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Remarks concerning the Lyapunov exponents of linear cocycles

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Dedicated to Professor Fabio Zanolin on the occasion of his 60th birthday

Abstract. *We impose a condition of pointwise convergence on the Lyapunov exponents of a d-dimensional cocycle over a compact metric minimal flow. This condition turns out to have significant consequences for the dynamics of the cocycle. We make use of such classical ODE techniques as the Lyapunov-Perron triangularization method, and the ergodic-theoretical techniques of Krylov and Bogoliubov.*

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1. Introduction

The question of the continuity properties of the Lyapunov exponents of a linear differential system under perturbation of the coefficient matrix is of intrinsic interest and is of importance in various applications. Many important results concerning this theme are due to the "Moscow school" centered around the Nemytskii seminar; we mention some representative papers ([3, 4, 26]) and refer especially to the book [5] by Bylov-Vinograd-Grobman-Nemytskii. In the works of the Moscow school, attention is not restricted to the Lyapunov exponents; other quantities such the upper and lower characteristic indexes and the Bohl exponent are also studied in a systematic way, both from the point of view of continuity and from that of intrinsic properties.

More recent work of Bochi-Viana [2] and of Bessa [1] permits one to make statements concerning the discontinuity of the Lyapunov exponents of certain topological/ergodic families of linear systems. The paper [1] adapts to the continuous setting certain important results of [2] for discrete cocycles. The basic object of study in [1, 2] is the set of Lyapunov exponents determined by the Oseledets theorem relative to a discrete or continuous cocycle and an ergodic measure defined on a compact metric flow. Generally speaking, it is shown that, if the cocycle does not admit a dominated splitting (a.k.a. an exponential separation), and if the Lyapunov exponents are not all equal, then those exponents do not vary continuously under C^0-perturbation of the cocycle. See also ([28, 30]) for results in this vein.

In a somewhat different vein, Furman [14] studied the case of a discrete cocycle over a strictly ergodic flow. He considered the time averages which define the maximal Lyapunov exponent of the cocycle; that exponent is well-defined by the subadditive ergodic theorem. He shows that, if the cocycle has dimension $d = 2$, and if the time averages converge uniformly with respect to the phase point of the flow, then the maximal Lyapunov exponent varies continuously if the cocycle is perturbed. If in addition the flow is equicontinuous, then the converse statement holds as well.

In the present paper, our point of departure is similar to that of [14], though we work with the usual Lyapunov exponents and not with the maximal exponent. We assume that, for each phase point in the flow, each Lyapunov exponent is defined by a true limit (and not by a non-convergent limit superior). Let $d \geq 2$ be the dimension of the cocycle. We show that, if the flow is minimal, and if the Oseledets spectrum of the cocycle is simple (i.e., consists of d distinct numbers), then the cocycle has the discrete spectrum property of Sacker and Sell. If $d = 2$, we do not need to assume that the Oseledets spectrum is simple (but need slightly more information concerning the limits defining the Lyapunov exponents). We are able to strengthen the continuity result of [14] in the sense that the compact metric flow is minimal but need not be strictly ergodic.

We wish to emphasize that our results will be proved by using quite classical techniques in the theory of linear differential and discrete systems. These include the method of Krylov and Bogoliubov for constructing invariant measures, and the Lyapunov-Perron triangularization procedure. We will also adapt a small part of that proof of the Oseledets theorem which is based on those methods. Beyond that, we will apply some specific results, including an ergodic oscillation result of [16], and two statements of [10] which concern smoothing of real cocycles and the untwisting of invariant vector bundles.

The paper is organized as follows. In Section 2 we prepare the ground by recalling the statement of the Oseledets theorem, and some elements of the spectral theory of Sacker and Sell for linear cocycles. In Section 3 we work out some consequences, regarding the continuity of Lyapunov exponents, of the hypothesis that a cocycle Φ have discrete spectrum. These results are (mostly) known, but perhaps not *well*-known.We also discuss a specific situation in which the results of [1, 2] imply the discontinuity of the Lyapunov exponents under a C^0-perturbation of the cycle.

In Section 4 we present our main result. We show that, if Φ is a cocycle over a compact minimal flow of dimension $d = 2$, and if the time averages which define its Lyapunov exponents all converge, then Φ has discrete spectrum. If the dimension d of Φ is greater than two, we encounter technical problems when attempting to prove the above result. We are, however, able to prove a theorem which has the following corollary. Suppose that $(\Omega, \{\tau_t\})$ is strictly ergodic with unique ergodic measure μ. Suppose that the cocycle Φ has simple Oseledets spectrum with respect to μ. Finally, suppose that the time averages which define the Lyapunov exponents of Φ all converge. Then Φ has discrete spectrum, and in fact the Sacker-Sell spectrum of Φ is simple. In classical language, this means that Φ has the Lillo property [23].

We finish this Introduction by listing some notational conventions which will be in force throughout the paper. First, the brackets $\langle\ ,\ \rangle$ will indicate the Euclidean inner product on $\mathbb{R}^d$. Second, the symbol $|\cdot|$ will denote a norm whose significance will be clear from the context if it is not explicitly defined. Third, we let $GL(\mathbb{R}^d)$ denote the set of invertible $d \times d$ matrices. Fourth, we let $\mathbb{L}(\mathbb{R}^d)$ denote the set of all $d \times d$ real matrices with the operator norm: if $A \in \mathbb{L}(\mathbb{R}^d)$, then $|A| = \sup\{|Ax| \mid x \in \mathbb{R}^d,\ |x| = 1\}$.

2. Preliminaries

In this section, we introduce basic concepts and results, and express in a precise way the issue to be discussed in this paper.

Let Ω be a compact metric space, and let T be either the reals ($T = \mathbb{R}$) or the integers ($T = \mathbb{Z}$). For each $t \in T$, let $\tau_t : \Omega \to \Omega$ be a continuous map. We say that the family $\{\tau_t \mid t \in T\}$ defines a *topological flow* on Ω if the following conditions are satisfied:

(i) $\tau_0(\omega) = \omega$ for all $\omega \in \Omega$;

(ii) $\tau_t \circ \tau_s = \tau_{t+s}$ for all $t, s \in T$;

(iii) the map $\tau : \Omega \times T \to \Omega : (t, \omega) \mapsto \tau_t(\omega)$ is continuous.

It is clear that, if these conditions are satisfied, then for each $t \in T$, the map $\tau_t : \Omega \to \Omega$ is a homeomorphism and $(\tau_t)^{-1} = \tau_{-t}$ $(t \in T)$. If $T = \mathbb{Z}$, then the topological flow $\{\tau_t \mid t \in \mathbb{Z}\}$ is generated by τ_1, in the sense that $\tau_n = (\tau_1)^n$ if $n > 0$ and $\tau_n = (\tau_{-1})^{-n}$ if $n < 0$. We will refer to a pair $(\Omega, \{\tau_t \mid t \in T\})$ consisting of a compact metric space Ω and a flow $\{\tau_t \mid t \in T\}$ on Ω as a compact metric flow.

Important examples of flows are obtained via the following construction. Let $\mathbb{T}^g = \mathbb{R}^g/\mathbb{Z}^g$ be the g-dimensional torus, and let $\gamma_1, \ldots, \gamma_g$ be rationally independent numbers. Let $\theta_1, \ldots, \theta_g$ be 1-periodic coordinates on $\mathbb{T}^g$. If $T = \mathbb{R}$ or $\mathbb{Z}$, set $\tau_t(\theta_1, \ldots, \theta_g) = (\theta_1 + t\gamma_1, \ldots, \theta_g + t\gamma_g)$ $(t \in T)$. Then $\{\tau_t \mid t \in T\}$ is a flow on $\mathbb{T}^g$, called a *Kronecker flow.*

A compact metric flow $(\Omega, \{\tau_t\})$ is called *minimal* or *Birkhoff recurrent* if Ω is nonempty and for each $\omega \in \Omega$, the *orbit* $\{\tau_t(\omega) \mid t \in T\}$ is dense in Ω. A Kronecker flow as defined above on $\Omega = \mathbb{T}^g$ is minimal. Actually a Kronecker flow satisfies a stronger property, namely that of *Bohr almost periodicity*: thus, in addition to minimality, there is a metric d on Ω, which is compatible with its topology, such that $d(\tau_t(\omega_1), \tau_t(\omega_2)) = d(\omega_1, \omega_2)$ for all points $\omega_1, \omega_2 \in \Omega$ and all $t \in T$. Clearly the Euclidean metric d on $\Omega = \mathbb{T}^g$ satisfies this last condition.

Let $(\Omega, \{\tau_t\})$ be a compact metric flow, and let μ be a regular Borel probability measure on Ω (thus in particular $\mu(\Omega) = 1$). The measure μ is called $\{\tau_t\}$-*invariant* if $\mu(\tau_t(B)) = \mu(B)$ for each Borel set $B \subset \Omega$ and $t \in T$. An invariant measure μ is called *ergodic* if it satisfies the following indecomposibility condition: whenever $B \subset \Omega$ is a Borel set and $\mu(\tau_t(B)\Delta B) = 0$ for all $t \in T$, there holds $\mu(B) = 0$ or $\mu(B) = 1$ (Δ = symmetric difference of sets).

A classical construction of Krylov and Bogoliubov ([20, 29]) shows that a compact metric flow $(\Omega, \{\tau_t\})$ always admits an ergodic measure μ. If $(\Omega, \{\tau_t\})$ is minimal and admits exactly one ergodic measure, then it is called *strictly ergodic*. A Kronecker flow $\{\tau_t\}$ on $\Omega = \mathbb{T}^g$ is strictly ergodic: the unique ergodic measure is the normalized Haar measure on $\mathbb{T}^g$.

Next we discuss cocycles. A T-*cocycle* over a compact metric flow $(\Omega, \{\tau_t\})$ with values in the general linear group $GL(\mathbb{R}^d)$ is a continuous map $\Phi : \Omega \times T \to GL(\mathbb{R}^d)$ such that:

(i) $\Phi(\omega, 0) = I$= identity for all $\omega \in \Omega$;

(ii) $\Phi(\omega, t+s) = \Phi(\tau_t(\omega), s)\Phi(\omega, t)$ for all $\omega \in \Omega$ and $t, s \in T$.

One obtains an important class of *real* cocycles ($T = \mathbb{R}$) from appropriate families of linear nonautonomous differential systems. Let $(\Omega, \{\tau_t\})$ be a compact metric real flow, and let $A : \Omega \to \mathbb{L}(\mathbb{R}^d)$ be a continuous function. Let $\Phi(\omega, t)$ be the fundamental matrix solution of the ODE

$$\frac{dx}{dt} = A(\tau_t(\omega))x \qquad (x \in \mathbb{R}^d); \tag{1_ω}$$

thus $\Phi(\omega, 0) = I$ and $\dfrac{d}{dt}\Phi(\omega, t) = A(\tau_t(\omega))\Phi(\omega, t)$ for all $\omega \in \Omega$ and $t \in T = \mathbb{R}$. It can be checked that Φ is a real cocycle.

