

# Verbal functions of a group

DANIELE TOLLER

**ABSTRACT.** *The aim of this paper is the study of elementary algebraic subsets of a group  $G$ , first defined by Markov in 1944 as the solution-set of a one-variable equation over  $G$ . We introduce the group of words over  $G$ , and the notion of verbal function of  $G$  in order to better describe the family of elementary algebraic subsets. The intersections of finite unions of elementary algebraic subsets are called algebraic subsets of  $G$ , and form the family of closed sets of the Zariski topology  $\mathfrak{Z}_G$  on  $G$ . Considering only some elementary algebraic subsets, one can similarly introduce easier-to-deal-with topologies  $\mathfrak{T} \subseteq \mathfrak{Z}_G$ , that nicely approximate  $\mathfrak{Z}_G$  and often coincide with it.*

**Keywords:** group of words, universal word, verbal function, (elementary, additively) algebraic subset, (partial) Zariski topology, centralizer topology, quasi-topological group topology.

**MS Classification 2010:** primary 54E15, 20F70; secondary: 57M07, 20K45.

## 1. Introduction

In 1944, Markov [20] introduced three special families of subsets of a group  $G$ , calling a subset  $X \subseteq G$ :

- (a) *elementary algebraic* if there exist an integer  $n > 0$ , elements  $g_1, \dots, g_n \in G$  and  $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ , such that

$$X = \{x \in G : g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} = e_G\}; \quad (1)$$

- (b) *additively algebraic* if  $X$  is a finite union of elementary algebraic subsets of  $G$ ;
- (c) *algebraic* if  $X$  is an intersection of additively algebraic subsets of  $G$ .

If  $G$  is a group, take  $x$  as a symbol for a variable, and denote by  $G[x] = G * \langle x \rangle$  the free product of  $G$  and the infinite cyclic group  $\langle x \rangle$  generated by  $x$ . We call  $G[x]$  the *group of words with coefficients in  $G$* , and its elements  $w$  are called *words in  $G$* . An element  $w \in G[x]$  has the form

$$w = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n}, \quad (2)$$

for an integer  $n \geq 0$ , elements  $g_1, \dots, g_n \in G$  and  $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ . The group  $G[x]$  is defined also via a universal property in Fact 2.1, and explicitly described in §2.2. Then an elementary algebraic subset  $X$  of  $G$  as in (1) will be denoted by  $E_w^G$  (or simply,  $E_w$ ), where  $w \in G[x]$  as in (2) is its defining word considered as an element of  $G[x]$ .

In particular, a word  $w \in G[x]$  determines its associated evaluation function  $f_w: G \rightarrow G$ , mapping  $g \mapsto w(g)$ , where  $w(g) \in G$  is obtained replacing  $x$  with  $g$  in (2) and taking products in  $G$ . We call *verbal* a function  $G \rightarrow G$  of the form  $f_w$ . Let us immediately show that some very natural functions  $G \rightarrow G$  are verbal.

EXAMPLE 1.1. 1. If  $g \in G$ , then one can consider the word  $w = g \in G[x]$ , so that  $f_w$  is the constant function  $g$  on  $G$ .

2. The identity map of  $G$  is the function  $f_x: g \mapsto g$ .

3. The inversion function of  $G$  is  $f_{x^{-1}}: g \mapsto g^{-1}$ .

4. More generally, for every integer  $n \in \mathbb{Z}$ , the word  $x^n \in G[x]$  determines the verbal function  $f_{x^n}: g \mapsto g^n$ .

5. The left translation in  $G$  by an element  $a \in G$  is the function  $f_{ax}: g \mapsto ag$ , and the right translation is the function  $f_{xa}: g \mapsto ga$ .

6. For an element  $a \in G$ , the word  $w = axa^{-1}$  determines the conjugation by  $a$ , as  $f_w: g \mapsto aga^{-1}$ .

In §3.2 we equip the set  $\mathcal{F}(G)$  of verbal functions of  $G$  with the pointwise product operation, making  $\mathcal{F}(G)$  a group. The surjection  $G[x] \rightarrow \mathcal{F}(G)$ , mapping  $w \mapsto f_w$ , shows that  $\mathcal{F}(G)$  is isomorphic to a quotient of  $G[x]$ .

We dedicate §3.3 to *monomials*, namely words of the form  $w = gx^m \in G[x]$ , for  $g \in G$  and  $m \in \mathbb{Z}$ . In the final §3.4 we consider abelian groups. We note that if  $G$  is abelian, and  $f \in \mathcal{F}(G)$ , then  $f = f_w$  for a monomial  $w \in G[x]$ . Then we describe  $\mathcal{F}(G)$  in Proposition 3.7.

In §4 we study the elementary algebraic subsets, which we redefine using verbal functions (Definition 4.1) as the subsets of the form

$$E_w = f_w^{-1}(\{e_G\}),$$

for  $w \in G[x]$ . In this sense one can consider  $E_w$  as the solution-set of the equation  $w(x) = e_G$  in  $G$ .

EXAMPLE 1.2. 1. For an element  $g \in G$ , let  $w = g^{-1}x \in G[x]$ . Then  $f_w: G \rightarrow G$  is the left translation by  $g^{-1}$  by Example 1.1, item 5, and  $E_w = \{g\}$ .

This shows that every singleton is an elementary algebraic subset of  $G$ .

2. If  $g \in G$ , then the centralizer

$$C_G(g) = \{h \in G \mid gh = hg\}$$

coincides with  $E_w$ , where  $w = gxg^{-1}x^{-1} \in G[x]$  (see also Example 3.2). Hence the centralizer  $C_G(g)$  is an elementary algebraic subset of  $G$ . Therefore, the centralizer  $C_G(S) = \bigcap_{g \in S} C_G(g)$  of any subset  $S$  of  $G$  is an algebraic subset. In particular, the center  $Z(G) = C_G(G)$  is an algebraic subset.

3. By Example 1.1, item 4, for every  $n \in \mathbb{N}$  the word  $x^n \in G[x]$  determines the verbal function  $f_{x^n} : g \mapsto g^n$ . Hence,  $E_w = G[n]$  by definition, where

$$G[n] = \{g \in G \mid g^n = e_G\}.$$

If  $G$  is abelian, then  $G[n]$  is a subgroup of  $G$ , called the  $n$ -socle of  $G$ . In the abelian case, these subsets (together with their cosets, of course) are all the non-empty elementary algebraic subsets of  $G$  (see (11)).

Then we see that the family of elementary algebraic subsets of  $G$  is stable under taking inverse image under verbal functions (Lemma 4.4). As a consequence, the translate of an elementary algebraic subset is an elementary algebraic subset (Example 4.5).

The family of algebraic subsets is stable under taking intersections and finite unions, and contains every finite subset (by Example 1.2, item 1), so is the family of closed sets of a  $T_1$  topology  $\mathfrak{Z}_G$  on  $G$ , the *Zariski topology of  $G$* . As a matter of fact, Markov did not explicitly introduce such a topology, that was first explicitly defined by Bryant in [7], as the *verbal topology of  $G$* . Here we keep the name Zariski topology, and the notation  $\mathfrak{Z}_G$ , for this topology, already used [3, 11, 12, 13, 15, 16].

In §5.2 and §5.3.1 we briefly also consider two other topologies on  $G$ , the *Markov topology*  $\mathfrak{M}_G$  and the *precompact Markov topology*  $\mathfrak{P}_G$ , introduced respectively in [12] and [13] as the intersections

$$\mathfrak{M}_G = \bigcap \{ \tau \mid \tau \text{ Hausdorff group topology on } G \},$$

$$\mathfrak{P}_G = \bigcap \{ \tau \mid \tau \text{ precompact Hausdorff group topology on } G \}.$$

The topology  $\mathfrak{M}_G$  was only implicitly introduced by Markov in the same paper [20], via the notion of *unconditionally closed subset* of  $G$ , namely a subset of  $G$  that is closed with respect to every Hausdorff group topology on  $G$ . Of course, the family of unconditionally closed subsets of  $G$  is the family of the closed sets of  $\mathfrak{M}_G$ . It can be directly verified from the definitions that

$$\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G. \tag{3}$$

Now we recall the definition of a *quasi-topological group*.

DEFINITION 1.3. *Let  $G$  be a group, and  $\tau$  a topology on  $G$ . The pair  $(G, \tau)$  is called quasi-topological group if for every  $a, b \in G$  the function  $(G, \tau) \rightarrow (G, \tau)$ , mapping  $x \mapsto ax^{-1}b$ , is continuous.*

Obviously,  $(G, \tau)$  is a quasi-topological group if and only if the inversion function  $f_{x^{-1}}$  and both the translations  $f_{gx}$  and  $f_{xg}$  are continuous, for every  $g \in G$ .

We dedicate §5 to quasi-topological groups, proving first some general results in §5.1. Then we use verbal functions in §5.2 to prove (3) and that all the pairs  $(G, \mathfrak{Z}_G)$ ,  $(G, \mathfrak{M}_G)$ ,  $(G, \mathfrak{P}_G)$  are quasi-topological groups (Corollary 5.7).

By Example 1.2, item 2, the center  $Z(G)$  of  $G$  is  $\mathfrak{Z}_G$ -closed. We extend this result proving that for every positive integer  $n$  also the  $n$ -th center  $Z_n(G)$  is  $\mathfrak{Z}_G$ -closed in Corollary 5.10, as a consequence of Theorem 5.9.

In §5.3, inspired by [3, 4, 5], we introduce the notion of a *partial Zariski topology*  $\mathfrak{T} \subseteq \mathfrak{Z}_G$  on  $G$ , namely a topology having *some* elementary algebraic subsets as a subbase for its closed sets. The aim of this definition is to study the cases when indeed the equality  $\mathfrak{T} = \mathfrak{Z}_G$  holds for some partial Zariski topology  $\mathfrak{T}$  on  $G$ , in order to have easier-to-deal-with subbases of  $\mathfrak{Z}_G$ .

For example, we introduce the *monomial topology*  $\mathfrak{T}_{mon}$  on a group  $G$  in Definition 5.19, whose closed sets are generated by the subsets  $E_w$ , for monomials  $w \in G[x]$ . We note that  $\mathfrak{T}_{mon} = \mathfrak{Z}_G$  when  $G$  is abelian (Example 5.20), and we prove that  $\mathfrak{T}_{mon}$  is the cofinite topology when  $G$  is nilpotent, torsion free (Corollary 5.21).

In §5.3.1 we recall a recent result from [3]. If  $X$  is an infinite set, we denote by  $S(X)$  the *symmetric group of  $X$* , consisting of the permutations of  $X$ . If  $\phi \in S(X)$ , its *support* is the subset  $\text{supp}(\phi) = \{x \in X \mid \phi(x) \neq x\} \subseteq X$ . We denote by  $S_\omega(X)$  the subgroup of  $S(X)$  consisting of the permutations having finite support. If  $G$  is a group with  $S_\omega(X) \leq G \leq S(X)$ , let  $\tau_p(G)$  denote the *point-wise convergence topology of  $G$* . The authors of [3] have introduced a partial Zariski topology  $\mathfrak{Z}'_G$  on  $G$  and proved that  $\mathfrak{Z}'_G = \mathfrak{Z}_G = \mathfrak{M}_G = \tau_p(G)$  (see Theorem 5.26).

We dedicate §5.4 to two others partial Zariski topologies, the *centralizer topologies*  $\mathfrak{C}_G$  and  $\mathfrak{C}'_G$ , defined as follows. Let  $\mathcal{C} = \{gC_G(a) \mid a, g \in G\}$ , and note that its members are elementary algebraic subsets by Example 1.2, item 2, and Example 4.5. Then  $\mathfrak{C}_G$  is the topology having  $\mathcal{C}$  as a subbase for its closed sets (Definition 5.15). If  $\mathcal{C}' = \mathcal{C} \cup \{\{g\} \mid g \in G\}$ , we similarly introduce  $\mathfrak{C}'_G$  as the topology having  $\mathcal{C}'$  as a subbase for its closed sets (Definition 5.28). See Proposition 5.30 for the first few properties of  $\mathfrak{C}_G$  and  $\mathfrak{C}'_G$ .

Finally, we see in §5.4.1 that every free non-abelian group  $F$  satisfies  $\mathfrak{C}_F = \mathfrak{C}'_F = \mathfrak{Z}_F$ . On the other hand, we consider a class of matrix groups  $H$  in §5.4.2, satisfying  $\mathfrak{C}_H \neq \mathfrak{C}'_H = \mathfrak{Z}_H$ .

