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Semi-free Circle Actions: the Multiplicative Structure

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Summary. - *In this paper we study the bordism groups of manifolds with semi-free S^1-actions, denoted by $SF_n(S^1)$. We study the multiplicative structure by using a J-homomorphism map. We also study the construction K, which gives a set of multiplicative generators, presenting an algebraic interpretation of this geometric construction. As an application, we analyze the homomorphisms $r_p : SF_*(S^1) \to SF_*(\mathbb{Z}_p)$ from the bordism group of semi-free S^1-actions on the bordism group of $\mathbb{Z}_p$-actions induced by the restriction functors.*

1. Introduction

In the early seventies equivariant bordism was a very active field of study. It has recently regained importance, mainly due to the latest studies done by E. Witten in which to know well the bordism of circle actions is very important.

The module structure of bordism of semi-free circle actions was done by F. Uchida (see [7]). The bordism groups of manifolds with semi-free S^1-actions are denoted by $SF_n(S^1)$. In Section 2 we present some known basic results about these groups and some important properties of the Smith homomorphism.

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In Section 3 we introduce the products in $N_*(S^1) = \oplus_{n\geq 0} N_n(S^1)$, where $N_n(S^1)$ is the bordism group of free S^1-actions on n-dimensional closed manifolds.

We study the multiplicative structure, in Section 5, by using a J-homomorphism map (analogous to the one introduced by J.M. Boardman in [2]), which is presented in Section 4.

We also study the construction K, which gives a set of multiplicative generators, and in the Section 6 we present an algebraic interpretation of this geometric construction.

Finally, as an application, in the last section we analyze the homomorphisms $r_p : SF_*(S^1) \to SF_*(\mathbb{Z}_p)$ from the bordism group of semi-free S^1-actions on the bordism group of $\mathbb{Z}_p$-actions induced by the restriction functors.

2. Semi-free S^1-actions

Let $SF_n(S^1)$ be the bordism group of semi-free S^1-actions on the n-dimensional closed manifolds, and let $SF_*(S^1) = \oplus_{n\geq 0} SF_n(S^1)$ be the N_*-module obtained. Analogously, we have the bordism groups $N_n(S^1)$ of free S^1-actions on n-dimensional closed manifolds and $N_*(S^1) = \oplus_{n\geq 0} N_n(S^1)$.

By $\{[S^{2n+1}, S^1]\}$ we shall denote the sphere with the standard circle action.

There is the Smith homomorphism

$$\Delta : N_*(S^1) \to N_*(S^1),$$

which is a N_*-module homomorphism with degree -2, which associates $[S^{2n+1}, S^1]$ to $[S^{2n-1}, S^1]$ (see [2]).

Proposition 2.1. *If $\Delta[M^{2n+1}, T] = 0$ for $[M^{2n+1}, T]$ in $N_{2n+1}(S^1)$, then there is an unique $[X^{2n}]$ in N_{2n} such that*

$$[M^{2n+1}, T] = [X^{2n}]\,[S^1, S^1].$$

Proposition 2.2. *Let $[M^{2n+1}, T]$ be a free S^1-action on the closed $(2n+1)$-manifold M^{2n+1} and $W^{2n+1} \subset M^{2n+1}$ a regular compact submanifold such that*
$W^{2n+1} \cup T(W^{2n+1}) = M^{2n+1}$ and $\partial W^{2n+1} = W^{2n+1} \cap T(W^{2n+1})$;
then $\Delta[M^{2n+1}, T] = [\partial W^{2n+1}, T|\partial W^{2n+1}]$.

Let $SF_*(S^1) \otimes_{N_*} N_*(S^1) \to N_*(S^1)$ be the pairing given by: if $[V^m, \tau] \in SF_m(S^1)$ and $[M^n, T] \in N_n(S^1)$, then we set

$$[V^m, \tau]\,[M^n, T] = [V^m \times M^n, \tau \times T]$$

since $[V^m \times M^n, \tau \times T]$ is a free S^1-action.

LEMMA 2.3. *If* $\gamma \in SF_{2m}(S^1)$ *and* $\alpha \in N_{2n+1}(S^1)$ *then* $\Delta(\gamma\alpha) = \gamma\Delta(\alpha)$.

Proof. Let $[M^{2n+1}, T]$ be a semi-free S^1-action on a closed $(2n+1)-$manifold. By (2.2), if $W^{2n+1} \subset M^{2n+1}$ is a compact regular submanifold such that $W \cup TW = M$ and $W \cap TW = \partial W$ then $\Delta[M^{2n+1}, T] = [\partial W^{2n+1}, T|_{\partial W^{2n+1}}]$.
Since $V^{2m} \times W^{2n+1} \subset V^{2m} \times M^{2n+1}$, we have

$$\begin{aligned}(V \times W) \cup T'(V \times W) &= (V \times W) \cup (\tau V \times TW)\\ &= (V \times W) \cup (V \times TW) = V \times (W \cup TW) = V \times M,\end{aligned}$$

where $T' = (\tau, T)$ and

$$\begin{aligned}\partial(V \times W) &= (\partial V \times W) \cup (V \times \partial W) = V \times \partial W = V \times (W \cap TW)\\ &= (V \times W) \cap (V \times TW) = (V \times W) \cap (\tau V \times TW).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\Delta[V \times M, \tau \times T] &= [\partial(V \times W), \tau \times T|_{\partial(V \times W)}]\\ &= [V \times \partial W, \tau \times T|_{V \times \partial W}] = [V, \tau]\,[\partial W, T|\partial W],\end{aligned}$$

that is, $\Delta(\delta\alpha) = \gamma\Delta(\alpha)$. □

Now let $\mathcal{B}_n(S^1)$ be the bordism group of semi-free S^1-actions on n-manifolds with boundary which are free on the boundary.

We have the N_*-module homomorphisms $j : SF_n(S^1) \to \mathcal{B}_n(S^1)$ which assigns $[M^n, T]$ to $[M^n, T]$, since $\partial M^n = \emptyset$; and $\partial : \mathcal{B}_n(S^1) \to N_n(S^1)$ which associates $[V^n, T]$ to $[\partial V^n, T|_{\partial V^n}]$.

THEOREM 2.4. *The sequence*

$$0 \to SF_n(S^1) \xrightarrow{j} \mathcal{B}_n(S^1) \to N_{n-1}(S^1) \to 0$$

is exact and it splits (see [5], 8.3*).*

Let $M_n(S^1) = \oplus_{k\geq 0}^{[n/2]} N_{n-2k}(BU_k)$ be the classifying space for the k-dimensional bundles, and $M_*(S^1) = \oplus_{n\geq 0} M_n(S^1)$.

