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Lower Bounds of Fučik Eigenvalues of the Weighted One Dimensional p-Laplacian

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Summary. - *In this paper we obtain a family of curves bounding the region which contains all the non trivial Fučik eigenvalues of the weighted one dimensional p laplacian with Neumann boundary conditions. We obtain different proofs of the isolation result of the trivial lines, and the existence of a gap at infinity between the first curve and the trivial lines. We also give a lower bound for the eigenvalues of the p-Laplacian with Neumann boundary conditions.*

1. Introduction

In this work we study the following weighted Fučik eigenvalue problem in a bounded interval (a, b),

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = r(x)|u(x)|^{p-2}[\alpha u^+(x) - \beta u^-(x)] \\ u'(a) = u'(b) = 0 \end{cases} \tag{1}$$

where $1 < p < \infty$, $r(x) \in L^\infty$ is allowed to change sign, $u^+ = \max(u, 0)$, $u^- = \max(-u, 0)$, and $\alpha, \beta \in I\!R$. A pair $(\alpha, \beta) \in I\!R^2$ is called a Fučik eigenvalue if Problem (1) has a nontrivial solution

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$u \in H^1(a,b)$, and u is called a Fučik eigenfunction. We call Σ the set of Fučik eigenvalues.

We also consider the related eigenvalue problem,

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = \mu r(x)|u(x)|^{p-2}u(x) \\ u'(a) = u'(b) = 0 \end{cases} \tag{2}$$

For positive and continuous weights $r(x)$, all eigenvalues form a countable set $\{\mu_k\}_{k\in\mathbb{N}}$, and the eigenfunction u_k corresponding to μ_k has exactly $k-1$ zeros, see [10, 15]. When the weight takes both positives and negatives values, there exist two sequences of positive and negative variational eigenvalues,

$$\sigma^+ = \{0 \le \mu_1^+ < \mu_2^+ < \dots\} \qquad \sigma^- = \{0 \ge \mu_1^- > \mu_2^- > \dots\},$$

which are obtained obtained by the Ljusternik Schnirelmann theory, see [5, 13]. However, it is not known if the totality of eigenvalues consists of these two sequences.

We consider first $r(x) > 0$ which implies $\sigma^- = \emptyset$, $\mu_1 = 0$. Let μ be any eigenvalue of Problem (2). It is clear that (μ,μ), $0 \times \mathbb{R}$ and $\mathbb{R} \times 0$ belongs to Σ. The lines $0 \times \mathbb{R}$ and $\mathbb{R} \times 0$ are called the trivial curves, and we call $\Sigma^* = \Sigma \setminus \{0 \times \mathbb{R} \cup \mathbb{R} \times 0\}$. When $r(x) \equiv 1$, it was proved in [7] that the two lines $0 \times \mathbb{R}$ and $\mathbb{R} \times 0$ are isolated in Σ. Also, the authors prove that the first curve Γ_1 is not asymptotic to zero, and remains bounded below by the first Dirichlet eigenvalue of an interval of length $2(b-a)$. The first non trivial curve Γ_1 is obtained as the first intersection point of Σ^* with a line parallel to the diagonal and passing through $(s,0)$ for each $s \in \mathbb{R}$, this construction was introduced in [8]. The problem with indefinite weights was considered in [1, 2]. We give here a different proof of the isolation result of the trivial lines and the existence of a gap at infinity between Γ_1 and the trivial lines. Our main result is the following:

THEOREM 1.1. *Let $\int_a^b r(x) = m$. There exists an hyperbolic type curve $y = f(x)$ such that $\beta \ge f(\alpha)$ for every Fučik eigenvalue $(\alpha,\beta) \in \Gamma_1$ of Problem (1). Moreover,*

$$\lim_{\alpha\to\infty} f(\alpha) = \frac{1}{m(b-a)^{p-1}}.$$

In certain sense, it is a lower bound for the Fučik eigenvalues. We also analyze the optimality of this curve, and we extend it to problems with indefinite weights. The proof is based on a Lyapunov type inequality for the Neumann boundary condition. This inequality was proved in [12] for $p = 2$ by using Ricatti equation techniques, we also give a different proof.

