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# Rendiconti dell’Istituto di Matematica dell'Università di Trieste - Volume 54 (2022) 

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## History

The journal Rendiconti dell'Istituto di Matematica dell'Università di Trieste was founded in 1969 by Arno Predonzan, with the aim of publishing original research articles in all fields of mathematics.
Rendiconti dell'Istituto di Matematica dell'Università di Trieste has been the first Italian mathematical journal to be published also on-line. The access to the electronic version of the journal is free. All published articles are available on-line.
In 2008 the Dipartimento di Matematica e Informatica, the owner of the journal, decided to renew it. The name of the journal however remained unchanged, but the subtitle An International Journal of Mathematics was added. The journal can be obtained by subscription, or by reciprocity with other similar journals. Currently more than 100 exchange agreements with mathematics departments and institutes around the world have been entered in.
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## Foreword

Volume 54 of our journal Rendiconti dell'Istituto di Matematica dell'Università di Trieste is divided in three sections. The first section consists of six articles submitted spontaneously to the journal, while the second and third section respectively contain the proceedings of the TAGSS Summer School 2021 "Hyperkähler and Prym Varieties: Classical and New Results" and those of the Conference "GO60 Pure \& Applied Algebraic Geometry celebrating Giorgio Ottaviani's 60th birthday". Section 2 has been edited with the collaboration of the organizers of the school, Valentina Beorchia, Barbara Fantechi and Ada Boralevi, while for Section 3 we benefited from the collaboration of Elena Angelini, Maria Chiara Brambilla and Daniele Faenzi. We gratefully acknowledge the valuable help of all the guest editors.

Michele Cirafici
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Section 1

# Orientation reversing finite abelian actions on $\mathbb{R} \mathbb{P}^{3}$ 

John Kalliongis and Ryo Ohashi


#### Abstract

We classify, up to equivalence, the orientation-reversing finite abelian actions on $\mathbb{R}^{3}$ and their quotient types. There are six different quotient types, and for each quotient type there is only one equivalence class. Descriptions of each action which represents an equivalence class are explicitly given.


Keywords: Finite group action, lens space, orbifold, orbifold handlebody, Heegaard decomposition.
MS Classification 2020: 57M10, 22E99, 57M05, 57M12, 57M60, 57S25, 57S30.

## 1. Introduction

The symmetries of manifolds have been an increasingly ubiquitous topic of study in low-dimensional topology (See for example [8, 9, 11, 14, 15, 23]). In [8], a complete classification (up to conjugation) for symmetries of the orientable and nonorientable 3-dimensional handlebodies of genus one is obtained. A similar classification is obtained in [11] for I-bundles over the projective space. In [9], the finite group actions on the lens space $L(p, q)$ which preserve a Heegaard decomposition were classified up to equivalence for $p>2$, by restricting these actions to an invariant Heegaard torus. However when $p=1$ or 2 , then an action on $L(p, q)$ may contain an element which when restricted to two different invariant Heegaard tori are not equivalent (See the examples in [9, p. 28]). To begin to address these questions for the case when $p=2$, in [12] we initiated the study of orientation preserving primary cyclic group actions on $L(2,1)=\mathbb{R}^{3}{ }^{3}$, and classified them up to equivalence. In [14], it was shown that the 3 -sphere and $\mathbb{R} \mathbb{P}^{3}$ are the only 3 -dimensional lens spaces $L(p, q)$ which admit orientation-reversing PL maps of period $4 k$ where $k \geq 1$, and in [15] no lens space other than the 3 -sphere $\mathbb{S}^{3}$ and $\mathbb{R} \mathbb{P}^{3}$ admits an orientation-reversing involution. In [10], a complete classification of orientation reversing geometric finite group actions on lens spaces $L(p, q)$ where $p>2$ and $q^{2} \equiv-1(\bmod p)$ is obtained if the action leaves a Heegaard torus invariant whose sides are exchanged by an orientation-reversing element.

In this paper, continuing the study for $p=2$, we consider the orientationreversing abelian actions on the three-dimensional projective space $\mathbb{R P}^{3}=$
$L(2,1)$, which is double covered by 3 -sphere $\mathbb{S}^{3}$. Note that the special orthogonal group $\mathrm{SO}(3)$ is isomorphic to $\mathbb{R}_{\mathbb{P}^{3}}$ (See [7] for details). The finite orientation reversing abelian actions on $\mathbb{R P}^{3}$ leave a Heegaard torus invariant while preserving its sides. Using this, we are able to classify, up to equivalence, these actions and compute their quotient spaces. In addition, an explicit construction is given of a standard action representing each equivalence class. Note that $\mathbb{R} \mathbb{P}^{3}$ is an elliptic 3 -manifold with a geometric structure, and we may assume by [5, Theorem E], which follows from Perelman's results in [16, 17, 18], that a finite action on $\mathbb{R} \mathbb{P}^{3}$ acts as a group of isometries. We work in the PL category.

A $G$-action on a manifold $X$ is a homomorphism $\varphi: G \rightarrow \operatorname{Homeo}_{P L}(X)$ where $\operatorname{Homeo}_{P L}(X)$ is the group of PL-homeomorphisms of $X$ and $\varphi$ is an injection. Two $G$-actions $\varphi$ and $\psi$ are equivalent if their images are conjugate in $\operatorname{Homeo}_{P L}(X)$. When $G$ is finite the quotient space is an orbifold which we denote by $X / \varphi$. We will assume $G$ is always finite.

Let $\varphi: G \rightarrow$ Homeo $_{P L}\left(\mathbb{R P}^{3}\right)$ be an orientation-reversing abelian action. We show (See Corollary 4.2) that there is a Heegaard torus (a separating torus whose closure of the two complementary components are solid tori) which is left invariant by the action whose sides are also preserved. The restriction to each invariant solid torus determines an orbifold quotient whose Euler number is zero. In [8] there is a complete list of all the handlebody orbifolds whose Euler number is zero. For any positive integer $n$, the orientable orbifolds are denoted by $(A 0, n)$ and $(B 0, n)$, while the non-orientable ones are denoted by $(A 1, n), \ldots,(A 3, n),(B 1, n), \ldots,(B 8, n)$. The orbifolds in the main theorem are obtained by identifying the boundaries of the non-orientable orbifolds via explicitly defined homeomorphisms. If $X$ and $Y$ are orbifolds and $\xi: \partial X \rightarrow \partial Y$ is a homeomorphism, denote by $O_{\xi}(X, Y)$ the orbifold obtained by identifying $\partial X$ to $\partial Y$ via $\xi$. These orbifolds, together with the maps $\xi$ and their fundamental groups are explicitly defined in the Appendix.

The main result in this paper, which appears as Theorem 6.1 in Section 6, is as follows:

Theorem 1.1. Let $\varphi: G \rightarrow$ Homeo $_{P L}\left(\mathbb{R P}^{3}\right)$ be an orientation-reversing finite abelian action. Then one of the following cases is true:

1) $G=\mathbb{Z}_{2^{b} m}$ where $b>1$, $m$ is odd and $\mathbb{R P}^{3} / \varphi$ is $O_{h_{1}^{-1}}\left(\left(B 5,2^{b-1} m\right),(A 1,2)\right)$;
2) $G=\mathbb{Z}_{2 m}, m$ is odd and $\mathbb{R P}^{3} / \varphi$ is $t O_{h_{2}^{-1}}((B 4, m),(A 3,1))$;
3) $G=\mathbb{Z}_{m} \times \mathbb{Z}_{2}$, $m$ even and $\mathbb{R P}^{3} / \varphi$ is $O_{h_{3}}((A 2,2),(B 3,, m))$;
4) $G=\mathbb{Z}_{4} \times \mathbb{Z}_{2}$, and $\mathbb{R} \mathbb{P}^{3} / \varphi$ is $O_{h_{4}}((B 2,2),(B 2,2))$;
5) $G=\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ and $\mathbb{R P}^{3} / \varphi$ is $O_{h_{5}}((B 6,2),(B 6,2))$;
6) $G=\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ and $\mathbb{R P}^{3} / \varphi$ is $O_{h_{6}}((B 7,1),(B 7,1))$.

Furthermore, in each individual case $i$ ), where $1 \leq i \leq 6, \varphi$ is equivalent to the Standard Quotient Type i Action.

The paper is organized as follows. Section 2 is devoted to some preliminary remarks and definitions concerning orbifolds, the Euler number, and Heegaard decomposition. The orbifolds $A(0, n)$ and $B(0, n)$ which cover all the nonorientable orbifolds of Euler number zero are defined, and the non-orientable orbifolds which are the union of these orbifolds and have finite fundamental group are listed. In Section 3, we define the standard abelian actions on $\mathbb{R} \mathbb{P}^{3}$, and identify their quotient types. We show in Section 4 that any orientationreversing abelian action on $\mathbb{R P}^{3}$ preserves a Heegaard torus. In Section 5, we investigate which orbifolds defined in Section 2 and the Appendix have a $\mathbb{Z}_{2}$-normal subgroup of their fundamental groups with abelian quotient, and whether they are covered by $\mathbb{R}^{3} \mathbb{P}^{3}$. Finally, we summarize the main results in Section 6. The Appendix contains the definition of each of the non-orientable orbifolds $(A 1, n), \ldots,(A 3, n),(B 1, n), \ldots,(B 8, n)$, the gluing maps identifying the boundaries of these orbifolds and their fundamental groups.

## 2. Orbifolds preliminaries, Heegaard decompositions with finite fundamental groups

Orbifolds were introduced and studied by Satake in [19, 20], and developed more fully by Thurston in [21]. Other good references include M.Yokoyama [22]; M. Boileau, S. Maillot and J. Porti [2]; S. Choi [3]; W. Dunbar [6]; D. Cooper, C.Hodgson and S. Kerchoff [4]. In this section we give brief preliminary notions about orbifolds, and refer the reader to the above references for more detail. We define the orientable orbifolds $(A 0, n)$ and $(B 0, n)$ which cover the non-orientable orbifolds of Euler number zero. In addition, we list which of the orbifolds having Euler number zero Heegaard decomposition have finite fundamental groups in Theorem 2.1.

An orbifold is a space which is the quotient space of $\mathbb{R}^{n}$ by a finite linear group. Consider $(\widetilde{U}, G)$ where $\widetilde{U}$ is an open subset of $\mathbb{R}^{n}$ and $G$ is a finite group of diffeomorphisms of $\widetilde{U}$. Let $U=\widetilde{U} / G$ be the quotient space and $\nu: \widetilde{U} \rightarrow U$ the quotient map. The quotient space $U$ is called a local model. If $G_{\tilde{x}}$ is the stabilizer for any $\tilde{x} \in \widetilde{U}$ and $G_{\tilde{x}} \neq 1$, then $\nu(\tilde{x})$ is called an exceptional point in $U$; it may be labelled with the order of $G_{\tilde{x}}$. An orbifold map $\psi$ between local models $U$ and $U^{\prime}$ consists of a pair $(\widetilde{\psi}, \gamma)$, where $\widetilde{\psi}: \widetilde{U} \rightarrow \widetilde{U}^{\prime}$ is a smooth map and $\gamma: G \rightarrow G^{\prime}$ is a group homomorphism such that $\widetilde{\psi}(g(\tilde{x}))=\gamma(g) \widetilde{\psi}(\tilde{x})$ for all $\tilde{x} \in \widetilde{U}$ and $g \in G$, and $\nu^{\prime} \widetilde{\psi}=\psi \nu$. An orbifold is a space which consists of local models glued together by orbifold maps. The set of exceptional points is referred to as the exceptional set or the singular locus. An orbifold $O$ with boundary $\partial O$ is define similarly by replacing $\mathbb{R}^{n}$ with the closed half space $\mathbb{R}_{+}^{n}$ to obtain local models for $x \in \partial O$. If $M$ is an $n$-manifold and $G$ is a group of diffeomorphisms which acts properly discontinuously on $M$ (for every compact subset $K \subset M$, the set $\{g \in G \mid g(K) \cap K \neq \emptyset\}$ is finite), then the quotient


Figure 1: $(A 0, n)$
space $M / G$ is an orbifold. The orbifolds $(A 0, n)$ and $(B 0, n)$, defined below, are good examples of 3 -dimensional orbifolds.

An orbifold handlebody $O$ is formed by gluing together orbifold 0 -handles (3-orbifolds covered by the 3 -ball $B^{3}$ ) and orbifold 1-handles (products with 2orbifolds covered by the disk $D^{2}$ ) so that the exceptional sets of the same type are identified. See [8] for more details. If the handlebody orbifold is orientable, then the underlying space is a handlebody. When there is a $n$-sheeted covering space $H \rightarrow O$ where $H$ is a handlebody, then the Euler number $\chi(O)=\frac{1}{n} \chi(H)$. See [4] for a more detailed description of the Euler number. An Euler number $1-g$ Heegaard decomposition of an orbifold $O$ is an ordered triple $\left(\Sigma, O_{1}, O_{2}\right)$ where $\Sigma \subset O$ is a closed 2-orbifold, $O_{i}$ is an orbifold handlebody having Euler number $1-g, \Sigma=\partial O_{i}=O_{1} \cap O_{2}$ and $O=O_{1} \cup O_{2}$.

In this paper we will be concerned with Euler number zero Heegaard decompositions where the orbifolds $O_{i}$, for $i=1,2$, will come from the list of the non-orientable orbifolds covered by $(A 0, n)$ and $(B 0, n)$. We now describe the orbifolds $(A 0, n)$ and $(B 0, n)$.

### 2.1. Orbifold $(A 0, n)$

We begin with the unit disk $D^{2}$ parameterized by $\left\{\rho e^{i \theta}=v \mid 0 \leq \rho \leq 1\right\}$. Let $V$ be the solid torus $S^{1} \times D^{2}$ and define a $\mathbb{Z}_{n}$-action on $V$ by $h(u, v)=\left(u, v e^{\frac{2 \pi i}{n}}\right)$. The orbifold quotient space $V /\langle h\rangle$ is denoted by $V(n)$ or $(A 0, n)$. This quotient space is a torus with a core of exceptional points of order $n$ (See Figure 1).

The orbifold fundamental group of $V(n)$ is

$$
\pi_{1}(V(n))=\left\langle l_{1}, m_{1} \mid\left[l_{1}, m_{1}\right]=1, m_{1}^{n}=1\right\rangle \simeq \mathbb{Z} \times \mathbb{Z}_{n}
$$



Figure 2: $(B 0, n)$

### 2.2. Orbifold $(B 0, n)$

Let $\tau: V(n) \rightarrow V(n)$ be the involution defined by $\tau(u, v)=(\bar{u}, \bar{v})$. The orbifold $V(n) /\langle\tau\rangle$ is denoted by $(B 0, n)$. Its underlying space is a 3-ball which has an exceptional set consisting of an embedded tree with five edges, one edge labeled with $n$ and the other four edges each labeled with 2 . The boundary is a Conway sphere with four cone points of order 2 (See Figure 2).

We obtain a covering map $\nu: V(n) \rightarrow(B 0, n)=V(n) /\langle\tau\rangle$ giving an exact sequence

$$
1 \rightarrow \pi_{1}(V(n)) \rightarrow \pi_{1}((B 0, n)) \rightarrow \mathbb{Z}_{2} \rightarrow 1
$$

which splits. Let $\nu_{*}\left(l_{1}\right)=l$ and $\nu_{*}\left(m_{1}\right)=m$. Since $\tau$ inverts both generators of $\pi_{1}(V(n))$, we obtain the following fundamental groups:

$$
\begin{aligned}
\pi_{1}((B 0, n))=\langle l, m, t| m^{n}=t^{2}=1, l m=m l, t l t^{-1}=l^{-1}, & \left.t m t^{-1}=m^{-1}\right\rangle \\
& =\operatorname{Dih}\left(\mathbb{Z} \times \mathbb{Z}_{n}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\pi_{1}(\partial(B 0, n))=\langle l, m, t| t^{2}=1, l m=m l, t l t^{-1}=l^{-1}, t m t^{-1} & \left.=m^{-1}\right\rangle \\
& =\operatorname{Dih}(\mathbb{Z} \times \mathbb{Z})
\end{aligned}
$$

In the Appendix we show that $(A 0, n)$ will double cover the non-orientable orbifolds $(A 1, n),(A 2, n),(A 3, n),(B 3, n),(B 4, n)$, and $(B 5, n)$; and the orbifold $(B 0, n)$ will double cover the non-orientable orbifolds $(B 1, n),(B 2, n),(B 6, n)$, $(B 7, n),(B 8, n)$. Furthermore these orbifolds are described there along with their fundamental groups. Recall that $O_{\xi}(X, Y)$ is the orbifold obtained by identifying $\partial X$ to $\partial Y$ via a homeomorphism $\xi: \partial X \rightarrow \partial Y$. The orbifolds $X$ and $Y$ will come from the list of non-orientable orbifolds whose boundaries are homeomorphic, and the gluing map $\xi=h_{i}$ for $1 \leq i \leq 7$ is defined in the Appendix. For groups $A$ and $B$, we use the notation $A \circ B$ to denote the semidirect product $A \rtimes B$, and use $A \circ_{-1} B$ to represent the specific action

| Orbifolds | Fundamental Group |
| :---: | :---: |
| $O_{h_{1}}((A 1, n),(B 5, m))$ | $\left\langle a, b \mid a^{2}=b^{2}, a^{2 m}=b^{2 m}=\left(b a^{-1}\right)^{n}=1\right\rangle \simeq \mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{2 m}$ |
| $O_{h_{2}}((A 3, n),(B 4, m))$ | $\begin{aligned} \langle a, b, c\| a^{n}=b^{2}=c^{2 m}=1, b a b^{-1}=a^{-1}, c a c^{-1} & =a^{-1} \\ \left.c b c^{-1}=b a\right\rangle & \simeq \operatorname{Dih}\left(\mathbb{Z}_{n}\right) \circ \mathbb{Z}_{2 m} \end{aligned}$ |
| $O_{h_{3}}((A 2, n),(B 3, m))$ | $\begin{aligned} \langle a, b, c\|[a, b]=[a, c]=1, a^{m}=b^{n}=c^{2}=1, c b c^{-1} & \left.=b^{-1}\right\rangle \\ & \simeq \operatorname{Dih}\left(\mathbb{Z}_{n}\right) \times \mathbb{Z}_{m} \end{aligned}$ |
| $O_{h_{4}}((B 2, n),(B 2, m))$ | $\begin{aligned} \langle a, b\| a^{2 n}=b^{2}=1, b a^{2} b^{-1}=a^{-2},(a b)^{2 m} & =1\rangle \\ & \simeq\left(\mathbb{Z}_{n} \circ-1 \mathbb{Z}_{2 m}\right) \circ \mathbb{Z}_{2} \end{aligned}$ |
| $O_{h_{5}}((B 6, n),(B 6, m))$ | $\begin{array}{r} \langle a, b, c, d\| a^{2}=b^{2}=c^{2}=(b c)^{n}=d^{2}=(a d)^{m}=1, a \leftrightarrow\{b, c\}, \\ d \leftrightarrow\{b, c\}\rangle \simeq \operatorname{Dih}\left(\mathbb{Z}_{n}\right) \times \operatorname{Dih}\left(\mathbb{Z}_{m}\right) \end{array}$ |
| $O_{h_{6}}((B 7, n),(B 7, m))$ | $\begin{aligned} \langle a, b, c\| a^{2}=b^{2 n}=\left(a b^{-1} a b\right)^{m}=c^{2}=1, & \left.a \leftrightarrow\left\{b^{2}, c\right\}, b^{c}=b^{-1}\right\rangle \\ & \simeq \operatorname{Dih}\left(\mathbb{Z}_{m}\right) \circ \operatorname{Dih}\left(\mathbb{Z}_{2 n}\right) \end{aligned}$ |
| $O_{h_{7}}((B 1, n),(B 8, m))$ | $\begin{aligned} \langle a, b, c\| a^{n}=b^{2}=c^{2}=1, b a b^{-1}=a^{-1},[a, c] & \left.=1,(c b)^{2 m}=1\right\rangle \\ & \simeq \mathbb{Z}_{n} \circ \operatorname{Dih}\left(\mathbb{Z}_{2 m}\right) \end{aligned}$ |

Table 1: Notation: $x^{y}=y x y^{-1}$, and if $x$ and $y$ commute we write $x \leftrightarrow y$.
$b a b^{-1}=a^{-1}$ for every $a \in A$ and $b \in B$. Thus the dihedral group $\operatorname{Dih}\left(\mathbb{Z}_{n}\right)=$ $\mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{2}$.

From [13], we have the following theorem:
Theorem 2.1. Let $X$ and $Y$ be any of the orbifolds $(A 1, n), \ldots,(A 3, n),(B 1, n)$, $\ldots,(B 8, n)$, and let $\xi: \partial X \rightarrow \partial Y$ be a homeomorphism. If $\pi_{1}\left(O_{\xi}(X, Y)\right)$ is finite, then $O_{\xi}(X, Y)$ is homeomorphic to one of the orbifolds listed in Table 1 with the corresponding fundamental group.

## 3. Standard orientation reversing abelian actions on $\mathbb{R} \mathbb{P}^{3}$

In this section, we will define some standard orientation reversing abelian actions on $\mathbb{R P}^{3}$. In addition, we calculate the quotient spaces of these actions, and the quotient spaces for their orientation preserving subgroups. These actions will be sorted by their quotient types, Quotient Type $i$ for $1 \leq i \leq 6$. A standard action with Quotient Type $i$ will be called the Standard Quotient Type $i$ Action. Since the later cases are similar to the previous cases, some of the details will be omitted.

We view $\mathbb{R P}^{3}=V_{1} \cup_{\alpha} V_{2}$ where the boundary $\partial V_{1}$ is identified with $\partial V_{2}$ by a homeomorphism $\alpha: \partial V_{1} \rightarrow \partial V_{2}$ defined by $\alpha\left(u_{1}, v_{1}\right)=\left(u_{2} v_{2}^{2}, u_{2} v_{2}\right)$ for $\left(u_{i}, v_{i}\right) \in V_{i}$.

Consider two orbifold solid tori $V(a)$ and $V(b)$, let $p$ and $q$ be relatively prime positive integers and choose $r, s \in \mathbb{Z}$ such that $r q-p s=-1$. Let $h: \partial V(a) \rightarrow \partial V(b)$ be the homeomorphism defined by $h(u, v)=\left(u^{r} v^{p}, u^{s} v^{q}\right)$. The orbifold $W(p, q ; a, b)$ is the orbifold obtained by identifying $\partial V(a)$ to $\partial V(b)$ via the homeomorphism $h$. The underlying space of $W(p, q, a, b)$, denoted by $|W(p, q, a, b)|$, is the lens space $L(p, q)$. As in the case of the lens space, the integers $p, q, a$ and $b$ determine the orbifold up to homeomorphism.

### 3.1. Quotient Type 1: $O_{h_{1}^{-1}}\left(\left(B 5,2^{b-1} m\right),(A 1,2)\right)$ with $b>1$ and $m$ odd.

Let $V_{1}=S^{1} \times D^{2}$, and define two homeomorphisms $f$ and $g$ on $V_{1}$ as follows: For $m$ a positive odd integer

$$
f\left(u_{1}, v_{1}\right)=\left(u_{1}, v_{1} e^{\frac{2 \pi i}{m}}\right), \quad \text { and } \quad g\left(u_{1}, v_{1}\right)=\left(\overline{u_{1}} e^{\frac{-2 \pi i}{2^{b-2}}}, u_{1} v_{1} e^{\frac{3(2 \pi i)}{2^{b}}}\right)
$$

Note that

$$
\begin{aligned}
g^{2}\left(u_{1}, v_{1}\right) & =g\left(\overline{u_{1}} e^{\frac{-2 \pi i}{2^{b-2}}}, u_{1} v_{1} e^{\frac{3(2 \pi i)}{2^{b}}}\right) \\
& =\left(\overline{\left.\left(\overline{u_{1}} e^{\frac{-2 \pi i}{b-2}}\right) e^{\frac{-2 \pi i}{b-2}},\left(\overline{u_{1}} e^{\frac{-2 \pi i}{b^{b-2}}}\right) u_{1} v_{1} e^{\frac{6(2 \pi i)}{2^{b}}}\right)}\right. \\
& =\left(u_{1}, v_{1} e^{\frac{-2 \pi i}{2^{b-2}}} e^{\frac{3(2 \pi i)}{2^{b-1}}}\right) \\
& =\left(u_{1}, v_{1} e^{\frac{2 \pi i}{2^{b-1}}}\right)
\end{aligned}
$$

It follows that $g$ is an orientation reversing homeomorphism with finite order $2^{b}$. Furthermore $f$ and $g$ commute, hence the two maps generate a $\mathbb{Z}_{m} \times \mathbb{Z}_{2^{b}}=\mathbb{Z}_{2^{b} m^{-}}$-action on $V_{1}$. We obtain an orbifold covering map $\eta_{1}: V_{1} \rightarrow$ $V_{1} /\langle f\rangle=V_{1}(m)$ defined by $\eta_{1}\left(u_{1}, v_{1}\right)=\left(u_{1}, v_{1}^{m}\right)$. The homeomorphism $g$ induces a homeomorphism $g_{1}$ on $V(m)$, and we may calculate $g_{1}$ as follows:

$$
\begin{aligned}
g_{1}\left(u_{1}, v_{1}\right)=\eta g\left(u_{1}, v_{1}^{\frac{1}{m}}\right) & =\eta\left(\overline{u_{1}} e^{\frac{-2 \pi i}{2^{b-2}}}, u_{1} v_{1}^{\frac{1}{m}} e^{\frac{3(2 \pi i)}{2^{b}}}\right) \\
& =\left(\overline{u_{1}} e^{\frac{-2 \pi i}{b-2}}, u_{1}^{m} v_{1} e^{\frac{3 m(2 \pi i)}{2^{b}}}\right) .
\end{aligned}
$$

We consider first the case where $b>1$. Thus we have a $\mathbb{Z}_{m} \times \mathbb{Z}_{2^{b}}=\mathbb{Z}_{2^{b} m^{-}}$ action where $b>1$ and $m$ is odd. It also follows that $g_{1}^{2}\left(u_{1}, v_{1}\right)=\left(u_{1}, v_{1} e^{\frac{2 m \pi i}{2 b-1}}\right)$ and $\left\langle g_{1}^{2}\right\rangle=\mathbb{Z}_{2^{b-1}}$. We obtain an orbifold covering $\lambda_{1}: V(m) \rightarrow V\left(2^{b-1} m\right)=$ $V(m) /\left\langle g_{1}^{2}\right\rangle$ defines by $\lambda_{1}\left(u_{1}, v_{1}\right)=\left(u_{1}, v_{1}^{2^{b-1}}\right)$. Further, $g_{1}$ induces an orientation reversing involution $g_{2}$ on $V\left(2^{b-1} m\right)$ which may be computed as follows:

$$
\begin{aligned}
g_{2}\left(u_{1}, v_{1}\right)=\lambda_{1} g_{1}\left(u_{1}, v_{1}^{\frac{1}{b-1}}\right) & =\lambda_{1}\left(\overline{u_{1}} e^{\frac{-2 \pi i}{2^{b-2}}}, u_{1}^{m} v_{1}^{\frac{1}{2-1}} e^{\frac{3 m(2 \pi i)}{2^{b}}}\right) \\
& =\left(\overline{u_{1}} e^{\frac{-2 \pi i}{b-2}},-u_{1}^{2^{b-1} m} v_{1}\right) .
\end{aligned}
$$

On the other hand, $g_{2}$ is an orientation reversing involution with two isolated fixed points, $\left(e^{\frac{-2 \pi i}{2 b-1}}, 0\right)$ and $\left(-e^{\frac{-2 \pi i}{2 b-1}}, 0\right)$. Thus by [13, Proposition 13] we see that $V\left(2^{b-1} m\right) /\left\langle g_{2}\right\rangle$ is the orbifold $\left(B 5,2^{b-1} m\right)$, and we have the following lemma.

Lemma 3.1. For any orbifold of the form $\left(B 5,2^{b-1} m\right)$ where $m$ is odd and $b>$ 1 , there exists a $\mathbb{Z}_{m} \times \mathbb{Z}_{2^{b}}$-action on the solid torus $V_{1}$ generated by $f\left(u_{1}, v_{1}\right)=$ $\left(u_{1}, v_{1} e^{\frac{2 \pi i}{m}}\right)$ and $g\left(u_{1}, v_{1}\right)=\left(\overline{u_{1}} e^{\frac{-2 \pi i}{2 b-2}}, u_{1} v_{1} e^{\frac{3(2 \pi i)}{2^{b}}}\right)$. The quotient type $V_{1} /\left(\mathbb{Z}_{m} \times\right.$ $\left.\mathbb{Z}_{2^{b}}\right)=\left(B 5,2^{b-1} m\right)$.

