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Recurrent Points of Continuous Functions on Connected Linearly Ordered Spaces

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Summary. - *Let L be a connected linearly ordered topological space and let f be a continuous function from L into itself. If $P(f)$ and $R(f)$ denote the set of periodic points and the set of recurrent points of f respectively, we show that the center of f is $cl_L P(f)$ and the depth of the center is at most 2. Furthermore we have $cl_L P(f) = cl_L R(f)$.*

1. Introduction

Let X be a topological space and let f be a continuous function from X into itself. A point $x \in X$ is said to be *a nonwandering point of f* if for any neighborhood U of x there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$. The set of nonwandering points of f will be denoted by $\Omega(f)$. If we define $\Omega_1(f) = \Omega(f), \quad \Omega_n(f) = \Omega(f\,|_{\Omega_{n-1}})$, $n \geq 2$, where $f\,|_{\Omega_{n-1}}$ means the restriction of f to $\Omega_{n-1}(f)$, then the set $\Omega_\infty(f) = \cap_{n=1}^{\infty} \Omega_n(f)$ is called *the center of f* and the minimal $n \in \mathbb{N} \cup \{\infty\}$ satisfying $\Omega_n(f) = \Omega_\infty(f)$ is called *the depth of the*

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center of f. Of course, $\Omega_\infty(f)$ can be the empty set if X is not compact.

As usual, $x \in X$ is said to be *a periodic point of f of period n* if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$ and $f^i(x) \neq x$ whenever $1 \leq i \leq n-1$. If $n = 1$ then x is called *a fixed point of f*. A point $x \in X$ is called *a recurrent point of f* if for every neighborhood U of x there exists $n \in \mathbb{N}$ such that $f^n(x) \in U$, i.e., there is a subset of the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ converging to x. $F(f), P(f), R(f)$ mean for the set of fixed points, periodic points and recurrent points respectively.

The notion of periodic point, recurrent point and center of a continuous function from a topological space into itself is one of the most important notions in Dynamical Systems. Usually, it is not an easy matter to determine the center and the depth of the center of f. In addition, the equality $cl_X P(f) = cl_X R(f)$ does not always hold. The most paradigmatic example arises when considering an irrational rotation f from the circle into itself, then $P(f) = \emptyset$ and $R(f)$ is the whole circle (see, for example, [4, Theorem 3.13]). However, for any continuous function f from the interval $[0,1]$ (endowed with the usual topology) into itself, it is well-known that $\Omega_2(f) = cl_X P(f) = cl_X R(f)$ and the depth of the center of f is at most 2 (see [3] and [8] for details). These results were extended in [10] to continuous functions from the n-od $= \{z \in \mathbb{C} : z^n \in [0,1]\}$ into itself and in [11] to general continuous tree functions. In this note we will state these results for continuous functions from a connected linearly ordered topological space into itself. The study of Dynamical Systems on connected linearly spaces was started in [9]. In general, the situation is different from the case of Dynamical Systems on the interval (see, for example, the interesting work [1]). However, we show here that the topology induced by a linear order permits us to adapt the techniques used in Dynamical Systems on the interval in order to improve the results expounded above.

We remind the reader that a *linearly ordered (topological) space* (abbreviated LOS) is a triple $(L, \leq, \mathcal{T})$ where $(L, \leq)$ is a linearly ordered set and $\mathcal{T}$ is the topology induced on L by the linear order $\leq$, i.e., a base for open sets in $\mathcal{T}$ is the family of all *open intervals* in L.

Given a LOS $(L, \leq, \mathcal{T})$, our notation for the intervals in $(L, \leq)$ is the usual one. For instance, if $x, y \in L$, $]x, y[$ we will denote the open interval $\{z \in L \mid x < z < y\}$ whenever $x < y$ and $]\leftarrow, y[$ the open interval $\{z \in L \mid z < y\}$. $\langle x, y\rangle$ stands either for the open interval $]x, y[$ if $x < y$ or for the open interval $]y, x[$ if $y < x$. When no confusion can result, we write L instead of $(L, \leq, \mathcal{T})$. We recall that a LOS is said to be *Dedekind complete* provided that every nonempty subset with an upper bound has a supremum (or, equivalently, every nonempty subset with a lower bound has an infimum). It is well-known that a LOS L is connected if and only if L is Dedekind complete and densely ordered, i.e., it is Dedekind complete and for each pair of elements $x, y \in L$ with $x < y$ there exists $z \in L$ such that $x < z < y$. The reader can find the standard references concerning linearly ordered spaces in [7]. Our notation and terminology are standard. The abbreviation $CLOS$ stands for *connected, linearly ordered space*. $cl_L A$ and $int_L B$ mean for the closure and the interior of a subset A in L respectively. Given a topological space L, we will denote by $C(L,\ L)$ the set of all continuous functions from L into itself. Undefined notions are usual ones as in [5] and [6].