We will prove the isolation of the first positive eigenvalue of problem (2), which enable us to define the second positive Neumann eigenvalue μ_2. Following the ideas of [3, 4], we will show that $\mu_2 = \mu_2^+$, and an associated eigenfunction has only one zero in (a, b). Hence, the Lyapunov inequality enable us to obtain a sharp lower bound for μ_2. We have the following theorem:

Theorem 1.2. *Let μ_2 be the second eigenvalue of Problem (2). Then,*

$$\frac{2^p}{(b-a)^{p-1}\int_a^b r^+(x)dx} \leq \mu_2, \tag{3}$$

where $r^+(x) = \max\{r(x), 0\}$.

We also prove the optimality of this bound:

Theorem 1.3. *Let $\varepsilon \in I\!R$ be a positive number. There exist a family of weight functions $r_\varepsilon \in L^\infty(a, b)$ satisfying $\int_a^b r_\varepsilon(x)dx = m$ such that*

$$\lim_{\varepsilon \to 0^+} \mu_{2,\varepsilon} = \frac{2^p}{(b-a)^{p-1}m}$$

where $\mu_{2,\varepsilon}$ is the second eigenvalue of

$$\begin{cases} -(|u'|^{p-2}u')' = \mu\, r_\varepsilon |u|^{p-2}u \\ u'(a) = u'(b) = 0. \end{cases} \tag{4}$$

The paper is organized as follows: Section 2 is devoted to the Lyapunov inequality, and the Neumann eigenvalue problem with indefinite weights, we prove Theorems 1.2 and 1.3, and we also consider the higher eigenvalues. In Section 3 we prove Theorem 1.1 and we extend it to indefinite weights.

2. A Lyapunov type Inequality and Neumann Eigenvalues

Let us consider the following quasilinear boundary value problem:

$$\begin{cases} -(|u'|^{p-2}u')' = s|u|^{p-2}u \\ u(a) = u'(b) = 0, \end{cases} \tag{5}$$

where $s \in L^\infty(a,b)$ is an integrable function, and $1 < p < \infty$. By a solution of problem (5) we understand a real valued function $u \in W^{1,p}(a,b)$, $u(a) = 0$ such that

$$\int_a^b |u'|^{p-2}u'v' = \int_a^b s|u|^{p-2}uv \tag{6}$$

for each $v \in W^{1,p}(a,b)$, $v(a) = 0$.

We have the following Lyapunov type inequality:

THEOREM 2.1. *Assume that the problem (5) has a positive solution. Then, the following inequality holds:*

$$\left(\frac{1}{b-a}\right)^{p-1} \leq \int_a^b s^+(x)dx.$$

Proof. Let c be a point in (a,b) where $u(x)$ is maximized. Clearly, by using Holder's inequality and the variational formulation (6),

$$\begin{aligned} u(c) &= \int_a^c u'(x)dx \\ &\leq (c-a)^{1/q}\left(\int_a^c |u'(x)|^p dx\right)^{1/p} \\ &\leq (b-a)^{1/q}\left(\int_a^b |u'(x)|^p dx\right)^{1/p} \\ &= (b-a)^{1/q}\left(\int_a^b s(x)|u(x)|^p dx\right)^{1/p} \\ &\leq (b-a)^{1/q}u(c)\left(\int_a^b s^+(x)dx\right)^{1/p}, \end{aligned}$$

and the result follows after cancelling $u(c)$ in both sides. □

As a corollary, we have the following inequality for solutions of the Neumann boundary value problem with indefinite weights:

Corollary 2.2. *Let $s \in L^\infty(a,b)$ be an integrable function. Let us suppose that the problem*

$$-(|u'|^{p-2}u')' = s|u|^{p-2}u \qquad u'(a) = u'(b) = 0$$

has a solution which changes sign in (a,b) once. Then

$$\frac{2^p}{(b-a)^{p-1}} \leq \int_a^b s^+(x)dx. \tag{7}$$

Proof. Let c be the zero of u in (a,b). Applying Theorem 2.1 in (a,c) and (c,b), we have:

$$\left(\frac{1}{c-a}\right)^{p-1} + \left(\frac{1}{b-c}\right)^{p-1} \leq \int_a^b s^+(x)dx.$$

