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A Note on Fixed Fuzzy Points for Fuzzy Mappings

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Summary. - *We prove a fixed fuzzy point theorem for fuzzy contraction mappings (in the S. Heilpern's sense) over a complete metric space, and as a consequence we obtain a fixed point theorem in the context of intuitionistic fuzzy sets.*

1. Introduction

After the introduction of the concept of a fuzzy set by Zadeh, several researches were conducted on the generalizations of the concept of a fuzzy set. The idea of intuitionistic fuzzy set is due to Atanassov [1], [2], [3] and recently Çoker [4] has defined the concept of intuitionistic fuzzy topological space which generalizes the concept of fuzzy topological space introduced by Chang [5]. Heilpern [6] introduced the concept of a fuzzy mapping and proved a fixed point theorem for fuzzy contraction mappings which is a generalization of the fixed point theorem for multivalued mappings of Nadler [7]. In this paper we give a fixed fuzzy point theorem for fuzzy contraction mappings over a complete metric space, which is a generalization of the given by S. Heilpern for fixed points. Then, we introduce the concept

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of intuitionistic fuzzy mapping and give an intuitionistic version of Heilpern's mentioned theorem.

2. Preliminaries

Let X be a nonempty set and $I = [0, 1]$. A fuzzy set of X is an element of I^X. For $A, B \in I^X$ we denote $A \subset B$ iff $A(x) \leq B(x)$, $\forall x \in X$.

Definition 2.1. *(Atanassov [3]) An intuitionistic fuzzy set (i-fuzzy set, for short) A of X is an object having the form $A =< A^1, A^2 >$ where $A^1, A^2 \in I^X$ and $A^1(x) + A^2(x) \leq 1$, $\forall x \in X$.*

We denote $IFS(X)$ the family of all i-fuzzy sets of X.

Remark 2.2. *If $A \in I^X$, then A is identified with the i-fuzzy set $< A, 1 - A >$ denoted by $[A]$.*

For $x \in X$ we write $\{x\}$ the characteristic function of the ordinary subset $\{x\}$ of X. For $\alpha \in]0, 1]$ the fuzzy point [8] x_α of X is the fuzzy set of X given by $x_\alpha(x) = \alpha$ and $x_\alpha(z) = 0$ if $z \neq x$. Now we give the following definition.

Definition 2.3. *Let x_α be a fuzzy point of X. We will say $< x_\alpha, 1- x_\alpha >$ is an i-fuzzy point of X and it will be denoted by $[x_\alpha]$. In particular $[x] =< \{x\}, 1 - \{x\} >$ will be called an i-point of X.*

Definition 2.4. *(Atanassov [3]) Let $A, B \in IFS(X)$. Then $A \subset B$ iff $A^1 \subset B^1$ and $B^2 \subset A^2$.*

Remark 2.5. *Notice $[x_\alpha] \subset A$ iff $x_\alpha \subset A^1$.*

Let (X, d) be a metric linear space. Recall the $\alpha - level$ A_α of $A \in I^X$ is defined by $A_\alpha = \{x \in X : A(x) \geq \alpha\}$ for each $\alpha \in]0, 1]$, and $A_0 = cl(\{x \in X : A(x) > 0\})$ where $cl(B)$ is the closure of B. Heilpern [6] called fuzzy mapping a mapping from the set X into a family $W(X) \subset I^X$ defined as follows: $A \in W(X)$ iff A_α is compact and convex in X for each $\alpha \in [0, 1]$ and $\sup\{A(x) : x \in X\} = 1$. In this context we give the following definitions.

DEFINITION 2.6. *Let* $A, B \in W(X)$, $\alpha \in [0,1]$. *Define*

$$p_\alpha(A,B) = \inf\{d(x,y) : x \in A_\alpha,\ y \in B_\alpha\},$$

$$D_\alpha = H(A_\alpha, B_\alpha),$$

$$D(A,B) = \sup_\alpha D_\alpha(A,B), \textit{ where } H \textit{ is the Hausdorff distance.}$$

For $x \in X$ we write $p_\alpha(x,B)$ instead of $p_\alpha(\{x\},B)$.

