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A note about the well-posedness of an Initial Boundary Value Problem for the heat equation in a layered domain

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Abstract. *Heat conduction in a layered domain with imperfect thermal contact interfaces is modeled by means of a system of elliptic or parabolic PDEs with suitable boundary and transmission conditions. Well-posedness of this problem is proved and a stability estimate of the solution is given.*

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1. Introduction

This note deals with heat conduction in a section of the layered body C. Assume that two or more layers are separated by *low conductivity imperfect interfaces.* According to the classification in [8], it means that a temperature jump is present between two adjacent layers while the heat flux is continuous.

Suppose that C is made by two slabs C^+ and C^-, with different thermal conductivities, separated by a very thin *low conductivity imperfect interface* D^ϵ ($\epsilon > 0$ represents a characteristic thickness of the interface). The limit process in which the thin "solid" interface D^ϵ shrinks to a two-dimensional set D^0, is widely studied in mathematical physics (see for example [6, 11]). The thermal properties of D^ϵ for $\epsilon \to 0$, are summarized in a parameter function $\tilde{h} : D^0 \to [0, \infty)$ called *thermal contact conductance*. The *thermal contact conductance* of the interface D^0 is a non-negative quantity related to the average of surface roughness effects in real objects. A detailed numerical modeling of roughness can be found in [10].

Consider the ideal framework in which the slabs are parallelepipeds. In particular, we focus on the intersection Ω between C and a plane π orthogonal to the interface, so that $S^0 = D^0 \cap \pi$, $\Omega^+ = C^+ \cap \pi$ and $\Omega^- = C^- \cap \pi$.

In mathematical terms, the temperature of a two-dimensional layered object like Ω is the solution of a system of two Initial Boundary Value Problems (IBVPs) for the heat equation coupled by means of suitable transmission conditions in which the function h (restriction of $\tilde{h}$ to S^0) plays the role of heat transfer coefficient. This system of IBVPs is the direct model underlying the inverse problem of identifying h from some additional data taken on the external boundary of the specimen (see for example [3, 7]). Existence and uniqueness of its solutions are proved in Theorem 3.1 and it supplements mathematical background in [7]. Though many results in applied sciences and engineering rely on this mathematical model (see for example [2, 3, 4, 7] to cite recent items), its well-posedness is always taken for granted and, consequently, a rigorous proof is bypassed. Here, a technique described in a recent paper about discontinuous Galerkin methods [1] is extended (from elliptic to parabolic; from layers of the same material to different materials) to prove existence and uniqueness of a weak solution of our system of IBVPs (see Section 3.2). A stability estimate is also derived in order to evaluate the sensitivity of the solution with respect to variations in the parameter h (see Section 3.3).

2. Geometry of the specimen. Notation.

We deal with the composite environment

$$\Omega = \Omega^+ \cup \Omega^- \cup S^0$$

where

$$\Omega^+ = (-L, L) \times (0, a_+), \qquad \Omega^- = (-L, L) \times (-a_-, 0).$$

The interface

$$S^0 = \{(x, y) : y = 0 \text{ and } x \in (-L, L)\}$$

opposes to heat transfer between Ω^+ and Ω^-.

Let $(0, T)$ be a "time interval" so that

$$Q^+ = \Omega^+ \times (0, T), \qquad Q^- = \Omega^- \times (0, T).$$

The thermal behavior of layers Ω^+ and Ω^- is determined by their conductivity (κ_+ and κ_-), density (ρ_+ and ρ_-) and specific heat (c_+ and c_-). The numbers $\alpha_\pm = \frac{\kappa_\pm}{\rho_\pm c_\pm}$ are the corresponding diffusivities.

The top boundary of Ω^+ is $S^+ = \{(x, y) : y = a_+ \text{ and } x \in (-L, L)\}$.

The bottom boundary of Ω^- is $S^- = \{(x, y) : y = -a_- \text{ and } x \in (-L, L)\}$.

We assume that the thermal contact conductance of S^0 takes the form $h(x) = h_0 + h_1(x, t)$ where h_0 is a positive real constant and h_1 is a non-negative function of class $C^0([-L, L] \times [0, T])$. Two examples in which h_1 describes, respectively, the deterioration of an insulating interface and the worsening of performances of an heat exchanger, are studied in [7].

In what follows, if u is a real function of two or more real variables, "u_q" means $\frac{\partial u}{\partial q}$ and "$u(q\pm)$" means $\lim_{\epsilon \to 0^+} u(q \pm \epsilon)$.