Now, the sum of the left hand side is minimized when both terms are equal, which gives

$$2\left(\frac{2}{b-a}\right)^{p-1} \leq \int_a^b s^+(x)dx. \qquad \square$$

Let us consider now the eigenvalue problem (2):

$$\begin{cases} -(|u'|^{p-2}u')' = \mu r|u|^{p-2}u \\ u'(a) = u'(b) = 0, \end{cases}$$

where $r \in L^\infty(a,b)$ is an integrable function with

$$meas\{x \in (a,b) : r(x) > 0\} \neq 0, \quad meas\{x \in (a,b) : r(x) < 0\} \neq 0.$$

The eigenvalues μ_k^+ are obtained by the Ljusternik Schnirelmann theory:

$$\mu_k^+ = \inf_{F \in C_k^{(a,b)}} \sup_{u \in F} \int_a^b |u'|^p \tag{8}$$

where

$$\begin{aligned} C_k^{(a,b)} &= \{C \subset M : C \text{ compact},\ C = -C,\ \gamma(C) \geq k\}, \\ M &= \{u \in W^{1,p}(a,b) : \int_a^b r|u|^p = 1\}, \end{aligned}$$

and $\gamma : \Sigma \to I\!N \cup \{\infty\}$ is the Krasnoselskii genus. The negatives eigenvalues σ^- coincides with the positive eigenvalues of the weight $-r$. The regularity results of [6, 14] imply that the solutions u are at least of class $C^{1,\alpha}_{loc}$, and satisfy the differential equation almost everywhere in (a, b).

We collect first some results related to the eigenvalues of problem (2) which can be found in [5, 11, 13]. However, the proof of the isolation of the first eigenvalue seems difficult to find, and we will prove it below.

Lemma 2.3. *If $\int_a^b r < 0$, then the eigenvalue of problem (2) admits a positive eigenvalue μ_1^+ with a positive eigenfunction, μ_1^+ is unique and simple, and the interval $(0, \mu_1^+)$ does not contain any eigenvalue. If $\int_a^b r < 0$, then $\mu_1^+ = 0$. If $\int_a^b r = 0$, then $\mu_1^+ = 0 = \mu_1^-$ is the unique eigenvalue with a positive eigenfunction.*

Remark 2.4. *We will prove now the isolation of the first eigenvalue for indefinite weights. For the Dirichlet boundary condition, this result can be found in [4]. However, the proof follows by using an estimate of the measure of the nodal domains of an eigenfunction u, a connected component of (a, b) $\{x \in (a, b) : u(x) = 0\}$. If N is a nodal domain of a Dirichlet eigenfunction with eigenvalue λ, then*

$$|N| \geq (C\lambda\|r\|_s)^{-\gamma} \tag{9}$$

where C, s, γ are positive constants depending only on p. Equation (9) was obtained by using the Sobolev immersion theorem. For the Neumann problem we cannot apply it on the nodal domains which reaches the boundary.

Lemma 2.5. *The eigenvalue μ_1^+ is isolated, that is, there exists $\delta > 0$ such that in the interval $(\mu_1^+; \mu_1^+ + \delta)$ there are no other eigenvalues of problem (2).*

