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A Note on Smooth Matrices of Constant Rank

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Summary. - *We show that, given a $\mathcal{C}^h$ time–varying matrix $A(t)$ of constant rank, there exists a $\mathcal{C}^h$ matrix $H(t)$ such that the rows of $H(t)A(t)$ are an orthonormal basis of the space spanned by the rows of $A(t)$. We present some consequences of this result and, in particular, we prove a version for $m \times n$ matrices of Doležal's Theorem.*

These results are not new, and references are given. All the proofs of the results stated in these references, with the exception of those based on the use of differential equations — which holds only for $h \geq 1$ —, find suitable $\mathcal{C}^h$ matrices defined on overlapping subsets of the domain and then patch them together without losing regularity and the other required properties. In our approach the patching needs to be done only for matrices consisting of one row *and all the remaining results are obtained by usual algebraic tools.*

1. Introduction

For every pair m, n of positive integers, $\mathbf{C}^{m\times n}$ denotes the space of the $m \times n$ matrices whose entries are elements of $\mathbf{C}$. If $A \in \mathbf{C}^{m\times n}$,

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the symbol $\langle A\rangle$ will denote the subspace of $\mathbf{C}^{1\times n}$ spanned by the rows of A.

Let $J \subset \mathbf{R}$ be an open interval, and let h be a non negative integer. A map

$$A \in \mathcal{C}^h(J, \mathbf{C}^{m\times n})$$

will be said a $\mathcal{C}^h$ *matrix on* J or a $\mathcal{C}^h$ $m \times n$ *matrix on* J.

Let A be a $\mathcal{C}^h$ $m \times n$ matrix on J. If there exists an integer k such that for every $t \in J$ it is $\operatorname{rank} A(t) = k$, we will say that A is a $\mathcal{C}^h$ $m \times n$ matrix of constant rank k on J.

In Section 2 of this paper, we prove (see Theorem 2.4) that if A is a $\mathcal{C}^h$ $m \times n$ matrix of constant rank $k \geq 1$ on J, then there exists a $\mathcal{C}^h$ $k \times m$ matrix H on J such that for every $t \in J$ the rows of $H(t)A(t)$ are an orthonormal basis of $\langle A(t)\rangle$. The main tool in the proof is a Lemma (see Lemma 2.2) which establishes that, given elements $a_1, \dots, a_m$ in $\mathcal{C}^h(J, \mathbf{C}^{1\times n})$ spanning, for every $t \in J$, a subspace of dimension at least one, there exists a linear combination $\omega_1 a_1 + \cdots + \omega_m a_m$ never vanishing on J with every $\omega_j \in \mathcal{C}^\infty(J, \mathbf{R})$.

Using Theorem 2.4, in Section 3 we prove various Corollaries. In particular we prove Corollary 3.1 — the existence of a $\mathcal{C}^h$ matrix of constant rank on J whose rows, for every $t \in J$, are an orthonormal basis of the orthogonal complement $\langle A(t)\rangle^\perp$ of $\langle A(t)\rangle$ in $\mathbf{C}^{1\times n}$ — and Corollary 3.2 — a version for $m \times n$ matrices of Doležal's Theorem.

These results are not new. The existence of $\mathcal{C}^h$ matrices of constant rank on J whose rows, for every $t \in J$, are bases of $\langle A(t)\rangle$ or of $\langle A(t)\rangle^\perp$ has been proved by Sibuya (Theorem 6 and Remark 3 of [13]), Kato (Chapter 2, Section 4, Paragraph 2 of [10]) and Gohberg et al. (Corollary 13.6.5 of [7]) — see also Rheinboldt (Section 3 of [11]), Evard (Theorem 8.2 of [4])and Evard and Jafari (Theorem 5 of [5]) for generalizations to matrices depending on more than a single real variable. Doležal's Theorem has been proved first in Doležal [3] and Weiss and Falb [15]; — for generalizations see Silverman and Bucy [14], Sen and Chidambara [12] and Grasse [8].

All the proofs of the cited results, with the exception of Kato (which finds these bases as the solution of a suitable differential equation), find suitable $\mathcal{C}^h$ matrices defined on overlapping subset of the domain and then patch them together without losing regularity and the other required properties. In our approach the patching needs to be done only for matrices consisting of *one row* — see Lemma 2.2 — and all the remaining results are obtained by usual algebraic tools.

Our approach allow us to prove that, given a $\mathcal{C}^h$ matrix A such that $\operatorname{rank} A(t)$ is *at least* k for every $t \in J$, there exists a $\mathcal{C}^h$ $k \times m$ matrix H such that, for every $t \in J$, the rows of $H(t)A(t)$ are an orthonormal set in $\langle A(t) \rangle$ — see Theorem 2.3. This result is somehow complementary to those of Silverman and Bucy [14], where a $n \times n$ matrix A such that $\operatorname{rank} A(t)$ is *at most* k for every t is considered.

Other factorizations of a $\mathcal{C}^h$ matrix have been considered (see Gingold and Hsieh [6], Dieci and Eirola [2] and Chern and Dieci [1]). Even if these results have been obtained without using Doležal's Theorem, such theorem may play a role in them. As an example, in the Appendix we show how the use of Doležal's Theorem may simplify the (nontrivial) proof of the existence of smooth SVD and complete QR factorizations given by Chern and Dieci (Theorem 2.4 and Corollary 2.5 in [1]).