3. Temperature of a two-layered domain with low conductivity interface

Assume that Ω is heated through S^+. The incoming heat flow is described by a function $\Phi \in C^0([-L, L] \times [0, T])$. The temperature of Ω is determined by solving a system of two IBVPs for the heat equation, respectively in Q^+ and Q^- connected by means of a set of transmission conditions through the interface (see Section 3.1). Exchange of heat between Ω and the external environment occurs through S^+ and S^- and it is modeled by means of Robin conditions with (positive) constant coefficients h_+ and h_- respectively. Vertical sides $x = -L$ and $x = L$ are assumed, for simplicity, thermally insulated. A temperature $U^M \geq 0$ is assumed for a fluid exchanging heat with S^+. The temperature of a fluid exchanging with S^- can be taken equal to zero without loss of generality. Initial temperature is given by the pair of functions $U^0_+ \in C^0(\overline{\Omega^+})$ and $U^0_- \in C^0(\overline{\Omega^-})$ (overbar means topological closure).

Analytical solutions (definitely not trivial) are known when the problem is one-dimensional, i.e. when U^0_+, U^0_-, h_1 and Φ are non-negative constants, also in presence of more than two layers [14]. If Ω^+ and Ω^- are made of the same material and $a_+ = a_-$, the system can be easily reduced to a single problem in Q^+ (or alternatively in Q^-) using the method of images [7].

3.1. A system of IBVPs for the heat equation

Since a single function from $\overline{\Omega}$ to $(0, \infty)$ is not suitable for representing the temperature of our specimen because it assumes two different values on S^0, we introduce the pair of functions

$$u^+ : Q^+ \to (0, \infty), \qquad u^- : Q^- \to (0, \infty)$$

with their extension to respective boundaries. The pair (u^+, u^-) must fulfill the following system of IBVPs

$$\begin{aligned}
&\rho_+ c_+ u^+_t = \kappa_+ \Delta u^+, && (x, y, t) \in Q^+, \\
&u^+(x, y, 0) = U^0_+(x, y), && (x, y) \in \Omega^+, \\
&\kappa_+ u^+_\nu(x, a_+, t) + h_+(u^+(x, a_+, t) - U^M) = \Phi(x, t), && x \in (-L, L), t \in (0, T), \\
&\kappa_+ u^+_\nu(-L, y, t) = \kappa_+ u^+_\nu(L, y, t) = 0, && y \in (0, a_+), t \in (0, T),
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
&\rho_- c_- u^-_t = \kappa_- \Delta u^-, && (x, y, t) \in Q^-, \\
&u^-(x, y, 0) = U^0_-(x, y), && (x, y) \in \Omega^-, \\
&\kappa_- u^-_\nu(x, -a_-, t) + h_- u^-(x, -a_-, t) = 0, && x \in (-L, L),\ t \in (0, T), \\
&\kappa_- u^-_\nu(-L, y, t) = \kappa_- u^-_\nu(L, y, t) = 0, && y \in (-a_-, 0),\ t \in (0, T),
\end{aligned} \tag{2}$$

coupled by means of the *transmission conditions*

$$\begin{aligned} &\kappa_+ u^+_\nu(x,0+,t) + h(x,t)(u^+(x,0+,t) - u^-(x,0-,t)) = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad x \in (-L,L),\ t \in (0,T)\,, \\ &\kappa_+ u^+_\nu(x,0+,t) = -\kappa_- u^-_\nu(x,0-,t), \qquad x \in (-L,L),\ t \in (0,T)\,. \end{aligned} \tag{3}$$

3.2. Main result: Existence and uniqueness of (u^+, u^-), weak solution of (1)-(3)

As for notation and basic theory of Sobolev spaces we refer to [13, Chapter 7]. In particular, we deal also with spaces involving time (see [13, Section 7.11.2]). Let H be a Hilbert space equipped with the norm $\|.\|_H$ and let H^* denote its dual space:

$$L^2(0,T;H) = \left\{ u : (0,T) \to H \ : \ u(t) \text{ measurable and } \int_0^T \|u(t)\|^2_H dt < \infty \right\},$$

$$C^0([0,T];H) = \left\{ u : [0,T] \to H \ : \ u(t) \text{ continuous and } \max_{[0,T]} \|u(t)\|_H < \infty \right\}.$$

THEOREM 3.1. *Suppose that:*

(i) h takes the form $h_0 + h_1$ where h_0 is a positive real constant and h_1 is a non-negative function of class $C^0([-L,L] \times [0,T])$;

(ii) $U^0_+ \in C^0(\overline{\Omega^+})$, $U^0_+ \in C^0(\overline{\Omega^-})$ and $\Phi \in C^0([-L,L] \times [0,T])$.