Actually the general real cocycle can be obtained in this way, up to "cohomology". We explain this. Let $(\Omega, \{\tau_t\})$ be a compact metric real flow, and let $\Psi : \Omega \times \mathbb{R} \to GL(\mathbb{R}^d)$ be a real cocycle. Then there exist continuous functions $A : \Omega \to \mathbb{L}(\mathbb{R}^d)$ and $F : \Omega \to GL(\mathbb{R}^d)$ such that, if $\Phi(\omega, t)$ is the cocycle generated by the family (1_ω) corresponding to $A(\cdot)$, then

$$\Psi(\omega, t) = F(\tau_t(\omega))\Phi(\omega, t)F(\omega)^{-1} \qquad (\omega \in \Omega,\ t \in \mathbb{R}).$$

See [10] for a proof; in fact one defines $F(\omega) = \dfrac{1}{\varepsilon}\displaystyle\int_0^\varepsilon \Phi(\omega, s)ds$ for sufficiently small ε. The function F is called a cohomology between Ψ and Φ. It turns out that the properties of a real cocycle which are of interest to us are preserved under a cohomology. So we will always be able to assume that the real cocycles which we study are derived from a family (1_ω) of linear differential systems in the manner described above.

An *integer* cocycle ($T = \mathbb{Z}$) is obtained from a nonautonomous difference equation, as follows. Set $A(\omega) = \Phi(\omega, 1)$, $\tau(\omega) = \tau_1(\omega)$, and consider

$$x_{n+1} = A(\tau^n(\omega))x_n \qquad (n \in \mathbb{Z},\ x \in \mathbb{R}^d). \tag{2_ω}$$

Then the family of difference equations (2_ω) generates the cocycle Φ in the sense that

$$\Phi(\omega, n) = A(\tau^{n-1}(\omega)) \dots A(\omega) \qquad n > 0,$$

$$\Phi(\omega, 0) = I,$$

$$\Phi(\omega, n) = A^{-1}(\tau^{n-1}(\omega)) \dots A^{-1}(\tau^{-1}(\omega)) \qquad n < 0$$

for all $\omega \in \Omega$. Note that an integer cocycle $\Phi(\omega, n)$ is determined once $\Phi(\omega, 1)$ is known.

Next let $T = \mathbb{R}$ or $\mathbb{Z}$, and let $(\Omega, \{\tau_t\})$ be a compact metric flow. Let $\Phi : \Omega \times T \to GL(\mathbb{R}^d)$ be a cocycle. We recall the definition and some basic properties of the Lyapunov exponents of Φ. Fix $\omega \in \Omega$. For each $0 \neq x \in \mathbb{R}^d$, let

$$\beta(\omega, x) = \limsup_{t\to\infty} \frac{1}{t} \ln |\Phi(\omega, t)x|.$$

The number $\beta(\omega, x)$ is called a Lyapunov exponent of Φ at ω. It is well-known that, as x varies over $\mathbb{R}^d \setminus \{0\}$, $\beta(\omega, x)$ takes on only finitely many values, say $\beta_1(\omega) \leq \beta_2(\omega) \leq \cdots \leq \beta_s(\omega)$ where $1 \leq s \leq d$. Moreover, for each $1 \leq r \leq s$, one has that $W_r(\omega) = \{0\} \cup \{0 \neq x \in \mathbb{R}^d \mid \beta(\omega, x) \leq \beta_r(\omega)\}$ is a vector subspace of $\mathbb{R}^d$. One says that $\{0\} = W_0(\omega) \subset W_1(\omega) \subset \cdots \subset W_s(\omega) = \mathbb{R}^d$ is the filtration associated to Φ at ω. Set $d_1 = \dim W_1(\omega), \dots, d_r = \dim W_r(\omega) - \dim W_{r-1}(\omega)$ $(2 \leq r \leq s)$; the integer d_r is called the multiplicity of $\beta_r(\omega)$ $(1 \leq r \leq s)$.

Continuing the discussion, we now define the upper Lyapunov exponent of Φ at ω to be

$$\beta_*(\omega) = \limsup_{t\to\infty} \frac{1}{t} \ln |\Phi(\omega, t)|.$$

It is clear that $\beta_s(\omega) \leq \beta_*(\omega)$ for each $\omega \in \Omega$. According to the regularity theory of Lyapunov [24], one has the following. Let d_r be the multiplicity of $\beta_r(\omega)$ $(1 \leq r \leq s)$, and suppose that $d_1\beta_1(\omega) + \cdots + d_s\beta_s(\omega) = \liminf_{t\to\infty} \frac{1}{t} \ln \det \Phi(\omega, t)$. Then $\beta_s(\omega) = \beta_*(\omega)$, and the limit $\lim_{t\to\infty} \frac{1}{t} \ln |\Phi(\omega, t)x|$ exists for each $0 \neq x \in \mathbb{R}^d$. One says that Φ is regular at ω. The regularity concept is important in the study of the stability of $x = 0$ relative to nonlinear perturbations of Φ.

There is a considerable body of Russian literature concerning the theory of the Lyapunov exponents, as well as other exponents related to a T-cocycle, namely the central exponents and the Bohl exponents. We will not discuss these important concepts, but refer the reader to [5].

It is useful to consider the Lyapunov exponents associated with the exterior products of the cocycle Φ. For this, let $\Lambda_1\mathbb{R}^d \cong \mathbb{R}^d$, $\Lambda_2\mathbb{R}^d, \ldots, \Lambda_d\mathbb{R}^d \cong \mathbb{R}$ be the exterior product spaces of $\mathbb{R}^d$. These spaces have natural inner products and norms induced by the Euclidean inner product and Euclidean norm in $\mathbb{R}^d$; (see [13, Chapter 1]). The cocycle Φ induces a cocycle with values in $GL(\mathbb{R}^d)$ for each $1 \le k \le d$, via the formula $\Lambda_k\Phi(\omega,t)(x_1 \wedge \cdots \wedge x_k) = \Phi(\omega,t)x_1 \wedge \cdots \wedge \Phi(\omega,t)x_k$. Each of these cocycles admits Lyapunov exponents which are analogues of these introduced above for Φ. In this paper, we will only make use of the upper Lyapunov exponents of these cocycles, which are determined as follows

$$\lambda_k(\omega) = \limsup_{t\to\infty} \frac{1}{t} \ln|\Lambda_k\Phi(\omega,t)| \qquad (\omega \in \Omega,\ 1 \le k \le d).$$

Of course, $\lambda_1(\omega) = \beta_*(\omega)$ and $\lambda_k(\omega) = \limsup\limits_{t\to\infty}\frac{1}{t}\ln\det\Phi(\omega,t)$.

Let us state a corollary of a result of Ruelle ([36, Proposition 1.3]).

Proposition 2.1. *Let $T = \mathbb{R}$ or $\mathbb{Z}$, let $(\Omega, \{\tau_t\})$ be a compact metric flow, and let $\Phi : \Omega \times T \to GL(n,\mathbb{R})$ be a T-cocycle. Let $\omega \in \Omega$. Suppose that, for each $k = 1, 2, \ldots, d$, the following limit exists:*

$$\lim_{t\to\infty} \frac{1}{t} \ln|\Lambda_k(\omega,t)| = \lambda_k(\omega).$$

Let $\beta_1(\omega) < \ldots < \beta_s(\omega)$ be the Lyapunov exponents of Φ at ω, and let $\{0\} = W_0(\omega) \subset W_1(\omega) \subset \cdots \subset W_s(\omega) = \mathbb{R}^d$ be the corresponding filtration. Then if $1 \le r \le s$ and if $0 \ne x \in W_r(\omega) \setminus W_{r-1}(\omega)$, one has

$$\lim_{t\to\infty} \frac{1}{t} \ln|\Phi(\omega,t)x| = \beta_r(\omega) \qquad (1 \le r \le s).$$

Thus the limit $\lim\limits_{t\to\infty} \frac{1}{t} \ln|\Phi(\omega,t)x|$ exists *for each $0 \ne x \in \mathbb{R}^d$.*

We now recall certain results concerning T-cocycles, namely the Oseledets theorem [31] and the spectral theorem of Sacker and Sell [38].

Theorem 2.2 (**Oseledets**). *Let $T = \mathbb{R}$ or $\mathbb{Z}$, let $(\Omega, \{\tau_t\})$ be a compact metric flow, and let μ be a $\{\tau_t\}$-ergodic measure on Ω. Let $\Phi : \Omega \times T \to GL(\mathbb{R}^d)$ be a T-cocycle over $(\Omega, \{\tau_t\})$. If $\omega \in \Omega$, let $\beta_1(\omega), \ldots, \beta_s(\omega)$ be the Lyapunov exponents of Φ at ω.*

There is a $\{\tau_t\}$-invariant μ-measurable subset $\Omega_1 \subset \Omega$ with $\mu(\Omega_1) = 1$, such that, if $\omega \in \Omega_1$, then $\mathbb{R}^d$ admits a direct sum decomposition

$$\mathbb{R}^d = V_1^{(m)}(\omega) \oplus V_2^{(m)}(\omega) \oplus \cdots \oplus V_s^{(m)}(\omega),$$

such that the following statements are valid. First, if $0 \neq x \in V_r^{(m)}(\omega)$, then

$$\lim_{t \to \pm\infty} \frac{1}{t} \ln |\Phi(\omega, t)x| = \beta_r(\omega);$$

note the two-sidedness of the limit. The dimension of $V_r^{(m)}(\omega)$ equals the multiplicity d_r of $\beta_r(\omega)$. Second, the number s and the multiplicities $d_1, \ldots, d_s$ do not depend on $\omega \in \Omega_1$, and moreover $\beta_r(\omega)$ is constant on Ω_1 $(1 \leq r \leq s)$. Third, the correspondence $\omega \mapsto V_r^{(m)}(\omega)$ is μ-measurable in the Grassmann sense $(1 \leq r \leq s)$. Fourth, the "measurable bundle"

$$V_r^{(m)} = \bigcup_{\omega \in \Omega_1} \left\{ (\omega, x) \mid x \in V_r^{(m)}(\omega) \right\}$$

is Φ invariant in the sense that, if $\omega \in \Omega_1$, $t \in T$ and $x \in V_r^{(m)}(\omega)$, then $(\tau_t(\omega), \Phi(\omega, t)x) \in V_r^{(m)}$.

This is not the most general form of the Oseledets theorem but it will be sufficient for our purposes. We note that the "μ-measurability" of $\omega \mapsto V_r^{(m)}(\omega)$ has the following meaning. – For each $\omega \in \Omega_1$, $V_r^{(m)}(\omega)$ defines an element of the Grassmannian manifold $Gr(d, d_r)$ of d_r-dimensional subspaces of $\mathbb{R}^d$; the mapping $\Omega_1 \mapsto Gr(d, d_r) : \omega \mapsto V_r^{(m)}(\omega)$ is μ-measurable. – The numbers $\beta_1 < \ldots < \beta_s$, which do not depend on $\omega \in \Omega_1$, are collectively referred to as the Oseledets spectrum or μ-spectrum of Φ.

