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Special Classes of Closed Four–Manifolds

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Dedicated to the memory of our dear friend Marco Reni

Summary. - *In this paper we present several results and state some open problems on the classification of topological and geometric structures of closed connected oriented (smooth) four–manifolds. In particular, we discuss many interesting classes of closed four–manifolds satisfying additional properties, that is, spin manifolds, manifolds with special homology (resp. homotopy), exact manifolds, geometric manifolds, and smooth manifolds. The results, some of them due to the authors and their collaborators, are obtained by using methods and techniques from algebraic and differential topology, and homological algebra.*

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1. Spin Manifolds

Let $\mathrm{SO}(n)$ be the special orthogonal linear group. The following facts are well-known:

1) $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$;

2) The universal covering space $\widetilde{\mathrm{SO}(n)}$ of $\mathrm{SO}(n)$ is $\mathrm{Spin}(n)$;

3) There is a fibration map

$$\mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \xrightarrow{\ \rho\ } \mathrm{SO}(n)$$

with fiber $\mathbb{Z}_2$;

4) The fibration map ρ induces a map between the classifying spaces

$$\mathrm{BSpin}(n) \longrightarrow \mathrm{BSO}(n) \sim G_n(\mathbb{R}^{n+m})$$

where $\mathrm{BSO}(n)$ is approximated by the set of all n-dimensional vector subspaces of $\mathbb{R}^{n+m}$.

Let $M^m \subset \mathbb{R}^{n+m}$ be a closed connected oriented smooth m-manifold which embeds smoothly in $\mathbb{R}^{n+m}$ ($m = 4$ in our case). We can now consider the Gauss map

$$\nu : M^m \to G_n(\mathbb{R}^{n+m}) \sim \mathrm{BSO}(n).$$

For any point $x \in M$, let $O_x(M)$ be the vector subspace orthogonal to the embedding $M^m \subset \mathbb{R}^{n+m}$, briefly called the *normal space* of M at x. Then $\nu(x)$ is the (linear) subspace of $\mathbb{R}^{n+m}$ passing through the origin and parallel with $O_x(M)$. The manifold M is *spin*, i.e. $w_2(M) = 0$, if and only if there exists a lifting $\bar{\nu}$ of the Gauss map ν such that the diagram

$$\begin{array}{ccc} M^m & \xrightarrow{\ \bar{\nu}\ } & \mathrm{BSpin}(n) \\ \| & & \downarrow \\ M^m & \xrightarrow[\ \nu\ ]{} & \mathrm{BSO}(n) \end{array}$$

commutes. Any lifting $\bar{\nu}$ of the Gauss map ν is called a *spin structure* on M.

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Let M_1^m and M_2^m be closed oriented spin smooth m-manifolds with spin structures

$$\begin{array}{ccc} M_i^m & \xrightarrow{\bar{\nu}_i} & \mathrm{BSpin}(n) \\ \| & & \downarrow \\ M_i^m & \xrightarrow[\nu_i]{} & \mathrm{BSO}(n) \end{array}$$

for $i = 1, 2$. Then M_1 is said to be *spin cobordant* to M_2 if there exist a compact connected oriented smooth $(m+1)$-manifold W^{m+1} with $\partial W = M_1 \cup (-M_2)$ and a spin structure on W

$$\begin{array}{ccc} W & \xrightarrow{\bar{\nu}_W} & \mathrm{BSpin}(n) \\ \| & & \downarrow \\ W & \xrightarrow[\nu_W]{} & \mathrm{BSO}(n) \end{array}$$

such that

$$\bar{\nu}_W|_{\partial W} = \bar{\nu}_1 \cup \bar{\nu}_2.$$

The *spin cobordism group* Ω_m^{Spin} is the set of all equivalence classes of closed oriented spin smooth m-manifolds modulo spin cobordant relation (see for example [37]).

THEOREM 1.1. *A closed oriented spin smooth* 4*-manifold* M *is null cobordant in* Ω_4^{Spin}*, i.e. is the boundary of a compact oriented spin smooth* 5*-manifold* W^5 *if and only if the signature of* M *vanishes.*

The Rohlin theorem gives an isomorphism

$$\Omega_4^{\mathrm{Spin}} \xrightarrow[\cong]{} \mathbb{Z}$$

which sends any cobordism class $[(M, \bar{\nu}_M)]$ to $\sigma(M)/16$ (here $\sigma(M)$ is the signature of M). In particular, the Kummer surface $K_4 = \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{C}P^3$ is a generator of Ω_4^{Spin}.

The following result was proved by Cappell and Shaneson [2] in 1979 (see also [22] and [43]).

THEOREM 1.2. *A closed oriented smooth* 4*-dimensional manifold embeds smoothly in Euclidean* 6*-space if and only if it is spin and has zero signature.*

Cochran investigated in [13] the question of when a spin closed smooth 4-manifold M with trivial signature also embeds smoothly in $\mathbb{R}^5$. The main result of [13] gives sufficient conditions in terms of $\pi_1(M)$, which work in a broad range of situations. For instance, $H_1(M)$ a product of at most two cyclic groups or $\pi_1(M)$ a free product of cyclic groups suffices. The proof is direct and was obtained by surgery of Kervaire–Milnor type [33] [36] [38], and is not an application of the usual surgically proved embedding results (see [48]). Furthermore, all geometrically "simple" 4-manifolds which embed in Euclidean 6-space (e.g. the topological product of two closed surfaces and the product of a closed 3-manifold with the circle) do indeed embed in $\mathbb{R}^5$. More generally, if $H_2(M)$ is finite or $H_2(\pi_1(M)) \cong 0$, then Theorem 4.1 of [13] gives necessary and sufficient conditions for M to embed in $\mathbb{R}^5$. This yields the first known examples of closed 4-manifolds with $\pi_1 \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ which embed in $\mathbb{R}^6$ but not in $\mathbb{R}^5$ (see [12]). Here we state Theorem 4.1 mentioned above (see [13, p. 259]).

Theorem 1.3. *Let M be a closed connected orientable smooth 4–manifold, and let $G = \pi_1(M)$. Then M embeds smoothly in $\mathbb{R}^5$ if and only if the following conditions hold:*

i) there exist finitely presented groups G_i and homomorphisms $g_i : G \to G_i$, $i = 1, 2$, such that the pushout of the diagram

$$\begin{array}{ccc} G & \xrightarrow{g_1} & G_1 \\ {\scriptstyle g_2}\downarrow & & \downarrow \\ G_2 & \longrightarrow & 1 \end{array}$$

is trivial;

ii) the g_i induce an isomorphism $H_1(G) \cong H_1(G_1) \oplus H_1(G_2)$, and an epimorphism $(\psi_1, \psi_2) : H_2(G) \to H_2(G_1) \oplus H_2(G_2)$;

iii) there exist spin structures σ_i on M and maps $f_i : M \to BG_i$ which induce the g_i and such that $[(M, \sigma_i, f_i)] = 0$ in $\Omega_4^{\mathrm{Spin}}(G_i)$ for any $i = 1, 2$;

iv) there exist one–half rank subgroups C and D such that $C \oplus D \cong H_2(M)$, $D \perp D$ with respect to the intersection pairing, and $C \subset \ker \psi_2$, $D \subset \mathrm{Ker}\, \psi_1$; and *v) either $H_2(G_1)$ torsion–free and $H_2(G_2) \cong 0$ or $H_2(G_1)$ and $H_1(G_2)$ torsion–free.*

For the definition of spin cobordism groups $\Omega_4^{\mathrm{Spin}}(G)$, where G is a group, we refer to [14], [39] and [40]. One consequence is (see [13, Theorem 6.2])

THEOREM 1.4. *Let M be a closed connected orientable smooth spin 4–manifold with trivial signature. Then if any of the following conditions hold, M will embed smoothly in $\mathbb{R}^5$.*

a) $H_1(M)$ is the direct sum of fewer than 3 cyclic groups;

b) $\pi_1(M)$ is a free product of any number of cyclic groups; or

c) $\pi_1(M) \cong G_1 \times G_2$, where

$$H_4(G_i) \cong H_3(G_i;\mathbb{Z}_2) \cong H_2(G_i;\mathbb{Z}) \cong \mathrm{Tor}(H_1(G_i),\mathbb{Z}_2) \cong 0$$

for any $i = 1, 2$.