Then:

(I) a weak solution (u^+, u^-) of problem (1)-(3) exists and it is unique, with $u^+ \in L^2(0,T,H^1(\Omega^+)) \cap C^0([0,T];L^2(\Omega^+))$ and $u^- \in L^2(0,T,H^1(\Omega^-)) \cap C^0([0,T];L^2(\Omega^-))$;

(II) $u^+_t \in L^2(0,T,H^1(\Omega^+)^*)$ and $u^-_t \in L^2(0,T,H^1(\Omega^-)^*)$;

(III) the energy estimate

$$\begin{aligned} &\rho_+ c_+ \|u^+(t)\|^2_0 + \rho_- c_- \|u^-(t)\|^2_0 + \kappa_+ \int_0^t \|u^+(\tau)\|^2_1 d\tau + \kappa_- \int_0^t \|u^-(\tau)\|^2_1 d\tau \\ &\quad \le e^{2\frac{\alpha_+}{L^2}t} (\rho_+ c_+ \|U^0_+\|^2_0 + \rho_- c_- \|U^0_-\|^2_0) \\ &\qquad + C(\rho_+, c_+, \kappa_+, \Omega^+, t) \max_{\tau \in [0,t]} \{ \int_{-L}^{L} (U^M h_+ + \Phi(x,\tau))^2 dx \} \end{aligned} \tag{4}$$

holds for almost all $t \in [0,T]$.

Proof. Step 1. A variational problem in a product Hilbert space.

It is convenient to write problem (1)-(3) in weak form. More precisely, for almost all $t \in [0,T]$, we must find $u^+(t)$ in $H^1(\Omega^+)$ and $u^-(t)$ in $H^1(\Omega^-)$ such that

$$\rho_+ c_+ \int_{\Omega^+} u_t^+(t) v^+ dxdy + \kappa_+ \int_{\Omega^+} \nabla u^+(t) \nabla v^+ dxdy + h_+ \int_{-L}^{L} u^+(t) v^+ dx \\ + \int_{-L}^{L} h(x,t)(u^+(t) - u^-(t)) v^+ dx = \int_{-L}^{L} (h_+ U^M + \Phi(x,t)) v^+ dx \,, \quad (5)$$

$$\rho_- c_- \int_{\Omega^-} u_t^-(t) v^- dxdy + \kappa_- \int_{\Omega^-} \nabla u^-(t) \nabla v^- dxdy + h_- \int_{-L}^{L} u^-(t) v^- dx \\ + \int_{-L}^{L} h(x,t)(u^-(t) - u^+(t)) v^- dx = 0 \,, \quad (6)$$

for all v^+ in $H^1(\Omega^+)$ and v^- in $H^1(\Omega^-)$.

Following [1], we define the (cartesian product) Hilbert space $V = H^1(\Omega^+) \times H^1(\Omega^-)$ equipped with the scalar product

$$(u,v)_V := \int_{\Omega^+} u^+ v^+ dxdy + \int_{\Omega^-} u^- v^- dxdy + L^2 \int_{\Omega^+} \nabla u^+ \nabla v^+ dxdy \\ + L^2 \int_{\Omega^-} \nabla u^- \nabla v^- dxdy \,.$$

The scale factor L^2 is required for dimensional reasons. Clearly $u = (u^+, u^-)$ and $v = (v^+, v^-)$ are in V while for all $w \in V$ the norm $\|w\|_V := \sqrt{(w,w)_V}$ is defined.

