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Complex Hamiltonian Equations and Hamiltonian Energy

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Summary. - *In the framework of Kaehlerian manifolds, we obtain complex Hamiltonian equations on momentum phase space T^*M. Also we conclude complex Hamiltonian equations via the Legendre transformation. Then, definiting complex Routhian function similar to real analogue, we calculate Hamiltonian energy function of the system associated to complex Routhian function.*

1. Introduction

The modern development of analytical mechanics in terms of intrinsical geometrical properties of differentiable manifolds shows that the dynamics of Lagrangian and Hamiltonian systems is characterized by a suitable vector field X defined on the tangent and cotangent bundles (phase-spaces of velocities and momentum) of a given configuration manifold. If M is an m-dimensional configuration manifold and $L : TM \to R$ a regular Lagrangian function then there is a unique vector field X on TM, called tangent bundle of M, such that $i_{X_L}\omega_L = dE_L$, where ω_L is the symplectic form and E_L is energy associated to Lagrangian function L. The vector field X is a *semispray* (a class of vector fields on the tangent bundle TM which in-

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terprets geometrically a system of second order differential equations or shortly second order differential equation) since its integral curves are the solutions of the Euler-Lagrange equations (X is called Euler-Lagrange vector field) and If $H : T^*M \to R$ is a regular Hamiltonian energy then there is a unique vector field X_H on T^*M,called cotangent bundle of M, such that $i_{X_H}\omega = dH$, where ω is the symplectic form and H energy (Hamiltonian energy or Hamiltonian function).

The vector field X_H is a semispray since its integral curves are the solutions of the Hamiltonian equations (X_H is called Hamiltonian vector field) [1, 2].

The paper is structured as follows. In second 2, we recall complex and Kaehlerian manifolds. Besides, we remind complex Euler-Lagrange and Hamiltonian equations [3, 4].

In second 3 we set complex Hamiltonian equations on momentum phase space T^*M. In second 4 we obtain complex Hamiltonian equations via the Legendre transformation on Kaehlerian manifold. In second 5, on the tangent bundle of Kaehlerian manifold TM, definiting Routhian function [5], we find Hamiltonian energy of a complex system. In second 6 we calculate complex coordinate of the system by means of Routhian function if complex coordinate is periodic. Hereafter, all mappings and manifolds are assumed to be differentiable of class C^∞ and the sum is taken over repeated indices. Also, we denote by $\mathcal{F}(TM)$ the set of complex functions on TM, by $\chi(TM)$ the set of complex vector fields on TM and by $\Lambda^1(TM)$ the set of complex 1-forms on TM.

2. Preliminaries

2.1. Complex and Kaehlerian manifolds

Let M be m-real dimensional configuration manifold. A tensor field J on TM is called an *almost complex structure* on TM if at every point p of TM, J is endomorphism of the tangent space $T_p(TM)$ such that $J^2 = -I$. A manifold TM with fixed almost complex structure J is called *almost complex manifold*. Let (x^i) and $(x^i,\, y^i)$ be a real coordinate system of M and TM, and $\{(\frac{\partial}{\partial x^i})_p, (\frac{\partial}{\partial y^i})_p\}$ and $\{(dx^i)_p, (dy^i)_p\}$ natural bases over $\mathbf{R}$ of tangent space and cotangent space of TM, respectively. Let TM be an almost complex manifold with fixed almost complex structure J. TM is called *complex manifold* if there exists an open covering $\{U\}$ of TM such that there exists a local coordinate system $\{(x^i,\, y^i) : 1 \leq i \leq m\}$ on each U, we have

$$J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}, \quad J(\frac{\partial}{\partial y^i}) = -\frac{\partial}{\partial x^i}. \tag{1}$$

