c is

$$\frac{\partial c}{\partial t} = \text{div} (A_{NN} \nabla c), \tag{39}$$

with uniform bounds on A_{NN} (see (23)), which gives good information on the qualitative behaviour of the total concentration c .

Moreover, also if $\Lambda \neq 0$, one can apply the stronger results of [1] for triangular matrixes A (Theorem 3).

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