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2. Main theorems and their proofs

If $A \in \mathbf{C}^{m \times n}$, the symbol A^* denotes the matrix $\bar{A}^T \in \mathbf{C}^{n \times m}$. In $\mathbf{C}^{m \times n}$ we will consider the canonical hermitian product defined by $A \bullet B = \operatorname{tr}(AB^*)$. If W is a subspace of $\mathbf{C}^{m \times n}$, then $W^\perp$ denotes the orthogonal complement of W in $\mathbf{C}^{m \times n}$.

LEMMA 2.1. *Let* $a_1, \ldots, a_m \in \mathcal{C}^0(J, \mathbf{C}^{1 \times n})$ *be such that for every* $t \in J$ *there exists* $i \in \{1, \ldots, m\}$ *such that* $a_i(t) \neq 0$.

Then, for every $t_0 \in J$ *there exist two sequences*

$$\cdots < t_{-2} < t_{-1} < t_0 \quad \textit{and} \quad t_0 < t_1 < t_2 < \cdots$$

(each of them may be finite and has at least two terms) such that

(a.1) $\sup\{ t_0, t_1, t_2, \ldots \} = \sup J$

(a.2) $\inf\{ \ldots, t_{-2}, t_{-1}, t_0 \} = \inf J$

(b.1) for every $\ell \geq 0$ *such that there exists* $t_{\ell+1}$*, there exists* $i_\ell \in \{1, \dots, m\}$ *such that*

$$a_{i_\ell}(t) \neq 0 \quad \textit{for every } t \in [t_\ell, t_{\ell+1})$$

(b.2) for every $\ell \leq 0$ *such that there exists* $t_{\ell-1}$*, there exists* $i_{\ell-1} \in \{1, \dots, m\}$ *such that*

$$a_{i_{\ell-1}}(t) \neq 0 \quad \textit{for every } t \in (t_{\ell-1}, t_\ell]$$

Proof. For every $\tau \in J$, define the real numbers $\theta^+(\tau)$ and $\theta^-(\tau)$ as follows: for every $i \in \{1, \dots, m\}$, let

$$\theta^{(i,+)}(\tau) = \begin{cases} \tau & \text{if } a_i(\tau) = 0 \\ \sup\{\sigma \in J : a_i(t) \neq 0 \text{ for every } t \in [\tau, \sigma)\} & \text{if } a_i(\tau) \neq 0 \end{cases}$$

and

$$\theta^{(i,-)}(\tau) = \begin{cases} \tau & \text{if } a_i(\tau) = 0 \\ \inf\{\sigma \in J : a_i(t) \neq 0 \text{ for every } t \in (\sigma, \tau]\} & \text{if } a_i(\tau) \neq 0 \end{cases}$$

then

$$\theta^+(\tau) = \max_{1 \leq i \leq m} \theta^{(i,+)}(\tau) \quad , \quad \theta^-(\tau) = \min_{1 \leq i \leq m} \theta^{(i,-)}(\tau).$$

By the assumptions it is: $\inf J \leq \theta^-(\tau) < \tau < \theta^+(\tau) \leq \sup J$.

Consider the following two procedures.

P^+ : let $t_\ell \in J$ where ℓ is a non-negative integer; define $t_{\ell+1} := \theta^+(t_\ell)$; if $t_{\ell+1} = \sup J$ then stop; if $t_{\ell+1} < \sup J$ then define $\ell := \ell + 1$ and apply P^+ to t_ℓ.

P^- : let $t_\ell \in J$ where ℓ is a non-positive integer; define $t_{\ell-1} := \theta^-(t_\ell)$; if $t_{\ell-1} = \inf J$ then stop; if $t_{\ell-1} > \inf J$ then define $\ell := \ell - 1$ and apply P^- to t_ℓ.

Let $t_0 \in J$, and consider the two sequences $t_0, t_1, t_2, \dots$ and $t_0, t_{-1}, t_{-2}, \dots$ obtained by applying to t_0 the procedures P^+ and P^- respectively. Observe that both sequences may be finite; however, at least t_1 and t_{-1} exist.

Obviously, properties (b.1) and (b.2) hold.

To prove (a.1), assume that there exists t_ℓ for every integer $\ell \geq 0$, otherwise the statement is obvious.

Let $\tau = \sup_{\ell \geq 0} t_\ell$ and assume, by contradiction, $\tau < \sup J$, so that $\tau \in J$. Let $i \in \{1, \dots, m\}$; for every $\ell \geq 0$ let $\eta_{i\ell} = \theta^{(i,+)}(t_\ell) \in [t_\ell, t_{\ell+1}]$, then $a_i(\eta_{i\ell}) = 0$, hence $\lim_{\ell \to \infty} a_i(\eta_{i\ell}) = 0$; moreover, since $\lim_{\ell \to \infty} \eta_{i\ell} = \tau$ and a_i is continuous, it is also $\lim_{\ell \to \infty} a_i(\eta_{i\ell}) = a_i(\tau)$; then $a_i(\tau) = 0$. As a consequence, $a_i(\tau) = 0$ for $i = 1, \dots, m$. This is a contradiction.