Let $z^i = x^i + \mathbf{i}\, y^i$, $\mathbf{i} = \sqrt{-1}$, $1 \leq i \leq m$, be a complex local coordinate system on a neighborhood U of any point p of TM. we define the vector fields and the dual covector fields

$$\begin{aligned} \left(\frac{\partial}{\partial z^i}\right)_p &= \frac{1}{2}\left\{\left(\frac{\partial}{\partial x^i}\right)_p - \mathbf{i}\left(\frac{\partial}{\partial y^i}\right)_p\right\} \\ \left(\frac{\partial}{\partial \overline{z}^i}\right)_p &= \frac{1}{2}\left\{\left(\frac{\partial}{\partial x^i}\right)_p + \mathbf{i}\left(\frac{\partial}{\partial y^i}\right)_p\right\}, \end{aligned} \tag{2}$$

$$\begin{aligned} \left(dz^i\right)_p &= \left(dx^i\right)_p + \mathbf{i}(dy^i)_p, \\ \left(d\overline{z}^i\right)_p &= \left(dx^i\right)_p - \mathbf{i}(dy^i)_p. \end{aligned} \tag{3}$$

which represent bases of the tangent space $T_p(TM)$ and cotangent space $T_p^*(TM)$ of TM, respectively. Then the endomorphism J acts on base (2) via

$$\left(\frac{\partial}{\partial z^i}\right) = \mathbf{i}\frac{\partial}{\partial z^i}, \quad J\left(\frac{\partial}{\partial \overline{z}^i}\right) = -\mathbf{i}\frac{\partial}{\partial \overline{z}^i}, \tag{4}$$

The dual endomorphism J^* of the cotangent space $T_p^*(TM)$ at any point p of manifold TM satisfies $J^{*2} = -I$ and is defined by

$$J^*(dz^i) = \mathbf{i}dz^i, \quad J^*(d\overline{z}^i) = -\mathbf{i}d\overline{z}^i. \tag{5}$$

Hermitian metric on an almost complex manifold with almost complex structure J is a Riemannian metric g on TM such that

$$g(JX, JY) = g(X, Y), \quad \forall X, Y \in \chi(TM). \tag{6}$$

An almost complex manifold TM with a Hermitian metric is called an almost Hermitian manifold. If TM is a complex manifold, then TM is called a Hermitian manifold. Let further TM be a 2m-dimensional almost Hermitian manifold with almost complex structure J and Hermitian metric g. The triple (TM, J, g) is called an *almost Hermitian structure*. Let (TM, J, g) be an almost Hermitian structure. The 2-form defined by

$$\Phi(X, Y) = g(X, JY), \quad \forall X, Y \in \chi(TM) \tag{7}$$

is called the Kaehlerian form of (TM, J, g). An almost Hermitian manifold is called *almost Kaehlerian* if its Kaehlerian form Φ is closed. If, moreover, TM is Hermitian, then TM is called a Kaehlerian manifold. Taking into consideration the above definitions , also we may say to be a *Kaehlerian manifold* of cotangent bundle T^*M.

2.2. Complex Euler-Lagrange and Hamiltonian equations

In this section, we remind complex Euler-Lagrange and complex Hamiltonian equations for classical mechanics structured on Kaehlerian manifold TM and T^*M given in [3, 4]. Let J be an almost complex structure on the Kaehlerian manifold and $(z^i, \overline{z}^i)$ its complex structures. We call to be the semispray the vector field ξ given by

$$\xi = \xi^i \frac{\partial}{\partial z^i} + \overline{\xi}^i \frac{\partial}{\partial \overline{z}^i}, \quad \xi^i = \dot{z}^i = \overline{z}^i, \quad \overline{\xi}^i = \dot{\xi}^i = \ddot{z}^i = \dot{\overline{z}}^i \quad 1 \leq i \leq m. \tag{8}$$

The vector field denoted by $V = J\xi$ and given by

$$J\xi = \mathbf{i}\xi^i \frac{\partial}{\partial z^i} - \mathbf{i}\overline{\xi}^i \frac{\partial}{\partial \overline{z}^i}, \tag{9}$$

is called *Liouville vector field* on the Kaehlerian manifold. We call *the kinetic energy* and *the potential energy of system* the maps given by $T, P : TM \to \mathbf{C}$ such that $T = \frac{1}{2} m_i (\overline{z}^i)^2 = \frac{1}{2} m_i (\dot{z}^i)^2, P = m_i \mathbf{g} h$, respectively, where m_i is mass of a mechanic system having m particles, $\mathbf{g}$ is the gravity acceleration and h is the origin distance of the a mechanic system on the Kaehlerian manifold. Then we call *Lagrangian function* the map $L : TM \to \mathbf{C}$ such that $L = T - P$ and also *the energy function* associated L the function given by $E_L = V(L) - L$. The *exterior product*(or *vertical contraction*) induced by J operator i_J defined by