At this point, we will extend $f$ and $g$ to $\mathbb{R P}^{3}$ and identify the quotient space. Let $V_{2}=S^{1} \times D^{2}$, and recall that $\mathbb{R} \mathbb{P}^{3}=V_{1} \cup_{\alpha} V_{2}$ where $\alpha: \partial V_{1} \rightarrow \partial V_{2}$ is a homeomorphism defined by $\alpha\left(u_{1}, v_{1}\right)=\left(u_{2} v_{2}^{2}, u_{2} v_{2}\right)$. Now $\alpha^{-1}\left(u_{2}, v_{2}\right)=$ $\left(u_{1}^{-1} v_{1}^{2}, u_{1} v_{1}^{-1}\right)$. We have the following:

$$
\begin{aligned}
\alpha f \alpha^{-1}\left(u_{2}, v_{2}\right) & =\alpha f\left(u_{1}^{-1} v_{1}^{2}, u_{1} v_{1}^{-1}\right) \\
& =\alpha\left(u_{1}^{-1} v_{1}^{2}, u_{1} v_{1}^{-1} e^{\frac{2 \pi i}{m}}\right) \\
& =\left(\left(u_{2}^{-1} v_{2}^{2}\right)\left(u_{2} v_{2}^{-1} e^{\frac{2 \pi i}{m}}\right)^{2},\left(u_{1}^{-1} v_{2}^{2}\right)\left(u_{2} v_{2}^{-1} e^{\frac{2 \pi i}{m}}\right)\right) \\
& =\left(u_{2} e^{\frac{4 \pi i}{m}}, v_{2} e^{\frac{2 \pi i}{m}}\right)
\end{aligned}
$$

Thus $f\left(u_{2}, v_{2}\right)=\left(u_{2} e^{\frac{4 \pi i}{m}}, v_{2} e^{\frac{2 \pi i}{m}}\right)$ is a fixed-point free map on $V_{2}$. Similarly extend $g$ to $\mathbb{R P}^{3}$ as follows:

$$
\begin{aligned}
\alpha g \alpha^{-1}\left(u_{2}, v_{2}\right) & =\alpha g\left(u_{1}^{-1} v_{1}^{2}, u_{1} v_{1}^{-1}\right) \\
& =\alpha\left(\overline{\left(u_{1}^{-1} v_{1}^{2}\right)} e^{\frac{-2 \pi i}{b^{b-2}}},\left(u_{1}^{-1} v_{1}^{2}\right)\left(u_{1} v_{1}^{-1}\right) e^{\frac{3(2 \pi i)}{2^{b}}}\right) \\
& =\alpha\left(\overline{\left(u_{1}^{-1} v_{1}^{2}\right)} e^{\frac{-2 \pi i}{b-2}}, v_{1} e^{\frac{3(2 \pi i)}{2^{b}}}\right) \\
& =\left(\left(\overline{\left(u_{2}^{-1} v_{2}^{2}\right)} e^{\frac{-2 \pi i}{2^{b-2}}}\right)\left(v_{2} e^{\frac{3(2 \pi i)}{2^{b}}}\right)^{2},\left(\overline{\left(u_{2}^{-1} v_{2}^{2}\right)} e^{\frac{-2 \pi i}{2^{b-2}}}\right)\left(v_{1} e^{\frac{3(2 \pi i)}{2^{b}}}\right)\right) \\
& =\left(u_{2} e^{\frac{2 \pi i}{b-1}}, u_{2} \overline{v_{2}} e^{\frac{-2 \pi i}{2^{b}}}\right) .
\end{aligned}
$$

In other words, $g\left(u_{2}, v_{2}\right)=\left(u_{2} e^{\frac{2 \pi i}{2^{b-1}}}, u_{2} \overline{v_{2}} e^{\frac{-2 \pi i}{2^{b}}}\right)$ where $b>1$.
In the mean time, we extend $\eta$ to $V_{2}$ to obtain a covering map $\eta_{2}: V_{2} \rightarrow$ $V_{2} /\langle f\rangle=V_{2}(1)$ defined by $\eta_{2}\left(u_{2}, v_{2}\right)=\left(u_{2}^{m}, u_{2}^{\frac{m-1}{2}} v_{2}\right)$. In addition, $g$ induces $g_{1}$ on $V_{2}(1)$ which may be computed as follows:

$$
\begin{aligned}
g_{1}\left(u_{2}, v_{2}\right) & =\eta_{2} g\left(u_{2}^{\frac{1}{m}}, u_{2}^{\frac{1-m}{2 m}} v_{2}\right) \\
& =\eta_{2}\left(u_{2}^{\frac{1}{m}} e^{\frac{2 \pi i}{b-1}}, u_{2}^{\frac{1}{m}}\left(u_{2}^{\frac{1-m}{2 m}} v_{2}\right)^{-1} e^{\frac{-2 \pi i}{2^{b}}}\right) \\
& =\eta_{2}\left(u_{2}^{\frac{1}{m}} \frac{2 \pi i}{e^{\frac{m-1}{b-1}}}, u_{2}^{\frac{m+1}{2 m}} v_{2}^{-1} e^{\frac{-2 \pi i}{2^{b}}}\right) \\
& =\left(u_{2} e^{\frac{2 \pi i m}{2^{-1}}}, u_{2}^{\frac{m-1}{2 m}} e^{\frac{2 \pi i(m-1)}{2^{b}}} u_{2}^{\frac{m+1}{2 m}} v_{2}^{-1} e^{\frac{-2 \pi i}{2^{b}}}\right) \\
& =\left(u_{2} e^{\frac{2 \pi i m}{2^{b-1}}}, u_{2} v_{2}^{-1} e^{\frac{2 \pi i(m-2)}{2^{b}}}\right) .
\end{aligned}
$$

Hence
$g_{1}\left(u_{2}, v_{2}\right)=\left(u_{2} e^{\frac{2 \pi i m}{2^{b-1}}}, u_{2} v_{2}^{-1} e^{\frac{2 \pi i(m-2)}{2^{b}}}\right)$ and $g_{1}^{2}\left(u_{2}, v_{2}\right)=\left(u_{2} e^{\frac{2 \pi i m}{2^{b-2}}}, v_{2} e^{\frac{2 \pi i m}{2^{b-1}}}\right)$.
Note that $\left\langle g_{1}^{2}\right\rangle=\mathbb{Z}_{2^{b-1}}$ and $g_{1}^{2^{b-1}}$ has as its fixed-point set the core $S^{1} \times\{0\}$. We obtain an orbifold covering map $\lambda_{2}: V_{2}(1) \rightarrow V_{1}(1) /\left\langle g_{1}^{2}\right\rangle=V_{2}(2)$ defined by $\lambda_{2}\left(u_{2}, v_{2}\right)=\left(u_{2}^{2^{b-2}}, u_{2}^{-1} v_{2}^{2}\right)$. Furthermore, $g_{1}$ induces an orientation reversing involution $g_{2}$ on $V_{2}(2)$ which we now compute below:

$$
\begin{aligned}
g_{2}\left(u_{2}, v_{2}\right) & =\lambda_{2} g_{1}\left(u_{2}^{\frac{1}{2 b-2}}, u_{2}^{\frac{1}{2 b-1}} v_{2}^{\frac{1}{2}}\right) \\
& =\lambda_{2}\left(u_{2}^{\frac{1}{2 b-2}} e^{\frac{2 \pi i m}{b^{b-1}}}, u_{2}^{\frac{1}{2 b-2}}\left(u_{2}^{\frac{1}{2 b-1}} v_{2}^{\frac{1}{2}}\right)^{-1} e^{\frac{2 \pi i(m-2)}{2^{b}}}\right) \\
& =\lambda_{2}\left(u_{2}^{\frac{1}{2 b-2}} e^{\frac{2 \pi i m}{b^{b-1}}}, u_{2}^{\frac{1}{2 b-1}} v_{2}^{\frac{-1}{2}} e^{\frac{2 \pi i(m-2)}{2^{b}}}\right) \\
& =\left(-u_{2}, v_{2}^{-1} e^{\frac{-2 \pi i}{2^{b-2}}}\right) .
\end{aligned}
$$

Since $g_{2}$ is a fixed-point free orientation reversing involution on $V_{2}(2)$, it follows by [13, Proposition 13] that $V_{2}(2) /\left\langle g_{2}\right\rangle$ is the orbifold $(A 1,2)$, and we have the following lemma.

Lemma 3.2. For any orbifold of the form ( $A 1,2$ ), there exists a $\mathbb{Z}_{m} \times \mathbb{Z}_{2^{b}}$ action on the solid torus $V_{2}$ where $m$ is odd and $b>1$, generated by $f\left(u_{2}, v_{2}\right)$ $=\left(u_{2} e^{\frac{4 \pi i}{m}}, v_{2} e^{\frac{2 \pi i}{m}}\right)$ and $g\left(u_{2}, v_{2}\right)=\left(u_{2} e^{\frac{2 \pi i}{2^{b-1}}}, u_{2} v_{2}^{-1} e^{\frac{-2 \pi i}{2^{\sigma}}}\right)$, such that $V_{2} /\left(\mathbb{Z}_{m} \times\right.$ $\left.\mathbb{Z}_{2^{b}}\right)=(A 1,2)$.

The next step is to compute the quotient space for the covering $\eta_{1} \cup$ $\eta_{2}:\left(V_{1} \cup_{\alpha} V_{2}\right) \rightarrow\left(V_{1} \cup_{\alpha} V_{2}\right) /\langle f\rangle=V_{1}(m) \cup_{\alpha_{1}} V_{2}(1)$. The matrix representations for $\eta_{1}$ and $\eta_{2}$ are $\left[\begin{array}{cc}1 & 0 \\ 0 & m\end{array}\right]$ and $\left[\begin{array}{cc}m & 0 \\ \frac{m-1}{2} & 1\end{array}\right]$ respectively. We compute the gluing map $\alpha_{1}: \partial V_{1}(m) \rightarrow \partial V_{2}(1)$ with matrix representation $\left[\begin{array}{cc}\mathbf{x} & \mathbf{y} \\ \mathbf{z} & \mathbf{w}\end{array}\right]$, by solving the equation

$$
\left[\begin{array}{cc}
\mathbf{x} & \mathbf{y} \\
\mathbf{z} & \mathbf{w}
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
0 & m
\end{array}\right]=\left[\begin{array}{cc}
m & 0 \\
\frac{m-1}{2} & 1
\end{array}\right]\left[\begin{array}{ll}
1 & 2 \\
1 & 1
\end{array}\right]
$$

We see that $\mathbf{x}=m, \mathbf{y}=2, \mathbf{z}=\frac{m+1}{2}$ and $\mathbf{w}=1$. Thus $\alpha_{1}\left(u_{1}, v_{1}\right)=$ $\left(u_{2}^{m} v_{2}^{2}, u_{2}^{\frac{m+1}{2}} v_{2}\right)$, the matrix representation for $\alpha_{1}$ is $\left[\begin{array}{cc}m & 2 \\ \frac{m+1}{2} & 1\end{array}\right]$ and the quotient space $\mathbb{R P}^{3} /\langle f\rangle=W(2,1 ; m, 1)$.

Finally, consider the orbifold covering $\lambda_{1} \cup \lambda_{2}: V_{1}(m) \cup_{\alpha_{1}} V_{2}(1) \rightarrow\left(V_{1}(m) \cup_{\alpha_{1}}\right.$ $\left.V_{2}(1)\right) /\left\langle g_{1}^{2}\right\rangle=V_{1}\left(2^{b-1} m\right) \cup_{\alpha_{2}} V_{2}(2)$, and identify the quotient space by computing the gluing map $\alpha_{2}$. The matrix representations for $\lambda_{1}$ and $\lambda_{2}$ are $\left[\begin{array}{cc}1 & 0 \\ 0 & 2^{b-1}\end{array}\right]$ and $\left[\begin{array}{cc}2^{b-2} & 0 \\ -1 & 2\end{array}\right]$ respectively. Solving a matrix equation similar to that above, we obtain $\alpha_{2}\left(u_{1}, v_{1}\right)=\left(u_{2}^{2^{b-2} m} v_{2}, u_{2}\right)$ with its matrix representation $\left[\begin{array}{cc}2^{b-2} m & 1 \\ 1 & 0\end{array}\right]$. Thus $W(2,1 ; m, 1) /\left\langle g_{1}^{2}\right\rangle=W\left(1,0 ; 2^{b-1} m, 2\right)$. The underlying space of $W\left(1,0 ; 2^{b-1} m, 2\right)$ is the 3 -sphere $S^{3}$ and the exceptional set is the Hopf link, with one exceptional set labeled with $2^{b-1} m$ and the other exceptional set labeled with 2.

Consequently, we can summarize the results above. Recall $\mathbb{R} \mathbb{P}^{3}=V_{1} \cup_{\alpha} V_{2}$ where $\alpha$ : $\partial V_{1} \rightarrow \partial V_{2}$ is a homeomorphism defined by $\alpha\left(u_{1}, v_{1}\right)=\left(u_{2} v_{2}^{2}, u_{2} v_{2}\right)$ for $\left(u_{i}, v_{i}\right) \in V_{i}$. Define homeomorphisms $f$ and $g$ on $\mathbb{R}^{3}$ as follows:

$$
\begin{aligned}
& f\left(u_{i}, v_{i}\right)= \begin{cases}\left(u_{1}, v_{1} e^{\frac{2 \pi i}{m}}\right), & \text { if } i=1 \\
\left(u_{2} e^{\frac{4 \pi i}{m}}, v_{2} e^{\frac{2 \pi i}{m}}\right), & \text { if } i=2\end{cases} \\
& g\left(u_{i}, v_{i}\right)= \begin{cases}\left(\overline{u_{1}} e^{\frac{-2 \pi i}{2^{b-2}}}, u_{1} v_{1} e^{\frac{3(2 \pi i)}{2^{b}}}\right), & \text { if } i=1 \\
\left(u_{2} e^{\frac{2 \pi i}{2^{b-1}}}, u_{2} \overline{v_{2}} e^{\frac{-2 \pi i}{2^{b}}}\right), & \text { if } i=2\end{cases}
\end{aligned}
$$

Theorem 3.3. Let $\varphi: \mathbb{Z}_{s} \rightarrow$ Homeo $_{P L}\left(\mathbb{R P}^{3}\right)$ be an action such that $s=2^{b} m$ where $b>1$ and $m$ is odd. Then $\varphi$ is equivalent to $\langle f\rangle \times\langle g\rangle=\mathbb{Z}_{m} \times$ $\mathbb{Z}_{2^{b}}=\mathbb{Z}_{2^{b} m}$, and the quotient space $\mathbb{R}^{3} / \varphi$ is homeomorphic to the orbifold $O_{h_{1}^{-1}}\left(\left(B 5,2^{b-1} m\right),(A 1,2)\right)$. Let $\varphi_{0}: \mathbb{Z}_{s / 2} \rightarrow$ Homeo $_{P L}\left(\mathbb{R P}^{3}\right)$ represent the restriction of $\varphi$ to the the orientation preserving subgroup. Then $\mathbb{R P}^{3} / \varphi_{0}$ is the orbifold $W\left(1,0 ; 2^{b-1} m, 2\right)$ whose underlying space is the 3 -sphere $S^{3}$, and the exceptional set is the Hopf link with one exceptional set labeled with $2^{b-1} m$ and the other exceptional set labeled with 2.

Proof. Let $\varphi: \mathbb{Z}_{s} \rightarrow$ Homeo $_{P L}\left(\mathbb{R} \mathbb{P}^{3}\right)$ be an action such that $s=2^{b} m$ where $b>1$ and $m$ is odd. By [14, Theorem A], there is only one such action up to equivalence. By construction, $\mathbb{R P}^{3} /\langle f, g\rangle=O_{\alpha_{3}}\left(\left(B 5,2^{b-1} m\right),(A 1,2)\right)$ for some gluing map $\alpha_{3}$. Since $O_{\alpha_{3}}\left(\left(B 5,2^{b-1} m\right),(A 1,2)\right)=O_{\alpha_{3}^{-1}}\left((A 1,2),\left(B 5,2^{b-1} m\right)\right)$, which by [13, Lemma 21] is homeomorphic to $O_{h_{1}}\left((A 1,2),\left(B 5,2^{b-1} m\right)\right)$, the result follows.

Next, we will treat the case where $b=1$, and so we will consider orientation reversing $\mathbb{Z}_{m} \times \mathbb{Z}_{2}=\mathbb{Z}_{2 m}$-actions on $\mathbb{R}^{3}{ }^{3}$ where $m$ is odd.

### 3.2. Quotient Type 2: $O_{h_{2}^{-1}}((B 4, m),(A 3,1))$ and $m$ odd

Substituting $b=1$ into the definition of $g$ defined in Quotient Type 1, we obtain the involution $h: V_{1} \rightarrow V_{1}$ defined by $h\left(u_{1}, v_{1}\right)=\left(\overline{u_{1}},-u_{1} v_{1}\right)$. It follows that $h$ is an orientation reversing involution which commutes with $f$ on $V_{1}$, and thus $\langle f\rangle \times\langle h\rangle=\mathbb{Z}_{m} \times \mathbb{Z}_{2}$. As above, if $\eta: V_{1} \rightarrow V_{1} /\langle f\rangle=V_{1}(m)$ is the covering, the induced map $\hat{h}$ on $V_{1}(m)$ is defined by $\hat{h}\left(u_{1}, v_{1}\right)=\left(\overline{u_{1}},-u_{1}^{m} v_{1}\right)$. The fixed-point set is $\{(1,0)\} \cup\left(\{-1\} \times D^{2}\right) \subseteq V_{1}(m)$. It follows by [13, Proposition 13] and the fixed-point set of $\hat{h}$, that $V_{1}(m) /\langle\hat{h}\rangle=(B 4, m)$.

The involution $h$ on $V_{2}$ is defined by $h\left(u_{2}, v_{2}\right)=\left(u_{2},-u_{2} \overline{v_{2}}\right)$, and the involution $\hat{h}$ on $V_{2}(1)$ is $\hat{h}\left(u_{2}, v_{2}\right)=\left(u_{2},-u_{2} \overline{v_{2}}\right)$. The fixed-point set consists of the set $\left\{\left(-e^{2 i \theta}, \rho e^{i \theta}\right) \mid 0 \leq \theta \leq 2 \pi, 0 \leq \rho \leq 1\right\} \subseteq V_{2}(1)$, which is a Möbius band. Hence by [13, Proposition 13], $V_{2}(1) /\langle\hat{h}\rangle=(A 3,1)$. We obtain the two lemmas below:

Lemma 3.4. For any orbifold of the form $(B 4, m)$ where $m$ is odd, there exists a $\mathbb{Z}_{m} \times \mathbb{Z}_{2}$-action on the solid torus $V_{1}$, generated by $f\left(u_{1}, v_{1}\right)=\left(u_{1}, v_{1} e^{\frac{2 \pi i}{m}}\right)$ and $h\left(u_{1}, v_{1}\right)=\left(\overline{u_{1}},-u_{1} v_{1}\right)$, such that $V_{1} /\left(\mathbb{Z}_{m} \times \mathbb{Z}_{2}\right)=(B 4, m)$.

Lemma 3.5. For any orbifold of the form (A3,1), there exists a $\mathbb{Z}_{m} \times \mathbb{Z}_{2}$-action on the solid torus $V_{2}$ where $m$ is odd, generated by $f\left(u_{2}, v_{2}\right)=\left(u_{2} e^{\frac{4 \pi i}{m}}, v_{2} e^{\frac{2 \pi i}{m}}\right)$ and $h\left(u_{2}, v_{2}\right)=\left(u_{2},-u_{2} \overline{v_{2}}\right)$, such that $V_{2} /\left(\mathbb{Z}_{m} \times \mathbb{Z}_{2}\right)=(A 3,1)$.

Let $f, h: \mathbb{R} \mathbb{P}^{3} \rightarrow \mathbb{R P}^{3}$ be homeomorphisms defined as follows:

$$
\begin{aligned}
& f\left(u_{i}, v_{i}\right)= \begin{cases}\left(u_{1}, v_{1} e^{\frac{2 \pi i}{m}}\right), & \text { if } i=1 \\
\left(u_{2} e^{\frac{4 \pi i}{m}}, v_{2} e^{\frac{2 \pi i}{m}}\right), & \text { if } i=2\end{cases} \\
& h\left(u_{i}, v_{i}\right)= \begin{cases}\left(\overline{u_{1}},-u_{1} v_{1}\right), & \text { if } i=1 \\
\left(u_{2},-u_{2} \overline{v_{2}}\right), & \text { if } i=2\end{cases}
\end{aligned}
$$

As a result of the above discussions, we obtain the following theorem.
Theorem 3.6. Let $\varphi: \mathbb{Z}_{s} \rightarrow$ Homeo $_{P L}\left(\mathbb{R} \mathbb{P}^{3}\right)$ be an action such that $s=2 m$ where $m$ is odd. Then $\varphi$ is equivalent to $\langle f\rangle \times\langle h\rangle=\mathbb{Z}_{m} \times \mathbb{Z}_{2}=\mathbb{Z}_{2 m}$, and the quotient space $\mathbb{R P}^{3} / \varphi$ is homeomorphic to the orbifold $O_{h_{2}^{-1}}((B 4, m),(A 3,1))$. Let $\varphi_{0}: \mathbb{Z}_{m} \rightarrow$ Homeo $_{P L}\left(\mathbb{R}^{3} \mathbb{P}^{3}\right)$ represent the restriction of $\varphi$ to the the orientation preserving subgroup. Then $\mathbb{R} \mathbb{P}^{3} / \varphi_{0}$ is the orbifold $W(2,1 ; m, 1)$, whose underlying space is $\mathbb{R}^{\mathbb{P}^{3}}$ with exceptional set a simple closed curve labeled with $m$.

Proof. Let $\varphi: \mathbb{Z}_{s} \rightarrow \operatorname{Homeo}_{P L}\left(\mathbb{R} \mathbb{P}^{3}\right)$ be an action such that $s=2 m$ where $m$ is odd. If $m>1$, then applying the Smith conjecture in [1] and [14, Theorem C], there is only one such action up to equivalence. If $m=1$, and hence
the action is an involution, applying [6] there is only one such action up to equivalence. By the above construction $\mathbb{R P}^{3} /\langle f, h\rangle=O_{\zeta}((B 4, m),(A 3,1))=$ $O_{\zeta^{-1}}((A 3,1),(B 4, m))$ for some gluing map $\zeta$. Since $O_{\zeta^{-1}}((A 3,1),(B 4, m))$ is homeomorphic to $O_{h_{2}}((A 3,1),(B 4, m))$ by [13, Lemma 23], the result follows.

### 3.3. Quotient Type 3: $O_{h_{3}}((A 2,2),(B 3,, m))$ where $m$ is even

Define homeomorphisms $f$ and $g$ on $\mathbb{R}^{3}$ as follows:

$$
\begin{aligned}
& f\left(u_{i}, v_{i}\right)= \begin{cases}\left(u_{1} e^{\frac{4 \pi i}{m}}, v_{1} e^{-\frac{2 \pi i}{m}}\right), & \text { if } i=1 \\
\left(u_{2}, v_{2} e^{\frac{2 \pi i}{m}}\right), & \text { if } i=2\end{cases} \\
& g\left(u_{i}, v_{i}\right)= \begin{cases}\left(u_{1}, \overline{u_{1} v_{1}}\right), & \text { if } i=1 \\
\left(\overline{u_{2}}, \overline{u_{2}} v_{2}\right), & \text { if } i=2\end{cases}
\end{aligned}
$$

A computation shows $f g=g f$, and so $\langle f, g\rangle$ defines a $\mathbb{Z}_{m} \times \mathbb{Z}_{2}$-action on $\mathbb{R} \mathbb{P}^{3}$. Furthermore, it can be shown that $\mathbb{R}^{3} / \varphi$ is the orbifold $O_{h^{\prime}}((A 2,2),(B 3,, m))$ for some homeomorphism $h^{\prime}$ between their boundaries.

Theorem 3.7. For $m$ even, the maps $f$ and $g$ define an action $\varphi: \mathbb{Z}_{m} \times \mathbb{Z}_{2} \rightarrow$ Homeo $_{P L}\left(\mathbb{R} \mathbb{P}^{3}\right)$ such that the quotient space $\mathbb{R}^{3} / \varphi$ is $O_{h_{3}}((A 2,2),(B 3, m))$. Let $\varphi_{0}: \mathbb{Z}_{m} \rightarrow$ Homeopl $\left(\mathbb{R}^{3} \mathbb{P}^{3}\right)$ represent the restriction of $\varphi$ to the the orientation preserving subgroup. Then $\mathbb{R}^{P} / \varphi_{0}$ is the orbifold $W\left(1, \frac{m}{2} ; 2, m\right)$ whose underlying space is the 3 -sphere $S^{3}$, and the exceptional set is the Hopf link with one exceptional set labeled with 2 and the other exceptional set labeled with $m$.

Proof. The proof is similar to Theorem 3.6, and uses the fact that by [13, Lemma 22], $O_{h^{\prime}}((A 2,2),(B 3, m))$ is homeomorphic to $O_{h_{3}}((A 2,2),(B 3, m))$.

