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A Note on Quasi-k-Spaces

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Summary. - *We prove that for a regular Hausdorff space X the following conditions are equivalent: (1) X is locally compact, (2) for each quasi-k-space Y, the product space $X \times Y$ is also a quasi-k-space.*

1. Introduction

Unless the contrary is explicitly stated, all topological spaces are assumed to be regular Hausdorff. Let Σ be a cover for a topological space X with topology τ. The family $\Sigma(\tau)$ of those subsets of X which intersect each $S \in \Sigma$ in an S-open set (i.e., open in S with the relative topology from τ) is a topology for X finer than τ. Now to each space X and to each cover Σ for X we may associate the space $\sigma(X)$, the same set of points topologized by $\Sigma(\tau)$. Let us call a space *a* Σ*-space* whenever $\sigma(X) = X$. If Σ is the cover of all countably compact (respectively, compact) subsets, Σ-spaces are called *quasi-k-spaces* (respectively, *k-spaces*).

The quasi-k-spaces and the k-spaces appear in several fields in General Topology and Functional Analysis. For instead, when studying compactness of function spaces in the topology of pointwise convergence [1] and in the theory of M-spaces introduced by K. Morita

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[4]. As it was showed by J. Nagata [6] a space X is a quasi-k-space (respectively, a k-space) if and only if X is a quotient space of a regular (respectively, paracompact) M-space.

In this note we are concerned with characterizing when a quasi-k-space satisfies that its product with every quasi-k-space is also a quasi-k-space. The similar question for k-spaces was solved by E. Michael [3]. He showed that $X \times Y$ is a k-space for every k-space Y if and only if X is locally compact. Our main result is to prove a similar one in the realm of quasi-k-spaces.

The terminology and notation are standard. If X is a topological space, Y is a set and $g : X \longrightarrow Y$ is an onto mapping, the strongest topology on Y making g continuous is called the *quotient topology* on Y. When Y is equipped with such a quotient topology, it is called a *quotient space* of X, and the inducing map g is called a *quotient map*. We denote by $\bigoplus_{\alpha\in A} X_\alpha$ *the disjoint topological sum* of a family $\{X_\alpha\}_{\alpha\in A}$ of topological spaces. A subset M of X is said to be quasi-k-closed (in X) provided that $M \cap K$ is closed in K for every countably compact subset K of X. Obviously, the definition of quasi-k-space can be reformulated in the following way: A space X is a quasi-k-space if every quasi-k-closed subset is closed. We remind the reader that a space X is locally countably compact if each point has a countably compact neighborhood. In the category of regular spaces each such space X has a base composed of countably compact neighborhoods at x for every $x \in X$. For terminology and notation not defined here and for general background see [2].

2. The results

First we shall prove a characterization of quasi-k-spaces that it will be used in the sequel (compare with [5], Theorem 1.2).

THEOREM 2.1. *Let X be a Hausdorff space. The following conditions are equivalent:*

1. *X is a quasi-k-space;*

2. *X is a quotient space of a disjoint topological sum of countably compact spaces;*

3. *X is a quotient space of a locally countably compact space.*

Proof. (1) $\Longrightarrow$ (2) Let $\mathcal{K}$ be the family defined as

$$\mathcal{K} = \{K \subset X \mid K \text{ is countably compact }\}$$

and consider the space $Y = \bigoplus_{K \in \mathcal{K}} K$. We shall prove that X is a quotient space of Y.

To see this, define the function φ from Y onto X by the requirement that $\varphi(x)$ be x whenever $x \in Y$. Beginning from the fact that X is a quasi-k-space , it is a routine matter to check that φ is a quotient map.

(2) $\Longrightarrow$ (3) It is clear.

(3) $\Longrightarrow$ (1) Let φ be a quotient map from a locally countably compact space Y onto X. Since φ is a quotient map, we need only show that $\varphi^{-1}(F)$ is closed in Y whenever F is quasi-k-closed in X. Suppose that there exists a quasi-k-closed subset (in X) F such that $\varphi^{-1}(F)$ is not closed in Y. We shall see that this leads us to a contradiction. Choose $y \in cl_Y \varphi^{-1}(F) \setminus \varphi^{-1}(F)$ and let V be a countably compact neighborhood of y in Y. φ being continuous, $\varphi(V)$ is countably compact and, consequently, $\varphi(V) \cap F$ is closed in $\varphi(V)$. On the other hand, as $\varphi(y) \notin F$, we can find an open set T such that $\varphi(y) \in T$ and

$$T \cap (\varphi(V) \cap F) = \emptyset.$$

Thus,

$$\varphi^{-1}(T) \cap \varphi^{-1}(\varphi(V) \cap F) = \emptyset. \qquad (\star)$$