This paper is a part of articles dedicated to the study of the Zariski topology of a group  $G$ , using the group of words  $G[x]$  and the group of verbal functions

$\mathcal{F}(G)$  of  $G$  as main tools, see [14, 15, 16, 17]. In particular, we develop the basic theory here.

We denote by  $\mathbb{Z}$  the group of integers, by  $\mathbb{N}_+$  the set of positive integers, and by  $\mathbb{N}$  the set of naturals. If  $n \in \mathbb{Z}$ , the cyclic subgroup it generates is  $n\mathbb{Z}$ , while the quotient group  $\mathbb{Z}/n\mathbb{Z}$  will be denoted by  $\mathbb{Z}_n$ .

If  $X$  is a set, and  $\mathcal{B} \subseteq \mathcal{P}(X)$  is a family of subsets of  $X$ , we denote by  $\mathcal{B}^\cup \subseteq \mathcal{P}(X)$  the family of finite unions of members of  $\mathcal{B}$ .

## 2. The group of words $G[x]$

### 2.1. The categorical aspect of $G[x]$

The group  $G[x]$  is determined by the universal property stated below.

**FACT 2.1.** *Let  $G$  be a group. Then there exist a unique (up to isomorphism) group  $G[x]$ , together with an injective group homomorphism  $i_G: G \rightarrow G[x]$ , satisfying the following universal property:*

*for every group  $\Gamma$ , for every group homomorphism  $\phi: G \rightarrow \Gamma$ , and for every  $\gamma \in \Gamma$ , there exists a unique group homomorphism  $\tilde{\phi}: G[x] \rightarrow \Gamma$  such that  $\tilde{\phi} \circ i_G = \phi$  and  $\tilde{\phi}(x) = \gamma$ .*

$$\begin{array}{ccc}
 & G[x] \ni x & \\
 i_G \nearrow & \downarrow \tilde{\phi} & \downarrow \\
 G & \xrightarrow{\phi} & \Gamma \ni \gamma
 \end{array}$$

From now on, we will identify  $G$  with  $i_G(G) \leq G[x]$ .

In the following example we illustrate a few particular cases when Fact 2.1 can be applied.

**EXAMPLE 2.2.** 1. Consider the identity map  $\text{id}_G: G \rightarrow G$ . By Fact 2.1, for every  $g \in G$  there exists a unique map  $\text{ev}_g: G[x] \rightarrow G$ , with  $\text{ev}_g \upharpoonright_G = \text{id}_G$  and  $\text{ev}_g(x) = g$ , that we call *evaluation map*. Then we define  $w(g) = \text{ev}_g(w)$  for every  $w \in G[x]$ .

$$\begin{array}{ccc}
 & G[x] \ni x & \\
 i_G \nearrow & \downarrow \text{ev}_g & \downarrow \\
 G & \xrightarrow{\text{id}_G} & G \ni g
 \end{array}$$

2. A  *$G$ -endomorphism of  $G[x]$*  is a group homomorphism  $\phi: G[x] \rightarrow G[x]$  such that  $\phi \circ i_G = i_G$ , i.e.  $\phi \upharpoonright_G = \text{id}_G$ , so that the following diagram

commutes:

$$\begin{array}{ccc} & & G[x] \\ & \nearrow i_G & \downarrow \phi \\ G & \xrightarrow{i_G} & G[x] \end{array} .$$

Then  $\phi$  is uniquely determined by the element  $w = \phi(x) \in G[x]$ , and now we show that every choice of  $w \in G[x]$  can be made, thus classifying the  $G$ -endomorphisms of  $G[x]$ . To this end, consider the map  $i_G: G \rightarrow G[x]$ . By Fact 2.1, for every  $w \in G[x]$  there exists a unique  $G$ -endomorphism  $\xi_w: G[x] \rightarrow G[x]$ , with  $x \mapsto w$ .

$$\begin{array}{ccc} & & G[x] \ni x \\ & \nearrow i_G & \downarrow \xi_w \\ G & \xrightarrow{i_G} & G[x] \ni w \end{array}$$

**PROPOSITION 2.3.** *Let  $\phi: G_1 \rightarrow G_2$  be a group homomorphism. Then there exists a unique group homomorphism  $F: G_1[x] \rightarrow G_2[x]$  such that  $F \upharpoonright_{G_1} = \phi$ ,  $F(x) = x$ . In particular, if  $\phi$  is surjective (resp., injective), then  $F$  is surjective (resp., injective).*

*Moreover, the following hold.*

1. *If  $H \leq G$  is a subgroup of  $G$ , then  $H[x] \leq G[x]$ .*
2. *If  $H \trianglelefteq G$  is a normal subgroup of  $G$ , and  $\bar{G} = G/H$ , then  $\bar{G}[x]$  is a quotient of  $G[x]$ .*

*Proof.* Composing  $\phi: G_1 \rightarrow G_2$  and the map  $i_{G_2}: G_2 \rightarrow G_2[x]$ , we obtain  $\psi = i_{G_2} \circ \phi: G_1 \rightarrow G_2[x]$ . Then apply Fact 2.1 and use the universal property of  $G_1[x]$  to get  $F = \tilde{\psi}: G_1[x] \rightarrow G_2[x]$  such that  $F(x) = x$  and  $F \circ i_{G_1} = i_{G_2} \circ \phi$ , i.e.  $F \upharpoonright_{G_1} = \phi$ .

$$\begin{array}{ccccc} & & & & G_1[x] \ni x \\ & & \nearrow i_{G_1} & & \downarrow \tilde{\psi} = F \\ G_1 & \xrightarrow{\phi} & G_2 & \xrightarrow{i_{G_2}} & G_2[x] \ni x \\ & \searrow \psi & & & \downarrow \end{array}$$

If  $\phi$  is surjective, then  $F$  is surjective too, as  $F(G_1[x])$  contains both  $x$  and  $\phi(G_1) = G_2$ , which generate  $G_2[x]$ .

In Remark 3.3, item 1, we will explicitly describe the map  $F$ , so that by (5) it will immediately follow that  $F$  is injective when  $\phi$  is injective.

1. In this case, the injection  $\phi: H \hookrightarrow G$  gives the injection  $F: H[x] \hookrightarrow G[x]$ .
2. The canonical projection  $\phi: G \rightarrow \overline{G}$  gives the surjection  $F: G[x] \rightarrow \overline{G}[x]$ .

□

The following corollary immediately follows from Proposition 2.3.

**COROLLARY 2.4.** *The assignment  $G \mapsto G[x]$ , and the canonical embedding  $G \xrightarrow{i_G} G[x]$ , define a pointed endofunctor  $\varpi: \mathbf{Gr} \rightarrow \mathbf{Gr}$  in the category of groups and group homomorphism. In other words, for every group homomorphism  $\phi: G_1 \rightarrow G_2$ , the following diagram commutes*

$$\begin{array}{ccc} G_1 & \xrightarrow{i_{G_1}} & G_1[x] \\ \phi \downarrow & & \downarrow \varpi(\phi) \\ G_2 & \xrightarrow{i_{G_2}} & G_2[x], \end{array}$$

where  $\varpi(\phi) = F$  is the map given by Proposition 2.3.

## 2.2. The concrete form of $G[x]$

Here we recall the concrete definition of  $G[x]$  in terms of products of the form (4) below that will be called *words*. In particular, if  $g \in G$ , then  $w = g \in G[x]$  will be called *constant word*, and we define its *length* to be  $l(w) = 0 \in \mathbb{N}$ . In the general case, for  $w \in G[x]$  there exist  $n \in \mathbb{N}$ , elements  $g_1, \dots, g_n, g_0 \in G$  and  $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ , such that

$$w = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} g_0. \tag{4}$$

Notice that in Markov's definition (1) of elementary algebraic subset of  $G$ , he was assuming the defining word  $w$  not to be constant (see Remark 4.2 for more details).

If  $g_i \neq e_G$  whenever  $\varepsilon_{i-1} = -\varepsilon_i$  for  $i = 2, \dots, n$ , we say that  $w$  is a *reduced word* in the free product  $G[x] = G * \langle x \rangle$  and we define the *length* of  $w$  by  $l(w) = n$ , where  $n \in \mathbb{N}$  is the least natural such that  $w$  is as in (4).

**DEFINITION 2.5.** If  $w \in G[x]$  is as in (4), we define the following notions.

- The *constant term* of  $w$  is  $\text{ct}(w) = w(e_G) = g_1 g_2 \cdots g_n g_0 \in G$ ;
- The *content* of  $w$  is  $\epsilon(w) = \sum_{i=1}^n \varepsilon_i \in \mathbb{Z}$ , which will also be denoted simply by  $\epsilon$  when no confusion is possible.

If  $w = g$ , then we define  $\epsilon(w) = 0$  and  $\text{ct}(w) = w(e_G) = g$ . We call *singular* a word  $w$  such that  $\epsilon(w) = 0$ . All constant words are singular by definition.

Given two elements  $g, h$  of a group  $G$ , recall that their commutator element is  $[g, h] = ghg^{-1}h^{-1} \in G$ . Note that  $[g, h] = e_G$  if and only if  $gh = hg$ , i.e.  $g$  and  $h$  commute. Then the *commutator subgroup*  $G'$  of  $G$  is the subgroup

$$G' = \langle [g, h] \mid g, h \in G \rangle$$

generated by all elements of  $G$  of the form  $[g, h]$ . It can easily be verified that if  $H \trianglelefteq G$  is a normal subgroup of  $G$ , then the quotient group  $G/H$  is abelian if and only if  $G' \leq H$ .

Both the functions  $\text{ct}: G[x] \rightarrow G$ , mapping  $w \mapsto \text{ct}(w)$ , and  $\epsilon: G[x] \rightarrow \mathbb{Z}$ , mapping  $w \mapsto \epsilon(w)$ , are surjective group homomorphisms. In particular,  $\text{ct}(G[x]') \leq G'$  and  $\epsilon(G[x]') \leq \mathbb{Z}' = \{0\}$ , so that  $G[x]' \leq \text{ct}^{-1}(G') \cap \ker(\epsilon)$ . In the following theorem, we prove the reverse inclusion.

**THEOREM 2.6.** *For every group  $G$ ,  $G[x]' = \text{ct}^{-1}(G') \cap \ker(\epsilon)$ .*

*Proof.* Let  $U = \text{ct}^{-1}(G') \cap \ker(\epsilon) = \{w \in G[x] \mid \text{ct}(w) \in G', \epsilon(w) = 0\}$ . Then  $G[x]' \subseteq U$  as we have noted above, and we prove the other inclusion by induction on  $l(w)$  for a word  $w \in U$ .

Let  $w \in G[x]$  and assume  $w \in U$ . We first consider the case when  $l(w) = 0$ , i.e.  $w = \text{ct}(w) \in G$  is a constant word, so that  $w \in G' \leq G[x]'$  and there is nothing to prove. So now let  $w \in U$  be as in (4), and note that  $\epsilon(w) = 0$  implies that  $n = l(w) > 0$  is even, so that for the base case we have to consider  $n = 2$ . Then  $w$  has the form  $w = g_1 x^\epsilon g_2 x^{-\epsilon} (g_1 g_2)^{-1} c$ , with  $c = \text{ct}(w) \in G'$ . Let  $g = g_1 g_2$ , and  $w_0 = [g_2^{-1}, x^\epsilon] \in G[x]'$ , so that  $w = g w_0 g^{-1} c = [g, w_0] w_0 c \in G[x]'$ .

Now assume  $n > 2$ . As  $\epsilon(w) = 0$ , we have  $\epsilon_{i+1} = -\epsilon_i$  for some  $1 \leq i \leq n-1$ . Then  $w = w_1 w_2 w_3$  for the words

$$\begin{aligned} w_1 &= g_1 x^{\epsilon_1} g_2 x^{\epsilon_2} \cdots g_{i-1} x^{\epsilon_{i-1}}, \\ w_2 &= g_i x^{\epsilon_i} g_{i+1} x^{\epsilon_{i+1}} (g_i g_{i+1})^{-1}, \\ w_3 &= (g_i g_{i+1}) g_{i+2} x^{\epsilon_{i+2}} \cdots g_n x^{\epsilon_n} g_0. \end{aligned}$$

As  $w_2 \in G[x]'$  by the base case, and  $w = [w_1, w_2] w_2 w_1 w_3$ , we only have to show that  $w_1 w_3 \in G[x]'$ . As

$$\text{ct}(w) = \text{ct}(w_1) \text{ct}(w_2) \text{ct}(w_3) = \text{ct}(w_1) e_G \text{ct}(w_3) = \text{ct}(w_1 w_3),$$

we have  $\text{ct}(w_1 w_3) \in G'$ , and similarly  $\epsilon(w_1 w_3) = 0$ . Then  $w_1 w_3 \in G[x]'$  by the inductive hypothesis.  $\square$

### 3. Verbal functions

#### 3.1. Definition and examples

**DEFINITION 3.1.** A word  $w \in G[x]$  determines the associated evaluation function  $f_w^G: G \rightarrow G$ . We will often write  $f_w$  for  $f_w^G$ . We call *verbal function* of  $G$



a function  $G \rightarrow G$  of the form  $f_w$ , and we denote by  $\mathcal{F}(G)$  the set of verbal functions on  $G$ .

