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Strong Artin-Rees Property in Rings of Dimension One and Two

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To Fabio Rossi, for his enthusiasm

Summary. - *Let* $(R, \mathbf{m})$ *be a local noetherian ring and let* $N \subseteq M$ *be two finitely generated* R*-modules such that the* $\dim M/N \leq 1$*. We give simple proof of the fact that there exists an integer* h *such that* $I^nM \cap N = I^{n-h}(I^hM \cap N)$*, for all* $n \geq h$ *and for all ideals* $I \subset R$*. We give upper bounds for such an integer* h*. Moreover, we give two examples of rings of dimension two where the property fails.*

1. Introduction

Let R be a noetherian ring, $I \subset R$ an ideal of R, and let $N \subseteq M$ be two finitely generated R-modules. By the Artin-Rees Lemma there exists an integer h depending on I, M and N such that

$$I^nM \cap N = I^{n-h}(I^hM \cap N), \quad \text{for all } n \geq h. \tag{1}$$

A weaker property is often used in the applications, namely

$$I^nM \cap N \subset I^{n-h}N, \quad \text{for all } n \geq h. \tag{2}$$

Much work has been done to determine whether h can be chosen uniformly, in the sense that (1) or (2) would be satisfied simultaneously for every ideal of a given family, see ([2], [3], [5], [4], [6], [10]).

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Following the definitions in [3], we say that the pair (N, M) has the *strong Artin-Rees property with respect to* $\mathcal{W}$ *with Artin-Rees number* h if (1) holds for all $I \in \mathcal{W}$. Notice that in this case, every integer bigger than h is an Artin-Rees number with respect to $\mathcal{W}$ for the pair (N, M). We denote by $\mathsf{ar}_R(N, M; \mathcal{W})$ the least of such integers.

When $\mathcal{W}$ is the family of all ideals, we say that the pair (N, M) has the *strong Artin-Rees property* and denote by $\mathsf{ar}_R(N, M)$ the least of the Artin-Rees numbers.

Planas-Vilanova [6] proves that any pair (N, M) with $\dim M/N \leq 1$ has the strong Artin-Rees property if R is an excellent ring. The proof comes down to the case of local rings. In this note we give a simpler proof of the strong Artin-Rees property over a one-dimensional local ring, with particular attention paid to upper bounds for $\mathsf{ar}_R(N, M)$. Such bounds find application in the study of other uniform Artin-Rees properties, see [7].

We also give an example of a family of ideals and two modules $N \subset M$ such that $\dim M/N = 2$ for which there exists no integer h such that (1) holds for all ideals in the family. If $\dim M/N = 3$ the strong Artin-Rees property was known to fail, see [10].

2. One-dimensional rings

In the rest of the paper $(R, \mathbf{m}, \mathsf{k})$ will denote a local noetherian ring with maximal ideal $\mathbf{m}$ and residue field k.

We first show that it is enough to study the strong Artin-Rees property with respect to the family of $\mathbf{m}$-primary ideals. For this, we first need a lemma.

LEMMA 2.1. *Let M be an R-module and let N_1, N_2 be two submodules of M. There exists an $h = h(N_1 + N_2 \subseteq M)$ such that*

$$N_1 \cap (N_2 + \mathbf{m}^n M) \subseteq (N_1 \cap N_2) + \mathbf{m}^{n-h} N_1,$$

for every $n \geq h$.