Analogously we prove statement (a.2). $\square$

Lemma 2.2. *Let $a_1, \dots, a_m \in \mathcal{C}^h(J, \mathbf{C}^{1 \times n})$ be such that*

$$\dim\langle a_1(t), \dots, a_m(t)\rangle \neq 0 \quad \textit{for every } t \in J.$$

There exist

$$\omega_1, \dots, \omega_m \in \mathcal{C}^\infty(J, \mathbf{R})$$

such that

$$\sum_{i=1}^{m} \omega_i(t) a_i(t) \neq 0 \quad \textit{for every } t \in J.$$

Proof. Let $t_0 \in J$, and let $\dots, t_{-2}, t_{-1}, t_0$ and $t_0, t_1, t_2, \dots$ be the sequences obtained applying Lemma 2.1 to $a_1, \dots, a_m$ and to t_0. In what follows we prove the statement assuming that there exists t_ℓ for every $\ell \in \mathbf{Z}$. The other cases have similar proof.

For every $\ell \in \mathbf{Z}$, let a_{i_ℓ} be as in (b.1) and (b.2) of Lemma 2.1, and choose $\epsilon_\ell > 0$ such that, defined

$$U_\ell = (t_\ell - \epsilon_\ell, t_{\ell+1}) \quad \text{for } \ell \geq 0 \quad , \quad U_\ell = (t_\ell, t_{\ell+1} + \epsilon_\ell) \quad \text{for } \ell < 0$$

it is

$$\inf U_\ell < \sup U_{\ell-1} < \inf U_{\ell+1} < \sup U_\ell$$

and

$$a_{i_\ell}(t) \neq 0 \quad \text{for every } t \in U_\ell.$$

For every $\ell \in \mathbf{Z}$, choose

$$\zeta_\ell \in \mathcal{C}^\infty(J, \mathbf{R})$$

such that

$$\zeta_\ell(t) \neq 0 \quad \text{for every } t \in U_\ell \quad , \quad \zeta_\ell(t) = 0 \quad \text{for every } t \in J \setminus U_\ell$$

and let

$$u_\ell = \zeta_\ell a_{i_\ell} \in \mathcal{C}^h(J, \mathbf{C}^{1\times n}).$$

Let $\alpha_0 = 1 \in \mathcal{C}^\infty(J, \mathbf{R})$; we prove by induction on ℓ that for every integer $\ell \geq 1$ there exist $\alpha_{-\ell}, \alpha_\ell \in \mathcal{C}^\infty(J, \mathbf{R})$ such that, defined

$$w_\ell = \sum_{\lambda=-\ell}^{\ell} \alpha_\lambda u_\lambda$$

it is

$$w_\ell(t) \neq 0 \quad \text{for every } t \in \cup_{\lambda=-\ell}^{\ell} U_\lambda.$$

For $\ell = 1$: if for every $t \in U_0 \cap U_1$ there exists $\beta(t) \in \mathbf{R}$ such that $u_0(t) + \beta(t)u_1(t) = 0$, then the function $\beta : U_0 \cap U_1 \to \mathbf{R}$ is a uniquely determined never vanishing $\mathcal{C}^h$ function, so that the sign of $\beta(t)$ is constant on $U_0 \cap U_1$; if $\beta > 0$, choose $\alpha_1 = -1 \in \mathcal{C}^\infty(J, \mathbf{R})$, if $\beta < 0$ choose $\alpha_1 = 1 \in \mathcal{C}^\infty(J, \mathbf{R})$.

If there exists $\tau \in U_0 \cap U_1$ such that for every $\beta \in \mathbf{R}$ it is $u_0(\tau) + \beta u_1(\tau) \neq 0$, let

$$\rho(t) = \begin{cases} ||u_0(t)||/||u_1(t)|| & t \in U_1 \\ 0 & t \geq \sup U_1 \end{cases} .$$

It is $\rho \in \mathcal{C}^h((\inf U_1, +\infty), \mathbf{R})$ and

$$\begin{aligned} &\rho(t) > 0 \quad t \in U_0 \cap U_1, \\ &\rho(t) = 0 \quad t \in [\sup U_0, +\infty), \\ &\lim\nolimits_{t \to \inf U_1} \rho(t) = +\infty. \end{aligned}$$

Let $b : U_0 \cap U_1 \to \mathbf{R}$ such that for every $t \in U_0 \cap U_1, u_0(t) + b(t)u_1(t)$ is the vector of minimum norm in the set of the vectors of the form $u_0(t) + \beta u_1(t), \beta \in \mathbf{R}$. Since for every $t \in U_0 \cap U_1$ we have:

$$b(t) = -\frac{u_0(t) \bullet u_1(t) + u_1(t) \bullet u_0(t)}{2\,||u_1(t)||^2}$$

then it is $b \in \mathcal{C}^h(U_0 \cap U_1, \mathbf{R})$.

There exists a neighborhood (c,d) of τ where the function $f \in \mathcal{C}^h(U_0 \cap U_1, \mathbf{R})$ defined by

$$f(t) = ||u_0(t) + b(t)u_1(t)||$$

is a not vanishing function. Therefore:

for every $t \in (c,d)$ and every $\beta \in \mathbf{R}$ it is $u_0(t) + \beta u_1(t) \neq 0$.