If  $w \in G[x]$  and  $g \in G$ , sometimes we also write  $w(g)$  for the element  $f_w(g) \in G$ . So *a priori*, if  $f$  is a verbal function, then  $f = f_w$  for a word  $w \in G[x]$  as in (4).

Besides the basic examples already given in Example 1.1, we will also consider verbal functions of the following form.

EXAMPLE 3.2. If  $\varepsilon \in \{\pm 1\}$ , and  $a \in G$ , the word  $w = [a, x^\varepsilon] = ax^\varepsilon a^{-1}x^{-\varepsilon} \in G[x]$  determines the verbal function  $f_w: g \mapsto [a, g^\varepsilon]$ . We will call *commutator verbal function* a function of this form.

Note that  $f_w: G \rightarrow G$  is the only map such that  $f_w \circ \text{ev}_g = \text{ev}_g \circ \xi_w$  for every  $g \in G$ , i.e. making the following diagram commute:

$$\begin{array}{ccc} G[x] & \xrightarrow{\xi_w} & G[x] \\ \text{ev}_g \downarrow & & \downarrow \text{ev}_g \\ G & \xrightarrow{f_w} & G. \end{array}$$

REMARK 3.3. 1. Let  $\phi: G_1 \rightarrow G_2$  be a group homomorphism, and

$$F = \varpi(\phi): G_1[x] \rightarrow G_2[x]$$

be as in Proposition 2.3. If  $w \in G[x]$  is as in (4), then

$$F: w \mapsto F(w) = \phi(g_1)x^{\varepsilon_1}\phi(g_2)x^{\varepsilon_2}\cdots\phi(g_n)x^{\varepsilon_n}\phi(g_0). \quad (5)$$

By (5), it immediately follows that  $F$  is injective when  $\phi$  is injective.

Moreover, one can easily see that  $\phi \circ f_w = f_{F(w)} \circ \phi$ , i.e. the following diagram commutes:

$$\begin{array}{ccc} G_1 & \xrightarrow{f_w} & G_1 \\ \phi \downarrow & & \downarrow \phi \\ G_2 & \xrightarrow{f_{F(w)}} & G_2. \end{array} \quad (6)$$

2. In particular, we will often consider the case when  $\phi$  is the canonical projection  $\pi: G \rightarrow G/N$ , if  $N$  is a normal subgroup of  $G$ . In this case, let  $\bar{G} = G/N$  be the quotient group, and for an element  $g \in G$ , let  $\bar{g} = \pi(g) \in \bar{G}$ . If  $w = g_1x^{\varepsilon_1}g_2x^{\varepsilon_2}\cdots g_nx^{\varepsilon_n}g_0 \in G[x]$ , let also  $\bar{w} = F(w) = \bar{g}_1x^{\varepsilon_1}\bar{g}_2x^{\varepsilon_2}\cdots\bar{g}_nx^{\varepsilon_n}\bar{g}_0 \in \bar{G}[x]$ . Then (6) (with  $\phi = \pi$ ) gives  $\pi \circ f_w = f_{\bar{w}} \circ \pi$ .

### 3.2. Universal words

The group operation on  $G[x]$  induces a group operation on  $\mathcal{F}(G)$  as follows. If  $w_1, w_2 \in G[x]$ , let  $w = w_1 w_2 \in G[x]$  be their product, and consider the verbal functions  $f_{w_1}, f_{w_2}, f_w \in \mathcal{F}(G)$ . Obviously,  $f_w$  is the pointwise product  $f_{w_1} \cdot f_{w_2}$  of  $f_{w_1}$  and  $f_{w_2}$ , namely the map  $f_w: g \mapsto f_{w_1}(g)f_{w_2}(g) = f_w(g)$ . With this operation,  $(\mathcal{F}(G), \cdot)$  is a group, with identity element the constant function  $e_{\mathcal{F}(G)}: g \mapsto e_G$  for every  $g \in G$ . If  $w^{-1}$  is the inverse of  $w \in G[x]$ , then the inverse of  $f_w \in \mathcal{F}(G)$  is  $f_{w^{-1}}$ , and will be denoted by  $(f_w)^{-1}$ .

For  $S \subseteq G$ , we denote by

$$(f_w)^{-1}(S) = \{(f_w)^{-1}(s) \mid s \in S\} = \{f_{w^{-1}}(s) \mid s \in S\} \subseteq G$$

the *image* of  $S$  under  $(f_w)^{-1} = f_{w^{-1}}$ , while  $f_w^{-1}(S) = \{g \in G \mid f_w(g) \in S\}$  will denote the *preimage* of  $S$  under  $f_w$ .

Consider the surjective group homomorphism  $\Phi_G: G[x] \rightarrow \mathcal{F}(G)$ ,  $w \mapsto f_w$ . Then  $\mathcal{F}(G) \cong G[x]/\mathcal{U}_G$ , where  $\mathcal{U}_G$  is the kernel

$$\mathcal{U}_G = \ker(\Phi_G) = \{w \in G[x] \mid \forall g \in G \ f_w(g) = e_G\} \leq G[x]. \quad (7)$$

**DEFINITION 3.4.** *If  $G$  is a group, and  $w \in G[x]$ , we say that  $w$  is a universal word for  $G$  if  $w \in \mathcal{U}_G$ .*

Note that a word  $w \in G[x]$  is universal exactly when  $E_w = G$ .

Recall that the *exponent*  $\exp(G)$  of a group  $G$  is the least common multiple, if it exists, of the orders of the elements of  $G$ . In this case,  $\exp(G) > 0$ . Otherwise, we conventionally define  $\exp(G) = 0$ . For example, every finite group  $G$  has positive exponent, and  $\exp(G)$  divides  $|G|$ .

**EXAMPLE 3.5.** 1. If  $w \in \mathcal{U}_G$ , then obviously  $\text{ct}(w) = f_w(e_G) = e_G$ .

2. If  $G$  has  $k = \exp(G) > 0$ , then  $w = x^k \in G[x]$  is a non-singular universal word for  $G$ , i.e.  $f_w \equiv e_G$  is the constant function.

The singular universal words will play a prominent role, so we set

$$\mathcal{U}_G^{\text{sing}} = \{w \in \mathcal{U}_G : \epsilon(w) = 0\} = \mathcal{U}_G \cap \ker \epsilon \leq G[x].$$

In particular, also  $\mathcal{U}_G^{\text{sing}}$  is a normal subgroup of  $G[x]$ .

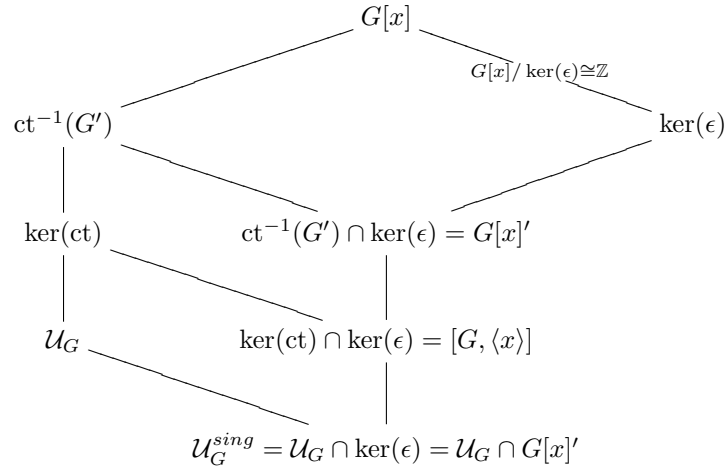
**REMARK 3.6.** Let  $[G, \langle x \rangle] = \langle [g, x^i] \mid g \in G, i \in \mathbb{Z} \rangle \leq G[x]$  be the subgroup of  $G[x]$  generated by all commutators  $[g, x^i] \in G[x]$ , for  $g \in G$  and  $i \in \mathbb{Z}$ . It can be easily verified that  $[G, \langle x \rangle] \subseteq \ker(\text{ct}) \cap \ker(\epsilon)$ . The other inclusion can

be proved by induction on  $l(w)$  of the words  $w \in \ker(\text{ct}) \cap \ker(\epsilon)$ , similarly to what we did in the proof of Theorem 2.6. Then

$$[G, \langle x \rangle] = \ker(\text{ct}) \cap \ker(\epsilon).$$

In particular,  $[G, \langle x \rangle]$  is a normal subgroup of  $G[x]$ , being the kernel of the natural surjective homomorphism  $G[x] \rightarrow G \times \langle x \rangle$  mapping  $w \mapsto (\text{ct}(w), x^{\epsilon(w)})$ .

Then, we have the following map of relevant subgroups of  $G[x]$  considered so far.



Using the normal subgroup  $\mathcal{U}_G$  of  $G[x]$ , we can define a congruence relation  $\approx$  on  $G[x]$  as follows: for a pair of words  $w_1, w_2 \in G[x]$ , we define  $w_1 \approx w_2$  if  $w_1 \mathcal{U}_G = w_2 \mathcal{U}_G$ . Then

$$w_1 \approx w_2 \text{ if and only if } \Phi_G(w_1) = \Phi_G(w_2), \text{ i.e., } f_{w_1} = f_{w_2}.$$

In particular, a word  $w$  is universal when  $w \approx e_{G[x]}$ , i.e.  $f_w$  is the constant function  $e_G$  on  $G$ . Note that the quotient group is  $G[x] / \approx = G[x] / \mathcal{U}_G \cong \mathcal{F}(G)$ .

A second monoid operation in  $\mathcal{F}(G)$  can be introduced as follows. If  $w$  is as in (4), and  $w_1 \in G[x]$ , one can consider the word

$$\xi_{w_1}(w) = g_1 w_1^{\epsilon_1} g_2 w_1^{\epsilon_2} \cdots g_n w_1^{\epsilon_n} g_0$$

obtained substituting  $w_1$  to  $x$  in  $w$  and taking products in  $G[x]$ . We shall also denote by  $w \circ w_1$  the word  $\xi_{w_1}(w)$ . On the other hand, one can consider the

usual composition of the associated verbal functions  $f_w, f_{w_1} \in \mathcal{F}(G)$ . Then this composition of words is compatible with the composition of functions, in the sense that

$$f_w \circ f_{w_1} = f_{w \circ w_1} \in \mathcal{F}(G).$$

With this operation,  $(\mathcal{F}(G), \circ)$  is a monoid, with identity element the identity function  $\text{id}_G = f_x$  of  $G$ , mapping  $\text{id}_G: g \mapsto g$  for every  $g \in G$ . Obviously,  $(\mathcal{F}(G), \circ)$  is a submonoid of the monoid  $(G^G, \circ)$  of all self-maps  $G \rightarrow G$ .

### 3.3. Monomials

Even if a group  $G$  has a quite simple structure (for example, is abelian), the group of words  $G[x]$  may be more difficult to study (for example,  $G[x]$  is never abelian, unless  $G$  is trivial). As we are more interested in its quotient group of verbal function  $\mathcal{F}(G)$ , it will be useful to consider some subset  $W \subseteq G[x]$  such that  $G[x] = W \cdot \mathcal{U}_G$ , i.e.  $\Phi_G(W) = \{f_w \mid w \in W\} = \mathcal{F}(G)$ , i.e.  $\mathcal{F}(G) = W/\approx$ . In the following §3.4 we will present such an appropriate subset  $W \subseteq G[x]$  in the case when  $G$  is abelian.