As general references for the algebraic and differential topology of 4-manifolds and surgery theory see [20], [22], [26], [34], [36], and [48]. Basic concepts and results of homological algebra can be found in [3].

2. Four-manifolds with vanishing second homology

We consider closed oriented smooth 4-manifolds M with second integral homology group $H_2(M;\mathbb{Z}) \cong 0$.

EXAMPLES 2.1. *1) Four-manifolds homotopy equivalent to the connected sum $Q = \#_k(\mathbb{S}^1 \times \mathbb{S}^3)$ (see [4], [6], [27], and [29]). In this case, we have also $\pi_2 \cong 0$.*

2) The boundary of a regular neighborhood of an acyclic connected 2-complex (arising from a finitely presented group) embedded in Euclidean 5-space.

3) Four-manifolds obtained from the standard 4-sphere $\mathbb{S}^4$ by surgery on 2-knots (see for example [26]). Let K be a 2-knot in $\mathbb{S}^4$. Then the closed 4-manifold obtained from $\mathbb{S}^4$ by surgery on K is defined as:

$$M(K) = (\mathbb{S}^4 \backslash \mathbb{S}^2 \times \mathring{D}^2) \cup_{\mathbb{S}^2 \times \mathbb{S}^1} (D^3 \times \mathbb{S}^1).$$

Of course, we have $\pi_1(M(K)) \cong \pi_1(K)$. It is known that $M(K)$ is aspherical if and only if $\pi_1(K)$ is an PD_4^+-group and the image of the fundamental class $[M(K)] \in H_4(M(K);\mathbb{Z})$ is non zero in $H_4(\pi_1(K);\mathbb{Z})$ (see for example [26]).

If $H_2(M;\mathbb{Z}) \cong 0$, then $H_1(M) \cong H^3(M) \cong FH_3(M)$ is a free abelian group (use Poincaré duality), i.e. $H_1(M) \cong \oplus_k \mathbb{Z}$. Applying the Universal Coefficient Theorem we get

$$0 \xrightarrow[\cong]{} \mathrm{Ext}_{\mathbb{Z}}(H_1(M),\mathbb{Z}_2) \longrightarrow H^2(M;\mathbb{Z}_2) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_2(M),\mathbb{Z}_2) \xrightarrow[\cong]{} 0,$$

hence $H^2(M;\mathbb{Z}_2) \cong 0$, i.e. M is a spin manifold. The signature of M vanishes since the integral intersection form of M is trivial (use $H_2(M;\mathbb{Z}) \cong 0$). So Theorem 1.1 and Theorem 1.2 apply in our case to give the following

THEOREM 2.2. *Let M be a closed oriented smooth 4-manifold with $H_2(M;\mathbb{Z}) \cong 0$. Then M embeds smoothly in $\mathbb{R}^6$. Moreover, there exists a compact connected oriented smooth spin 5-manifold W^5 such that $M = \partial W$.*

By a sequence of surgeries we can always assume that W is simply-connected. For this, we observe that a surgery along an embedding

$$\varphi : \mathbb{S}^1 \times D^4 \to W^5$$

does not disturb the boundary $\partial W = M$ and the property $w_2 = 0$. So we obtain a spin simply-connected 5-manifold W with $\partial W = M$.

The following result was proved in [10].

THEOREM 2.3. *If $H_2(M;\mathbb{Z}) \cong 0$, then we can do surgery on the group $H_2(W;\mathbb{Z}) \cong \pi_2(W)$. In other words, there is a spin simply-connected smooth 5-manifold W with $\partial W = M$ and $\pi_2(W) \cong 0$.*

Applying the homology exact sequence of the pair (W, M) we get:

i) $H_5(W) \cong 0$ since $\partial W = M$ is nonempty;

ii) $H_4(W) \cong H^1(W, \partial W) \cong 0$ since we have the exact sequence:

$$H^0(W) \xrightarrow[\cong]{} H^0(M) \longrightarrow H^1(W,\partial W) \longrightarrow H^1(W) \cong 0$$

iii) The isomorphism $H_3(W) \cong H^2(W, \partial W)$ and the exact sequence:

$$0 \cong H^1(W) \longrightarrow H^1(M) \xrightarrow[\cong]{} H^2(W, M) \longrightarrow H^2(W) \cong 0$$

imply that $H_3(W) \cong H^1(M) \cong H_1(M) \cong \oplus_k \mathbb{Z}$.

Hence W has the homotopy type of a wedge $\vee_k \mathbb{S}^3$. Now let us consider the double

$$DW = W \cup_M W.$$

We have:

1) $\pi_1(DW) \cong 0$;

2) $H_2(DW) \cong H_1(M) \cong \oplus_k \mathbb{Z}$;

3) DW is spin.

It follows from Barden's classification theorem of closed simply-connected smooth 5-manifolds [1] (see also [45]) that DW is diffeomorphic to the connected sum $\#_k(\mathbb{S}^2 \times \mathbb{S}^3)$.

Summarizing, we have the following result [10]

Theorem 2.4. *Let M be a closed oriented smooth 4-manifold with $H_2(M; \mathbb{Z}) \cong 0$.*

1) Then M bounds a compact smooth spin oriented 5-manifold W which is homotopy equivalent to a wedge $\vee_k \mathbb{S}^3$ of 3-spheres, where k is the rank of $H_1(M)$;

2) Furthermore, M embeds smoothly in the connected sum of k copies of $\mathbb{S}^2 \times \mathbb{S}^3$ (hence in $\mathbb{R}^6$).

Example 2.5.

$$Q = \#_k(\mathbb{S}^1 \times \mathbb{S}^3),$$

$$W = \#_k(D^2 \times \mathbb{S}^3),$$

$$DW \cong \#_k(\mathbb{S}^2 \times \mathbb{S}^3).$$

In this case, we have also a compact 5-manifold $V = \#_k(\mathbb{S}^1 \times D^4)$ such that $\partial V = Q$ and $\pi_1(V) \cong *_k\mathbb{Z} \cong \pi_1(Q)$. Moreover, we have

$$X = W \cup_Q V = \mathbb{S}^5,$$

hence Q embeds smoothly in $\mathbb{S}^5$ (or $\mathbb{R}^5$).

QUESTION 2.6. *What can we say for a generic* 4*-manifold* M *with* $H_2(M;\mathbb{Z}) \cong 0$*?*

We are going to construct a compact smooth 5-manifold V with $\partial V = M$, $\pi_1(V) \cong \pi_1(M)$, and $H_2(V;\mathbb{Z}) \cong 0$. For this, we need the hypothesis $f_*([M]) = 0$, where

$$f : M \to B\pi_1(M)$$

is the classifying map for the universal covering, and $[M] \in H_4(M;\mathbb{Z})$ is the fundamental class of M. Under this condition, there is a compact smooth 5-manifold V with boundary $\partial V = M$, and $\pi_1(V) \cong \pi_1(M)$. Now it is impossible in general to kill $\pi_2(V)$. But after a finite sequence of surgeries of Kervaire-Milnor type, we can simplify $H_2(V;\mathbb{Z})$. In fact, we can always assume that $H_2(V;\mathbb{Z})$ is isomorphic to either 0, $\mathbb{Z}_2$, or $\mathbb{Z}$ (for the proof see [10]).