### 3.4. Quotient Type 4: $O_{h_{4}}((B 2,2),(B 2,, 2))$

Define homeomorphisms $\theta$ and $\tau$ on $\mathbb{R} \mathbb{P}^{3}$ as follows:

$$
\begin{aligned}
& \theta\left(u_{i}, v_{i}\right)= \begin{cases}\left(-\overline{u_{1}}, u_{1} v_{1}\right), & \text { if } i=1 \\
\left(-u_{2},-u_{2} \overline{v_{2}}\right), & \text { if } i=2\end{cases} \\
& \tau\left(u_{i}, v_{i}\right)= \begin{cases}\left(\overline{u_{1}}, \overline{v_{1}}\right), & \text { if } i=1 \\
\left(\overline{u_{2}}, \overline{v_{2}}\right), & \text { if } i=2\end{cases}
\end{aligned}
$$

A computation shows $\theta^{4}=i d=\tau^{2}$ and $\theta \tau=\tau \theta$, and so $\langle\theta, \tau\rangle$ defines a $\mathbb{Z}_{4} \times \mathbb{Z}_{2^{-}}$ action on $\mathbb{R P}^{3}$. We remark that letting $b=2$ in the definition of $g$ in Quotient

Type 1 of this section, also gives a $\mathbb{Z}_{4}$-action which is conjugate to $\theta$ by the homeomorphism:

$$
k\left(u_{i}, v_{i}\right)= \begin{cases}\left(i u_{1}, v_{1}\right), & \text { if } i=1 \\ \left(i u_{2}, i v_{2}\right), & \text { if } i=2\end{cases}
$$

Observe that $\theta^{2}\left(u_{i}, v_{i}\right)=\left(u_{i},-v_{i}\right)$, and we have a covering map $\nu: \mathbb{R} \mathbb{P}^{3} \rightarrow$ $\mathbb{R P}^{3} /\left\langle\theta^{2}\right\rangle=(A 0,2) \cup_{\alpha_{1}}(A 0,2)$ where $\nu\left(u_{i}, v_{i}\right)=\left(u_{i}, v_{i}^{2}\right)$; the matrix corresponding to $\alpha_{1}$ is $\left[\begin{array}{ll}1 & 1 \\ 2 & 1\end{array}\right]$. The induced maps $\bar{\theta}$ and $\bar{\tau}$ on $(A 0,2) \cup_{\alpha_{1}}(A 0,2)$ are defined by

$$
\begin{aligned}
& \bar{\theta}\left(u_{i}, v_{i}\right)=\left\{\begin{array}{ll}
\left(-\overline{u_{1}}, u_{1}^{2} v_{1}\right), & \text { if } i=1 \\
\left(-u_{2}, u_{2}^{2} \overline{v_{2}}\right), & \text { if } i=2
\end{array}\right. \text { and } \\
& \bar{\tau}\left(u_{i}, v_{i}\right)=\left(\overline{u_{i}}, \overline{v_{i}}\right) .
\end{aligned}
$$

Moding out by the action of $\bar{\tau}$, we obtain a covering map $\nu_{1}:(A 0,2) \cup_{\alpha_{1}}$ $(A 0,2) \rightarrow(B 0,2) \cup_{\overline{\alpha_{1}}}(B 0,2)$. Now $\bar{\theta}$ induces an involution on $(B 0,2) \cup_{\overline{\alpha_{1}}}$ $(B 0,2)$, whose quotient is $O_{f^{\prime}}((B 2,2),(B 2,2))$ for some homeomorphism $f^{\prime}: \partial(B 2,2) \rightarrow \partial(B 2,2)$. We obtain the result below.
THEOREM 3.8. Let $\varphi: \mathbb{Z}_{4} \times \mathbb{Z}_{2} \rightarrow$ Homeo $_{P L}\left(\mathbb{R} \mathbb{P}^{3}\right)$ be an action such that $\varphi\left(\mathbb{Z}_{4} \times \mathbb{Z}_{2}\right)=\langle\theta, \tau\rangle$. Then the quotient space $\mathbb{R} \mathbb{P}^{3} / \varphi$ is homeomorphic to the orbifold $O_{h_{4}}((B 2,2),(B 2,2))$. Let $\varphi_{0}: \mathbb{Z}_{4} \times \mathbb{Z}_{2} \rightarrow$ Homeo $_{P L}\left(\mathbb{R}^{3} \mathbb{P}^{3}\right)$ represent the restriction of $\varphi$ to the the orientation preserving subgroup. Then $\mathbb{R P}^{3} / \varphi_{0}$ is the orbifold $O_{\overline{\alpha_{1}}}((B 0,2),(B 0,2))$, where $\overline{\alpha_{1}}$ is uniquely determined by the matrix

$$
\left[\begin{array}{ll}
1 & 1 \\
2 & 1
\end{array}\right]
$$

Proof. The quotient space $\mathbb{R}^{3} / \varphi$ has a finite fundamental group. The result now follows by the above construction, Theorem 11 and Lemma 25 in [13].
3.5. Quotient Type 5: $O_{h_{5}}((B 6,2),(B 6,2))$

Define homeomorphisms $f, g$ and $h$ on $\mathbb{R P}^{3}$ as follows:

$$
\begin{aligned}
& f\left(u_{i}, v_{i}\right)= \begin{cases}\left(\overline{u_{1}}, \overline{v_{1}}\right), & \text { if } i=1 \\
\left(\overline{u_{2}}, \overline{v_{2}}\right), & \text { if } i=2\end{cases} \\
& g\left(u_{i}, v_{i}\right)= \begin{cases}\left(u_{1}, \overline{u_{1} v_{1}}\right), & \text { if } i=1 \\
\left(\overline{u_{2}}, \overline{u_{2}} v_{2}\right), & \text { if } i=2\end{cases} \\
& h\left(u_{i}, v_{i}\right)= \begin{cases}\left(u_{1},-v_{1}\right), & \text { if } i=1 \\
\left(u_{2},-v_{2}\right), & \text { if } i=2\end{cases}
\end{aligned}
$$

It follows that $\langle f, g, h\rangle$ is a $\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$-action on $\mathbb{R} \mathbb{P}^{3}$. We may choose a covering map $\nu: \mathbb{R P}^{3} \rightarrow \mathbb{R P}^{3} /\langle h\rangle$ defined by $\nu\left(u_{1}, v_{1}\right)=\left(u_{1},-u_{1} v_{1}^{2}\right)$ and $\nu\left(u_{2}, v_{2}\right)=$ $\left(u_{2}, u_{2}^{-1} v_{2}^{2}\right)$. Then $\mathbb{R P}^{3} /\langle h\rangle$ is the orbifold $V(2) \cup_{r_{1}} V(2)=W(1,0 ; 2,2)$ where $r_{1}: V(2) \rightarrow V(2)$ is defined by $r_{1}\left(u_{1}, v_{1}\right)=\left(-v_{2}, u_{2}\right)$. The induced maps $f_{1}$ and $g_{1}$ on $W(1,0 ; 2,2)$ are defined by

$$
f_{1}\left(u_{i}, v_{i}\right)= \begin{cases}\left(\overline{u_{1}}, \overline{v_{1}}\right), & \text { if } i=1 \\ \left(\overline{u_{2}}, \overline{v_{2}}\right), & \text { if } i=2\end{cases}
$$

and

$$
g_{1}\left(u_{i}, v_{i}\right)= \begin{cases}\left(u_{1}, \overline{v_{1}}\right), & \text { if } i=1 \\ \left(\overline{u_{2}}, v_{2}\right), & \text { if } i=2\end{cases}
$$

Now, $W(1,0 ; 2,2) /\left\langle f_{1}\right\rangle=O_{r_{2}}((B 0,2),(B 0,2))$ for some gluing map $r_{2}: \partial(B 0,2) \rightarrow \partial(B 0,2)$. The map $r_{2}$ is an order 4 rotation which permutes the cone points of order 2 on $\partial(B 0,2)$. It follows that for the induced map $g_{2}$ on the orbifold $O_{r_{2}}((B 0,2),(B 0,2))$ we obtain $O_{r_{2}}((B 0,2),(B 0,2)) /\left\langle g_{3}\right\rangle=$ $O_{r_{3}}((B 6,2),(B 6,2))$. Summarizing we have the following theorem:

Theorem 3.9. The maps $f, g$ and $h$ define an action $\varphi: \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} \rightarrow$ Hoтeo $_{P L}\left(\mathbb{R P}^{3}\right)$ such that the quotient space $\mathbb{R P}^{3} / \varphi$ is $O_{h_{5}}((B 6,2),(B 6,2))$. Let $\varphi_{0}: \mathbb{Z}_{2} \times \mathbb{Z}_{2} \rightarrow$ Homeo $_{P L}\left(\mathbb{R} \mathbb{P}^{3}\right)$ represent the restriction of $\varphi$ to the the orientation preserving subgroup. Then $\mathbb{R}^{3} / \varphi_{0}$ is the orbifold $O_{r_{2}}((B 0,2),(B 0,2))$ where $\overline{r_{2}}$ is uniquely determined by the matrix $\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]$.
Proof. The quotient space $\mathbb{R}^{3} / \varphi$ is the orbifold $O_{r_{3}}((B 6,2),(B 6,2))$ for some map $r_{3}$. By [13, Lemma 26], this orbifold is homeomorphic to $O_{r}((B 6,2),(B 6,2))$ which completes the proof.
3.6. Quotient Type 6: $O_{h_{6}}((B 7,1),(B 7,1))$

Define homeomorphisms $f$ and $g$ on $\mathbb{R P}^{3}$ as follows:

$$
\begin{aligned}
& f\left(u_{i}, v_{i}\right)= \begin{cases}\left(\overline{u_{1}},-u_{1} v_{1}\right), & \text { if } i=1 \\
\left(u_{2},-u_{2} \overline{v_{2}}\right), & \text { if } i=2\end{cases} \\
& g\left(u_{i}, v_{i}\right)= \begin{cases}\left(u_{1}, \overline{u_{1} v_{1}}\right), & \text { if } i=1 \\
\left(\overline{u_{2}}, \overline{u_{2}} v_{2}\right), & \text { if } i=2\end{cases}
\end{aligned}
$$

We see that $\langle f, g\rangle$ is a $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$-action on $\mathbb{R P}^{3}$ and $f g\left(u_{i}, v_{i}\right)=\left(\overline{u_{i}},-\overline{v_{i}}\right)$. Let $\eta: \mathbb{R} \mathbb{P}^{3} \rightarrow \mathbb{R P}^{3} /\langle f g\rangle$ be an orbifold covering map and note that the quotient space is $O_{\widehat{\alpha}}((B 0,1),(B 0,1))$ for some homeomorphism $\widehat{\alpha}: \partial(B 0,1) \rightarrow \partial(B 0,1)$. Let $\widehat{g}$ be the induced involution on $O_{\widehat{\alpha}}((B 0,1),(B 0,1))$.

The fixed-point set of $\left.f g\right|_{\partial V_{i}}$ is $\operatorname{Fix}\left(\left.f g\right|_{\partial V_{i}}\right)=\{(1, i),(1,-i),(-1, i),(-1,-i)\}$. It follows that $\left.g\right|_{\partial V_{i}}$ fixes two elements of $\operatorname{Fix}\left(\left.f g\right|_{\partial V_{i}}\right)$, and exchanges the other two.

This implies that $O_{\widehat{\alpha}}((B 0,1),(B 0,1)) /\langle\widehat{g}\rangle$ is the orbifold $O_{r^{\prime}}((B 7,1),(B 7,1))$ for some homeomorphism $r^{\prime}: \partial(B 7,1) \rightarrow \partial(B 7,1)$. As a result, we obtain the following theorem:

Theorem 3.10. The maps $f$ and $g$ define an action $\varphi: \mathbb{Z}_{2} \times \mathbb{Z}_{2} \rightarrow$ Homeo $_{P L}\left(\mathbb{R} \mathbb{P}^{3}\right)$ such that the quotient space $\mathbb{R}^{3} / \varphi$ is the orbifold $O_{h_{6}}((B 7,1),(B 7,1))$. Let $\varphi_{0}: \mathbb{Z}_{2} \rightarrow$ Homeo $_{P L}\left(\mathbb{R}^{3} \mathbb{P}^{3}\right)$ represent the restriction of $\varphi$ to the the orientation preserving subgroup. Then $\mathbb{R}^{3} / \varphi_{0}$ is the orbifold $O_{\bar{\alpha}}((B 0,1),(B 0,1))$ where $\bar{\alpha}$ is uniquely determined by the matrix $\left[\begin{array}{ll}1 & 2 \\ 1 & 1\end{array}\right]$.
Proof. The quotient space $\mathbb{R}^{3} / \varphi$ is the orbifold $O_{r^{\prime}}((B 7,1),(B 7,1))$ for some homeomorphism $r^{\prime}$. Since the fundamental group of the quotient space is finite, it follows by [13, Lemma 27] that this orbifold is homeomorphic to $O_{h_{6}}((B 7,1),(B 7,1))$, completing the proof.

## 4. Splitting orientation-reversing abelian actions on $\mathbb{R P}^{3}$

In this section, we will show that any abelian orientation reversing action on $\mathbb{R P}^{3}$ splits and preserves the sides of the splitting. An action $\varphi: G \rightarrow$ Homeo $_{P L}\left(\mathbb{R P}^{3}\right)$ is said to split if there is a Heegaard torus $T$ such that $\varphi(g)(T)=$ $T$ for all $g \in G$. If in addition, each complementary component of the Heegaard torus is invariant under the action, then we say $\varphi$ preserves the sides of the splitting.

Theorem 4.1. Let $\varphi: G \rightarrow$ Homeo $_{P L}\left(\mathbb{R} \mathbb{P}^{3}\right)$ be a finite action which contains an orientation reversing element $j \in \varphi(G)$, such that $\langle j\rangle$ is a normal subgroup of $\varphi(G)$. If $j$ is an involution, assume $\varphi(G) /\langle j\rangle$ is not the symmetric group $S_{4}$ or the alternating groups $A_{4}$ and $A_{5}$. Then $\varphi$ splits and preserves the sides of the splitting.

Proof. The element $j$ generates a cyclic group $\mathbb{Z}_{2^{b} m}$ where $m$ is odd. Since the orientation preserving subgroup of $\mathbb{Z}_{2^{b} m}$ has index two and generated by $j^{2}$, it follows that $b \geq 1$.

Suppose first that $b>1$. By Theorem 3.3, $\langle j\rangle$ is conjugate to the group $\langle f\rangle \times\langle g\rangle=\mathbb{Z}_{m} \times \mathbb{Z}_{2^{b}}=\mathbb{Z}_{2^{b} m}$. Conjugating all of $\varphi(G)$ by this element, we may assume $\langle j\rangle=\langle f\rangle \times\langle g\rangle$. Furthermore, the quotient space $\mathbb{R}^{3} /\langle j\rangle=$ $\left(B 5,2^{b-1} m\right) \cup_{h_{1}^{-1}}(A 1,2)$. Let $\nu: \mathbb{R P}^{3} \rightarrow \mathbb{R P}^{3} /\langle j\rangle$ be the covering map. The core in $V_{2}$ is $S^{1} \times\{0\}$, and $\nu\left(S^{1} \times\{0\}\right)$ is the exceptional set $\gamma$ in $(A 1,2)$, which is a simple closed curve labeled with the integer 2 . The induced action $\varphi(G) /\langle j\rangle=H$ on $\mathbb{R}^{3} /\langle j\rangle$ must leave $\gamma$ invariant. Let $U$ be an $H$-invariant
regular neighborhood of $\gamma$. Now $U$ is the orbifold $(A 0,2)$, which lifts to a $\varphi(G)$-invariant solid torus $\widetilde{U}$, containing the core. Its boundary $\partial \widetilde{U}$ is a $\varphi(G)$ invariant Heegaard torus whose sides are preserved by $\varphi(G)$.

Assume $b=1$. By Theorem 3.6, $\langle j\rangle$ is conjugate to $\langle f\rangle \times\langle h\rangle=\mathbb{Z}_{m} \times$ $\mathbb{Z}_{2}=\mathbb{Z}_{2 m}$, and we may assume as above that $\langle j\rangle=\langle f\rangle \times\langle h\rangle$. If $m \neq 1$, then $\mathbb{R P}^{3} /\langle f\rangle=(A 0, m) \cup_{\alpha_{1}}(A 0,1)$, where the matrix for $\alpha_{1}$ is $\left[\begin{array}{cc}m & 2 \\ \frac{m+1}{2} & 1\end{array}\right]$. (See computation following Lemma 3.2.) The orbifold ( $A 0, m$ ) contains an exceptional set consisting of a simple closed curve labeled with an $m$. Letting $H$ be the quotient group $\varphi(G) /\langle f\rangle$, it follows that $H$ must leave the exceptional set invariant. The exceptional set lifts to the core in $V_{1}$, and the proof follows as above.

Now suppose $m=1$. In this case $\mathbb{R P}^{3} /\langle h\rangle=(B 4,1) \cup_{h_{2}^{-1}}(A 3,1)$, and we again let $\nu: \mathbb{R}^{3} \rightarrow(B 4,1) \cup_{h_{2}^{-1}}(A 3,1)$ be the covering map. The exceptional set consists of a point in $(B 4,1)$, a projective plane $P$ with $P \cap(B 4,1)$ a mirrored disk and $P \cap(A 3,1)$ a mirrored Möbius band. For the core in $S^{1} \times\{0\}$ in $V_{2}$, it follows that $\nu\left(S^{1} \times\{0\}\right)$ is an orientation reversing element in the mirrored Möbius band. Letting $H$ be $\varphi(G) /\langle h\rangle$, the projective plane must be left invariant by $H$. Since $\varphi(G) /\langle h\rangle$ is neither $S_{4}, A_{4}$ nor $A_{5}$, it follows by [11, Theorem 7.2] that $\left.H\right|_{P}$ leaves an orientation reversing loop invariant. Since the lift of this loop is isotopic to the core in $V_{2}$, the proof follows as above.

We obtain the following corollary:
Corollary 4.2. Let $\varphi: G \rightarrow$ Homeo $_{P L}\left(\mathbb{R} \mathbb{P}^{3}\right)$ be an orientation reversing abelian action. Then $\varphi$ splits and preserves the sides of the splitting.

## 5. Orbifolds covered by $\mathbb{R} \mathbb{P}^{3}$

In this section, we will identify which of the non-orientable orbifolds listed in Theorem 2.1 as defined in the Appendix may be covered by $\mathbb{R} \mathbb{P}^{3}$ and identify the subgroup corresponding to the covering. The orbifolds in Section 2 are: $O_{h_{1}}((A 1, n),(B 5, m)), \quad O_{h_{2}}((A 3, n),(B 4, m)), \quad O_{h_{3}}((A 2, n),(B 3, m))$, $O_{h_{4}}((B 2, n),(B 2, m)), \quad O_{h_{5}}((B 6, n),(B 6, m)), \quad O_{h_{6}}((B 7, n),(B 7, m))$, $O_{h_{7}}((B 1, n),(B 8, m))$.

It will be convenient to apply the following proposition and corollary, which essentially follow from orbifold covering space theory. The reader is referred to the paper of M. Yokoyama [22] for a good elucidation of orbifold theory.

Proposition 5.1. Let $O$ be a 3-dimensional orbifold, $W$ a 3-dimensional suborbifold and $i: W \hookrightarrow O$ the inclusion map. Suppose $G$ is a subgroup of $\pi_{1}(O)$ and $H=i_{*}^{-1}(G) \leq \pi_{1}(W)$. Let $\eta: \widetilde{O} \rightarrow O$ and $\lambda: \widetilde{W} \rightarrow W$ be the cover-
ings corresponding to $G$ and $H$ respectively. If $\widetilde{W}$ is an orbifold, then $\widetilde{O}$ is an orbifold.

Proof. Let $L$ be a component of $\eta^{-1}(W)$ and note that $p=\left.\eta\right|_{L}: L \rightarrow W$ is a covering map. From standard covering space theory $p_{*}\left(\pi_{1}(L)\right)=i_{*}^{-1}\left(\eta_{*}\left(\pi_{1}(\widetilde{O})\right)\right)$. Since $i_{*}^{-1}\left(\eta_{*}\left(\pi_{1}(\widetilde{O})\right)\right)=H$, we have $p_{*}\left(\pi_{1}(L)\right)=H$, and this equals $\lambda_{*}\left(\pi_{1}(\widetilde{W})\right)$. Thus there is an orbifold homeomorphism $f: \widetilde{W} \rightarrow L \subset \widetilde{O}$, implying $L$ and therefore $\widetilde{O}$ is an obifold.

Corollary 5.2. Let $O$ be a 3-dimensional orbifold, W a 3-dimensional suborbifold and $i: W \hookrightarrow O$ the inclusion map. Let $G$ be a subgroup of $\pi_{1}(O)$ containing an element $i_{*}(\alpha)$ where $\alpha \in \pi_{1}(W)$, and let $\widetilde{O}$ be the covering of $O$ corresponding to $G$. Suppose the covering translation on the universal covering space of $W$ associated with $\alpha$ has a fixed point. Then $\widetilde{O}$ is an orbifold.

Proof. Let $H=i_{*}^{-1}(G) \leq \pi_{1}(W)$ and note that $\alpha \in H$. Let $U$ be the universal covering space of $W$. Since the covering translation associated with $\alpha$ has a fixed point, it follows that $U / H=\widetilde{W}$ is an orbifold, which is the covering of $W$ corresponding to $H$. The result now follows by Proposition 5.1.

### 5.1. Quotient Type 1: $O_{h_{1}}((A 1, n),(B 5, m))$

From the Appendix the orbifold fundamental group of $O_{h_{1}}((A 1, n),(B 5, m))$ is

$$
\begin{aligned}
\pi_{1}\left(O_{h_{1}}((A 1, n),(B 5, m))\right) & =\left\langle a, b \mid a^{2}=b^{2}, a^{2 m}=b^{2 m}=\left(b a^{-1}\right)^{n}=1\right\rangle \\
& =\left\langle b a^{-1}\right\rangle \circ_{-1}\langle a\rangle=\mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{2 m}
\end{aligned}
$$

Furthermore, the elements $a$ and $b$ in $\pi_{1}((A 1, n))$ acting on the universal covering space $\mathbb{R} \times D^{2}$ of $(A 1, n)$ are defined by $a(t, v)=\left(t-\frac{1}{2}, \bar{v}\right)$ and $b(t, v)=$ $\left(t-\frac{1}{2}, \bar{v} e^{\frac{2 \pi i}{n}}\right)$.

Note that as elements in either $\pi_{1}((A 1, n))$ or $\pi_{1}\left(O_{h_{1}}((A 1, n),(B 5, m))\right)$ they are orientation reversing.

Proposition 5.3. Let $H$ be a normal subgroup of $\pi_{1}\left(O_{h_{1}}((A 1, n),(B 5, m))\right)=$ $\mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{2 m}$ isomorphic to $\mathbb{Z}_{2}$, and let $Q=\pi_{1}\left(O_{h_{1}}((A 1, n),(B 5, m))\right) / H$ be the quotient group. Suppose $n \neq 1$ and $Q$ is abelian. Then one of the following is true:

1) $n=2$, either $H=\left\langle b a^{m-1}\right\rangle$ or $\left\langle b a^{-1}\right\rangle$, and $Q=\mathbb{Z}_{2 m}$;
2) $n=2, H=\left\langle a^{m}\right\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{m}$;
3) $n=4, H=\left\langle\left(b a^{-1}\right)^{2}\right\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{2 m}$.

Proof. Recall from Section 2 that $a\left(b a^{-1}\right) a^{-1}=\left(b a^{-1}\right)^{-1}$. Let $w=b a^{-1}$, and suppose $H=\left\langle w^{s} a^{t}\right\rangle$ where $0 \leq s<n$ and $0 \leq t<2 m$.

Assume first that $s$ and $t$ are both non-zero. Since $H$ is normal, $w^{s} a^{t}$ $=a\left(w^{s} a^{t}\right) a^{-1}=w^{-s} a^{t}$, which implies $w^{2 s}=1$ or $s=\frac{n}{2}$. Note that $1=$ $\left(w^{n / 2} a^{t}\right)^{2}=a^{2 t}$, for either $t$ even or odd. This implies $t=m$, and thus $H=\left\langle w^{n / 2} a^{m}\right\rangle$.

Suppose $m$ is odd. Then

$$
w^{n / 2} a^{m}=w\left(w^{n / 2} a^{m}\right) w^{-1}=w^{(n / 2+1)} a^{m} w^{-1} a^{-m} a^{m}=w^{(n / 2+2)} a^{m}
$$

This implies $w^{2}=1$, and thus $n=2$. We now suppose $m$ is even. Consider the group $Q=\pi_{1}\left(O_{h_{1}}((A 1, n),(B 5, m))\right) / H$. Since $Q$ is abelian, $w H$ $=(a H)(w H)(a H)^{-1}=w^{-1} H$, which implies $w^{2} \in\left\langle w^{n / 2} a^{m}\right\rangle$. If $w^{2} \neq 1$, then $w^{2}=w^{n / 2} a^{m}$, or $w^{(4-n) / 2}=a^{m}$. But this contradicts the semi-direct product property that $\langle w\rangle \cap\langle a\rangle=\{1\}$, and so $w^{2}=1$ and $n=2$. In either case $H=\left\langle\left(b a^{-1}\right)^{n / 2} a^{m}\right\rangle=\left\langle\left(b a^{-1}\right) a^{m}\right\rangle=\left\langle b a^{m-1}\right\rangle$. Furthermore, $Q=$ $\pi_{1}\left(O_{h_{1}}((A 1, n),(B 5, m))\right) / H=\langle a, b| a^{2}=b^{2}, a^{2 m}=b^{2 m}=\left(b a^{-1}\right)^{2}=$ $\left.1, b a^{m-1}=1\right\rangle=\left\langle a \mid a^{2 m}=1\right\rangle \simeq \mathbb{Z}_{2 m}$.

Suppose $s=0$. Then $\mathbb{Z}_{2} \simeq H=\left\langle a^{t}\right\rangle$, and $t=m$. A similar argument as above shows that if $m$ is either even or odd, then $n=2$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{m}$. Now suppose $t=0$ and $\mathbb{Z}_{2} \simeq H=\left\langle w^{s}\right\rangle$. It follows that $s=n / 2$, and $Q=\mathbb{Z}_{n / 2} \circ_{-1} \mathbb{Z}_{2 m}$. In order for $Q$ to be abelian, either $n=2, H=\langle w\rangle$ and $Q=\mathbb{Z}_{2 m}$, or $n=4, H=\left\langle w^{2}\right\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{2 m}$.

Proposition 5.4. Let $\varphi: G \rightarrow$ Homeo $_{P L}\left(\mathbb{R} \mathbb{P}^{3}\right)$ be a finite action such that the quotient space $\mathbb{R P}^{3} / \varphi$ is the orbifold $O_{h_{1}}((A 1, n),(B 5, m))$. Then $n \neq 1$.

Proof. Let $\nu: \mathbb{R P}^{3} \rightarrow \mathbb{R P}^{3} / \varphi=O_{h_{1}}((A 1, n),(B 5, m))$ be the covering map, and note that $\nu_{*}\left(\pi_{1}\left(\mathbb{R}^{3}\right)\right)$ is a normal subgroup of $\pi_{1}\left(O_{h_{1}}((A 1, n),(B 5, m))\right)$ isomorphic to $\mathbb{Z}_{2}$ of finite index. Suppose $n=1$, and therefore $\pi_{1}\left(O_{h_{1}}((A 1, n),(B 5, m))\right) \simeq \mathbb{Z}_{2 m}$. This implies that $G \simeq \mathbb{Z}_{m}$, and therefore $m \neq 1$. Thus the maximum order of every exceptional point in $\mathbb{R P}^{3} / \varphi$ is $m$. However, $O_{h_{1}}((A 1, n),(B 5, m))$ has two cone points of order $2 m$, giving a contradiction. Thus $n \neq 1$.

Corollary 5.5. Let $\varphi: G \rightarrow \operatorname{Homeoo}_{P L}\left(\mathbb{R} \mathbb{P}^{3}\right)$ be a finite abelian action such that the quotient space $\mathbb{R P}^{3} / \varphi$ is the orbifold $O_{h_{1}}((A 1, n),(B 5, m))$. Then the following is true:

1) The action is conjugate to the Standard Quotient Type 1 Action;
2) $n=2$, and $m=2^{b-1} m_{0}$ where $m_{0}$ is odd and $b>1$;
3) $G \simeq \mathbb{Z}_{m_{0}} \times \mathbb{Z}_{2^{b}}=\mathbb{Z}_{2 m}$;
4) The covering corresponds to the subgroup $\left\langle b a^{m-1}\right\rangle$.

Proof. By Proposition 5.4, $n \neq 1$.
Let $\nu: \mathbb{R}^{P^{3}} \rightarrow \mathbb{R}^{3} / \varphi=O_{h_{1}}((A 1, n),(B 5, m))$ be the covering map, and note that $\nu_{*}\left(\pi_{1}\left(\mathbb{R P}^{3}\right)\right)=H$ is a $\mathbb{Z}_{2}$ normal subgroup. By assumption $Q=$
$\pi_{1}\left(O_{h_{1}}((A 1, n),(B 5, m))\right) / H=G$ is an abelian group. We now apply Proposition 5.3 and consider each case separately.

Suppose $n=2, H=\left\langle b a^{m-1}\right\rangle$ or $\left\langle b a^{-1}\right\rangle$, and $Q=\mathbb{Z}_{2 m}$. Since both $b a^{m-1}$ and $a^{m}$ are orientation reversing elements when $m$ is odd, and $\mathbb{R} \mathbb{P}^{3}$ is orientable, it follows that $H=\left\langle b a^{m-1}\right\rangle$ or $\left\langle b a^{-1}\right\rangle$ and $m$ is even. Viewing $b a^{-1}$ as an element in $\pi_{1}((A 1, n))$, we note that $b a^{-1}$ has a fixed point as an action on the universal covering space. Therefore by Corollary 5.2, the covering of $O_{h_{1}}((A 1, n),(B 5, m))$ corresponding to $\left\langle b a^{-1}\right\rangle$ is not a manifold. The case where $n=2, H=\left\langle a^{m}\right\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{m}$ is eliminated in a similar way. This is done by recalling that $a$ in $\pi_{1}((A 1, n))$ is identified with $x$ in $\pi_{1}((B 5, m))$, where $x$ acting on the universal covering space is defined by $x(t, v)=\left(-t, v e^{\frac{\pi i}{m}}\right)$. Since this map has a fixed point, this case is also eliminated using Corollary 5.2. Now suppose $n=4, H=\left\langle\left(b a^{-1}\right)^{2}\right\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{2 m}$. Note that $\left(b a^{-1}\right)^{2}(t, v)=(t,-v)$, and therefore has a fixed point eliminating this case also.

Thus, the only possible case is $n=2, H=\left\langle b a^{m-1}\right\rangle$ where $m$ is even and $G=\mathbb{Z}_{2 m}$. Write $m=2^{b-1} m_{0}$ where $b>1$ and $m_{0}$ is odd. By Theorem 3.3, $\varphi$ is conjugate to the Standard Quotient Type 1 Action, which is a $\mathbb{Z}_{m_{0}} \times \mathbb{Z}_{2^{b}}=$ $\mathbb{Z}_{2 m}$-action on $\mathbb{R P}^{3}$ with quotient space $O_{h_{1}}((A 1, n),(B 5, m))$, completing the proof.