$$i_J \omega(Z_1, Z_2, \ldots, Z_r) = \sum_{i=1}^{r} \omega(Z_1, \ldots, JZ_i, \ldots, Z_r), \tag{10}$$

where $\omega \in \wedge^r TM$, $Z_i \in \chi(TM)$. The exterior vertical derivation d_J is defined by

$$d_J = [i_J, d] = i_J d - d i_J, \tag{11}$$

where d is the usual exterior derivation. For almost complex structure J determined by (4), the closed Kaehlerian form is the closed 2-form given by $\Phi_L = -dd_J L$ such that

$$d_J = \mathbf{i}\frac{\partial}{\partial z^i}dz^i - = \mathbf{i}\frac{\partial}{\partial \overline{z}^i}d\overline{z}^i : \mathcal{F}(TM) \to \wedge^1 TM \tag{12}$$

and where $\mathcal{F}(TM)$ the set of complex functions on TM and $\Lambda^1(TM)$ the set of complex 1-forms on TM. Let TM be Kaehlerian manifold with closed Kaehlerian form Φ_L, and the vector field V is Liouville vector field and the function $L : TM \to \mathbf{C}$ is a Lagrangian on TM. Since the map $TM_{\Phi_L} : \chi(TM) \to \wedge^1(TM)$ such that $TM_{\Phi_L}(\xi) = i_\xi \Phi_L$ is an isomorphism, there exists a unique vector ξ on TM such that $i_\xi \Phi_L = dE_L$. We call ξ on TM as a *Lagrangian vector field* associated energy (or Lagrangian function) L on Kaehlerian manifold TM with closed Kaehlerian form Φ_L. (TM, Φ_L, ξ) (or $TM, \Phi_L, L)$ is called a Lagrangian system on Kaehlerian manifold TM. Let the curve $\alpha : \mathbf{C} \to TM$ be integral curve of ξ. we may give complex Euler-Lagrange equations on Kaehlerian manifold TM:

$$\mathbf{i}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial z^i}\right) - \frac{\partial L}{\partial z^i} = 0 \quad \mathbf{i}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \overline{z}^i}\right) + \frac{\partial L}{\partial \overline{z}^i} = 0, \tag{13}$$

whose solutions are the paths of the semispray ξ,where $\overline{z}^i = \dot{z}^i$. Let T^*M be the Kaehlerian manifold and $(z_i, \overline{z}_i), 1 \leq i \leq m$ its complex coordinates. Let almost complex structure J^* and Liouville form λ given by $J^*(dz_i) = \mathbf{i}dz_i$, $J^*(d\overline{z}_i) = -\mathbf{i}d\overline{z}_i$ and by $\lambda = (J^*(\omega)) = \frac{1}{2}\mathbf{i}(-z_i d\overline{z}_i + \overline{z}_i dz_i)$ such that $\omega = \frac{1}{2}(z_i d\overline{z}_i + \overline{z}_i dz_i)$ complex 1-form on T^*M. If $\Phi = -d\lambda$ is closed Kaehlerian form, then Φ is also a symplectic structure on T^*M. Let T^*M be Kaehlerian manifold with closed Kaehlerian form and the function $H : T^*M \to \mathbf{C}$ a Hamiltonian on T^*M. Since the map $T^*M_\Phi : \chi(T^*M) \to \wedge^1(T^*M)$ such that $\chi(T^*M)$ is sets of complex vector fields on T^*M and $\wedge^1(T^*M)$ is sets of anti-symmetric complex 1-forms on T^*M is an isomorphism, there exists a unique vector Z_H on T^*M such that $i_{Z_H}\Phi = dH$. We call Z_H on T^*M as a *Hamiltonian vector field* associated Hamiltonian energy H on Kaehlerian manifold with closed Kaehlerian form. (T^*M, Φ, Z_H) (or $T^*M, \Phi, H)$ is called a Hamiltonian system on Kaehlerian manifold with closed Kaehlerian form. Let T^*M be Kaehlerian manifold with closed Kaehlerian form Φ. Hamiltonian vector field Z_H on Kaehlerian manifold with closed Kaehlerian form Φ is given by