Now we consider

$$\begin{array}{ccc} m = \min_{(\inf U_1, \tau]} \rho(t) & , & M = \max_{[\tau, \sup U_0]} \rho(t) \\ \mu = 2M - \frac{m}{2} & , & \lambda = \frac{m}{2} + \frac{\mu}{2} \end{array}$$

and the sequence $(\phi_\nu)_{\nu \in \mathbf{N}}, \phi_\nu \in \mathcal{C}^\infty(\mathbf{R}, (\frac{m}{2}, 2M))$ defined by

$$\phi_\nu(t) = \lambda + \frac{\mu}{\pi} \arctan\left[\nu(t - \tau)\right].$$

It is

$$\begin{array}{l} \lim_{\nu\to\infty} \phi_\nu(d) = \lambda + \frac{\mu}{2} = 2M \\ \lim_{\nu\to\infty} \phi_\nu(c) = \lambda - \frac{\mu}{2} = \frac{m}{2} \end{array}$$

as a consequence, there exists $\nu_0 \in \mathbf{N}$ such that

$$\phi_{\nu_0}(t) < \rho(t) \quad \text{for } \inf U_1 < t \leq c \quad , \quad \phi_{\nu_0}(t) > \rho(t) \quad \text{for } t \geq d.$$

Assuming $\alpha_1 = \phi_{\nu_0} \in \mathcal{C}^\infty(J, \mathbf{R})$ we have

$$\begin{array}{l} \alpha_1(t) > 0 \quad \text{for } t \in J, \\ \alpha_1(t) < \rho(t) \quad \text{for } t \in (\inf U_1, c], \\ \alpha_1(t) > \rho(t) \quad \text{for } t \in [d, \sup J). \end{array}$$

The same argument allows us to choose α_{-1}.

Once obtained $\alpha_{-\ell}, \ldots, \alpha_\ell$ for a $\ell \geq 1$, $\alpha_{-(\ell+1)}, \alpha_{\ell+1}$ may be obtained from

$$\cup_{\lambda=-\ell}^{\ell} U_\lambda, U_{-(\ell+1)}, U_{\ell+1}, w_\ell, u_{-(\ell+1)}, u_{\ell+1}$$

in the same way α_{-1}, α_1 have been obtained from

$$U_0, U_{-1}, U_1, u_0, u_{-1}, u_1.$$

For $i = 1, \dots, m$, let

$$\omega_i = \sum_{\ell \text{ s.t. } i_\ell = i} \zeta_\ell \alpha_\ell$$

and observe that the sum is well defined since for every $t \in J$ at most two terms are not zero. It is immediately seen that $\omega_1, \dots, \omega_m$ verify the statement. $\square$

THEOREM 2.3. *Let A be a $\mathcal{C}^h$ $m \times n$ matrix on J, and let $k \geq 1$ be an integer.*

If $\operatorname{rank} A(t) \geq k$ for every $t \in J$, then there exists $H \in \mathcal{C}^h(J, \mathbf{C}^{k\times m})$ such that for every $t \in J$ the rows of $H(t)A(t)$ are an orthonormal set in $\langle A(t) \rangle$.

Proof. By induction on k.

If $k = 1$, by Lemma 2.2 there exist $\omega_1, \dots, \omega_m \in \mathcal{C}^\infty(J, \mathbf{R})$ such that

$$(\omega_1(t), \dots, \omega_m(t))A(t) \neq 0 \quad \text{for every } t \in J.$$

Let

$$\omega = ||(\omega_1, \dots, \omega_m)A||^{-1}(\omega_1, \dots, \omega_m) \in \mathcal{C}^h(J, \mathbf{C}^{1\times m}).$$

Then $H = \omega$ verifies the statement.

Assume the statement proved for an integer $k \geq 1$, and let $\operatorname{rank} A(t) \geq k + 1$ for every $t \in J$.

Let ω be as above and let $b_1 = \omega A \in \mathcal{C}^h(J, \mathbf{C}^{1\times n})$ so that $||b_1(t)|| = 1$ for every $t \in J$. Observe that for every $t \in J$ it is

$$\langle A(t) \rangle = \langle b_1(t) \rangle \oplus \{a \in \langle A(t) \rangle \ : \ b_1(t) \bullet a = 0\}$$

and

$$\begin{aligned}\{a \in \langle A(t) \rangle \ : \ b_1(t) \bullet a = 0\} &= \langle A(t) - A(t)b_1^*(t)b_1(t) \rangle = \\ &= \langle (I - A(t)b_1^*(t)\omega(t))A(t) \rangle.\end{aligned}$$

Moreover $(I - Ab_1^*\omega)A$ is a $\mathcal{C}^h$ matrix on J and for every $t \in J$ its rank is at least k. The statement for $k + 1$ is then obtained by applying the induction to the matrix $(I - Ab_1^*\omega)A$. $\square$

The following statement is an immediate consequence of Theorem 2.3.

Theorem 2.4. *Let A be a $\mathcal{C}^h$ $m \times n$ matrix of constant rank k on J, and let $k \geq 1$.*

There exists $H \in \mathcal{C}^h(J, \mathbf{C}^{k\times m})$ such that for every $t \in J$ the rows of $H(t)A(t)$ are an orthonormal basis of $\langle A(t)\rangle$.