The Oseledets theorem is a basic result in the theory of real or discrete cocycles. It has been proved using two distinct approaches. One method of proof uses the triangularization technique of Lyapunov-Perron; see [18, 31]. The other approach makes use of the subadditive ergodic theorem of Kingman [15, 36]. Both methods offer advantages and important information.

Next we review some aspects of the Sacker-Sell spectral theory, which taken together can be thought of as a continuous analogue of the Oseledets theory. First recall that a T-cocycle Φ over a compact metric flow $(\Omega, \{\tau_t\})$ is said to have an *exponential dichotomy* if there are positive constants $k > 0$, $\gamma > 0$ and a continuous, projection-valued function $\omega \mapsto P_\omega = P_\omega^2 : \Omega \to \mathbb{L}(\mathbb{R}^d)$ such that the following estimates hold:

$$|\Phi(\omega, t) P_\omega \Phi(\omega, s)^{-1}| \leq k e^{-\gamma(t-s)} \qquad t \geq s$$

$$|\Phi(\omega, t)(I - P_\omega) \Phi(\omega, s)^{-1}| \leq k e^{\gamma(t-s)} \qquad t \leq s$$

for all $\omega \in \Omega$ and $t, s \in T$.

The following basic theorem was proved by Sacker-Sell [37] and Selgrade [39]. Recall that a compact metric flow $(\Omega, \{\tau_t\})$ is said to be chain recurrent [7] if for each $\omega \in \Omega$, $\varepsilon > 0$ and $T > 0$, there are points $\omega = \omega_0, \omega_1, \dots, \omega_N = \omega$ and times $t_1 > T, \dots, t_N > T$ such that $d(\tau_{t_i}(\omega_{i-1}), \omega_i) \leq \varepsilon$ $(1 \leq i \leq N)$. A minimal flow $(\Omega, \{\tau_t\})$ is chain recurrent.

Theorem 2.3. *Suppose that the compact metric flow $(\Omega, \{\tau_t\})$ is chain recurrent, where $t \in T = \mathbb{R}$ or $\mathbb{Z}$. Let $\Phi : \Omega \times T \to GL(\mathbb{R}^d)$ be a T-cocycle. Suppose that, for each $\omega \in \Omega$, the condition $\sup_{t \in T} |\Phi(\omega, t)x| < \infty$ implies that $x = 0$; i.e., the cocycle Φ admits no nontrivial "bounded orbits". Then Φ admits an exponential dichotomy over Ω.*

Let us define the *dynamical* (or *Sacker-Sell*) spectrum σ_Φ of the T-cocycle Φ over the compact metric flow $(\Omega, \{\tau_t\})$ to be $\{\lambda \in \mathbb{R} \mid$ the translated cocycle $e^{\lambda t}\Phi(\omega, t)$ does not admit an exponential dichotomy over $\Omega\}$. Let us also recall that a compact metric flow $(\Omega, \{\tau_t\})$ is said to be invariantly connected [21] if Ω cannot be expressed as the union of two nonempty disjoint compact invariant subsets. We state the spectral theorem of Sacker-Sell.

Theorem 2.4 ([38]). *Let $(\Omega, \{\tau_t\})$ be a compact metric invariantly connected flow, where $T = \mathbb{R}$ or $\mathbb{Z}$. Let $\Phi : \Omega \times T \to GL(\mathbb{R}^d)$ be a T-cocycle. Then the dynamical spectrum σ_Φ of Φ is a disjoint union of finitely many compact intervals:*

$$\sigma_\Phi = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_q, b_q]$$

where $1 \leq q \leq d$ and $-\infty < a_1 \leq b_1 < a_2 \leq \ldots < a_q \leq b_q < \infty$. To each interval $[a_p, b_p]$ there corresponds a Φ-invariant topological vector subbundle $V_p^{(c)} \subset \Omega \times \mathbb{R}^d$ with the property that

$$(\omega, x) \in V_p^{(c)} \quad \text{and} \quad x \neq 0$$

$$\Updownarrow$$

$$a_p \leq \liminf_{t\to\infty} \frac{1}{t} \ln |\Phi(\omega, t)x| \leq \limsup_{t\to\infty} \frac{1}{t} \ln |\Phi(\omega, t)x| \leq b_p$$

$$\text{and}$$

$$a_p \leq \liminf_{t\to-\infty} \frac{1}{t} \ln |\Phi(\omega, t)x| \leq \limsup_{t\to-\infty} \frac{1}{t} \ln |\Phi(\omega, t)x| \leq b_p$$

$(1 \leq p \leq q)$.

One has further

$$\Omega \times \mathbb{R}^d = V_1^{(c)} \oplus V_2^{(c)} \oplus \cdots \oplus V_q^{(c)} \qquad \text{(Whitney sum).}$$

We will emphasize the following concept:

DEFINITION 2.5. *Let $T = \mathbb{R}$ or $\mathbb{Z}$, let $(\Omega, \{\tau_t\})$ be a compact metric flow, and let Φ be a T-cocycle over $(\Omega, \{\tau_t\})$. Suppose that $(\Omega, \{\tau_t\})$ is invariantly connected. Then Φ is said to have* discrete spectrum *if each spectral interval $[a_p, b_p]$ reduces to a point: $a_p = b_p$ for each $1 \leq p \leq q$.*

The discrete spectrum concept is related to but weaker than that of the "Lillo property" [23]. See [19] in this regard.

In Section 3, we will state and prove some results to the effect that, if a T-cocycle Φ has discrete spectrum, then its Lyapunov exponents vary continuously under perturbation of Φ. We claim no particular originality for these results as many statements of this type appear in the literature; e.g., [4, 26]. We do wish to emphasize our use of the Krylov-Bogoliubov method in our proofs, and the fact that one result (Proposition 3.4) appears to be more general than most. We also note that quite recent papers [1, 2, 14] have taken up the theme of the continuity of Lyapunov exponents, so it may not be inappropriate if we do so as well.

In Section 4, we give conditions which are sufficient in order that a cocycle Φ have discrete spectrum. One of our results (Theorem 4.4) generalizes a result of Furman [14] when $d = 2$.

To our knowledge, the connection between the expressibility of $\beta(\omega, x)$ as a limit for all $\omega \in \Omega$, $0 \neq x \in \mathbb{R}^d$, and the discrete spectrum property has not received much attention in the literature. However that may be, the said connection has turned out to be important in the spectral theory of quasicrystals. In this context $d = 2$. For example, in the paper [8] by Damanik-Lenz, the authors use the so-called avalance principle and detailed properties of certain strictly ergodic shift flows to verify that $\lim\limits_{t\to\infty} \dfrac{1}{t} \ln |\Phi(\omega, t)|$ exists for all $\omega \in \Omega$. One can then use Proposition 2.1 to show that $\beta(\omega, x)$ is expressible as a limit for all $\omega \in \Omega$, $0 \neq x \in \mathbb{R}^2$. In [8], the authors use the Furman result mentioned above to show that Φ has discrete spectrum; that result is subsumed in ours. They go on to show that, for certain quasicrystals, the spectrum of the associated Schrödinger operator has zero Lebesgue measure and is purely singular and continuous.

Perhaps our results will be useful in the study of higher-dimensional spectral problems of Atkinson type. We plan to investigate this issue in future work.

3. Discrete spectrum and Lyapunov exponents

In this section, we derive some continuity results for the Lyapunov exponents of a T-cocycle Φ ($T = \mathbb{R}$ or $\mathbb{Z}$) when Φ has discrete spectrum. As stated above, we make no claims concerning the originality of these results, as there is a very substantial literature on the subject. On the other hand, we think it is appropriate to present them here since they generalize some theorems in the recent literature. Also our proofs differ from some others in our systematic use of the classical Krylov-Bogoliubov method.

We begin the discussion with a simple consequence of Theorem 2.4.

PROPOSITION 3.1. *Let $T = \mathbb{R}$ or $\mathbb{Z}$, let $(\Omega, \{\tau_t\})$ be a compact metric flow which is invariantly connected, and let $\Phi : \Omega \times T \to GL(\mathbb{R}^d)$ be a T-cocycle. Suppose that the dynamical spectrum σ_Φ of Φ is discrete:*

$$\sigma_\Phi = \{a_0 < a_2 < \ldots < a_q\} \qquad (1 \leq q \leq d).$$

Then for each $\omega \in \Omega$ and $0 \neq x \in \mathbb{R}^d$ the limits $\lim_{t\to\pm\infty} \frac{1}{t} \ln |\Phi(\omega, t)x|$ exist. In fact, if $(\omega, x) \in V_p^{(c)}$ then $\lim_{t\to\pm\infty} \frac{1}{t} \ln |\Phi(\omega, t)x| = a_p$ $(1 \leq p \leq q)$, while if $x \notin V_p^{(c)}(\omega)$ for all $p = 1, 2, \ldots, q$, then $\lim_{t\to\infty} \frac{1}{t} \ln |\Phi(\omega, t)x| = a_m$ and $\lim_{t\to-\infty} \frac{1}{t} \ln |\Phi(\omega, t)x| = a_l$ where $l \leq m$ and $x \in V_l^{(c)}(\omega) \oplus \cdots \oplus V_m^{(c)}(\omega)$. .

Actually, if one restricts attention to the dynamics of Φ on a subbundle $V_p^{(c)}$, then the limits defining the Lyapunov exponents converge uniformly, in a sense which we now make precise. We first consider real cocycles, and carry out a preliminary discussion concerning them.

Let $\mathbb{L}$ be the usual projective space of lines through the origin in $\mathbb{R}^d$, so that $\mathbb{L}$ is a compact $(d-1)$-dimensional manifold. Let $\mathbb{B} = \Omega \times \mathbb{L}$. We assume that the $\mathbb{R}$-cocycle $\Phi = \Phi(\omega, t)$ is defined by the family of linear ordinary differential equations

$$x' = A(\tau_t(\omega))x \qquad \omega \in \Omega,\ x \in \mathbb{R}^d \tag{3_ω}$$

where $A : \Omega \to \mathbb{L}(\mathbb{R}^d)$ is a continuous function. Define a flow $\{\hat{\tau}_t \mid t \in \mathbb{R}\}$ on $\mathbb{B}$ by setting $\hat{\tau}_t(\omega, l) = (\tau_t(\omega), \Phi(\omega, t)l)$ for $\omega \in \Omega$, $l \in \mathbb{L}$. Then define $f : \mathbb{B} \to \mathbb{R} : f(\omega, l) = \langle A(\omega)x, x\rangle / \langle x, x\rangle$ where $0 \neq x \in l$. It is easy to check that, if $x \in \mathbb{R}^d$ has norm 1, and if $l \in \mathbb{L}$ is the line containing x, then

$$\ln |\Phi(\omega, t)x| = \int_0^t f(\hat{\tau}_t(\omega, l))ds. \tag{4}$$

This formula allows one to use ergodic theory (in particular the method of Krylov-Bogoliubov) to study the limiting expressions which define Lyapunov exponents.