But, as $y \in cl_Y \varphi^{-1}(F)$, there is $z \in Y$ satisfying

$$z \in \big(\varphi^{-1}(T) \cap V\big) \cap \varphi^{-1}(F).$$

So, $\varphi(z) \in \varphi(V) \cap F$. Therefore,

$$z \in \varphi^{-1}(T) \cap \varphi^{-1}(\varphi(V) \cap F).$$

This is contrary to condition $(\star)$ and the proof is complete. □

Given an ordinal number α, the symbol $W(\alpha)$ stands for the set of all ordinal numbers less than α. When viewed as a topological space this has the usual order topology. As usual, ω denotes the first infinite ordinal number. If X is a non locally compact space at a point x_0, E. Michael constructed in [3] a space $\mathcal{K}(X)$ associated with X in the following way: let $\{U_i\}_{i\in I}$ be a base of non-compact closed neighborhoods of the point x_0. For each $i \in I$, since U_i is not compact, there are a limit ordinal number $\eta(i)$ and a well-ordered (by inclusion) family $\left\{F_j^i\right\}_{j<\eta(i)}$ of closed subsets of U_i such that

$$\bigcap \left\{F_j^i \mid j < \eta(i)\right\} = \emptyset.$$

Consider now the k-space $Z = \bigoplus_{i\in I} W(\eta(i)+1)$. The Michael space $\mathcal{K}(X)$ is defined as the quotient space of Z obtained by identifying all points $\{\eta(i)\}_{i\in I}$ with a point y_0. Since a quotient space of a k-space is also a k-space ([2], Theorem 3.3.23), $\mathcal{K}(X)$ is a k-space (and, *a fortiori*, a quasi-k-space). We need the following important property of $\mathcal{K}(X)$.

THEOREM 2.2. *Let X be a non locally compact space at a point x_0. If K is a countably compact subset of $\mathcal{K}(X)$, then K meets at most finitely many elements of the family $\{W(\eta(i))\}_{i\in I}$.*

Proof. Let K be a subset of $\mathcal{K}(X)$ such that there exists a sequence $\{i_n\}_{n<\omega}$ in I such that K meets $W(\eta(i_n))$ for all $n < \omega$. We shall show that K is not countably compact.

Choose, for each $n < \omega$, an $\alpha_n \in K \cap W(\eta(i_n))$. We shall prove that the sequence $\{\alpha_n\}_{n<\omega}$ does not admit any cluster point in Y. For this in turn, it suffices to check that the point y_0 is not a cluster point of $\{\alpha_n\}_{n<\omega}$. As $\eta(i_n)$ is a limit ordinal, there exists an open set V_n for each $n<\omega$ such that

$$\alpha_n \notin V_n\ , \quad \eta(i_n) \in V_n.$$

Let D_n be the set defined as follows:

$$D_n = \{\lambda \in W(\eta(i_n)) \mid \lambda \in V_n\} \cup \{y_0\}$$

and consider the open neighborhood D of $\{y_0\}$,

$$D = \left(\bigcup_{n<\omega} D_n\right) \cup E$$

where $E = \bigcup\{W(\eta(i)) : i \neq i_n \text{ for every } n<\omega\}$. It is clear that D does not meet $\{\alpha_n\}_{n<\omega}$ as was to be proved. □

We determine next when a space X satisfies that $X \times Y$ is a quasi-k-space for each quasi-k-space Y. The following lemma is well-known; a proof can be extracted from [2], Corollary 3.10.14.

LEMMA 2.3. *The product space $X \times Y$ of a locally compact space X and a locally countably compact space Y is locally countably compact.*

THEOREM 2.4. *Let X be a regular Hausdorff space. The following assertions are equivalent:*

1. *X is locally compact;*
2. *If Y is a quasi-k-space , then so is $X \times Y$.*

Proof. (1) $\Longrightarrow$ (2) Let Y be a quasi-k-space . According to Theorem 2.1, we can find a locally countably compact space Z such that Y is a quotient space of Z. Let φ be a quotient map from Z onto Y. By [7], Lemma 4 the function f from $X \times Z$ onto $X \times Y$ defined as

$$f = id_X \times \varphi$$

(where id_X stands for the identity map on X) is a quotient map. As, by Lemma 2.3, the space $X \times Z$ is locally countably compact, the result holds by condition (3) in Theorem 2.1.

(2) $\Longrightarrow$ (1) Let X be a non locally compact space at x_0. We shall construct a quasi-k-space Y such that $X \times Y$ is not a quasi-k-space . To see this, let $\{U_i\}_{i\in I}$ be a base for closed neighborhoods of the point x_0 and consider, for each U_i, $i \in I$, a family $\{F^i_\alpha\}_{\alpha<\eta(i)}$ of nonempty closed sets of U_i satisfying the same conditions as in Michael's construction. Let $Y = \mathcal{K}(X)$ be the Michael space associated with this family. We shall prove that $X \times Y$ is not a quasi-k-space. For this end, given $i \in I$ and $\mu \in W(\eta(i))$, let M^i_μ be the closed set defined as $M^i_\mu = \bigcap_{\lambda<\mu} F^i_\mu$. Since the family $\{F^i_\alpha\}_{\alpha<\eta(i)}$ is well-ordered by inclusion, the set M^i_μ is nonempty. Now, for each $i \in I$, let $H_i = \bigcup_{\lambda<\eta(i)} \{M^i_\lambda \times \{\lambda\}\}$. Because $\bigcap_{\mu<\eta(i)} M^i_\mu = \emptyset$, it is easy to check that each H_i is a closed set. We shall complete the proof by showing that $H = \bigcup_{i\in I} H_i$ is a quasi-k-closed, non closed set in $X \times Y$. In fact, for each $i \in I$,

$$H \cap \{X \times (W(\eta(i)) \cup \{y_0\})\} = H_i,$$

and, by Theorem 2.2, H is quasi-k-closed. On the other hand, for each neighborhood $U \times V$ of (x_0, y_0), we can find $i \in I$ such that $U_i \subset U$ and, consequently, if $\mu \in V \cap W(\eta(i))$,

$$(U \times V) \cap H_i \neq \emptyset.$$

Thus, $(x_0, y_0) \in cl_{X\times Y} H \setminus H$ as was to be proved. $\square$

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