Proof. Let us assume by contradiction that there exists a sequence of eigenvalues of (2), $\mu^n \to \mu_1^+$, and let u^n be an eigenfunction of μ^n satisfying $\int_a^b r|u^n|^p = 1$. Now, since

$$0 < \int_a^b |(u^n)'|^p = \mu^n \int_a^b r|u^n|^p,$$

the set $\{u^n\}_n$ is bounded in $W^{1,p}(a,b)$, and there exist a subsequence (still denoted $\{u^n\}_n$) and $u \in W^{1,p}(a,b)$ such that $u^n \to u$ weakly in $W^{1,p}(a,b)$, and $\int_a^b r|u|^p = 1$. Hence,

$$\int_a^b |u'|^p \leq \liminf_{n\to\infty} \mu^n \int_a^b r|u^n|^p = \mu_1^+.$$

We conclude that u is an eigenfunction associated to μ_1^+, and we can assume that $u > 0$ (the case $u < 0$ is similar). Since $u^n \to u$ in measure, and u^n changes sign, we have

$$|\Omega_n^-| \to 0,$$

where $\Omega_n^- = \{x \in (a,b) : u^n(x) < 0\}$. From equation (9), u^n cannot have interior nodal domains, since the measure of the interior nodal domains is bounded by below. Hence, Ω_n^- contains at least one boundary point. Let us assume that $a \in \Omega_n^-$ (the case $b \in \Omega_n^-$ is similar). We have $(a, c_n) \subset \Omega_n^-$, where c_n is the first zero of u^n, and Theorem 2.1 gives

$$\frac{1}{(c_n - a)^{p-1} \int_a^{c_n} r^+(x)dx} \leq \mu^n.$$

Clearly, if $|\Omega_n^-| \to 0$, then μ^n goes to infinity, which contradicts the fact that $\mu^n \to \mu_1^+$. □

Since μ_1^+ is isolated and there exist other eigenvalues, it makes sense to define the second eigenvalue μ_2 of problem (2) as:

$$\mu_2 = \min\{\mu \in I\!R : \mu > \mu_1^+ \text{ and } \mu \text{ is an eigenvalue}\}.$$

Proposition 2.6. *The eigenvalue μ_2 coincides with the second variational eigenvalue μ_2^+ given by the Lyusternik Schnirelman theory.*

Proof. Let u be an eigenfunction of μ_2. Since u changes sign, let us define $w_1 = k.u^+$ and $w_2 = h.u^-$, where we choose $k, h \in I\!R$ such that $\int_a^c r|w_1|^p = \int_c^b r|w_2|^p = 1$. The set

$$F_2 = \{s.w_1 + t.w_2 : s, t \in I\!R, \int_a^b r|s.w_1 + t.w_2|^p = 1\}$$

satisfy $\gamma(F_2) \geq 2$, since they are linearly independent, and is an admissible set in the variational characterization of μ_2^+. Moreover,

$$\int_a^b r|s.w_1 + t.w_2|^p = s^p \int_a^b r|w_1|^p + t^p \int_a^b r|w_2|^p,$$

which gives $|s|^p + |t|^p = 1$.

Hence, from (8), we have

$$\mu_2^+ = \inf_{F \in C_2^{(a,b)}} \sup_{u \in F} \int_a^b |u'|^p \leq \sup_{u \in F} \int_a^b |u'|^p \leq (|s|^p + |t|^p)\mu_2 = \mu_2$$

and the other inequality follows from the definition of μ_2. □

PROPOSITION 2.7. *Any eigenfunction corresponding to μ_2 has only one zero.*

Proof. Let c be the first zero in (a, b) of an eigenfunction u_2 corresponding to μ_2. The first eigenfunction v of problem

$$\begin{cases} -(|v'|^{p-2}v')' = \lambda r|v|^{p-2}v \\ v(c) = v'(b) = 0, \end{cases}$$

is of one sign and is simple (the proof follows by using the Picone's identity, as in [4]). Since the restriction of u_2 to (c, b) is a solution with $\lambda = \mu_2$, the first eigenvalue λ_1 satisfies $\lambda_1 \leq \mu_2$. If $\lambda_1 = \mu_2$, then $v = u_2$, and in this case, u_2 has no zeros in (c, b).