PROPOSITION 3.2. *Let $(\Omega, \{\tau_t,\ t \in \mathbb{R}\})$ be a compact metric invariantly connected flow, and let $\Phi : \Omega \times \mathbb{R} \to GL(\mathbb{R}^d)$ be a real cocycle. Let $[a_p, b_p]$ be the p-th interval in the dynamical spectrum σ_Φ of Φ, and let the corresponding* *spectral subbundle be $V_p^{(c)}$ $(1 \le p \le q)$. Suppose that $[a_p, b_p]$ degenerates to a point for some $p \in \{1, 2, \dots, q\}$: thus $a_p = b_p$. Then*

$$\lim_{t \to \pm\infty} \frac{1}{t} \ln |\Phi(\omega, t)x| = a_p$$

where the limit is uniform with respect to pairs $(\omega, x) \in V_p^{(c)}$ with $|x| = 1$.

Proof. It follows from (4) that it is sufficient to prove that $\frac{1}{t}\int_0^t f(\hat{\tau}_s(b))ds$ converges uniformly to a_p with respect to $b = (\omega, l) \in \mathbb{B}_p = \{(\omega, l) \mid l \subset V_p^{(c)}(\omega)\}$. We do this by using arguments of the classical Krylov-Bogoliubov type (see, e.g., [29]).

Suppose for contradiction that, for some $\varepsilon > 0$, there exist a sequence $\{t_n\} \subset \mathbb{R}$ with $|t_n| \to \infty$ and a sequence $\{b_n = (\omega_n, l_n)\} \subset \mathbb{B}_p$ such that

$$\left| \frac{1}{t_n} \int_0^{t_n} f(\hat{\tau}_s(b_n))ds - a_p \right| \ge \varepsilon \qquad (n = 1, 2, \dots).$$

Let $C(\mathbb{B}_p)$ be the space of continuous, real-valued functions on $\mathbb{B}_p$ with the uniform norm. Let $\mathcal{F} \subset C(\mathbb{B}_p)$ be a countable dense set: $\mathcal{F} = \{f_1, f_2, \dots, f_k, \dots\}$ with $f_1 = f$. Using a Cantor diagonal argument, we can determine a subsequence $\{t_m\}$ of $\{t_n\}$ such that

$$\lim_{m \to \infty} \frac{1}{t_m} \int_0^{t_m} f_k(\hat{\tau}_s(b_m))ds$$

exists for $k = 1, 2, \dots$. Call the limit $\nu_*(f_k)$ $(1 \le k < \infty)$. One shows easily that ν_* extends to a bounded nonnegative linear functional on $C(\mathbb{B}_p)$, which we also denote by ν_*. It is clear that $\nu_*(c) = c$ for each constant function c on $\mathbb{B}_p$. This functional is $\{\hat{\tau}_t\}$-invariant in the sense that $\nu_*(g \circ \hat{\tau}_t) = \nu_*(g)$ for each $g \in C(\mathbb{B}_p)$ and each $t \in \mathbb{R}$. Using the Riesz representation theorem, one can find a $\{\hat{\tau}_t\}$-invariant measure ν on $\mathbb{B}_p$ such that

$$\left| \int_{\mathbb{B}_p} f d\nu - a_p \right| \ge \varepsilon.$$

We claim that there exists a $\{\hat{\tau}_t\}$-ergodic measure e on $\mathbb{B}_p$ such that

$$\left| \int_{\mathbb{B}_p} f de - a_p \right| \ge \varepsilon.$$

To see this, use the Krein-Mil'man theorem to represent the weak-$*$ compact convex set I of $\{\hat{\tau}_t\}$-invariant linear functionals on $\mathbb{B}_p$ as the closed convex hull of its set E of extreme points. It is easy to see that $e_* \in E$ if and only if its associated measure e is ergodic. By the Choquet representation theorem [35]:

$$\int_{\mathbb{B}_p} f d\nu = \int_E \left(\int_{\mathbb{B}_p} f de_* \right) dm(e_*)$$

where m is the representing measure of ν_* on E. It is now clear that e can be found.

Changing notation, let ν be a $\{\hat{\tau}_t\}$-ergodic measure on $\mathbb{B}_p$ such that

$$\left| \int_{\mathbb{B}_p} f d\nu - a_p \right| \geq \varepsilon.$$

By the Birkhoff ergodic theorem there is a set $\mathbb{B}_* \subset \mathbb{B}_p$ of full ν-measure such that, if $b_* \in \mathbb{B}_*$, then

$$\frac{1}{t} \int_0^t f(\hat{\tau}_s(b_*)) ds \to \int_{\mathbb{B}_p} f d\nu \neq a_p$$

as $t \to \infty$. This contradicts Proposition 3.1 and completes the proof of Proposition 3.2. $\square$

Remark 3.3. *(a) We can prove the $T = \mathbb{Z}$-analogue of Proposition 3.2 in the following way. Set $A(\omega) = \Phi(\omega, 1)$, then define $f_* : \mathbb{B}_p \to \mathbb{R} : f_*(\omega, l) = \frac{1}{2} \ln\langle A(\omega)x, A(\omega)x\rangle$ for each $(\omega, l) \in \mathbb{B}_p$ and $x \in l$, $|x| = 1$. One can check that Proposition 3.2 and its proof remain valid if one considers an integer cocycle Φ and if f is substituted with the above function f_*.*

(b) Let $T = \mathbb{R}$ or $\mathbb{Z}$, let $(\Omega, \{\tau_t\})$ be an invariantly connected compact metric flow, and let $\Phi : \Omega \times T \to GL(\mathbb{R}^d)$ be a T-cocycle for which the hypotheses of Proposition 3.2 are valid. Let $\Phi_*(\omega, t)$ be the restriction of $\Phi(\omega, t)$ to $V_p^{(c)}$, so that for each $\omega \in \Omega$ and $t \in T$ one has the linear transformation $\Phi_*(\omega, t) : V_p^{(c)}(\omega) \to V_p^{(c)}(\tau_t(\omega))$. Define the norm $|\Phi_*(\omega, t)|$ in the usual way. Then

$$\lim_{t \to \pm\infty} \frac{1}{t} \ln |\Phi_*(\omega, t)| = a_p,$$

where the limit is uniform in $\omega \in \Omega$. This statement is a consequence of Proposition 3.2, because for each $\omega \in \Omega$ and $t \in \mathbb{R}$, there exists a unit vector $x \in V_p^{(c)}(\omega)$ such that $|\Phi_*(\omega, t)| = |\Phi_*(\omega, t)x|$.

(c) By combining Propositions 3.1 and 3.2, one obtains a continuity result for the Lyapunov exponents of Φ with respect to variation of $\omega \in \Omega$. In fact, let $\{\beta_1(\omega), \ldots, \beta_s(\omega)\}$ be the Lyapunov exponents of Φ at ω, with

multiplicities $d_1, \ldots, d_s$. If the hypotheses of Propositions 3.1 and 3.2 are valid, then the multiplicities and the exponents $\beta_r(\omega)$ themselves do not depend on $\omega \in \Omega$.

Now we consider another type of continuity result for the Lyapunov exponents of a T-cocycle Φ. We will see that it is possible to vary the matrix function $A(\omega)$ in a non-uniform way, and still retain continuous variation of the exponents. We formulate a result along these lines which illustrate the power of a perturbation theorem due to Sacker and Sell ([38]; see also Palmer [34]).

For this, let Ω be the g-torus $\mathbb{T}^g = \mathbb{R}^g/\mathbb{Z}^g$. Let $\gamma_1, \ldots, \gamma_g$ be rationally independent numbers. Consider the Kronecker flow $\{\tau_t\}$ on $\mathbb{T}^g$ defined by $\gamma = (\gamma_1, \ldots, \gamma_g)$. Thus if $\omega \in \mathbb{R}^g/\mathbb{Z}^g$, then $\tau_t(\omega) = \omega + \gamma t$ $(t \in \mathbb{R})$.

Next, let $A : \mathbb{T}^g \to \mathbb{L}(\mathbb{R}^d)$ be a continuous function. Let $\Phi(\omega, t)$ be the cocycle defined by the family of differential systems (1_ω):

$$x' = A(\tau_t(\omega))x.$$

Suppose that Φ has discrete spectrum; $\sigma_\Phi = \{a_1 < a_2 < \ldots < a_q\}$.

Let $\gamma^{(n)}$ be a sequence in $\mathbb{R}^g$ such that $\gamma^{(n)} \to \gamma$. Each $\gamma^{(n)}$ defines a flow $\{\tau_t^{(n)}\}$ on $\mathbb{T}^g$ via the formula $\tau_t^{(n)}(\omega) = \omega + \gamma^{(n)} t$. However these flows need not be minimal because we do not assume that the components $\gamma_1^{(n)}, \ldots, \gamma_g^{(n)}$ of $\gamma^{(n)}$ are rationally independent. Let $\Phi^{(n)}(\omega, t)$ be the cocycle generated by the family of linear systems

$$x' = A(\tau_t^{(n)}(\omega))x.$$

Note that, if $\gamma^{(n)} \neq \gamma$ for $n = 1, 2, \ldots$, then $A(\tau_t^{(n)}(\omega))$ certainly does not converge uniformly in $t \in \mathbb{R}$ to $A(\tau_t(\omega))$ $(\omega \in \Omega)$. Nevertheless we have the following result.

Proposition 3.4. *For each $\omega \in \Omega$ and $n \geq 1$, let $\{\beta_r^{(n)}(\omega) \mid 1 \leq r \leq s = s(n)\}$ be the Lyapunov exponents of $\Phi^{(n)}$. Also let $\beta_*^{(n)}(\omega)$ be the upper Lyapunov exponent of $\Phi^{(n)}$ at ω $(\omega \in \Omega, n \geq 1)$.*

Given $\varepsilon > 0$, there exists $n_0 \geq 1$ such that, if $n \geq n_0$, then each Lyapunov exponent $\beta_r^{(n)}(\omega)$ is in the ε-neighborhood of σ_Φ $(\omega \in \Omega)$ and $\beta_*^{(n)}(\omega)$ is in the ε-neighborhood of a_q.

We sketch the proof of Proposition 3.4. Let $\mathcal{C} = \{c : \mathbb{R} \to \mathbb{L}(\mathbb{R}^d) \mid c$ is continuous and bounded$\}$ with the the topology of uniform convergence on compact sets. Introduce the Bebutov (translation) flow $\{\hat{\tau}_t\}$ on $\mathcal{C}$: thus $\hat{\tau}_t c(\cdot) = c(\cdot + t)$ for each $t \in \mathbb{R}$ and $c \in \mathcal{C}$.

Next let $U \subset \mathbb{R}^g$ be a compact neighborhood of γ. For each $\hat{\gamma} \in U$ and each $\omega \in \Omega$, set $c(t, \omega, \hat{\gamma}) = A(\omega + \hat{\gamma} t)$ $(t \in \mathbb{R})$. Set $C_{\hat{\gamma}} = \{c(\cdot, \omega, \hat{\gamma}) \mid \omega \in \Omega\} \subset \mathcal{C}$, and further set $C = \bigcup\{C_{\hat{\gamma}} \mid \hat{\gamma} \in U\} \subset \mathcal{C}$. It can be checked that C is a compact, $\{\hat{\tau}_t\}$-invariant subset of $\mathcal{C}$ which is invariantly connected.