Obviously, $\operatorname{rank} H(t) = k$ *for every $t \in J$.*

3. Some Corollaries and Doležal's Theorem

In this Section we use Theorem 2.4 to prove some Corollaries. In particular, we give a new proof of Doležal's Theorem (see Corollary 3.2).

Corollary 3.1. *Let A be a $\mathcal{C}^h$ $m \times n$ matrix of constant rank $k < n$ on J. There exists a $\mathcal{C}^h$ $(n-k) \times n$ matrix Ω of constant rank $n-k$ on J such that for every $t \in J$ the rows of $\Omega(t)$ are an orthonormal basis of $\langle A(t)\rangle^{\perp}$.*

In particular, for every $t \in J$ it is $\langle A(t)\rangle^{\perp} = \langle \Omega(t)\rangle$.

Proof. If $k = 0$, $\Omega = I \in \mathcal{C}^h(J, \mathbf{C}^{n\times n})$ verifies the statement.

Otherwise, by Theorem 2.4 there exists $B \in \mathcal{C}^h(J, \mathbf{C}^{k\times n})$ such that for every $t \in J$ the rows of $B(t)$ are an orthonormal basis of $\langle A(t)\rangle$.

It is easily seen that for every $t \in J$ and for every $x \in \mathbf{C}^{1\times n}$ it is

$$\text{“orthogonal projection of } x \text{ on } \langle A(t)\rangle\text{”} = xB^*(t)B(t)$$

hence

$$\text{“normal component of } x \text{ with respect to } \langle A(t)\rangle\text{”} = x(I - B^*(t)B(t)).$$

As a consequence, for every $t \in J$, it is

$$\langle A(t)\rangle^{\perp} = \langle I - B^*(t)B(t)\rangle.$$

Since $I - B^*B$ is a $\mathcal{C}^h$ matrix of constant rank $n-k$ on J, by Theorem 2.4 there exists $H \in \mathcal{C}^h(J, \mathbf{C}^{(n-k)\times n})$ such that for every $t \in J$ the rows of $H(t)(I - B^*(t)B(t))$ are an orthonormal basis of $\langle I - B^*(t)B(t)\rangle$.

Then $\Omega = H(I - B^*B)$ verifies the statement. $\square$

COROLLARY 3.2 (DOLEŽAL'S THEOREM). *Let A be a $\mathcal{C}^h$ $m \times n$ matrix of constant rank k on J and let $1 \leq k < m$.*

There exist a $\mathcal{C}^h$ $m \times m$ matrix H and a $\mathcal{C}^h$ $k \times n$ matrix A_1 of constant rank k on J such that for every $t \in J$ it is

$$H(t) \text{ non singular} \quad , \quad A(t) = H(t) \begin{array}{c} \left[\begin{array}{c} A_1(t) \\ 0 \end{array} \right] \end{array} \begin{array}{l} k \\ m-k \end{array} .$$

Proof. By Theorem 2.4 there exists $H_1 \in \mathcal{C}^h(J, \mathbf{C}^{k \times m})$ of constant rank k on J such that for every $t \in J$ the rows of $H_1(t)A(t)$ are a basis of $\langle A(t) \rangle$. Let $A_1 = H_1 A \in \mathcal{C}^h(J, \mathbf{C}^{k \times n})$; A_1 is obviously a $\mathcal{C}^h$ $k \times n$ matrix of constant rank k on J.

Since $A^* \in \mathcal{C}^h(J, \mathbf{C}^{n \times m})$ and for every $t \in J$ it is $\operatorname{rank} A^*(t) = k$, by Corollary 3.1 there exists $H_2 \in \mathcal{C}^h(J, \mathbf{C}^{(m-k) \times m})$ of constant rank $m - k$ on J such that for every $t \in J$ it is $\langle A^*(t) \rangle^\perp = \langle H_2(t) \rangle$.

Obviously

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \in \mathcal{C}^h(J, \mathbf{C}^{m \times m}) \quad , \quad \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} A = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} \begin{array}{l} k \\ m-k \end{array} .$$

Moreover

$$\begin{bmatrix} H_1(t) \\ H_2(t) \end{bmatrix} \text{ is non singular for every } t \in J.$$

Indeed: let $t \in J, x_1 \in \mathbf{C}^{1 \times k}$ and $x_2 \in \mathbf{C}^{1 \times (m-k)}$ be such that $(x_1, x_2)H(t) = 0$; then

$$x_1 H_1(t) = -x_2 H_2(t), \quad x_1 H_1(t) A(t) = -x_2 H_2(t) A(t), \quad x_1 A_1(t) = 0.$$

Since $\operatorname{rank} A_1(t) = k$, it is $x_1 = 0$; since $\operatorname{rank} H_2(t) = m - k$, it is $x_2 = 0$.