### 5.2. Quotient Type 2 : $O_{h_{2}}((A 3, n),(B 4, m))$

The orbifold fundamental group of $\pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right)$ is

$$
\begin{aligned}
\langle a, b, c| a^{n}=b^{2}=c^{2 m}=1, b a b^{-1}= & \left.a^{-1}, c a c^{-1}=a^{-1}, c b c^{-1}=b a\right\rangle \\
& =\left(\langle a\rangle \circ_{-1}\langle b\rangle\right) \circ\langle c\rangle \simeq \operatorname{Dih}\left(\mathbb{Z}_{n}\right) \circ \mathbb{Z}_{2 m}
\end{aligned}
$$

The maps $a, b$ and $c$ are defined on the universal covering space of $(A 3, n)$ by $a(t, v)=\left(t, v e^{\frac{2 \pi i}{n}}\right), b(t, v)=(t, \bar{v})$ and $c(t, v)=\left(t+\frac{1}{2}, \bar{v} e^{\frac{-\pi i}{n}}\right)$. Note that $b$ and $c$ are orientation reversing elements when viewed as elements of $\pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right)$.

Proposition 5.6. Let $H \simeq \mathbb{Z}_{2}$ be a normal subgroup of $\pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right)$, and let $Q=\pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right) / H$ be the quotient group. If $Q$ is abelian, then one of the following is true:

1) $n=1, H=\left\langle c^{m}\right\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{m}$;
2) $n=2, H=\langle a\rangle$, and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{2 m}$;
3) $n=1, H=\langle b\rangle$ or $H=\left\langle b c^{m}\right\rangle$, and $Q=\mathbb{Z}_{2 m}$.

Proof. Recall that $c a c^{-1}=a^{-1}, c b c^{-1}=b a$ and $c^{2}$ commutes with both $a$ and b. As the orbifold fundamental group is a semi-direct product, we may write $H=\left\langle a^{s} b^{\epsilon} c^{t}\right\rangle$ where $0 \leq s<n, \epsilon=0$ or 1 , and $0 \leq t<2 m$.

We first assume $\epsilon=0$, and so $H=\left\langle a^{s} c^{t}\right\rangle$. Since $H$ is normal, $a^{s} c^{t}=$ $c a^{s} c^{t} c^{-1}=a^{-s} c^{t}$, which indicates $a^{2 s}=1$. Observe that this implies $1=$ $\left(a^{s} c^{t}\right)^{2}=c^{2 t}$ whether $t$ is either even or odd. There are three cases to consider depending on the values of $s$ and $t$.

Suppose $s=0$, and therefore $t \neq 0$. Since $c^{2 t}=1$, it follows that $t=m$ and $H=\left\langle c^{m}\right\rangle$. If $n=1$, then $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{m}$ proving 1) in the statement of the proposition. We now assume $n \neq 1$. If $m$ is odd, then $b c^{m} b^{-1}=b(b a)^{-1} c^{m}=$ $a c^{m}$, showing $H=\left\langle c^{m}\right\rangle$ is not a normal subgroup. If $m$ is even, then $H=\left\langle c^{m}\right\rangle$ is a normal subgroup. However $Q=\left(\mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{2}\right) \circ \mathbb{Z}_{m}$ is not abelian, removing this case from consideration.

Assume $t=0$, and so $s \neq 0$ and $n \neq 1$. Furthermore since $a^{2 s}=1$, it follows that $s=n / 2$ and $H=\left\langle a^{n / 2}\right\rangle$. Obviously, $Q=\left(\mathbb{Z}_{n / 2} \circ-1 \mathbb{Z}_{2}\right) \circ \mathbb{Z}_{2 m}$ is abelian, if $n=2$. Thus $H=\langle a\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{2 m}$ proving 2$)$.

Next, we assume $s \neq 0$ and $t \neq 0$, and therefore $n \neq 1$ and $H=\left\langle a^{n / 2} c^{m}\right\rangle$. We claim that the quotient $Q$ is not abelian, and thus this case does not occur. If $m$ is odd, then by normality of $H$, we have $a^{n / 2} c^{m}=b\left(a^{n / 2} c^{m}\right) b^{-1}=$ $a^{-n / 2} b c^{m} b^{-1}=a^{-n / 2} b(b a)^{-1} c^{m}=a^{-n / 2} b a^{-1} b^{-1} c^{m}=a^{-n / 2} a c^{m}$. This implies $a=1$ giving a contradiction. Thus $m$ must be even which we now assume. Since $c^{m}$ commutes with every element, it follows that the subgroup $L=\left\langle a^{n / 2} c^{m}, c^{m}\right\rangle$ is also a normal subgroup of $\pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right)$. We obtain an injection $\pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right) / L \rightarrow \pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right) / H=Q$. Now $\pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right) / L$ is

$$
\begin{aligned}
\langle a, b, c| a^{n}=b^{2}=1, b a b^{-1}=a^{-1}, c a c^{-1}=a^{-1}, c b c^{-1}= & \left.b a, c^{m}=1, a^{n / 2}=1\right\rangle \\
& =\left(\mathbb{Z}_{n / 2} \circ_{-1} \mathbb{Z}_{2}\right) \circ \mathbb{Z}_{m}
\end{aligned}
$$

If $Q$ is abelian, then so is $\pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right) / L$. This implies $n=2$ and $H=\left\langle a c^{m}\right\rangle$. As a consequence, we must have

$$
Q=\pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right) / H=\left\langle b, c \mid b^{2}=1, c^{2 m}=1, c b c^{-1}=b c^{m}\right\rangle
$$

which is not abelian.
On the other hand, if $\epsilon=1$, then our $\mathbb{Z}_{2}$ normal subgroup is written as $H=\left\langle a^{s} b c^{t}\right\rangle$. Assume first that $s=0$, and thus $H=\left\langle b c^{t}\right\rangle$. By the normality condition, $b c^{t}=c\left(b c^{t}\right) c^{-1}=b a c^{t}$, which implies $1=a$ and hence $n=1$. Furthermore, $1=\left(b c^{t}\right)^{2}=c^{2 t}$. Hence $H=\left\langle b c^{t}\right\rangle$ where $t=0$ or $m$. As a result, $\pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right)=\mathbb{Z}_{2} \times \mathbb{Z}_{2 m}$ with $H=\langle b\rangle$ or $H=\left\langle b c^{m}\right\rangle$. In both cases, $Q=\mathbb{Z}_{2 m}$ proving 3).

We now suppose $s \neq 0$, and so $n \neq 1$. Since $H \unlhd \pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right)$, we have $a^{s} b c^{t}=c\left(a^{s} b c^{t}\right) c^{-1}=a^{-s}(b a) c^{t}=a^{-s-1} b c^{t}$, which implies $a^{2 s+1}=1$. Thus $n$ is odd and $s=(n-1) / 2$. In addition, $1=\left(a^{s} b c^{t}\right)^{2}=c^{2 t}$ whether $t$ is even or odd. Hence $t=0$ or $m$ and $H=\left\langle a^{(n-1) / 2} b c^{t}\right\rangle$. If $t$ is even, then again by normality $a^{(n-1) / 2} b c^{t}=b\left(a^{(n-1) / 2} b c^{t}\right) b^{-1}=a^{(1-n) / 2} b c^{t}$, showing $a^{n-1}=1$.

However the order of $a$ is $n$, giving a contradiction. Thus we may assume $t$ is odd, therefore $t=m \geq 1$, and $H=\left\langle a^{(n-1) / 2} b c^{m}\right\rangle$ with $m$ odd. The group $Q=\pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right) / H=\mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{2 m}$, and since $n>1$ and $m \geq 1$ are both odd, this group cannot be abelian. This completes the proof.

Corollary 5.7. Let $\varphi: G \rightarrow$ Homeo $_{P L}\left(\mathbb{R} \mathbb{P}^{3}\right)$ be a finite abelian action such that the quotient space $\mathbb{R}^{3} / \varphi$ is the orbifold $O_{h_{2}}((A 3, n),(B 4, m))$. Then the following is true:

1) The action is conjugate to the Standard Quotient Type 2 Action;
2) $n=1$ and $m$ is odd;
3) $G=\mathbb{Z}_{2 m}$;
4) The covering corresponds to the subgroup $\left\langle b c^{m}\right\rangle$.

Proof. Let $\nu: \mathbb{R P}^{3} \rightarrow \mathbb{R}^{3} / \varphi=O_{h_{2}}((A 3, n),(B 4, m))$ be the covering map. Note that $\nu_{*}\left(\pi_{1}\left(\mathbb{R P}^{3}\right)\right)=H$ is a $\mathbb{Z}_{2}$ normal subgroup of $\pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right)$. Furthermore, the quotient $Q=\pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right) / H$ is isomorphic to the group $G$. Furthermore, 2) may also be excluded by Corollary 5.2 since the element $a \in \pi_{1}((A 3, n))$ has a fixed point.

We now consider 1) of Proposition 5.6. Since $c$ is orientation reversing, it follows that $m$ must be even. Recall that $c$ is identified with $z x \in \pi_{1}((B 4, m))$, and $c^{2}=(z x)^{2}=y$. Thus $c^{m}=y^{\frac{m}{2}}$. The element $y \in \pi_{1}((B 4, m))$ acts on the universal covering space as $y(t, v)=\left(t, v e^{\frac{2 \pi i}{m}}\right)$. Since this map has a fixed point, again by Corollary 5.2 we exclude this case. As for case 3 ), since $b$ is an orientation reversing element, this leaves us with only $H=\left\langle b c^{m}\right\rangle$ and $G=\mathbb{Z}_{2 m}$. Here $m$ must be odd to guarantee an orientation preserving element. Applying Theorem 3.6, $\varphi$ is conjugate to the Standard Quotient Type 2 Action, which is $\mathbb{Z}_{2 m}$-action on $\mathbb{R P}^{3}$ with quotient type $O_{h_{2}}((A 3,1),(B 4, m))$.

### 5.3. Quotient Type 3: $O_{h_{3}}((A 2, n),(B 3, m))$

From Section 2, the orbifold fundamental group of $O_{h_{3}}((A 2, n),(B 3, m))$ is

$$
\begin{aligned}
& \pi_{1}\left(O_{h_{3}}((A 2, n),(B 3, m))\right) \\
& \quad=\left\langle a, b, c \mid[a, b]=[a, c]=1, a^{m}=b^{n}=c^{2}=1, c b c^{-1}=b^{-1}\right\rangle \\
& \quad=\left(\langle b\rangle \circ_{-1}\langle c\rangle\right) \times\langle a\rangle=\operatorname{Dih}\left(\mathbb{Z}_{n}\right) \times \mathbb{Z}_{m}
\end{aligned}
$$

The elements $a, b$ and $c$ in $\pi_{1}((A 2, n))$ acting on the universal covering space are defined by $a(t, v)=(t+1, v), b(t, v)=\left(t, v e^{\frac{2 \pi i}{n}}\right)$ and $c(t, v)=(t, \bar{v})$.
Proposition 5.8. Let $H \simeq \mathbb{Z}_{2}$ be a normal subgroup of
$\pi_{1}\left(O_{h_{3}}((A 2, n),(B 3, m))\right)$, and let $Q=\pi_{1}\left(O_{h_{3}}((A 1, n),(B 5, m))\right) / H$ be the quotient group. Then one of the following is true where $\epsilon=0$ or 1 :

1) If $m$ and $n$ are both not equal to 1 , then $m, n$ are both even and $H=\left\langle b^{n / 2} c^{\epsilon} a^{m / 2}\right\rangle$. If either $Q$ is abelian or $\epsilon=1$, then $n=2$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{m} ;$
2) If $n=1$ and $m \neq 1$ is odd, then $H=\langle c\rangle$ and $Q=\mathbb{Z}_{m}$;
3) If $n=1$ and $m \neq 1$ is even, then $H$ is either $\langle c\rangle,\left\langle a^{m / 2}\right\rangle$ or $\left\langle c a^{m / 2}\right\rangle$, with quotient group $Q$ isomorphic to $\mathbb{Z}_{m}, \mathbb{Z}_{2} \times \mathbb{Z}_{m / 2}$ or $\mathbb{Z}_{m}$;
4) If $n=2$ and $m=1$, then $H$ is either $\langle b\rangle,\langle c\rangle$ or $\langle b c\rangle$ and $Q=\mathbb{Z}_{2}$;
5) If $n>2$ and $m=1$, then $n$ is even, $H=\left\langle b^{n / 2}\right\rangle$, and $Q=\operatorname{Dih}\left(\mathbb{Z}_{n / 2}\right)$.

Proof. The subgroup $H=\left\langle b^{s} c^{\epsilon} a^{t}\right\rangle$ where $0 \leq s<n, \epsilon=0$ or 1 and $0 \leq t<m$. We will assume first that $m$ and $n$ are both not equal to 1 . Since $H$ is normal, it follows that $b^{s} c^{\epsilon} a^{t}=c\left(b^{s} c^{\epsilon} a^{t}\right) c^{-1}=b^{-s} c^{\epsilon} a^{t}$. This implies $b^{2 s}=1$ or $s=n / 2$. Observe that $1=\left(b^{s} c^{\epsilon} a^{t}\right)^{2}$ is equal to $a^{2 t}$ if either $\epsilon=0$ or 1 . Hence $t=m / 2$, and in either case $H=\left\langle b^{n / 2} c^{\epsilon} a^{m / 2}\right\rangle$. Suppose first that $\epsilon=0$, and thus $H=\left\langle b^{n / 2} a^{m / 2}\right\rangle$. If follows that $H$ is normal, and so no new information is obtained. If $Q$ is abelian, then $b H=(c H)(b H)(c H)^{-1}=c b c^{-1} H=b^{-1} H$. Hence $b^{2} \in\left\langle b^{n / 2} c^{\epsilon} a^{m / 2}\right\rangle$. If $b^{2} \neq 1$, then $b^{2}=b^{n / 2} a^{m / 2}$ or $b^{(4-n) / 2}=a^{m / 2}$, giving a contradiction. Thus $b^{2}=1, n=2$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{m}$. Suppose $\epsilon=1$. Again by normality of $H$, it follows that $b^{n / 2} c a^{m / 2}=b\left(b^{n / 2} c a^{m / 2}\right) b^{-1}$ $=b^{n / 2} b^{2} c a^{m / 2}$, implying again that $b^{2}=1$ and proving 1).

Since $\pi_{1}\left(O_{h_{3}}((A 2, n),(B 3, m))\right)$ is isomorphic to $\mathbb{Z}_{2} \times \mathbb{Z}_{m}$ in 2$)$ and 3$)$ and isomorphic to $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ in 4), the results follow easily. For 5), $\pi_{1}\left(O_{h_{3}}((A 2, n),(B 3, m))\right)=\operatorname{Dih}\left(\mathbb{Z}_{n}\right), n>2$ and $H=\left\langle b^{s} c^{\epsilon}\right\rangle$. If $\epsilon=0$, it follows that $H=\left\langle b^{n / 2}\right\rangle$. However if $\epsilon=1$, it follows by normality that $b^{s} c=b\left(b^{s} c\right) b^{-1}=b b^{s} c$, which implies $b^{2}=1$ and $n=2$. This contradicts $n>2$.

Corollary 5.9. Let $\varphi: G \rightarrow \operatorname{Homeo}_{P L}\left(\mathbb{R} \mathbb{P}^{3}\right)$ be a finite abelian action such that the quotient space $\mathbb{R P}^{3} / \varphi$ is homeomorphic to $O_{h_{3}}((A 2, n),(B 3, m))$. Then the following is true:

1) The action is conjugate to the Standard Quotient Type 3 Action;
2) $n=2$ and $m$ is even;
3) $G=\mathbb{Z}_{2} \times \mathbb{Z}_{m}$;
4) The covering corresponds to the subgroup $\left\langle b a^{m / 2}\right\rangle$.

Proof. Suppose $\varphi: G \rightarrow \operatorname{Homeo}_{P L}\left(\mathbb{R}^{3}\right)$ is a finite abelian action such the quotient space $\mathbb{R P}^{3} / \varphi=O_{h_{3}}((A 2, n),(B 3, m))$, and let $\nu: \mathbb{R P}^{3} \rightarrow$ $O_{h_{3}}((A 2, n),(B 3, m))$ be the covering map with $\nu_{*}\left(\pi_{1}\left(\mathbb{R}^{P}\right)\right)=H$. If $n=$ $m=1$, then $\pi_{1}\left(O_{h_{3}}((A 2,1),(B 3,1))\right) \simeq \mathbb{Z}_{2}$, giving a contradiction. Now $H$ is a normal subgroup of $\pi_{1}\left(O_{h_{3}}((A 2, n),(B 3, m))\right)$ which is isomorphic to $\mathbb{Z}_{2}$ and corresponds to an orientation preserving element of order 2. The quotient group $Q=\pi_{1}\left(O_{h_{3}}((A 2, n),(B 3, m))\right) / H$ is abelian. The element $c \in \pi_{1}\left(O_{h_{3}}((A 2, n),(B 3, m))\right)$ is represented by an orientation reversing element, and therefore 2) in Proposition 5.8 is eliminated. Furthermore, since $a$ and $b$ are orientation preserving, $H$ cannot be generated by $\left\langle b^{n / 2} c a^{m / 2}\right\rangle$, $\left\langle c a^{m / 2}\right\rangle$ or $\left\langle b^{n / 2} c\right\rangle$.

Suppose $m$ and $n$ are both not equal to 1. Then by 1) in Proposition 5.8, $n=2, H=\left\langle b a^{m / 2}\right\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{m}$. We will show that the 3)-5) in Proposition 5.8 may also be eliminated.

Suppose $n=1, m \neq 1$ is even, and since $H$ must be an orientation preserving subgroup, $H=\left\langle a^{m / 2}\right\rangle$ by 3) in Proposition 5.8. Recall that the element $a$ is identified with the element $y$ in $\pi_{1}((B 3, m))$, and $y(t, v)=\left(t, e^{\frac{2 \pi i}{m}}\right)$ has a fixed point. This eliminates 3 ) by Corollary 5.2.

Suppose $n \neq 1$ is even, and $m=1$. Thus by 4) and 5) in Proposition 5.8, $H=\langle b\rangle$ or $H=\left\langle b^{n / 2}\right\rangle$, and $Q=\mathbb{Z}_{2}$ or $Q=\operatorname{Dih}\left(\mathbb{Z}_{n / 2}\right)$ respectively. In the latter case, in order for $Q$ to be abelian, $n$ must be 2 or 4 . Since $b$ has a fixed point, 4) and 5) are also eliminated by Corollary 5.2.

Since any regular covering of $O_{h_{3}}((A 2, n),(B 3, m))$ by $\mathbb{R}^{3}$ corresponds to the subgroup $\left\langle b a^{m / 2}\right\rangle$, any such action is conjugate to the Standard Quotient Type 3 Action on $\mathbb{R P}^{3}$, which is $\mathbb{Z}_{2} \times \mathbb{Z}_{m}$.

### 5.4. Quotient Type 4: The orbifold $O_{h_{4}}((B 2, n),(B 2, m))$

The orbifold fundamental group of $O_{h_{4}}((B 2, n),(B 2, m))$ is

$$
\begin{aligned}
\pi_{1}\left(O_{h_{4}}((B 2, n),(B 2, m))\right) & =\left\langle a, b \mid a^{2 n}=b^{2}=1, b a^{2} b^{-1}=a^{-2},(a b)^{2 m}=1\right\rangle \\
& =\left(\left\langle a^{2}\right\rangle \circ_{-1}\langle a b\rangle\right) \circ\langle b\rangle=\left(\mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{2 m}\right) \circ \mathbb{Z}_{2}
\end{aligned}
$$

The maps $a$ and $b$ are defined on the universal covering space of $(B 2, n)$ by $a(t, v)=\left(-t+\frac{1}{2}, v e^{\frac{\pi i}{n}}\right), b(t, v)=(-t, \bar{v})$.

Proposition 5.10. Let $H \simeq \mathbb{Z}_{2}$ be a normal subgroup of
$\pi_{1}\left(O_{h_{4}}((B 2, n),(B 2, m))\right)$ generated by orientation preserving elements such that the quotient group $Q=\pi_{1}\left(O_{h_{4}}((B 2, n),(B 2, m))\right) / H$ is an abelian group. Then one of the following is true:

1) $n=m=2, H=\left\langle a^{2}(a b)^{2}\right\rangle$ and $Q=\mathbb{Z}_{4} \times \mathbb{Z}_{2}$;
2) $n=1, m=2, H=\left\langle(a b)^{2}\right\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{2}$;
3) $n=2, m=1, H=\left\langle a^{2}\right\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{2}$;
4) $n=m=1, H=\langle b\rangle$ and $Q=\mathbb{Z}_{2}$.

Proof. It is convenient to let $x=a^{2}, y=a b$ and $z=b$. Note that $y x y^{-1}=x^{-1}$, $z x z^{-1}=x^{-1}$ and $z y z^{-1}=x^{-1} y^{-1}$. Let $H=\left\langle x^{s} y^{t} z^{\epsilon}\right\rangle$ where $0 \leq s<n$, $0 \leq t<2 m$ and $\epsilon=0$ or 1 . Since $y$ is orientation reversing, $x$ and $z$ are orientation preserving and $H$ is generated by orientation preserving elements, it follows that $t$ must be even. This implies $x y^{t} x^{-1}=y^{t}$ and $z y^{t} z^{-1}=y^{-t}$.

Case I: $H=\left\langle x^{s} y^{t}\right\rangle$.
Assume first that $s \neq 0$ and $t \neq 0$. Since $1=\left(x^{s} y^{t}\right)^{2}=x^{2 s} y^{2 t}$, we obtain $s=\frac{n}{2}, t=m$, and thus $H=\left\langle x^{\frac{n}{2}} y^{m}\right\rangle$. It follows that $\left\langle x^{\frac{n}{2}} y^{m}\right\rangle$ is a normal subgroup, and thus no new information is obtained. Being that
$\pi_{1}\left(O_{h_{4}}((B 2, n),(B 2, m))\right) / H$ is abelian, we have that $x H=y H x H y^{-1} H=$ $x^{-1} H$. This implies $x^{2} \in\left\langle x^{\frac{n}{2}} y^{m}\right\rangle$. The equation $x^{2}=x^{\frac{n}{2}} y^{m}$ is impossible, and so $x^{2}=1$ showing $n=2$. This shows that $\pi_{1}\left(O_{h_{4}}((B 2, n),(B 2, m))\right)=$ $\left(\mathbb{Z}_{2} \times \mathbb{Z}_{2 m}\right) \circ \mathbb{Z}_{2}$, and so $Q=\pi_{1}\left(O_{h_{4}}((B 2, n),(B 2, m))\right) / H=\mathbb{Z}_{2 m} \circ \mathbb{Z}_{2}$. The action given in the quotient group $Q$ is $z y z^{-1}=y^{m-1}$. In order for $Q$ to be abelian, $m=2$, showing 1 ) in the statement of the proposition.

Suppose $s=0, t \neq 0$, and so $H=\left\langle y^{t}\right\rangle$. We will show that this gives 2) in the proposition. It follows that $t=m$ which is even, and so $H$ is always normal giving no new information. Now $Q=\left(\mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{m}\right) \circ \mathbb{Z}_{2}$ where $z y z^{-1}=x^{-1} y^{-1}$. In order for $Q$ to be abelian, $x=1$ and so $n=1$, and $m=2$.

Next assume that $s \neq 0, t=0$ and so $H=\left\langle x^{s}\right\rangle$. We obtain $s=\frac{n}{2}$ and $H=\left\langle x^{\frac{n}{2}}\right\rangle$. Furthermore, $Q=\left(\mathbb{Z}_{\frac{n}{2}} \circ_{-1} \mathbb{Z}_{2 m}\right) \circ \mathbb{Z}_{2}$. In order for $Q$ to be abelian, $n=2$ and $m=1$, showing 3 ) in the statement of the proposition.
Case II: $H=\left\langle x^{s} y^{t} z\right\rangle$.
Suppose $t \neq 0$. It follows that $1=\left(x^{s} y^{t} z\right)^{2}$, giving no new information. A computation shows $y\left(x^{s} y^{t} z\right) y^{-1}=x^{-s+1} y^{t+2} z$, which must equal $x^{s} y^{t} z$. Therefore $y^{2}=1$ and $m=1$, contradicting $t$ even.

Assume $t=0, s \neq 0$ and $n \neq 1$, and thus $H=\left\langle x^{s} z\right\rangle$. A computation shows $y\left(x^{s} z\right) y^{-1}=x^{-s+1} y^{2} z$, which must equal $x^{s} z$ implying $x^{2 s-1}=1$. In addition, we must also have $x^{s} z=z\left(x^{s} z\right) z^{-1}=x^{-s} z$, giving $x^{2 s}=1$. This implies $x=1$, contradicting $n \neq 1$.

We now assume $t=0, s=0$, and thus $H=\langle z\rangle$. By normality, $z=y z y^{-1}$ $=y^{2} x z$, which implies $x=1$ and $y^{2}=1$. Thus $n=1$ and $m=1$, giving us 4) of the proposition and completing the proof.

Corollary 5.11. Let $\varphi: G \rightarrow$ Homeo $_{P L}\left(\mathbb{R}^{( } \mathbb{P}^{3}\right)$ be a finite abelian action such that the quotient space $\mathbb{R}^{3} / \varphi$ is the orbifold $O_{h_{4}}((B 2, n),(B 2, m))$. Then the following is true:

1) The action is conjugate to the Standard Quotient Type 4 Action;
2) $n=m=2$;
3) $G=\mathbb{Z}_{4} \times \mathbb{Z}_{2}$;
4) The covering corresponds to the subgroup $\left\langle a^{2}(a b)^{2}\right\rangle$.

Proof. Let $\nu: \mathbb{R P}^{3} \rightarrow \mathbb{R}^{3} / \varphi$ be an orbifold covering map and $\nu_{*}\left(\pi_{1}\left(\mathbb{R}^{3}\right)\right)=$ $H, \quad$ a $\quad \mathbb{Z}_{2}$-normal subgroup of $\pi_{1}\left(O_{h_{4}}((B 2, n),(B 2, m))\right)$ with quotient $\pi_{1}\left(O_{h_{4}}((B 2, n),(B 2, m))\right) / H$ an abelian group. Applying Proposition 5.10, suppose 2) or 3) holds. Then $\mathbb{R P}^{3} / \varphi$ is either $O_{h_{4}}((B 2,1),(B 2,2))$ or $O_{h_{4}}((B 2,2),(B 2,1))$ and $G=\mathbb{Z}_{2} \times \mathbb{Z}_{2}$. In either case, there is a cone point of order 4 in the quotient space $\mathbb{R P}^{3} / \varphi$. This would imply that there is an element in $G$ of order 4, giving a contradiction. Since $b$ defined on the universal covering space of $(B 2, n)$ has a fixed point, 4$)$ is also eliminated by Corollary 5.2. This leaves 1 ). Now any regular covering of $O_{h_{4}}((B 2, n),(B 2, m))$ by $\mathbb{R} \mathbb{P}^{3}$ cor-
responds to the subgroup $\left\langle a^{2}(a b)^{2}\right\rangle$. Therefore, any such action is conjugate to the Standard Quotient Type 4 Action on $\mathbb{R} \mathbb{P}^{3}$, which is $\mathbb{Z}_{4} \times \mathbb{Z}_{2}$.

### 5.5. Quotient Type 5: The orbifold $O_{h_{5}}((B 6, n),(B 6, m))$

Recall the orbifold fundamental group of $\pi_{1}\left(O_{h_{5}}((B 6, n),(B 6, m))\right)$ is

$$
\begin{aligned}
&\langle a, b, c, d| a^{2}=b^{2}=c^{2}=(b c)^{n}=d^{2}=1 \\
& {\left.[a, b]=[a, c]=[b, d]=[c, d]=1,(a d)^{m}=1\right\rangle } \\
&=\left(\langle b c\rangle \circ_{-1}\langle c\rangle\right) \times\left(\langle a d\rangle \circ_{-1}\langle a\rangle\right)=\left(\mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{2}\right) \times\left(\mathbb{Z}_{m} \circ_{-1} \mathbb{Z}_{2}\right) .
\end{aligned}
$$

We note that the generators $a, b, c$ and $d$ are all orientation reversing elements. The maps on the universal covering space of $(B 6, n)$ are defined as follows: $a(t, v)=(-t, v), b(t, v)=(t, \bar{v})$, and $c(t, v)=\left(t, \bar{v} e^{\frac{-2 \pi i}{n}}\right)$ and $d(t, v)=(-t-$ $1, v)$.