Define a cocycle $\hat{\Phi}$ on C in the following way: $\hat{\Phi}(c,t)$ is the fundamental matrix solution of the linear differential equation $x' = c(t)x$ ($c \in C,\ t \in \mathbb{R},\ x \in \mathbb{R}^d$). Let $C_\gamma = \{t \mapsto A(\omega + \gamma t) \mid \omega \in \Omega\} \subset C$; it can be checked that the dynamical spectrum of the restriction $\hat{\Phi}_\gamma = \hat{\Phi}|_{C_\gamma \times \mathbb{R}}$ equals σ_Φ. Similarly, let $C_{\gamma_n} = \{t \mapsto A(\omega + \gamma_n t) \mid \omega \in \Omega\}$. Then the dynamical spectrum of the restriction $\hat{\Phi}_n = \hat{\phi}|_{C_{\gamma_n} \times \mathbb{R}}$ of $\hat{\Phi}$ to $C_{\gamma_n} \times \mathbb{R}$ equals $\sigma_{\Phi^{(n)}}$.

We are now in a position to apply the perturbation Theorem 6 of [38]. According to this theorem, there is a neighborhood $W \subset C$ of C_γ with the property that, if C_* is a $\{\hat{\tau}_t\}$-invariant subset of W, then the dynamical spectrum of $\hat{\Phi}_{C_*}$ is contained in the ε-neighborhood of $\sigma_{\hat{\Phi}_\gamma} = \sigma_\Phi = \{a_1 < a_2 < \ldots < a_q\}$. Now if n is sufficiently large, then $C_\gamma \subset W$. So the remarks of the preceding paragraph and Proposition 3.1 imply that the thesis of Proposition 3.4 is true.

Remark 3.5. *Let $T = \mathbb{Z}$, let $A : \Omega = \mathbb{T}^g \to GL(\mathbb{R}^d)$ be a continuous map, let $\gamma \in \mathbb{R}^g$ have rationally independent components, and let $\Phi(\omega,t)$ be the cocycle generated by the family of difference equations*

$$x_{t+1} = A(\omega + \gamma t)x_t \qquad (\omega \in \Omega,\ t \in \mathbb{Z}).$$

Similarly, let $\Phi^{(n)}(\omega,t)$ be the cocycle generated by the family

$$x_{t+1} = A(\omega + \gamma^{(n)} t)x_t \qquad (\omega \in \Omega,\ t \in \mathbb{Z})$$

where $\gamma^{(n)} \in \mathbb{R}^g$ ($n = 1, 2, \ldots$). Then Proposition 3.4 is true as stated for Φ and $\Phi^{(n)}$. The proof is practically identical to that given above for real cocycles (one must introduce a discrete Bebutov flow, and one must note that [38, Theorem 6] holds also for integer cocycles).

We have shown that the discrete spectrum condition has significant consequences for the convergence of the limits which define the Lyapunov exponents, and for the continuity of those Lyapunov exponents. Our results can be viewed as generalizations of [14, Theorem 3].

If the discrete spectrum condition does not hold, then one cannot expect the Lyapunov exponents of Φ to vary continuously when Φ is subjected to a C^0-perturbation. We indicate a concrete result along these lines, the proof of which uses important theorems of Bochi-Viana [2] and Bessa [1]. These papers were motivated by a well-known conjecture of Mañe [25].

Let $T = \mathbb{R}$ or $\mathbb{Z}$. Let $(\Omega, \{\tau_t\})$ be a compact metric flow which is strictly ergodic with unique ergodic measure μ. Thus for example it can be a Kronecker flow as defined in Section 2.

Let $\Phi : \Omega \times T \to GL(\mathbb{R}^d)$ be a cocycle over $(\Omega, \{\tau_t\})$. Suppose that the dynamical spectrum σ_Φ of Φ is a single interval: $\sigma_\Phi = [a, b]$. Suppose that

$a < b$. Let $\{\beta_1 < \ldots < \beta_s\}$ be the Oseledets spectrum of Φ with respect to μ, and let $V_1^{(m)}, \ldots, V_s^{(m)}$ be the corresponding Oseledets bundles.

According to the results of [1] and [2], there is a C^0-residual set $\{\Psi\}$ of $GL(\mathbb{R}^d)$-valued cocycles over $(\Omega, \{\tau_t\})$ for which one of the following alternatives holds.

(i) The Oseledets spectrum of Ψ reduces to a single point;

(ii) The Oseledets bundles give rise to a dominated splitting (or exponential separation) of Ψ over $(\Omega, \{\tau_t\})$.

Moreover, it is shown that, if Ψ does not admit a dominated splitting, then an arbitrarily small C^0-perturbation of Ψ has property (i). See also [28, 30] for related results. We will not define the concept of dominated splitting/exponential separation here. For this we refer to [1, 2] or to the older literature on exponential separation (e.g., [3, 4, 5, 32, 33]).

Now, one can use a Krylov-Bogoliubov argument to show that, if the Oseledets bundles of Ψ give rise to a dominated splitting, then the dynamical spectrum σ_Ψ of Ψ consists of at least two disjoint intervals. We omit the proof, but note that it uses the hypothesis that $(\Omega, \{\tau_t\})$ admits just one ergodic measure.

Returning to the cocycle Φ, one can use another Krylov-Bogoliubov argument to show that the endpoints a and b of $\sigma_\Phi = [a, b]$ are in the Oseledets spectrum; see [18]. But an arbitrarily small C^0-perturbation of Φ has the property that its Oseledets spectrum reduces to a single point. This implies that the Lyapunov exponents of Φ cannot vary continuously if Φ is varied in the C^0-sense.

4. Consequences of convergence

In this section, we consider a problem which is inverse to that taken up in Section 3. Namely, suppose that Φ is a cocycle over a compact metric flow $(\Omega, \{\tau_t\})$, and suppose that $\lim_{t\to\infty} \frac{1}{t} \ln |\Phi(\omega, t)|$ exists for all $\omega \in \Omega$ and all $0 \neq x \in \mathbb{R}^d$. We ask if the cocycle Φ has discrete spectrum. In general this is not true, as the following example shows.

Example 4.1. *Let Ω be the annulus $0 < \alpha \leq r \leq \beta$, $0 \leq \theta \leq 2\pi$ in the plane $\mathbb{R}^2$ with polar coordinates (r, θ). Let $a : \Omega \to \mathbb{R}$ be a continuous function such that the correspondence $r \mapsto \int_0^{2\pi} a(r, \bar{\theta}) d\bar{\theta}$ takes on more than one value. Consider the family of one-dimensional ODEs*

$$x' = a(r, \theta + t)x \qquad x \in \mathbb{R} \tag{5_ω}$$

where $\omega = (r,\theta) \in \Omega$. *The family* (5_ω) *has the form of the family* (1_ω) *if we put* $\tau_t(r,\theta) = (r,\theta+t)$ *for* $t \in \mathbb{R}$ *and* $(r,\theta) \in \Omega$. *It it clear that the cocycle* Φ *which is determined by equations* (5_ω) *has the form*

$$\Phi(\omega,t) = \exp\left(\int_0^t a(r,\theta+s)ds\right) \qquad (\omega = (r,\theta) \in \Omega,\ t \in \mathbb{R}).$$

We see that, if $\omega = (r,\theta) \in \Omega$ *and* $0 \neq x \in \mathbb{R}$, *then* $\lim_{t\to\infty} \frac{1}{t} \ln|\Phi(\omega,t)x|$ *exists and equals* $\frac{1}{2\pi}\int_0^{2\pi} a(r,\bar\theta)d\bar\theta$. *This integral traces out a nondegenerate interval* I *as* r *varies from* α *to* β. *It turns out that* I *is the dynamical spectrum of the family* (5_ω).

This example is in fact "too simple" and only indicates that we must specify our inverse problem in a more detailed way. So let us suppose that $(\Omega,\{\tau_t\})$ is minimal, and that, for each $\omega \in \Omega$ and each $0 \neq x \in \mathbb{R}^d$, the limit $\lim_{t\to\infty} \frac{1}{t} \ln|\Phi(\omega,t)x|$ exists. We ask: does Φ have discrete spectrum?

This question has an affirmative answer if $d = 1$. It may well be that the answer is still affirmative if $d \geq 2$. We have not been able to prove this, however. Here is what we can and will do.

(1) If $d \geq 2$, we suppose (in addition to the conditions already listed) that, for each ergodic measure μ on Ω, the corresponding Oseledets spectrum $\{\beta_1(\mu) < \beta_2(\mu) < \ldots < \beta_s(\mu)\}$ is simple. That is, $s = d$, or equivalently all the multiplicities d_r are equal to 1 ($1 \leq r \leq s = d$: see Theorem 2.2). Under these conditions, we will show that Φ has discrete spectrum. In fact, it will turn out that the numbers $\beta_1(\mu) = \beta_1, \ldots, \beta_d(\mu) = \beta_d$ do not depend on the choice of the ergodic measure μ, and that $\sigma_\Phi = \{\beta_1 < \beta_2 < \ldots < \beta_d\}$. Thus in particular Φ satisfies the classical Lillo property [23].

(2) If $d = 2$, we make no a priori hypothesis regarding the Oseledets spectrum: we suppose that $(\Omega,\{\tau_t\})$ is minimal, and that, for each $\omega \in \Omega$ and each $0 \neq x \in \mathbb{R}^2$, the limits $\lim_{t\to\infty} \frac{1}{t} \ln|\Phi(\omega,t)x|$ and $\lim_{t\to-\infty} \frac{1}{t} \ln|\Phi(\omega,t)x|$ exist (they need not be equal). We will prove that, subject to these hypotheses, Φ has discrete spectrum. As noted in the Introduction, we generalize a result of Furman [14], who assumes that $(\Omega,\{\tau_t\})$ is strictly ergodic. He uses certain properties of the projective flow defined by Φ when $d = 2$. See also [17] in this regard.

To our knowledge, our inverse problem has not been frequently discussed in the literature. We point out that the hypothesis concerning the existence of the limits $\lim_{t\to\infty} \frac{1}{t} \ln |\Phi(\omega, t)x|$ (and $\lim_{t\to-\infty} \frac{1}{t} \ln |\Phi(\omega, t)x|$ in point (2)) is rather delicate since no uniformity is assumed. We also remark that there are various results in the literature to the effect that the set of cocycles over a given compact metric flow which have simple Oseledets spectrum is dense in various topologies. See, e.g., [12, 27].

After these preliminary remarks, we express point (1) in a formal statement:

THEOREM 4.2. *Let $T = \mathbb{R}$ or $\mathbb{Z}$. Let $(\Omega, \{\tau_t\})$ be a compact metric minimal flow, and let $\Phi : \Omega \times T \to GL(\mathbb{R}^d)$ be a T-cocycle over $(\Omega, \{\tau_t\})$. Suppose that, for every $\{\tau_t\}$-ergodic measure μ on Ω, the Oseledets spectrum is simple. This means that it consists of d distinct points $\beta_1 < \ldots < \beta_d$ (which may depend on μ). Suppose that, for each $\omega \in \Omega$ and $0 \neq x \in \mathbb{R}^d$, the limit $\lim_{t\to\infty} \frac{1}{t} \ln |\Phi(\omega, t)x|$ exists. Then the dynamical spectrum σ_Φ of Φ consists of d distinct points (and in particular is discrete).*

Note that, if $(\Omega, \{\tau_t\})$ is minimal, then it is invariantly connected and chain recurrent. So the results stated in Section 2 will be available to us in the proof of Theorem 4.2, to which we now turn.