2. The results

It is not a hard matter to check that a continuous function from a $CLOS$ into itself can admit no fixed points. For instead, the function f from the real line $\mathbb{R}$ (endowed with the usual topology) into itself defined by the requirement that $f(x)$ be $x+1$ whenever $x \in \mathbb{R}$ admits no recurrent points. We begin by stating the relationship between fixed points and recurrent points in the realm of $CLOS$.

LEMMA 2.1. *Let L be a $CLOS$ and let $f \in C(L,\ L)$. If there are $x, y \in L$ such that $f(x) \leq x$ and $f(y) \geq y$, then f has a fixed point in $cl_L(\langle x, y\rangle)$.*

Proof. Let x, y be such that $f(x) < x$ and $f(y) > y$. Suppose that f has no fixed points in $cl_L(\langle x, y\rangle)$. Define

$$A = \{z \in cl_L(\langle x, y\rangle) : f(z) < z\},$$

$$B = \{z \in cl_L(\langle x, y\rangle) : f(z) > z\}.$$

Then A and B are nonempty pairwise disjoint subsets of $cl_L(\langle x, y\rangle)$ such that $cl_L(\langle x, y\rangle) = A \cup B$. We shall show that A and B are closed subsets of $cl_L(\langle x, y\rangle)$. For this in turn, let $z \in cl_L(\langle x, y\rangle)$ be a point such that there exists a net $\{z_\delta\}_{\delta \in D} \subset A$ converging to z. Since, for each $\delta \in D$, $f(z_\delta) < z_\delta$, it is a routine matter to check that $f(z) \leq z$. On the other hand, since f has no fixed points in $cl_L(\langle x, y\rangle)$, we have that $f(z) < z$. Thus, A is a closed subset of $cl_L(\langle x, y\rangle)$. An argument similar to this one proves that B is also a closed subset of $cl_L(\langle x, y\rangle)$. So, the connected set $cl_L(\langle x, y\rangle)$ is the union of two pairwise disjoint closed subsets, a contradiction. Thus, there exists $z \in cl_L(\langle x, y\rangle)$ satisfying that $f(z) = z$ and the proof is complete. □

The following lemma is an easy, but useful, consequence of Lemma 2.1. The argument necessary to prove it was extracted from the similar situation on the interval (see, for example, [2, Lemma 4.14]). From now on, we shall make no special mention of the well-known fact that x is a periodic point of a function $f \in C(L, L)$ if and only if x is a fixed point of f^n for some $n \geq 1$.

Lemma 2.2. *Let L be a CLOS and let $f \in C(L, L)$. Let U be an open interval which does not contain periodic points. If $x \in U \cap \Omega(f)$, then $f^n(x) \notin U$ for every $n \geq 1$.*

Proof. Suppose there exists $n \geq 1$ such that $f^n(x) \in U$. We can assume, without loss of generality, that $x < f^n(x)$. Then, since f^n is continuous, there are pairwise disjoint open subintervals of U, U_1 and U_2 satisfying

$$x \in U_1,\ f^n(x) \in U_2,\ f^n(U_1) \subset U_2.$$

As x is nonwandering, we can find $y \in U_1$ and $m \geq n$ so that also $f^m(y) \in U_1$. Obviously, $f^m(y) < f^n(x)$. Set $z = f^n(y)$. Then $z > f^l(z)$ where $l = m - n$.

On the other hand, if $y < f^{nk}(y)$ for some $k \geq 1$, then $y < f^{nk+1}(y)$, since f^{nk} does not have a fixed point in $[y, f^n(y)] \subset U$. So, it follows by induction on $k \geq 1$ that $y < f^{nk}(y)$ for each $k \geq 1$. In particular, $y < f^{ln}(y)$. But, in the same way, $f^l(z) > z$ implies that $f^{ln}(z) > z$. Therefore, by Lemma 2.1, there exists $w \in \langle y, z \rangle$ such that $f^{ln}(w) = w$ which is a contradiction. □

Corollary 2.3. *Let L be a CLOS. For each $f \in C(L, L)$, the following conditions are equivalent:*

1. $F(f) \neq \emptyset$;
2. $R(f) \neq \emptyset$;
3. $\Omega(f) \neq \emptyset$;
4. $P(f) \neq \emptyset$.

Proof. The implications (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are clear. So, we need only check (3) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (1).

(3) $\Longrightarrow$ (4) Let $x \in \Omega(f)$. Since L is connected, L is an open interval. As $f^n(x) \in L$ for every $n \geq 1$, by Lemma 2.2, L contains periodic points.