We define $w_1 = k.u_2$ in (a, c) and zero in (c, b), and $w_2 = h.v$ in (c, b) and zero in (a, c), with $k, h \in \mathbb{R}$ such that $\int_a^c r|w_1|^p = \int_c^b r|w_2|^p = 1$. As before, the set

$$F_2 = \{s.w_1 + t.w_2 : s, t \in \mathbb{R}, |s|^p + |t|^p = 1\}$$

satisfies $\gamma(F_2) \geq 2$, and is an admissible set in the variational characterization of μ_2. Hence, from (8), we have

$$\begin{aligned} \mu_2 &= \inf_{F \in C_2^{(a,b)}} \sup_{u \in F} \int_a^b |u'|^p \\ &\leq \sup_{u \in F} \int_a^c |u'|^p + \sup_{u \in F} \int_c^b |u'|^p \\ &\leq s^p \mu_2 + t^p \lambda \end{aligned}$$

Now, if u_2 has another zero, $\lambda < \mu_2$, which gives the contradiction:

$$\mu_2 \leq s^p \mu_2 + t^p \lambda < (s^p + t^p)\mu_2 = \mu_2,$$

and the proof is finished. □

We prove now Theorem 1.2.

Proof of Theorem 1.2: Let u_2 be an eigenfunction of μ_2, which has only one zero in (a, b). The Lyapunov inequality of Corollary (2.2) with $s(x) = \mu_2 r(x)$ gives the desired result. □

In order to prove the optimality of the lower bound (3), we need the following results for the one dimensional Steklov eigenvalue problem:

LEMMA 2.8. *Let $\tau_1 = 0$ be the first eigenvalue of the Steklov problem*

$$\begin{cases} -(|v'(x)|^{p-2}v'(x))' &= 0 \\ -|v'(a)|^{p-2}v'(a) &= \tau|v(a)|^{p-2}v(a) \\ |v'(b)|^{p-2}v'(b) &= \tau|v(b)|^{p-2}v(b) \end{cases} \tag{10}$$

Then, $\tau_1 = 0$ is the unique eigenvalue with a positive associated eigenfunction, which is simple and isolated. The second eigenvalue and the corresponding eigenfunction are given by

$$\tau_2 = \frac{2^{p-1}}{(b-a)^{p-1}}, \qquad \textit{and} \qquad v_2 = x - \frac{b+a}{2}. \tag{11}$$

The one dimensional case could be solved explicitly by integrating equation (10). The general case for the Steklov eigenvalue problem in $\Omega \subset I\!R^n$,

$$\begin{cases} -div(|\nabla u|^{p-2}\nabla u) &= 0 & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \eta} &= \tau V(x)|u|^{p-2}u & \text{on } \partial\Omega \end{cases}$$

was considered in [9], where the proof of the simplicity and isolation of τ_1 could be found. The eigenvalues are characterized variationally as

$$\tau_k(\Omega) = \inf_{F \in C_k^\Omega} \sup_{u \in F} \int_\Omega |u'|^p \tag{12}$$

where

$$\begin{aligned} C_k^{\Omega} &= \{C \subset M^{\Omega} : C \text{ compact },\ C = -C,\ \gamma(C) \geq k\}, \\ M^{\Omega} &= \{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} V|u|^p = 1\}. \end{aligned}$$

For the one dimensional case, we have $M^{(a,b)} = \{u \in W^{1,p}(a,b) : |u(a)|^p + |u(b)|^p = 1\}$.

We prove now the optimality the lower bound (3).

Proof of Theorem 1.3: Let $\varepsilon > 0$. We introduce the weights functions r_ε,

$$r_\varepsilon = \begin{cases} \frac{m}{2\varepsilon} & \text{for } x \in [a, a+\varepsilon] \\ 0 & \text{for } x \in [a+\varepsilon, b-\varepsilon] \\ \frac{m}{2\varepsilon} & \text{for } x \in [b-\varepsilon, b] \end{cases}$$

and we consider the set $C_2 = span\{w_1, w_2\} \cap \{u \in W^{1,p}(a,b) : \int_a^b r_\varepsilon |u|^p = 1\}$, where

$$w_1 = \begin{cases} x - \frac{b+a}{2} & x \in [a, \frac{b+a}{2}] \\ 0 & x \in [\frac{b+a}{2}, b] \end{cases}, \quad w_2 = \begin{cases} 0 & x \in [a, \frac{b+a}{2}] \\ x - \frac{b+a}{2} & x \in [\frac{b+a}{2}, b] \end{cases}$$