Before beginning the proof of Theorem 4.2, we describe several convenient constructions. Let Φ be a T-cocycle over $(\Omega, \{\tau_t\})$, let $\sigma_\Phi = [a_1, b_1] \cup \cdots \cup [a_q, b_q]$, and let $V_1^{(c)}, \ldots, V_q^{(c)}$ be the corresponding spectral subbundles of Theorem 2.4. These are topological vector subbundles of $\Omega \times \mathbb{R}^d$, of fiber dimension $1 \leq d_1, \ldots, d_q$ where $d_1 + \cdots + d_q = d$. They need not be topologically trivial; i.e., they need not be equivalent to product bundles $\Omega \times \mathbb{R}^{d_p}$, $1 \leq p \leq q$.

However, it is explained in [10] how these bundles can be trivialized via an appropriate cohomology. We explain the relevant constructions of [10].

Let us recall that a minimal flow $(\hat{\Omega}, \{\hat{\tau}_t\})$ is said to be an extension of the minimal flow $(\Omega, \{\tau_t\})$ if there is a continuous map $\pi : \hat{\Omega} \to \Omega$ such that $\pi \circ \hat{\tau}_t = \tau_t \circ \pi$ for all $t \in T$ (one says that π is a flow homomorphism). Using the minimality of $(\Omega, \{\tau_t\})$ one sees that π must be surjective.

The cocycle Φ can be lifted to a cocycle $\hat{\Phi}$ on $\hat{\Omega}$ via the formula $\hat{\Phi}(\hat{\omega}, t) = \Phi(\pi(\hat{\omega}), t)$ $(\hat{\omega} \in \hat{\Omega},\ t \in T)$. Moreover the bundles $V_1^{(c)}, \ldots, V_q^{(c)}$ lift to $\hat{\Omega}$ via the usual pullback construction. Call the lifted bundles $\hat{V}_1^{(c)}, \ldots, \hat{V}_q^{(c)}$; they are $\hat{\Phi}$-invariant and it is easy to see that they are the spectral subbundles of $\hat{\Phi}$. Let us write $\hat{V}_p^{(c)}(\hat{\omega}) = \hat{V}_p^{(c)} \cap (\{\hat{\omega}\} \times \mathbb{R}^d)$ for the fiber of $\hat{V}_p(c)$ at $\hat{\omega} \in \hat{\Omega}$.

Next let $\mathcal{O}(d)$ be the group of orthogonal $d \times d$ matrices. According to [10, Theorem 4.5], one can find a minimal extension $(\hat{\Omega}, \{\hat{\tau}_t\})$ of $(\Omega, \{\tau_t\})$ together with a continuous map $F : \hat{\Omega} \to \mathcal{O}(d)$ such that, if $\tilde{V}_p^{(c)}(\hat{\omega}) = F(\hat{\omega})\hat{V}_p^{(c)}(\hat{\omega})$, then the bundle $\tilde{V}_p^{(c)} = \bigcup_{\hat{\omega}\in\hat{\Omega}} \tilde{V}_p^{(c)}(\hat{\omega})$ is a product bundle. In fact, let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$. For each $p \in \{2, 3, \ldots, q\}$, let us identify $\mathbb{R}^{d_p}$ with the span of the set of unit vectors $\{e_{d_1+\dots d_{p-1}+1},\dots,e_{d_1+\dots+d_p}\}$; if $p=1$ we identify $\mathbb{R}^{d_1}$ with $Span\{e_1,\dots,e_{d_1}\}$. Then F can be chosen so that $\tilde{V}_p^{(c)}=\Omega\times\mathbb{R}^{d_p}$ $(1\le p\le q)$.

Define the cocycle $\tilde{\Phi}$ by

$$\tilde{\Phi}(\hat{\omega},t)=F(\tilde{\tau}_t(\hat{\omega}))\hat{\Phi}(\hat{\omega},t)F(\hat{\omega})^{-1} \qquad (\hat{\omega}\in\hat{\Omega},\ t\in T);$$

thus $\tilde{\Phi}$ is cohomologous to the cocycle Φ via the cohomology F. We see that $\tilde{\Phi}$ admits the spectral decomposition $\tilde{V}_1^{(c)}=\Omega\times\mathbb{R}^{d_1},\dots,\tilde{V}_q^{(c)}=\Omega\times\mathbb{R}^{d_q}$.

We conclude that, to prove Theorem 4.2, there is no loss of generality in assuming that the spectral subbundles of Φ are product bundles: $V_p^{(c)}=\Omega\times\mathbb{R}^{d_p}$ $(1\le p\le q)$. This is equivalent to saying that there is no loss of generality in assuming that: (i) Φ has block-diagonal form:

$$\Phi=\begin{pmatrix}\Phi_1 & & 0\\ & \ddots & \\ 0 & & \Phi_q\end{pmatrix} \tag{6}$$

where Φ_p is a T-cocycle over $(\Omega,\{\tau_t\})$ with values in $GL(\mathbb{R}^d)$, and (ii) the dynamical spectrum of Φ_p is the single interval $[a_p,b_p]$ $(1\le p\le q)$. (The reader is warned that, if $(\Omega,\{\tau_t\})$ is strictly ergodic, then the extension $(\hat{\Omega},\{\hat{\tau}_t\})$ of the above construction need not be strictly ergodic.)

We pass to a second construction. Say that Φ is *upper triangular* if $\Phi=(\Phi_{ij})$ where $\Phi_{ij}=0$ if $i>j$ and $\Phi_{ii}>0$ $(1\le i\le d)$. Our construction will give rise to a cohomology between a suitable lifted version of Φ, and an upper triangular cocycle.

Let $\mathcal{O}(d)$ be the group of orthogonal $d\times d$ matrices. If $u_0\in\mathcal{O}(d)$, then $\Phi(\omega,t)u_0$ can be uniquely decomposed in the form

$$\Phi(\omega,t)u_0=U(\omega,u_0,t)\Delta(\omega,u_0,t) \qquad (\omega\in\Omega,\ t\in T)$$

where $U\in\mathcal{O}(d)$ and Δ is upper triangular with positive diagonal elements. This follows from the Gram-Schmidt decomposition of $\Phi(\omega,t)u_0$. It turns out that, if one sets $\hat{\tau}_t(\omega,u_0)=(\tau_t(\omega),U(\omega,u_0,t))$ then $\{\hat{\tau}_t\mid t\in T\}$ is a flow on $\Omega\times\mathcal{O}(d)$, and Δ is a $\{\hat{\tau}_t\}$-cocycle.

Note that, if Φ has a block diagonal structure as in (6), then U and Δ have corresponding block-diagonal structures.

Next let $\hat{\Omega}\subset\Omega\times\mathcal{O}(d)$ be a minimal $\{\hat{\tau}_t\}$-subflow (such a subflow exists by Zorn's Lemma). Then the projection $\pi:\hat{\Omega}\to\Omega:(\omega,u_0)\mapsto\omega$ is continuous, and $\pi\circ\hat{\tau}_t=\tau_t\circ\pi$. We introduce the lifted cocycle $\hat{\Phi}:\hat{\Omega}\times T\to GL(\mathbb{R}^d):\hat{\Phi}(\omega,t)=\Phi(\pi(\hat{\omega}),t)$ where $\hat{\omega}=(\omega,t)\in\hat{\Omega}$. Note that the map $F:\hat{\Omega}\to\mathcal{O}(d):F(\omega,u_0)=u_0$ defines a cohomology between $\hat{\Phi}$ and Δ. In fact, $F(\hat{\tau}_t(\hat{\omega}))\Delta(\hat{\omega})F(\hat{\omega})^{-1}=\hat{\Phi}(\hat{\omega},t)$ for $\hat{\omega}=(\omega,t)\in\hat{\Omega}$ and $t\in T$.

Our third and final construction was already discussed in Section 2. Namely, assume that $T = \mathbb{R}$. Then there exists a continuous function $A : \Omega \to \mathbb{L}(\mathbb{R}^d)$ such that Φ is cohomologous to the cocycle generated by the family of linear ODEs (1_ω):

$$x' = A(\tau_t(\omega))x.$$

We observe that, if a given cocycle Φ has a block-triangular form as in the first construction, then the coefficient matrix $A(\cdot)$ in (1_ω) may be chosen to have the corresponding block-diagonal form. Moreover, if Φ has an upper triangular from as in the second construction, then $A(\cdot)$ may be chosen to have the corresponding upper triangular form.

We assume until further notice that $T = \mathbb{R}$. Using the above constructions, we see that by introducing a suitable minimal extension of $(\Omega, \{\tau_t\})$, and by introducing a suitable cohomology, it can be arranged that Φ satisfies the following conditions.

HYPOTHESES 4.3. (a) *The cocycle Φ is generated by a family of linear ODEs*

$$x' = A(\tau_t(\omega))x \qquad \omega \in \Omega,\ x \in \mathbb{R}^d \tag{7_ω}$$

where the matrix function $A(\cdot)$ has block-diagonal form: $A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_q \end{pmatrix}$.

(b) *If Φ_p is the cocycle over $(\Omega, \{\tau_t\})$ which is generated by the family $x' = A_p(\tau_t(\omega))x$, then the dynamical spectrum σ_p of Φ_p is the single interval $[a_p, b_p]$ $(1 \le p \le q)$.*

(c) *Each matrix function A_p is upper triangular $(1 \le p \le q)$.*

It can be shown that, if Φ and Ψ are cohomologous cocycles, and if Φ satisfies the hypotheses of Theorem 4.2, then so does Ψ. It can also be shown that, if Φ and Ψ are cohomologous, and if Φ satisfies the thesis of Theorem 4.2, then so does Ψ.

We pass to the proof of Theorem 4.2 in the case when $T = \mathbb{R}$. According to the above constructions and remarks, we can assume that Φ satisfies any or all of Hypotheses 4.3 (a)–(c), when it is appropriate to do so.

We proceed by induction on the dimension d of the cocycle Φ. Suppose that $d = 1$. There is no loss of generality in assuming that Φ is generated by a family of one dimensional systems of the form (1_ω). The family (1_ω) has the form $x' = A(\tau_t(\omega))x$ where $A : \Omega \to \mathbb{R}$ is a continuous scalar function. Using the hypothesis concerning the existence of the limits which define the Lyapunov exponents of Φ, we see that $\lim_{t\to\infty} \frac{1}{t}\int_o^t A(\tau_s(\omega))ds$ exists for all $\omega \in \Omega$.