Proof. By the Artin-Rees Lemma there exists h such that for every $n \geq h$ the following holds:

$$\mathbf{m}^n M \cap (N_1 + N_2) = \mathbf{m}^{n-h}(\mathbf{m}^h M \cap (N_1 + N_2)) \subset \mathbf{m}^{n-h}(N_1 + N_2). \quad (3)$$

Then the following holds for $n \geq h$:

$$\begin{aligned} N_1 \cap (N_2 + \mathbf{m}^n M) &= N_1 \cap (N_2 + (\mathbf{m}^n M \cap (N_1 + N_2)) \\ &= N_1 \cap (N_2 + \mathbf{m}^{n-h}(\mathbf{m}^h M \cap (N_1 + N_2))) \\ &\subseteq N_1 \cap (N_2 + \mathbf{m}^{n-h}(N_1 + N_2)) \\ &\subseteq N_1 \cap (N_2 + \mathbf{m}^{n-h} N_1) \\ &\subseteq N_1 \cap N_2 + \mathbf{m}^{n-h} N_1. \end{aligned}$$ □

Remark 2.2. *Notice that if h is an integer that satisfies Lemma 2.1, then every bigger integer does as well.*

Proposition 2.3. *Let M be an R-module and $N \subset M$ a submodule. Let $\mathcal{W}$ be the family of $\mathbf{m}$-primary ideals. Assume that (N, M) has the strong uniform Artin-Rees property with respect to $\mathcal{W}$. Then $\mathsf{ar}_R(N, M) \leq \mathsf{ar}_R(N, M; \mathcal{W})$.*

Proof. Let $h_0 = \mathsf{ar}(N, M; \mathcal{W})$ and assume by contradiction that there exists $I \subset R$ and $n \geq h_0$ such that $I^{n-h_0}(I^{h_0} M \cap N) \neq I^n M \cap N$.

On the other hand, for all $h >> 0$ and for such a fixed n and h_0, the inclusions below hold. Inclusion (4) holds by the definition of h_0, inclusions (5) and (6) hold by expanding the powers of $(I + \mathbf{m}^h)$.

$$\begin{aligned} I^n M \cap N &\subseteq (I + \mathbf{m}^h)^n M \cap N, && (4) \\ &\subseteq (I + \mathbf{m}^h)^{n-h_0}((I + \mathbf{m}^h)^{h_0} M \cap N), && (5) \\ &\subseteq I^{n-h_0}((I + \mathbf{m}^h)^{h_0} M \cap N) + \mathbf{m}^h M, && (6) \\ &\subseteq I^{n-h_0}((I^{h_0} + \mathbf{m}^h) M \cap N) + \mathbf{m}^h M, && (7) \\ &= I^{n-h_0}((I^{h_0} M + \mathbf{m}^h M) \cap N) + \mathbf{m}^h M. && (8) \end{aligned}$$

Let h_1 be an integer depending on $(I^{h_0} M + N) \subseteq M$ that satisfies Lemma 2.1 with $N_1 = N$, $N_2 = I^{h_0} M$. By Remark 2.2, we may assume $h_1 \geq n - h_0$ and obtain

$$(I^{h_0} M + \mathbf{m}^h M) \cap N \subseteq (I^{h_0} M \cap N) + \mathbf{m}^{h-h_1} M,$$

for every $h > h_1$. So for $n \geq h_0$ and $h > h_1$, with $h_1 \geq n - h_0$,

$$\begin{aligned} & I^{n-h_0}((I^{h_0} M + \mathbf{m}^h M) \cap N) + \mathbf{m}^h M && (9) \\ &\subseteq I^{n-h_0}(I^{h_0} M \cap N + \mathbf{m}^{h-h_1} M) + \mathbf{m}^h M, && (10) \\ &\subseteq I^{n-h_0}(I^{h_0} M \cap N) + \mathbf{m}^{h-h_1+n-h_0} M + \mathbf{m}^h M, && (11) \\ &\subseteq I^{n-h_0}(I^{h_0} M \cap N) + \mathbf{m}^{h-h_1+n-h_0} M. && (12) \end{aligned}$$

Putting together the right and the left end of the chain of inclusions (3)–(11), we obtain that

$$I^n M \cap N \subseteq I^{n-h_0}(I^{h_0} M \cap N) + \mathbf{m}^{h-h_1+n-h_0} M,$$

for every $h > h_1$. By taking the intersection of the right side of the inclusion over all $h > h_1$, we can conclude $I^n M \cap N \subseteq I^{n-h_0}(I^{h_0} M \cap N)$. Since the reverse inclusion always holds, there is equality $I^{n-h_0}(I^{h_0} M \cap N) = I^n M \cap N$, which contradicts the assumption. □

We also need another kind of reduction, see for example [3, (2.4)].