A word of the form  $w = gx^m$ , for  $g \in G$  and  $m \in \mathbb{Z}$ , is called a *monomial*. One can associate a monomial to an arbitrary word  $w = g_1x^{\varepsilon_1}g_2x^{\varepsilon_2} \cdots g_nx^{\varepsilon_n}g_0 \in G[x]$  as follows, letting

$$w_{ab} = \text{ct}(w)x^{\varepsilon(w)} = g_1g_2 \cdots g_n g_0 x^{\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n} \in G[x]. \quad (8)$$

The monomials in  $G[x]$  do not form a subgroup unless  $G$  is trivial. Nevertheless, one can “force” them to form a group, by taking an appropriate *quotient* of  $G[x]$ . Indeed, recall the surjective homomorphism  $G[x] \rightarrow G \times \langle x \rangle$  mapping  $w \mapsto (\text{ct}(w), x^{\varepsilon(w)})$  considered in Remark 3.6. Then the group  $G \times \langle x \rangle$  “parametrizes” in the obvious way all monomials of  $G[x]$  (although the group operation is not the one from  $G[x]$ ).

### 3.4. A leading example: the abelian case

A case when  $\mathcal{F}(G)$  has a very transparent description is that of abelian groups. Let  $(G, +, 0_G)$  be an abelian group. While  $G[x]$  is not abelian in any case, its quotient  $\mathcal{F}(G)$  becomes indeed abelian, and so we keep additive notation also to denote words  $w \in G[x]$ . Remind that we really are interested only in the evaluation function  $f_w \in \mathcal{F}(G)$  associated to  $w$ , and to its preimage  $E_w^G = f_w^{-1}(\{0_G\}) = \{g \in G \mid f_w(g) = 0_G\}$  (see Definition 4.1).

Then,  $w \approx w_{ab} = \text{ct}(w) + \varepsilon(w)x$  for every word  $w \in G[x]$ , and in particular, letting

$$W = \{w_{ab} \mid w \in G[x]\} = \{g + nx \mid g \in G, n \in \mathbb{Z}\} \subseteq G[x],$$

we have  $W/\approx = G[x]/\approx$ , so that

$$\mathcal{F}(G) = \{f_{g+nx} \mid g \in G, n \in \mathbb{Z}\}.$$

For these reasons, when  $G$  is abelian, we will only consider the monomials  $w \in W$ . These observations are heavily used in computing  $\mathcal{F}(G)$  for an abelian group  $G$  (hence also  $\mathbb{E}_G$ , see Example 4.3).

Note that the surjective homomorphism  $\Psi_G: G[x] \rightarrow G \times \mathbb{Z}$ , mapping  $w \mapsto (\text{ct}(w), \epsilon(w))$ , has kernel  $\ker(\Psi_G) = \ker(\text{ct}) \cap \ker(\epsilon) = G[x]'$  by Theorem 2.6, so that  $G[x]/G[x]' \cong G \times \mathbb{Z}$ . So, if one considers the quotient  $G[x]/G[x]'$ , the canonical projection  $G[x] \rightarrow G[x]/G[x]'$  is exactly  $w \mapsto w_{ab} = \text{ct}(w) + \epsilon(w)x$ . Moreover, being  $\mathcal{F}(G) \cong G[x]/\mathcal{U}_G$  abelian, we have that  $\mathcal{U}_G \geq G[x]'$ , so  $\mathcal{F}(G) \cong \frac{G[x]/G[x]'}{G[x]'/\mathcal{U}_G}$  is a quotient of  $G \times \mathbb{Z}$ .

Here we give an explicit description of the group  $\mathcal{F}(G)$ .

**PROPOSITION 3.7.** *If  $G$  is an abelian group, then:*

$$\mathcal{F}(G) \cong \begin{cases} G \times \mathbb{Z} & \text{if } \exp(G) = 0, \\ G \times \mathbb{Z}_n & \text{if } \exp(G) = n > 0. \end{cases}$$

*Proof.* Let  $n = \exp(G) \in \mathbb{N}$ . Note that  $\Psi'_G: G \times \mathbb{Z} \rightarrow \mathcal{F}(G)$ , mapping  $(g, k) \rightarrow f_{g+kx}$ , is a surjective group homomorphism, and that  $(g, k) \in \ker(\Psi'_G)$  if and only if  $w = g + kx \in \mathcal{U}_G$ .

In this case,  $g = \text{ct}(w) = 0_G$  by Example 3.5, item 1, so that  $w = kx$ . If  $n = 0$ , then  $k = 0$ . If  $n > 0$ , then either  $k = 0$ , or  $k \neq 0$  and  $n \mid k$ . In any case,  $k \in n\mathbb{Z}$ .

This proves  $\ker(\Psi'_G) = \{0_G\} \times n\mathbb{Z}$ . □

#### 4. Elementary algebraic subsets

This section is focused on the family  $\mathbb{E}_G \subseteq \mathcal{P}(G)$ , consisting of preimages  $f_w^{-1}(\{e_G\})$ , rather than on the group  $G[x]$ , consisting of words  $w$ , or its quotient  $\mathcal{F}(G)$ , consisting of verbal functions  $f_w$ . We begin recalling Markov's definition of elementary algebraic subset of a group, using the terminology of verbal functions.

**DEFINITION 4.1.** *If  $w \in G[x]$ , we let*

$$E_w^G = f_w^{-1}(\{e_G\}) = \{g \in G \mid f_w(g) = e_G\} \subseteq G,$$

*we call  $E_w^G$  elementary algebraic subset of  $G$ , and we denote it simply by  $E_w$  when no confusion is possible. We denote by  $\mathbb{E}_G = \{E_w \mid w \in G[x]\} \subseteq \mathcal{P}(G)$  the family of elementary algebraic subsets of  $G$ .*

*According to Markov's definition on page 71, if  $X \subseteq G$ , we call it:*

- additively algebraic if  $X$  is a finite union of elementary algebraic subsets of  $G$ , i.e. if  $X \in \mathbb{E}_G^\cup$ ;
- algebraic if  $X$  is an intersection of additively algebraic subsets of  $G$ .

Then the algebraic subsets form the family of  $\mathfrak{Z}_G$ -closed sets, and  $\mathbb{E}_G$  is a subbase for the  $\mathfrak{Z}_G$ -closed sets; while the additively algebraic subsets are exactly the members of  $\mathbb{E}_G^\cup$ , and are a base for the  $\mathfrak{Z}_G$ -closed sets.

REMARK 4.2. If  $w = g \in G$  is a constant word, then either  $E_w = G$  or  $E_w = \emptyset$  (depending on whether  $g = e_G$  or  $g \neq e_G$ ). For this reason, in studying the family  $\mathbb{E}_G$  and the topology  $\mathfrak{Z}_G$ , there is no harm in assuming  $w$  not to be constant, i.e. to be as in Markov's definition (1) of elementary algebraic subset (see also §4.1).

Let us fix a group  $G$ . We will now consider the iterated images of  $G$  under  $\varpi^n$ , for  $n \in \mathbb{N}_+$ , and to this end we need to introduce a countable set of variables  $\{x_n \mid n \in \mathbb{N}_+\}$ . Then applying  $\varpi$  we obtain the following diagram:

$$G \xrightarrow{\varpi} G[x_1] \xrightarrow{\varpi} (G[x_1])[x_2] \xrightarrow{\varpi} ((G[x_1])[x_2])[x_3] \xrightarrow{\varpi} \dots \quad (9)$$

If  $n \in \mathbb{N}_+$ , we let  $G_n = G[x_1, \dots, x_n] = \varpi^n(G)$ , and it can be proved that if  $\sigma \in S_n$ , then

$$G_n \cong G[x_{\sigma(1)}, \dots, x_{\sigma(n)}].$$

Every  $w = w(x_1, \dots, x_n) \in G_n$  determines the associated evaluation function of  $n$  variables over  $G$ , that we denote by  $f_w: G^n \rightarrow G$ , in analogy with Definition 3.1.

Finally, one can define  $E_w \subseteq G^n$  as the preimage  $E_w = f_w^{-1}(\{e_G\})$ , and consider the family  $\{E_w \mid w \in G_n\}$  as a subbase for the closed sets of a topology on  $G^n$ .

These observations are the basis of a theory of algebraic geometry over groups, recently started with [6] and developed in a series of subsequent papers. In this work, we focus on the case when  $n = 1$ , considering only verbal functions  $f_w: G \rightarrow G$  of one variable, and elementary algebraic subsets  $E_w \subseteq G$ .

EXAMPLE 4.3 (**A leading example: the abelian case II**). Let  $G$  be an abelian group (see §3.4). Then the elementary algebraic subset of  $G$  determined by  $f_{g+nx}$  is

$$E_{g+nx} = \begin{cases} \emptyset & \text{if } g + nx = 0_G \text{ has no solution in } G, \\ G[n] + x_0 & \text{if } x_0 \text{ is a solution of } g + nx = 0_G. \end{cases} \quad (10)$$

On the other hand, if  $n \in \mathbb{Z}$ , and  $g \in G$ , then  $G[n] + g = E_{nx-ng}$ . So the non-empty elementary algebraic subsets of  $G$  are exactly the cosets of the  $n$ -socles of  $G$ :

$$\mathbb{E}_G \setminus \{\emptyset\} = \{G[n] + g \mid n \in \mathbb{N}, g \in G\}. \quad (11)$$

Then  $\mathbb{E}_G^{\cup}$  is the family of all the  $\mathfrak{Z}_G$ -closed subsets of an abelian group  $G$ . In other words, *every algebraic subset of  $G$  is additively algebraic*.

It follows from Remark 4.2 and (10) that if  $G$  is abelian, and  $w \in G[x]$  is singular, then either  $E_w = G$  or  $E_w = \emptyset$ .

There are easy examples showing that in general none of the elementary algebraic subsets  $E_{x^n}^G = G[n]$  need to be a coset of a subgroup. See for example [17], where we show a class of groups  $G$  such that the subgroup generated by  $G[n]$  is the whole group  $G$ , for every  $n \in \mathbb{N}_+$ .

Now we prove that the inverse image of an elementary algebraic subset under a verbal function is still an elementary algebraic subset.

LEMMA 4.4. *For every group  $G$ , the family  $\mathbb{E}_G$  is stable under taking inverse image under verbal functions.*

*Proof.* For every pair  $w, w' \in G[x]$ , consider the verbal function  $f_w$  and the elementary algebraic subset  $E_{w'}$ . Then

$$f_w^{-1}E_{w'} = f_w^{-1}f_{w'}^{-1}(\{e_G\}) = (f_{w'} \circ f_w)^{-1}(\{e_G\}) = f_{w' \circ w}^{-1}(\{e_G\}) = E_{w' \circ w}, \quad (12)$$

so that  $f_w^{-1}E_{w'} \in \mathbb{E}_G$ .  $\square$

As a first application of Lemma 4.4, we see that the translate of an elementary algebraic subset is still an elementary algebraic subset.

EXAMPLE 4.5. 1. By Example 1.1, item 5, the left translation in  $G$  by an element  $g \in G$  is the verbal function  $f_{gx}$ , and so  $gS = f_{g^{-1}x}^{-1}(S)$  for every subset  $S \subseteq G$ . In particular, by (12) we have

$$gE_w = f_{g^{-1}x}^{-1}(E_w) = E_{w \circ g^{-1}x}. \quad (13)$$

Similarly,  $E_w g = E_{w \circ xg^{-1}}$ . Note that  $\epsilon(w \circ g^{-1}x) = \epsilon(w) = \epsilon(w \circ xg^{-1})$ .

2. If  $a \in G$ , then  $C_G(a) = E_w$ , for the word  $w = axa^{-1}x^{-1} \in G[x]$  by Example 1.2, item 2. By (13), its left coset determined by an element  $g \in G$  is  $gC_G(a) = E_{w_1}$  for

$$w_1 = w \circ (g^{-1}x) = a(g^{-1}x)a^{-1}(g^{-1}x)^{-1} = ag^{-1}xa^{-1}x^{-1}g.$$

Note also that, for  $w_2 = gw_1g^{-1} = (gag^{-1})xa^{-1}x^{-1}$ , we have  $E_{w_2} = E_{w_1} = gC_G(a)$ .

On the other hand,  $C_G(a)g = gg^{-1}C_G(a)g = gC_G(g^{-1}ag)$ , so that

$$\mathcal{C} = \{gC_G(a) \mid a, g \in G\}$$

is the family of all cosets of one-element centralizers in  $G$ . By the above observations,

$$\mathcal{C} = \{E_w \mid \exists a, g \in G \ w = (gag^{-1})xa^{-1}x^{-1}\} \subseteq \mathbb{E}_G.$$

#### 4.1. Further reductions

As already noted above, to study  $\mathcal{F}(G)$  it is sufficient to consider a subset  $W \subseteq G[x]$  such that  $\mathcal{F}(G) = \Phi_G(W) = W/\approx$ . Since our effort is really devoted to the study of the Zariski topology  $\mathfrak{Z}_G$  on a group  $G$ , hence to the family  $\mathbb{E}_G$ , a further reduction is also possible as follows.