Now the flow $(\Omega, \{\tau_t\})$ is by assumption minimal, so one can use an oscillation result of Johnson [16] to show that the quantity $\bar{a} = \lim_{t\to\infty} \frac{1}{t}\int_0^t A(\tau_s(\omega))ds$

does not depend on $\omega \in \Omega$, and the limit is uniform in ω. Moreover $\bar{a} = \lim_{t\to-\infty} \frac{1}{t}\int_0^t A(\tau_s(\omega))ds$, where again the limit is uniform in $\omega \in \Omega$. One can now check directly that the dynamical spectrum of Φ satisfies $\sigma_\Phi = \{\bar{a}\}$; i.e., it is discrete.

Next, suppose that Theorem 4.2 is valid for all continuous $\mathbb{R}$-cocycles of dimension $\leq d-1$, over all minimal flows $(\Omega, \{\tau_t\})$. We suppose without loss of generality that our given cocycle Φ satisfies Hypotheses 4.3 (a) and (b). Suppose first that the number of diagonal blocks of the (d-dimensional) matrix function $A(\cdot)$ is at least 2. Each block $A_1, \ldots, A_q$ then has dimension $\leq d-1$. So by the induction hypothesis, the family

$$x' = A_p(\tau_t(\omega))x \qquad (\omega \in \Omega,\ x \in \mathbb{R}^d)$$

has discrete spectrum ($1 \leq p \leq q$). By Hypotheses 4.3 (2), this spectrum is the singleton $\{a_p\}$, and it follows that the cocycle Φ has discrete spectrum: $\sigma_\Phi = \{a_1, \ldots, a_q\}$. So Theorem 4.2 is proved in this case.

We now assume that $q = 1$, which means that the spectrum σ_Φ of Φ consists of a single interval $[a, b]$. We must show that $a = b$. We assume w.l.o.g. that Hypotheses $(a), (b)$ and (c) are valid. The matrix function $A(\cdot)$ has values in $\mathbb{L}(\mathbb{R}^d)$ and is upper triangular.

Let us write

$$A(\omega) = \begin{pmatrix} A_*(\omega) & a_{1d}(\omega) \\ & \\ 0 & a_{dd}(\omega) \end{pmatrix}$$

where A_* takes values in $\mathbb{L}(\mathbb{R}^{d-1})$ and is upper triangular. Consider the family of subsystems

$$y' = A_*(\tau_t(\omega))y \qquad \omega \in \Omega,\ y \in \mathbb{R}^{d-1}. \tag{8_ω}$$

Note that a solution $y(t)$ of (8_ω) determines a solution $x(t) = \begin{pmatrix} y(t) \\ x_n(t) \end{pmatrix}$ of (7_ω) by setting $x_n(t) = 0$; that is, $x(t) = \begin{pmatrix} y(t) \\ 0 \end{pmatrix}$ is a solution of (7_ω) if and only if $y(t)$ is a solution of (8_ω).

We see that the family (8_ω) has the property that $\lim_{t\to\infty} \frac{1}{t} \ln |y(t)|$ exists whenever $y(t)$ is a nonzero solution of equation (8_ω) $(\omega \in \Omega)$. By the induction hypothesis, the dynamical spectrum σ_* of the family (8_ω) is discrete, say

$$\sigma_* = \{\alpha_1 < \alpha_2 < \ldots < \alpha_j\}$$

where $1 \leq j \leq d-1$. By Proposition 2.2, the set of Lyapunov exponents of (8_ω) is exactly $\{\alpha_1, \ldots, \alpha_j\}$ for each $\omega \in \Omega$. Moreover, if d_i is the multiplicity of α_i for $1 \leq i \leq j$, then $d_1 + \cdots + d_j = d-1$.

Now, for each ergodic measure μ on Ω, the Oseledets spectrum of the family (8_ω) is contained in the dynamical spectrum σ_* of that family. Moreover, the Oseledets spectrum equals the set of averages

$$\left\{\int_\Omega a_{ii}(\omega)d\mu(\omega) \mid 1 \le i \le d-1\right\}$$

of the diagonal elements of A_*; see [18, 31]. By hypothesis, the μ-Oseledets spectrum of Φ is simple, and therefore the μ-Oseledets spectrum of the cocycle Φ_* generated by equations (8_ω) is also simple. Using the fact that $\sigma_* = \{\alpha_1 < \alpha_2 < \ldots < \alpha_j\}$, we see that each multiplicity $d_i = 1$, and that $\sigma_* = \{\alpha_1 < \alpha_2 < \ldots < \alpha_{d-1}\}$ consists of $d-1$ distinct real numbers. It is clear that these numbers are just a reordered version of the numbers $\left\{\int_\Omega a_{ii}(\omega)d\mu(\omega) \mid 1 \le i \le d-1\right\}$. One can show (by applying Proposition 3.2, or by carrying out a "secondary" induction on j, $1 \le j \le d-1$) that $\int_\Omega a_{ii}(\omega)d\mu(\omega)$ does not depend on the choice of the $\{\tau_t\}$-ergodic measure μ, if $1 \le j \le d-1$.

We must now study the significance of the numbers $\int_\Omega a_{dd}(\omega)d\mu(\omega)$ as μ ranges over the set of $\{\tau_t\}$-ergodic measures on Ω. To do this, it is convenient to introduce a projective flow. The construction is quite similar to that carried out in the proof of Theorem 3.2 above. Let $\mathbb{L}$ be the $(d-1)$-dimensional manifold of lines through the origin in $\mathbb{R}^d$. Let $\mathbb{B} = \Omega \times \mathbb{L}$, and define a flow $\{\hat{\tau}_t\}$ on $\mathbb{B}$ by setting $\hat{\tau}_t(\omega, l) = (\tau_t(\omega), \Phi(\omega, t)l)$ $(\omega \in \Omega,\ l \in \mathbb{L})$. Define $f : \mathbb{B} \to \mathbb{R} : f(\omega, l) = \langle A(\omega)x, x\rangle / \langle x, x\rangle$ if $0 \ne x \in l$. Then if $x(t)$ is a solution of (7_ω), and if $l \in \mathbb{L}$ is the line containing $x(0) \ne 0$, then

$$\int_0^t f(\hat{\tau}_s(\omega, l))ds = \ln \frac{|x(t)|}{|x(0)|}. \tag{9}$$

By the hypothesis concerning the existence of the limits defining the Lyapunov exponents, and by (9), one has that the limit $\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\hat{\tau}_s(b))ds$ exists for each $b \in \mathbb{B}$. Let us denote the limit by $f_*(b)$. Since f_* is the pointwise limit of a sequence of continuous functions, it admits a residual set of continuity points [6]. Let b_* be a point of continuity of f_*. For each $\varepsilon > 0$, there is an open neighborhood $U = U(\varepsilon) \subset \Omega \times \mathbb{L}$ of b_* such that, if $b \in U$, then $|f_*(b) - f_*(b_*)| < \varepsilon$. There is no loss of generality in assuming that $U = U_1 \times U_2$ where $U_1 \subset \Omega$ and $U_2 \subset \mathbb{L}$ are open sets. There is also no loss of generality in assuming that U does not intersect the $\{\hat{\tau}_t\}$-invariant set $\mathbb{B}_1 = \{(\omega, l) \in \mathbb{B} \mid l \subset \mathbb{R}^{d-1} \subset \mathbb{R}^d\}$.

For each $\omega \in \Omega$, there is a real number $\beta_*(\omega)$ such that the set of Lyapunov exponents of equation (7_ω) equals $\{\alpha_1, \alpha_2, \ldots, \alpha_{d-1}, \beta_*(\omega)\}$. Let $\beta_{\max}(\omega) = \max\{\alpha_1,\ \ldots, \alpha_{d-1}, \beta_*(\omega)\}$ be the largest Lyapunov exponent of (7_ω). Write the continuity point b_* of f_* in the form $b_* = (\omega_*, l_*)$. It follows from the continuity of f_* at b_* that $f_*(b_*)$ equals $\beta_{\max}(\omega_*)$. In fact, this is a consequence of the observation that, if $\bar{\beta}(\omega_*)$ is the maximum of the Lypunov exponents of (7_{ω_*}) which are distinct from $\beta_{\max}(\omega_*)$, then $\Big\{x \in \mathbb{R}^d \mid \ x = 0$ or $\lim\limits_{t\to\infty} \frac{1}{t} \ln |\Phi(\omega_*, t)x| \le \bar{\beta}(\omega_*)\Big\}$ is a proper vector subspace of $\mathbb{R}^d$, so its complement in $\mathbb{R}^d$ is open and dense. This means that there is an open dense subset $W \subset \mathbb{L}$ such that, if $l \in W$, then $\lim\limits_{t\to\infty} \int_0^t f(\hat{\tau}_s(\omega_*, l) ds = \beta_{\max}(\omega_*)$.

Recall that we are working under the hypothesis that σ_Φ is a single interval $[a, b]$. Using Theorem 2.4, we see that the numbers $\alpha_1, \dots, \alpha_{d-1}$ all lie in $[a, b]$. Suppose for the time being that b is greater than α_{d-1}.

According to a result of [18], there is a $\{\tau_t\}$-ergodic measure μ on Ω for which b is a Lyapunov exponent of Φ, for μ-a.a. $\omega \in \Omega$. By Theorem 2.4, we have that $\beta_{\max}(\omega) = b$ for μ-a.a. $\omega \in \Omega$. Fix a point $\bar{\omega} \in \Omega$ such that $\beta_{\max}(\bar{\omega}) = b$. If $x \in \mathbb{R}^d$, we write $x = \begin{pmatrix} y \\ x_d \end{pmatrix}$ where $y \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$. Let x be a vector such that $\lim\limits_{t\to\infty} \frac{1}{t} \ln |\Phi(\bar{\omega}, t)x| = b$. Writing $\Phi(\bar{\omega}, t)x = \Phi(\bar{\omega}, t) \begin{pmatrix} y \\ x_d \end{pmatrix} = \begin{pmatrix} y(t) \\ x_d(t) \end{pmatrix}$, and using the fact that $b > \alpha_{d-1} = \max\{\alpha_i \mid 1 \le i \le d-1\}$, we see that $x_d \ne 0$, and that $\lim\limits_{t\to\infty} \frac{1}{t} \ln |x_d(t)| = b$. (For later use, we note that $b = \lim\limits_{t\to\infty} \frac{1}{t} \ln |x_d(t)| = \lim\limits_{t\to\infty} \frac{1}{t} \int_0^t a_{dd}(\tau_s(\bar{\omega}) ds.$)

One checks that, if $\begin{pmatrix} y \\ x_d \end{pmatrix}$ is *any* vector with $x_d \ne 0$, then $\lim\limits_{t\to\infty} \frac{1}{t} \ln |\Phi(\bar{\omega}, t)x| = b$.

Return to the continuity point $(\omega_*, l_*) \in \mathbb{B}$ of f_* which was introduced previously. Let $\varepsilon > 0$, and choose $U(\varepsilon) = U = U_1 \times U_2$ as before. Let $\bar{\omega}$ be the point of the preceding two paragraphs. Since $(\Omega, \{\tau_t\})$ is minimal, the positive semiorbit $\{\tau_t(\bar{\omega}) \mid t \ge 0\}$ is dense in Ω, hence it enters U_1. Using the fact that U does not intersect $\mathbb{B}$ together with the result of the previous paragraph, we can find a vector $\begin{pmatrix} y \\ x_d \end{pmatrix} \in \mathbb{R}^d$, whose projective image l lies in U_2, such that $\lim\limits_{t\to\infty} \frac{1}{t} \ln |\Phi(\omega_*, t)x| = b$. This means that $|f_*(\omega_*, l_*) - b| \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have that $f_*(\omega_*, l_*) = b$.