Lemma 2.4. *Let $(R, \mathbf{m}, \mathsf{k})$ be a local noetherian ring. The extension $R \longrightarrow R[x]_{\mathbf{m}R[x]}$ is faithfully flat and $R[x]_{\mathbf{m}R[x]}$ has an infinite residue field.*

Let $R \to S$ be a faithfully flat extension. Let M be an R-module and $N \subset M$ a submodule. If $(N \otimes_R S, M \otimes_R S)$ has the strong uniform Artin-Rees property then $\mathsf{ar}_R(N \subseteq M) \leq \mathsf{ar}_S(N \otimes_R S \subseteq M \otimes_R S)$.

Proposition 2.5. *Suppose $(R, \mathbf{m}, \mathsf{k})$ is a one-dimensional local noetherian ring with infinite residue field. There exists an integer $r = r(R)$, such that for every $\mathbf{m}$-primary ideal I there exists $y \in I$ so that $I^n = yI^{n-1}$, for every $n \geq r$.*

Proof. First suppose that R is Cohen-Macaulay and let e be the multiplicity of the ring. By [8, Chapter 3, (1.1)], we have that $\mu(I) \leq e$, where $\mu(I)$ denotes the minimal number of generators of I and I is an arbitrary $\mathbf{m}$-primary ideal. Therefore, $\mu(I^e) \leq e < e+1$. Hence, by [8, Chapter 2, (2.3)], there exists $y \in I$ such that $I^e = yI^{e-1}$, so that for every $n \geq e$ we have $I^n = yI^{n-1}$. Set r to be e.

Next suppose $\operatorname{depth}(R) = 0$, and let $0 = q_1 \cap q_2 \cdots \cap q_{s+1}$ be a minimal primary decomposition of 0 where q_{s+1} is $\mathbf{m}$-primary and set $J = q_1 \cap q_2 \cdots \cap q_s$. Then R/J is Cohen-Macaulay and there exists a h_0 such that $\mathbf{m}^{h_0} J = 0$. Let e_1 be the multiplicity of R/J. Then, by the above case, there exists a $y \in I$ such that for every $n \geq e_1$ we have $I^n \subseteq yI^{n-1} + J$ and hence $I^n \subseteq yI^{n-1} + I^n \cap J$, for every $n > e_1$. By [3, (4.2)], there exists a h_1, depending just on R and J, such that for every $n \geq h_1$ and every ideal $I \subset R$ we have

$I^n \cap J \subset I^{n-h_1}J$. Hence, for every $n \geq r = \max\{e_1, h_0 + h_1\}$ one has the following inclusions:

$$\begin{aligned} I^n &\subseteq yI^{n-1} + I^n \\ &\subseteq yI^{n-1} + I^n \cap J \\ &\subseteq yI^{n-1} + I^{n-h_1}J \\ &\subseteq yI^{n-1} + \mathbf{m}^{h_0}J = yI^{n-1}. \square \end{aligned}$$

We are now ready to prove the main theorem. If M is a finite length module we denote by $\ell(M)$ its length.

