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A Fixed Point Theorem for Fuzzy Contraction Mappings

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Summary. - *In this paper, we give a fixed point theorem for fuzzy contraction mappings in quasi-pseudo-metric spaces which is a generalization of the corresponding one for metric spaces given by S. Heilpern.*

1. Introduction

S. Heilpern [1] introduced the concept of a fuzzy mapping, i.e., mapping from an arbitrary set to a certain subfamily of fuzzy sets in a metric linear space X. He proved a fixed point theorem for fuzzy contraction mappings which is a generalization of the fixed point theorem for multivalued mappings of Nadler [3] arising from the set-representation of fuzzy sets [4]. In this paper we extend the result of Heilpern to quasi- pseudo-metric spaces which are left K-sequentially complete.

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2. Preliminaries

The set of positive integers is denoted by $\mathbb{N}$. Recall that a nonnegative real valued function d defined on a nonempty set X is said to be a quasi-pseudo-metric provided it satisfies the following properties: for every $x,\ y,\ z\ \in X$,
$d(x,z) \leq d(x,y) + d(y,z)$
$d(x,x) = 0$.
The set $B_\varepsilon(x) = \{y \in X : d(x,y) < \varepsilon\}$ is the d-ball with centre x and radius $\varepsilon > 0$. The topology $\mathcal{T}(d)$, having as a base the family of all d-balls $B_\varepsilon(x)$ with $x \in X$ and $\varepsilon > 0$, is the topology on X induced by d. (X, d) is called a quasi-pseudo-metric space, if d is a quasi-pseudo-metric on X and we will suppose it is endowed with the topology $\mathcal{T}(d)$, in the following.

If d is a quasi-pseudo-metric on X, then d^{-1}, defined by $d^{-1}(x,y) = d(y,x)$ whenever $x, y \in X$, is also a quasi-pseudo-metric on X. We will denote B_ε^{-1} the d^{-1}-ball with centre x and radius $\varepsilon > 0$. Only if confusion is possible we write d-closed or d^{-1}-closed, for example, to distinguish the topological concept in (X, d) or (X, d^{-1}). We will denote $\min(d, d^{-1})$ by $d \wedge d^{-1}$. We will make use of the following notion, which has been studied by various authors under different names (see e.g. [2], [5]).

A sequence (x_n) in a quasi-pseudo-metric space (X, d) is called left K-Cauchy if for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $d(x_r, x_s) < \varepsilon$ for all $r, s \in \mathbb{N}$ with $k \leq r \leq s$. A quasi-pseudo-metric space (X, d) is said to be left K-sequentially complete if each left K-Cauchy sequence in X converges (with respect to the topology $\mathcal{T}(d)$).

Let x be a point in X and A a nonempty subset of X. We define the distance $d(x, A)$ from x to A by

$$d(x, A) = \inf\{d(x,a) : a \in A\}.$$

Thus $d(x, A) = 0$ iff $x \in clA$, the closure of A in X.

Now let A and B be nonempty subsets of X. We define the distance $d(A, B)$ from A to B by

$$d(A, B) = \inf\{d(x,y) : x \in A,\ y \in B\}$$

and clearly $d(A,B) \neq d(B,A)$ in general. Now, we define the *Hausdorff separation* of A from B by

$$d_H(A,B) = \sup\{d(a,B) : a \in A\}.$$

Thus we have $d_H(A,B) \geq 0$ with $d_H(A,B) = 0$ iff $A \subset clB$. In addition, the triangle inequality

$$d_H(A,C) \leq d_H(A,B) + d_H(B,C)$$

holds for all nonempty subsets A, B and C of X. In general, however $d_H(A,B) \neq d_H(B,A)$.

We define the *Hausdorff distance*, deduced from the quasi-pseudometric d, between nonempty subsets A and B of X by

$$H(A,B) = \max\{d_H(A,B),\ d_H(B,A)\}.$$

This is now symmetric in A and B. Consequently, $H(A,B) \geq 0$ with $H(A,B) = 0$ iff $clA = clB$, $H(A,B) = H(B,A)$ and $H(A,C) \leq H(A,B) + H(B,C)$ for any nonempty subsets A, B and C of X. When d is a metric on X, clearly H is the usual Hausdorff distance.