Next, let $\varepsilon > 0$, and let $\omega \in \Omega$ be any point of Ω. Again the positive semiorbit $\{\tau_t(\omega) \mid t \ge 0\}$ enters U_1. So there exists a vector $x = \begin{pmatrix} y \\ x_d \end{pmatrix} \in \mathbb{R}^d$ with $x_d \ne 0$ such that $\left| \lim\limits_{t\to\infty} \frac{1}{t} \ln |\Phi(\omega, t)x| - f_*(\omega_*, l_*) \right| \le \varepsilon$. Hence $\left| \lim\limits_{t\to\infty} \frac{1}{t} \ln |\Phi(\omega, t)x| - b \right| \le \varepsilon$. At this point choose $0 < \varepsilon < \frac{1}{2}(b - \alpha_{d-1})$, and write $\Phi(\omega,t)x = \begin{pmatrix} y(t) \\ x_d(t) \end{pmatrix}$. It can be checked that the limit $\lim_{t\to\infty} \frac{1}{t} \ln |x_d(t)|$ exists and is $\geq b - \varepsilon > \alpha_{d-1} + \varepsilon$.

Now, $\lim_{t\to\infty} \frac{1}{t} \ln |x_d(t)| = \lim_{t\to\infty} \frac{1}{t} \int_0^t a_{dd}(\tau_s(\omega))ds$. We are able to conclude that the limit $\lim_{t\to\infty} \frac{1}{t} \int_0^t a_{dd}(\tau_s(\omega))ds$ exists for all $\omega \in \Omega$. The limit equals b if $\omega = \bar{\omega}$. By the oscillation result of [16], $\int_\Omega a_{dd} d\mu = b$ for all ergodic measures μ on Ω. By using a Krylov-Bogoliubov argument, one proves that $\lim_{t\to\infty} \frac{1}{t} \int_0^t a_{dd}(\tau_s(\omega))ds = b$, and the limit is uniform in $\omega \in \Omega$.

Let $\alpha_{d-1} < \lambda < b$. Let us show that $\lambda \notin \sigma_\Phi$. Using Theorem 2.3, we see that it is sufficient to show that, if $\omega \in \Omega$ and $0 \neq x \in \mathbb{R}^d$, then $e^{\lambda t}\Phi(\omega,t)x$ is not bounded in $-\infty < t < \infty$. To do this, note first that, if $x = \begin{pmatrix} y \\ 0 \end{pmatrix} \in \mathbb{R}^d$, then $|e^{-\lambda t}\Phi(\omega,t)x| \to \infty$ as $t \to -\infty$, because $\sigma_* = \{\alpha_1, \ldots, \alpha_{d-1}\}$. On the other hand, if $x = \begin{pmatrix} y \\ x_d \end{pmatrix}$ with $x_d \neq 0$, and if $x_d(t)$ is defined by $\Phi(\omega,t)x = \begin{pmatrix} y(t) \\ x_d(t) \end{pmatrix}$, then $|x_d(t)| \to \infty$ as $t \to \infty$. So in fact $\lambda \notin \sigma_\Phi$.

However, σ_Φ is by hypothesis the interval $[a,b]$, and we know that $\alpha_{d-1} \in \sigma_\Phi$. So we have arrived at a contradiction, and must conclude that $b \leq \alpha_{d-1}$.

There remains to study the situation when $b \leq \alpha_{d-1}$. For this, let us first recall that, if $1 \leq i \leq d-1$, then $\int_\Omega a_{ii} d\mu$ does not depend on the choice of the ergodic measure μ on Ω. Second, we recall that, if μ is an ergodic measure on Ω, then the corresponding Oseledets spectrum equals $\left\{ \int_\Omega a_{11} d\mu, \ldots, \int_\Omega a_{dd} d\mu \right\}$. By hypothesis, the Oseledets spectrum is simple for each $\{\tau_t\}$-ergodic measure μ on Ω. So $\int_\Omega a_{dd} d\mu < \alpha_{d-1}$ for each such μ. Let us define $\bar{\alpha} = \sup \left\{ \int_\Omega a_{dd} d\mu \;\middle|\; \mu \text{ is a } \{\tau_t\}\text{-ergodic measure on } \Omega \right\}$. We claim that $\bar{\alpha} < \alpha_{d-1}$. Here is a sketch of the proof. Since the set $\{\nu\}$ of $\{\tau_t\}$-invariant measures on Ω is compact and convex in the weak-$*$ topology, and since μ is an extreme point of $\{\nu\}$ if and only if μ is ergodic, we can use the Choquet theorem [35] to show that $\int_\Omega a_{dd} d\nu \leq \alpha_{d-1}$ for each $\{\tau_t\}$-invariant measure ν on Ω. If $\bar{\alpha}$ equals α_{d-1}, then the weak-$*$ compactness of $\{\nu\}$ allows us to find an invariant measure ν on Ω such that $\int_\Omega a_{dd} d\mu = \alpha_{d-1}$. Using the Choquet theorem again, we determine an ergodic measure μ on Ω such that $\int_\Omega a_{dd} d\mu = \alpha_{d-1}$. This is not possible, so indeed $\bar{\alpha} < \alpha_{d-1}$.

We can now use a Krylov-Bogoliubov argument to prove the following statement: Let $\varepsilon > 0$; then there exists $T > 0$ such that, if $t \le -T$ and $\omega \in \Omega$, then $\frac{1}{t}\int_0^t a_{dd}(\tau_s(\omega))ds \le \bar{\alpha} + \varepsilon$.

Next choose $\lambda \in (\bar{\alpha}, \alpha_{d-1})$ such that $\lambda > \alpha_{d-2}$. We claim that λ is not in the spectrum σ_Φ of Φ. As before, it is sufficient to show that, if $\omega \in \Omega$ and $0 \ne x \in \mathbb{R}^d$, then $e^{-\lambda t}\Phi(\omega, t)x$ is unbounded on $-\infty < t < \infty$. So let $x = \begin{pmatrix} y \\ 0 \end{pmatrix}$ where $y \in \mathbb{R}^{d-1}$. Then $e^{-\lambda t}\Phi(\omega, t)x$ is unbounded because $\lambda \notin \sigma_* = \{\alpha_1, \dots, \alpha_{d-1}\}$. On the other hand, if $x = \begin{pmatrix} y \\ x_d \end{pmatrix}$ with $x_d \ne 0$, then $e^{-\lambda t}\Phi(\omega, t)x$ is unbounded as $t \to -\infty$.

We conclude as before that σ_Φ cannot be an interval, which contradicts the assumption that $\sigma_\Phi = [a, b]$. This completes the proof of Theorem 4.2 in the case $T = \mathbb{R}$.

There remains to prove Theorem 4.2 in the case when $T = \mathbb{Z}$. One can do this by following the steps of the above proof for $T = \mathbb{R}$. The proof when $T = \mathbb{Z}$ is actually somewhat simpler, since one need not effect a cohomology which transforms the cocycle Φ into the cocycle defined by a family of differential systems (1_ω). We omit the details. □

We finish the paper with a discussion of the case $d = 2$. We are able to strengthen Theorem 4.2 in the sense that we do not need the hypothesis of simple Oseledets spectrum. On the other hand, we need the convergence of the time averages which define the Lyapunov exponents at $t = -\infty$.

THEOREM 4.4. *Let $T = \mathbb{R}$ or $\mathbb{Z}$, and let $(\Omega, \{\tau_t\})$ be a minimal flow. Let Φ be a T-cocycle over $(\Omega, \{\tau_t\})$ with values in $GL(\mathbb{R}^2)$. Suppose that, for each $\omega \in \Omega$ and $0 \ne x \in \mathbb{R}^2$, the limits*

$$\lim_{t\to\infty} \frac{1}{t} \ln |\Phi(\omega, t)x|, \qquad \lim_{t\to-\infty} \frac{1}{t} \ln |\Phi(\omega, t)x|$$

both exist (they may or may not be equal). Then Φ has discrete spectrum.

Proof. We consider the case $T = \mathbb{R}$. There is no loss of generality in assuming that Hypotheses 4.3 $(a), (b)$ and (c) are satisfied. In particular the spectrum σ_Φ consists of a single interval; $\sigma_\Phi = [a, b]$ with $a \le b$.

Let us write equations (7_ω) in the form

$$x' = \begin{pmatrix} a_{11}(\tau_s(\omega)) & a_{12}(\tau_s(\omega)) \\ 0 & a_{22}(\tau_s(\omega)) \end{pmatrix} x,$$

where $x \in \mathbb{R}^2$. It follows from the hypothesis concerning the existence of the limits that, for each $\omega \in \Omega$, the limit $\lim_{t\to\infty} \frac{1}{t}\int_0^t a_{11}(\tau_s(\omega))ds$ exists. By [16], there is a real number $\bar{a}_1$ such that $\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t a_{11}(\tau_s(\omega))ds = \bar{a}_1$, where the limits are uniform in $\omega \in \Omega$.

It follows that $\bar{a}_1 = \int_\Omega a_{11} d\mu$ for each $\{\tau_t\}$-ergodic measure on Ω. Now, by [18] there is an ergodic measure μ_a on Ω such that a is an element of the μ_a-Oseledets spectrum. Similarly, there is an ergodic measure μ_b on Ω such that b is an element of the μ_b-Oseledets spectrum. Therefore $\bar{a}_1 \in \{a, b\}$.

Suppose first that $\bar{a}_1 = a$, and assume for contradiction that $b > a$. Then we can argue as in the proof of Theorem 4.2 to show that $\int_\Omega a_{22} d\mu = b$ for every ergodic measure μ on Ω and so $\frac{1}{t}\int_0^t a_{22}(\tau_s(\omega))ds = b$ uniformly in $\omega \in \Omega$. Again, arguing as in the proof of Theorem 4.2, one shows that, if $\lambda \in (a, b)$, then λ is not in σ_Φ. This is a contradiction, so $b = a$ and in fact Φ has discrete spectrum.

If $\bar{a}_1 = b$, then we use the hypothesis that $\lim_{t\to-\infty} \frac{1}{t} \ln|\Phi(\omega,t)x|$ exists for all $\omega \in \Omega$ and all $0 \neq x \in \mathbb{R}^2$. One assumes for contradiction that $a < b$, then repeats the steps of the proof of Theorem 4.2, using the negative-time Lyapunov exponents $\lim_{t\to-\infty} \frac{1}{t} \ln|\Phi(\omega,t)x|$ in place of the positive-time exponents $\lim_{t\to\infty} \frac{1}{t} \ln|\Phi(\omega,t)x|$. The end result is that, if $a < \lambda < b$. then $\lambda \notin \sigma_\Phi$. So one again concludes that σ_Φ is discrete.

$\square$

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