THEOREM 2.5. *There is the isomorphism*

$$F : \mathcal{B}_n(S^1) \to \oplus_{k\geq 0}^{[n/2]} N_{n-2k}(BU_k)$$

given by: $[M^n, T]$ *is associated to* $\sum_k [\nu^k \to F^{n-2k}]$ *where* F^{n-2k} *is the fixed point set component and* ν^k *is the normal bundle corresponding.*

THEOREM 2.6. $M_*(S^1)$ *is a graded polynomial algebra on* N_* *with generators the classes* $[\lambda \to CP(n)], n \geq 0$, *i.e., the canonic line bundles over* $CP(n)$.

THEOREM 2.7. $N_*(S^1)$ *is a* N_* *module with base given by the elements* $\{[S^{2n+1}, S^1]\}_{n\geq 0}$.

REMARK. Any element in $N_{2n+1}(S^1)$ can be written in a unique way as

$$\sum_{r=0}^{n} [X^{2r}][S^{2(n-r)+1}, S^1],$$

and any element in $N_{2n}(S^1)$ can be written in a unique way as

$$\sum_{r=0}^{n-1} [X^{2r+1}]\, [S^{2(n-r)-1}, S^1].$$

Let i in $M_2(S^1)$ be a semi-free S^1-action on the unitary disk given by the scalar product, then $\partial(i) = [S^1, S^1]$ and we have the following lemma.

LEMMA 2.8. *The diagram*

$$\begin{array}{ccc} M_{2k+2}(S^1) & \overset{\partial}{\to} & N_{2k+1}(S^1) \\ i \uparrow & & \uparrow \\ M_{2k}(S^1) & \overset{\partial}{\to} & N_{2k-1}(S^1) \end{array}$$

is commutative.

3. Products in $\oplus_{n\geq 0} N_{2n+1}(S^1)$.

We define a product in $\oplus_{n\geq 0} N_{2n+1}(S^1)$ the following way: suppose that $\alpha \in N_{2n+1}(S^1)$ and $\beta \in N_{2m+1}(S^1)$. Let α' be in $M_{2n+2}(S^1)$ and $\beta' \in M_{2m+2}(S^1)$ such that $\partial(\alpha') = \alpha$ and $\partial(\beta') = \beta$. Let be $K = \max(m, n)$ and $k = \min(m, n)$. We define

$$\alpha\beta = \Delta^{K+1}(\alpha'\beta') \in N_{2k+1}(S^1).$$

In case $n = m$, this define a ring structure with unity on $N_{2n+1}(S^1)$, where the unity element is $[S^{2n+1}, S^1]$.

LEMMA 3.1. *If $n < m$ then $\alpha\Delta\beta = \alpha\beta$ and if $n \geq m$ then $\Delta(\alpha\beta) = \alpha\Delta\beta$.*

Proof. Let us consider $n < m$ and $\beta'' \in M_{2m}(S^1)$ with $\partial(\beta'') = \Delta\beta$. Then $\Delta(\partial(\beta'' i) + \beta) = 0$. By (2.1), there is a unique $[V^{2m}]$ in N_{2m} such that $\partial(\beta'' i) + \beta = [V^{2m}]\ [S^1, S^1]$. Then, $\partial(\beta'' i) + [V^{2m}]\ [S^1, S^1] = \beta$.
Since $\partial(i) = [S^1, S^1]$, we have $\partial(\beta'' i + [V^{2m}]i) = \beta$. Thus,

$$\begin{aligned} \alpha\beta &= \Delta^{m+1}\partial(\alpha'\beta'' i + [V^{2m}]\alpha' i) = \Delta^{m+1}\partial(\alpha'\beta'' i) + \\ &\quad + \Delta^{m+1}\partial([V^{2m}]\alpha' i) \\ &= \Delta^{m}\partial(\alpha'\beta'') + [V^{2m}]\Delta^{m}\partial(\alpha') = \alpha\Delta\beta + [V^{2m}]\Delta^{m}\alpha. \end{aligned}$$

Now, since $n < m$ and $\alpha \in N_{2n+1}(S^1)$, we have that $\Delta^m\alpha = 0$.
If $n \geq m$, we have $\partial(\beta'') = \Delta\beta$. Then $\Delta(\partial(\beta'' i + \beta)) = 0$. So there is $[V^{2m}] \in N_{2m}$ such that $\partial(\beta'' i) + [V^{2m}]\ [S^1, S^1] = \beta$.
Since $\partial(i) = [S^1,\ S^1]$, we obtain $\partial(\beta'' i + [V^{2m}]i) = \beta$. Hence

$$\begin{aligned} \Delta(\alpha\beta) &= \Delta^{n+2}\partial(\alpha'\beta'' i + [V^{2m}]\alpha' i) = \Delta^{n+1}\partial(\alpha'\beta'' + [V^{2m}]\alpha') \\ &= \Delta^{n+1}\partial(\alpha'\beta'') + [V^{2m}]\Delta^{n+1}\partial(\alpha') \\ &= \alpha\Delta\beta + [V^{2m}]\Delta^{n+1}\alpha. \end{aligned}$$

Since $n \geq m$ and $\alpha \in N_{2n+1}(S^1)$, we have that $\Delta^{n+1}\alpha = 0$.
Therefore, $\Delta(\alpha\beta) = \alpha\Delta\beta$. □

LEMMA 3.2. *For any pair α, β we have $\Delta(\alpha\beta) = \Delta\alpha\Delta\beta$.*

Proof. Suppose that $n < m$. Then $\alpha\beta = \alpha\Delta\beta$ by (3.1). Thus, $m - 1 \geq n$ and $\Delta(\alpha\Delta\beta) = \Delta\alpha\Delta\beta$. Therefore, $\Delta(\alpha\beta) = \Delta\alpha\Delta\beta$.
Finally, if $n = m$, $\Delta(\alpha\beta) = \alpha\Delta\beta$ and $\Delta\alpha\Delta\beta = \alpha\Delta\beta$. Hence, $\Delta(\alpha\beta) = \Delta\alpha\Delta\beta$. □

Lemma 3.3. *If α, β and $\delta \in N_{2n+1}(S^1)$ then $\alpha(\beta\delta) = (\alpha\beta)\delta$.*

Proof. Choose α', β' and $\delta' \in M_{2n+2}(S^1)$ such that $\partial(\alpha') = \alpha$, $\partial(\beta') = \beta$ and $\partial(\delta') = \delta$. Then by (3.1)

$$\alpha(\beta\delta) = \alpha\Delta^{n+1}\partial(\beta'\delta') = \alpha\Delta(\Delta^n\partial(\beta'\delta')) = \alpha\Delta^n\partial(\beta'\delta').$$