REMARK 2.1. *Given a quasi-pseudometric* $d : X \times X \longrightarrow R^+$ *let* $\rho = \max\{d, d^{-1}\}$. *Then* ρ *is obviously a pseudometric on* X. *Moreover, it is easy to notice that the Hausdorff distance* $H(A,B) = H_d(A,B)$ *determined by* d *and the Hausdorff distance* $H_\rho(A,B)$ *determined by the pseudometric* ρ *coincide.*

A fuzzy set on X is an element of I^X where $I = [0,1]$. The $\alpha - level$ set A_α of a fuzzy set A on X is defined as

$$A_\alpha = \{x \in X : A(x) \geq \alpha\} \text{ for each } \alpha \in]0,1],$$
$$A_0 = cl(\{x \in X : A(x) > 0\}).$$

For $x \in X$ we denote by $\{x\}$ the characteristic function of the ordinary subset $\{x\}$ of X.

DEFINITION 2.2. *Let* (X,d) *be a quasi-pseudo-metric space. We define the family of fuzzy sets on* X, $W^*(X)$, *as follows:*

$$W^*(X) = \{A \in I^X : A_1 \text{ is nonempty } d-\text{closed and } d^{-1}-\text{compact}\}.$$

For a metric linear space (X,d), in [1] it is defined the family $W(X)$ of fuzzy sets on X, as follows, $A \in W(X)$ iff A_α is compact and convex in X for each $\alpha \in [0,1]$ and $\sup_{x\in X} A(x) = 1$. Clearly, A_α is closed for $\alpha \in [0,1]$ and it is easy to verify that A_1 is nonempty. Then, in a metric linear space (X,d) we have the following inclusions: $W(X) \subset W^*(X) \subset I^X$.

For working with a similar notation to [1] we introduce the next definition.

DEFINITION 2.3. . *Let (X,d) be a quasi-pseudo-metric space and let $A,\ B \in W^*(X),\ \alpha \in [0,1]$. Then we define:*

$$p_\alpha(A,B) = \inf\{d(x,y) : x \in A_\alpha,\ y \in B_\alpha\} = d(A_\alpha, B_\alpha)$$

$$D_\alpha(A,B) = H(A_\alpha, B_\alpha)$$

where H is the Hausdorff distance deduced from the quasi-pseudo-metric d on X;

$$D(A,B) = \sup_\alpha D_\alpha(A,B)$$

Notice that p_α is non-decreasing function of α, and then $p_1(A,B) = d(A_1,B_1)$.

The following definition is more general than the one given in [1].

DEFINITION 2.4. . *Let X and Y be an arbitrary set and a quasi-pseudo-metric space, respectively. F is said to be a fuzzy mapping if F is a mapping from the set X into $W^*(Y)$.*

DEFINITION 2.5. . *Let $A,\ B \in I^X$. As usual in fuzzy theory, we denote $A \subset B$ when $A(x) \leq B(x)$, for each $x \in X$. We say x is a* **fixed point** *of the mapping $F : X \longrightarrow I^X$, if $\{x\} \subset F(x)$.*

We will use the following three lemmas, whose proofs we omit, given for a quasi-pseudo-metric space (X,d). They were given in metric version (the first one modified) by Heilpern [1], for the family $W(X)$.

LEMMA 2.6. . *Let $x \in X$ and $A \in W^*(X)$. Then $\{x\} \subset A$ if and only if $p_1(x,A) = 0$.*

LEMMA 2.7. . *$p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$, for any $x, y \in X$, $A \in W^*(X)$.*

LEMMA 2.8. . *If $\{x_0\} \subset A$ then $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ for each $A, B \in W^*(X)$.*

We will need the following lemma.

LEMMA 2.9. . *Suppose $K \neq \emptyset$ is compact in the quasi-pseudo-metric space (X, d^{-1}). If $z \in X$, then there exists $k_0 \in K$ such that $d(z, K) = d(z, k_0)$.*

Proof. Let A be a nonempty subset of X. From $d(z, x) \leq d(z, y) + d(y, x)$ whenever $x, y, z \in X$, we conclude, taking the infimum of the last expresion for $z \in A$, that

$$d(A, x) \leq d(A, y) + d(y, x) \tag{1}$$

We will see that $d(A, x)$ is a d^{-1}-lower-semicontinous (lsc) function of X. Let $x_0 \in X$ and $\varepsilon > 0$. By (1) we have $d(A, y) \geq d(A, x_0) - d(y, x_0)$ and then for $y \in B_\varepsilon^{-1}(x_0)$ we have $d(A, y) > d(A, x_0) - \varepsilon$ and so $d(A, x)$ is a d^{-1}-lsc function.

In particular if A is the one-point set $\{z\}$, the function $d(z, k)$ is a d^{-1}-lsc function of $k \in K$, and since K is d^{-1}-compact then there exists $k_0 \in K$ such that $d(z, k_0) = \min\{d(z, k) : k \in K\}$, i.e, $d(z, k_0) = d(z, K)$.