The matrices

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}^{-1} \quad \text{and} \quad A_1$$

verify the statement. □

COROLLARY 3.3. *Let A be a $\mathcal{C}^h$ $m \times n$ matrix of constant rank k on J and let $1 \leq k < m$. Then* *(a) there exist $U \in \mathcal{C}^h(J, \mathbf{C}^{m\times m})$ and a $\mathcal{C}^h$ $k \times n$ matrix M of constant rank k on J such that for every $t \in J$ the matrix $U(t)$ is unitary and*

$$U(t)A(t) = \begin{bmatrix} M(t) \\ 0 \end{bmatrix}.$$

Moreover

(b) if $k < n$ there exist also $V \in \mathcal{C}^h(J, \mathbf{C}^{n\times n})$ and $B \in \mathcal{C}^h(J, \mathbf{C}^{k\times k})$ such that for every $t \in J$ the matrix $V(t)$ is unitary, the matrix $B(t)$ is non singular and

$$U(t)A(t)V(t) = \begin{bmatrix} B(t) & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof. By Doležal's Theorem applied to A, there exist a $\mathcal{C}^h$ $m \times m$ matrix W on J and a $\mathcal{C}^h$ $k \times n$ matrix N of constant rank k on J such that for every $t \in J$ it is

$$W(t) \text{ non singular} \quad , \quad W(t)A(t) = \begin{bmatrix} N(t) \\ 0 \end{bmatrix}.$$

Let $w_1, \ldots, w_m \in \mathcal{C}^h(J, \mathbf{C}^{1\times m})$ be the rows of W; the usual Gram-Schmidt procedure applied to the family $w_m, \ldots, w_1$ allows us to find matrices $T, U \in \mathcal{C}^h(J, \mathbf{C}^{m\times m})$ such that for every $t \in J$ the matrix $T(t)$ is upper triangular and non singular, the matrix $U(t)$ is unitary, and $T(t)W(t) = U(t)$. Hence for every $t \in J$ it is

$$U(t)A(t) = T(t)\begin{bmatrix} N(t) \\ 0 \end{bmatrix} = \begin{bmatrix} M(t) \\ 0 \end{bmatrix}.$$

Obviously M is a $\mathcal{C}^h$ $k \times n$ matrix of constant rank k on J. This proves statement (a).

Let $k < n$. Statement (a) applied to M^* gives a $\mathcal{C}^h$ $n \times n$ matrix V^* and a $\mathcal{C}^h$ $k \times k$ matrix B^* such that for every $t \in J$ the matrix $V^*(t)$ is unitary, the matrix $B^*(t)$ is non singular and

$$V^*(t)M^*(t) = \begin{bmatrix} B^*(t) \\ 0 \end{bmatrix}.$$

Hence for every $t \in J$ it is

$$U(t)A(t)V(t) = \begin{bmatrix} M(t) \\ 0 \end{bmatrix} V(t) = \begin{bmatrix} B(t) & 0 \\ 0 & 0 \end{bmatrix}.$$

This proves statement (b). □

COROLLARY 3.4. *Let A be a $\mathcal{C}^h$ $m \times n$ matrix of constant rank on J. Let $a \in \mathcal{C}^h(J, \mathbf{C}^{1\times n})$ be such that $a(t) \in \langle A(t)\rangle$ for every $t \in J$.*

Then there exists $\omega \in \mathcal{C}^h(J, \mathbf{C}^{1\times m})$ such that $a = \omega A$.

Proof. If for every $t \in J$ it is $\operatorname{rank} A(t) = 0$, then $a = 0$ and hence $\omega = 0$ verifies the statement. If for every $t \in J$ it is $\operatorname{rank} A(t) = k \geq 1$, by Theorem 2.4 there exists $H \in \mathcal{C}^h(J, \mathbf{C}^{k\times m})$ such that for every $t \in J$ the rows of $H(t)A(t)$ are an orthonormal basis of $\langle A(t)\rangle$.

Obviously, for every $t \in J$ there exists a unique $\alpha(t) \in \mathbf{C}^{1\times k}$ such that $a(t) = \alpha(t)H(t)A(t)$. A straightforward argument proves that $\alpha \in \mathcal{C}^h(J, \mathbf{C}^{1\times k})$. Hence it is sufficient to choose $\omega = \alpha H$. □

COROLLARY 3.5. *Let A, A_1 be $\mathcal{C}^h$ matrices of dimension $m\times n, m_1\times n$ and of constant rank k, k_1 on J such that*

$$\langle A_1(t)\rangle \subset \langle A(t)\rangle \quad \textit{for every } t \in J,$$
$$0 < k_1 < k.$$

There exist $\mathcal{C}^h$ matrices Λ_1, Λ_2 of dimension $k_1 \times m, (k - k_1) \times m$ of constant rank $k_1, k - k_1$ on J such that for every $t \in J$ the rows of $\Lambda_1(t)A(t)$ are an orthonormal basis of $\langle A_1(t)\rangle$ and the rows of

$$\begin{bmatrix} \Lambda_1(t) \\ \Lambda_2(t) \end{bmatrix} A(t)$$

are an orthonormal basis of $\langle A(t)\rangle$.

In particular, for every $t \in J$ it is

$$\operatorname{rank} \Lambda_1(t)A(t) = k_1, \quad \operatorname{rank} \Lambda_2(t)A(t) = k - k_1, \quad \operatorname{rank} \begin{bmatrix} \Lambda_1(t) \\ \Lambda_2(t) \end{bmatrix} = k$$

and

$$\langle A_1(t)\rangle = \langle \Lambda_1(t)A(t)\rangle,$$
$$\langle A(t)\rangle = \langle A_1(t)\rangle \oplus \langle \Lambda_2(t)A(t)\rangle.$$

Proof. By Theorem 2.4 (respectively: by Theorem 2.4 and Corollary 3.4) there exists a $\mathcal{C}^h$ matrix H (respectively: H_1) such that for every $t \in J$ the rows of the matrix $H(t)A(t)$ (respectively: $H_1(t)A(t)$) are an othonormal basis of $\langle A(t)\rangle$ (respectively: of $\langle A_1(t)\rangle$). Let $B = HA \in \mathcal{C}^h(J, \mathbf{C}^{k\times n})$ and $B_1 = H_1 A \in \mathcal{C}^h(J, \mathbf{C}^{k_1\times n})$.