Thus, successively, we obtain $\alpha(\beta\delta) = \alpha\partial(\beta'\delta')$. Hence, by definition,

$$\begin{aligned}\alpha\partial(\beta'\delta') &= \Delta^{2n+1+1}\partial(\alpha'(\beta'\delta'))=\Delta^{2n+2}\Delta((\alpha'\beta')\delta')\\ &= \partial(\alpha'\beta')\delta=\Delta^{n+1}\partial(\alpha'\beta')\delta=(\alpha\beta)\delta.\end{aligned}$$

□

Theorem 3.4. *The product $\alpha\beta$ defines a ring structure on $\oplus_{n\geq 0}N_{2n+1}(S^1)$.*

Proof. We must show that the associative law is true in general.
Let $\alpha \in N_{2n+1}(S^1)$, $\beta \in N_{2m+1}(S^1)$ and $\delta \in N_{2p+1}(S^1)$.
We can suppose $m \leq p$, then, by (3.1), we have $\alpha(\beta\delta) = \alpha(\beta\Delta^{p-m}\delta)$.
Since $\beta\Delta^{p-m}\delta = \beta\Delta(\Delta^{p-m-1}\delta) = \beta\Delta^{p-m-1}\delta = \beta\Delta(\Delta^{p-m-2}\delta) = \beta\Delta^{p-m-2}\delta = \ldots = \beta\delta$, we have to consider:
Case I. If $n \leq m$, then

$$\begin{aligned}\alpha(\beta\delta) &= \alpha\Delta^{m-n}(\beta\delta) = \alpha\Delta^{m-n}(\beta\Delta^{p-m}\delta)\\ &= \alpha(\Delta^{m-n}\beta\Delta^{m-n}\Delta^{p-m}\delta)\\ &= \alpha(\Delta^{m-n}\beta\Delta^{p-n}\delta) = (\alpha\Delta^{m-n}\beta)\Delta^{p-n}\delta = (\alpha\beta)\delta.\end{aligned}$$

Case II. If $n > m$, then

$$\begin{aligned}\alpha(\beta\delta) &= \Delta\alpha(\beta\delta) = (\Delta^{n-m}\alpha)(\beta\delta) = (\Delta^{n-m}\alpha)(\beta\Delta^{p-m}\delta)\\ &= (\Delta^{n-m}\alpha\beta)\Delta^{p-n}\delta = (\alpha\beta)\delta.\end{aligned}$$

□

Remark. For $\alpha \in N_{2n+1}(S^1)$, since

$$\partial(i^{m+1}) = [S^{2m+1}, S^1] \text{ and } \Delta^{m+1}\partial(\alpha' i^{m+1}) = \partial(\alpha') = \alpha,$$

we have that $\alpha[S^{2m+1}, S^1] = \alpha$ for $n \leq m$.

If $n > m$, we have

$$\alpha[S^{2m+1}, S^1] = \Delta^{n+1}\partial(\alpha' i^{m+1}) = \Delta^{n-m}\alpha.$$

In particular, $N_{2n+1}(S^1)$ is a subring of $\oplus_{n\geq 0}N_{2n+1}(S^1)$ with unity $[S^{2n+1}, S^1]$, and

$$\Delta : N_{2n+1}(S^1) \to N_{2n-1}(S^1)$$

is a ring homomorphism.

4. The stable bordism homomorphism

We are going to introduce the ring

$$\mathcal{R} = \stackrel{invlim}{\longrightarrow} (N_{2n-1}(S^1) \stackrel{\Delta}{\leftarrow} N_{2n+1}(S^1)).$$

An element in $\mathcal{R}$ is a sequence $\{\alpha_n\}_0^\infty$ such that $\alpha_n \in N_{2n+1}(S^1)$ and $\Delta(\alpha_n) = \alpha_{n-1}$, $\forall n \geq 1$.

We define the homomorphism $\overline{J} : M_*(S^1) \to \mathcal{R}$, where

$$M_*(S^1) = \oplus_{k\geq 0}\oplus_{j=0}^{k} N_{2k-2j}(BU_j),$$

in the following way: let be $A \in M_{2k}(S^1)$, then

$$\overline{J}_n(A) = \Delta^k \partial(Ai^{n+1}).$$

Observe that $\overline{J}_n(A)$ belongs to $N_{2n+1}(S^1)$, and that the sequence $\{\overline{J}_n(A)\}_0^\infty$ belongs to $\mathcal{R}$, for

$$\Delta\overline{J}_n(A) = \Delta^{k+1}\partial(Ai^{n+1}) = \Delta^k\partial(Ai^n) = \overline{J}_{n-1}(A).$$

Finally, we define $\overline{J}(A) = \{\overline{J}_n(A)\}_0^\infty$.

Note that $\overline{J}_n(A) = \Delta^{k-n-1}\partial(A)$, if $k \geq n+1$.

The homomorphism $\overline{J}$ is stable with respect to the multiplication by i, that is, $\overline{J}(Ai) = \overline{J}(A)$.

THEOREM 4.1. *The stable homomorphism* $\overline{J} : \oplus_{k\geq 0} \oplus_{j=0}^{k} N_{2k-2j}(BU_j) \to \mathcal{R}$ *is multiplicative.*

Proof. One needs to consider only pairs A, B in $M_{2k}(S^1)$ and components $0 \leq n < k$, since $\overline{J}$ is stable. In this case,

$$\overline{J}_n(A)\overline{J}_n(B) = \Delta^{k-n-1}\partial(A)\Delta^{k-n-1}\partial(B) = \Delta^{k-n-1}(\partial(A)\partial(B)).$$

By definition,

$$\begin{aligned}\partial(A)\partial(B) &= \Delta^k \partial(AB) = \Delta^{2k}\partial(ABi^k) \\ &= \Delta^{2k}\partial(ABi^{n-1+1}) = \overline{J}_{k-1}(AB).\end{aligned}$$

Therefore,

$$\begin{aligned}\overline{J}_n(A)\overline{J}_n(B) &= \Delta^{k-n-1}(\partial(A)\partial(B)) \\ &= \Delta^{k-n-1}(\overline{J}_{k-1}(AB)) = \overline{J}_n(AB).\end{aligned}$$

□

Taking the ring

$$M_*(S^1) = \oplus_{k\geq 0}\oplus_{j=0}^{k} N_{2k-2j}(BU_j),$$

we are going to consider the quotient ring F obtained factoring by the ideal of the elements of the form $A+Ai$, where A belongs to $M_*(S^1)$. There is a natural induced ring homomorphism $\overline{J} : F \to \mathcal{R}$.