The closed smooth 5-manifold

$$X = W \cup_M V$$

satisfies the following properties:

i) $\pi_1(X) \cong 0$ by Van Kampen's theorem since $\pi_1(V) \cong \pi_1(M)$;

ii) $H_2(X)$ is isomorphic to either $\mathbb{Z}$, $\mathbb{Z}_2$, or the trivial group;

iii) X is diffeomorphic to either X_∞, X_{-1} or $\mathbb{S}^5$ (by Barden's classification theorem).

Recall that X_{-1} and X_∞ are constructed as follows. Let $B \to \mathbb{S}^2$ be the non-trivial D^3-bundle, and let B^* be the same with opposite orientation. Then we have $H_2(\partial B) \cong \mathbb{Z} \oplus \mathbb{Z}$ with generators p and q. In fact, ∂B is homeomorphic to $\mathbb{C}P^2 \# (-\mathbb{C}P^2)$. Let

$$g_{-1} : \partial B \to \partial B$$

and

$$g_\infty : \partial B \to \partial B$$

be diffeomorphisms such that

$$(g_{-1})_*(p) = p, \qquad (g_{-1})_*(q) = -q,$$

and $(g_\infty)_*$ is the identity on $H_2(\partial B)$. Then we define:

$$X_{-1} = B \cup_{g_{-1}} B^*$$

and

$$X_\infty = B \cup_{g_\infty} B^*.$$

It follows that $H_2(X_{-1}) \cong \mathbb{Z}_2$ and $H_2(X_\infty) \cong \mathbb{Z}$.

The following result was proved in [10]:

THEOREM 2.7. *Let M be a closed oriented smooth 4-manifold with $H_2(M;\mathbb{Z}) \cong 0$. Suppose that $f_*[M] = 0$, where $f : M \to B\pi_1(M)$ is the classifying map for the universal covering, and $[M] \in H_4(M;\mathbb{Z})$ is the fundamental class. Then M embeds smoothly either in $\mathbb{S}^5$ (hence in $\mathbb{R}^5$), X_{-1}, or X_∞.*

We complete the section with three open problems.

PROBLEM 2.8: Construct examples of closed connected smooth 4-manifolds M with $H_2(M;\mathbb{Z}) \cong 0$ which embed smoothly in one of the above 5-manifolds but not in the other two.

PROBLEM 2.9: Construct examples of closed connected smooth 4-manifolds M with $H_2(M;\mathbb{Z}) \cong 0$ which do not embed in $\mathbb{R}^5$.

PROBLEM 2.10: Classify up to homotopy equivalence (resp. up to homeomorphism, up to diffeomorphism) closed smooth oriented 4-manifolds with $H_2 \cong 0$.

A partial result concerning Problem 2.10 was proved in [4] (see also [8]):

THEOREM 2.11. *Let M^4 be a closed orientable topological 4-manifold with $\pi_1 \cong *_k\mathbb{Z}$ (free group on k free generators), and $H_2 \cong 0$. Then M is homotopy equivalent (resp. s-cobordant) to $\#_k(\mathbb{S}^1 \times \mathbb{S}^3)$. In particular, a closed orientable topological 4-manifold M is TOP homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^3$ if and only if $\pi_1(M) \cong \mathbb{Z}$ and $H_2 \cong 0$ (or equivalently, $\chi(M) = 0$).*

3. Four-manifolds with special third homotopy

We first consider the class of closed topological 4-manifolds M^4 (possibly nonorientable) with $\pi_3(M) \cong 0$.

The following result was proved in [5]:

Theorem 3.1. *Let M^4 be a closed connected topological 4-manifold such that $\pi_3(M) \cong 0$. If $\pi_1(M)$ is finite, then M is TOP homeomorphic to either $\mathbb{S}^4$, $\mathbb{C}P^2$, $*\mathbb{C}P^2$ (the Chern manifold), $\mathbb{R}P^4$ or the unique non-smoothable homotopy $\mathbb{R}P^4$ (fake $\mathbb{R}P^4$). If $\pi_1(M)$ is infinite, then M is aspherical.*

Remark 3.2. *If the Borel conjecture holds in dimension 4 (homotopy equivalent aspherical 4-manifolds are TOP homeomorphic), Theorem 3.1 yields a complete list of all homeomorphy types of closed 4-manifolds with $\pi_3 \cong 0$. Recall that the conjecture is true for example for π_1 poly-(cyclic or finite), i.e. π_1 admits a finite composition serie whose factors are all infinite cyclic or finite cyclic.*

To prove the theorem above, let us first assume that $\widetilde{M}$ is compact, i.e. $\pi_1(M)$ is finite. Then $\widetilde{M}^{(3)} = \widetilde{M}\backslash \mathring{B}^4$ is homotopy equivalent to a wedge $\vee_r \mathbb{S}^2_i$.

Since the homomorphism

$$\oplus_r \pi_3(\mathbb{S}^2_i) \to \pi_3(\widetilde{M}^{(3)}) \cong \pi_3(\vee_r \mathbb{S}^2_i)$$

is injective, the exact homotopy sequence

$$\pi_4(\widetilde{M}, \widetilde{M}^{(3)}) \cong H_4(\widetilde{M}, \widetilde{M}^{(3)}) \cong \mathbb{Z} \longrightarrow \pi_3(\widetilde{M}^{(3)}) \longrightarrow 0$$

implies that $r \leq 1$. Recall that $\pi_3(\widetilde{M}) \cong \pi_3(M) \cong 0$. Therefore, $\widetilde{M}^{(3)}$ is homotopy equivalent to either zero or $\mathbb{S}^2$. Then $\widetilde{M}$ is homotopy equivalent to either $\mathbb{S}^4$, $\mathbb{C}P^2$ or $*\mathbb{C}P^2$. By Freedman's theorem [19] [20], $\widetilde{M}$ is TOP homeomorphic to one of these manifolds. Therefore, the only possibilities for M are finite quotients of $\mathbb{S}^4$, $\mathbb{C}P^2$, or $*\mathbb{C}P^2$. So M must be TOP homeomorphic to either $\mathbb{S}^4$, $\mathbb{C}P^2$, $*\mathbb{C}P^2$, $\mathbb{R}P^4$ or the unique non-smoothable homotopy $\mathbb{R}P^4$ (fake $\mathbb{R}P^4$) by [20] and [44].