(4) $\Longrightarrow$ (1) Let $x \in P(f)$. Suppose, without loss of generality, that the period of x is $n > 1$ and $x < f(x)$. Then it must exists $z \in \{f^k(x)\}_{k=1}^{n-1}$ such that $z > f(z)$. By Lemma 2.1, f has a fixed point in $cl_L \langle x, z \rangle$. □

The following theorem is the starting point of our results on recurrent points in $CLOS$. First we need the following lemmas. The first one was proved in [5, Theorem 1], and the second one is the version in the realm of $CLOS$ of Lemma 4 in [3]. We recall that a point $x \in L$ is called *an eventually periodic point of f* provided that x is not a periodic point (of f) and there exists $n \geq 1$ such that $f^n(x)$ is a periodic point (of f).

Lemma 2.4. *Let X be a (Hausdorff) topological space.*
If $f \in C(X, X)$, then $R(f^n) = R(f)$ for each $n \in \mathbb{N}$.

Lemma 2.5. *Let L be a CLOS and let $f \in C(L, L)$. If an open interval U contains eventually periodic points and no periodic points, then $U \cap R(f) = \emptyset$.*

Proof. Let $x \in U$ be an eventually periodic point of f such that $f^n(x) \in P(f)$ for some $n > 1$. Let us find a positive integer m such that $m \geq n \geq 1$ and $f^{n+m}(x) = f^n(x)$. Suppose, without loss of generality, that $f^m(x) > x$. Then

$$f^{km}(x) = f^m(x) > x \text{ for all } k \geq 1.$$

We will prove that

$$f^{km}(y) > y \text{ for all } y \in U \text{ and for all } k \geq 1.$$

If it is not the case, there exist $y \in U$ and $k \geq 1$ such that $f^{km}(y) < y$. By Lemma 2.1, there exists a fixed point of f^{km} in $\langle x, y \rangle$. This is contrary to the hypothesis.

Next, we will show that U contains no recurrent points of f^m. In fact, let $W = \{y \in U \mid f^{km}(y) \in U \text{ for some } k \geq 1\}$. Obviously, if $y \in U \backslash W$, y is not a recurrent point of f^m. On the other hand, if $y \in W$ and $r = \min\{k \geq 1 \mid f^{km}(y) \in U\}$, we have that $U \cap]f^{rm}(y), \rightarrow [$ is an open neighborhood of y missing $\{f^{km}(y)\}_{k \geq 1}$. So, we have obtained that $U \cap R(f^m) = \emptyset$. By Lemma 2.4, $U \cap R(f) = \emptyset$ and the proof is complete. □

THEOREM 2.6. *Let L be a CLOS and let $f \in C(L, L)$. If $x \in \Omega(f) \setminus cl_L P(f)$, then $x \notin cl_L R(f)$.*

Proof. If $P(f) = \emptyset$ the result is a consequence of the Corollary 2.3. So, suppose that $P(f) \neq \emptyset$. Let $x \in L \setminus cl_L P(f)$. According to [6, 3O.1], $L \setminus cl_L P(f)$ is expressible in a unique way as a union of disjoint maximal open intervals,

$$L \setminus cl_L(P(f)) = \bigsqcup_{\alpha \in \Lambda} I_\alpha .$$

Let us take $\alpha_0 \in \Lambda$ such that $x \in I_{\alpha_0}$. Since x is nonwandering, there exists $n \in \mathbb{N}$ such that $f^n(I_{\alpha_0}) \cap I_{\alpha_0} \neq \emptyset$. There are two possibilities:

(1) $f^n(I_{\alpha_0}) \subseteq I_{\alpha_0}$. In this case, since I_{α_0} does not meet $P(f)$, the function

$$f^n : I_{\alpha_0} \to I_{\alpha_0}$$

has no fixed points. Suppose, without loss of generality, that $f^n(x) < x$. So, because I_{α_0} is a $CLOS$, Lemma 2.1 says us that

$$f^n(y) < y \quad \text{for all} \quad y \in I_{\alpha_0}.$$

But $f^{kn}(x) \in I_{\alpha_0}$ for all $k \in \mathbb{N}$ and, consequently,

$$f^{kn}(x) < f^{(k-1)n}(x) < x \quad \text{for all} \quad k \geq 2.$$

Applying again Lemma 1, we obtain that $f^{kn}(y) < f^{(k-1)}(y) < y$ for all $k \geq 2$ and all $y \in I_{\alpha_0}$. Now, let $y \in I_{\alpha_0}$ and consider $r = \min\{k \mid f^{kn}(y) \in I_{\alpha_0}\}$. Define the open neighborhood of y, V as:

$$V = \begin{cases} I_{\alpha_0} & \text{if } k = 0, \\ I_{\alpha_0} \cap]f^{kn}(y), \to [& \text{if } k \geq 1. \end{cases}$$

It is clear that $V \cap \{f^{nk}(y)\}_{k \geq 1} = \{y\}$. So, y is not a recurrent point of f^n. By Lemma 2.4, y is not a recurrent point of f. We have just proved that $I_{\alpha_0} \cap R(f) = \emptyset$. Thus $x \notin cl_L R(f)$.