Let $\Lambda_1 = H_1$.

Let $B_2 = A(I - B_1^* B_1)$. It is easily seen that for every $t \in J$ it is

$$\{a \in \langle A(t)\rangle \ : \ a \perp \langle A_1(t)\rangle\} = \{a \in \langle A(t)\rangle \ : \ a \perp \langle B_1(t)\rangle\} = \langle B_2(t)\rangle$$

and

$$\langle A(t)\rangle = \langle A_1(t)\rangle \oplus \langle B_2(t)\rangle.$$

Since B_2 is a $\mathcal{C}^h$ $m \times n$ matrix of constant rank $k - k_1$ on J, by Theorem 2.4 and Corollary 3.4 there exists a unique $\mathcal{C}^h$ matrix Λ' such that for every $t \in J$ the rows of the $\mathcal{C}^h$ matrix $B_2' = \Lambda' B = \Lambda' HA$ are an orthonormal basis of $\langle B_2(t)\rangle$; let $\Lambda_2 = \Lambda' H$.

It is easily seen that Λ_1, Λ_2 verify the statement. □

Corollary 3.6. *Let A_1, A_2 be $\mathcal{C}^h$ matrices of dimension $m_1 \times n$, $m_2 \times n$ and of constant rank k_1, k_2 on J.*

Let

$$\dim\langle A_1(t)\rangle \cap \langle A_2(t)\rangle = k \geq 1 \quad \textit{for every } t \in J.$$

Then there exist $\mathcal{C}^h$ matrices Λ_1, Λ_2 of dimension $k \times m_1$ and $k \times m_2$ respectively such that $\Lambda_1 A_1 = \Lambda_2 A_2$ and for every $t \in J$ the rows of $\Lambda_1(t)A_1(t) = \Lambda_2(t)A_2(t)$ are an orthonormal basis of $\langle A_1(t)\rangle \cap \langle A_2(t)\rangle$.

In particular, for every $t \in J$ it is

$$\langle A_1(t)\rangle \cap \langle A_2(t)\rangle = \langle \Lambda_1(t)A_1(t)\rangle$$

and

$$\langle A_1(t)\rangle \cap \langle A_2(t)\rangle = \langle \Lambda_2(t)A_2(t)\rangle.$$

Proof. Since for every $t \in J$ it is

$$\langle A_1(t)\rangle \cap \langle A_2(t)\rangle = (\langle A_1(t)\rangle^{\perp} + \langle A_2(t)\rangle^{\perp})^{\perp},$$

by the assumptions, Corollary 3.1 and Theorem 2.4, there is a $\mathcal{C}^h$ $k \times n$ matrix Ω of constant rank on J such that for every $t \in J$ the rows of $\Omega(t)$ are an orthonormal basis of $\langle A_1(t)\rangle \cap \langle A_2(t)\rangle$.

By Corollary 3.4 applied first to A_1 and each row of Ω, and then to A_2 and each row of Ω, there exist $\mathcal{C}^h$ matrices Λ_1 and Λ_2 of dimension $k \times m_1$ and $k \times m_2$ respectively such that $\Omega(t) = \Lambda_1(t)A_1(t) = \Lambda_2(t)A_2(t)$. □

4. Appendix

In this Appendix we consider the following statement proved in [1] and used to show the existence of smooth SVD and complete QR factorizations of a $\mathcal{C}^h$ matrix (Theorem 2.4 and Corollary 2.5 of [1]).

THEOREM 4.1 (SEE [1], THEOREM 2.4). *Let A be a $\mathcal{C}^h$ $m \times n$ matrix of constant rank k on $\mathbf{R}$ and let $m \geq n$ and $1 \leq k \leq n$.*

Then there exist $U \in \mathcal{C}^h(\mathbf{R}, \mathbf{C}^{m\times m})$, $V \in \mathcal{C}^h(\mathbf{R}, \mathbf{C}^{n\times n})$ and $S^+ \in \mathcal{C}^h(\mathbf{R}, \mathbf{C}^{k\times k})$ such that for every $t \in J$ the matrices $U(t)$ and $V(t)$ are unitary, the matrix $S^+(t)$ is hermitian positive definite and

$$U(t)^*A(t)V(t) = \begin{bmatrix} S^+(t) & 0 \\ 0 & 0 \end{bmatrix}.$$

This statement has been proved in [1] as a nontrivial consequence of a result by Sibuya on a block diagonalization of $\mathcal{C}^h$ matrices. To show that Doležal's Theorem may play a role in smooth SVD and complete QR factorizations, we prove that Theorem 4.1 may be easily obtained (through Corollary 3.3) by Doležal's Theorem as follows.