Let us assume now that $\widetilde{M}$ is not compact, i.e. $\pi_1(M)$ is infinite. Then we have $H_1(\widetilde{M}) \cong H_4(\widetilde{M}) \cong 0$, and $H_2(\widetilde{M}) \cong \pi_2(\widetilde{M}) \cong \pi_2(M)$. Suppose $\varphi : \mathbb{S}^2 \to \widetilde{M}$ represents a generator of $H_2(\widetilde{M}) \cong \pi_2(\widetilde{M})$. We are going to prove that φ is null homotopic, and hence $H_2(\widetilde{M}) \cong 0$. Since $\pi_3(M) \cong 0$, the composition

$$\mathbb{S}^3 \xrightarrow{\ \eta\ } \mathbb{S}^2 \xrightarrow{\ \varphi\ } \widetilde{M}$$

is homotopic to zero, where η is the Hopf map. Then φ extends to a map

$$\phi : \mathbb{S}^2 \cup_\eta D^4 = \mathbb{C}P^2 \to \widetilde{M}.$$

Now we have $H_3(\widetilde{M}) \cong 0$ by the commutativity of the following diagram

$$\begin{array}{ccccc}
\pi_4(\widetilde{M}^{(4)}, \widetilde{M}^{(3)}) & \xrightarrow{\theta} & \pi_3(\widetilde{M}^{(3)}) & \longrightarrow & \pi_3(\widetilde{M}^{(4)}) \cong 0 \\
{\scriptstyle \partial_4}\big\downarrow & & \big\downarrow{\scriptstyle i_*} & & \\
\pi_3(\widetilde{M}^{(3)}, \widetilde{M}^{(2)}) & = & \pi_3(\widetilde{M}^{(3)}, \widetilde{M}^{(2)}) & & \\
{\scriptstyle \partial_3}\big\downarrow & & \big\downarrow{\scriptstyle k_*} & & \\
\pi_2(\widetilde{M}^{(2)}, \widetilde{M}^{(1)}) & \xleftarrow[j_*]{} & \pi_2(\widetilde{M}^{(2)}) & \longleftarrow & \pi_2(\widetilde{M}^{(1)}) \cong 0.
\end{array}$$

Here we have used the isomorphisms $\pi_3(\widetilde{M}^{(4)}) \cong \pi_3(M) \cong 0$. In fact, we obtain isomorphisms

$$H_3(\widetilde{M}) = \operatorname{Ker} \partial_3 / \operatorname{Im} \partial_4 \cong \operatorname{Ker} k_* / \operatorname{Im} i_* \cong 0.$$

Observe that $H_3(\widetilde{M}) \cong H_4(\widetilde{M}) \cong 0$ yield $H^4(\widetilde{M}) \cong 0$ by the Universal Coefficient sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Ext}_{\mathbb{Z}}(H_3(\widetilde{M}), \mathbb{Z}) & \longrightarrow & H^4(\widetilde{M}) & \longrightarrow & \\
 & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(H_4(\widetilde{M}), \mathbb{Z}) & \longrightarrow & 0. & &
\end{array}$$

Let us consider the induced homomorphism

$$\phi^* : H^*(\widetilde{M}) \to H^*(\mathbb{C}P^2).$$

We prove that $\phi^q = 0$ for any $q > 0$. Let $x \in H^2(\widetilde{M})$ with $\phi^2(x) = u \in H^2(\mathbb{C}P^2)$. Since $H^4(\widetilde{M}) \cong 0$, we have $x^2 = 0$, and whence $u^2 = (\phi^2(x))^2 = 0$ in $H^4(\mathbb{C}P^2) \cong \mathbb{Z}$. This gives $u = 0$, i.e. $\phi^2 = 0$. Thus, $\phi^q = 0$ for any $q > 0$. Hence φ is homotopic to zero, i.e. $H_2(\widetilde{M}) \cong 0$. So $\widetilde{M}$ is contractible, i.e. M is aspherical.

The above discussion suggests in a natural way the following:

Problem 3.3: Classify the homotopy (resp. homeomorphism, diffeomorphism) type of closed (smooth) connected 4-manifolds with special third homotopy; for example, $\pi_3 \cong \mathbb{Z}, \ldots, \mathbb{Z}^\infty$. Which are the admissible fundamental groups for such manifolds?

Examples are given by closed 4-manifolds M with universal covering space $\widetilde{M} = \mathbb{S}^3 \times \mathbb{R}$ (hence $\pi_2 \cong 0$ and $\pi_3 \cong \mathbb{Z}$), which were considered in [25]. In this case, the fundamental group $\pi = \pi_1(M)$ is an extension of $\mathbb{Z}$ or the infinite dihedral group $D = \mathbb{Z}_2 * \mathbb{Z}_2$ by a (maximal) finite normal subgroup F. Moreover, F has cohomological period dividing 4, and the associate covering space M_F is a finite orientable Poincaré complex of (formal) dimension 3.

If $\pi/F \cong D$, then M is nonorientable and π acts trivially on $\pi_3(M)$.

If $\pi/F \cong \mathbb{Z}$, and $\tau : M_F \to M_F$ is a generator of the group of covering transformations, then M is homotopy equivalent to the mapping torus $M(\tau)$ (for more details see [25] and [26]).

4. Exact manifolds

Let us consider closed smooth connected oriented 4-manifolds M such that $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$. Following [31] and [32], a *leaf* of M is a bicollared 3-submanifold V of M which represents a generator of $H_3(M;\mathbb{Z}) \cong H^1(M;\mathbb{Z}) \cong \mathbb{Z}$. We can specify an orientation of V uniquely by the orientation of M and a generator of $H^1(M;\mathbb{Z})$, and always take V connected.

A closed smooth 4-manifold M with $\pi_1 \cong \mathbb{Z}$ is *split* or *TOP-split*, respectively, if it is diffeomorphic or homeomorphic to the connected sum $(S^1 \times S^3)\# M_1$, where M_1 is the unique simply-connected 4-manifold obtained from M by a 2-handle surgery killing $\pi_1(M)$.

Kawauchi introduced in [31] and [32] the concept of *exact* 4-*manifold*, i.e. a closed connected smooth 4-manifold M with first integral homology group $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$ which admits an *exact* leaf, a special leaf satisfying a certain exactness condition (see Theorem 4.1). This concept was motivated by an attempt to correct an error in [30] on the TOP-splittability for closed 4-manifolds with $\pi_1 \cong \mathbb{Z}$. In fact, Hambleton and Teichner [23] gave an example of a non-TOP-splittable topological 4-manifold with $\pi_1 \cong \mathbb{Z}$ and Witt index zero, and showed that every 4-manifold with $\pi_1 \cong \mathbb{Z}$ and Witt index ≥ 3 is necessarily TOP-splittable, where the *Witt index* of a 4-manifold M with $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$ is defined to be the integer $(\beta_2(M) - |\sigma(M)|)/2$.

In [32] it was proved a criterion of TOP-splittability for closed

4-manifolds with $\pi_1 \cong \mathbb{Z}$ (see Corollary 4.4). This result shows that the exactness on closed 4-manifolds M with $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$ is a generalization of the TOP-splittability on 4-manifolds with $\pi_1 \cong \mathbb{Z}$. Some different criteria of TOP-splittability (also for closed 4-manifolds with more general fundamental groups) were given in [6]–[7] and [20]. In [23] it was constructed a non-TOP-splittable topological 4-manifold with $\pi_1 \cong \mathbb{Z}$. However, as stated in [32], it appears unknown whether there exists a non-TOP-splittable *smooth* closed 4-manifold with $\pi_1 \cong \mathbb{Z}$.