5. The structure of $\mathcal{R}$

Since $N_*(S^1)$ is a N_*-module with base given by $\{[S^{2n+1}, S^1]\}_{n\geq 0}$, we can describe the product in $N_{2n+1}(S^1)$ directly.

If $\alpha = \sum_{r=0}^{n} [X^{2r}]\,[S^{2(n-r)+1}, S^1]$ and $\beta = \sum_{r=0}^{n} [Y^{2r}]\,[S^{2(n-r)+1}, S^1]$, consider $\alpha' = \sum_{r=0}^{n} [X^{2r}]\, i^{n-r+1}$, and $\beta' = \sum_{r=0}^{n} [Y^{2r}]\, i^{n-r+1}$, then

$$\alpha'\beta' = \sum_{r=0}^{n} (\sum_{r+s=l} [X^{2r}]\,[Y^{2s}]) i^{2(n+1)-l},$$

and

$$\alpha\beta = \Delta^{n+1}\partial(\alpha'\beta') = \sum_{r=0}^{n} (\sum_{r+s=l} [X^{2r}]\,[Y^{2s}])[S^{2n+1-l}, S^1].$$

We can use this to identify $\mathcal{R}$ with the ring $N_*^{(2n)}(\theta)$ of formal power series on the graduated ring $N_*^{(2n)}$. We assign $\{\alpha_n\}_0^\infty$ to $\sum_0^\infty [X^{2r}]\theta^r$, where

$$\alpha_n = \sum_0^n [X^{2r}]\,[S^{2(n-r)+1}, S^1].$$

We have $\Delta\alpha_{n+1} = \alpha_n$, and the product $\alpha_n\beta_n$ is given by the formal power series multiplication. Thus, we have $\overline{J} : F \to N(\theta)$.

LEMMA 5.1. *The image of*

$$SF_{2m}(S^1) \to M_{2m}(S^1) \to F \to N(\theta)$$

is the ideal generated by θ^m*, and*

$$\overline{J}([M^{2m}, T]) = [M^{2m}]\theta^m + \text{ terms with power bigger than } m.$$

Proof. Let $[M^{2m}, T]$ be given, consider the normal bundle to the fixed point set $[\xi \to F]$ in $M_{2m}(S^1)$. So, $\partial([\xi \to F]) = 0$ in $N_{2m-1}(S^1)$, and

$$\overline{J}_n([\xi \to F]) = \Delta^{m-n-1}\partial([\xi \to F]) = 0 \text{ for } 0 \le n \le m-1.$$

Thus, $\overline{J}_m([\xi \to F]) = \Delta^m\partial([\xi \to F]i^{m+1})$ in $N_{2m+1}(S^1)$. Consequently, $\overline{J}_m([\xi \to F]) = \partial([\xi \to F]i) = [S(\xi \oplus 1_C), S^1]$.
Since $\Delta([S(\xi \oplus 1_C), S^1]) = 0$, then, by (2.1), there is a unique $[X^{2m}]$ such that $[S(\xi \oplus 1_C), S^1] = [X^{2m}]\,[S^1, S^1]$.
Next, since $[CP(\xi \oplus 1_C)] = [M^{2m}]$, we have that $[S(\xi \oplus 1_C), S^1] = [M^{2m}]\,[S^1, S^1]$.
Thus $\overline{J}([\xi \to F]) = [M^{2m}]\theta^m +$ terms with power bigger than m. □

LEMMA 5.2. *Let* $[\xi \to F]$ *be in* $M_{2m}(S^1)$*. If* $\overline{J}([\xi \to F]) = \beta\theta^m +$ *terms with power bigger than* m*, then there is a manifold with semi-free* S^1*-action* $[M^{2m}, T]$ *such that* β *is in the class of* M^{2m} *and* $[\xi \to F]$ *is the normal bundle to the fixed point set of* $[M^{2m}, T]$*.*

Proof. We have $0 = \overline{J}_{m-1}([\xi \to F]) = \Delta^{m-m+1-1}\partial([\xi \to F]) = \partial([\xi \to F])$. On the other hand, $\partial([\xi \to F]) = [S(\xi), S^1]$. Therefore, $[S(\xi), S^1] = 0$.
Now, suppose that $\partial[V^{2m}, \tau] = [S(\xi), S^1]$, where $[V^{2m}, \tau]$ is in N_{2m} (S^1). And consider $M^{2m} = (D(\xi) \cup V^{2m})/S(\xi) \equiv \partial V^{2m}$ and $T = S \cup \tau$.
The normal bundle to the fixed point set of $[M^{2m}, T]$ is $\xi \to F$, hence $\beta = [M^{2m}]$. □

LEMMA 5.3. *Let* 1_C *be in* $M_2(S^1)$. *Then* $\overline{J}(1_C) = 1$.

Proof. Since $\overline{J}_n(1_C) = \Delta\partial(1_C i^n) = \Delta\partial(i^{n+2}) = \Delta[S^{2n+3}, S^1] = [S^{2n+1}, S^1] \in N_{2n+1}(S^1)$, for all $n \geq 0$, we have that $\overline{J}(1_C) = 1$. □

LEMMA 5.4. *Let* $[V^{2s}]$ *be in* N_{2s} *and* $A \in M_{2k}$. *Then* $\overline{J}([V^{2s}]A) = [V^{2s}]\theta^s\overline{J}(A)$.

Proof. We have

$$\begin{aligned}\overline{J}_n([V^{2s}]A) &= \Delta^{s+k}\partial([V^{2s}])Ai^{n+1}) \\ &= \Delta^{s+k}[V^{2s}]\partial(Ai^{n+1}) = [V^{2s}]\Delta^{s+k}\partial(Ai^{n+1}).\end{aligned}$$

Since $\Delta^{s+k}\partial(Ai^{n+1})$ is in $N_{2n+1-2s}(S^1)$, and $2(n-s)+1$ is odd, we have

$$\Delta^{s+k}\partial(Ai^{n+1}) = \sum_{r=0}^{n-s}[X^{2r}][S^{2(n-s-r)+1}, S^1].$$

Therefore,

$$\begin{aligned}\overline{J}_n([V^{2s}]A) &= [V^{2s}]\sum_{r=0}^{n-s}[X^{2r}]\,[S^{2(n-s-r)+1}, S^1] \\ &= \sum_{r=0}^{n-s}[X^{2r}][V^{2s}][S^{2(n-s-r)+1}, S^1].\end{aligned}$$

Thus,

$$\overline{J}_n([V^{2s}]A) = \sum_{r=0}^{\infty}[X^{2r}][V^{2s}]\theta^{s+r} = [V^{2s}]\theta^s\sum_{r=0}^{\infty}[X^{2r}]\theta^r = [V^{2s}]\theta^s\overline{J}(A).$$

□

Now, we are going to define the operator $\mathcal{K} : SF_*(S^1) \to SF_*(S^1)$.