Theorem 2.6. *Let (R, m, k) be a one-dimensional local ring. Then every pair (N, M), with $N \subset M$, has the strong uniform Artin-Rees property, and* $\mathsf{ar}_R(N, M) \leq \max\{r, \ell(\mathrm{H}^0_{\mathbf{m}}(M/N))\} + \ell(\mathrm{H}^0_{\mathbf{m}}(M/N))$, *where $r = r(R)$ is an integer as in Proposition 2.5.*

Proof. By Lemma 2.4, $\mathsf{ar}_R(N \subseteq M) \leq \mathsf{ar}_S(N \otimes_R S \subseteq M \otimes_R S)$, for any ring extension $R \to S$; thus we may assume that R has infinite residue field. Let I be an $\mathbf{m}$-primary ideal. Set $h_1 = \ell(\mathrm{H}^0_{\mathbf{m}}(M/N))$ and $h = \max\{r, \ell(\mathrm{H}^0_{\mathbf{m}}(M/N))\} + \ell(\mathrm{H}^0_{\mathbf{m}}(M/N))$.

Assume first that M/N is Cohen-Macaulay. By Proposition 2.5 we can choose $y \in I$ such that y is a non-zerodivisor in M/N, such that for $n > h = r$,

$$\begin{aligned} I^n M \cap N &= yI^{n-1}M \cap N, \\ &\subseteq y(I^{n-1}M \cap N), \quad \text{by the property of } y, \\ &\subseteq I(I^{n-1}M \cap N), \quad \text{since } y \in I. \end{aligned}$$

Now suppose that M/N is not Cohen-Macaulay and let $M'/N = H^0_m(M/N)$. For every $n \geq h$ and every $I \subseteq R$ we have:

$$\begin{aligned} I^n M \cap N &= I^n M \cap M' \cap N, \text{ since } N \subset M', \\ &= I^{n-r}(I^r M \cap M') \cap N, \text{ since } M/M' \text{ is Cohen-Macaulay}, \\ &\subseteq I^{n-r}(I^r M \cap M'), \\ &= I^{n-r-h_1} I^{h_1}(I^r M \cap M'), \text{ since } n - r \geq h_1, \\ &= I^{n-r-h_1}(I^{h_1}(I^r M \cap M') \cap N), \text{ since } I^{h_1}M' \subset N, \\ &\subseteq I^{n-r-h_1}(I^{r+h_1}M \cap M' \cap N), \\ &\subseteq I^{n-h}(I^h M \cap N), \end{aligned}$$

where the last containment follows since $r + h_1 \leq h$ and in general $I^{n'}(I^{h'}M \cap N) \subseteq I^{n'-s}(I^{s+h'}M \cap N)$ for all n', h', and $s \leq n'$. $\square$

3. Relation Type

Let $I = (x_1, \ldots, x_n)$ be an ideal in R. Map the polynomial ring, with the standard grading, $R[X_1, \ldots, X_n]$ onto the Rees algebra $R[It]$ by sending X_i to $x_i t$. Let L be the kernel of this map. Then L is an homogeneous ideal and the relation type of I is defined to be the minimum integer h such that the ideal L can be generated by elements of degree less than or equal to h. It is denoted by reltype(I). This number does not depend on the choice of the minimal generators of the ideal I.

Let I be an $\mathbf{m}$-primary ideal in a Cohen-Macaulay local ring, it holds that reltype$(I) \leq e$, where e is the multiplicity of R, see [1].

The following lemma had been proved by Wang in [9] for parameters ideals. The same argument applies for every ideal, we include it here for simplicity.

Lemma 3.1. *Let* $(R, \mathbf{m}, \mathsf{k})$ *be a local ring and* J *be an ideal of* R*; let* $\bar{R}$ *denote* R/J*. Let* $I = (x_1, \ldots, x_m)$ *be an ideal of* R *and suppose that* reltype$(I\bar{R}) \leq h$*, for some* $h > 0$*. Then for every* $n > h$*,*