For a leaf V of a 4-manifold M with $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$, let M_V be the 4-manifold obtained from M by splitting it along V. The boundary ∂M_V is the disjoint union $(-V^+) \cup V^-$ of the two copies $V^\pm$ of V under the natural identifications

$$i^\pm : V \cong V^\pm \subset \partial M_V.$$

Let $\widetilde{M}$ be the infinite cyclic connected covering space of M. For a subspace A of M, let $\widetilde{A}$ denote the preimage of A under the covering projection $p : \widetilde{M} \to M$. For a generator t of the infinite cyclic covering transformation group of $\widetilde{M}$, the pair

$$(\widetilde{M}_V, (-\widetilde{V}^+) \cup \widetilde{V}^-)$$

is the disjoint union of the covering translations

$$(t^i M_V, (-t^i V^+) \cup t^i V^-)$$

$(i = 0, \pm 1, \pm 2, \dots)$ of the pair $(M_V, (-V^+) \cup V^-)$ by identifying $(M_V, (-V^+) \cup V^-)$ with a lift of it to $\widetilde{M}$, and $\widetilde{V}$ is the disjoint union of $t^i V$ $(i = 0, \pm 1, \pm 2, \dots)$ under the identification of V with $-V^+ = t^{-1}V^-$ in $\widetilde{M}$. From a technical reason, we often regard M_V as $M_V = \mathrm{cl}(M \backslash V \times [-1,1])$ for a bi-collar neighborhood of V in M. Following [31] and [32], let us denote the torsion part of an abelian group G by $\mathrm{t}G$ and the torsion-free group $G/\mathrm{t}G$ by $\mathrm{b}G$. Let $\Lambda = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$ be the integral Laurent polynomial ring. The homology $H_*(\widetilde{M};\mathbb{Z})$ is naturally regarded as a finitely generated Λ-module. For a Λ-module H, let TH denote the Λ-torsion part of H, and let BH denote the Λ-torsion-free part H/TH. Let $E^q(H) = \mathrm{Ext}^q_\Lambda(H, \Lambda)$.

For a leaf V of a closed smooth 4-manifold M with $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$, the image of the natural homomorphism

$$\widetilde{i}_* : H_2(\widetilde{V};\mathbb{Z}) \to BH_2(\widetilde{M};\mathbb{Z})$$

is called the *leaf submodule* of $BH_2(\widetilde{M};\mathbb{Z})$ on $\widetilde{V}$, and it is denoted by $X(\widetilde{V})$.

The Λ-intersection form

$$S : H_2(\widetilde{M};\mathbb{Z}) \times H_2(\widetilde{M};\mathbb{Z}) \to \Lambda$$

is defined by

$$S(x,y) = \sum_{i=-\infty}^{+\infty} s(t^i x, y)t^i$$

where s denotes the integral intersection form on $H_2(\widetilde{M};\mathbb{Z})$. Then S induces a non-degenerate (but not necessarily non-singular) Λ-Hermitian form

$$BH_2(\widetilde{M};\mathbb{Z}) \times BH_2(\widetilde{M};\mathbb{Z}) \to \Lambda$$

which is also called the Λ-intersection form, and denoted by S.

Kawauchi gave in [31] and [32] an algebraic characterization of an exact leaf (see Theorem 4.1), and an algebraic characterization of an exact 4-manifold (see Theorem 4.2). Moreover, he proved the stable existence of an exact 4-manifold (see Theorem 4.5).

THEOREM 4.1. *For a leaf V of a closed smooth 4-manifold M with $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$, the following conditions are equivalent:*

1) The natural semi-exact sequence

$$0 \longrightarrow tH_2(\widetilde{M},\widetilde{V};\mathbb{Z}) \xrightarrow{\widetilde{\partial}} tH_1(\widetilde{V};\mathbb{Z}) \xrightarrow{\widetilde{i}_*} tH_1(\widetilde{M};\mathbb{Z})$$

induced from the homology exact sequence of the pair $(\widetilde{M},\widetilde{V})$ is exact;

2) The natural semi-exact sequence

$$0 \longrightarrow tH_2(\widetilde{M},\widetilde{M}_V;\mathbb{Z}) \xrightarrow{\widetilde{\partial}'} tH_1(\widetilde{M}_V;\mathbb{Z}) \xrightarrow{\widetilde{i}'_*} tH_1(\widetilde{M};\mathbb{Z})$$

induced from the homology exact sequence of the pair $(\widetilde{M},\widetilde{M}_V)$ is exact;

3) There is a splitting

$$BH_2(\widetilde{M};\mathbb{Z}) \cong P \oplus X(\widetilde{V}) \oplus Y$$

such that $X(\widetilde{V})$ *is the leaf submodule on* $\widetilde{V}$ *and the following properties (3.1)-(3.3) are satisfied:*

(3.1) The direct summands P *and* $X(\widetilde{V}) \oplus Y$ *are orthogonal, i.e.* $P \perp (X(\widetilde{V}) \oplus Y)$*, with respect to*

$$S : BH_2(\widetilde{M};\mathbb{Z}) \times BH_2(\widetilde{M};\mathbb{Z}) \to \Lambda.$$

(3.2) There is a Λ*-basis for* P *which are represented by 2-cycles in* M_V *and on which the restriction*

$$S|_P : P \times P \to \Lambda$$

is represented by an integer matrix with determinant ± 1*.*

(3.3) The direct summand Y *is* Λ*-free, and* $X(\widetilde{V})^{\perp} = X(\widetilde{V})$*, i.e.* $X(\widetilde{V})$ *is self-orthogonal, with respect to the restriction*

$$S|_{X(\tilde{V})\oplus Y} : (X(\widetilde{V}) \oplus Y) \times (X(\widetilde{V}) \oplus Y) \to \Lambda.$$

Following [31] [32], a closed connected smooth 4-manifold M with $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$ is said to be *exact* if there is an exact leaf V of M, i.e. V satisfies one of the conditions (1)-(3) of Theorem 4.1. Kawauchi showed in [32] that the exactness on a closed 4-manifold M with $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$ can be algebraically characterized. We now state his theorem.

Theorem 4.2. *A closed connected smooth 4-manifold* M *with first integral homology group* $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$ *is exact if and only if there is a splitting* $P \oplus X \oplus Y$ *of* $BH_2(\widetilde{M};\mathbb{Z})$ *such that*

1) P *is orthogonal to* $X \oplus Y$ *with respect to*

$$S : BH_2(\widetilde{M};\mathbb{Z}) \times BH_2(\widetilde{M};\mathbb{Z}) \to \Lambda;$$

2) There is a Λ*-basis for* P *on which the restriction*

$$S|_P : P \times P \to \Lambda$$

is represented by an integer matrix with determinant ± 1; *and*

3) The direct summand Y *is a free* Λ*-module and* $X^{\perp} = X$ *with respect to the restriction*

$$S|_{X \oplus Y} : (X \oplus Y) \times (X \oplus Y) \to \Lambda.$$

In this case, we have a connected exact leaf V of M with $X(\widetilde{V}) = X$ and a Λ-basis for P consisting of 2-cycles in $M_V \subset \widetilde{M}$. Note that Theorems 4.1 and 4.2 hold for topological 4-manifolds M with $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$ as well as the following corollary holds for topological 4-manifolds with $\pi_1 \cong \mathbb{Z}$ because the punctured manifold of every closed connected 4-manifold is smoothable (see [20] and [22]).

Corollary 4.3. *A closed connected smooth* 4*-manifold* M *with fundamental group* $\pi_1(M) \cong \mathbb{Z}$ *is exact if and only if it is TOP-splittable (or equivalently, if and only if the signature of* M *is trivial).*

We now present the proof of Corollary 4.3 given in [32]. Let M be a closed connected smooth 4-manifold with $\pi_1(M) \cong \mathbb{Z}$. Then $H_2(\widetilde{M};\mathbb{Z})$ is proved to be a free Λ-module (for details see the quoted paper). If M is TOP-splittable, then we can take $P = H_2(\widetilde{M};\mathbb{Z}) \cong \pi_2(M)$ and $X = Y = 0$. Conversely, assume that M is exact. Then we have a splitting $H_2(\widetilde{M};\mathbb{Z}) = P \oplus X \oplus Y$ as in Theorem 4.2. Using the fact that $H_2(\widetilde{M};\mathbb{Z})$ is Λ-free, we obtain that X is Λ-projective and hence Λ-free. Then it follows that the Λ-intersection form S on $H_2(\widetilde{M};\mathbb{Z})$ is represented by a symmetric integer matrix A. Let M_1 be the simply connected 4-manifold obtained from M by a surgery killing a generator of $\pi_1(M) \cong \mathbb{Z}$. It is proved in [30] that M and $(S^1 \times S^3)\#M_1$ are homology cobordant. This implies that M and $(S^1 \times S^3)\#M_1$ have the same Kirby-Siebenmann invariant (because it is a cobordism invariant) and have the same Λ-intersection form represented by the symmetric integer matrix A. By [20], there is a homeomorphism from M to $(S^1 \times S^3)\#M_1$, and so M is TOP-splittable.