As an example to introduce this reduction, consider the abelian group  $G = \mathbb{Z} \times \mathbb{Z}_2$ , and the verbal functions  $f_w, f_{w'} \in \mathcal{F}(G)$ , associated to  $w = 2x, w' = 4x \in G[x]$ . Then  $f_w \neq f_{w'}$ , and yet  $E_w = f_w^{-1}(\{0_G\}) = \{0_{\mathbb{Z}}\} \times \mathbb{Z}_2 = f_{w'}^{-1}(\{0_G\}) = E_{w'}$ .

Another example of a more general property could be the following: consider a word  $w \in G[x]$ , and its inverse  $w^{-1} \in G[x]$ . Obviously  $f_{w^{-1}} = (f_w)^{-1} \neq f_w$  in general, but for an element  $g \in G$  we have  $f_{w^{-1}}(g) = e_G$  if and only if  $f_w(g) = e_G$ . In particular,

$$E_{w^{-1}} = f_{w^{-1}}^{-1}(\{e_G\}) = f_w^{-1}(\{e_G\}) = E_w.$$

So  $E_w = E_{w^{-1}}$  in every group  $G$ , and in Remark 4.6 below we slightly generalize this result.

So we introduce another equivalence relation  $\sim$  on  $G[x]$  defined as follows: for a pair of words  $w_1, w_2 \in G[x]$ , we define  $w_1 \sim w_2$  if  $E_{w_1} = E_{w_2}$ . Obviously,  $w \approx w'$  implies  $w \sim w'$ .

For example, as noted above  $w \sim w^{-1}$  for every  $w \in G[x]$ .

REMARK 4.6. Let  $w \in G[x]$ , and  $s \in \mathbb{Z}$ . Consider the element  $w^s \in G[x]$ , and note that  $\epsilon(w^s) = s\epsilon(w)$  and  $w^s(g) = (w(g))^s$  for every  $g \in G$ . Hence,  $E_{w^s} = \{g \in G \mid (w(g))^s = e_G\} = f_w^{-1}(G[s])$  is the preimage of  $G[s]$  under  $f_w$ .

In particular, if  $G[s] = \{e_G\}$ , then  $E_{w^s} = E_w$ , i.e.  $w \sim w^s$ .

Then, in describing  $\mathbb{E}_G$ , we can restrict ourselves to a subset  $W \subseteq G[x]$  of representants with respect to the equivalence  $\sim$ , that is such that the quotient set  $W/\sim = G[x]/\sim$ . For example, if  $W \subseteq G[x]$  satisfies  $\mathcal{F}(G) = \{f_w \mid w \in W\}$ , that is  $G[x]/\approx = W/\approx$ , then  $G[x]/\sim = W/\sim$ . As we have seen in §3.4, in the *abelian* case the set  $W = \{gx^n \in G[x] \mid n \in \mathbb{N}, g \in G\}$  satisfies  $G[x]/\approx = W/\approx$ , so that this  $W$  will do.

Finally, note that  $f_w(g) = e_G$  if and only if  $f_{awa^{-1}}(g) = e_G$  holds for every  $a \in G$ , so that  $w \sim awa^{-1}$ , and  $\epsilon(awa^{-1}) = \epsilon(w)$ . Then, in describing  $\mathbb{E}_G$ , there is no harm in assuming that a word  $w = g_1x^{\epsilon_1}g_2x^{\epsilon_2} \cdots g_nx^{\epsilon_n}g_0 \in G[x]$  has  $g_0 = e_G$  (or  $g_1 = e_G$ ); indeed, *from now on, we will often consider exclusively words  $w$  of the form*

$$w = g_1x^{\epsilon_1}g_2x^{\epsilon_2} \cdots g_nx^{\epsilon_n} \in G[x]. \quad (14)$$

LEMMA 4.7. *Let  $v \in G[x]$ . Then  $v \sim w$  for a word  $w \in G[x]$  as in (14), with  $\epsilon(w) = |\epsilon(v)| \geq 0$ .*



*Proof.* By Remark 4.6, we have that  $v \sim v^{-1}$ , and  $\epsilon(v^{-1}) = -\epsilon(v)$ , so that we can assume  $\epsilon(v) \geq 0$ .

Then, by the above discussion,  $v \sim w$  for a word  $w$  as in (14), and with  $\epsilon(w) = \epsilon(v)$ .  $\square$

## 5. Quasi-topological group topologies

Let  $X$  be a set, and  $\lambda$  be an infinite cardinal number. We denote by  $[X]^{<\lambda}$  the family of subsets of  $X$  having size strictly smaller than  $\lambda$ .

As the family  $\mathcal{B} = [X]^{<\lambda} \cup \{X\}$  is stable under taking finite unions and arbitrary intersections, it is the family of closed sets of a topology on  $X$ , denoted by  $co\text{-}\lambda_X$ . For example, taking  $\lambda = \omega$ , one obtains the *cofinite topology*

$$cof_X = co\text{-}\omega_X.$$

For example, for every infinite cardinal number  $\lambda$ , the space  $(G, co\text{-}\lambda_G)$  is a  $T_1$  quasi-topological group. In particular, if  $G$  is infinite,  $(G, cof_G)$  is a  $T_1$ , non-Hausdorff (being Noetherian) quasi-topological group. So if  $G$  is infinite, then  $(G, cof_G)$  is not a topological group. We will use the topologies  $co\text{-}\lambda_G$  on  $G$  as a source of counter-examples in Example 5.5.

### 5.1. General results

In what follows we give some general results for quasi-topological groups. For a reference on this topic, see for example [2].

**THEOREM 5.1.** *Let  $(G, \tau)$  be a quasi-topological group.*

(a) *If  $S \subseteq G$ , then the  $\tau$ -closure of  $S$  is*

$$\bar{S} = \bigcap_{U \in \mathcal{V}_\tau(e_G)} U \cdot S = \bigcap_{V \in \mathcal{V}_\tau(e_G)} S \cdot V.$$

(b) *If  $H$  is a subgroup with non-empty interior, then  $H$  is open.*

(c) *A finite-index closed subgroup of  $G$  is open.*

(d) *The closure of a (normal) subgroup is a (normal) subgroup.*

*Proof.* To prove (a), (b) and (c) one only needs the inversion and shifts to be continuous, so proceed as in the case of topological groups.

(d) Let  $H$  be a subgroup of  $G$ , and  $\bar{H}$  be its  $\tau$ -closure. We have to show that  $\bar{H}$  is a subgroup, that is:  $\bar{H}^{-1} \subseteq \bar{H}$  and  $\bar{H} \cdot \bar{H} \subseteq \bar{H}$ .

The hypothesis that the inversion function is  $\tau$ -continuous guarantees that  $\bar{H}^{-1} \subseteq \overline{H^{-1}} = \bar{H}$ .

In the same way, for every  $h \in H$ , the left translation by  $h$  in  $G$  is  $\tau$ -continuous, so  $h \cdot \bar{H} \subseteq \overline{h \cdot H} = \bar{H}$ ; as this holds for every  $h \in H$ , we get  $H \cdot \bar{H} \subseteq \bar{H}$ .

Now consider the right traslation in  $G$  by an element  $c \in \overline{H}$ . It is  $\tau$ -continuous, so

$$\overline{H} \cdot c \subseteq \overline{H \cdot c} \subseteq \overline{H \cdot \overline{H}} \subseteq \overline{\overline{H}} = \overline{H}.$$

From the above inclusion, we finally deduce  $\overline{H} \cdot \overline{H} \subseteq \overline{H}$ .

Composing translations we obtain that also conjugations are  $\tau$ -continuous; so if  $H$  is a normal subgroup and  $g \in G$ , then

$$g \cdot \overline{H} \cdot g^{-1} \subseteq \overline{g \cdot H \cdot g^{-1}} = \overline{H}.$$

□

Let  $(G, \tau)$  be a quasi-topological group, and  $N$  be a normal subgroup of  $G$ . Consider the quotient group  $\overline{G} = G/N$  and the canonical map  $\pi: (G, \tau) \rightarrow \overline{G}$ . The quotient topology  $\overline{\tau}$  of  $\tau$  on  $\overline{G}$  is the final topology of  $\pi$ , namely  $\overline{\tau} = \{A \subseteq \overline{G} \mid \pi^{-1}(A) \in \tau\}$ . Then, the following results hold.

**PROPOSITION 5.2.** *If  $(G, \tau)$  is a quasi-topological group, then  $(\overline{G}, \overline{\tau})$  is a quasi-topological group, and the map  $\pi: (G, \tau) \rightarrow (\overline{G}, \overline{\tau})$  is continuous and open. In particular,  $\overline{\tau} = \{\pi(X) \subseteq \overline{G} \mid X \in \tau\}$ .*

*Proof.* Proceed as in the case of topological groups to verify that  $(\overline{G}, \overline{\tau})$  is a quasi-topological group.

We prove that  $\pi$  is open. Let  $A \in \tau$ , and note that  $\pi(A) \in \overline{\tau}$  if and only if  $\pi^{-1}\pi(A) \in \tau$ . As

$$\pi^{-1}\pi(A) = A \cdot N = \bigcup_{n \in N} A \cdot n,$$

and  $(\overline{G}, \overline{\tau})$  is a quasi-topological group, we are done. □

**PROPOSITION 5.3.** *If  $(G, \tau)$  is a quasi-topological group, then the following are equivalent.*

- (1)  $N$  is  $\tau$ -closed;
- (2)  $\{e_{\overline{G}}\}$  is  $\overline{\tau}$ -closed;
- (3)  $\overline{\tau}$  is a  $T_1$  topology.

*Proof.* (1)  $\Rightarrow$  (2). Let  $N$  be  $\tau$ -closed. We are going to prove that  $A = \overline{G} \setminus \{e_{\overline{G}}\}$  is  $\overline{\tau}$ -open, and note that this holds if and only if  $\pi^{-1}(A)$  is  $\tau$ -open. As  $\pi^{-1}(A) = G \setminus N$  is  $\tau$ -open by assumption, we are done.

(2)  $\Rightarrow$  (3) holds as  $(\overline{G}, \overline{\tau})$  is a quasi-topological group by Proposition 5.2.

(3)  $\Rightarrow$  (1). If  $\overline{\tau}$  is a  $T_1$  topology, in particular  $\{e_{\overline{G}}\}$  is  $\overline{\tau}$ -closed, so that  $N = \pi^{-1}(\{e_{\overline{G}}\})$  is  $\tau$ -closed, being  $\pi$  continuous. □

## 5.2. Verbal functions

We will now characterize which topologies on a group make it a quasi-topological group, in term of continuity of an appropriate family of verbal functions.

LEMMA 5.4. *Let  $G$  be a group, and  $\tau$  a topology on  $G$ . Then  $(G, \tau)$  is a quasi-topological group if and only if  $f_w : (G, \tau) \rightarrow (G, \tau)$  is continuous for every word of the form  $w = gx^\varepsilon \in G[x]$ , with  $g \in G$  and  $\varepsilon = \pm 1$ .*

*In particular, if a topology  $\sigma$  on a group  $G$  makes continuous every verbal function, then  $(G, \sigma)$  is a quasi-topological group. If  $\sigma$  is also  $T_1$ , then  $\mathfrak{Z}_G \subseteq \sigma$ .*

*Proof.* Let  $\iota$  denote the inversion function of  $G$ . If  $(G, \tau)$  is a quasi-topological group, then every verbal function of the form  $f_{gx}$ , being a left translation, is  $\tau$ -continuous. Then also every  $f_{gx^{-1}} = f_{gx} \circ \iota$  is  $\tau$ -continuous.

For the converse, let  $\tau$  be a topology on  $G$  such that  $f_w : (G, \tau) \rightarrow (G, \tau)$  is continuous for every word  $w = gx^\varepsilon$ , with  $g \in G$  and  $\varepsilon = \pm 1$ . Then items 3 and 5 in Example 1.1 show that the inversion and the left translations are verbal functions of this form, hence are  $\tau$ -continuous. Finally, the right translation by an element  $g$  is  $f_{xg} = f_{x^{-1}} \circ f_{g^{-1}x^{-1}}$ .