$$I^n \cap J = I^{n-h}(I^h \cap J).$$

Proof. Let $n \geq h$ and let $x \in I^n \cap J$. Then there exists a polynomial F in $R[X_1, \ldots, X_m]$, homogeneous of degree n, such that $F(x_1, \ldots, x_m) = x$. Modulo J, $\bar{F}$ is a relation on the $\bar{x}_i$'s, so by hypothesis there are polynomials G_i of degree h, and H_i, of degree $n-h$, such that $\bar{F} = \sum \bar{G}_i \bar{H}_i$ in $\bar{R}[X_1, \ldots, X_m]$ and $\bar{G}_i$ are relations on the $\bar{x}_i$. Therefore, $F = \sum G_i H_i + K$ for some $K \in R[X_1, \ldots, X_m]$ of degree n and coefficients in J. Since:

$$\begin{aligned} K(x_1, \ldots, x_m) &\in JI^n \subset I^{n-h}(I^h \cap J), \\ G_i(x_1, \ldots, x_m) &\in I^h \cap J, \quad \text{and} \\ H_i(x_1, \ldots, x_m) &\in I^{n-h}, \end{aligned}$$

this shows that the element $x = F(x_1, \ldots, x_m)$ is in $I^{n-h}(I^h \cap J)$. $\square$

LEMMA 3.2. *Let* $(R, \mathbf{m}, \mathsf{k})$ *a noetherian local ring. If* J *is an ideal of* R *such that* $\dim(R/J) \leq 1$ *then* (J, R) *has the strong Artin-Rees property.*

Proof. If $\dim(R/J) = 0$ then there exists a power of the maximal ideal $\mathbf{m}^h \subset J$. Therefore, for $n > h$ and for every ideal I we have the following:

$$I^n \cap J = I^n = II^{n-1} = I(I^{n-1} \cap J).$$

Assume $\dim(R/J) = 1$. By Lemma 2.1 it is enough to show that (J, R) has the strong Artin-Rees property with respect to the family of $\mathbf{m}$-primary ideals. Suppose that R/J is Cohen-Macaulay; then the conclusion holds by section 3 and by Lemma 3.1.

Suppose R/J has dimension one and it is not Cohen-Macaulay. Let $J \subset J'$ such that R/J' is Cohen-Macaulay and let h_0 such that $\mathbf{m}^{h_0} J' \subset J$. By the Cohen-Macaulay case there exists an Artin-Rees number $h_1 = h_1(J' \subset R)$. We may assume $h_1 > h_0$. Let $h = h_1 + h_0$.For every $n \geq h$, the inequalities below follow.

$$\begin{aligned}
I^n \cap J &= I^n \cap J' \cap J, && \text{since } J \subseteq J', \\
&= I^{n-h_1}(I^{h_1} \cap J') \cap J, && \text{by definition of } h_1, \\
&\subseteq I^{n-h_1}(I^{h_1} \cap J') \\
&= I^{n-h_1-h_0} I^{h_0}(I^{h_1} \cap J') \\
&= I^{n-h_1-h_0}(I^{h_0}(I^{h_1} \cap J') \cap J), && \text{since } I^{h_0} J' \subseteq J, \\
&\subseteq I^{n-h}(I^h \cap J' \cap J) \\
&= I^{n-h}(I^h \cap J). && \square
\end{aligned}$$

PROPOSITION 3.3. *Let* $(R, \mathbf{m}, \mathsf{k})$ *be a local noetherian ring. Let* M *be an* R*-module and* $N \subseteq M$ *a submodule. Let* $J \subset \operatorname{ann}(M/N)$ *be an ideal of* R*. If* (J, R) *and* $(N/JM, M/JM)$ *have the strong uniform Artin-Rees property, then*

$$\mathsf{ar}_R(N, M) \leq \max\{\mathsf{ar}_R(J, R), \mathsf{ar}_{R/J}(N/JM, M/JM)\}.$$

In particular, if $\dim(M/N) = 1$ *then* $\mathsf{ar}_R(N, M)$ *is bounded above by*

$$\max\{\mathsf{ar}_R(J, R), \max\{r(R/J), \ell(\mathrm{H}^0_{\mathbf{m}}(M/N))\} + \ell(\mathrm{H}^0_{\mathbf{m}}(M/N))\}.$$