By Corollary 4.3, the non-TOP-splittable topological closed 4-manifold with $\pi_1 \cong \mathbb{Z}$ and Witt index zero, given by Hambleton and Teichner in [23], is not exact.

The following stable existence theorem on exact 4-manifolds, proved in [32], also holds for topological 4-manifolds with $H_1 \cong \mathbb{Z}$.

THEOREM 4.4. *For every closed connected smooth 4-dimensional manifold M with $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$, there is a non-negative integer n such that the connected sum $M\#n(S^2 \times S^2)$ is exact.*

Finally, it was proved in [31] that any closed connected smooth 4-manifold M with $H_1(M;\mathbb{Z}) \cong \mathbb{Z}$ is exact if it is homology cobordant to a splittable 4-manifold, $\sigma(M) = 0$, and the first Alexander polynomial has no root of unity as a root.

The above discussion suggests in a natural way the following:

PROBLEM 4.5: Construct some examples in the category TOP (resp. DIFF) of exact 4-manifolds which are non-TOP-splittable (resp. non-DIFF-splittable).

PROBLEM 4.6: Classify the homotopy (resp. homeomorphism, diffeomorphism) type of closed exact (smooth) 4-manifolds.

5. Geometric manifolds

Let X be a simply-connected smooth n-manifold with a complete homogeneous Riemannian metric such that its isometry group G acts transitively on X, and contains discrete subgroups (*lattices*) Γ which act freely on X, and such that the quotient X/Γ has finite volume. A closed connected n-manifold M is said to be *geometric* in the sense of Thurston, or equivalently, an X-*manifold* if it is a quotient of type X/Γ as defined above. It is well-known that every closed 1- or 2-manifold is geometric, and there are just the classical geometries of constant curvature: Euclidean $\mathbb{E}^1$ and $\mathbb{E}^2$, spherical $\mathbb{S}^2$, and hyperbolic $\mathbb{H}^2$. In dimension 3, there are exactly 8 geometries: in addition to $\mathbb{E}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$, we have also the products $\mathbb{S}^2 \times \mathbb{E}^1$ and $\mathbb{H}^2 \times \mathbb{E}^1$, the twisted products $\widetilde{\mathrm{SL}}$ and Nil^3, and the solvable Lie geometry Sol^3. Much current research on 3-manifold topology has been guided by Thurston's Geometrization Conjecture, i.e. *every closed irreducible* 3-*manifold has a canonical finite decomposition into geometric pieces*. This conjecture has been established for many classes of 3-manifolds, but there is no a complete proof. The main problem is still the classification of all hyperbolic 3-manifolds. The 4-dimensional geometries are exactly 19, and were worked out by Filipkiewicz [16] in his PhD thesis (see also [46] and [47]). One of them is in fact an infinite family of closely related geometries $\mathrm{Sol}^4_{m,n}$, including $\mathrm{Sol}^3 \times \mathbb{E}^1$, and one has no compact realization (only finite volume) but plays an important role in geometric decompositions. Here we give the complete list of the 4-dimensional geometries, i.e. the corresponding model spaces:

compact: $\mathbb{S}^4$, $\mathbb{C}P^2$, and $\mathbb{S}^2 \times \mathbb{S}^2$;
mixed spherical: $\mathbb{S}^3 \times \mathbb{E}^1$, $\mathbb{S}^2 \times \mathbb{E}^2$, and $\mathbb{S}^2 \times \mathbb{H}^2$;
solvable Lie: $\mathrm{Nil}^3 \times \mathbb{E}^1$, Nil^4, $\mathrm{Sol}^4_{m,n}$, Sol^4_0, and Sol^4_1
(note that $\mathrm{Sol}^3 \times \mathbb{E}^1 = \mathrm{Sol}^4_{m,n}$ for all $m \geq 1$);
Euclidean: $\mathbb{E}^4$
mixed aspherical: $\mathbb{H}^2 \times \mathbb{E}^2$, $\widetilde{\mathrm{SL}} \times \mathbb{E}^1$, and $\mathbb{H}^3 \times \mathbb{E}^1$;
semisimple: $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^4$, and $\mathbb{H}^2(\mathbb{C})$;
noncompact: $\mathbb{F}^4$ (tangent bundle of $\mathbb{H}^2$)

Some questions arise in a natural way:

PROBLEM 5.1: When is a closed connected 4-manifold homeomorphic, diffeomorphic, homotopy equivalent or s-cobordant to a geometric 4-manifold?

PROBLEM 5.2: Classify the homotopy (resp. homeomorphism, diffeomorphism) type of all geometric manifolds? Which are the admissible fundamental groups for the manifolds admitting a geometry of a specified model?

There are many results in the current literature concerning the problems above (see for example [24] and [26]), but there is no complete classification. For example, in [28] it was investigated when a 4-manifold which fibres over an aspherical closed surface admits a geometry.

In order to formulate an analogue of Thurston's conjecture, Hillman introduced in [28] the following definition. A closed n-manifold M has a *geometric decomposition* if it may be split along a finite collection $\mathcal{S} = \{S_i\}$ of disjoint 2-sided hypersurfaces S_i (*cusps*) such that the components (*pieces*) of the complement $M \setminus \cup \mathcal{S}$, are complete geometric manifolds of finite volume. Of course, the decomposition may not be unique.

The following is the main result proved in [28]:

THEOREM 5.3. *If a closed* 4*-manifold* M *admits a geometric decomposition, then either*

i) M *is geometric; or*

ii) M *has a double covering which is an* $\mathbb{S}^2 \times \mathbb{H}^2$*-manifold; or*

iii) the components of $M \backslash \cup \mathcal{S}$ *have geometry* $\mathbb{H}^4$, $\mathbb{H}^3 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^2$ *or* $\widetilde{\mathrm{SL}} \times \mathbb{E}^1$*; or*

iv) the components of $M \backslash \cup \mathcal{S}$ *have geometry* $\mathbb{H}^2(\mathbb{C})$ *, or* $\mathbb{F}^4$*; or*

v) the components of $M \backslash \cup \mathcal{S}$ *all have geometry* $\mathbb{H}^2 \times \mathbb{H}^2$*.*

In cases (iii), (iv) or (v) the Euler characteristic of M *is nonnegative, and in cases (iii) or (iv)* M *is aspherical.*

Thus except for geometries $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$, and perhaps $\widetilde{\mathrm{SL}} \times \mathbb{E}^1$ no closed geometric manifold has a proper (i.e. the set of cusps is nonempty) geometric decomposition. This means that there is no hope of obtaining a direct analogue of Thurston's program in dimension 4. However, one can settle Problems 5.1 and 5.2 above for the class of 4-manifolds which admit geometric decompositions.