Proposition 5.12. Let $H \simeq \mathbb{Z}_{2}$ be a normal subgroup of
$\pi_{1}\left(O_{h_{5}}((B 6, n),(B 6, m))\right)$ generated by orientation preserving elements such that the quotient group $Q=\pi_{1}\left(O_{h_{5}}((B 6, n),(B 6, m))\right) / H$ is an abelian group. Then the following is true:

1) $n=1$ or 2 and $m=2$ or $4, H=\left\langle(a d)^{\frac{m}{2}}\right\rangle$ and $Q$ is either $\mathbb{Z}_{2} \times \mathbb{Z}_{2}, \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ or $\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} ;$
2) $n=2$ or 4 and $m=1$ or $2, H=\left\langle(b c)^{\frac{n}{2}}\right\rangle$ and $Q$ is either $\mathbb{Z}_{2} \times \mathbb{Z}_{2}, \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ or $\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} ;$
3) $n=m=2$, $H=\langle(b c)(a d)\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$;
4) $n=m=2, H=\langle c d\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$;
5) $n=2$ and $m=1$ or $2, H=\langle b a\rangle$ and $Q$ is either $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ or $\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$;
6) $n=1$ or 2 , $m=1$ or $2, H=\langle c a\rangle$ and $Q$ is either $\mathbb{Z}_{2}, \mathbb{Z}_{2} \times \mathbb{Z}_{2}$, or
$\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} ;$
7) $n=m=2, H=\langle b d\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$.

Proof. The group $H=\left\langle(b c)^{s} c^{\epsilon_{1}}(a d)^{t} a^{\epsilon_{2}}\right\rangle$ where $0 \leq s<n, 0 \leq t<m$ and $\epsilon_{i}=0$ or 1 . Since both ( $b c$ ) and (ad) are orientation preserving, $a$ and $c$ are both orientation reversing, the two cases that need to be considered are $\epsilon_{1}=\epsilon_{2}=0$ or $\epsilon_{1}=\epsilon_{2}=1$.
Case I: $H=\left\langle(b c)^{s}(a d)^{t}\right\rangle$.
Suppose $s=0$ and $t \neq 0$. This implies that $H=\left\langle(a d)^{\frac{m}{2}}\right\rangle$ and the quotient group $Q \simeq\left(\mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{2}\right) \times\left(\mathbb{Z}_{\frac{m}{2}} \circ_{-1} \mathbb{Z}_{2}\right)$. If $Q$ is abelian, we must have $n=1$ or 2 and $m=2$ or 4 .

If $s \neq 0$ and $t=0$, then $H=\left\langle(b c)^{\frac{n}{2}}\right\rangle$. The quotient group $Q \simeq\left(\mathbb{Z}_{\frac{n}{2}} \circ_{-1}\right.$ $\left.\mathbb{Z}_{2}\right) \times\left(\mathbb{Z}_{m} \circ_{-1} \mathbb{Z}_{2}\right)$, and thus if $Q$ is abelian $n=2$ or 4 and $m=1$ or 2 .

We now assume $s \neq 0$ and $t \neq 0$. Since $H=\left\langle(b c)^{s}(a d)^{t}\right\rangle \simeq \mathbb{Z}_{2}$, and $b c$ and $a d$ commute, we have $1=\left((b c)^{s}(a d)^{t}\right)^{2}=(b c)^{2 s}(a d)^{2 t}$. This implies $s=\frac{n}{2}$,
$t=\frac{m}{2}$ and $H=\left\langle(b c)^{\frac{n}{2}}(a d)^{\frac{m}{2}}\right\rangle$. Clearly $H$ is normal. In the abelian quotient $Q$, we have $b H=b c H c H=c H b c H=c b c H$, which implies $(b c)^{2} \in H$. This could only happen if $(b c)^{2}=1$; hence $n=2$. Similarly $d H=a H a d H=a d H a H$ $=a d a H$, which implies $(a d)^{2} \in H$ and $m=2$. Thus $H=\langle(b c)(a d)\rangle$ and $n=m=2$.
Case II: $H=\left\langle(b c)^{s} c(a d)^{t} a\right\rangle$.
Suppose $s=0$ and $t \neq 0$. In this case $H=\left\langle c(a d)^{t} a\right\rangle$. By normality, $c(a d)^{t} a$ $=(a d)\left[c(a d)^{t} a\right](a d)^{-1}=c(a d)^{t}(a d)^{2} a$, which implies $(a d)^{2}=1, m=2$ and $t=1$. Likewise, $c(a d)^{t} a=(b c)\left[c(a d)^{t} a\right](b c)^{-1}=(b c)^{2} c(a d)^{t} a$ shows $(b c)^{2}=1$ and $n=2$. Therefore in this case, $\pi_{1}\left(O_{h_{5}}((B 6,2),(B 6,2))\right)$ is abelian and $H=\langle c d\rangle$.

If $s \neq 0$ and $t=0$, then $H=\left\langle(b c)^{s} c a\right\rangle$. Conjugating the generator by $b c$ and using the argument from the previous case, shows that $(b c)^{2}=1$, and therefore $n=2, s=1$ and $H=\langle b a\rangle$. In order for $Q$ to be abelian $m=1$ or 2 .

We consider the case where $s=t=0$ and $H=\langle c a\rangle$. Suppose $n \neq 1$. By computing, we obtain $(b c)(c a)(b c)^{-1}=b c b a$, and by normality this must equal $c a$. Hence we obtain $(b c)^{2}=1$ which implies $n=2$. Similarly, if $m \neq 1$, then $(a d)(c a)(a d)^{-1}=c a(a d)^{-1}(d a)=c a(a d)^{-2}$. By normality, this must equal $c a$, and thus $(a d)^{-2}=1$ implying $m=2$. We conclude that $n=1$ or 2 , and $m=1$ or 2 .

Finally consider the case $s \neq 0$ and $t \neq 0$ and $H=\left\langle(b c)^{s} c(a d)^{t} a\right\rangle$. By conjugating the generator by $b c$ and $a d$, we conclude as above that $n=m=2$. Thus the group is abelian and $H=\langle b d\rangle$.

Corollary 5.13. Let $\varphi: G \rightarrow$ Homeo $_{P L}\left(\mathbb{R} \mathbb{P}^{3}\right)$ be a finite abelian action such that the quotient space $\mathbb{R}^{3} / \varphi$ is the orbifold $O_{h_{5}}((B 6, n),(B 6, m))$. Then the following is true:

1) The action is conjugate to the Standard Quotient Type 5 Action;
2) $n=m=2$;
3) $G=\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$;
4) The covering corresponds to the subgroup $\langle(b c)(a d)\rangle$.

Proof. We obtain the following maps: $(b c)(t, v)=\left(t, v e^{\frac{2 \pi i}{n}}\right),(c d)(t, v)=(-t-$ $\left.1, \bar{v} e^{-\frac{2 \pi i}{n}}\right),(b a)(t, v)=(-t, \bar{v}),(c a)(t, v)=\left(-t, \bar{v} e^{-\frac{2 \pi i}{n}}\right)$ and $(b d)(t, v)=(-t-$ $1, \bar{v})$. Note that all these maps have fixed points, and therefore 2) and 4) - 7) in Proposition 5.12 may be excluded by Corollary 5.2.

The element $a d$ in $\pi_{1}((B 6, n))$ is identified (See Appendix) with the element $y z$ in $\pi_{1}((B 6, m))$, and $(y z)(t, v)=\left(t, v e^{\frac{2 \pi i}{n}}\right)$ which has a fixed point. Thus 1) in Proposition 5.12 is excluded like the others above. Hence, the only remaining case in Proposition 5.12 is 3 ). Since any regular covering of $O_{h_{5}}((B 6, n),(B 6, m))$ by $\mathbb{R}^{3}$ corresponds to the subgroup $\langle(b c)(a d)\rangle$, any such action is conjugate to the Standard Quotient Type 5 Action on $\mathbb{R P}^{3}$ which is $\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$.

### 5.6. Quotient Type 6: The orbifold $O_{h_{6}}((B 7, n),(B 7, m))$

The fundamental group of $O_{h_{6}}((B 7, n),(B 7, m))$ is

$$
\begin{aligned}
\langle a, b, c| & \left.a^{2}=b^{2 n}=\left(a b^{-1} a b\right)^{m}=c^{2}=1,\left[a, b^{2}\right]=[a, c]=1, c b c^{-1}=b^{-1}\right\rangle \\
& =\left(\left\langle a b^{-1} a b\right\rangle \circ_{-1}\langle a\rangle\right) \circ\left(\langle b\rangle \circ_{-1}\langle c\rangle\right)=\left(\mathbb{Z}_{m} \circ_{-1} \mathbb{Z}_{2}\right) \circ\left(\mathbb{Z}_{2 n} \circ_{-1} \mathbb{Z}_{2}\right) .
\end{aligned}
$$

The maps $a, b$ and $c$ act on the universal covering space of $(B 7, n)$ as follows: $a(t, v)=(-t-1, v), b(t, v)=\left(-t, v e^{\frac{\pi i}{n}}\right), c(t, v)=(t, \bar{v})$. Note also that $b a b^{-1}=$ $b^{-1} a b$. For convenience if we let $d=a b^{-1} a b$, then $b d b^{-1}=d^{-1}, b a b^{-1}=a d$ and $c d c^{-1}=d$.

Proposition 5.14. Let $H \simeq \mathbb{Z}_{2}$ be a normal subgroup of
$\pi_{1}\left(O_{h_{6}}((B 7, n),(B 7, m))\right)$ generated by orientation preserving elements such that the quotient group $Q=\pi_{1}\left(O_{h_{6}}((B 7, n),(B 7, m))\right) / H$ is an abelian group. Then the following is true:

1) $n=2, m=1, H=\left\langle b^{2}\right\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$;
2) $n=1, m=2, H=\langle d\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$;
3) $n=m=1, H$ is one of the groups $\langle a c\rangle,\langle a b\rangle,\langle b c\rangle$ and $Q=\mathbb{Z}_{2} \times \mathbb{Z}_{2}$.

Proof. The group $H=\left\langle d^{s} a^{\epsilon_{1}} b^{t} c^{\epsilon_{2}}\right\rangle \simeq \mathbb{Z}_{2}$ where $0 \leq s<m, 0 \leq t<2 n$ and $\epsilon_{i}=0$ or 1 . Since $a, b$ and $c$ are orientation reversing elements, it follows that $d$ is orientation preserving. Since $H$ is an orientation preserving subgroup, we have the following cases to consider: I) $t$ is even, and either $\epsilon_{1}=\epsilon_{2}=0$ or $\left.\epsilon_{1}=\epsilon_{2}=1, \mathrm{II}\right) t$ is odd, and either $\epsilon_{1}=1$ and $\epsilon_{2}=0$ or $\epsilon_{1}=0$ and $\epsilon_{2}=1$.

Case I: $t$ is even.
We consider first the situation when $\epsilon_{1}=\epsilon_{2}=0$, and thus $H=\left\langle d^{s} b^{t}\right\rangle$. Assume $s \neq 0$ and $t \neq 0$. Since $t$ is even, it follows that $b^{t}$ commutes with $d$, and thus $1=\left(d^{s} b^{t}\right)^{2}=d^{2 s} b^{2 t}$. This implies $s=\frac{m}{2}, t=n$ and $H=\left\langle d^{\frac{m}{2}} b^{n}\right\rangle$. One can verify that $H$ is indeed a normal subgroup. Since the quotient $Q$ is abelian, we have $d H=(b H)(d H)(b H)^{-1}=d^{-1} H$, or $d^{2} \in\left\langle d^{\frac{m}{2}} b^{n}\right\rangle$. This is impossible unless $d^{2}=1$. Thus $m=2$, the fundamental group $\pi_{1}\left(O_{h_{6}}((B 7,2),(B 7, m))\right)$ $=\left(\mathbb{Z}_{2} \times \mathbb{Z}_{2}\right) \circ\left(\mathbb{Z}_{2 n} \circ_{-1} \mathbb{Z}_{2}\right)$ and $H=\left\langle d b^{n}\right\rangle$. Now $Q=\mathbb{Z}_{2} \times\left(\mathbb{Z}_{2 n} \circ_{-1} \mathbb{Z}_{2}\right)$, which is not abelian unless $n=1$. However in this case $t=n$ is even giving a contradiction, and so this subcase cannot happen. Therefore, in this case either $s=0$ or $t=0$. Suppose $s=0$, and thus $H=\left\langle b^{t}\right\rangle$. It follows that $t=n$ and $H$ is always a normal subgroup. Furthermore, $Q=\left(\mathbb{Z}_{m} \circ_{-1} \mathbb{Z}_{2}\right) \circ\left(\mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{2}\right)$ being abelian implies $m=1$ and $n=2$, and thus $\pi_{1}\left(O_{h_{6}}((B 7,2),(B 7,1))\right)=$ $\mathbb{Z}_{2} \times\left(\mathbb{Z}_{4} \circ_{-1} \mathbb{Z}_{2}\right)$. A similar argument shows that if $t=0$, then $n=1, m=2$, $H=\langle d\rangle$ and $\pi_{1}\left(O_{h_{6}}((B 7,2),(B 7,1))\right)=\left(\mathbb{Z}_{2} \times \mathbb{Z}_{2}\right) \circ\left(\mathbb{Z}_{2} \times \mathbb{Z}_{2}\right)$.

Assume $\epsilon_{1}=\epsilon_{2}=1$ and hence $H=\left\langle d^{s} a b^{t} c\right\rangle$. If $s=0$ and $H=\left\langle a b^{t} c\right\rangle$, then we always have $1=\left(a b^{t} c\right)^{2}$ giving no new information. By normality, $a b^{t} c=b\left(a b^{t} c\right) b^{-1}=a d b^{t+2} c$, which implies $m=1, b^{2}=1$ and $n=1$. Since
$t$ is even, we must have $t=0$. Thus $n=m=1, \pi_{1}\left(O_{h_{6}}((B 7,1),(B 7,1))\right)=$ $\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ and $H=\langle a c\rangle$. We now suppose $t=0$ and $H=\left\langle d^{s} a c\right\rangle$. It always follows that $1=\left(d^{s} a c\right)^{2}$. By normality, $d^{s} a c=b\left(d^{s} a c\right) b^{-1}=d^{-s-1} a b^{2} c=$ $d^{-s-1} b^{2} a c$. This implies $b^{2}=1, n=1$ and $s=\frac{m-1}{2}$. Conjugating by $a$ yields $d^{\frac{m-1}{2}} a c=a\left(d^{\frac{m-1}{2}} a c\right) a^{-1}=d^{\frac{1-m}{2}} a c$, which implies $d=1$ and $m=1$. The two outcomes give us 3) in the statement of the theorem. We now suppose $s \neq 0$ and $t \neq 0$. In this case it always follows that $1=\left(d^{s} a b^{t} c\right)^{2}$, so we do not obtain any new information. By normality, we must have $d^{s} a b^{t} c=b\left(d^{s} a b^{t} c\right) b^{-1}=$ $d^{-s-1} a b^{t+2} c$, which implies $s=\frac{m-1}{2}, b^{2}=1$ and $n=1$. Since $t$ is even, $t=0$ giving a contradiction.
Case II: $t$ is odd.
We suppose $\epsilon_{1}=1$ and $\epsilon_{2}=0$, and thus $H=\left\langle d^{s} a b^{t}\right\rangle$. If $s=0$ and thus $H=\left\langle a b^{t}\right\rangle$, then $1=\left(a b^{t}\right)^{2}=d b^{2 t}$. This implies $d=1, m=1$ and $t=n$. Thus the orbifold fundamental group is $\mathbb{Z}_{2} \times\left(\mathbb{Z}_{2 n} \circ_{-1} \mathbb{Z}_{2}\right)$ and $Q=\mathbb{Z}_{2 n} \circ_{-1} \mathbb{Z}_{2}$. Now $Q$ is abelian only if $n=1$. Thus $n=m=1, \pi_{1}\left(O_{h_{6}}((B 7,1),(B 7,1))\right)=$ $\mathbb{Z}_{2} \times\left(\mathbb{Z}_{2} \times \mathbb{Z}_{2}\right)$ and $H=\langle a b\rangle$. Suppose $s \neq 0$ and $H=\left\langle d^{s} a b^{t}\right\rangle$. Since $H \simeq \mathbb{Z}_{2}$, $1=\left(d^{s} a b^{t}\right)^{2}=d^{2 s+1} b^{2 t}$, so $s=\frac{m-1}{2}$ implying $m$ is odd, and $t=n$ which is also odd. Thus $H=\left\langle d^{\frac{m-1}{2}} a b^{n}\right\rangle$, and one can check that this is always a normal subgroup. Suppose $m \neq 1$. Since $Q$ is abelian, we have $d H=(b H)(d H)(b H)^{-1}$ $=d^{-1} H$, implying $d^{2} \in H$. It follows that $d^{2}=1$ and $m=2$. However $m$ is odd giving a contradiction. Thus $m=1, \pi_{1}\left(O_{h_{6}}((B 7, n),(B 7,1))\right)=$ $\mathbb{Z}_{2} \times\left(\mathbb{Z}_{2 n} \circ_{-1} \mathbb{Z}_{2}\right)$ and $H=\left\langle a b^{n}\right\rangle$. Now $Q=\mathbb{Z}_{2 n} \circ_{-1} \mathbb{Z}_{2}$, which is abelian only if $n=1$. Thus in this case $n=m=1$ to obtain $\pi_{1}\left(O_{h_{6}}((B 7,1),(B 7,1))\right)=$ $\mathbb{Z}_{2} \times\left(\mathbb{Z}_{2} \times \mathbb{Z}_{2}\right)$ and $H=\langle a b\rangle$.

Assume now that $\epsilon_{1}=0$ and $\epsilon_{2}=1$, and thus $H=\left\langle d^{s} b^{t} c\right\rangle$. If $s=0$ and $H=\left\langle b^{t} c\right\rangle$, then it always follows that $1=\left(b^{t} c\right)^{2}$. By normality, $b^{t} c=$ $a\left(b^{t} c\right) a^{-1}=d b^{t} c$. Thus $d=1, m=1$ and the orbifold fundamental group is $\mathbb{Z}_{2} \times\left(\mathbb{Z}_{2 n} \circ_{-1} \mathbb{Z}_{2}\right)$. Again by normality, $b^{t} c=c\left(b^{t} c\right) c^{-1}=b^{-t} c$, implying $t=n$. Furthermore, $b^{n} c=b\left(b^{n} c\right) b^{-1}=b^{n+2} c$. This implies $b^{2}=1$ and $n=1$. Thus $n=m=1$ so that $\pi_{1}\left(O_{h_{6}}((B 7,1),(B 7,1))\right)=\mathbb{Z}_{2} \times\left(\mathbb{Z}_{2} \times \mathbb{Z}_{2}\right)$ and $H=\langle b c\rangle$. We now suppose $s \neq 0$. A computation shows that $\left(d^{s} b^{t} c\right)^{2}=1$ is always true. Suppose $m \neq 1$. By normality, $d^{s} b^{t} c=a\left(d^{s} b^{t} c\right) a^{-1}=d^{-s+1} b^{t} c$, and thus $s=\frac{m+1}{2} \neq 0$ and $H=\left\langle d^{\frac{m+1}{2}} b^{t} c\right\rangle$. Again by normality, we have $d^{\frac{m+1}{2}} b^{t} c$ $=b\left(d^{\frac{m+1}{2}} b^{t} c\right) b^{-1}=d^{\frac{-m-1}{2}} b^{t+2} c$, which implies $d^{m+1}=1$. Hence $d=1$ and $m=1$, contradicting the fact that $m \neq 1$. Hence $m=1$ and $H=\left\langle b^{t} c\right\rangle$. Using normality, we have $b^{t} c=b\left(b^{t} c\right) b^{-1}=b^{t+2} c$, or $b^{2}=1$. Thus $n=1$, $\pi_{1}\left(O_{h_{6}}((B 7,1),(B 7,1))\right)=\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ and $H=\langle b c\rangle$.

Corollary 5.15. Let $\varphi: G \rightarrow$ Homeo $_{P L}\left(\mathbb{R P}^{3}\right)$ be a finite abelian action such that the quotient space $\mathbb{R}^{3} \mathbb{P}^{3} / \varphi$ is the orbifold $O_{h_{6}}((B 7, n),(B 7, m))$. Then the following is true:

1) The action is conjugate to the Standard Quotient Type 6 Action;
2) $n=m=1$;
3) $G=\mathbb{Z}_{2} \times \mathbb{Z}_{2}$;
4) The covering corresponds to the subgroup $\langle a b\rangle$.

Proof. Note that in 1) of Proposition 5.14, $b^{2}(t, v)=(t,-v)$, and in 3) $a c(t, v)=$ $(-t-1, \bar{v})$ and $b c(t, v)=(-t,-\bar{v})$. Since they have fixed points, these cases are excluded by Corollary 5.2. In 2) of Proposition 5.14, $d=a b^{-1} a b \in \pi_{1}((B 7,1))$ is identified with $y^{2} \in \pi_{1}((B 7,2))$ (See Appendix). Since $y(t, v)=\left(-t, v e^{\frac{\pi i}{m}}\right)$, we see that $y^{m}$ has a fixed point, and we may exclude this case. This leaves only the 3 ) where $n=m=1$ and the subgroup $\langle a b\rangle$. Since any regular covering of $O_{h_{6}}((B 7, n),(B 7, m))$ by $\mathbb{R}^{3} \mathbb{P}^{3}$ corresponds to the subgroup $\langle a b\rangle$, any such action is conjugate to the Standard Quotient Type 6 Action on $\mathbb{R} \mathbb{P}^{3}$ which is $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$.

### 5.7. Quotient Type 7: The orbifold $O_{h_{7}}((B 1, n),(B 8, m))$

Recall that the orbifold fundamental group is

$$
\begin{aligned}
& \pi_{1}\left(O_{h_{7}}((B 1, n),(B 8, m))\right) \\
& \quad=\left\langle a, b, c \mid a^{n}=b^{2}=c^{2}=1, b a b^{-1}=a^{-1},[a, c]=1,(c b)^{2 m}=1\right\rangle \\
& =\langle a\rangle \circ\left(\langle c b\rangle \circ_{-1}\langle c\rangle\right)=\mathbb{Z}_{n} \circ \operatorname{Dih}\left(\mathbb{Z}_{2 m}\right) .
\end{aligned}
$$

It follows that $(c b) a(c b)^{-1}=a^{-1}$. From the Appendix, that maps $a, b, c$ on the universal covering space of $(B 1, n)$ are defined as follows: $a(t, v)=\left(t, v e^{2 \pi i / n}\right)$, $b(t, v)=(-t, \bar{v})$, and $c(t, v)=\left(\frac{1}{2}-t, v\right)$.

Proposition 5.16. Let $H \simeq \mathbb{Z}_{2}$ be a normal subgroup of
$\pi_{1}\left(O_{h_{7}}((B 1, n),(B 8, m))\right)$ generated by orientation preserving elements such that the quotient group $Q=\pi_{1}\left(O_{h_{7}}((B 1, n),(B 8, m))\right) / H$ is an abelian group. Then the following is true:

1) $n=2, m=1, H$ is $\langle a\rangle$, $\langle b\rangle$ or $\langle a b\rangle$ and $Q$ is $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$;
2) $n=4, m=1, H=\left\langle a^{2}\right\rangle$ and $Q$ is $\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$;
3) $n=1$ or 2 , $m=2, H=\left\langle(c b)^{2}\right\rangle$ and $Q$ is either $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ or $\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$.

Proof. The subgroup $H=\left\langle a^{s}(c b)^{t} c^{\epsilon}\right\rangle$ where $0 \leq s<n, 0 \leq t<2 m$ and $\epsilon=0$ or 1 . Since only $c$ is orientation reversing, the elements $c b$ and $c$ are orientation reversing. Thus there are two cases to consider, $t$ even and $\epsilon=0$, or $t$ odd and $\epsilon=1$.

Case I: $t$ is even, $\epsilon=0$, and thus $H=\left\langle a^{s}(c b)^{t}\right\rangle$.
Since $t$ is even, we have $1=\left(a^{s}(c b)^{t}\right)^{2}=a^{2 s}(c b)^{2 t}$. Suppose first that $t=0$ and $s \neq 0$, and thus $H=\left\langle a^{s}\right\rangle$. It follows that $s=\frac{n}{2}, H$ is normal and $Q=\mathbb{Z}_{\frac{n}{2}} \circ\left(\mathbb{Z}_{2 m} \circ_{-1} \mathbb{Z}_{2}\right)$. In order for $Q$ to be abelian, we must have $\frac{n}{2}=1$ or 2 and $m=1$. This gives us 1) and 2) in the statement of the proposition.

Suppose $t \neq 0$ and $s=0$, and so $H=\left\langle(c b)^{t}\right\rangle$. We see that $t=m$ which is even, and thus $a$ and $(c b)^{m}$ commute. This implies $H$ is always normal. Now $Q=\mathbb{Z}_{n} \circ\left(\mathbb{Z}_{m} \circ_{-1} \mathbb{Z}_{2}\right)$, which is only abelian if both $n$ and $m$ equal 1 or 2 . Since $m$ is even, $m=2$. and this gives 3 ). Assume next that $t \neq 0$ and $s \neq 0$. It follows that $s=\frac{n}{2}$ and $t=m$ where $m$ is even. A check shows that $H=\left\langle a^{\frac{n}{2}}(c b)^{m}\right\rangle$ is always normal. Since $Q$ is abelian, we must have $a^{-1} H=$ $(c b H)(a H)(c b H)^{-1}=a H$, which implies $a^{2} \in\left\langle a^{\frac{n}{2}}(c b)^{m}\right\rangle$. This is impossible unless $a^{2}=1$, and thus $n=2$. This shows $H=\left\langle a(c b)^{m}\right\rangle$, and $Q=\mathbb{Z}_{2 m} \circ_{-1} \mathbb{Z}_{2}$. This can only be abelian if $m=1$, which contradicts $m$ being even. So this sub-case cannot happen.
Case II: $t$ is odd, $\epsilon=1$, and thus $H=\left\langle a^{s}(c b)^{t} c\right\rangle$.
It is always the case that $\left(a^{s}(c b)^{t} c\right)^{2}=1$, since $t$ is odd. Suppose $s=0$, and so $H=\left\langle(c b)^{t} c\right\rangle$. Now $a(c b)^{t} c a^{-1}=a^{2}(c b)^{t} c$, which must equal $(c b)^{t} c$ by normality. This implies $n=2$. Furthermore, $(c b)\left((c b)^{t} c\right)(c b)^{-1}=(c b)^{t+2} c$, which by normality must equal $(c b)^{t} c$. This implies $(c b)^{2}=1$ and $m=1$. Thus $\pi_{1}\left(O_{h_{7}}((B 1,2),(B 8,1))\right)=\mathbb{Z}_{2} \times\left(\mathbb{Z}_{2} \times \mathbb{Z}_{2}\right)$ and $H=\langle(c b) c\rangle=\langle b\rangle$, giving 1). We now assume $s \neq 0$, and thus $H=\left\langle a^{s}(c b)^{t} c\right\rangle$. A computation shows $a\left(a^{s}(c b)^{t} c\right) a^{-1}=a^{s+2}(c b)^{t} c$ and $(c b)\left(\left(a^{s}(c b)^{t} c\right)(c b)^{-1}=a^{-s}(c b)^{t+2} c\right.$. By normality, it must be the case that $a^{2}=1$ and $(c b)^{2}=1$. Thus $n=2$ and $m=1$, which implies $H=\langle a(c b) c\rangle=\langle a b\rangle$, giving us 1$)$.

Corollary 5.17. There is no abelian action on $\mathbb{R} \mathbb{P}^{3}$, whose quotient space is the orbifold $O_{h_{7}}((B 1, n),(B 8, m))$.

Proof. By Proposition 5.16, we need only consider the subgroups listed there. Observe that the maps $a, a^{2}, b$ and $a b$ have fixed points in the universal cover of $(B 1, n)$. So these cases may be excluded by Corollary 5.2. The remaining case to consider is the subgroup generated by $(c b)^{2}$ where $n=1$ or 2 and $m=2$. From the Appendix, we see that the elements $c$ and $b$ in $\pi_{1}((B 1, n))$ are identified with the elements $y$ and $y z$ in $\pi_{1}((B 8,2))$ respectively. Thus $(c b)^{2}$ is identified with $\left(y^{2} z\right)^{2}$. The maps $y$ and $z$ acting on the universal cover $\mathbb{R} \times D^{2}$ of $(B 8,2)$ are defined by $y(t, v)=(t, \bar{v})$ and $z(t, v)=\left(1-t, v e^{\frac{\pi i}{2}}\right)$. Since $y^{2}=1$, we have $(c b)^{2}$ identified with $z^{2}$. Note that $z^{2}$ has a fixed point. Again applying Corollary 5.2 proves the result.