which is admissible in the characterization (8) of the second variational Neumann eigenvalue. Let $w = sw_1 + tw_2 \in C_2$. We have:

$$1 = s^p \int_a^\varepsilon \frac{m}{2\varepsilon}|w_1|^p + t^p \int_\varepsilon^b \frac{m}{2\varepsilon}|w_2|^p = \frac{s^p + t^p}{2} \int_a^b r_\varepsilon \left| x - \frac{b+a}{2} \right|^p,$$

since $\int_a^\varepsilon |w_1|^p = \int_\varepsilon^b |w_2|^p$. Now,

$$\int_a^b |w'|^p = s^p \int_a^{\frac{b+a}{2}} |w_1'|^p + t^p \int_{\frac{b+a}{2}}^b |w_2'|^p = \frac{b-a}{2}(s^p + t^p),$$

and replacing $s^p + t^p$, we obtain

$$\int_a^b |w'|^p = \frac{(b-a)}{\int_a^b r_\varepsilon \left| x - \frac{b+a}{2} \right|^p} = \frac{(b-a)}{\int_a^b r_\varepsilon |v_2|^p},$$

where $v_2 = x - \frac{b+a}{2}$ is the second Steklov eigenfunction. Hence,

$$\mu_{2,\varepsilon} = \inf_{F \in C_2^{(a,b)}} \sup_{u \in F} \int_a^b |u'|^p \leq \frac{(b-a)}{\int_a^b r_\varepsilon |v_2|^p} \tag{13}$$

Let $\delta_c(x)$ be the delta function at c. We have

$$\int_a^b r_\varepsilon(x)|v_2|^p \to \frac{m}{2}\int_a^b (\delta_a(x)+\delta_b(x))|v_2|^p dx$$

as $\varepsilon \to 0^+$. Here,

$$\int_a^b (\delta_a(x)+\delta_b(x))|v_2|^p dx = |v_2(a)|^p + |v_2(b)|^p = 2\left(\frac{b-a}{2}\right)^p.$$

Now, since Theorem 1.2 gives $\frac{2^p}{m(b-a)^{p-1}} \leq \mu_{2,\varepsilon}$, we obtain as $\varepsilon \to 0^+$ in equation (13),

$$\mu_{2,\varepsilon} \to \frac{2^p}{m(b-a)^{p-1}},$$

which gives the optimality of the lower bound. □

Remark 2.9. *Let us note that the characterizations of the second Neumann and Steklov variational eigenvalues (8) and (12) coincides for the singular weight r in $[a,b]$ given by*

$$r = \frac{m}{2}\delta_a(x) + \frac{m}{2}\delta_b(x),$$

since replacing v by u_2 in equation (6) we obtain

$$\begin{aligned}\int_a^b |u_2'|^p &= \mu_2^+ \int_a^b \frac{m}{2}(\delta_a(x)+\delta_b(x))|u_2|^p \\ &= \mu_2^+ \frac{m}{2}(|u_2(a)|^p + |u_2(b)|^p)\end{aligned}$$

Finally, we consider the higher Neumann eigenvalues, obtained by the Lyusternik Schnirelman method or not. We have:

Theorem 2.10. *Let $\tilde{\mu}_k$ be a positive eigenvalue of Problem (2) such that the associate eigenfunction has $k-1$ zeros in (a,b). Then,*

$$\frac{2^p(k-1)^p}{(b-a)^{p-1}\int_a^b r(x)dx} \leq \tilde{\mu}_k.$$

Proof. Let u_k be an associate eigenfunction to $\tilde{\mu}_k$. Let $x_1 < x_2 < \ldots < x_{k-1}$ be the zeros of u_k in (a,b). Let c_j be a maximum of $|u_k(x)|$ in (x_j, x_{j+1}), $c_0 = a$ and $c_{k-1} = b$. We apply the Lyapunov inequality in each interval $(c_j - 1, c_j)$, $1 \le j \le k-1$, which gives

$$\sum_{j=1}^{k-1} \frac{2^p}{(c_j - c_{j-1})^{p-1}} \le \tilde{\mu}_k \sum_{j=1}^{k-1} \left(\int_{c_{j-1}}^{c_j} r^+(x)dx \right) \le \tilde{\mu}_k \int_a^b r^+(x)dx.$$