6. Smooth manifolds

In this section we recall some basic results concerning with smooth and PL structures on topological n-manifolds according to [35] [37] [42] (even if our attention is devoted principally to dimension 4). Let DIFF(n) be the groups of diffeomorphisms of $\mathbb{R}^n$, PL(n) the group of invertible PL maps of $\mathbb{R}^n$, and TOP(n) the group of homeomorphisms of $\mathbb{R}^n$. The limit spaces under the inclusions $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ are denoted by DIFF, PL, and TOP. These are topological groups which have (as for Lie groups) classifying spaces BDIFF, BPL, and BTOP, respectively (note that BDIFF = BO since the stable orthogonal group O is homotopy equivalent to DIFF). There are canonical fibrations BDIFF $\to$ BTOP, BPL $\to$ BTOP, and BDIFF $\to$ BPL whose fibers are denoted by TOP / DIFF, TOP / PL, and PL / DIFF, respectively. Any n-manifold M^n (with or without boundary) carries an n-dimensional tangent bundle which is topological, PL, or smooth according to the structure of M. Suppose that M has an $\mathcal{M}$-structure ($\mathcal{M}$ = TOP, PL, or DIFF). The $\mathcal{M}$ -tangent bundle of M is represented by a classifying map $\tau_M^{\mathcal{M}} : M \to \mathrm{B}\,\mathcal{M}$. Let us consider now one of the natural inclusions $\mathcal{M} \subset \mathcal{N}$. The main result of higher dimensional smoothing theory states that a lifting of $\tau_M^{\mathcal{N}}$ (relative to the boundary) to $\mathrm{B}\,\mathcal{M}$ corresponds to an $\mathcal{M}$-structure on M. In other words, we must have a commutative (up to homotopy) diagram:

$$\begin{array}{ccc} M^n & \xrightarrow{\tau_M^{\mathcal{M}}} & \mathrm{B}\,\mathcal{M} \\ \| & & \downarrow \\ M^n & \xrightarrow[\tau_M^{\mathcal{N}}]{} & \mathrm{B}\,\mathcal{N}. \end{array}$$

The following basic result of smoothing theory can be found for example in [20, Theorem 8.3B, p. 119].

THEOREM 6.1. *Suppose $\mathcal{M} \subset \mathcal{N}$ is one of the natural inclusions, and if $\mathcal{N} = \mathrm{TOP}$, then $n \geq 5$. Let M^n be an n-dimensional $\mathcal{N}$-manifold with an $\mathcal{M}$-structure on its boundary. Then the isotopy classes of $\mathcal{M}$-structures on M extending the one given on ∂M correspond bijectively to homotopy classes of liftings (relative to the boundary) of the stable $\mathcal{N}$-tangent bundle to $\mathrm{B}\,\mathcal{M}$.*

In this case, the homotopy classes of $[M, \mathcal{N}/\mathcal{M}]$ represents the $\mathcal{M}$-structures, up to isotopy, on M (rel ∂M) which are compatible with the $\mathcal{N}$-structure. Recall that two $\mathcal{M}$ -structures Σ and Σ' on M are said to be *isotopic* if there is a topological isotopy $H : M \times I \to M$ such that $H_0 : M \times 0 \to M$ is the identity, and $H_1 : M \times 1 \to M$ is an $\mathcal{M}$-isomorphism from M_Σ onto $M_{\Sigma'}$. By Theorem 6.1, the smoothing theory in higher dimensions is reduced to obstruction theory. Given an $\mathcal{N}$-tangent bundle on M, there are obstructions in $H^{i+1}(M, \partial M; \pi_i(\mathcal{N}/\mathcal{M}))$ to the existence of a refinement to an $\mathcal{M}$-structure. Since the fiber PL / DIFF of the canonical map BDIFF $\to$ BPL is 6-connected, it follows that smooth and PL structures are the same, up to isotopy, for any dimension $n \leq 6$ (hence we do not distinguish between the two types of structures). The fiber TOP / PL of the canonical map BPL $\to$ BTOP is an Eilenberg-Mac Lane space $K(\mathbb{Z}_2, 3)$. So the unique obstruction to smoothing a topological manifold M^n (rel ∂M) for $n \geq 5$ is a cohomology class $k(M) \in H^4(M, \partial M; \mathbb{Z}_2)$, called the *Kirby-Siebenmann invariant* of M. Unfortunately, the vanishing of $k(M)$ is necessary but not sufficient in general to smoothing a topological 4-manifold

M^4. However, $M^4 \times \mathbb{R}$ is smoothable (relative to the boundary) if and only if $k(M) = 0$. This does not imply in general that M^4 itself is smoothable. If M^4 is not compact, the obstruction group is trivial, and hence $M^4 \times \mathbb{R}$ is smoothable in this case.

The following result, due to Quinn [42], states that any topological 4-manifold admits smooth structures in the complement of a discrete set.

THEOREM 6.2. *Let M^4 be a topological 4-manifold, and let M_0 be the manifold obtained from M by removing an interior point from each compact component. Then M_0 has a smooth structure.*

Unfortunately, the smooth structure on M_0 does not extend in general to M. There are well–known counterexamples [42] arising from the results obtained in [15] and [19]. Freedman proved in [19] that any symmetric unimodular $\mathbb{Z}$–form can be realized as intersection form of a closed simply–connected topological 4-manifold. Successively, Donaldson has shown in [15] that if the intersection form of a smooth simply–connected closed 4-manifold is positive definite, then it is equivalent (over $\mathbb{Z}$) to the standard form (represented by the identity matrix). The integer matrices E_8 and $2E_8 = E_8 \oplus E_8$ are unimodular, symmetric, and positive definite; but they are not equivalent (over $\mathbb{Z}$) to the standard form. Let $\parallel E_8 \parallel$ and $\parallel 2E_8 \parallel$ be the closed simply-connected topological 4-manifolds realizing E_8 and $2E_8$, respectively, as their intersection forms. The manifold $\parallel E_8 \parallel$ is not smoothable since its Kirby-Siebenmann invariant is 1 (mod 2) (recall that if the intersection form λ_M of M is even, then $k(M) = (1/8)\,\mathrm{sign}(\lambda_M)$ (mod 2), and that $\mathrm{sign}(E_8) = 8$). The manifold $\parallel 2E_8 \parallel$ has trivial (mod 2) Kirby-Siebenmann invariant (use $\mathrm{sign}(2E_8) = 16$), but it is not smoothable by Donaldson' s theorem. However, Theorem 6.2 says that removing a closed 4-cell from $\parallel E_8 \parallel$ (resp. $\parallel 2E_8 \parallel$) yields a smooth manifold. Following [42], an *almost smoothing* of a compact topological 4-manifold M is a smooth structure in the complement of a discrete set, i.e. there is a finite set $\{x_1, \ldots, x_r\}$ of points in M (called the *singular points* of the almost smoothing) such that $M \setminus \{x_1, \ldots, x_r\}$ is smooth. If x is a singular point, then there exists an open regular neighbourhood $U = U(x)$ of x in M which is homeomorphic to $\mathbb{R}^4$ (assume that x maps to $0 \in \mathbb{R}^4$ under this homeomorphism). The almost smooth structure of M induces a smooth structure on the complement $U\backslash\{x\} \cong \mathbb{R}^4\backslash\{0\} \cong \mathbb{S}^3 \times (0,\infty)$, called the *end* of the singularity. Two ends (of different singular points) are said to be *equivalent* if there is a diffeomorphism of the smooth structures "near $\mathbb{S}^3 \times \{0\}$", i.e. a diffeomorphism from $\mathbb{S}^3 \times (0,\epsilon)$ onto $\mathbb{S}^3 \times (0,\epsilon')$ for some sufficiently small ϵ, $\epsilon' > 0$. Examples of ends and singular points are given by the so-called displacements of the standard 2-sphere $\mathbb{C}P^1 = \mathbb{S}^2$ in $\mathbb{C}P^2$ (see for example [20], [41], and [42]). Analogous constructions can be obtained by considering the displacements of the standard wedge $\mathbb{S}^2 \vee \mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{S}^2$ (see [17] and [20]). It is well-known that the complement of the standard 2-sphere $\mathbb{C}P^1 = \mathbb{S}^2$ in $\mathbb{C}P^2$ is diffeomorphic to $\mathbb{R}^4$ (endowed with the standard smooth structure). Let us consider a homeomorphism $d : \mathbb{C}P^2 \to \mathbb{C}P^2$ which is topologically isotopic to the identity map. Then the image $\mathbb{S}^2_d$ of $\mathbb{C}P^1 = \mathbb{S}^2$ under d is called a *displacement* of $\mathbb{S}^2$ in $\mathbb{C}P^2$. Of course, the complement of $\mathbb{S}^2_d$ in $\mathbb{C}P^2$ is still TOP homeomorphic to $\mathbb{R}^4$. Moreover, $\mathbb{C}P^2\backslash\mathbb{S}^2_d$ admits a smooth structure since it is open in $\mathbb{C}P^2$; but, this structure may not be diffeomorphic to the standard one. Following [42], a singular point of an almost smoothing of M is said to be *resolvable* if its end is equivalent to the end of a certain displacement $\mathbb{S}^2_d \subset \mathbb{C}P^2$.