Let $[M^n, T]$ be a semi-free S^1-action. Consider $D^2 \times M^n$ and the actions T_1 and T_2 defined by:

$$\begin{aligned}&T_1 : (t,(z,m)) \mapsto (tz, m), \quad \text{and} \\ &T_2 : (t,(z,m)) \mapsto (tz, T(t,m)).\end{aligned}$$

Restricting T_1 and T_2 to $S^1 \times M^n$, we get the induced actions $[S^1 \times M^n, T_1]$ and $[S^1 \times M^n, T_2]$.

There exists a diffeomorphism $\varphi : [S^1 \times M^n, T_1] \to [S^1 \times M^n, T_2]$ given by $(s, x) \mapsto (s, T(s, x))$. Thus, the action $[S^1 \times M^n, T_1]$ is equivariantly diffeomorphic to $[S^1 \times M^n, T_2]$.

Taking the disjoint union $[D^2 \times M^n, T_1] \cup [D^2 \times M^n, T_2]$, we can get the closed manifold M^{n+2} and the S^1-action τ_1 on M^{n+2}, using the identification of $[S^1 \times M^n, T_1]$ with $[S^1 \times M^n, T_2]$ through φ. We define

$$\mathcal{K}[M^n, T] = [M^{n+2}, \tau_1].$$

The fixed point set of τ_1 is $F_1 = \text{Fix }(T) \cup M^n$, where Fix (T) is the fixed point set of T and the normal bundle to Fix (T) is $\eta \oplus 1_C \to$ Fix (T), η being the normal bundle to the Fix (T) on M^n. Moreover, the normal bundle to M^n is the trivial complex bundle 1_C.

Using an inductive process, one can assign to $[M^n, T]$ a sequence of semi-free S^1-actions $[V(n, k), \tau_k]$. To do this, consider $[V(n, 0), \tau_0]$ $= [M^n, T]$ and $[V(n, 1), \tau_1] = \mathcal{K}[M^n, T]$. Now, we get $[V(n, 2), \tau_2]$ applying the above construction to $[V(n, 1), \tau_1]$. Thus, applying this construction, successively, k times, we get $[V(n, k), \tau_k]$, where the fixed point set is

$$F_k = \text{Fix}(T) \cup (\cup_0^{k-1} V(n, j)).$$

Furthermore, the normal bundle η_k to F_k is $\eta \oplus 1_C^k$, where η is the normal bundle to the Fix (T) on M^n, and $1_C^k = \oplus_0^{k-1} 1_C^{k-j}$, with 1_C^{k-j} the trivial complex bundle over $V(n, j)$.

Lemma 5.5. *If η is the normal bundle to the fixed point set of $[M^n, T]$ then* $[V(n, k)] = [CP(\eta \oplus 1_C^{k+1})] + \sum_{j=0}^{k-1} [CP(k - j)][V(n, j)]$.

Lemma 5.6. *Let λ_n be the canonical complex line bundle over $CP(n)$. Then*

$$\overline{J}(\lambda_n) = 1 + \sum_{i=0}^{\infty} [V(n + 1, i)] \theta^{n+i+1}.$$

Proof. Let T_0 be a semi free S^1-action on $CP(n + 1)$ defined by T_0: $[z_0, z_1, \ldots, z_n] \mapsto [sz_0, z_1, \ldots, z_n]$, where s belongs to S^1. The normal bundle to the fixed point set is $\lambda \oplus 1_C^{(n+1)}$, where λ is the canonic complex line bundle to $CP(n)$ and $1_C^{(n+1)}$ is the trivial complex bundle to $CP(0)$.

Now, we are going to consider the manifolds with semi free S^1-action $[V(n+1,k),\tau_k]$, where $[V(n+1,0),\tau_0]=[CP(n+1),T_0]$.
The fixed point set of $[V(n+1,k),\tau_k]$ is

$$[\nu\oplus 1_C^k\to F]+[1_C^k\to V(n+1,0)]+ \\ +[1_C^{k-1}\to V(n+1,1)]+\ldots+[1_C\to V(n+1,k-1)],$$

where $\nu\to F$ is the bundle

$$(1_C^{(n+1)}\to CP(0))\cup(\lambda\to CP(n)).$$

Taking $k=2$, we see that the fixed point set of $[V(n+1,2),\tau_2]$ is

$$[\nu\oplus 1_C^2\to F]+[1_C^2\to V(n+1,0)]+[1_C\to V(n+1,1)].$$

Since

$$\overline{J}[V(n+1,2),\tau_2] \\ =[V(n+1,2)]\theta^{n+3}+\text{ terms with power bigger than }(n+3)$$

and

$$\overline{J}[V(n+1,2),\tau_2] \\ =\overline{J}[\nu\oplus 1_C^2\to F]+\overline{J}[1_C^2\to V(n+1,0)]+\overline{J}[1_C\to V(n+1,1)],$$

we get

$$[V(n+1,2)]\theta^{n+2}+\text{ terms with power bigger than }n+2 \\ =\overline{J}[\nu\to F]+[V(n+1,0)]\theta^{n+1}+[V(n+1,1)]\theta^{n+2}.$$

Therefore,

$$\begin{aligned}\overline{J}[\nu\to F] = {} & [V(n+1,0)]\theta^{n+1}+[V(n+1,1)]\theta^{n}+ \\ & +[V(n+1,2)]\theta^{n+3}+ \\ & +\text{ terms with power bigger than }n+3.\end{aligned}$$

On the other hand,

$$\overline{J}[\nu\to F]=\overline{J}[1_C^{n+1}\to CP(0)]+\overline{J}(\lambda_n).$$

Hence,

$$\begin{aligned}\overline{J}(\lambda_n) &= 1 + [V(n+1,0)]\theta^{n+1} + [V(n+1,1)]\theta^{n+2} + \\ &\quad + [V(n+1,2)]\theta^{n+3} + \\ &\quad \text{terms with power bigger than } n+3.\end{aligned}$$

Finally, in general we have the result, that is,

$$\overline{J}(\lambda_n) = 1 + \sum_{i=0}^{\infty}[V(n+1,i)]\theta^{n+i+1}.$$

□

6. Algebraic interpretation of the geometric construction $\mathcal{K}$

As in [1], in this section we are going to express $SF_*(S^1)$ as a direct sum of certain submodules.