For the last part, note that if  $\sigma$  is  $T_1$ , then  $\{e_G\}$  is  $\sigma$ -closed. If moreover every  $f_w$  is  $\sigma$ -continuous, then also every  $E_w = f_w^{-1}(\{e_G\})$  is  $\sigma$ -closed. As  $\mathbb{E}_G$  is a subbase for the  $\mathfrak{Z}_G$ -closed sets, we conclude  $\mathfrak{Z}_G \subseteq \sigma$ .  $\square$

EXAMPLE 5.5. Let  $(G, \tau)$  be a  $T_1$  quasi-topological group. By Lemma 5.4, every verbal function in  $\{f_{gx^\varepsilon} \mid g \in G, \varepsilon = \pm 1\}$  is  $\tau$ -continuous. We shall see that not every verbal function need to be  $\tau$ -continuous.

To this end, recall that the space  $(G, co\text{-}\lambda_G)$  is a  $T_1$  quasi-topological group for every infinite cardinal number  $\lambda$ .

So let  $\omega \leq \lambda < \kappa = |G|$ , note that  $co\text{-}\lambda_G$  is not the discrete topology on  $G$ , and consider  $\tau = co\text{-}\lambda_G$ .

1. Let  $G$  be a group having a non-central element  $a$  such that  $|C_G(a)| \geq \lambda$  (for example, the group  $G = \oplus_{\kappa} S_3$  will do). Then let  $w = [a, x] \in G[x]$ , and consider the commutator verbal function  $f_w \in \mathcal{F}(G)$ . As  $f_w^{-1}(\{e_G\}) = C_G(a) \neq G$ , we have that  $f_w$  is not  $\tau$ -continuous.
2. Let  $G$  be a non-abelian group such that  $|G[2]| \geq \lambda$  (also in this case the group  $G = \oplus_{\kappa} S_3$  considered above will do). Then let  $w = x^2 \in G[x]$ , and consider the verbal function  $f_w \in \mathcal{F}(G)$ . As  $f_w^{-1}(\{e_G\}) = G[2] \neq G$ , we have that  $f_w$  is not  $\tau$ -continuous.

In the following results we prove that  $(G, \mathfrak{Z}_G)$ ,  $(G, \mathfrak{M}_G)$  and  $(G, \mathfrak{P}_G)$  are quasi-topological groups.

PROPOSITION 5.6. *For every group  $G$ , the following hold.*

1. Every verbal function is  $\mathfrak{Z}_G$ -continuous.
2. The pair  $(G, \mathfrak{Z}_G)$  is a quasi-topological group.
3.  $\mathfrak{Z}_G$  is the initial topology of the family of all verbal functions  $\{f: G \rightarrow (G, \mathfrak{Z}_G) \mid f \in \mathcal{F}(G)\}$ .

*Proof.* 1. Follows from the fact that  $\mathbb{E}_G$  is a subbase for the  $\mathfrak{Z}_G$ -closed subsets of  $G$ , and from Lemma 4.4.

2. Immediately follows by Lemma 5.4 and item 1.

3. Also follows by item 1. □

**COROLLARY 5.7.** *Every group topology on a group  $G$  makes continuous every verbal function of  $G$ . In particular  $\mathfrak{M}_G$  and  $\mathfrak{P}_G$  make continuous every verbal function of  $G$ , so  $\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G$ , and all the three are quasi-topological group topologies.*

*Proof.* As a verbal function is a composition of products and inversions, it is continuous with respect to every group topology. The same is true for  $\mathfrak{M}_G$  and  $\mathfrak{P}_G$ , which are intersections of group topologies, then Lemma 5.4 applies. □

If  $N$  is a normal subgroup of a group  $G$ , and  $\overline{G} = G/N$  is the quotient group, consider the quotient topology  $\overline{\mathfrak{Z}}_G$  on  $\overline{G}$ . In the following proposition we prove that every verbal function  $(\overline{G}, \overline{\mathfrak{Z}}_G) \rightarrow (\overline{G}, \overline{\mathfrak{Z}}_G)$  of  $\overline{G}$  is continuous.

**PROPOSITION 5.8.** *Let  $N$  be a normal subgroup of a group  $G$ , and let  $\overline{G} = G/N$ . Then the quotient topology  $\overline{\mathfrak{Z}}_G$  makes continuous every verbal function of  $\overline{G}$ .*

*Proof.* Let  $v = \overline{g}_1 x^{\varepsilon_1} \overline{g}_2 x^{\varepsilon_2} \cdots \overline{g}_n x^{\varepsilon_n} \in \overline{G}[x]$ , and we have to prove that

$$f_v: (\overline{G}, \overline{\mathfrak{Z}}_G) \rightarrow (\overline{G}, \overline{\mathfrak{Z}}_G)$$

is continuous. If  $w = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} \in G[x]$ , then  $v = \overline{w}$ , and in the notation of Remark 3.3, item 2, the following diagram commutes.

$$\begin{array}{ccc} (G, \mathfrak{Z}_G) & \xrightarrow{f_w} & (G, \mathfrak{Z}_G) \\ \pi \downarrow & & \downarrow \pi \\ (\overline{G}, \overline{\mathfrak{Z}}_G) & \xrightarrow{f_{\overline{w}}} & (\overline{G}, \overline{\mathfrak{Z}}_G). \end{array} \quad (15)$$

As  $f_w$  is continuous and  $\overline{\mathfrak{Z}}_G$  is the final topology of the canonical projection  $\pi: (G, \mathfrak{Z}_G) \rightarrow \overline{G}$ , also  $f_{\overline{w}}$  is continuous. □

The main result of this subsection is the following theorem characterizing the normal subgroups  $N$  of  $G$  such that the canonical projection  $\pi: (G, \mathfrak{Z}_G) \rightarrow (\overline{G}, \mathfrak{Z}_{\overline{G}})$  is continuous, where  $\overline{G} = G/N$ .

**THEOREM 5.9.** *Let  $N$  be a normal subgroup of a group  $G$ , and let  $\overline{G} = G/N$ . Then the following conditions are equivalent:*

- (1)  $N$  is  $\mathfrak{Z}_G$ -closed;
- (2)  $\overline{\mathfrak{Z}}_G$  is a  $T_1$  topology;
- (3)  $\mathfrak{Z}_{\overline{G}} \subseteq \overline{\mathfrak{Z}}_G$ ;
- (4) the canonical map  $\pi: (G, \mathfrak{Z}_G) \rightarrow (\overline{G}, \mathfrak{Z}_{\overline{G}})$  is continuous.

*Proof.* (1)  $\Leftrightarrow$  (2) follows by Proposition 5.3 and Proposition 5.6, item 2.

(2)  $\Rightarrow$  (3) follows by Proposition 5.8 and Lemma 5.4.

(3)  $\Rightarrow$  (4). In this case, the map  $\text{id}: (\overline{G}, \overline{\mathfrak{Z}}_G) \rightarrow (\overline{G}, \mathfrak{Z}_{\overline{G}})$  is continuous, and so also the composition

$$(G, \mathfrak{Z}_G) \xrightarrow{\pi} (\overline{G}, \overline{\mathfrak{Z}}_G) \xrightarrow{\text{id}} (\overline{G}, \mathfrak{Z}_{\overline{G}}).$$

(4)  $\Rightarrow$  (1) holds as  $\{e_{\overline{G}}\}$  is  $\mathfrak{Z}_{\overline{G}}$ -closed and  $N = \pi^{-1}(\{e_{\overline{G}}\})$ . □

As an application of Theorem 5.9, we prove in the following corollary that every  $n$ -th center  $Z_n(G)$  of  $G$  is  $\mathfrak{Z}_G$ -closed, where  $Z_n(G) \leq G$  is defined inductively as follows, for  $n \in \mathbb{N}_+$ . Let  $Z_1(G) = Z(G)$ . Consider the quotient group  $G/Z(G)$ , its center  $Z(G/Z(G))$ , and its preimage  $Z_2(G) \leq G$  under the canonical projection  $\pi: G \rightarrow G/Z(G)$ . Proceed by induction to define an ascending chain of characteristic subgroups  $Z_n(G)$ .

**COROLLARY 5.10.** *For every group  $G$ , and every positive integer  $n$ , the subgroup  $Z_n(G)$  is  $\mathfrak{Z}_G$ -closed.*

*Proof.* The center  $Z(G) = Z_1(G)$  is  $\mathfrak{Z}_G$ -closed by Example 1.2, item 2. If  $\overline{G} = G/Z(G)$ , then the projection  $\pi: (G, \mathfrak{Z}_G) \rightarrow (\overline{G}, \mathfrak{Z}_{\overline{G}})$  is continuous by Theorem 5.9. As  $Z(\overline{G})$  is  $\mathfrak{Z}_{\overline{G}}$ -closed, we have that  $Z_2(G) = \pi^{-1}(Z(\overline{G}))$  is  $\mathfrak{Z}_G$ -closed.

Proceed by induction to get the thesis. □

**REMARK 5.11.** Corollary 5.10 can also be proved observing that it is possible to define by induction  $Z_1(G) = Z(G)$  and, for an integer  $i \geq 1$ , and  $x \in G$ , note that  $x \in Z_{i+1}(G)$  if and only if  $[g, x] = gxg^{-1}x^{-1} \in Z_i(G)$  for every  $g \in G$ . Equivalently, if  $w_g = [g, x] \in G[x]$ , then

$$Z_{i+1}(G) = \bigcap_{g \in G} \{x \in G \mid [g, x] \in Z_i(G)\} = \bigcap_{g \in G} f_{w_g}^{-1}(Z_i(G)).$$

As  $f_{w_g}$  is  $\mathfrak{Z}_G$ -continuous by Proposition 5.6, item 1, for every  $g \in G$ , and  $Z_i(G)$  is  $\mathfrak{Z}_G$ -closed by inductive hypothesis, we deduce that  $Z_{i+1}(G)$  is  $\mathfrak{Z}_G$ -closed.

### 5.3. Partial Zariski topologies

Given a subset  $W \subseteq G[x]$ , we consider the family  $\mathcal{E}(W) = \{E_w^G \mid w \in W\} \subseteq \mathbb{E}_G$  of elementary algebraic subsets of  $G$  determined by the words  $w \in W$ . Then, following [4] and [5], we consider the topology  $\mathfrak{T}_W$  having  $\mathcal{E}(W)$  as a subbase for its closed sets.

EXAMPLE 5.12. 1. Note that  $\mathcal{E}(G[x]) = \mathbb{E}_G$ , so  $\mathfrak{T}_{G[x]} = \mathfrak{J}_G$ .

2. Taking  $W = \{gx \mid g \in G\}$ , one obtains that  $\mathcal{E}(W) = \{\{g\} \mid g \in G\}$ , so that  $\mathfrak{T}_W = \text{cof}_G$ .

LEMMA 5.13. *Let  $W \subseteq G[x]$ , and assume that  $gw \in W$ , for every  $w \in W$  and every  $g \in G$ . Then  $\mathfrak{T}_W$  is the initial topology of the family of verbal functions  $\{f_w : G \rightarrow (G, \text{cof}_G) \mid w \in W\}$ .*

*Proof.* If  $\tau$  is such initial topology, then  $\mathcal{F} = \{f_w^{-1}(\{g\}) = f_{g^{-1}w}^{-1}(\{e_G\}) \mid w \in W, g \in G\}$  is a subbase for the  $\tau$ -closed sets.

By assumption,  $\mathcal{F}$  coincides with  $\mathcal{E}(W) = \{E_w = f_w^{-1}(\{e_G\}) \mid w \in W\}$ , so that  $\tau = \mathfrak{T}_W$ .  $\square$

In particular,  $\mathfrak{J}_G$  can be equivalently defined as the initial topology of the family of all verbal functions  $\{f : G \rightarrow (G, \text{cof}_G) \mid f \in \mathcal{F}(G)\}$ .

EXAMPLE 5.14. Let  $a, b \in G$ , and  $w = bxa x^{-1} = ba[a^{-1}, x] \in G[x]$ . Note that  $E_w \neq \emptyset$  if and only if there exists an element  $g \in G$  such that  $b = ga^{-1}g^{-1}$ , i.e.  $b$  and  $a^{-1}$  are conjugated elements in  $G$ . In this case,  $w = (ga^{-1}g^{-1})xa x^{-1}$ .

In particular, letting  $V = \{bxa x^{-1} \mid a, b \in G\} \subseteq G[x]$  and

$$W_{\mathfrak{C}} = \{[g, a][a, x] = (gag^{-1})xa^{-1}x^{-1} \mid a, g \in G\} \subseteq V, \quad (16)$$

we obtain that  $\mathcal{E}(V) \setminus \emptyset = \mathcal{E}(W_{\mathfrak{C}}) \subseteq \mathcal{E}(V)$ , so that  $\mathfrak{T}_V = \mathfrak{T}_{W_{\mathfrak{C}}}$ . Moreover, by Example 4.5, item 2, we have

$$\mathcal{E}(W_{\mathfrak{C}}) = \{gC_G(a) \mid a, g \in G\}. \quad (17)$$

DEFINITION 5.15. *Given a group  $G$ , we denote by  $\mathfrak{C}_G$  the topology  $\mathfrak{T}_{W_{\mathfrak{C}}}$ , for  $W_{\mathfrak{C}} \subseteq G[x]$  as in (16). We call  $\mathfrak{C}_G$  the centralizer topology of  $G$ .*

We will study  $\mathfrak{C}_G$  in more details in §5.4.