## 6. Main results

In the last section of this paper, we summarize the main results.
Theorem 6.1. Let $\varphi: G \rightarrow$ Homeo $_{P L}\left(\mathbb{R} \mathbb{P}^{3}\right)$ be an orientation reversing finite abelian action. Then one of the following cases is true:

1) $G=\mathbb{Z}_{2^{b} m}$ where $b>1$, $m$ is odd and $\mathbb{R}^{3} / \varphi$ is $O_{h_{1}^{-1}}\left(\left(B 5,2^{b-1} m\right),(A 1,2)\right)$;
2) $G=\mathbb{Z}_{2 m}$, $m$ is odd and $\mathbb{R P}^{3} / \varphi$ is $t O_{h_{2}^{-1}}((B 4, m),(A 3,1))$;
3) $G=\mathbb{Z}_{m} \times \mathbb{Z}_{2}, m$ even and $\mathbb{R P}^{3} / \varphi$ is $O_{h_{3}}((A 2,2),(B 3,, m))$;
4) $G=\mathbb{Z}_{4} \times \mathbb{Z}_{2}$, and $\mathbb{R} \mathbb{P}^{3} / \varphi$ is $O_{h_{4}}((B 2,2),(B 2,2))$;
5) $G=\mathbb{Z}_{2} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2}$ and $\mathbb{R} \mathbb{P}^{3} / \varphi$ is $t O_{h_{5}}((B 6,2),(B 6,2))$;
6) $G=\mathbb{Z}_{2} \times \mathbb{Z}_{2}$ and $\mathbb{R P}^{3} / \varphi$ is $t O_{h_{6}}((B 7,1),(B 7,1))$.

Furthermore, in each individual case $i)$, where $1 \leq i \leq 6, \varphi$ is equivalent to the Standard Quotient Type i Action.

Proof. Let $\varphi: G \rightarrow$ Homeo $_{P L}\left(\mathbb{R P}^{3}\right)$ be an orientation reversing finite abelian action. By Corollary 4.2, $\varphi$ splits and preserves the sides of the splitting. Write $\mathbb{R P}^{3}=V_{1} \cup V_{2}$, where each $V_{i}$ for $i=1,2$ is a $\varphi(G)$-invariant solid torus. The non-orientable 3-orbifold $V_{i} / \varphi(G)$ is one of the orbifolds $(A 1, n), \ldots,(A 3, n)$, $(B 1, n), \ldots,(B 8, n)$. This implies that if $\nu: \mathbb{R P}^{3} \rightarrow \mathbb{R}^{3} / \varphi(G)$ is the orbifold covering, then $\mathbb{R}^{3} / \varphi(G)$ is $O_{\xi}(X, Y)$ where $X$ and $Y$ are any of the orbifolds $(A 1, n), \ldots,(A 3, n),(B 1, n), \ldots,(B 8, n)$ and $\xi: \partial X \rightarrow \partial Y$ is some homeomorphism. Since $\nu_{*}\left(\pi_{1}\left(\mathbb{R P}^{3}\right)\right)$ has finite index in $\pi_{1}\left(O_{\xi}(X, Y)\right)$, it follows that $\pi_{1}\left(O_{\xi}(X, Y)\right)$ is finite. By Theorem 2.1, $O_{\xi}(X, Y)$ is one of the seven orbifolds listed in the chart. Corollary 5.17 states there is no orientation reversing finite abelian action on $\mathbb{R P}^{3}$ whose quotient space is the orbifold $O_{h_{7}}((B 1, n),(B 8, m))$, thus excluding the seventh orbifold in the chart. Applying Corollaries $5.5,5.7,5.9,5.11,5.13$ and 5.15 to the first six orbifolds proves the result.

## Appendix

In this Appendix, we will define the orbifolds $(A 1, n), \ldots,(A 3, n),(B 1, n), \ldots$, $(B 8, n)$ along with their fundamental groups. Since the fundamental groups of each boundary surjects onto the fundamental group of their orbifolds, we use the same letters for both presentations of the fundamental groups. In addition, if $X$ and $Y$ are orbifolds from this list having homeomorphic boundaries, we will identify the orbifolds $O_{\xi}(X, Y)$ obtained by identifying $\partial X$ to $\partial Y$ via $\xi$ which have finite fundamental groups.

In describing the orbifolds $O_{\xi}(X, Y)$, we will give the details for $X=(A 1, n)$ and $Y=(B 5, m)$ by providing the definition of the lift of the gluing map $\xi: \partial X \rightarrow \partial Y$ to the universal cover of each boundary component. In addition, we obtain a description of the lift of the gluing map on the orientable covers $\partial V(n)$ and $\partial V(m)$ of $\partial(A 1, n)$ and $\partial(B 5, m)$ respectively. For subsequent orbifolds, we just describe the lift of the gluing map to the covers $\partial V(n)$ and $\partial V(m)$ and refer the reader to [13] for the details.

We start by considering the orbifolds which are double covered by $(A 0, n)$. It will be convenient to define the 2-dimensional orbifolds $D^{2}(n)$ and $\Delta(n)$. Let $r_{o}$ be a rotation and $r_{e}$ be a reflection on $D^{2}$, defined by $r_{o}\left(\rho e^{i \theta}\right)=\rho e^{i(\theta+2 \pi / n)}$ and $r_{e}\left(\rho e^{i \theta}\right)=\rho e^{-i \theta}$. Now $D^{2} /\left\langle r_{o}\right\rangle$ is the orbifold $D^{2}(n)$ whose underlying space is a disk, and has a cone point of order $n$ in its center. The map $r_{e}$
induces a reflection $\overline{r_{e}}$ on $D^{2}(n)$, and $D^{2}(n) /\left\langle\overline{r_{e}}\right\rangle$ is the orbifold denoted by $\Delta(n)$. We may be parameterize it as $\left\{\rho e^{i \theta} \mid 0 \leq \rho \leq 1,0 \leq \theta \leq \pi\right\}$ where the point $(0,0)$ is a coner-reflector point of order $n$, and $\left\{\rho e^{i \theta} \mid \theta=0\right.$ or $\left.\pi\right\}$ is the set of mirror points.

In addition, we need to define the 3 -dimensional orbifolds $B^{3}(n), C\left(\mathbb{P}^{2}, 2 n\right)$ and $Z_{n}^{h}$. Let $\mathbb{B}^{3}=\left\{(x, y, z) \mid x^{2}+y^{2}+z^{2} \leq 1\right\}$ and for any point $(x, y, z) \in \mathbb{B}^{3}$ using spherical coordinates, we have $x=\rho \sin \phi \cdot \cos \theta, y=\rho \sin \phi \cdot \sin \theta$ and $z=\rho \cos \phi$ where $0 \leq \rho \leq 1$. We begin by defining a rotation of order $n$ on $\mathbb{B}^{3}$ as follows:

$$
r(x, y, z)=\left(\rho \sin \phi \cdot \cos \left(\theta+\frac{2 \pi}{n}\right), \rho \sin \phi \cdot \sin \left(\theta+\frac{2 \pi}{n}\right), \rho \cos \phi\right) .
$$

Note that $r$ fixes the line segment $\left\{(x, y, z) \in \mathbb{B}^{3} \mid x=0, y=0,-1 \leq z \leq 1\right\}$.
We define the antipodal map $i$ on $\mathbb{B}^{2}$ by $i(x, y, z)=(-x,-y,-z)$. In terms of the spherical coordinate system, $i(x, y, z)=(\rho \sin (\phi+\pi) \cdot \cos \theta, \rho \sin (\phi+$ $\pi) \cdot \sin \theta, \rho \cos (\phi+\pi))$. Observe that $i \circ r \circ i^{-1}=r$.

Let $\mathbb{B}^{3}(n)$ be the orbifold $\mathbb{B}^{3} /\langle r\rangle$, which is a 3 -ball with an arc of exceptional points of order $n$. The induced involution on $\mathbb{B}^{3}(n)$ is designated by $\bar{i}$, and denote $C\left(\mathbb{P}^{2}, 2 n\right)$ to be the 3-orbifold $\mathbb{B}^{3}(n) /\langle\bar{i}\rangle$. The underlying space of $C\left(\mathbb{P}^{2}, 2 n\right)$ is the cone over the projective plane $\mathbb{P}^{2}$, which is $\mathbb{P}^{2} \times[0,1] /(w, 0) \simeq *$, where $*$ indicates a point. The exceptional set consists of an arc where all points except one endpoint have order $n$, and the other endpoint has order $2 n$. The boundary of this orbifold, $\partial\left(C\left(\mathbb{P}^{2}, 2 n\right)\right)$, consists of a projective plane with one cone point of order $n$.

Let $Z_{n}^{h}$ be the orbifold $\mathbb{B}^{3}(n) /\left\langle r_{e}\right\rangle$ where the reflection $r_{e}: \mathbb{B}^{3}(n) \rightarrow \mathbb{B}^{3}(n)$ is defined by $r_{e}(x, y, z)=(x, y,-z)$. The underlying space of $Z_{n}^{h}$ is a 3 -ball, with a half of its boundary is a mirrored disk, together with an arc of exceptional points each of order $n$ except for one endpoint meeting this mirrored disk at a point of order $2 n$. Let $s: B^{3} \rightarrow B^{3}$ be the spin involution about the $y$-axis which we defined by $s(x, y, z)=(\rho \sin (\phi+\pi) \cdot \cos (-\theta), \rho \sin (\phi+\pi) \cdot \sin (-\theta), \rho \cos (\phi+$ $\pi)$ ). Notice that $s r s^{-1}=r^{-1}$, and thus $s$ induces an involution $\bar{s}$ on $B^{3}(n)$. Let $B^{3}(n, 2,2)=B^{3}(n) /\langle\bar{s}\rangle$. The underlying space of $B^{3}(n, 2,2)$ is a ball with a properly embedded tree having three edges meeting at one point of order $2 n$, with two of the edges labeled with a 2 and the remaining edge labeled with an $n$.

Orbifold $(A 1, n)$ : It will be convenient to view the elements of the fundamental groups as acting on the universal covering spaces of $\partial(A 1, n)$ and $(A 1, n)$. Let $\mathbb{R}^{2}$ be the plane and define $\tilde{a}, \tilde{b}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ by $\tilde{a}(t, s)=\left(t-\frac{1}{2},-s\right)$ and $\tilde{b}(t, s)=$ $\left(t-\frac{1}{2},-s+\frac{1}{n}\right)$. Note that $\tilde{a}^{2}=\tilde{b}^{2}$, and $\left(\tilde{a} \tilde{b}^{-1}\right)^{n}(t, s)=(t, s-1)$. If $\eta: \mathbb{R}^{2} \rightarrow$ $\mathbb{R}^{2} /\left\langle\left(\tilde{a} \tilde{b}^{-1}\right)^{n}\right\rangle=\mathbb{R} \times S^{1}$, then $\eta(t, s)=\left(t, e^{2 \pi i s}\right)$ and the induced maps $a, b: \mathbb{R} \times$ $S^{1} \rightarrow \mathbb{R} \times S^{1}$ are defined by $a(t, v)=\left(t-\frac{1}{2}, \bar{v}\right)$ and $b(t, v)=\left(t-\frac{1}{2}, \bar{v} e^{\frac{2 \pi i}{n}}\right)$. These maps extend to $\mathbb{R} \times D^{2}$, which is the universal covering of $(A 1, n)$,
and we will use the same labels for the extensions. We obtain a covering $p_{1}: \mathbb{R} \times D^{2} \rightarrow \mathbb{R} \times D^{2} /\left\langle a^{2}, a b^{-1}\right\rangle=V(n)$ defined by $p_{1}\left(t, \rho e^{i \theta}\right)=\left(e^{2 \pi i t}, \rho e^{i n \theta}\right)$. The induced map $a_{1}$ on $V(n)$ is defined by $a_{1}(u, v)=(-u, \bar{v})$ and $V(n) /\left\langle a_{1}\right\rangle=$ $(A 1, n)$. The orbifold $(A 1, n)$ is a solid Klein bottle with a simple closed curve core of exceptional points of type $n$. The boundary $\partial(A 1, n)$ is a Klein bottle with fundamental group $\pi_{1}(\partial(A 1, n))=\left\langle a, b \mid a^{2}=b^{2}\right\rangle$. Since $a\left(a b^{-1}\right) a^{-1}=$ $a^{2} b^{-1} a^{-1}=b^{2} b^{-1} a^{-1}=b a^{-1}=\left(a b^{-1}\right)^{-1}$, it follows that $a b^{-1}$ is a meridian curve. Thus the orbifold fundamental group of $(A 1, n)$ is

$$
\begin{aligned}
\pi_{1}((A 1, n)) & =\left\langle a, b \mid a^{2}=b^{2},\left(a b^{-1}\right)^{n}=1\right\rangle \quad \text { and } \\
\pi_{1}(\partial(A 1, n)) & =\left\langle a, b \mid a^{2}=b^{2}\right\rangle
\end{aligned}
$$

Orbifold $(B 5, m)$ : Let $D_{1}$ and $D_{2}$ be two disjoint disks in $\partial \mathbb{B}^{3}(m)$ containing the exceptional points. We consider the orbifold $C\left(\mathbb{P}^{2}, 2 m\right) \cup \mathbb{B}^{3}(m) \cup$ $C\left(\mathbb{P}^{2}, 2 m\right)=(B 5, m)$ where we glue $D_{1}$ to the boundary of one copy of $C\left(\mathbb{P}^{2}, 2 m\right)$ and $D_{2}$ to the boundary of the other copy of $C\left(\mathbb{P}^{2}, 2 m\right)$ so that the exceptional sets match up. Furthermore, $\partial(B 5, m)$ is a Klein bottle whose fundamental group surjects to the orbifold fundamental group of $(B 5, m)$. It can be seen that if $f: V(m) \rightarrow V(m)$ is the map defined by $f(u, v)=(\bar{u},-v)$, then $V(m) /\langle f\rangle=(B 5, m)$.

We view the generators of the fundamental groups acting on the universal covering space. Let $\tilde{x}, \tilde{y}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ be defined by $\tilde{x}(t, s)=\left(-t, s-\frac{1}{2 m}\right)$ and $\tilde{y}(t, s)=\left(-t+1, s-\frac{1}{2 m}\right)$. Observe that $\tilde{x}^{2}=\tilde{y}^{2}$ and $\tilde{x}^{-2 m}(t, s)=(t, s+1)$. We obtain a covering $\eta: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2} /\left\langle\tilde{x}^{-2 m}\right\rangle=\mathbb{R} \times S^{1}$ defined by $\eta(t, s)=\left(t, e^{2 \pi i s}\right)$. The induced maps $x$ and $y$ on $\mathbb{R} \times S^{1}$ are defined by $x(t, v)=\left(-t, v e^{\frac{-\pi i}{m}}\right)$ and $y(t, v)=\left(-t+1, v e^{\frac{-\pi i}{m}}\right)$. These maps extend to $\mathbb{R} \times D^{2}$, which is the universal covering of $(B 5, m)$. Let $p_{1}: \mathbb{R} \times D^{2} \rightarrow \mathbb{R} \times D^{2} /\left\langle y x^{-1}, x^{-2}\right\rangle=V(m)$ be the covering map defined by $p_{1}(t, v)=\left(e^{2 \pi i t}, v^{m}\right)$. The induced map $x_{1}: V(m) \rightarrow$ $V(m)$ is defined by $x_{1}(u, v)=(\bar{u},-v)$, and $V(m) /\left\langle x_{1}\right\rangle=(B 5, m)$. The orbifold fundamental group of $(B 5, m)$ is

$$
\begin{aligned}
\pi_{1}((B 5, m)) & =\left\langle x, y \mid x^{2}=y^{2}, x^{2 m}=y^{2 m}=1\right\rangle \quad \text { and } \\
\pi_{1}(\partial(B 5, m)) & =\left\langle x, y \mid x^{2}=y^{2}\right\rangle .
\end{aligned}
$$

Orbifold $O_{h_{1}}((A 1, n),(B 5, m))$ : Recall that the maps defining the fundamental groups $\tilde{a}, \tilde{b}, \tilde{x}, \tilde{y}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ are defined as follows: $\tilde{a}(t, s)=\left(t-\frac{1}{2},-s\right)$, $\tilde{b}(t, s)=\left(t-\frac{1}{2},-s+\frac{1}{n}\right), \tilde{x}(t, s)=\left(-t, s-\frac{1}{2 m}\right)$ and $\tilde{y}(t, s)=\left(-t+1, s-\frac{1}{2 m}\right)$. We obtain covering maps $\lambda_{1}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2} /\left\langle\tilde{a}^{2}, \tilde{a} \tilde{b}^{-1}\right\rangle=T_{1}=\partial(A 0, n)$ and $\lambda_{2}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2} /\left\langle\tilde{x}^{-2}, \tilde{y} \tilde{x}^{-1}\right\rangle=T_{2}=\partial(B 5, m)$ defined by $\lambda_{1}(t, s)=\left(e^{2 \pi i t}, e^{2 \pi i n s}\right)$ and $\lambda_{2}(t, s)=\left(e^{2 \pi i t}, e^{2 \pi i m s}\right)$ respectively. The induced maps $a_{1}$ on $T_{1}$ and $x_{1}$ on $T_{2}$ are defined by $a_{1}(u, v)=(-u, \bar{v})$ and $x_{1}(u, v)=(\bar{u},-v)$ respectively. We obtain covering maps $\mu_{1}: T_{1} \rightarrow T_{1} /\left\langle a_{1}\right\rangle=\partial(A 1, n)$ and $\mu_{2}: T_{2} \rightarrow T_{2} /\left\langle x_{1}\right\rangle=$ $\partial(B 5, m)$.

Define a map $\tilde{h}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ by $\tilde{h}(t, s)=\left(n s, \frac{t}{m}\right)$, and note that $\tilde{h}^{-1}(t, s)=$ $\left(m s, \frac{t}{n}\right)$. We compute $\tilde{h} \tilde{a} \tilde{h}^{-1}(t, s)=\tilde{h} \tilde{a}\left(m s, \frac{t}{n}\right)=\tilde{h}\left(m s-\frac{1}{2},-\frac{t}{n}\right)=(-t, s-$ $\left.\frac{1}{2 m}\right)=\tilde{x}(t, s)$. A similar computation shows that $\tilde{h} \tilde{b} \tilde{h}^{-1}=\tilde{y}$. Thus $\tilde{h}$ projects to maps $h_{1}$ and $h$ making the following diagram commute:


A computation shows that $\tilde{h}_{1}(u, v)=(v, u)$ for any $(u, v) \in T_{1}$.
When we identify $\partial(A 1, n)$ to $\partial(B 5, m)$ via $h_{1}$, the generators are identified by $a=x$ and $b=y$. It follows that the orbifold fundamental group of $O_{h_{1}}((A 1, n),(B 5, m))$ is

$$
\begin{aligned}
\pi_{1}\left(O_{h_{1}}((A 1, n),(B 5, m))\right) & =\left\langle a, b \mid a^{2}=b^{2}, a^{2 m}=b^{2 m}=\left(b a^{-1}\right)^{n}=1\right\rangle \\
& =\left\langle b a^{-1}\right\rangle \circ_{-1}\langle a\rangle=\mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{2 m}
\end{aligned}
$$

We note that both $a$ and $b$ are orientation reversing elements.
Orbifold $(A 3, n)$ : The orbifold $(A 3, n)$ is

$$
(\Delta(n) \times[0,1]) /\left(\rho e^{i \theta}, 0\right) \simeq\left(\rho e^{i(-\theta-\pi)}, 1\right)
$$

and the underlying space of $(A 3, n)$ is a solid Klein bottle. The boundary of the underlying space consists of two Mobius strips, one of which is mirrored containing an orientation reversing circle of cone points of orders $n$.

The universal covering space of $(A 3, n)$ is $\mathbb{R} \times D^{2}$, and the covering transformation maps $a, b, c$ on $\mathbb{R} \times D^{2}$ are defined as follows: $a\left(t, \rho e^{i \theta}\right)=\left(t, \rho e^{i\left(\theta+\frac{2 \pi}{n}\right)}\right)$, $b\left(t, \rho e^{i \theta}\right)=\left(t, \rho e^{-i \theta}\right)$ and $c\left(t, \rho e^{i \theta}\right)=\left(t+\frac{1}{2}, \rho e^{i\left(-\theta-\frac{\pi}{n}\right)}\right)$. A computation shows the following: $c a c^{-1}=a^{-1}, c b c^{-1}=b a, b a b^{-1}=a^{-1}, a^{n}=b^{2}=1$. Hence the group generated by these elements is $\operatorname{Dih}\left(\mathbb{Z}_{n}\right) \circ \mathbb{Z}=\left(\langle a\rangle \circ_{-1}\langle b\rangle\right) \circ\langle c\rangle$.

Define an orbifold covering map $p_{1}: \mathbb{R} \times D^{2} \rightarrow \mathbb{R} \times D^{2} /\left\langle a, c^{2}\right\rangle=\widetilde{V}(n)$ by $p_{1}\left(t, \rho e^{i \theta}\right)=\left(e^{2 \pi i t}, \rho e^{i n \theta}\right)$. The maps $b$ and $c$ induce maps $b_{1}$ and $c_{1}$ respectively on $\widetilde{V}(n)$, and it can be shown using the covering map $p_{1}$ that $b_{1}(u, v)=(u, \bar{v})$ $\underset{\sim}{\operatorname{Van}}\left(c_{1}(u, v)=(-u,-\bar{v})\right.$. Observe that $b_{1} c_{1}(u, v)=(-u,-v)$. Let $p_{2}: \widetilde{V}(n) \rightarrow$ $\widetilde{V}(n) /\left\langle b_{1} c_{1}\right\rangle=V(n)$ be the orbifold covering map, and note that $p_{2}(u, v)=$
$\left(u^{2}, u v\right)$. We see that $b_{1}$ induces a map $b_{2}$ on $V(n)$ defined by $b_{2}(u, v)=$ $p_{2} b_{1}\left(u^{1 / 2}, u^{-1 / 2} v\right)=(u, u \bar{v})$. It follows that $V(n) /\left\langle b_{2}\right\rangle=(A 3, n)$ and the fundamental group $\pi_{1}((A 3, n))=\left(\langle a\rangle \circ_{-1}\langle b\rangle\right) \circ\langle c\rangle=\operatorname{Dih}\left(\mathbb{Z}_{n}\right) \circ \mathbb{Z}$.

The boundary of $(A 3, n)$ is a mirrored Mobius band $m \ddot{M}$, and its fundamental group $\pi_{1}(m \ddot{M})=\left(\langle a\rangle \circ_{-1}\langle b\rangle\right) \circ\langle c\rangle=\left(\mathbb{Z} \circ_{-1} \mathbb{Z}_{2}\right) \circ \mathbb{Z}$. It may be convenient to write $\pi_{1}(m \ddot{M})=(\langle b\rangle *\langle b a\rangle) \circ\langle c\rangle=\left(\mathbb{Z}_{2} * \mathbb{Z}_{2}\right) \circ \mathbb{Z}$ where $c b c^{-1}=b a$ and $c(b a) c^{-1}=b$. Thus, the orbifold fundamental group of $(A 3, n)$ is

$$
\begin{aligned}
\pi_{1}((A 3, n)) & =\left\langle a, b, c \mid a^{n}=b^{2}=1, b a b^{-1}=a^{-1}, c a c^{-1}=a^{-1}, c b c^{-1}=b a\right\rangle \\
& =\left(\mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{2}\right) \circ \mathbb{Z} \text { and } \\
\pi_{1}(\partial(A 3, n)) & =\left\langle b, b a, c \mid c b c^{-1}=b a, c(b a) c^{-1}=b\right\rangle=\left(\mathbb{Z}_{2} * \mathbb{Z}_{2}\right) \circ \mathbb{Z} .
\end{aligned}
$$

Orbifold $(B 4, m)$ : The orbifold $(B 4, m)$ is $C\left(\mathbb{P}^{2}, 2 m\right) \cup \mathbb{B}^{3}(m) \cup Z_{m}^{h}$ where the exceptional sets of order $m$ match up. The boundary $\partial(B 4, m)$ is a mirrored Mobius band. The covering translations on the universal covering space $\mathbb{R} \times \mathbb{D}^{2}$ of $(B 4, m)$ are defined as follows: $x(t, v)=(t+1, v), y(t, v)=\left(t, v e^{\frac{2 \pi i}{m}}\right)$, $z(t, v)=\left(-t, v e^{\frac{2 \pi i t}{m}}\right)$. The element $z$ is an orientation reversing element. Define an orbifold covering map $p_{1}: \mathbb{R} \times D^{2} \rightarrow \mathbb{R} \times D^{2} /\langle x, y\rangle=V(m)$ by $p_{1}\left(t, \rho e^{i \theta}\right)=\left(e^{2 \pi i t}, \rho e^{i m \theta}\right)$. Then $z$ induces a map $z_{1}: V(m) \rightarrow V(m)$ defined by $z_{1}(u, v)=(\bar{u}, u v)$. The quotient space $V(m) /\left\langle z_{1}\right\rangle$ is the orbifold $(B 4, m)$ and its fundamental group is

$$
\begin{aligned}
\pi_{1}((B 4, m)) & =\left\langle x, y, z \mid[x, y]=1, y^{m}=z^{2}=1, z x z^{-1}=x^{-1} y, z y z^{-1}=y\right\rangle \\
& =\left(\mathbb{Z} \times \mathbb{Z}_{m}\right) \circ \mathbb{Z}_{2} \text { and } \\
\pi_{1}(\partial(B 4, m)) & =\left\langle x^{-2} y z, z, z x \mid(z x)\left(x^{-2} y z\right)(z x)^{-1}=z,(z x) z(z x)^{-1}=x^{-2} y z\right\rangle \\
& =\left(\left\langle x^{-2} y z\right\rangle *\langle z\rangle\right) \circ\langle z x\rangle=\left(\mathbb{Z}_{2} * \mathbb{Z}_{2}\right) \circ \mathbb{Z} .
\end{aligned}
$$

Orbifold $O_{h_{2}}((A 3, n),(B 4, m))$ : Recall that $\partial V(n) /\left\langle b_{2}\right\rangle=\partial(A 3, n)$ and $V(m) /\left\langle z_{1}\right\rangle=\partial(B 4, m)$. Define $\tilde{h}_{2}: \partial V(n) \rightarrow \partial V(m)$ by $\tilde{h}_{2}(u, v)=\left(\bar{u} v^{2}, u \bar{v}\right)$ and observe that $\tilde{h}_{2}^{-1}(u, v)=\left(u v^{2}, u v\right)$. Since $\tilde{h}_{2} b_{2} \tilde{h}_{2}^{-1}=z_{1}$, we obtain the following commutative diagram:

where $\mu_{i}$ are the quotient maps.
We use the map $h_{2}: \partial(A 3, n) \rightarrow \partial(B 4, m)$ to define $O_{h_{2}}((A 3, n),(B 4, m))$. It follows by [13], that the generators are identified by $b=z, b a=x^{-2} y z, c=z x$
and $c^{2}=y$, and so the orbifold fundamental group $\pi_{1}\left(O_{h_{2}}((A 3, n),(B 4, m))\right)$ is

$$
\begin{aligned}
\langle a, b, c| a^{n}=b^{2}=c^{2 m}=1, b a b^{-1}= & \left.a^{-1}, c a c^{-1}=a^{-1}, c b c^{-1}=b a\right\rangle \\
& =(\langle a\rangle \circ-1\langle b\rangle) \circ\langle c\rangle=\operatorname{Dih}\left(\mathbb{Z}_{n}\right) \circ \mathbb{Z}_{2 m} .
\end{aligned}
$$

The elements $b$ and $c$ are orientation reversing elements in the fundamental group.
Orbifold (A2,n): Define the maps $a, b, c: \mathbb{R} \times D^{2} \rightarrow \mathbb{R} \times D^{2}$ as follows: $a(t, v)=$ $\overline{(t+1, v), b(t, v)}=\left(t, v e^{\frac{2 \pi i}{n}}\right), c(t, v)=(t, \bar{v})$. We obtain an orbifold covering $\operatorname{map} p_{1}: \mathbb{R} \times D^{2} \rightarrow \mathbb{R} \times D^{2} /\langle a, b\rangle=V(n)$ defined by $p_{1}\left(t, \rho e^{i \theta}\right)=\left(e^{2 \pi i t}, \rho e^{i n \theta}\right)$. Then $c$ induces an involution $c_{1}: V(n) \rightarrow V(n)$ defined by $c_{1}(u, v)=(u, \bar{v})$. There is an orbifold covering map $\mu_{1}: V(n) \rightarrow V(n) /\left\langle c_{1}\right\rangle=(A 2, n)$. The orbifold $(A 2, n)=S^{1} \times \Delta(n)$, has underlying space a solid torus with boundary $\partial(A 2, n)$ a mirrored annulus. The orbifold fundamental group of $(A 2, n)$ is

$$
\begin{aligned}
\pi_{1}((A 2, n)) & =\left\langle a, b, c \mid[a, b]=[a, c]=1, b^{n}=1, c b c^{-1}=b^{-1}, c^{2}=1\right\rangle \\
& =\left(\langle b\rangle \circ_{-1}\langle c\rangle\right) \times\langle a\rangle=\operatorname{Dih}\left(\mathbb{Z}_{n}\right) \times \mathbb{Z} \text { and } \\
\pi_{1}(\partial(A 2, n)) & =\left\langle a, b, c \mid[a, b]=[a, c]=1, c b c^{-1}=b^{-1}, c^{2}=1\right\rangle \\
& =\operatorname{Dih}(\mathbb{Z}) \times \mathbb{Z} .
\end{aligned}
$$

Orbifold $(B 3, m)$ : On $\mathbb{R} \times D^{2}$ define maps $x, y$ and $z$ by $x(t, v)=(t+1, v)$, $y(t, v)=\left(t, v e^{\frac{2 \pi i}{m}}\right)$ and $z(t, v)=(-t, v)$. As above, we obtain an orbifold covering map $p_{2}: \mathbb{R} \times D^{2} \rightarrow \mathbb{R} \times D^{2} /\langle x, y\rangle=V(m)$ defined by $p_{1}\left(t, \rho e^{i \theta}\right)=$ $\left(e^{2 \pi i t}, \rho e^{i m \theta}\right)$.