We sum (5) and (6) and obtain the variational problem:

for almost all $t \in [0,T]$, find $u(t) \in V$ such that

$$\langle u_t(t), v \rangle + a(u(t), v) = \int_{-L}^{L} (h_+ U^m + \Phi(x,t)) v^+(x,a) dx \\ \textit{for all } v \in V \textit{ with } \; u^{\pm}(0) = U^0_{\pm}.$$

Here,

$$\langle u_t(t), v \rangle = \rho_+ c_+ \int_{\Omega^+} u_t^+(t) v^+ dxdy + \rho_- c_- \int_{\Omega^-} u_t^-(t) v^- dxdy$$

denotes a suitably weighted duality pairing between V^* and V while

$$a(u(t),v) = \kappa_+ \int_{\Omega^+} \nabla u^+(t)\nabla v^+ dxdy + \kappa_- \int_{\Omega^-} \nabla u^-(t)\nabla v^- dxdy$$
$$+ h_+ \int_{-L}^{L} u^+(t)v^+ dx + h_- \int_{-L}^{L} u^-(t)v^- dx$$
$$+ \int_{-L}^{L} h(x,t)(u^+(t) - u^-(t))(v^+ - v^-)dx$$

is a bilinear form on $V \times V$.

Step 2. Existence and uniqueness of the solution.

We recall that, if $w = (w^+, w^-) \in V$, the trace inequality (see [5, Theorem 1.5.1.10])

$$\int_{S^\pm \cup S^0} |w^\pm|^2 \le c(\Omega^\pm)\|w\|_V^2 \tag{7}$$

holds. It follows from the constructive proof in [5] that $c(\Omega^\pm) < 2(1 + \frac{3}{a_\pm})$ (not optimal). Continuity of the bilinear form a follows from Schwarz inequality and (7). Indeed, we have

$$|a(u(t),v)| \le K\|u(t)\|_V \|v\|_V \,, \tag{8}$$

where $K = \max\{\kappa_\pm\} + \max\{h_+, h_-, \max_{[-L,L]\times[0,T]} h\} \max\{c(\Omega^\pm)\}$.

Since h_+, h_- and $\min_{[-L,L]\times[0,T]} h$ are positive, there are two positive constants $\lambda = \max\{\kappa_+, \kappa_-\}$ and $\gamma = \min\{\kappa_+, \kappa_-\}$ such that

$$a(u(t),u(t)) + \lambda\|u(t)\|_0^2 \ge \gamma\|u(t)\|_V^2,$$

i.e. the bilinear form a is weakly coercive (see [12, Section 11.1.1]). Hence, existence of a solution $u \in L^2(0,T,V) \cap C^0([0,T];V)$ of (1)-(3) and its uniqueness follow from [12, Theorem 11.1.1] (see also [9, Theorem 5.3]). Energy estimate (4) is derived straightforwardly following [12] . □

Remark 3.2. The energy estimate does not account for heat exchange through the boundaries. In the special case in which $h_+ = h_- = 0$, it is likely to be optimal when the interface is insulating ($h = 0$) or highly conductive ($(u^+ - u^-)^2 \approx 0$.

Remark 3.3. Well posedness can be proved also in the stationary case in which the temperature u solves a system of BVPs for the Laplace equation. If we assume Dirichlet conditions on S^+ and S^- instead of Robin ones, we can use the same procedure of this section. Observe that, in the stationary case, the required coercivity of the bilinear form a follows from the Poincaré inequality (see [12, Section 1.3]).

3.3. Stability of the solution with respect to the parameters

The same technique used in deriving the energy estimate leads to the evaluation of the sensitivity of the solution with respect to the interface conductance h.

THEOREM 3.4. *Let $\tilde{u}$ denote the unique solution of (1)-(3) when the conductance in S^0 is $h + \delta h$ (with $h + \delta h \geq h_0$) and set $\delta u := \tilde{u} - u$. Assume that all other parameters ($\alpha_\pm$, $\kappa_\pm$, $h_\pm$, Φ) remain unchanged. We have the local stability estimate:*

$$\frac{\|\delta u\|_{L^2([0,T],V)}}{\|u\|_{L^2([0,T],V)}} \leq K \max_{[-L,L]\times[0,T]} |\delta h|.$$

Proof. Subtract

$$\langle u_t(t), \delta u\rangle + a_h(u(t), \delta u) = \int_{-L}^{L} (h_+ U^m + \Phi(x,t))\delta u^+(x,a)dx\,, \quad u^\pm(0) = U^0_\pm$$

from

$$\langle \tilde{u}_t(t), \delta u\rangle + a_{h+\delta h}(\tilde{u}(t), \delta u) = \int_{-L}^{L} (h_+ U^m + \Phi(x,t))\delta u^+(x,a)dx\,, \quad \tilde{u}^\pm(0) = U^0_\pm$$

where we have stressed the dependence of the bilinear form on conductance h at the interface. We have

$$\langle \delta u_t(t), \delta u\rangle + a_{h+\delta h}(u(t) + \delta u, \delta u) - a_h(u(t), \delta u) = 0\,, \quad \delta u^\pm(0) = 0.$$