Proof. The second statement follows from the first and Theorem 2.6. For the first part, let $h = \max\{\mathsf{ar}_R(J,R), \mathsf{ar}_{R/J}(N/JM, M/JM)\}$. Let $\phi : R^m \to M$, a surjection of a free module onto M. Denote by $K = \ker(\phi)$ and by $L = \phi^{-1}(N)$, the pre-image of the submodule $N \subset M$. Then, as shown in [2], it is enough to show that there exists a h such that for every $n \geq h$ and for every ideal $I \subset R$, we have $I^n R^m \cap L = I^{n-h}(I^h R^m \cap L)$. Therefore, without loss of generality we may assume M is a free module.

Since $h \geq \mathsf{ar}_{R/J}(N/JM, M/JM)$, for every $n \geq h$ and for every ideal I, we have $I^n M \cap N \subset I^{n-h}(I^h M \cap N) + JM$. Therefore,

$$I^n M \cap N \subset I^{n-h}(I^h M \cap N) + JM \cap I^n M = I^{n-h}(I^h M \cap N) + (I^n \cap J)M,$$

where the last equality holds since M is a free module. Since $h \geq \mathsf{ar}_R(J,R)$, we have $I^n \cap J = I^{n-h}(I^h \cap J)$. Hence,

$$\begin{aligned} I^n M \cap N &= I^{n-h}(I^h M \cap N) + I^{n-h}(I^h \cap J)M \\ &= I^{n-h}(I^h M \cap N) + I^{n-h}(I^h M \cap JM) \\ &\subset I^{n-h}(I^h M \cap N), \quad \text{since } JM \subseteq N. \end{aligned}$$ □

4. Two-dimensional rings

The following example (see [10]), shows that the uniform Artin-Rees property does not hold if $\dim M/N = 2$.

Example. Let $R = \mathsf{k}[x,y,z]/(z^2)$. Consider the following family of ideals in R:

$$I_n = (x^n, y^n, x^{n-1}y + z),$$

for every $n \in \mathbb{N}$. Let J the ideal generated by z.

We want to show that $I_n(I_n^{n-1} \cap J) \neq I_n^n \cap J$, for every $n \geq 2$. In particular we will show that

$$x^{(n-1)^2} y^{n-1} z \in I_n^n \cap J \quad \text{but} \quad x^{(n-1)^2} y^{n-1} z \notin I_n(I_n^{n-1} \cap J).$$

Denote $x^{(n-1)^2} y^{n-1} z$ by ξ.

The ideal I_n is a homogeneous ideal if we assign degree one to x and y and degree n to z. With such a grading, ξ has degree $(n-1)^2+$

$n-1+n=n^2$. Since $x^{(n-1)^2}y^{n-1}z=(x^{n-1}y+z)^n-(x^n)^{n-1}y^n\in I_n^n$ the first claim holds.

Suppose $x^{(n-1)^2}y^{n-1}z\in I_n(I_n^{n-1}\cap J)$. This remains true modulo $(x^{(n-1)^2+1},y^n)$. The ideal I_n^{n-1} modulo $(x^{(n-1)^2+1},y^n)R$ is generated by

$$\{x^{n(n-1-i)}(x^{n-1}y+z)^i\mid i=0,1,\dots,n-1\}.$$

Moreover,

$$\begin{aligned} x^{n(n-1-i)}(x^{n-1}y+z)^i &= x^{n(n-1-i)}(x^{(n-1)i}y^i+x^{(n-1)(i-1)}y^{i-1}z)\\ &= x^{n^2-n-i}y^i+x^{n^2-2n-i+1}y^{i-1}z. \end{aligned}$$