DEFINITION 2.7. *Let* X *be a metric space and* $\alpha \in [0,1]$. *Consider the following family* $W_\alpha(X)$*:*

$$W_\alpha(X) = \{A \in I^X\ :\ A_\alpha \textit{ is nonempty and compact}\}.$$

Now, we define the family $\Phi_\alpha(X)$ *of i-fuzzy sets of* X *as follows:*

$$\Phi_\alpha(X) = \{A \in IFS(X)\ :\ A^1 \in W_\alpha(X)\}$$

Clearly, for $\alpha \in I$, $W(X) \subset \Phi_\alpha(X)$ in the sense of Remark 2.2.

We will use the following lemmas which are adequate modifications of the ones given in [6] for the family $W(X)$, when (X,d) is a metric space.

LEMMA 2.8. *Let* $x \in X$ *and* $A \in W_\alpha(X)$. *Then* $x_\alpha \subset A$ *if* $p_\alpha(x,A) = 0$.

LEMMA 2.9. $p_\alpha(x,A) \leq d(x,y) + p_\alpha(y,A)$, *for* $x, y \in X$, $A \in W_\alpha(X)$.

LEMMA 2.10. *If* $x_\alpha \subset A$, *then* $p_\alpha(x,B) \leq D_\alpha(A,B)$, *for each* $A, B \in W_\alpha(X)$

3. Fixed fuzzy point theorem

In mathematical programming, problems are expressed as optimizing some goal function given certain constraints and there are real-life problems that consider multiple objectives. Generally, it is very difficult to get a feasible solution that carries us to the optimum of all the objective functions. A possible method of resolution that is quite useful is one using Fuzzy Sets. The idea is to relax the pretenses of optimization by means of a subjective gradation which can be modelled into fuzzy membership functions μ_i . If $F = \cap\mu_i$ the objective will be to search x such that max $F = F(x)$. If $\max F = 1$, then there exists x such that $F(x) = 1$, but if $\max F = \alpha, \alpha \in]0,1[$ the solution of the multiobjective optimization is a fuzzy point x_α and $F(x) = \alpha$.

In a more general sense than the one given by Heilpern, a mapping $F : X \longrightarrow I^X$ is a fuzzy mapping over X ([9]) and $(F(x))(x)$ is the fixed degree of x for F. In this context we give the following definition.

Definition 3.1. *Let x_α be a fuzzy point of X. We will say that x_α is a fixed fuzzy point of the fuzzy mapping F over X if $x_\alpha \subset F(x)$ (i.e., the fixed degree of x is at least α). In particular, and according to [6] , if $\{x\} \subset F(x)$ we say that x is a fixed point of F.*

The next proposition is a generalization of [6] Theorem 3.1.

Theorem 3.2. *Let $\alpha \in]0,1]$ and let (X,d) be a complete metric space. Let F be a fuzzy mapping from X into $W_\alpha(X)$ satisfying the following condition: there exists $q \in]0,1[$ such that*

$$D_\alpha(F(x), F(y)) \leq qd(x,y), \text{ for each } x, y \in X.$$

Then there exists $x \in X$ such that x_α is a fixed fuzzy point of F. In particular if $\alpha = 1$, then x is a fixed point of F.

Proof. Let $x_0 \in X$. Since $(F(x_0))_\alpha \neq \emptyset$, then there exists $x_1 \in X$ such that $x_1 \in (F(x_0))_\alpha$. there exists $x_2 \in (F(x_0))_\alpha$. Since $(F(x_1))_\alpha$ is a nonempty compact subset of X, then there exists $x_2 \in (F(x_1))_\alpha$ such that

$$d(x_1, x_2) = d(x_1, (F(x_1))_\alpha) \leq D_\alpha(F(x_0), F(x_1))$$

by Lemma 2.10. By induction we construct a sequence (x_n) in X such that $x_n \in (F(x_{n-1}))_\alpha$ and $d(x_n, x_{n+1}) \leq D_\alpha(F(x_{n-1}), F(x_n))$.

In a similar way to the proof of [6] Theorem 3.1., it is proved that (x_n) is a Cauchy sequence. Suppose (x_n) converges to $x \in X$. Now, by Lemmas 2.9, 2.10 we have

$$p_\alpha(x,F(x)) \leq d(x,x_n)+p_\alpha(x_n,F(x)) \leq d(x,x_n)+D_\alpha(F(x_{n-1}),F(x))$$
$$\leq d(x,x_n)+qd(x_{n-1},x)$$

In consequence $p_\alpha(x,F(x)) = 0$ and by Lemma 2.8, $x_\alpha \subset F(x)$. □

REMARK 3.3. *Theorem 3.1 of [6] states the existence of a fixed point of the fuzzy mapping* $F : X \longrightarrow W(X)$ *whenever* $D(F(x),F(y)) \leq qd(x,y)$ *for each* $x,y \in X$, *being* $q \in]0,1[$. *Now, in the light of the above theorem, it is clear that the condition* $D(F(x),F(y)) \leq qd(x,y)$ *can be weakened to* $D_1(F(x),F(y)) \leq qd(x,y)$.