Let $[CP(n), T_0]$ be in $SF_{2n}(S^1)$, where $T_0 : [z_0, \dots, z_n] \mapsto [sz_0, z_1, \dots, z_n]$, $s \in S^1$. The fixed point set of this semi free S^1-action is $(1_C^n \to CP(0)) \cup (\lambda \to CP(n-1))$. Therefore, one can write $[CP(n), T_0] = \alpha_n + \alpha_1^n$, where $\alpha_n = [CP(n-1), \lambda]$. Thus, $\alpha_n = [CP(n), T_0] + \alpha_1^n$. Since $M_*(S^1)$ is a polynomial algebra over N_* generated by the elements α_n in $M_{2n}(S^1)$, for $n \geq 1$, we have that the elements $[CP(n), T_0]$, for $n \geq 2$, together $\alpha_1 = i$ in $M_2(S^1)$, constitute another system of polynomial generators for $M_*(S^1)$. We are going to denote by Q the polynomial subalgebra of $SF_*(S^1)$ generated by $[CP(n), T_0]$; and by Q_{2m} the intersection $Q \cap SF_{2m}(S^1)$. Thus, $M_*(S^1) = Q[i]$. By (5.1), $SF_{2m}(S^1)$ consists of polynomials P in i over Q for which $\overline{J}(P)$ belongs to the ideal generated by θ^m.

Lemma 6.1. *Given an element y in $M_{2m}(S^1)$, there is an unique polynomial F in $N_*[i]$, with no constant terms, such that $y + F$ belongs to $SF_{2m}(S^1)$.*

Proof. Since $M_*(S^1) = Q[i]$, we have that $y = q_0 + q_1 i + q_2 i^2 + \dots + q_{m-1} i^{m-1} + q_m i^m$, where q_j is in $Q_{2(m-j)}$.
Thus, since $\overline{J}(i) = 1$ and $\overline{J}$ is stable, we have that

$$\begin{aligned}\overline{J}(y) &= \overline{J}(q_0) + \overline{J}(q_1 i) + \overline{J}(q_2 i^2) + \dots + \overline{J}(q_{m-1} i^{m-1}) + \overline{J}(q_m i^m) \\ &= \overline{J}(q_0) + \overline{J}(q_1) + \dots + \overline{J}(q_{m-1}) + \overline{J}(q_m).\end{aligned}$$

Since q_j belongs to $Q_{2(m-j)} = Q \cap SF_{2m}(S^1)$, then
$\overline{J}(q_j) = [W_{2j}^{2(m-j)}]\theta^{m-j} + [W_{2j}^{2(m-j)+2}]\theta^{m-j+1} + \ldots + [W_{2j}^{2(m-1)}]\theta^{m-1} + [W_{2j}^{2m}]\theta^{m}+$ terms with power bigger than m, where $[W_{2j}^{2(m-j)}]$ is in $N_{2(m-j)}$. Thus,
$\overline{J}(y) = [W_{2m}^{0}] + ([W_{2m}^{2}] + [W_{2(m-1)}^{2}])\theta + \ldots + (\sum_{k=0}^{j}[W_{2(m-k)}^{2j}])\theta^{j} + \ldots + (\sum_{k=0}^{m-1}[W_{2(m-k)}^{2(m-1)}])\theta^{m-1} + (\sum_{k=0}^{m}[W_{2(m-k)}^{2m}])\theta^{k}+$ terms with power bigger than m.
Let

$$F = [W_{2m}^{0}]i^{m} + ([W_{2m}^{2}] + [W_{2(m-1)}^{2}])i^{m-1} + \ldots \\ \ldots + (\sum_{k=0}^{j}[W_{2(m-k)}^{2j}])i^{m-j} + \ldots + (\sum_{k=0}^{m-1}[W_{2(m-k)}^{2(m-1)}])i$$

be in $N_*[i]$. Then, we have
$\overline{J}(y+F) = (\sum_{k=0}^{m}[W_{2(m-k)}^{2m}])\theta^{m}+$ terms with power bigger than m, that is, $\overline{J}(y+F)$ is in the ideal generated by θ^m. Therefore, by (5.1), $y+F$ belongs to $SF_{2m}(S^1)$. □

Let $\varepsilon : Q \to N_*$ be the aumentation homomorphism. Thus, we have the following lemma.

Lemma 6.2. *Let q_j be in $SF_{2(m-j)}(S^1)$, for $0 \leq j \leq m$. Then $\mathcal{K}q_j$ is in $SF_{2(m-j+1)}(S^1)$, where we denote by $\mathcal{K}q_j$ the element $i(q_j + [W_j^{2(m-j)}])$, and $[W_j^{2(m-j)}] = \varepsilon(q_j)$.*

Proof. Since i_{q_j} is in $M_{2(m-j+1)}(S^1)$, there is an unique element F in $N_*[i]$, with no constant term, such that $iq_j + F$ belongs to $S_{2(m-j+1)}(S^1)$.
Now, we are going to verify that $F = [W_{2j}^{2(m-j)}]\theta^{m-j+1}+$ terms with power bigger than $m-j+1$.
We know that $\overline{J}(q_j) = [W_{2j}^{2(m-j)}]\theta^{m-j} + [W_{2j}^{2(m-j+1)}]\theta^{m-j+1}+$ terms with power bigger than $m-j+1$.
Thus, $\overline{J}(iq_j + [W_{2j}^{2(m-j)}]i) = [W_{2j}^{2(m-j+1)}]\theta^{m-j+1}+$ terms with power bigger than $m-j+1$.
Therefore, $\overline{J}(\mathcal{K}q_j)$ is in the ideal generated by θ^{m-j+1}, and this fact imply that $\mathcal{K}q_j$ belongs to $SF_{2(m-j+1)}(S^1)$. □

REMARK 6.3. Denoting by $\mathcal{K}^2 q_j$ the element $i\mathcal{K}q_j$, by (6.2), we have that $\mathcal{K}^2 q_j$ is in $SF_{2(m-j+2)}(S^1)$. Thus, successively, $\mathcal{K}^n q_j = i\mathcal{K}^{n-1} q_j$ belongs to $SF_{2(m-j+n)}(S^1)$, and $\mathcal{K}^n q_j = i^n q_j + [W_{2j}^{2(m-j)}]i^n$.