(2) $f^n(I_{\alpha_0}) \cap (L \setminus I_{\alpha_0}) \neq \emptyset$. In this case, beginning from the form of I_{α_0} we need to distinguish three cases: (i) $I_{\alpha_0} =]p, q[$ for some $p, q \in L$, (ii) $I_{\alpha_0} =] \leftarrow, q[$ for some $q \in L$, (iii) $I_{\alpha_0} =]p, \to [$ for some $p \in L$. (Notice that $P(f) = \emptyset$ whenever I_{α_0} is the whole L).

We only prove the Case (i). The other ones follow in a similar way. So, suppose that there exist $p, q \in L$ such that $I_{\alpha_0} =]p, q[$.

We will start by showing that $p, q \in cl_L P(f)$. We will only do for p, the proof for q being analogous. In fact, if $p \notin cl_L P(f)$, there is a neighborhood $[x, y]$ of p such that $[x, y] \cap P(f) = \emptyset$. Consider now the open interval $]x, q[$. Then $]x, q[\cap P(f) = \emptyset$ and $]x, q[\subset L \setminus cl_L P(f)$. This contradicts that I_{α_0} be maximal. Thus, $p \in cl_L P(f)$.

Now, as $I_{\alpha_0} \cap P(f) = \emptyset$, we can apply Lemma 2.1 in order to suppose, without loss of generality, that $f^n(x) < x$ for each $x \in I_{\alpha_0}$. The first step of the proof is to show that $q \notin f^n(I_{\alpha_0})$. In fact, if $q \in f^n(I_{\alpha_0})$, there is $y \in I_{\alpha_0}$ such that $f^n(y) = q < y$ which contradicts that q is the least upper bound of I_{α_0}.

Next, since $f^n(I_{\alpha_0}) \cap (L \setminus I_{\alpha_0}) \neq \emptyset$, $f^n(I_{\alpha_0}) \cap I_{\alpha_0} \neq \emptyset$ and $q \notin f^n(I_{\alpha_0})$, it is an easy matter to check that $p \in int_L f^n(I_{\alpha_0})$ and, consequently, we can choose a neighborhood V of p such that $V \subset f^n(I_{\alpha_0})$. As $p \in cl_L P(f)$, there exists $y \in I_{\alpha_0}$ such that $f^n(y) \in P(f)$. So, I_{α_0} contains eventually periodic points and the result follows from Lemma 2.5. □

THEOREM 2.7. *Let L be a CLOS and let $f \in C(L, L)$. Then $cl_L R(f) = cl_L P(f)$.*

Proof. Obviously, $cl_L P(f) \subset cl_L R(f)$. Conversely, let $x \in L \setminus cl_L P(f)$. If x is a nonwandering point for f, by Theorem 2.6, $x \notin cl_L R(f)$. On the other hand, if $x \notin \Omega(f)$, there exists a neighborhood U of x such that $f^n(U) \cap U = \emptyset$ for each $n \in \mathbb{N}$. So, U contains no recurrent points and the proof is complete. □

As a consequence of the Theorem 2.7 we can obtain the following result due to Coven and Hadlund (see [3, Theorem 1]).

COROLLARY 2.8. *For each continuous function f from the unit interval I into itself, $cl_I P(f) = cl_I R(f)$.*

We close by turning our attention to the center and the depth of the center of a function $f \in C(L, L)$.

THEOREM 2.9. *Let L be a CLOS and let $f \in C(L, L)$. Then $\Omega_2(f) = cl_L P(f)$. Hence the depth of the center is at most 2 and the center is $cl_L P(f)$.*

Proof. Clearly $cl_L P(f) \subset \Omega_n(f)$ for each $n \in \mathbb{N}$. So, we need only check that $\Omega_2(f) \subset cl_L P(f)$. To see this, suppose that $x \in \Omega(f|_{\Omega(f)})$ and let U be an open interval containing x. Since $U \cap \Omega(f)$ is open in $\Omega(f)$, we have that $(U \cap \Omega(f)) \cap f^n(U \cap \Omega(f)) \neq \emptyset$ for some $n > 1$, that is, there exists $y \in U \cap \Omega(f)$ such that $f^n(y) \in U \cap \Omega(f)$. Hence, by Lemma 2.2, $U \cap P(f) \neq \emptyset$. Thus, $x \in cl_L P(f)$. □

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