$$Z_H = \frac{1}{\mathbf{i}} \frac{\partial H}{\partial \overline{z}_i} \frac{\partial}{\partial z_i} - \frac{1}{\mathbf{i}} \frac{\partial H}{\partial z_i} \frac{\partial}{\partial \overline{z}_i}, \quad 1 \leq i \leq m \tag{14}$$

on T^*M. Let $\{z_i, \overline{z}_i : 1 \leq i \leq m\}$ be the complex coordinates in the Kaehlerian manifold. Suppose that the curve

$$\alpha : I \subset \mathbf{C} \to T^*M \tag{15}$$

Now, from $Z_H(\alpha(t)) = \dot{\alpha}$, then we infer the following equations

$$\frac{dz_i}{dt} = \frac{1}{\mathbf{i}} \frac{\partial H}{\partial \overline{z}_i}, \quad \frac{d\overline{z}_i}{dt} = -\frac{1}{\mathbf{i}} \frac{\partial H}{\partial z_i}. \tag{16}$$

which are called complex Hamiltonian equations on Kaehlerian manifold T^*M.

3. Complex Hamiltonian equations on momentum phase space

In this section, we obtain complex Hamiltonian equations for classical mechanics structured on momentum space T^*M that is 2m- dimensional cotangent bundle of an m-dimensional configuration manifold M. Let T^*M be the momentum space and $(z^i, \overline{z}_i = \frac{\partial L}{\partial z^i}), 1 \leq i \leq m$ its complex coordinates, where L is Lagrangian function. Let almost complex structure J^* and Liouville form λ give by $J^*(dz^i) = \mathbf{i}dz^i$, $J^*(d\overline{z}_i) = -\mathbf{i}d\overline{z}_i$ and by $\lambda = (J^*(\omega)) = \frac{1}{2}\mathbf{i}(-z^i d\overline{z}_i + \overline{z}_i dz^i)$ such that $\omega = \frac{1}{2}(z^i d\overline{z}_i + \overline{z}_i dz^i)$ complex 1-form on T^*M. If $\Phi = -d\lambda$ is closed Kaehlerian form, then Φ is also a symplectic structure on T^*M.

Definition 3.1. *Let T^*M be momentum space with closed Kaehlerian form Φ and the function $H : T^*M \to \mathbf{C}$ a Hamiltonian on T^*M. Since the map $T^*M_\Phi : \chi(T^*M) \to \wedge^1(T^*M)$ such that $\chi(T^*M)$ is* *sets of complex vector fields on T^*M and $\wedge^1(T^*M)$ is sets of anti-symmetric complex 1-forms on T^*M is an isomorphism, there exists a unique vector Z_H on T^*M such that $i_{Z_H}\Phi = dH$. We call Z_H on T^*M as a Hamiltonian vector field associated energy (or Hamiltonian function) H on momentum space T^*M with closed Kaehlerian form Φ. (T^*M, Φ, Z_H) (or $T^*M, \Phi, H)$ is called a Hamiltonian system on momentum space T^*M with closed Kaehlerian form Φ.*

Proposition 3.2. *Let T^*M be momentum space with closed Kaehlerian form Φ. Hamiltonian vector field Z_H onmomentum space T^*M with closed Kaehlerian form Φ is given by*

$$Z_H = \frac{1}{\mathbf{i}}\frac{\partial H}{\partial \overline{z}_i}\frac{\partial}{\partial z^i} - \frac{1}{\mathbf{i}}\frac{\partial H}{\partial z^i}\frac{\partial}{\partial \overline{z}_i}, \quad 1 \leq i \leq m \tag{17}$$

on T^*M.