LEMMA 6.4. *Let q_j be in $SF_{2(m-j)}(S^1) \cap Q$, $j = 0, \dots, m$. If $y = q_0 + q_1 i + q_2 i^2 + \dots + q_{m-1} i^{m-1} + q_m i^m$ belongs to $SF_{2m}(S^1)$, then $y = q_0 + \mathcal{K}q_1 + \mathcal{K}^2 q_2 + \dots + \mathcal{K}^m q_m$ and $\varepsilon(q_j) = 0$.*

Proof. We can write

$$y = q_0 + \mathcal{K}q_1 + \mathcal{K}^2 q_2 + \dots + \mathcal{K}^{m-1} q_{m-1} + \mathcal{K}^m q_m + [W_2^{2(m-1)}]i + \\ + [W_4^{2(m-2)}]i^2 + \dots + [W_{2(m-1)}^2]i^{m-1} + [W_{2m}^0]i^m,$$

where $[W_{2j}^{2(m-j)}] = \varepsilon(q_j)$.
Since y is in $SF_{2m}(S^1)$, and $q_0, \ \mathcal{K}q_1, \ \mathcal{K}^2 q_2, \dots, \mathcal{K}^m q_m$ belong to $SF_{2m}(S^1)$, we must have $(\sum_{j=1}^m [W_{2j}^{2(m-j)}]i^j)$ in $SF_{2m}(S^1)$.
But we also have $\overline{J}([W_{2j}^{2(m-j)}]i^j) = [W_{2j}^{2(m-j)}]\theta^{m-j}$ and $m - j < m$, then we conclude that $W_{2j}^{2(m-j)}$, $j = 1, \dots, m$, are boundary manifolds.
Thus, $(\sum_{j=1}^m [W_{2j}^{2(m-j)}]i^j) = 0$.
Therefore, $y = q_0 + \mathcal{K}q_1 + \mathcal{K}^2 q_2 + \dots + \mathcal{K}^m q_m$. □

Denoting by $Q^+_{2(m-j)} = ker(Q_{2(m-j)} \to N_{2(m-j)})$, we have the following theorems.

THEOREM 6.5. *$SF_{2m}(S^1)$ is the direct sum of Q_{2m} and $\mathcal{K}^m Q^+_{2(m-n)}$, for $m \geq n > 0$; and $\mathcal{K}^n$ embeds $Q^+_{2(m-n)}$ in $SF_{2m}(S^1)$.*

THEOREM 6.6. *The N_*- module $SF_*(S^1)$ is the direct sum of Q and N_*-submodules $\mathcal{K}^n Q^+$, for $n > 0$; and $\mathcal{K}^n$ embeds Q^+ in $SF_*(S^1)$, where $Q^+ = ker(Q \to N_*)$.*

7. $\mathbb{Z}_P$-actions, p an odd prime

We denote by $[M^n, T]$ a closed manifold M^n together a p-periodic diffeomorphism T, p an odd prime. We have the following bordism groups: the bordism group of free $\mathbb{Z}_p$-actions $N_*(\mathbb{Z}_p)$, the bordism group of semi free $\mathbb{Z}_p$-actions $SF_*(\mathbb{Z}_p)$, and the bordism group of semi free $\mathbb{Z}_p$-actions on manifolds with boundary which is free on the boundary $M_*(\mathbb{Z}_p)$.

THEOREM 7.1. *We have the exact sequence*

$$\ldots \to N_n(\mathbb{Z}_p) \to SF_n(\mathbb{Z}_p) \xrightarrow{j_*} M_n(\mathbb{Z}_p) \to N_{n-1}(\mathbb{Z}_p) \to \ldots$$

where $M_n(\mathbb{Z}_p) = \oplus N_{n-2k}(BU(k_1) \times \ldots \times BU(k_{(p-1)/2}))$ *(see* [4], 38.3.*)*

Consider the homomorphism $\overline{\varepsilon} : N_*(\mathbb{Z}_p) \to N_*$ defined by $\overline{\varepsilon}[M^n, T] = [M^n/T]$. Let $\widetilde{N}_*(\mathbb{Z}_p)$ be the reduced group, i.e., $\widetilde{N}_*(\mathbb{Z}_p) = ker\,\overline{\varepsilon}$.

THEOREM 7.2. *The sequence*

$$0 \to N_n \xrightarrow{i_*} SF_n(\mathbb{Z}_p) \xrightarrow{j_*} M_n(\mathbb{Z}_p) \xrightarrow{\partial} \widetilde{N}_{n-1}(\mathbb{Z}_p) \to 0$$

is exact. The homomorphism i_* *is defined by* $i_*[M^n] = [M^n \times \mathbb{Z}_p,\ 1 \times \sigma]$, *where* σ *is the* p*-periodic map which permutes the elements of* $\mathbb{Z}_p$.

THEOREM 7.3. $\widetilde{N}_*(\mathbb{Z}_p)$ *is an* N_**-module generated by the elements* $\{[S^{2k-1}, \rho]\}$, *where* $\rho = 2\pi i/p$.

The N_*-modules $SF_*(\mathbb{Z}_p)$ and $M_*(\mathbb{Z}_p)$ are graduated rings with multiplication induced by the cartesian product

$$[M_0, T_0][M_1, T_1] = [M_0 \times M_1, T_0 \times T_1].$$

Denoting by I the image of i_*, then I is the ideal of $SF_*(\mathbb{Z}_p)$ generated by $[\mathbb{Z}_p, \sigma]$, since j_* is a ring homomorphism. Therefore $\widehat{SF}_*(\mathbb{Z}_p) = SF_*(\mathbb{Z}_p)/I$ is a ring and we have the following theorem.

THEOREM 7.4. *The sequence*

$$0 \to \widehat{SF}_*(\mathbb{Z}_p) \xrightarrow{j_*} M_*(\mathbb{Z}_p) \xrightarrow{\partial} \widetilde{N}_*(\mathbb{Z}_p) \to 0$$

is exact.

Consider the set $\{\overline{\alpha}_{2k-1} : k = 1, 2, \ldots\}$ of generators of $\widetilde{N}_*(\mathbb{Z}_p)$, where $\overline{\alpha}_{2k-1} = [S^{2k-1}, \rho]$ and $\rho = \exp(2\pi i/p)$. There are closed manifolds M^{4k}, $k = 1, 2, \ldots$, where $\overline{\beta}_{2k-1} = p\overline{\alpha}_{2k-1} + [M^4]\overline{\alpha}_{2k-5} + [M^8]\overline{\alpha}_{2k-9} + \ldots = 0$, for $k = 1, 2, \ldots$, in $\widetilde{N}_*(\mathbb{Z}_p)$. Thus, $\widetilde{N}_*(\mathbb{Z}_p)$ is isomorphic as a N_*-module to the quotient of the N_*-free module generated by the elements $\overline{\alpha}_1$, $\overline{\alpha}_3$, $\overline{\alpha}_5, \ldots$, by the submodule generated by $\overline{\beta}_1$, $\overline{\beta}_3$, $\overline{\beta}_5, \ldots$.