The induced involution $z_{1}: V(m) \rightarrow V(m)$ is defined by $z_{1}(u, v)=(\bar{u}, v)$. There is an orbifold covering map $\mu_{2}: V(m) \rightarrow V(m) /\left\langle z_{1}\right\rangle=(B 3, m)$. The orbifold quotient $(B 3, m)$, has underlying space $D^{2} \times I$ with both $D^{2} \times\{0\}$ and $D^{2} \times\{1\}$ being mirrored, and an exceptional set $\{0\} \times I$ of order $m$. The boundary $\partial(B 3, m)$ is a mirrored annulus. The orbifold fundamental group of $(B 3, m)$ is

$$
\begin{aligned}
\pi_{1}((B 3, m)) & =\left\langle x, y, z \mid[x, y]=1, y^{m}=1,[y, z]=1, z x z^{-1}=x^{-1}, z^{2}=1\right\rangle \\
& =\left(\langle x\rangle \circ_{-1}\langle z\rangle\right) \times\langle y\rangle=\operatorname{Dih}(\mathbb{Z}) \times \mathbb{Z}_{m} \text { and } \\
\pi_{1}(\partial(B 3, m)) & =\left\langle x, y, z \mid[x, y]=[y, z]=1, z x z^{-1}=x^{-1}, z^{2}=1\right\rangle \\
& =\operatorname{Dih}(\mathbb{Z}) \times \mathbb{Z} .
\end{aligned}
$$

Orbifold $O_{h_{3}}((A 2, n),(B 3, m))$ : Recall that $\partial V(n) /\left\langle c_{1}\right\rangle=\partial(A 2, n)$ and $\overline{\partial V(m) /\left\langle z_{1}\right\rangle=\partial(B 3, m) \text {. Define } \tilde{h}_{3}: \partial V(n) \rightarrow \partial V(m) \text { by } \tilde{h}_{3}(u, v)=(v, u), ~\left(\tilde{h}_{3}\right)}$ and observe that $\tilde{h}_{3}^{-1}=\tilde{h}_{3}$. Since $\tilde{h}_{3} c_{1} \tilde{h}_{3}^{-1}=z_{1}$, we obtain the following
commutative diagram:

where $\mu_{i}$ are the quotient maps. Identify $\partial(A 2, n)$ to $\partial(B 3, m)$ via $h_{3}$ to obtain the orbifold $O_{h_{3}}((A 2, n),(B 3, m))$. It follows by [13] that that the generators are identified by $a=y, b=x$ and $c=z$. Hence the orbifold fundamental group is

$$
\begin{aligned}
& \pi_{1}\left(O_{h_{3}}((A 2, n),(B 3, m))\right) \\
& \quad=\left\langle a, b, c \mid[a, b]=[a, c]=1, a^{m}=b^{n}=c^{2}=1, c b c^{-1}=b^{-1}\right\rangle \\
& \quad=\left(\langle b\rangle \circ_{-1}\langle c\rangle\right) \times\langle a\rangle=\operatorname{Dih}\left(\mathbb{Z}_{n}\right) \times \mathbb{Z}_{m}
\end{aligned}
$$

The element $c$ is an orientation reversing element.
Orbifold $(B 2, n)$ : The orbifold $(B 2, n)=B^{3}(n, 2,2) \cup C\left(\mathbb{P}^{2}, 2 n\right)$ where a disk in $\partial\left(B^{3}(n, 2,2)\right)$ containing the exceptional point of order $n$ is identified to a disk in $\partial\left(C\left(\mathbb{P}^{2}, 2 n\right)\right)$ containing the exceptional point of order $n$.

Define maps $a$ and $b$ on $\mathbb{R} \times D^{2}$ by $a\left(t, \rho e^{i \theta}\right)=\left(-t+\frac{1}{2}, \rho e^{i\left(\theta+\frac{\pi}{n}\right)}\right)$ and $b\left(t, \rho e^{i \theta}\right)=\left(-t, \rho e^{-i \theta}\right)$. The map $a$ is orientation reversing. It is easy to check that $a^{2}\left(t, \rho e^{i \theta}\right)=\left(t, \rho e^{i\left(\theta+\frac{2 \pi}{n}\right)}\right)$ and $(a b)^{2}\left(t, \rho e^{i \theta}\right)=\left(t+1, \rho e^{i \theta}\right)$, hence we have relations $a^{2 n}=b^{2}=1$ and $b a^{2} b^{-1}=a^{-2}$. The manifold $\mathbb{R} \times D^{2}$ is the universal covering of $(B 2, n)$, which can be seen by means of the following sequence of coverings. First, let $p: \mathbb{R} \times D^{2} \rightarrow \mathbb{R} \times D^{2} /\left\langle a^{2},(a b)^{2}\right\rangle=V(n)$ be defined by $p\left(t, \rho e^{i \theta}\right)=\left(e^{2 \pi i t}, \rho e^{i n \theta}\right)$. The induced maps $a_{1}$ and $b_{1}$ on $V(n)$ are defined by $a_{1}(u, v)=(-\bar{u},-v)$ and $b_{1}(u, v)=(\bar{u}, \bar{v})$. Secondly, we have a covering map $p_{1}: V(n) \rightarrow V(n) /\left\langle b_{1}\right\rangle=(B 0, n)$, and $a_{1}$ induces the anti-podal map $a_{2}$ on $(B 0, n)$. Finally we obtain a covering map $\mu:(B 0, n) \rightarrow(B 0, n) /\left\langle a_{2}\right\rangle=$ $(B 2, n)$. The orbifold fundamental group of $(B 2, n)$ is

$$
\begin{aligned}
\pi_{1}((B 2, n)) & =\left\langle a, b \mid a^{2 n}=b^{2}=1, b a^{2} b^{-1}=a^{-2}\right\rangle \\
& =\left(\left\langle a^{2}\right\rangle \circ\langle a b\rangle\right) \circ\langle b\rangle=\left(\mathbb{Z}_{n} \circ \mathbb{Z}\right) \circ \mathbb{Z}_{2} \quad \text { and } \\
\pi_{1}(\partial(B 2, n)) & =\left\langle a, b \mid b^{2}=1, b a^{2} b^{-1}=a^{-2}\right\rangle \\
& =\left(\left\langle a^{2}\right\rangle \circ\langle a b\rangle\right) \circ\langle b\rangle=(\mathbb{Z} \circ \mathbb{Z}) \circ \mathbb{Z}_{2} .
\end{aligned}
$$

The boundary of $(B 2, n)$ is a projective plane with two cone points each of order 2 (See Figure 3).


Figure 3: $\partial(B 2, n)$

Orbifold $O_{h_{4}}((B 2, n),(B 2, m))$ : We use the letters $x$ and $y$ to denote the generators of $\pi_{1}((B 2, m))$, and note that the definitions are identical with $n$ replaced by $m$. Thus $\mathbb{R} \times D^{2} /\left\langle a^{2},(a b)^{2}\right\rangle=V(n)$ and $\mathbb{R} \times D^{2} /\left\langle x^{2},(x y)^{2}\right\rangle=$ $V(m)$; and $V(n) /\left\langle b_{1}\right\rangle=(B 0, n)$, and $V(m) /\left\langle y_{1}\right\rangle=(B 0, m)$. Define a map $\tilde{h}_{4}: \partial V(n) \rightarrow \partial V(m)$ by $\tilde{h}_{4}(u, v)=(-\bar{v}, u)$ and observe that $\tilde{h}_{4}^{-1}(u, v)=$ $(v,-\bar{u})$. A computation shows $\tilde{h}_{4} a_{1} \tilde{h}_{4}^{-1}=y_{1} x_{1}^{-1}$ and $\tilde{h}_{4} b_{1} \tilde{h}_{4}^{-1}=y_{1}$. Thus $\tilde{h}_{4}$ induces maps $\widehat{h}_{4}$ and $h_{4}$ making the following diagram commute:


By identifying $\partial(B 2, n)$ to $\partial(B 2, m)$ via $h_{4}$, it follows by [13] that the generators are related by $a=y x^{-1}$ and $b=x y x$. It follows that $a b=x$ and $a^{2} b=y$. Thus the orbifold fundamental group is

$$
\begin{aligned}
\pi_{1}\left(O_{h_{4}}((B 2, n),(B 2, m))\right) & =\left\langle a, b \mid a^{2 n}=b^{2}=1, b a^{2} b^{-1}=a^{-2},(a b)^{2 m}=1\right\rangle \\
& =\left(\left\langle a^{2}\right\rangle \circ_{-1}\langle a b\rangle\right) \circ\langle b\rangle=\left(\mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{2 m}\right) \circ \mathbb{Z}_{2}
\end{aligned}
$$

The element $a$ is an orientation reversing element.
Orbifold $(B 6, n)$ : Define maps on $\mathbb{R} \times D^{2}$ by $a\left(t, \rho e^{i \theta}\right)=\left(-t, \rho e^{i \theta}\right), b\left(t, \rho e^{i \theta}\right)$ $=\left(t, \rho e^{-i \theta}\right), c\left(t, \rho e^{i \theta}\right)=\left(t, \rho e^{i\left(-\theta-\frac{2 \pi}{n}\right)}\right)$ and $d\left(t, \rho e^{i \theta}\right)=\left(-t-1, \rho e^{i \theta}\right)$. Let $p: \mathbb{R} \times D^{2} \rightarrow \mathbb{R} \times D^{2} /\langle a d, b c\rangle=V(n)$ be defined by $p\left(t, \rho e^{i \theta}\right)=\left(e^{2 \pi i t}, \rho e^{i n \theta}\right)$. Then $a$ and $b$ induce maps $a_{1}$ and $b_{1}$ on $V(n)$ defined by $a_{1}(u, v)=(\bar{u}, v)$ and $b_{1}(u, v)=(u, \bar{v})$. Furthermore, there is a covering map $p_{1}: V(n) \rightarrow$ $V(n) /\left\langle a_{1} b_{1}\right\rangle=(B 0, n)$ and $b_{1}$ induces a reflection $b_{2}$ on $(B 0, n)$ through a disk containing the exceptional set. Modding out by $b_{2}$ we obtain the final covering map $\mu:(B 0, n) \rightarrow(B 0, n) /\left\langle b_{2}\right\rangle=(B 6, n)$. The orbifold fundamental


Figure 4: $\partial(B 6, n)$
group of $(B 6, n)$ is

$$
\begin{aligned}
& \pi_{1}((B 6, n))=\langle a, b, c, d| a^{2}=b^{2}=c^{2}=(b c)^{n}=d^{2}=1 \\
&\quad[a, b]=[a, c]=[b, d]=[c, d]=1\rangle \\
&=\left(\langle b c\rangle \circ_{-1}\langle c\rangle\right) \times(\langle a\rangle *\langle d\rangle)=\left(\mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{2}\right) \times\left(\mathbb{Z}_{2} * \mathbb{Z}_{2}\right) \text { and } \\
& \pi_{1}(\partial(B 6, n))=\langle a, b, c, d| a^{2}=b^{2}=c^{2}=d^{2}=1, \\
&\quad[a, b]=[a, c]=[b, d]=[c, d]=1\rangle \\
&=\left(\langle b c\rangle \circ_{-1}\langle c\rangle\right) \times(\langle a\rangle *\langle d\rangle)=\left(\mathbb{Z} \circ_{-1} \mathbb{Z}_{2}\right) \times\left(\mathbb{Z}_{2} * \mathbb{Z}_{2}\right) .
\end{aligned}
$$

The boundary of $(B 6, n)$ is a mirrored disk with four cone points of order two on the mirror (See Figure 4).

Orbifold $O_{h_{5}}((B 6, n),(B 6, m))$ : As above, we use the letters $x, y, z$ and $w$ to denote the generators of $\pi_{1}((B 6, m))$ where the definitions are identical with $m$ replacing $n$. Thus $\mathbb{R} \times D^{2} /\langle a d, b c\rangle=V(n)$ and $\mathbb{R} \times D^{2} /\langle x w, y z\rangle=V(m)$. Define a homeomorphism $\tilde{h}_{5}: \partial V(n) \rightarrow \partial V(m)$ by $\tilde{h_{5}}(u, v)=(-v, u)$. A computation shows $\tilde{h}_{5} a_{1} \tilde{h}_{5}^{-1}=y_{1}$ and $\tilde{h}_{5} b_{1} \tilde{h}_{5}^{-1}=x_{1}$. The map $\tilde{h}_{5}$ induces maps $\widehat{h}_{5}$ and $h_{5}$ making the following diagram commute:


Thus when identifying $\partial(B 6, n)$ to $\partial(B 6, m)$ via $h_{5}$, it follows by [13] that the generators are identified by $a=y, b=x w x, c=x$ and $d=z$, and the orbifold


Figure 5: $\partial(B 7, n)$
fundamental of group is

$$
\begin{aligned}
& \pi_{1}\left(O_{h_{5}}((B 6, n),(B 6, m))\right) \\
& \quad=\langle a, b, c, d| a^{2}=b^{2}=c^{2}=(b c)^{n}=d^{2}=1, \\
& \left.\quad[a, b]=[a, c]=[b, d]=[c, d]=1,(a d)^{m}=1\right\rangle \\
& \quad=\left(\langle b c\rangle \circ_{-1}\langle c\rangle\right) \times\left(\langle a d\rangle \circ_{-1}\langle a\rangle\right)=\left(\mathbb{Z}_{n} \circ_{-1} \mathbb{Z}_{2}\right) \times\left(\mathbb{Z}_{m} \circ_{-1} \mathbb{Z}_{2}\right) .
\end{aligned}
$$

Note that the elements $a, b, c$ and $d$ are orientation reversing.
Orbifold $(B 7, n)$ : Define maps $a, b, c$ on $\mathbb{R} \times D^{2}$ as follows: $a\left(t, \rho e^{i \theta}\right)=(-t-$ $\overline{\left.1, \rho e^{i \theta}\right), b\left(t, \rho e^{i \theta}\right)}=\left(-t, \rho e^{i\left(\theta+\frac{\pi}{n}\right)}\right)$ and $c\left(t, \rho e^{i \theta}\right)=\left(t, \rho e^{-i \theta}\right)$. Let $p: \mathbb{R} \times$ $D^{2} \rightarrow \mathbb{R} \times D^{2} /\left\langle b^{2},(b a)\right\rangle=V(n)$ be the covering map defined by $p\left(t, \rho e^{i \theta}\right)=$ $\left(e^{2 \pi i t}, \rho e^{i(n \theta+\pi t)}\right)$. The induced maps $a_{1}$ and $c_{1}$ on $V(n)$ are $a_{1}(u, v)=(\bar{u},-\bar{u} v)$ and $c_{1}(u, v)=(u, u \bar{v})$. Observe that $a_{1} c_{1}(u, v)=(\bar{u},-\bar{v})$, and thus there is a covering map $p_{1}: V(n) \rightarrow V(n) /\left\langle a_{1} c_{1}\right\rangle=(B 0, n)$. If $a_{2}$ be the induced map on $(B 0, n)$, then we have another covering $\mu:(B 0, n) \rightarrow(B 0, n) /\left\langle a_{2}\right\rangle=(B 7, n)$. Since $a$ and $b^{2}$ commute, we have $b^{-1} a b=b a b^{-1}$, hence the orbifold fundamental group of $(B 7, n)$ is

$$
\begin{aligned}
\pi_{1}((B 7, n)) & =\left\langle a, b, c \mid a^{2}=b^{2 n}=c^{2}=1,\left[a, b^{2}\right]=[a, c]=1, c b c^{-1}=b^{-1}\right\rangle \\
& =\left\langle a, b a b^{-1}\right\rangle \circ\left(\langle b\rangle \circ_{-1}\langle c\rangle\right)=\left(\mathbb{Z}_{2} * \mathbb{Z}_{2}\right) \circ\left(\mathbb{Z}_{2 n} \circ_{-1} \mathbb{Z}_{2}\right) \text { and } \\
\pi_{1}(\partial(B 7, n)) & =\left\langle a, b, c \mid a^{2}=c^{2}=1,\left[a, b^{2}\right]=[a, c]=1, c b c^{-1}=b^{-1}\right\rangle \\
& =\left\langle a, b a b^{-1}\right\rangle \circ\left(\langle b\rangle \circ_{-1}\langle c\rangle\right)=\left(\mathbb{Z}_{2} * \mathbb{Z}_{2}\right) \circ\left(\mathbb{Z}^{\circ} \circ_{-1} \mathbb{Z}_{2}\right) .
\end{aligned}
$$

The boundary of $(B 7, n)$ is a mirrored disk with two cone points on the mirror and one cone point in the interior (See Figure 5).

Orbifold $O_{h_{6}}((B 7, n),(B 7, m))$ : We use the letters $x, y$ and $z$ to denote the generators of $\pi_{1}((B 7, m))$ where the definitions are identical with $n$ replaced with $m$. As above we obtain covering maps $\mathbb{R} \times D^{2} \rightarrow \mathbb{R} \times D^{2} /\left\langle b^{2},(b a)\right\rangle=V(n)$ and $\mathbb{R} \times D^{2} \rightarrow \mathbb{R} \times D^{2} /\left\langle y^{2},(y x)\right\rangle=V(m)$. The induced maps on $V(n)$ and $V(m)$ are denoted by $a_{1}, c_{1}$ and $x_{1}, z_{1}$ respectively.

Define a homeomorphism $\tilde{h}_{6}: \partial V(n) \rightarrow \partial V(m)$ by $\tilde{h_{6}}(u, v)=\left(-\bar{u} v^{2}, \bar{u} v\right)$.

The map $\tilde{h}_{6}$ induces maps $\widehat{h}_{6}$ and $h_{6}$ making the following diagram commutes.


Identifying $\partial(B 7, n)$ to $\partial(B 7, m)$ via $h_{6}$ to obtain the orbifold $O_{h_{6}}((B 7, n),(B 7, m))$, it follows by [13] that the generators are identified by $a=z, b=z y x$, and $c=y x y^{-1}$. Furthermore, the fundamental group $\pi_{1}\left(O_{h_{6}}((B 7, n),(B 7, m))\right)$ is

$$
\begin{aligned}
&\langle a, b, c|\left.a^{2}=b^{2 n}=\left(a b^{-1} a b\right)^{m}=c^{2}=1,\left[a, b^{2}\right]=[a, c]=1, c b c^{-1}=b^{-1}\right\rangle \\
& \quad=\left(\left\langle a b^{-1} a b\right\rangle \circ_{-1}\langle a\rangle\right) \circ\left(\langle b\rangle \circ_{-1}\langle c\rangle\right)=\left(\mathbb{Z}_{m} \circ_{-1} \mathbb{Z}_{2}\right) \circ\left(\mathbb{Z}_{2 n} \circ_{-1} \mathbb{Z}_{2}\right) .
\end{aligned}
$$

Note that the elements $a, b$ and $c$ are orientation reversing.
Orbifold $(B 1, n)$ : The orbifold $B^{3}(n, 2,2) \cup Z_{n}^{h}$ where a disk in $\partial\left(B^{3}(n, 2,2)\right)$ containing the exceptional point of order $n$ is identified with a disk in $\partial\left(Z_{n}^{h}\right)$ containing the exceptional point of order $n$.

Define maps $a, b, c$ on $\mathbb{R} \times D^{2}$ as follows: $a(t, v)=\left(t, v e^{2 \pi i / n}\right), b(t, v)=$ $(-t, \bar{v})$, and $c(t, v)=\left(\frac{1}{2}-t, v\right)$. The manifold $\mathbb{R} \times D^{2}$ is the universal cover of $(B 1, n)$. Let $p_{1}: \mathbb{R} \times D^{2} \rightarrow \mathbb{R} \times D^{2} /\left\langle(c b)^{2}, a\right\rangle=V(n)$ be defined by $p_{1}\left(t, \rho e^{i \theta}\right)=$ $\left(e^{2 \pi i t}, \rho e^{i n \theta}\right)$. Then $b$ and $c$ induce involutions $b_{1}$ and $c_{1}$ respectively on $V(n)$, where $b_{1}(u, v)=(\bar{u}, \bar{v})$ and $c_{1}(u, v)=(-\bar{u}, v)$. Now $V(n) /\left\langle b_{1}\right\rangle=(B 0, n)$; and $c_{1}$ induces an orientation reversing involution $c_{2}$ on $(B 0, n)$. The quotient space $(B 0, n) /\left\langle c_{2}\right\rangle$ is the orbifold $(B 1, n)$. The orbifold fundamental group of $(B 1, n)$ is

$$
\begin{aligned}
\pi_{1}((B 1, n)) & =\left\langle a, b, c \mid a^{n}=b^{2}=c^{2}=1, b a b^{-1}=a^{-1}, c a c^{-1}=a\right\rangle \\
& =\langle a\rangle \circ(\langle b\rangle *\langle c\rangle)=\mathbb{Z}_{n} \circ\left(\mathbb{Z}_{2} * \mathbb{Z}_{2}\right) \text { and } \\
\pi_{1}(\partial(B 1, n)) & =\left\langle a, b, c \mid b^{2}=c^{2}=1, b a b^{-1}=a^{-1}, c a c^{-1}=a\right\rangle \\
& =\langle a\rangle \circ(\langle b\rangle *\langle c\rangle)=\mathbb{Z} \circ\left(\mathbb{Z}_{2} * \mathbb{Z}_{2}\right) .
\end{aligned}
$$

Orbifold $(B 8, m)$ : Define orientation reversing maps $x, y$ and $z$ on $\mathbb{R} \times D^{2}$ by $x(t, v)=\left(-t, v e^{\frac{\pi i}{m}}\right), y(t, v)=(t, \bar{v})$ and $z(t, v)=\left(1-t, v e^{\frac{\pi i}{m}}\right)$. Now $\mathbb{R} \times$ $D^{2}$ is the universal covering of $(B 8, m)$. Note that $x z^{-1}(t, v)=(t-1, v)$. Let $p_{1}: \mathbb{R} \times D^{2} \rightarrow \mathbb{R} \times D^{2} /\left\langle x z^{-1}, z^{2}\right\rangle=V(m)$ be defined by $p_{1}\left(t, \rho e^{i \theta}\right)=$
( $\left.e^{2 \pi i t}, \rho e^{i m \theta}\right)$. We obtain induced maps $x_{1}$ and $y_{1}$ on $V(m)$ defined as follows: $x_{1}(u, v)=(\bar{u},-v)$ and $y_{1}(u, v)=(u, \bar{v})$. Since $x_{1} y_{1}(u, v)=(\bar{u},-\bar{v})$, this implies $V(m) /\left\langle x_{1} y_{1}\right\rangle=(B 0, m)$. Furthermore, if $y_{2}$ is the induced map on $(B 0, m)$, then $y_{2}$ is a reflection through a disk that does not contain the cone points of order 2 in the boundary. The orbifold $(B 8, m)=(B 0, m) /\left\langle y_{2}\right\rangle$. The orbifold fundamental group of $(B 8, m)$ is

$$
\begin{aligned}
\pi_{1}((B 8, m))= & \langle x, y, z| x^{2 m}=y^{2}=z^{2 m}=1, y x y^{-1}=x^{-1} \\
& \left.y z y^{-1}=z^{-1}, x^{2}=z^{2}\right\rangle \\
= & \left\langle x z^{-1}\right\rangle \circ\left(\langle x\rangle \circ{ }_{-1}\langle y\rangle\right)=\mathbb{Z} \circ\left(\mathbb{Z}_{2 m} \circ_{-1} \mathbb{Z}_{2}\right) \text { and } \\
\pi_{1}(\partial(B 8, m))= & \left\langle x, y, z \mid y^{2}=1, y x y^{-1}=x^{-1}, y z y^{-1}=z^{-1}, x^{2}=z^{2}\right\rangle \\
= & \left\langle x z^{-1}\right\rangle \circ\left(\langle x\rangle \circ{ }_{-1}\langle y\rangle\right)=\mathbb{Z} \circ\left(\mathbb{Z} \circ{ }_{-1} \mathbb{Z}_{2}\right) \\
= & \left\langle x z^{-1}\right\rangle \circ(\langle x y\rangle *\langle y\rangle)=\mathbb{Z} \circ\left(\mathbb{Z}_{2} * \mathbb{Z}_{2}\right) .
\end{aligned}
$$

Orbifold $O_{h_{7}}((B 1, n),(B 8, m))$ : As above we obtain covering maps $\mathbb{R} \times D^{2} \rightarrow$ $\overline{\mathbb{R}} \times D^{2} /\left\langle(c b)^{2}, a\right\rangle=V(n)$ and $\mathbb{R} \times D^{2} \rightarrow \mathbb{R} \times D^{2} /\left\langle x z^{-1}, z^{2}\right\rangle=V(m)$. The induced maps on $V(n)$ are $a_{1}, b_{1}$ and $c_{1}$, and the induced maps on $V(m)$ are $x_{1}$ and $y_{1}$.

Define a homeomorphism $\tilde{h}_{7}: V(n) \rightarrow V(m)$ by $\tilde{h}_{7}(u, v)=(v,-i u)$. The map $\tilde{h}_{7}$ induced maps $\widehat{h}_{7}$ and $h_{7}$ making the following diagram commute:


When we identify $\partial(B 1, n)$ to $\partial(B 8, m)$ via $h_{7}$, we obtain the orbifold $O_{h_{7}}((B 1, n),(B 8, m))$. By [13] the generators are identified by $a=\left(x z^{-1}\right)^{-1}=$ $z x^{-1}, b=y x$ and $c=y$ and the fundamental group is

$$
\begin{aligned}
\pi_{1}\left(O_{h}((B 1, n),(B 8, m))\right) & =\langle a, b, c| a^{n}=b^{2}=c^{2}=1, b a b^{-1}=a^{-1}, \\
& \left.\quad[a, c]=1,(c b)^{2 m}=1\right\rangle \\
& =\langle a\rangle \circ\left(\langle b\rangle *\langle c\rangle /\left\langle(c b)^{2 m}\right\rangle\right) \\
& =\langle a\rangle \circ\left(\langle c b\rangle \circ_{-1}\langle c\rangle /\left\langle(c b)^{2 m}\right\rangle\right) \\
& =\mathbb{Z}_{n} \circ \operatorname{Dih}\left(\mathbb{Z}_{2 m}\right) .
\end{aligned}
$$

The elements $c$ is orientation reversing.

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# A note about the well-posedness of an Initial Boundary Value Problem for the heat equation in a layered domain 

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#### Abstract

Heat conduction in a layered domain with imperfect thermal contact interfaces is modeled by means of a system of elliptic or parabolic PDEs with suitable boundary and transmission conditions. Well-posedness of this problem is proved and a stability estimate of the solution is given.


Keywords: Heat equation, composite domain.
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## 1. Introduction

This note deals with heat conduction in a section of the layered body $C$. Assume that two or more layers are separated by low conductivity imperfect interfaces. According to the classification in [8], it means that a temperature jump is present between two adjacent layers while the heat flux is continuous.