Proof. Let T^*M be momentum space with closed Kaehlerian form Φ. Consider that Hamiltonian vector field Z_H associated Hamiltonian energy H is given by

$$Z_H = Z^i\frac{\partial}{\partial z^i} + \overline{Z}_i\frac{\cdot}{\partial}\partial\overline{z}_i, \quad 1 \leq i \leq m. \tag{18}$$

Let $\{z^i, \overline{z}_i : 1 \leq i \leq m\}$ be the complex coordinates in the momentum space. Suppose that the curve

$$\alpha : I \subset \mathbf{C} \to TM \tag{19}$$

be an integral curve of Hamiltonian vector field Z_H, i.e.,

$$Z_H(\alpha(t)) = \dot{\alpha}, \ t \in I. \tag{20}$$

In the local coordinates we have

$$\begin{aligned} \alpha(t) &= (z^i(t), \overline{z}_i(t)), & (21)\\ \dot{\alpha}\ (t) &= \frac{dz^i}{dt}\frac{\partial}{\partial z^i} + \frac{d\overline{z}_i}{dt}\frac{\partial}{\partial \overline{z}_i}. & (22)\end{aligned}$$

For the closed Kaehlerian form Φ on TM, we have

$$\Phi = -d\lambda = -d(\frac{1}{2}\mathbf{i}(-z^i d\overline{z}_i + \overline{z}_i dz^i)) = -\mathbf{i}d\overline{z}_i \wedge dz^i. \tag{23}$$

From the isomorphism given in 3.1, we calculate by

$$\iota_{Z_H}\Phi = i_{Z_H}(-d\lambda) = -\mathbf{i}\overline{Z}_i dz^i + \mathbf{i}Z^i d\overline{z}_i. \tag{24}$$

On the other hand, we obtain as

$$dH = \frac{\partial H}{\partial z^i}dz^i + \frac{\partial H}{\partial \overline{z}_i}d\overline{z}_i \tag{25}$$

the differential of Hamiltonian energy. From $i_{Z_H}\Phi = dH$, we find as

$$Z_H = \frac{1}{\mathbf{i}}\frac{\partial H}{\partial \overline{z}_i}\frac{\partial}{\partial z^i} - \frac{1}{\mathbf{i}}\frac{\partial H}{\partial z^i}\frac{\partial}{\partial \overline{z}_i}, \quad 1 \leq i \leq m,$$

the Hamiltonian vector field on momentum space T^*M with closed Kaehlerian form Φ. □

Now, from $Z_H(\alpha(t)) = \dot{\alpha}$, then we infer the following equations

$$\frac{dz^i}{dt} = \frac{1}{\mathbf{i}}\frac{\partial H}{\partial \overline{z}_i}, \quad \frac{d\overline{z}_i}{dt} = -\frac{1}{\mathbf{i}}\frac{\partial H}{\partial z^i} \tag{26}$$

which are called complex Hamiltonian equations on momentum space T^*M. Also, if we set complex coordinates $(\overline{z}^i, z_i = \frac{\partial L}{\partial \overline{z}^i}), 1 \leq i \leq m$, on momentum space T^*M, where L is Lagrangian function, and $\overline{z}^i$ is conjugate of z^i, we have the complex Hamiltonian equations given by

$$\frac{d\overline{z}^i}{dt} = \frac{1}{\mathbf{i}}\frac{\partial H}{\partial z_i}, \quad \frac{dz_i}{dt} = -\frac{1}{\mathbf{i}}\frac{\partial H}{\partial \overline{z}^i}. \tag{27}$$

4. Complex Hamiltonian equations via Legendre transformation

We may say that the *complex Legendre transformation* is the mapping Leg determined by L given by

$$\begin{array}{rcl} Leg : TM & \rightarrow & T^*M \\ (z^i, \overline{z}^i) & \rightarrow & (z^i, \overline{z}_i), \\ \dfrac{\partial L}{\partial z^i} & = & \overline{z}_i. \end{array}$$

Given by (13) complex Euler-Lagrange equations on Kaehlerian manifold TM. Suppose the following as Hamiltonian energies associated to Lagrangian function

$$\begin{array}{rclrclrcl} H(\overline{z}_i, z^i, t) & = & \overline{z}_i\,\dot{z}^i - L(z^i, \overline{z}^i, t), & \dfrac{\partial L}{\partial z^i} & = & \overline{z}_i, & \dfrac{dz^i}{dt} & = & \dot{z}^i \\ H(z_i, \overline{z}^i, t) & = & z_i\,\dot{\overline{z}}^i - L(z^i, \overline{z}^i, t), & \dfrac{\partial L}{\partial \overline{z}^i} & = & z_i, & \dfrac{d\overline{z}^i}{dt} & = & \dot{\overline{z}}^i \end{array} \tag{28}$$