Using the change of variable $\delta w^+ = e^{-\beta_+ t}\delta u^+$ and $\delta w^- = e^{-\beta_- t}\delta u^-$ with $\beta_\pm = \frac{\alpha_\pm}{L^2}$, we have

$$\begin{aligned}\langle \tilde{u}_t(t), \delta u\rangle = {} & e^{2\beta_+ t}\frac{\rho_+ c_+}{2}\frac{d}{dt}\|\delta w^+\|_0^2 + e^{2\beta_- t}\frac{\rho_- c_-}{2}\frac{d}{dt}\|\delta w^-\|_0^2 \\ & + e^{2\beta_+ t}\kappa_+\|\delta w^+\|_0^2 + e^{2\beta_- t}\kappa_-\|\delta w^-\|_0^2.\end{aligned}$$

Since

$$\begin{aligned} a_{h+\delta h}(u(t) + \delta u, \delta u) - a_h(u(t), \delta u) = {} & \kappa_+ e^{2\beta_+ t}\int_{\Omega^+} |\nabla \delta w^+(t)|^2 dxdy \\ & + e^{2\beta_- t}\kappa_- \int_{\Omega^-} |\nabla \delta w^-(t)|^2 dxdy + e^{2\beta_+ t} h_+ \int_{-L}^{L} \delta w^+(t)^2 dx \\ & + e^{2\beta_- t} h_- \int_{-L}^{L} \delta w^-(t)^2 dx + \int_{-L}^{L} (h+\delta h)(e^{\beta_+ t}\delta w^+ - e^{\beta_- t}\delta w^-)^2 dx \\ & + \int_{-L}^{L} \delta h (e^{\beta_+ t} w^+(t) - e^{\beta_- t} w^-(t))(e^{\beta_+ t}\delta w^+ - e^{\beta_- t}\delta w^-)dx \end{aligned}$$

we have

$$
\begin{aligned}
&e^{2\beta_+ t}\frac{\rho_+ c_+}{2}\frac{d}{dt}\|\delta w^+\|_0^2 + e^{2\beta_- t}\frac{\rho_- c_-}{2}\frac{d}{dt}\|\delta w^-\|_0^2 \\
&\quad + e^{2\beta_+ t}\kappa_+\|\delta w^+\|_0^2 + e^{2\beta_- t}\kappa_-\|\delta w^-\|_0^2 + \kappa_+ e^{2\beta_+ t}\|\nabla\delta w^+(t)\|_0^2 \\
&\quad + e^{2\beta_- t}\kappa_-\|\nabla\delta w^-(t)\|_0^2 + e^{2\beta_+ t}h_+\int_{-L}^{L}\delta w^+(t)^2 dx \\
&\quad + e^{2\beta_- t}h_-\int_{-L}^{L}\delta w^-(t)^2 dx + \int_{-L}^{L}(h+\delta h)(e^{\beta_+ t}\delta w^+ - e^{\beta_- t}\delta w^-)^2 dx \\
&\quad + \int_{-L}^{L}\delta h(e^{\beta_+ t}w^+(t) - e^{\beta_- t}w^-(t))(e^{\beta_+ t}\delta w^+ - e^{\beta_- t}\delta w^-)dx = 0.
\end{aligned}
\tag{9}
$$

Set $\beta_m := \min\{\beta_-, \beta_+\}$, $\beta_M := \max\{\beta_-, \beta_+\}$ and $\kappa_m = \min\{\kappa_-, \kappa_+\}$ and evaluate

$$
\begin{aligned}
&\int_{-L}^{L}|\delta h||(u^+(t) - u^-(t))(\delta u^+ - \delta u^-)|dx \\
&\quad \le \int_{-L}^{L}|\delta h|(|u^+(t)||\delta u^+| + |u^+(t)||\delta u^-| + |u^-(t)||\delta u^+| + |u^-(t)||\delta u^-|dx \\
&\quad \le \max_{[-L,L]\times[0,T]}|\delta h|(c(\Omega^+)\|u^+(t)\|_1\|\delta u^+\|_1 + \sqrt{c(\Omega^+)c(\Omega^-)}\|u^+(t)\|_1\|\delta u^-\|_1 \\
&\qquad + \sqrt{c(\Omega^+)c(\Omega^-)}|u^-(t)\|_1\|\delta u^+\|_1 + c(\Omega^-)\|u^-(t)\|_1\|\delta u^-\|_1) \\
&\quad \le \max_{[-L,L]\times[0,T]}|\delta h|e^{\beta_M t}\max\{c(\Omega^+), c(\Omega^-)\}\|\delta w\|_1\|w\|_1\,.
\end{aligned}
$$