The next example illustrates when the above theorem has certain advantages if compared with Heilpern's (Theorem 3.1 of [6]).

EXAMPLE 3.4. *Take* $a,b,c \in]-\infty,+\infty[$, *such that* $a < b < c$. *Let* $X = \{a,b,c\}$ *and* $d : X \times X \longrightarrow [0,+\infty[$ *the Euclidean metric. Let* $\alpha \in]0,0.5[$ *and suppose* $F : X \longrightarrow I^X$ *defined by*

$$F(a)(x) = \begin{cases} 1 & \text{if } x=a \\ 2\alpha & \text{if } x=b \\ \alpha/2 & \text{if } x=c \end{cases} \qquad F(b)(x) = \begin{cases} 1 & \text{if } x=a \\ \alpha & \text{if } x=b \\ \alpha/2 & \text{if } x=c \end{cases}$$

$$F(c)(x) = \begin{cases} 1 & \text{if } x=a \\ \alpha & \text{if } x=b \\ 0 & \text{if } x=c \end{cases}$$

Then

$$F(a)_1 = F(b)_1 = F(c)_1 = \{a\},\ F(a)_\alpha = F(b)_\alpha = F(c)_\alpha = \{a,b\},$$
$$F(a)_{\frac{\alpha}{2}} = F(b)_{\frac{\alpha}{2}} = \{a,b,c\} \text{ and } F(c)_{\frac{\alpha}{2}} = \{a,b\}\,.$$

Consequently

$$D_1(F(x),F(y)) = H(F(x)_1,F(y)_1) = 0,$$
$$D_\alpha(F(x),F(y)) = H(F(x)_\alpha,F(y)_\alpha) = 0,\ \forall x,y \in X\ .$$

By Theorem 3.2 there exists a fixed fuzzy point x_1 (a fixed point) and a fixed fuzzy point x_α of the fuzzy mapping F. We can see by the definition of F that a is a fixed point and b_α is a fuzzy point. Nevertheless Heilpern's theorem is not useful in this example because $D_{\frac{\alpha}{2}}(F(a), F(c)) = H(\{a,b,c\}, \{a,b\}) \geq d(a,c)$ and then $D(F(a), F(c)) = \sup_r H(F(a)_r, F(c)_r) \geq d(a,c)$.

4. Fixed i-fuzzy point

DEFINITION 4.1. *An i-fuzzy mapping over X is a mapping F from X into $IFS(X)$. We will say that $[x_\alpha]$ is a fixed i-fuzzy point of F if $[x_\alpha] \subset F(x)$. In particular we will say that $[x]$ is a fixed i-point of F if $[x] \subset F(x)$.*

DEFINITION 4.2. *Let (X,d) be a metric space and $0 < \alpha \leq 1$. For $A, B \in \Phi_\alpha(X)$ we define $D^*_\alpha(A,B) = max\{D_\alpha(A^1, B^1), D_\alpha(1 - A^2, 1 - B^2)\}$. Clearly, D^*_α is a pseudometric on $\Phi_\alpha(X)$.*

REMARK 4.3. *Let $A, B \in W_\alpha(X)$. If we consider $[A], [B] \in \Phi_\alpha(X)$ then*

$$D^*_\alpha([A],[B]) = \max\{D_\alpha(A,B), D_\alpha(1-(1-A), 1-(1-B)) = D_\alpha(A,B)$$

Now, as a consequence of Theorem 3.2 we have the following corollary.

COROLLARY 4.4. *Let (X,d) be a complete metric space and let F be an i-fuzzy mapping from X into $\Phi_\alpha(X)$, $\alpha \in]0,1]$, satisfying the following condition: There exists $q \in]0,1[$ such that*

$$D^*_\alpha(F(x), F(y)) \leq qd(x,y), \text{ for each } x, y \in X$$

Then there exists $x \in X$ such that $[x_\alpha]$ is a fixed i-fuzzy point of F. In particular if $\alpha = 1$, then $[x]$ is a fixed i-point of F.

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