The following is the main theorem of [42]; it explains the importance of the Kirby-Siebenmann invariant for characterizing the singular points of an almost smoothing of a compact topological 4-manifold.

THEOREM 6.3. *Any compact connected topological* 4*-manifold* M *has an almost smoothing such that:*

(i) If $k(M) = 0$ *(mod* 2*), then all the singular points are resolvable;*

(ii) If $k(M) = 1$ *(mod* 2*), all but one are resolvable, and the exceptional one has an end which is equivalent to the smooth structure of Freedman's fake* $\mathbb{S}^3 \times \mathbb{R}$ *(constructed in [18]).*

Theorem 6.3 combined with Donaldson's theorem [15] permits to obtain a very nice proof of the existence of some exotic smooth structures on $\mathbb{R}^4$ (for details see Corollary 1.4 of [42]). For a construction of an infinite set of exotic $\mathbb{R}^4$'s we refer to [21].

We now report the nice proof of Theorem 6.3 given by Quinn in [42]. The first step is to reduce to the $k = 0$ case by introducing a singularity if $k = 1$. Let M^4 be a compact connected topological 4-manifold. Then the Kirby-Siebenmann invariant $k(M)$ can be computed as follows. Take an almost smoothing of M with singular points $x_1, \dots, x_r$. The end at x_i is a smoothing of $\mathbb{S}^3 \times \mathbb{R}$. Perturb the canonical projection $\mathbb{S}^3 \times \mathbb{R} \to \mathbb{R}$ to be smoothly transverse to $0 \in \mathbb{R}$, and let N_i be the inverse image. Then N_i is an orientable smooth 3-manifold which bounds a framed smooth 4-manifold W_i as $\Omega_3^{\text{framed}} \cong 0$. Then we have

$$k(M) = \sum_{i=1}^{r} \frac{1}{8} \operatorname{sign}(W_i) \pmod 2.$$

Suppose now $k(M) = 1$ (mod 2). Use the smoothing of Freedman's fake $\mathbb{S}^3 \times \mathbb{R}$ (see [18]) to define an almost smoothing of a neighbourhood of a point $p \in M$, with p corresponding to the $+\infty$ end of $\mathbb{S}^3 \times \mathbb{R}$. As above, make the projection $\mathbb{S}^3 \times \mathbb{R} \to \mathbb{R}$ transverse to $0 \in \mathbb{R}$. Let N_p be the inverse image of $0 \in \mathbb{R}$, and M' the complement of the inverse image of $(0, +\infty)$. Then M' is a compact 4-manifold. An almost smoothing of M' fits together with the almost smoothing of the neighbourhood of p to give an almost smoothing of M. Then we have $k(M') + (1/8)\operatorname{sign}(W_p) = k(M)$ (mod 2), where the smooth 4-manifold W_p has boundary N_p as above. Since $\operatorname{sign}(W_p) = 8$ by [18], it follows that $k(M') = 0$ (mod 2). Thus a resolvable almost smoothing of M' extends to an almost smoothing of M which satisfies (ii) of the statement. Suppose now $k(M) = 0$ (mod 2), and consider a smooth structure on $M \backslash p$ by Theorem 6.2. Let D^4 be a 4-ball in M with center p. The smooth product structure on $(M \backslash \frac{1}{2}D^4) \times \mathbb{R} \subset (M \backslash p) \times \mathbb{R}$ extends to a smooth structure on all of $M \times \mathbb{R}$ because $k(M)$ vanishes. The projection $M \times \mathbb{R} \to \mathbb{R}$ is smoothly transverse to $0 \in \mathbb{R}$ on $(M \backslash \frac{1}{2}D^4) \times \mathbb{R}$. Approximate it rel $(M \backslash \frac{1}{2}D^4) \times \mathbb{R}$ to be transverse to 0 on all of $M \times \mathbb{R}$. The inverse image N of $0 \in \mathbb{R}$ is a smooth 4-manifold which is topologically homeomorphic to a connected sum $M \# P$. Now P embeds in Euclidean 5-space, so its tangent bundle τ_P is stably trivial. This implies that the first Pontrjagin class of P vanishes, hence $\operatorname{sign}(P) = 0$ by the Hirzebruch formula. We can always assume that P is simply-connected by a surgery argument. Applying Freedman's classification theorem of closed simply-connected TOP 4-manifolds [19] [20], we get that P is

TOP homeomorphic to $h(\mathbb{S}^2 \times \mathbb{S}^2)$ for some $h > 0$. Therefore we have a smooth structure on $M \# h(\mathbb{S}^2 \times \mathbb{S}^2)$. Since $(\mathbb{S}^2 \times \mathbb{S}^2)\#\mathbb{C}P^2$ is diffeomorphic to $2\mathbb{C}P^2\#(-\mathbb{C}P^2)$ (see for example [37]), we obtain that $N\#\mathbb{C}P^2$ is diffeomorphic to $M\#i\mathbb{C}P^2\#j(-\mathbb{C}P^2)$ for some i and j. This defines a resolution of M since $N\#\mathbb{C}P^2$ is a smooth 4-manifold which covers $M \cong (M\#i\mathbb{C}P^2\#j(-\mathbb{C}P^2))/(i+j)\mathbb{S}^2$. Moreover, the smooth structures near the copies of $\mathbb{S}^2$ are equivalent to those of displacements of the standard $\mathbb{S}^2 = \mathbb{C}P^1$ in $\mathbb{C}P^2$ (see [42] for more details). This finishes the proof of Theorem 6.3.

We complete the section with some open problems which are related to the topics of smoothing.

PROBLEM 6.4: Classify all the smooth structures on closed smoothable 4-manifolds with special fundamental group (trivial, finite, torsion free, free [4] [6] [27] [29], surface group [9], poly-(finite or cyclic) [20], elementary amenable [26], etc.), or with special higher homotopy (resp. homology).

PROBLEM 6.5: Classify all the smooth structures on closed smoothable 4-manifolds which belong to one of the special classes presented in the paper.

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