But $n^2-n-i\geq(n-1)^2+1$ for $i\leq n-2$. Therefore, I_{n-1}^n modulo $(x^{(n-1)^2+1},y^n)$ is generated by

$$\{x^{(n-1)^2}y^{n-1}+x^{(n-1)(n-2)}y^{(n-2)}z,\ x^{n^2-2n-i+1}y^{i-1}z\mid i=1,\dots,n-2\}.$$

Let

$$\begin{aligned} f&=x^{(n-1)^2}y^{n-1}+x^{(n-1)(n-2)}y^{(n-2)}z,\\ g_i&=x^{n^2-2n-i+1}y^{i-1}z. \end{aligned}$$

Let $hf+\sum h_ig_i$ be a homogeneous element of $I_n^{n-1}\cap J$ that appears in the expression of ξ as element of $I_n(I_n^{n-1}\cap J)$. By degree reasons we can assume h is not a constant polynomial.

Let $m(x,y,z)$ be a homogeneous monomial of h. If z does not divide m, then

$$\begin{aligned} &m(x,y,z)f=m'(x,y)x^{(n-1)(n-2)+1}y^{(n-2)}z\\ \text{or}\quad &m(x,y,z)f=m'(x,y)x^{(n-1)(n-2)}y^{(n-2)+1}z; \end{aligned}$$

if z does divide m then $m(x,y,z)f=m'(x,y)x^{(n-1)^2}y^{n-1}z$, with m' possibly a unit. By a degree counting we can see that $\deg(hf)\geq n^2-n+1$. Therefore, for every element $a\in I_{n-1}$ we have $\deg(ahf)>n^2=\deg(\xi)$. This shows a contradiction.

The following example refines the above in that now R is a reduced ring.

Example. Let $R = \mathsf{k}[x,y,z]/xz$. Consider the following family of ideals:

$$I_n = (x^n, y^n, x^{n-1}y + z^n),$$

for every $n \in \mathbb{N}$. Let $J = (z)$. Again, we claim that $I_n(I_n^{n-1} \cap J) \neq I_n^n \cap J$ for every $n \geq 1$. We will show that

$$z^{n^2} \in I_n^n \cap J \quad \text{but} \quad z^{n^2} \notin I_n(I_n^{n-1} \cap J).$$

Indeed, $z^{n^2} = (x^{n-1}y + z^n)^n - (x^n)^{n-1}y^n \in I_n^n$ and trivially $z^{n^2} \in J$.

On the other hand I_n^{n-1} is generated by:

$$\{x^{n(n-1)}, x^{(n-1)^2}y^{n-1} + z^{n(n-1)}, \\ y^n L, x^{(n-1)^2+i}y^{n-1-i} \mid i = 1, \dots n-1\},$$

for some ideal L in R. Notice that if $z^{n^2} \in I_n(I_n^{n-1} \cap J)$ then this also holds modulo y^n. Moreover, if a homogeneous element

$$f(x,y)x^{n(n-1)} + g(x,y,z)(x^{(n-1)^2}y^{n-1} + z^{n(n-1)}) \\ + \sum_{i=1}^{n-1} h_i(x,y)x^{(n-1)^2+i}y^{n-1-i}$$

is in J, writing $g(x,y,z) = g''(x,y) + zg'(x,y,z)$, we see that

$$f(x,y)x^{n(n-1)} + g''(x,y)x^{(n-1)^2}y^{n-1} + \sum h_i(x,y)x^{(n-1)^2+i}y^{n-1-i} = 0.$$

But if this is the case, since $xz = 0$ in R, we have

$$fx^{n(n-1)} + g(x^{(n-1)^2}y^{n-1} + z^{n(n-1)}) + \sum h_i x^{(n-1)^2+i} = zg'z^{n(n-1)}. \tag{13}$$

But $zg'z^{n(n-1)}$ is an homogeneous element of degree at least $n^2 - n + 1$ and multiplication by any element in I_n increases the degree by n. Therefore, any element in $I_n(I_n^n \cap J)$ has degree at least $n^2 + 1$ while z^{n^2} has degree strictly smaller.

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