□

3. Fixed point theorem

Now, we prove a fixed point theorem for fuzzy contraction mappings in quasi-pseudo-metric spaces.

THEOREM 3.1. *Let (X, d) be a left K-sequentially complete quasi-pseudo-metric space, and F be a fuzzy mapping from X to $W^*(X)$ satisfying the following condition: there exists $q \in]0, 1[$, such that*

$$D(F(x), F(y)) \leq q\,(d \wedge d^{-1})(x, y) \text{ for each } x, y \in X.$$

Then there exists $x^* \in X$ such that $\{x^*\} \subset F(x^*)$.

Proof. Let $x_0 \in X$ and $\{x_1\} \subset F(x_0)$. By Lemma 2.9 there exists $x_2 \in X$ such that $\{x_2\} \subset F(x_1)$ and $d(x_1, x_2) \leq d(x_1, (F_(x_1))_1)$ since $(F(x_1))_1$ is d^{-1}- compact. We have

$$d(x_1\, ,\ x_2) \leq d(x_1, (F(x_1))_1) \leq H(x_1, (F(x_1))_1) \leq D(F(x_0), F(x_1)).$$

Continuing in this way we produce a sequence (x_n) in X such that $\{x_n\} \subset F(x_{n-1})$ and $d(x_n, x_{n+1}) \leq D(F(x_{n-1}), F(x_n))$ for each $n \in \mathbb{N}$. We will prove that (x_n) is a left K-Cauchy sequence.

$$d(x_1, x_2) \leq D(F(x_0), F(x_1)) \leq q\,(d \wedge d^{-1})(x_0, x_1) \leq q\, d(x_0, x_1)$$

and

$$\begin{aligned} d(x_k, x_{k+1}) &\leq D(F(x_{k-1}), F(x_k)) \leq q\,(d \wedge d^{-1})(x_{k-1}, x_k) \\ &\leq q\, d(x_{k-1}, x_k) \leq q^k\, d(x_0, x_1), \text{ for } k = 0, 1, 2 \ldots \end{aligned}$$

For $n < m$ we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=0}^{m-n-1} d(x_{n+i}, x_{n+i+1}) \leq \sum_{i=n}^{m-1} q^i\, d(x_0, x_1) \\ &\leq \frac{q^n}{1-q}\, d(x_0, x_1) \end{aligned}$$

whenever $q \in]0, 1[$ and then (x_n) is a left K-Cauchy sequence, since q^n converges to 0 as $k \to \infty$. Then, since X is left K-sequentially complete in X, there exists $x^* \in X$ such that $\lim_n x_n = x^*$.

Now, by Lemma 2.7

$$p_1(x^*, F(x^*)) \leq d(x^*, x_n) + p_1(x_n, F(x^*))$$

.

Then by Lemma 2.8 (compare with [1]):

$$\begin{aligned} p_1(x^*, F(x^*)) &\leq d(x^*, x_n) + D_1(x_n, F(x^*)) \\ &\leq d(x^*, x_n) + D(F(x_{n-1}), F(x^*)) \\ &\leq d(x^*, x_n) + q\,(d \wedge d^{-1})(x_{n-1}, x^*) \\ &\leq d(x^*, x_n) + q\, d(x^*, x_{n-1}). \end{aligned}$$

Now, $d(x^*, x_n)$ and $d(x^*, x_{n-1})$ converge to 0 as $n \to \infty$. Hence, by Lemma 2.6 we conclude that $\{x^*\} \subset F(x^*)$. □

When d is a complete metric on X, we get the following result of Heilpern [1]

COROLLARY 3.2. *Let X be a complete metric linear space and F be a fuzzy mapping from X to $W(X)$ satisfying the following condition: there exists $q \in]0,1[$ such that*

$$D(F(x), F(y)) \leq qd(x,y) \text{ for each } x, y \in X.$$

Then, there exists $x^* \in X$ such that $\{x^*\} \subset F(x^*)$.

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REFERENCES

[1] S. HEILPERN, *Fuzzy mappings and fixed point theorem*, J. Math. Anal. Appl. **83** (1981), 566–569.
[2] J.C. KELLY, *Bitopological spaces*, Proc. London Math. Soc. **13** (1963), 71–89.
[3] S. B. NADLER, *Multivalued contraction mappings*, Pacific J. Math. **30** (1969), 475–488.
[4] H.T. NGUYEN, *A note on the extension principle for fuzzy sets*, J. Math Anal. Appl. **64** (1978), 369–380.
[5] I.L. REILLY, P.V. SUBRAHMANYAM, AND M.K. VAMANAMURTHY, *Cauchy sequences in quasi-pseudometric spaces*, Mh. Math. **93** (1982), 127–140.

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