By disregarding the third line in (9), which is made of positive terms, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\langle\delta w(t), \delta w(t)\rangle + \kappa_m\|w\|_1^2 \\
&\qquad \le \max_{[-L,L]\times[0,T]}|\delta h|e^{(\beta_M-\beta_m)T}\max\{c(\Omega^+), c(\Omega^-)\}\|\delta w\|_1\|w\|_1.
\end{aligned}
$$

Integrating on t both sides of the inequality, we get

$$
\begin{aligned}
&\frac{1}{2}\|\delta u\|_0^2 + \kappa_m\int_0^T\|\delta u\|_1^2 dt \\
&\qquad\qquad \le \max_{[-L,L]\times[0,T]}|\delta h|e^{2\beta_M T}\max\{c(\Omega^+), c(\Omega^-)\}\int_0^T\|\delta u\|_1\|u\|_1 dt.
\end{aligned}
$$

Applying the Schwarz inequality to the integral on the right hand side we have

$$\sqrt{\frac{\int_0^T \|\delta u\|_1^2 dt}{\int_0^T \|u\|_1^2 dt}} \leq K \max_{[-L,L]\times[0,T]} |\delta h| \,,$$

where $K = \frac{e^{2\beta_M T}}{\kappa_m} \max\{c(\Omega^+), c(\Omega^-)\}$. □

Conclusions

We have proved that a system of parabolic equations that model heat conduction in a layered domain is a well-posed problem under very natural hypotheses. The proof comes from the weak formulation of the problem in a suitable product Hilbert space. This result helps with the construction of rigorous foundations of the inverse problem studied in [3, 7].

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References

[1] A. Cangiani, E.H. Georgoulis, and Y.A. Sabawi, *Adaptive discontinuous galerkin methods for elliptic interface problems*, Math. Comp. **87** (2018), 2675–2707.

[2] F. Ciampa, P. Mahmoodi, F Pinto, and M. Meo, *Recent advances in active infrared thermography for non-destructive testing of aerospace components*, Sensors **18** (2018), no. 2, 1–37.

[3] M.J. Colaco and C.J.S. Alves, *A backward reciprocity function approach to the estimation of spatial and transient thermal contact conductance in double-layered materials using non-intrusive measurements*, Numerical Heat Transfer **68** (2015), no. 2, 117–132.

[4] J. Gonzalez, A. Bedoya, A. Mendioroz, and A. Salazar, *Measuring the thermal resistance of vertical interfaces separating two different media using infrared thermography*, J. Thermal Sciences **135** (2019), 410–416.

[5] P. Grisvard, *Elliptic problems in nonsmooth domain*, Pitman, Boston, 1985.

[6] Z. Hashin, *Thin interphase/imperfect interface in conduction.*, J. Applied Physics **89** (2001), 2261–2267.

[7] G. Inglese and R. Olmi, *Identification of time-varying inaccessible thermal conductance from data at the boundary*, Appl. Anal. (2020), 1–15, DOI: 10.1080/00036811.2020.1761017.

[8] A. Javili, S. Kaessmair, and P. Steinmann, *General imperfect interfaces*, Comput. Methods Appl. Mech. Engrg. **275** (2014), 76–97.

[9] J-L. Lions and E. Magenes, *Problémes aux limites non homogénes et applications*, Dunod, Paris, 1968.
[10] M.V. Murashov and S.D. Panin, *Numerical modelling of contact heat transfer problem with work hardened rough surfaces*, Internat. J. Heat Mass Transfer **90** (2015), 72–80.
[11] R. Perrussel and C. Poignard, *Asymptotic expansion of steady-state potential in a high contrast medium with a thin resistive layer*, Appl. Math. Comp. **221** (2013), 48–65.
[12] A. Quarteroni and A. Valli, *Numerical approximation of partial differential equations*, Springer, Berlin, 1997.
[13] S. Salsa, *Partial differential equations in action*, Springer, Milano, 2008.
[14] N.E. Sheils, *Multilayer diffusion in a composite medium with imperfect contact*, Appl. Math. Model. **46** (2017), 450–464.

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