Now, we consider the following diagram

$$\begin{array}{ccccccccc} 0 & \to & SF_*(S^1) & \xrightarrow{j} & M_*(S^1) & \to & N_*(S^1) & \to 0 \\ & & \downarrow r_p & & \downarrow r'_p & & \downarrow r''_p & \\ 0 & \to & \widehat{SF}_*(\mathbb{Z}_p) & \xrightarrow{j_*} & M_*(\mathbb{Z}_p) & \to & \widetilde{N}_*(\mathbb{Z}_p) & \to 0 \end{array}$$

where r_p is the homomorphism sending the S^1-action $[M, S]$ to the restriction $\mathbb{Z}_p$- action.

Since $N_*(S^1)$ is an N_*-free module generated by $\alpha_{2k-1} = [S^{2k-1}, S]$, where S is the S^1-action on S^{2k-1} given by $s(t, (z_0, \ldots, z_{2k-1})) = (tz_0, tz_1, \ldots, tz_{2k-1})$, $t \in S^1$; and $M_*(S^1)$ is a polynomial algebra generated by $\lambda_i : [\lambda \to CP(i)]$, we are going to define a N_*-submodule B of $M_*(S^1)$ in the following way: the kernel of r''_p is the N_*-free module generated by $\beta_k = p\alpha_{2k-1} + [M^4]\alpha_{2k-5} + [M^8]\alpha_{2k-9} + \ldots$, where $r''_p(\beta_k) = \overline{\beta}_{2k-1} = 0$ in $\widetilde{N}_*(\mathbb{Z}_p)$. Let $\widehat{\beta}_k$ be defined by: $\widehat{\beta}_k = p\lambda_0^k + [M^4]\lambda_0^{k-2} + [M^8]\lambda_0^{k-4} + \ldots$ in $M_*(S^1)$, and let B be the N_*-free module generated by $\widehat{\beta}_1$, $\widehat{\beta}_3, \widehat{\beta}_5, \ldots$, which is a submodule of $M_*(S^1)$. Since $\overline{J}(\widehat{\beta}_k) = 1 + [M^4]\theta^2 + [M^8]\theta^4 + \ldots$, where $\overline{J}$ is the Boardman homomorphism, and denoting by x_n a basic n-dimensional element of N_*, we have that

$$\begin{aligned} &\overline{J}(x_{2k-2}\widehat{\beta}_1) = x_{2k-2}\theta^{k-1} + \ldots \\ &\overline{J}(x_{2k-6}\widehat{\beta}_3) = x_{2k-6}\theta^{k-3} + x_{2k-6}[M^4]\theta^{k-1} + \ldots \\ &\overline{J}(x_{2k-10}\widehat{\beta}_5) = x_{2k-10}\theta^{k-5} + x_{2k-10}[M^4]\theta^{k-3} + \ldots \\ &\vdots \\ &\overline{J}(x_4\widehat{\beta}_{k-2}) = x_4\theta^2 + x_4[M^4]\theta^4 + \ldots \end{aligned}$$

So, any combination of the elements above has power of $\theta \leq k - 1$. Therefore, it follows that there isn't element in $SF_{2k}(S^1)$ with image nozero in B. Hence, we have that $j(SF_*(S^1)) \cap B = (0)$.

THEOREM 7.5. *The homomorphism $r_p : SF_*(S^1) \to \widehat{SF}_* \mathbb{Z}_p)$ is 1-1.*

Proof. we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \to & SF_*(S^1) & \xrightarrow{j} & M_*(S^1) & \xrightarrow{\simeq} & \oplus N_k(BU(r)) \\ & & \downarrow r_p & & \downarrow r'_p & & \downarrow r \\ 0 & \to & SF_*(\mathbb{Z}_p) & \xrightarrow{j_*} & M_*(\mathbb{Z}_p) & \xrightarrow{\simeq} & \oplus N_*(BU(r_1) \times \ldots \times BU(r_k)) \end{array}$$

where r is given by inclusion on the first factor. Therefore, r is 1 - 1. This imply that r'_p is 1 - 1,and finally r_p is 1 - 1, since $r'_p \circ j = j_* \circ r_p$ and $j,\ j_*$ are 1 - 1. □

REMARK 7.6. For $p = 3$, we have that r'_p is an isomorphism because

$$M_*(S^1) \simeq \oplus N_*(BU(r)),\ M_*(\mathbb{Z}_3) \simeq \oplus N_*(BU(r))$$

and the map r is an isomorphism.
Thus, since $B \cap \overline{J}(SF_*(S^1)) = (0)$, we have that $j_*(\widehat{SF}_*(\mathbb{Z}_3)) \simeq j(SF_*(S^1)) \oplus B$, and $\widehat{SF}_*(\mathbb{Z}_3) \simeq j(SF_*(S^1)) \oplus B$, since j_* is 1 - 1.
Therefore

$$SF_*(\mathbb{Z}_3) \simeq N_* \oplus j(SF_*(S^1)) \oplus B.$$

We are going to denote by $(\mathbb{Z}_p)^k_n$ the set of classes in the bordism group N_n which are represented by a n-manifold which is the fixed point set of a closed $(n+k)$-manifold with a semi-free $\mathbb{Z}_p$-action. We have that $(\mathbb{Z}_p)^k_n$ is a subgroup of N_n, $(\mathbb{Z}_0)^0_n \simeq N_n$ and $(\mathbb{Z}_p)^k_* = \oplus_{n \geq 0} (\mathbb{Z}_p)^k_n$ is an ideal of N_*.

THEOREM 7.7. *We have that $(\mathbb{Z}_p)^2_n \simeq N_n$.*

Proof. Let $\widehat{\beta}_k = p\lambda_0^k + [M^4]\lambda_0^{k-2} + [M^8]\lambda_0^{k-4} + \ldots$ be in $M_*(\mathbb{Z}_p)$. Since $\partial(\widehat{\beta_k})$ in $\tilde{N}_*(\mathbb{Z}_p)$ is a boundary, we have that $\widehat{\beta}_k$ belongs to the image of j_*. In particular, $\widehat{\beta}_1 = p\lambda_0$ belongs to the image of j_*. Therefore $[x_n]p\lambda_0$ belongs to the image of j_*, where x_n is a n-dimensional generator, $n \neq 2^r - 1$, of N_*.
Thus, there is a semi free $\mathbb{Z}_p$-action $[M^{n+2}, T]$ with fixed point set $[x_n] + \ldots + [x_n]$ (p-times), where p is an odd prime.
Since $[x_n] + \ldots + [x_n] = [x_n]$, we have that $[x_n]$ is in $(\mathbb{Z}_p)^2_n$. Hence, $(\mathbb{Z}_p)^2_n \simeq N_n$. □

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