Now, the sum on the left hand side is minimized when all the summands coincides, which gives the lower bound,

$$2^p(k-1)\left(\frac{k-1}{b-a}\right)^{p-1} \le \tilde{\mu}_k \int_a^b r^+(x)dx,$$

which completes the proof. □

3. The Fučik eigenvalues

In this section we consider the Fučik eigenvalue problem (1),

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = r(x)(\alpha|u(x)|^{p-2}u^+(x) - \beta|u(x)|^{p-2}u^-(x)) \\ u'(a) = u'(b) = 0 \end{cases}$$

where $r(x) \in L^\infty$ is a positive integrable function. Let us recall Theorem 1.1:

THEOREM 1.1. *Let $\int_a^b r(x) = m$. There exist an hyperbolic type curve $y = f(x)$ such that $\beta \ge f(\alpha)$ for every Fučik eigenvalue $(\alpha, \beta) \in \Gamma_1$ of Problem (1). Moreover,*

$$\lim_{\alpha \to \infty} f(\alpha) = \frac{1}{m(b-a)^{p-1}}.$$

Proof. Let us suppose that (α, β) is a Fučik eigenvalue of Problem (1) such that the associate eigenfunction u has only one zero c in (a,b). Hence, by using Theorem 2.1, we have:

$$\alpha \ge \frac{1}{(c-a)^{p-1} \int_a^c r(x)dx} \tag{14}$$

$$\beta \geq \frac{1}{(b-c)^{p-1} \int_c^b r(x)dx} \tag{15}$$

From Equation (14), we have

$$\int_c^b r(x)dx = m - \int_a^c r(x)dx \leq m - \frac{1}{\alpha(c-a)^{p-1}}$$

and replacing in Equation (15) gives

$$\beta \geq \frac{1}{m(b-c)^{p-1}} \frac{\alpha(c-a)^{p-1}}{\alpha(c-a)^{p-1} - 1/m}.$$

Equivalently,

$$\beta \geq \frac{1}{m(b-c)^{p-1}} \left(1 + \frac{1}{m\alpha(c-a)^{p-1} - 1}\right). \tag{16}$$

We define the function

$$f(x) = \frac{1}{m(b-a)^{p-1}} \left(1 + \frac{1}{m(b-a)^{p-1}x - 1}\right)$$

and the theorem is proved. □

Corollary 3.1. *All the non-trivial Fučik eigenvalues for the Neumann boundary condition are contained in the region:*

$$\left\{(\alpha, \beta) \in I\!R^2 \, : \, \beta \geq \frac{1}{m(b-a)^{p-1}} \left(1 + \frac{1}{m(b-a)^{p-1}\alpha - 1}\right)\right\} \tag{17}$$

Remark 3.2. *Let us note that Equation (16) gives also a lower bound for the location of the zero c, for a given Fučik eigenvalue (α, β) In fact, the inequality shows that c cannot be too close to a or b. We have*

$$\lim_{c \to a^+} \left[\frac{1}{m(b-c)^{p-1}} \left(1 + \frac{1}{m\alpha(c-a)^{p-1} - 1}\right)\right] = \infty$$

and the same is valid when $c \to b^-$.

Our next result shows that Inequalities (14) and (15) are sharp.

THEOREM 3.3. *Let $c \in (a,b)$ be fixed. Let $\varepsilon > 0$. Then, there exist a weight $r_{\alpha\beta,\varepsilon}(x)$ with $\int_a^b r_{\alpha\beta,\varepsilon}(x) = m$ such that $(\alpha,\beta) \in \Sigma^*$ and*

$$\alpha - \frac{1}{(c-a)^{p-1}\int_a^c r_{\alpha\beta,\varepsilon}(x)dx} \le \varepsilon \tag{18}$$

$$\beta - \frac{1}{(b-c)^{p-1}\int_c^b r_{\alpha\beta,\varepsilon}(x)dx} \le \varepsilon. \tag{19}$$

Also, the region (17) cannot be improved when $\alpha \to \infty$.