Taking differential of functions the above, we have

$$\begin{array}{rcl} dH & = & \dot{z}^i\,d\overline{z}_i - \dfrac{\partial L}{\partial z^i}dz^i - \dfrac{\partial L}{\partial t}dt, \\ dH & = & \dot{\overline{z}}^i\,dz_i - \dfrac{\partial L}{\partial \overline{z}^i}d\overline{z}^i - \dfrac{\partial L}{\partial t}dt. \end{array} \tag{29}$$

Besides, complete differential Hamiltonian energies given by variables $(\overline{z}_i, z^i, t)$ and $(z_i, \overline{z}^i, t)$ are

$$\begin{array}{rcl} dH & = & \dfrac{\partial H}{\partial \overline{z}_i}d\overline{z}_i + \dfrac{\partial H}{\partial z^i}dz^i + \dfrac{\partial H}{\partial t}dt, \\ dH & = & \dfrac{\partial H}{\partial z_i}dz_i + \dfrac{\partial H}{\partial \overline{z}^i}d\overline{z}^i + \dfrac{\partial H}{\partial t}dt. \end{array} \tag{30}$$

From (29) and (30), we calculate

$$\begin{array}{rcl} \dot{z}^i & = & \dfrac{\partial H}{\partial \overline{z}_i}, \quad \dfrac{\partial L}{\partial z^i} = -\dfrac{\partial H}{\partial z^i}, \dfrac{\partial L}{\partial t} = -\dfrac{\partial H}{\partial t}, \\ \dot{\overline{z}}^i & = & \dfrac{\partial H}{\partial z_i}, \quad \dfrac{\partial L}{\partial \overline{z}^i} = -\dfrac{\partial H}{\partial \overline{z}^i}, \dfrac{\partial L}{\partial t} = -\dfrac{\partial H}{\partial t}. \end{array} \tag{31}$$

Consequently, by means of (13) and (31), we conclude complex Hamiltonian equations on Kaehlerian manifold T^*M :

$$\frac{d\overline{z}_i}{dt} = -\frac{1}{\mathbf{i}}\frac{\partial H}{\partial z^i}\ \frac{dz_i}{dt} = \frac{1}{\mathbf{i}}\frac{\partial H}{\partial \overline{z}^i}. \tag{32}$$

5. Hamiltonian energy of system by means of Routhian function

Let $T(TM)$ and $T^*(TM)$ be tangent and cotangent bundle of Kaehlerian manifold TM. Then we define a mapping as follows:

$$\begin{aligned} T(TM) &\rightarrow T^*(TM) \\ (z^i, \omega^i, \overline{z}^i, \overline{\omega}^i) &\rightarrow (z^i, \omega^i, z_i = \frac{\partial L}{\partial \overline{z}^i}, \overline{\omega}^i) \end{aligned} \tag{33}$$

The differential of Lagrangian function $L(z^i, \omega^i, \overline{z}^i, \overline{\omega}^i)$ on $T(TM)$ is

$$\begin{aligned} dL &= \frac{\partial L}{\partial z^i} dz^i + \frac{\partial L}{\partial \overline{z}^i} d\overline{z}^i + \frac{\partial L}{\partial \omega^i} d\omega^i + \frac{\partial L}{\partial \overline{\omega}^i} d\overline{\omega}^i \\ &= \overline{z}_i dz^i + z_i d\overline{z}^i + \frac{\partial L}{\partial \omega^i} d\omega^i + \frac{\partial L}{\partial \overline{\omega}^i} d\overline{\omega}^i. \end{aligned} \tag{34}$$

Then we have

$$d(L - z_i \overline{z}^i) = \overline{z}_i dz^i - \overline{z}^i dz_i + \frac{\partial L}{\partial \omega^i} d\omega^i + \frac{\partial L}{\partial \overline{\omega}^i} d\overline{\omega}^i. \tag{35}$$