By definition, the family  $\mathcal{C} = \{gC_G(a) \mid a, g \in G\}$  is a subbase for the  $\mathfrak{C}_G$ -closed subsets of  $G$ . On the other hand, one can consider the topology  $\mathcal{C}$  generates taking its members as open sets, i.e. the coarsest topology  $\mathcal{T}_G$  on  $G$  such that  $gC_G(a)$  is  $\mathcal{T}_G$ -open, for every  $a, g \in G$ . The topology  $\mathcal{T}_G$  has been introduced by Taïmanov in [22] and is now called the *Taïmanov topology* of  $G$ . See for example [10] for a recent work on this topic.

DEFINITION 5.16. *The Taïmanov topology  $\mathcal{T}_G$  on a group  $G$  is the topology having the family of the centralizers of the elements of  $G$  as a subbase of the filter of the neighborhoods of  $e_G$ .*

It is easy to check that  $\mathcal{T}_G$  is a group topology, and for every element  $g \in G$  the subgroup  $C_G(g)$  is a  $\mathcal{T}_G$ -open (hence, closed) subset of  $G$ . So note that  $\mathfrak{C}_G \subseteq \mathcal{T}_G$  in general (see Lemma 5.33 for a sufficient condition on  $G$  to have  $\mathfrak{C}_G = \mathcal{T}_G$ ).

Note that  $\overline{\{e_G\}}^{\mathcal{T}_G} = Z(G)$ , so  $\mathcal{T}_G$  need not be Hausdorff.

LEMMA 5.17 ([10, Lemma 4.1]). *If  $G$  is a group, then the following hold for  $\mathcal{T}_G$ .*

1.  $\mathcal{T}_G$  is Hausdorff if and only if  $G$  is center-free.
2.  $\mathcal{T}_G$  is indiscrete if and only if  $G$  is abelian.

REMARK 5.18. If  $S \subseteq G$ , let

$$C(S) = \{[g, a][a, x] = (gag^{-1})xa^{-1}x^{-1} \mid g \in G, a \in S\} \subseteq G[x],$$

$$D(S) = \{[xcx^{-1}, b] \mid b, c \in S\} \subseteq G[x].$$

For example,  $C(G) = W_{\mathfrak{C}}$  as in (16), so that  $\mathfrak{T}_{C(G)} = \mathfrak{C}_G$ .

In [3], the authors introduced two *restricted Zariski topologies*  $\mathfrak{Z}'_G, \mathfrak{Z}''_G$  on a group  $G$ , that in our notation are respectively  $\mathfrak{Z}'_G = \mathfrak{T}_{C(G[2]) \cup D(G[2])}$ , and  $\mathfrak{Z}''_G = \mathfrak{T}_{C(G[2])}$ . Obviously,  $\mathfrak{Z}''_G \subseteq \mathfrak{Z}'_G \subseteq \mathfrak{Z}_G$  and  $\mathfrak{Z}''_G \subseteq \mathfrak{C}_G$  hold for every group  $G$ . See also Theorem 5.26.

In the following definition we introduce the partial Zariski topology  $\mathfrak{T}_{mon}$  determined by the monomials. Note that by Lemma 4.7 there is no harm in considering only the monomials with non-negative content. Moreover, by Remark 4.2 we can indeed consider only positive-content monomials.

DEFINITION 5.19. *If  $M = \{gx^n \mid g \in G, n \in \mathbb{N}_+\} \subseteq G[x]$  is the family of the monomials with positive content, then we denote by  $\mathfrak{T}_{mon}$  the topology having  $\mathcal{E}(M)$  as a subbase for its closed sets, and we call it the monomial topology of  $G$ .*

Note that  $gx \in M$  for every  $g \in G$ , so that  $\mathfrak{T}_{mon}$  is  $T_1$  topology.

EXAMPLE 5.20. Let  $G$  be abelian. We have seen in §3.4 that  $w \approx w_{ab}$  for every  $w \in G[x]$ . As in studying  $E_w$  we can assume  $\epsilon(w) \geq 0$  by Lemma 4.7, this shows that  $\mathfrak{T}_{mon} = \mathfrak{Z}_G$ .

In [21], Ol'shanskij built the first example of a countable group  $G$  with  $\mathfrak{Z}_G = \delta_G$ , so that  $\mathfrak{Z}_G = \mathfrak{M}_G = \delta_G$ . A closer look at his proof reveals that really also  $\mathfrak{T}_{mon} = \delta_G$  for such a group  $G$ .

Recall that a group  $G$  is said to *satisfy the cancellation law* if  $x^n = y^n$  implies  $x = y$ , for every  $n \in \mathbb{N}_+$  and  $x, y \in G$ . Here we recall also that a group  $G$  is called:

- *nilpotent* if  $Z_n(G) = G$  for some  $n \in \mathbb{N}_+$ ,
- *torsion-free* if every element has infinite order.

It is a classical result due to Chernikov, that if  $G$  is a nilpotent, torsion-free group, then  $G$  satisfies the cancellation law.

**COROLLARY 5.21.** *If  $G$  satisfies the cancellation law, then  $\mathfrak{T}_{mon} = \text{cof}_G$ . In particular,  $\mathfrak{T}_{mon} = \text{cof}_G$  for every nilpotent, torsion-free group  $G$ .*

*Proof.* It suffices to prove that  $E_w$  has at most one element, for every monomial  $w = gx^m \in G[x]$  with  $m > 0$ . So let  $w = gx^m$ , and assume  $a \in E_w$ , so that  $a^m = g^{-1}$ . Then  $E_w = \{p \in G \mid p^m = a^m\}$ , so that  $E_w = \{a\}$ .

If  $G$  is a nilpotent, torsion-free group, then Chernikov's result applies.  $\square$

### 5.3.1. Permutation groups

In what follows,  $X$  is an infinite set. For a subgroup  $G \leq S(X)$  of the permutation group of  $X$ , recall that  $\tau_p(G)$  denote the *point-wise convergence topology* of  $G$ . Then  $\tau_p(G)$  is a Hausdorff group topology, so that  $\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \tau_p(G)$  for every group  $G \leq S(X)$ . The following classic result was proved by Gaughan in 1967.

**THEOREM 5.22** ([18]). *Let  $G = S(X)$ . Then  $\tau_p(G)$  is contained in every Hausdorff group topology on  $G$ .*

In particular, it follows from Theorem 5.22 that  $\mathfrak{M}_{S(X)} = \tau_p(S(X))$  is itself a Hausdorff group topology.

Ten years after Gaughan's Theorem 5.22, Dierolf and Schwanengel (unaware of his result) proved the following:

**THEOREM 5.23** ([9]). *Let  $S_\omega(X) \leq G \leq S(X)$ . Then  $\tau_p(G)$  is a minimal Hausdorff group topology.*

Although Theorem 5.23 provides new results for groups  $S_\omega(X) \leq G \leq S(X)$ , Theorem 5.22 gives a much stronger result for the whole group  $S(X)$ . That is why Dikranjan conjectured the following.

**CONJECTURE 5.24** ([19]). *Let  $S_\omega(X) \leq G \leq S(X)$ . Then  $\mathfrak{M}_G = \tau_p(G)$ .*



The following question was raised by Dikranjan and Shakhmatov (see Theorem 5.22).

QUESTION 5.25 ([11]). *Does  $\mathfrak{M}_{S(X)}$  coincide with  $\mathfrak{Z}_{S(X)}$ ?*

It has recently turned out that Dikranjan’s conjecture is true, and Dikranjan-Shakhmatov’s question has a positive answer. It has been proved in [3] that  $\mathfrak{Z}_G = \mathfrak{M}_G$  is the pointwise convergence topology for all subgroups  $G$  of infinite permutation groups  $S(X)$ , that contain the subgroup  $S_\omega(X)$  of all permutations of finite support.

THEOREM 5.26 ([3]). *If  $S_\omega(X) \leq G \leq S(X)$ , then  $\mathfrak{Z}_G'' \subsetneq \mathfrak{Z}'_G = \mathfrak{Z}_G = \mathfrak{M}_G = \tau_p(G)$ .*

As a corollary of Theorem 5.26, the same authors have obtained the following answer to another question posed by Dikranjan and Shakhmatov.

COROLLARY 5.27 ([3]). *The class of groups  $G$  satisfying  $\mathfrak{Z}_G = \mathfrak{M}_G$  is not closed under taking subgroups.*

*Proof.* Let  $H$  be a group such that  $\mathfrak{Z}_H \neq \mathfrak{M}_H$ , embed it in  $G = S(H)$ , and apply Theorem 5.26 to conclude that  $\mathfrak{Z}_G = \mathfrak{M}_G$ . □

### 5.4. Centralizer topologies

In this subsection, we study two partial Zariski topologies. The first one is the topology  $\mathfrak{C}_G$  introduced in Definition 5.15. As we shall see in Proposition 5.30, item 3, the topology  $\mathfrak{C}_G$  is not  $T_1$  in general, so  $\mathfrak{C}_G \neq \mathfrak{Z}_G$  in general. As  $\mathfrak{C}_G \subseteq \mathfrak{Z}_G$ , we can still consider the coarsest  $T_1$  topology  $\mathfrak{C}'_G$  on  $G$  such that

$$\mathfrak{C}_G \subseteq \mathfrak{C}'_G \subseteq \mathfrak{Z}_G.$$

The cofinite topology  $cof_G$  being the coarsest  $T_1$  topology on  $G$ , in the following definition we introduce the  $T_1$ -refinement topology  $\mathfrak{C}'_G$  of  $\mathfrak{C}_G$ .

DEFINITION 5.28. *The  $T_1$  centralizer topology  $\mathfrak{C}'_G$  on a group  $G$  is the supremum (in the lattice of all topologies on  $G$ )*

$$\mathfrak{C}'_G = \mathfrak{C}_G \vee cof_G.$$

REMARK 5.29. Then  $\mathfrak{C}'_G$  is  $T_1$ , and  $\mathfrak{C}_G \subseteq \mathfrak{C}'_G \subseteq \mathfrak{Z}_G$ , so that  $\mathfrak{C}_G = \mathfrak{C}'_G$  if and only if  $\mathfrak{C}_G$  is  $T_1$ .

Let  $W = W_{\mathfrak{C}} \cup \{gx \mid g \in G\}$ . Then  $\mathfrak{C}'_G = \mathfrak{T}_W$ , as

$$\mathcal{E}(W) = \mathcal{E}(W_{\mathfrak{C}}) \cup \{\{g\} \mid g \in G\} = \{gC_G(a) \mid a, g \in G\} \cup \{\{g\} \mid g \in G\},$$

where the second equality follows from (17). Obviously,  $\mathfrak{C}'_G = \mathfrak{T}_{W'}$  also for  $W' = \{axbx^{-1} \mid a, b \in G\} \cup \{xg \mid g \in G\}$ .

In what follows, we denote by  $\iota_G$  the indiscrete topology on  $G$ , namely

$$\iota_G = \{\emptyset, G\} \subseteq \mathcal{P}(G).$$

Here follows some easy-to-establish properties of the centralizer topologies  $\mathfrak{C}_G$  and  $\mathfrak{C}'_G$ .

PROPOSITION 5.30. *Let  $G$  be a group. Then the following hold.*

1. *Both the pair  $(G, \mathfrak{C}_G)$  and  $(G, \mathfrak{C}'_G)$  are quasi-topological groups.*
2.  *$\overline{\{e_G\}}^{\mathfrak{C}_G} = Z(G)$ .*
3.  *$\mathfrak{C}_G$  is  $T_1$  (so  $\mathfrak{C}_G = \mathfrak{C}'_G$ ) if and only if  $Z(G) = \{e_G\}$ , while  $\mathfrak{C}_G = \iota_G$  if and only if  $G = Z(G)$  is abelian.*
4. *If  $H \leq G$ , then  $\mathfrak{C}_H \subseteq \mathfrak{C}_G \upharpoonright_H$  and  $\mathfrak{C}'_H \subseteq \mathfrak{C}'_G \upharpoonright_H$ .*

*Proof.* (1) is straightforward.