We omit the proof, which is similar to the one given in Theorem 1.3. Indeed, from suitable approximations of

$$r_{\alpha\beta}(x) = t\delta_a(x) + (m-t)\delta_b(x),$$

with $0 < t < m$, we obtain the optimality of Inequalities (14), (15).

REMARK 3.4. *Corollary 3.1 gives the isolation of the trivial lines, and an uniform estimate of the gap at infinity for positive weights with fixed integral m.*

Let us note that the inequality

$$\beta \ge \frac{1}{m(b-a)^{p-1}}$$

obtained when $\alpha \to \infty$ cannot be improved, being the optimal lower bound of the first eigenvalue of the mixed problem

$$\begin{cases} -(|u'|^{p-2}u')' = \mu r|u|^{p-2}u \\ u(a) = u'(b) = 0 \end{cases}$$

which follows immediately from Lemma 2.1.

We consider the extension of Theorem 1.1 to indefinite weights. When $r(x)$ changes sign, the Fučik eigenvalue problem for the p-laplacian was considered in [1, 2]. In the quadrants $I\!R^+ \times I\!R^+$ and $I\!R^- \times I\!R^-$ there exist two families of curves in Σ. In the quadrants $I\!R^+ \times I\!R^-$ and $I\!R^- \times I\!R^+$, the number of curves can be finite, and depends on the number of sign changes of $r(x)$.

Let $\int_a^b r^+(x) = m^+$, $\int_a^b r^-(x) = m^-$ and $\int_a^b |r(x)| = m$. We are thus led to the following strengthening of Theorem 1.1:

THEOREM 3.5. *Let* $(\alpha, \beta) \in \Sigma^*$, $m^+ > 0, m^- > 0$, *and* $\alpha > 0$.

(i) If $\beta > 0$, *then there exists a curve* $y = f^+(x)$,

$$f^+(x) = \frac{1}{m^+(b-a)^{p-1}} \left(1 + \frac{1}{m^+(b-a)^{p-1}x + 1}\right),$$

such that $\beta \geq f^+(\alpha)$.

(ii) If $\beta < 0$, *then there exists a curve* $y = f^\pm(x)$,

$$f^\pm(x) = \frac{-1}{m(b-a)^{p-1}} \left(1 + \frac{1}{m(b-a)^{p-1}x - 1}\right),$$

such that $\beta \leq f^\pm(\alpha)$.

Proof. The proof of Theorem 1.1 could be easily adapted to this case, taking $r^+(x)$ for part (i).

For the part (ii), in quadrant $I\!R^+ \times I\!R^-$ we obtain it from the inequalities given by Theorem 2.1, replacing the intervals of positivity and negativity by (a, b):

$$\alpha \geq \frac{1}{(b-a)^{p-1}m^+} \qquad \text{and} \qquad \beta \leq \frac{-1}{(b-a)^{p-1}m^-},$$

since

$$\frac{1}{(b-a)^{p-1}|\beta|} \leq m^- = m - m^+ \leq m - \frac{1}{(b-a)^{p-1}\alpha},$$

which gives,

$$\beta \leq \frac{-\alpha}{m(b-a)^{p-1}\alpha - 1} = \frac{-1}{m(b-a)^{p-1}} \left(1 + \frac{1}{m(b-a)^{p-1}\alpha - 1}\right),$$

and the proof is finished. □

REMARK 3.6. *In much the same way, for* $\alpha < 0$ *we deduce the existence of two curves* $f^-(x), f^\mp(x)$ *by considering the weight* $-r(x)$.

REMARK 3.7. *It is possible to obtain curves defining regions which contains the eigenvalues corresponding to eigenfunctions with a prescribed number of nodal domains, by using the lower bounds in Theorem 2.10.*

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