LEMMA 4.2. *Let B be a $\mathcal{C}^h$ $k \times k$ matrix on $\mathbf{R}$ such that for every $t \in \mathbf{R}$ the matrix $B(t)$ is non singular.*

Then there exist matrices $Q, S \in \mathcal{C}^h(\mathbf{R}, \mathbf{C}^{k\times k})$, such that for every $t \in \mathbf{R}$ the matrix $Q(t)$ is unitary, the matrix $S(t)$ is hermitian positive definite and

$$B(t) = Q(t)S(t).$$

Proof. Let $\mathcal{H}$ be the set of all hermitian $k \times k$ matrices, and consider it (in the obvious way) as a normed vector space of dimension k^2 on $\mathbf{R}$. Let $\mathcal{P}$ be the open subset of $\mathcal{H}$ of all hermitian positive definite

$k \times k$ matrices. For every $M \in \mathcal{P}$, let $\sqrt{M}$ be the unique element of $\mathcal{P}$ such that $(\sqrt{M})^*\sqrt{M} = M$. The argument of the proof of Theorem "Derivative of the square root" on page 23 of [9] proves that the map

$$\sqrt{\bullet} : \mathcal{P} \to \mathcal{P}$$

is a $\mathcal{C}^\infty$-map.

For every $t \in \mathbf{R}$ let $S(t) = \sqrt{B(t)^*B(t)}$ and $Q(t) = B(t)S(t)^{-1}$. The above argument proves that $S \in \mathcal{C}^h(\mathbf{R}, \mathbf{C}^{k\times k})$, hence also $Q \in \mathcal{C}^h(\mathbf{R}, \mathbf{C}^{k\times k})$; moreover, for every $t \in \mathbf{R}$ it is $B(t) = Q(t)S(t)$, and the matrix $Q(t)$ is unitary. $\square$

We can now give the proof of Theorem 4.1.

Proof. There exist matrices $U_1 \in \mathcal{C}^h(\mathbf{R}, \mathbf{C}^{m\times m})$, $V \in \mathcal{C}^h(\mathbf{R}, \mathbf{C}^{n\times n})$ and $B \in \mathcal{C}^h(\mathbf{R}, \mathbf{C}^{k\times k})$ such that for every $t \in \mathbf{R}$ the matrices $U_1(t)$ and $V(t)$ are unitary, the matrix $B(t)$ is non singular and

(i) whenever $k < n \leq m$ it is

$$U_1(t)A(t)V(t) = \begin{bmatrix} B(t) & 0 \\ 0 & 0 \end{bmatrix},$$

(ii) whenever $k = n < m$ it is

$$U_1(t)A(t)V(t) = \begin{bmatrix} B(t) \\ 0 \end{bmatrix},$$

(iii) whenever $k = n = m$ it is

$$U_1(t)A(t)V(t) = B(t).$$

The existence of matrices verifying (i) follows by Corollary 3.3, (ii) follows by (a) of Corollary 3.3 and assuming $V = I$, and (iii) is obvious assuming $U_1 = V = I$.

Then, the statement follows applying Lemma 4.2 to B. $\square$

References

[1] J. Chern and L. Dieci, *Smoothness and periodicity of some matrix decompositions*, SIAM J. Matrix Anal. Appl. **22** (2000), 772–792.

[2] L. Dieci and T. Eirola, *On smooth decompositions of matrices*, SIAM J. Matrix Anal. Appl. **20** (1999), 800–819.

[3] V. Doležal, *The existence of a continuous basis of a certain linear subspace of E_r which depends on a parameter*, Casopis pro pestování matematiki **89** (1964), 466–468.

[4] J.-Cl. Evard, *On the existence of bases of class C^p of the kernel and the image of a matrix function*, Linear Algebra Appl. **135** (1990), 33–67.

[5] J.-Cl. Evard and F. Jafari, *The set of all $m \times n$ rectangular real matrices of rank r is connected by analytic regular arcs*, Proceedings AMS **120** (1994), 413–419.

[6] H. Gingold and P. Hsieh, *Globally analytic triangularization of a matrix function*, Linear Algebra Appl. **169** (1992), 75–101.

[7] I. Gohberg, P. Lancaster, and L. Rodman, *Invariant Subspaces of Matrices with Applications*, Wiley, New York, 1986.

[8] K.A. Grasse, *A vector-bundle version of a theorem of V. Dolezal*, Linear Algebra Appl. **392** (2004), 45–59.

[9] M.E. Gurtin, *An Introduction to Continuum Mechanics*, Academic Press, New York, 1981.

[10] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1976.

[11] W.C. Rheinboldt, *On the computation of multi-dimensional solution manifolds of parametrized equations*, Numer. Math. **53** (1988), 165–181.

[12] P. Sen and M.R. Chidambara, *Dolezal's theorem revisited*, Math. Systems Theory **13** (1979), 67–79.

[13] Y. Sibuya, *Some global properties of matrices of functions of one variable*, Math. Annalen **161** (1965), 67–77.

[14] L.M. Silverman and R.S. Bucy, *Generalizations of a theorem of Dolezal*, Math. Systems Theory **4** (1970), 334–339.

[15] L. Weiss and P.L. Falb, *Doležal's theorem, linear algebra with continuously parametrized elements, and time varying systems*, Math. Systems Theory **3** (1969), 67–75.

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