(2). As  $Z(G) = \bigcap_{g \in G} C_G(g)$  is  $\mathfrak{C}_G$ -closed, one only has to verify that every  $\mathfrak{C}_G$ -closed subset containing  $e_G$  must also contain  $Z(G)$ .

(3). Immediately follows from items (1) and (2).

(4). To prove that  $\mathfrak{C}_H \subseteq \mathfrak{C}_G \upharpoonright_H$ , it suffices to note that for every element  $h \in H$  we have that  $C_H(h) = C_G(h) \cap H$  is a  $\mathfrak{C}_G \upharpoonright_H$ -closed subset of  $H$ .

To prove the inclusion  $\mathfrak{C}'_H \subseteq \mathfrak{C}'_G \upharpoonright_H$ , note that  $\text{cof}_H = \text{cof}_G \upharpoonright_H$ , so that

$$\mathfrak{C}'_H = \mathfrak{C}_H \vee \text{cof}_H \subseteq \mathfrak{C}_G \upharpoonright_H \vee \text{cof}_G \upharpoonright_H \subseteq (\mathfrak{C}_G \vee \text{cof}_G) \upharpoonright_H = \mathfrak{C}'_G \upharpoonright_H.$$

□

We shall see in §5.4.1 that every free non-abelian group  $F$  satisfies  $\mathfrak{C}_F = \mathfrak{C}'_F = \mathfrak{Z}_F$ . On the other hand, we consider a class of matrix groups  $H$  in §5.4.2, satisfying  $\mathfrak{C}_H \neq \mathfrak{C}'_H = \mathfrak{Z}_H$ .

EXAMPLE 5.31. Let us show that the inclusion  $\mathfrak{C}_H \subseteq \mathfrak{C}_G \upharpoonright_H$  in Proposition 5.30, item 4, may be proper. To this end, it will suffice to consider a group  $G$  having an abelian, non-central subgroup  $H$ , so that

$$\iota_H = \mathfrak{C}_H \subsetneq \mathfrak{C}_G \upharpoonright_H.$$

Indeed,  $\iota_H = \mathfrak{C}_H$  holds by Proposition 5.30, item 3, as  $H$  is abelian, while  $\emptyset \neq Z(G) \cap H \subsetneq H$  is a  $\mathfrak{C}_G \upharpoonright_H$ -closed subset of  $H$ .

PROPOSITION 5.32. *Let  $G$  be a group,  $\overline{G} = G/Z(G)$ , and  $\tau$  be the initial topology on  $G$  of the canonical projection map*

$$\pi: G \rightarrow (\overline{G}, \text{cof}_{\overline{G}}). \quad (18)$$

*Then  $\tau \subseteq \mathfrak{C}_G$ .*

*Moreover,  $\mathfrak{C}_G = \tau$  if and only if for every  $g \in G \setminus Z(G)$  the index  $[C_G(g) : Z(G)]$  is finite.*

*Proof.* As the family of singletons of  $\overline{G}$  is a subbase for the  $\text{cof}_{\overline{G}}$ -closed sets, and  $\pi^{-1}(\{gZ(G)\}) = gZ(G)$  is  $\mathfrak{C}_G$ -closed for every  $g \in G$  by Proposition 5.30, items 1 and 2, we immediately obtain  $\tau \subseteq \mathfrak{C}_G$ .

For the reverse inclusion, we have that  $\mathfrak{C}_G \subseteq \tau$  if and only if  $C_G(g)$  is  $\tau$ -closed for every  $g \in G$  by Proposition 5.30, item 1. As  $C_G(g) = G$  is certainly  $\tau$ -closed for an element  $g \in Z(G)$ , it is sufficient to consider the case when  $C_G(g)$  is  $\tau$ -closed for every  $g \in G \setminus Z(G)$ .

Finally note that, if  $g \in G \setminus Z(G)$ , then  $G \geq C_G(g) \geq Z(G)$ . So  $C_G(g) = \pi^{-1}(\pi(C_G(g)))$  is  $\tau$ -closed exactly when  $\pi(C_G(g)) = C_G(g)/Z(G)$  is finite.  $\square$

Recall that a group  $G$  is called an *FC-group* if the index  $[G : C_G(F)]$  is finite for every  $F \in [G]^{<\omega}$ , or equivalently if  $[G : C_G(g)]$  is finite for every  $g \in G$ .

Now we prove that the centralizer topology  $\mathfrak{C}_G$  and the Taimanov topology  $\mathcal{T}_G$  coincide on an FC-group  $G$ .

LEMMA 5.33. *If  $G$  is an FC-group, then  $\mathfrak{C}_G = \mathcal{T}_G$ .*

*Proof.* The inclusion  $\mathfrak{C}_G \subseteq \mathcal{T}_G$  holds for every group, so we prove the reverse one. To this end, it suffices to prove that  $C_G(F)$  is a  $\mathfrak{C}_G$ -neighborhood of  $e_G$ , for every  $F \in [G]^{<\omega}$ . So let  $F \in [G]^{<\omega}$ , and note that  $C_G(F)$  is a finite-index subgroup as  $G$  is an FC-group. As  $(G, \mathfrak{C}_G)$  is a quasi-topological group by Proposition 5.30, item 1, we can apply Theorem 5.1 (c) to conclude that  $C_G(F)$  is  $\mathfrak{C}_G$ -open.  $\square$

### 5.4.1. The Zariski topology of free non-abelian groups

Let  $F$  be a free non-abelian group, and let

$$\mathcal{B} = \{\{f\}, fC_F(g) \mid f, g \in F\} \subseteq \mathbb{E}_F.$$

By Remark 5.29, the family  $\mathcal{B}$  is a subbase for the  $\mathfrak{C}'_F$ -closed subsets.

PROPOSITION 5.34 ([8, Theorem 5.3]). *Arbitrary intersections of proper elementary algebraic subsets of  $F$  are elements of  $\mathcal{B}^\cup$ .*

In the original statement of Proposition 5.34, the authors used the family

$$\{\{f\}, fC_F(g)h \mid f, g, h \in F\}$$

instead of  $\mathcal{B}$ . Recall that  $fC_F(g)h = fhC_F(h^{-1}gh)$ , so that really the two families coincide.

THEOREM 5.35. *If  $F$  is a free non-abelian group, then  $\mathfrak{C}_F = \mathfrak{C}'_F = \mathfrak{Z}_F$ .*

*Proof.* It trivially follows from Proposition 5.34 that  $\mathbb{E}_F \subseteq \mathcal{B}^\cup$ , so that  $\mathcal{B} \subseteq \mathbb{E}_G$  yields  $\mathbb{E}_F^\cup = \mathcal{B}^\cup$ . Then  $\mathfrak{Z}_F = \mathfrak{C}'_F$ , while  $\mathfrak{C}_F = \mathfrak{C}'_F$  holds by Proposition 5.30, item 3.  $\square$

### 5.4.2. The Zariski topology of Heisenberg groups

If  $n$  is a positive integer, and  $K$  is an infinite field, the  $n$ -th Heisenberg group with coefficients in  $K$  is the matrix group

$$H = H(n, K) = \left\{ \begin{pmatrix} 1 & x_1 & \cdots & x_n & y \\ & 1 & & 0 & z_1 \\ & 0 & \ddots & & \vdots \\ & 0 & & 1 & z_n \\ & 0 & & 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+2}(K) \mid x_1, \dots, x_n, z_1, \dots, z_n, y \in K \right\}.$$

As  $Z(H) \cong K$  is not trivial, Proposition 5.30, item 3, implies  $\mathfrak{C}_H \neq \mathfrak{C}'_H$ .

In [16], we have computed the Zariski topology of  $H(1, K)$ . It follows from [16, Remark 6.9] that  $\mathfrak{Z}_{H(1, K)} = \mathfrak{C}'_{H(1, K)}$  when  $\mathrm{char} K \neq 2$ .

If  $\mathrm{char} K \neq 2$ , then it can also be proved using the same techniques that  $\mathfrak{C}'_H = \mathfrak{Z}_H$ , so that

$$\mathfrak{C}_H \neq \mathfrak{C}'_H = \mathfrak{Z}_H$$

for every  $n \in \mathbb{N}_+$ .

## Acknowledgements

We gratefully thank Dikran Dikranjan for the valuable corrections and suggestions on the first draft of this paper. We also thank the referee for the careful reading and the useful comments.

## REFERENCES

- [1] J. F. ADAMS, *Lectures on Lie groups*, W. A. Benjamin Inc., New York-Amsterdam (1969), xii+182.
- [2] A. ARHANGEL'SKIĬ AND M. TKACHENKO, *Topological groups and related structures*, Atlantis Studies in Mathematics, 1. Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2008), xiv+781.
- [3] T. BANAKH, I. GURAN, AND I. PROTASOV, *Algebraically determined topologies on permutation groups*, *Topology Appl.* **159** (2012), 2258–2268.
- [4] T. BANAKH AND I. PROTASOV, *Zariski topologies on groups*, preprint (<http://arxiv.org/abs/1001.0601>).
- [5] T. BANAKH, I. PROTASOV, AND O. SIPACHEVA, *Topologization of sets endowed with an action of a monoid*, *Topology Appl.*, <http://dx.doi.org/10.1016/j.topol.2014.02.041>.
- [6] G. BAUMSLAG, A. MYASNIKOV, AND V. REMESLENNIKOV, *Algebraic geometry over groups. I: Algebraic sets and ideal theory*, *J. Algebra* **219** (1999), 16–79.
- [7] R.M. BRYANT, *The verbal topology of a group*, *J. Algebra* **48** (1977), 340–346.

- [8] I.M. CHISWELL AND V. REMESLENNIKOV, *Equations in free groups with one variable. I*, J. Group Theory **3** (2000), 445–466.
- [9] S. DIEROLF AND U. SCHWANENGEL, *Un exemple d'un groupe topologique  $q$ -minimal mais non precompact*, Bull. Sci. Math. (2) **101:3** (1977), 265–269.
- [10] D. DIKRANJAN AND A. GIORDANO BRUNO, *Arnautov's problems on semitopological isomorphisms*, Applied General Topology **10** (2009), 85–119.
- [11] D. DIKRANJAN AND D. SHAKHMATOV, *Selected topics from the structure theory of topological groups*, Open problems in topology II, E. Perl. Ed., Elsevier (2007), 389–406.
- [12] D. DIKRANJAN AND D. SHAKHMATOV, *Reflection principle characterizing groups in which unconditionally closed sets are algebraic*, J. Group Theory **11** (2008), 421–442.
- [13] D. DIKRANJAN AND D. SHAKHMATOV, *The Markov-Zariski topology of an abelian group*, J. Algebra **324** (2010), 1125–1158.
- [14] D. DIKRANJAN AND D. TOLLER, *The universal exponents of a group*, work in progress.
- [15] D. DIKRANJAN AND D. TOLLER, *Markov's problems through the looking glass of Zariski and Markov topologies*, Ischia Group Theory 2010, Proc. of the Conference, World Scientific Publ. Singapore (2012), 87–130.
- [16] D. DIKRANJAN AND D. TOLLER, *The Markov and Zariski topologies of some linear groups*, Topology Appl. **159** (2012), 2951–2972.
- [17] D. DIKRANJAN AND D. TOLLER, *Productivity of the Zariski topology on groups*, Comment. Math. Univ. Carolin. **54** (2013), 219–237.
- [18] E.D. GAUGHAN, *Topological group structures of infinite symmetric groups*, Proc. Nat. Acad. Sci. U.S.A. **58** (1967), 907–910.
- [19] G. LUKÁCS, *Report on the open problems session*, Topology Appl. **159** (2012), 2476–2482.
- [20] A.A. MARKOV, *On unconditionally closed sets*, Comptes Rendus Doklady AN SSSR (N.S.) **44** (1944), 180–181 (in Russian).
- [21] A.YU. OL'SHANSKIJ, *A note on countable non-topologizable groups*, Vestnik Mosk. Gos. Univ. Mat. Mekh. **3** (1980), 103.
- [22] A.D. TAĪMANOV, *Topologizable groups. II.*, Sibirsk. Mat. Zh. **19** (1978), 1201–1203, 1216 (in Russian; English translation: Siberian Math. J. **19** (1978), 848–850).

Author's address:

Daniele Toller  
Dipartimento di Matematica e Informatica,  
Università di Udine,  
Via delle Scienze, 206  
33100 Udine, Italy  
E-mail: [toller.daniele@gmail.com](mailto:toller.daniele@gmail.com)

Received April 12, 2013

Revised April 15, 2014