We may definite Routhian function as follows:

$$R(z^i, z_i, \omega^i, \overline{\omega}^i) = z_i \overline{z}^i - L. \tag{36}$$

where are velocity $\overline{z}^i = \dot{z}^i$ and momentum $z_i = \frac{\partial L}{\partial \overline{z}^i}$. Taking the differential of (36), one may calculate

$$dR = -\overline{z}_i dz^i + \overline{z}^i dz_i - \frac{\partial L}{\partial \omega^i} d\omega^i - \frac{\partial L}{\partial \overline{\omega}^i} d\overline{\omega}^i. \tag{37}$$

Otherwise we write

$$dR = \frac{\partial R}{\partial z^i} dz^i + \frac{\partial R}{\partial z_i} dz_i + \frac{\partial R}{\partial \omega^i} d\omega^i + \frac{\partial R}{\partial \overline{\omega}^i} d\overline{\omega}^i. \tag{38}$$

From (37) and (38), one may find the equalities

$$\overline{z}^i = \frac{\partial R}{\partial z_i}, \quad \overline{z}_i = -\frac{\partial R}{\partial z^i}, \tag{39}$$

and

$$\frac{\partial L}{\partial \omega^i} = -\frac{\partial R}{\partial \omega^i}, \quad \frac{\partial L}{\partial \overline{\omega}^i} = -\frac{\partial R}{\partial \overline{\omega}^i}. \tag{40}$$

Keeping in mind complex Euler-Lagrange equations given by (13), we obtain

$$\mathbf{i}\frac{\partial}{\partial t}\left(\frac{\partial R}{\partial \omega^i}\right)-\frac{\partial R}{\partial \omega^i}=0, \quad \mathbf{i}\frac{\partial}{\partial t}\left(\frac{\partial R}{\partial \overline{\omega}^i}\right)+\frac{\partial R}{\partial \overline{\omega}^i}=0. \tag{41}$$

Thus, Routhian function is respectively Hamiltonian energy and Lagrangian function with respect to coordinates z^i and $(\omega^i, \overline{\omega}^i)$. Being Hamiltonian energy of system, it may write

$$H=\overline{z}^i\frac{\partial L}{\partial z^i}+\overline{\omega}^i\frac{\partial L}{\partial \overline{\omega}^i}-L=\overline{z}^i z_i+\overline{\omega}^i\frac{\partial L}{\partial \overline{\omega}^i}-L. \tag{42}$$

By means of (36) and (40), Hamiltonian energy of system using Routhian function we calculate

$$H=R-\overline{\omega}^i\frac{\partial R}{\partial \overline{\omega}^i}. \tag{43}$$

6. Corollary

If some coordinates are periodic, it is benefit to use Routhian function. Because, both Lagrangian function and Routhian function are not dependent z^i, then z^i are periodic. Hence Routhian function is only $z_i, \omega^i, \overline{\omega}^i$. For z^i are periodic, momentum z_i are constant. If it is written constant value instead of momentum, we have equations being constant only coordinates $\omega^i, \overline{\omega}^i$ the following as:

$$\begin{aligned}\mathbf{i}\frac{\partial}{\partial t}\left(\frac{\partial R(z_i,\omega^i,\overline{\omega}^i)}{\partial \omega^i}\right)-\frac{\partial R(z_i,\omega^i,\overline{\omega}^i)}{\partial \omega^i} &= 0,\\ \mathbf{i}\frac{\partial}{\partial t}\left(\frac{\partial R(z_i,\omega^i,\overline{\omega}^i)}{\partial \overline{\omega}^i}\right)+\frac{\partial R(z_i,\omega^i,\overline{\omega}^i)}{\partial \overline{\omega}^i} &= 0.\end{aligned} \tag{44}$$

Thus, from the above equations, one may find $\omega^i(t)$ and $\overline{\omega}^i(t)$. Consequently, writing in (44) the values of $\omega^i(t)$ and $\overline{\omega}^i(t)$ and finding integral of $\overline{z}^i=\frac{\partial R(z_i,\omega^i,\overline{\omega}^i)}{\partial z_i}$ we calculate $z^i(t)$.

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