Suppose that $C$ is made by two slabs $C^{+}$and $C^{-}$, with different thermal conductivities, separated by a very thin low conductivity imperfect interface $D^{\epsilon}(\epsilon>0$ represents a characteristic thickness of the interface). The limit process in which the thin "solid" interface $D^{\epsilon}$ shrinks to a two-dimensional set $D^{0}$, is widely studied in mathematical physics (see for example [6, 11]). The thermal properties of $D^{\epsilon}$ for $\epsilon \rightarrow 0$, are summarized in a parameter function $\tilde{h}: D^{0} \rightarrow[0, \infty)$ called thermal contact conductance. The thermal contact conductance of the interface $D^{0}$ is a non-negative quantity related to the average of surface roughness effects in real objects. A detailed numerical modeling of roughness can be found in [10].

Consider the ideal framework in which the slabs are parallelepipeds. In particular, we focus on the intersection $\Omega$ between $C$ and a plane $\pi$ orthogonal to the interface, so that $S^{0}=D^{0} \cap \pi, \Omega^{+}=C^{+} \cap \pi$ and $\Omega^{-}=C^{-} \cap \pi$.

In mathematical terms, the temperature of a two-dimensional layered object like $\Omega$ is the solution of a system of two Initial Boundary Value Problems (IBVPs) for the heat equation coupled by means of suitable transmission conditions in which the function $h$ (restriction of $\tilde{h}$ to $S^{0}$ ) plays the role of heat
transfer coefficient. This system of IBVPs is the direct model underlying the inverse problem of identifying $h$ from some additional data taken on the external boundary of the specimen (see for example $[3,7]$ ). Existence and uniqueness of its solutions are proved in Theorem 3.1 and it supplements mathematical background in [7]. Though many results in applied sciences and engineering rely on this mathematical model (see for example $[2,3,4,7]$ to cite recent items), its well-posedness is always taken for granted and, consequently, a rigorous proof is bypassed. Here, a technique described in a recent paper about discontinuous Galerkin methods [1] is extended (from elliptic to parabolic; from layers of the same material to different materials) to prove existence and uniqueness of a weak solution of our system of IBVPs (see Section 3.2). A stability estimate is also derived in order to evaluate the sensitivity of the solution with respect to variations in the parameter $h$ (see Section 3.3).

## 2. Geometry of the specimen. Notation.

We deal with the composite environment

$$
\Omega=\Omega^{+} \cup \Omega^{-} \cup S^{0}
$$

where

$$
\Omega^{+}=(-L, L) \times\left(0, a_{+}\right), \quad \Omega^{-}=(-L, L) \times\left(-a_{-}, 0\right)
$$

The interface

$$
S^{0}=\{(x, y): y=0 \text { and } x \in(-L, L)\}
$$

opposes to heat transfer between $\Omega^{+}$and $\Omega^{-}$.
Let $(0, T)$ be a "time interval" so that

$$
Q^{+}=\Omega^{+} \times(0, T), \quad Q^{-}=\Omega^{-} \times(0, T)
$$

The thermal behavior of layers $\Omega^{+}$and $\Omega^{-}$is determined by their conductivity ( $\kappa_{+}$and $\kappa_{-}$), density ( $\rho_{+}$and $\rho_{-}$) and specific heat ( $c_{+}$and $c_{-}$). The numbers $\alpha_{ \pm}=\frac{\kappa_{ \pm}}{\rho_{ \pm} c_{ \pm}}$are the corresponding diffusivities.

The top boundary of $\Omega^{+}$is $S^{+}=\left\{(x, y): y=a_{+}\right.$and $\left.x \in(-L, L)\right\}$.
The bottom boundary of $\Omega^{-}$is $S^{-}=\left\{(x, y): y=-a_{-}\right.$and $\left.x \in(-L, L)\right\}$.
We assume that the thermal contact conductance of $S^{0}$ takes the form $h(x)=h_{0}+h_{1}(x, t)$ where $h_{0}$ is a positive real constant and $h_{1}$ is a non-negative function of class $C^{0}([-L, L] \times[0, T])$. Two examples in which $h_{1}$ describes, respectively, the deterioration of an insulating interface and the worsening of performances of an heat exchanger, are studied in [7].

In what follows, if $u$ is a real function of two or more real variables, " $u_{q}$ " means $\frac{\partial u}{\partial q}$ and " $u(q \pm)$ " means $\lim _{\epsilon \rightarrow 0^{+}} u(q \pm \epsilon)$.

## 3. Temperature of a two-layered domain with low conductivity interface

Assume that $\Omega$ is heated through $S^{+}$. The incoming heat flow is described by a function $\Phi \in C^{0}([-L, L] \times[0, T])$. The temperature of $\Omega$ is determined by solving a system of two IBVPs for the heat equation, respectively in $Q^{+}$and $Q^{-}$ connected by means of a set of transmission conditions through the interface (see Section 3.1). Exchange of heat between $\Omega$ and the external environment occurs through $S^{+}$and $S^{-}$and it is modeled by means of Robin conditions with (positive) constant coefficients $h_{+}$and $h_{-}$respectively. Vertical sides $x=-L$ and $x=L$ are assumed, for simplicity, thermally insulated. A temperature $U^{M} \geq 0$ is assumed for a fluid exchanging heat with $S^{+}$. The temperature of a fluid exchanging with $S^{-}$can be taken equal to zero without loss of generality. Initial temperature is given by the pair of functions $U_{+}^{0} \in C^{0}\left(\overline{\Omega^{+}}\right)$and $U_{-}^{0} \in$ $C^{0}\left(\overline{\Omega^{-}}\right)$(overbar means topological closure).

Analytical solutions (definitely not trivial) are known when the problem is one-dimensional, i.e. when $U_{+}^{0}, U_{-}^{0}, h_{1}$ and $\Phi$ are non-negative constants, also in presence of more than two layers [14]. If $\Omega^{+}$and $\Omega^{-}$are made of the same material and $a_{+}=a_{-}$, the system can be easily reduced to a single problem in $Q^{+}$(or alternatively in $Q^{-}$) using the method of images [7].

### 3.1. A system of IBVPs for the heat equation

Since a single function from $\bar{\Omega}$ to $(0, \infty)$ is not suitable for representing the temperature of our specimen because it assumes two different values on $S^{0}$, we introduce the pair of functions

$$
u^{+}: Q^{+} \rightarrow(0, \infty), \quad u^{-}: Q^{-} \rightarrow(0, \infty)
$$

with their extension to respective boundaries. The pair $\left(u^{+}, u^{-}\right)$must fulfill the following system of IBVPs

$$
\begin{array}{ll}
\rho_{+} c_{+} u_{t}^{+}=\kappa_{+} \Delta u^{+}, & (x, y, t) \in Q^{+} \\
u^{+}(x, y, 0)=U_{+}^{0}(x, y), & (x, y) \in \Omega^{+} \\
\kappa_{+} u_{\nu}^{+}\left(x, a_{+}, t\right)+h_{+}\left(u^{+}\left(x, a_{+}, t\right)-U^{M}\right)=\Phi(x, t), & x \in(-L, L), t \in(0, T), \\
\kappa_{+} u_{\nu}^{+}(-L, y, t)=\kappa_{+} u_{\nu}^{+}(L, y, t)=0, & y \in\left(0, a_{+}\right), t \in(0, T), \tag{1}
\end{array}
$$

and

$$
\begin{array}{ll}
\rho_{-} c_{-} u_{t}^{-}=\kappa_{-} \Delta u^{-}, & (x, y, t) \in Q^{-}, \\
u^{-}(x, y, 0)=U_{-}^{0}(x, y), & (x, y) \in \Omega^{-}, \\
\kappa_{-} u_{\nu}^{-}\left(x,-a_{-}, t\right)+h_{-} u^{-}\left(x,-a_{-}, t\right)=0, & x \in(-L, L), t \in(0, T),  \tag{2}\\
\kappa_{-} u_{\nu}^{-}(-L, y, t)=\kappa_{-} u_{\nu}^{-}(L, y, t)=0, & y \in\left(-a_{-}, 0\right), t \in(0, T),
\end{array}
$$

coupled by means of the transmission conditions

$$
\begin{align*}
& \kappa_{+} u_{\nu}^{+}(x, 0+, t)+h(x, t)\left(u^{+}(x, 0+, t)-u^{-}(x, 0-, t)\right)=0 \\
& x \in(-L, L), t \in(0, T),  \tag{3}\\
& \kappa_{+} u_{\nu}^{+}(x, 0+, t)=-\kappa_{-} u_{\nu}^{-}(x, 0-, t), x \in(-L, L), t \in(0, T) .
\end{align*}
$$

### 3.2. Main result: Existence and uniqueness of $\left(u^{+}, u^{-}\right)$, weak solution of (1)-(3)

As for notation and basic theory of Sobolev spaces we refer to [13, Chapter 7]. In particular, we deal also with spaces involving time (see [13, Section 7.11.2]). Let $H$ be a Hilbert space equipped with the norm $\|\cdot\|_{H}$ and let $H^{*}$ denote its dual space:

$$
\begin{aligned}
& L^{2}(0, T ; H)=\left\{u:(0, T) \rightarrow H: u(t) \text { measurable and } \int_{0}^{T}\|u(t)\|_{H}^{2} d t<\infty\right\} \\
& C^{0}([0, T] ; H)=\left\{u:[0, T] \rightarrow H: u(t) \text { continuous and } \max _{[0, T]}\|u(t)\|_{H}<\infty\right\}
\end{aligned}
$$

## Theorem 3.1. Suppose that:

(i) $h$ takes the form $h_{0}+h_{1}$ where $h_{0}$ is a positive real constant and $h_{1}$ is a non-negative function of class $C^{0}([-L, L] \times[0, T])$;
(ii) $U_{+}^{0} \in C^{0}\left(\overline{\Omega^{+}}\right), U_{+}^{0} \in C^{0}\left(\overline{\Omega^{-}}\right)$and $\Phi \in C^{0}([-L, L] \times[0, T])$.

Then:
(I) a weak solution ( $u^{+}, u^{-}$) of problem (1)-(3) exists and it is unique, with $u^{+} \in L^{2}\left(0, T, H^{1}\left(\Omega^{+}\right)\right) \cap C^{0}\left([0, T] ; L^{2}\left(\Omega^{+}\right)\right)$and $u^{-} \in L^{2}\left(0, T, H^{1}\left(\Omega^{-}\right)\right) \cap$ $C^{0}\left([0, T] ; L^{2}\left(\Omega^{-}\right)\right) ;$
(II) $u_{t}^{+} \in L^{2}\left(0, T, H^{1}\left(\Omega^{+}\right)^{*}\right)$ and $u_{t}^{-} \in L^{2}\left(0, T, H^{1}\left(\Omega^{-}\right)^{*}\right)$;
(III) the energy estimate

$$
\begin{align*}
& \rho_{+} c_{+}\left\|u^{+}(t)\right\|_{0}^{2}+\rho_{-} c_{-}\left\|u^{-}(t)\right\|_{0}^{2}+\kappa_{+} \int_{0}^{t}\left\|u^{+}(\tau)\right\|_{1}^{2} d \tau+\kappa_{-} \int_{0}^{t}\left\|u^{-}(\tau)\right\|_{1}^{2} d \tau \\
& \leq e^{2 \frac{\alpha_{+}}{L^{2}} t}\left(\rho_{+} c_{+}\left\|U_{+}^{0}\right\|_{0}^{2}+\rho_{-} c_{-}\left\|U_{-}^{0}\right\|_{0}^{2}\right) \\
& \quad+C\left(\rho_{+}, c_{+}, \kappa_{+}, \Omega^{+}, t\right) \max _{\tau \in[0, t]}\left\{\int_{-L}^{L}\left(U^{M} h_{+}+\Phi(x, \tau)\right)^{2} d x\right\} \tag{4}
\end{align*}
$$

holds for almost all $t \in[0, T]$.

Proof. Step 1. A variational problem in a product Hilbert space.
It is convenient to write problem (1)-(3) in weak form. More precisely, for almost all $t \in[0, T]$, we must find $u^{+}(t)$ in $H^{1}\left(\Omega^{+}\right)$and $u^{-}(t)$ in $H^{1}\left(\Omega^{-}\right)$such that

$$
\begin{gather*}
\rho_{+} c_{+} \int_{\Omega^{+}} u_{t}^{+}(t) v^{+} d x d y+\kappa_{+} \int_{\Omega^{+}} \nabla u^{+}(t) \nabla v^{+} d x d y+h_{+} \int_{-L}^{L} u^{+}(t) v^{+} d x \\
+\int_{-L}^{L} h(x, t)\left(u^{+}(t)-u^{-}(t)\right) v^{+} d x=\int_{-L}^{L}\left(h_{+} U^{M}+\Phi(x, t)\right) v^{+} d x  \tag{5}\\
\rho_{-} c_{-} \int_{\Omega^{-}} u_{t}^{-}(t) v^{-} d x d y+\kappa_{-} \int_{\Omega^{-}} \nabla u^{-}(t) \nabla v^{-} d x d y+h_{-} \int_{-L}^{L} u^{-}(t) v^{-} d x \\
+\int_{-L}^{L} h(x, t)\left(u^{-}(t)-u^{+}(t)\right) v^{-} d x=0 \tag{6}
\end{gather*}
$$

for all $v^{+}$in $H^{1}\left(\Omega^{+}\right)$and $v^{-}$in $H^{1}\left(\Omega^{-}\right)$.
Following [1], we define the (cartesian product) Hilbert space $V=H^{1}\left(\Omega^{+}\right) \times$ $H^{1}\left(\Omega^{-}\right)$equipped with the scalar product

$$
\begin{aligned}
(u, v)_{V}:=\int_{\Omega^{+}} & u^{+} v^{+} d x d y+\int_{\Omega^{-}} u^{-} v^{-} d x d y+L^{2} \int_{\Omega^{+}} \nabla u^{+} \nabla v^{+} d x d y \\
& +L^{2} \int_{\Omega^{-}} \nabla u^{-} \nabla v^{-} d x d y
\end{aligned}
$$

The scale factor $L^{2}$ is required for dimensional reasons. Clearly $u=\left(u^{+}, u^{-}\right)$ and $v=\left(v^{+}, v^{-}\right)$are in $V$ while for all $w \in V$ the norm $\|w\|_{V}:=\sqrt{(w, w)_{V}}$ is defined.

We sum (5) and (6) and obtain the variational problem: for almost all $t \in[0, T]$, find $u(t) \in V$ such that

$$
\begin{aligned}
& \left\langle u_{t}(t), v\right\rangle+a(u(t), v)=\int_{-L}^{L}\left(h_{+} U^{m}+\Phi(x, t)\right) v^{+}(x, a) d x \\
& \quad \text { for all } v \in V \text { with } u^{ \pm}(0)=U_{ \pm}^{0} .
\end{aligned}
$$

Here,

$$
\left\langle u_{t}(t), v\right\rangle=\rho_{+} c_{+} \int_{\Omega^{+}} u_{t}^{+}(t) v^{+} d x d y+\rho_{-} c_{-} \int_{\Omega^{-}} u_{t}^{-}(t) v^{-} d x d y
$$

denotes a suitably weighted duality pairing between $V^{*}$ and $V$ while

$$
\begin{aligned}
& a(u(t), v)= \kappa_{+} \\
& \int_{\Omega^{+}} \nabla u^{+}(t) \nabla v^{+} d x d y+\kappa_{-} \int_{\Omega^{-}} \nabla u^{-}(t) \nabla v^{-} d x d y \\
&+ h_{+} \int_{-L}^{L} u^{+}(t) v^{+} d x+h_{-} \int_{-L}^{L} u^{-}(t) v^{-} d x \\
&+\int_{-L}^{L} h(x, t)\left(u^{+}(t)-u^{-}(t)\right)\left(v^{+}-v^{-}\right) d x
\end{aligned}
$$

is a bilinear form on $V \times V$.
Step 2. Existence and uniqueness of the solution.
We recall that, if $w=\left(w^{+}, w^{-}\right) \in V$, the trace inequality (see [5, Theorem 1.5.1.10])

$$
\begin{equation*}
\int_{S^{ \pm} \cup S^{0}}\left|w^{ \pm}\right|^{2} \leq c\left(\Omega^{ \pm}\right)\|w\|_{V}^{2} \tag{7}
\end{equation*}
$$

holds. It follows from the constructive proof in [5] that $c\left(\Omega^{ \pm}\right)<2\left(1+\frac{3}{a_{ \pm}}\right)$(not optimal). Continuity of the bilinear form $a$ follows from Schwarz inequality and (7). Indeed, we have

$$
\begin{equation*}
|a(u(t), v)| \leq K\|u(t)\|_{V}\|v\|_{V} \tag{8}
\end{equation*}
$$

where $K=\max \left\{\kappa_{ \pm}\right\}+\max \left\{h_{+}, h_{-}, \max _{[-L, L] \times[0, T]} h\right\} \max \left\{c\left(\Omega^{ \pm}\right)\right\}$.
Since $h_{+}, h_{-}$and $\min _{[-L, L] \times[0, T]} h$ are positive, there are two positive constants $\lambda=\max \left\{\kappa_{+}, \kappa_{-}\right\}$and $\gamma=\min \left\{\kappa_{+}, \kappa_{-}\right\}$such that

$$
a(u(t), u(t))+\lambda\|u(t)\|_{0}^{2} \geq \gamma\|u(t)\|_{V}^{2}
$$

i.e. the bilinear form $a$ is weakly coercive (see [12, Section 11.1.1]). Hence, existence of a solution $u \in L^{2}(0, T, V) \cap C^{0}([0, T] ; V)$ of (1)-(3) and its uniqueness follow from [12, Theorem 11.1.1] (see also [9, Theorem 5.3]). Energy estimate (4) is derived straightforwardly following [12] .

Remark 3.2. The energy estimate does not account for heat exchange through the boundaries. In the special case in which $h_{+}=h_{-}=0$, it is likely to be optimal when the interface is insulating $(h=0)$ or highly conductive ( $u^{+}-$ $\left.u^{-}\right)^{2} \approx 0$.
Remark 3.3. Well posedness can be proved also in the stationary case in which the temperature $u$ solves a system of BVPs for the Laplace equation. If we assume Dirichlet conditions on $S^{+}$and $S^{-}$instead of Robin ones, we can use the same procedure of this section. Observe that, in the stationary case, the required coercivity of the bilinear form $a$ follows from the Poincaré inequality (see [12, Section 1.3]).

### 3.3. Stability of the solution with respect to the parameters

The same technique used in deriving the energy estimate leads to the evaluation of the sensitivity of the solution with respect to the interface conductance $h$.

Theorem 3.4. Let $\tilde{u}$ denote the unique solution of (1)-(3) when the conductance in $S^{0}$ is $h+\delta h$ (with $h+\delta h \geq h_{0}$ ) and set $\delta u:=\tilde{u}-u$. Assume that all other parameters $\left(\alpha_{ \pm}, \kappa_{ \pm}, h_{ \pm}, \Phi\right)$ remain unchanged. We have the local stability estimate:

$$
\frac{\|\delta u\|_{L^{2}([0, T], V)}}{\|u\|_{L^{2}([0, T], V)}} \leq K \max _{[-L, L] \times[0, T]}|\delta h| .
$$

Proof. Subtract

$$
\left\langle u_{t}(t), \delta u\right\rangle+a_{h}(u(t), \delta u)=\int_{-L}^{L}\left(h_{+} U^{m}+\Phi(x, t)\right) \delta u^{+}(x, a) d x, \quad u^{ \pm}(0)=U_{ \pm}^{0}
$$

from

$$
\left\langle\tilde{u}_{t}(t), \delta u\right\rangle+a_{h+\delta h}(\tilde{u}(t), \delta u)=\int_{-L}^{L}\left(h_{+} U^{m}+\Phi(x, t)\right) \delta u^{+}(x, a) d x, \quad \tilde{u}^{ \pm}(0)=U_{ \pm}^{0}
$$

where we have stressed the dependence of the bilinear form on conductance $h$ at the interface. We have

$$
\left\langle\delta u_{t}(t), \delta u\right\rangle+a_{h+\delta h}(u(t)+\delta u, \delta u)-a_{h}(u(t), \delta u)=0, \quad \delta u^{ \pm}(0)=0
$$

Using the change of variable $\delta w^{+}=e^{-\beta_{+} t} \delta u^{+}$and $\delta w^{-}=e^{-\beta_{-} t} \delta u^{-}$with $\beta_{ \pm}=\frac{\alpha_{ \pm}}{L^{2}}$, we have

$$
\begin{gathered}
\left\langle\tilde{u}_{t}(t), \delta u\right\rangle=e^{2 \beta_{+} t} \frac{\rho_{+} c_{+}}{2} \frac{d}{d t}\left\|\delta w^{+}\right\|_{0}^{2}+e^{2 \beta_{-} t} \frac{\rho_{-} c_{-}}{2} \frac{d}{d t}\left\|\delta w^{-}\right\|_{0}^{2} \\
+e^{2 \beta_{+} t} \kappa_{+}\left\|\delta w^{+}\right\|_{0}^{2}+e^{2 \beta_{-} t} \kappa_{-}\left\|\delta w^{-}\right\|_{0}^{2} .
\end{gathered}
$$

Since

$$
\begin{aligned}
& a_{h+\delta h}(u(t)+\delta u, \delta u)-a_{h}(u(t), \delta u)=\kappa_{+} e^{2 \beta_{+} t} \int_{\Omega^{+}}\left|\nabla \delta w^{+}(t)\right|^{2} d x d y \\
&+ e^{2 \beta-t} \kappa_{-} \int_{\Omega^{-}}\left|\nabla \delta w^{-}(t)\right|^{2} d x d y+e^{2 \beta_{+} t} h_{+} \int_{-L}^{L} \delta w^{+}(t)^{2} d x \\
&+e^{2 \beta_{-} t} h_{-} \int_{-L}^{L} \delta w^{-}(t)^{2} d x+\int_{-L}^{L}(h+\delta h)\left(e^{\beta_{+} t} \delta w^{+}-e^{\beta_{-} t} \delta w^{-}\right)^{2} d x \\
& \quad+\int_{-L}^{L} \delta h\left(e^{\beta_{+} t} w^{+}(t)-e^{\beta_{-} t} w^{-}(t)\right)\left(e^{\beta_{+} t} \delta w^{+}-e^{\beta-t} \delta w^{-}\right) d x
\end{aligned}
$$

we have

$$
\begin{align*}
& e^{2 \beta_{+} t} \frac{\rho_{+} c_{+}}{2} \frac{d}{d t}\left\|\delta w^{+}\right\|_{0}^{2}+e^{2 \beta_{-} t} \frac{\rho_{-} c_{-}}{2} \frac{d}{d t}\left\|\delta w^{-}\right\|_{0}^{2} \\
& \quad+e^{2 \beta_{+} t} \kappa_{+}\left\|\delta w^{+}\right\|_{0}^{2}+e^{2 \beta_{-} t} \kappa_{-}\left\|\delta w^{-}\right\|_{0}^{2}+\kappa_{+} e^{2 \beta_{+} t}\left\|\nabla \delta w^{+}(t)\right\|_{0}^{2} \\
& \quad+e^{2 \beta-t} \kappa_{-}\left\|\nabla \delta w^{-}(t)\right\|_{0}^{2}+e^{2 \beta_{+} t} h_{+} \int_{-L}^{L} \delta w^{+}(t)^{2} d x  \tag{9}\\
& \quad+e^{2 \beta_{-} t} h_{-} \int_{-L}^{L} \delta w^{-}(t)^{2} d x+\int_{-L}^{L}(h+\delta h)\left(e^{\beta_{+} t} \delta w^{+}-e^{\beta-t} \delta w^{-}\right)^{2} d x \\
& \quad+\int_{-L}^{L} \delta h\left(e^{\beta_{+} t} w^{+}(t)-e^{\beta-t} w^{-}(t)\right)\left(e^{\beta_{+} t} \delta w^{+}-e^{\beta-t} \delta w^{-}\right) d x=0 .
\end{align*}
$$

Set $\beta_{m}:=\min \left\{\beta_{-}, \beta_{+}\right\}, \beta_{M}:=\max \left\{\beta_{-}, \beta_{+}\right\}$and $\kappa_{m}=\min \left\{\kappa_{-}, \kappa_{+}\right\}$and evaluate

$$
\begin{aligned}
& \int_{-L}^{L}|\delta h|\left|\left(u^{+}(t)-u^{-}(t)\right)\left(\delta u^{+}-\delta u^{-}\right)\right| d x \\
& \leq \int_{-L}^{L}|\delta h|\left(\left|u^{+}(t)\right|\left|\delta u^{+}\right|+\left|u ^ { + } ( t ) \left\|\delta u ^ { - } \left|+\left|u ^ { - } ( t ) \left\|\delta u ^ { + } \left|+\left|u^{-}(t) \| \delta u^{-}\right| d x\right.\right.\right.\right.\right.\right.\right. \\
& \leq \max _{[-L, L] \times[0, T]}|\delta h|\left(c\left(\Omega^{+}\right)\left\|u^{+}(t)\right\|_{1}\left\|\delta u^{+}\right\|_{1}+\sqrt{c\left(\Omega^{+}\right) c\left(\Omega^{-}\right)}\left\|u^{+}(t)\right\|\left\|_{1}\right\| \delta u^{-} \|_{1}\right. \\
&\left.\quad+\sqrt{c\left(\Omega^{+}\right) c\left(\Omega^{-}\right)} \mid u^{-}(t)\left\|_{1}\right\| \delta u^{+}\left\|_{1}+c\left(\Omega^{-}\right)\right\| u^{-}(t)\left\|_{1}\right\| \delta u^{-} \|_{1}\right) \\
& \leq \max _{[-L, L] \times[0, T]}|\delta h| e^{\beta_{M} t} \max \left\{c\left(\Omega^{+}\right), c\left(\Omega^{-}\right)\right\}\|\delta w\|_{1}\|w\|_{1} .
\end{aligned}
$$

By disregarding the third line in (9), which is made of positive terms, we obtain

$$
\begin{aligned}
\frac{1}{2} \frac{d}{d t} & \langle\delta w(t), \delta w(t)\rangle+\kappa_{m}\|w\|_{1}^{2} \\
& \leq \max _{[-L, L] \times[0, T]}|\delta h| e^{\left(\beta_{M}-\beta_{m}\right) T} \max \left\{c\left(\Omega^{+}\right), c\left(\Omega^{-}\right)\right\}\|\delta w\|_{1}\|w\|_{1} .
\end{aligned}
$$

Integrating on $t$ both sides of the inequality, we get

$$
\begin{aligned}
\frac{1}{2}\|\delta u\|_{0}^{2}+\kappa_{m} & \int_{0}^{T}\|\delta u\|_{1}^{2} d t \\
& \leq \max _{[-L, L] \times[0, T]}|\delta h| e^{2 \beta_{M} T} \max \left\{c\left(\Omega^{+}\right), c\left(\Omega^{-}\right)\right\} \int_{0}^{T}\|\delta u\|_{1}\|u\|_{1} d t
\end{aligned}
$$

Applying the Schwarz inequality to the integral on the right hand side we have

$$
\sqrt{\frac{\int_{0}^{T}\|\delta u\|_{1}^{2} d t}{\int_{0}^{T}\|u\|_{1}^{2} d t}} \leq K \max _{[-L, L] \times[0, T]}|\delta h|
$$

where $K=\frac{e^{2 \beta_{M} T}}{\kappa_{m}} \max \left\{c\left(\Omega^{+}\right), c\left(\Omega^{-}\right)\right\}$.

## Conclusions

We have proved that a system of parabolic equations that model heat conduction in a layered domain is a well-posed problem under very natural hypotheses. The proof comes from the weak formulation of the problem in a suitable product Hilbert space. This result helps with the construction of rigorous foundations of the inverse problem studied in $[3,7]$.

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# Unbounded generalizations of the Fuglede-Putnam theorem 

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#### Abstract

In this note, we prove and disprove several generalizations of unbounded versions of the Fuglede-Putnam theorem.


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## 1. Essential background

All operators considered here are linear but not necessarily bounded. If an operator is bounded and everywhere defined, then it belongs to $B(H)$ which is the algebra of all bounded linear operators on $H$ (see [14] for its fundamental properties).

Most unbounded operators that we encounter are defined on a subspace (called domain) of a Hilbert space. If the domain is dense, then we say that the operator is densely defined. In such case, the adjoint exists and is unique.

Let us recall a few basic definitions about non-necessarily bounded operators. If $S$ and $T$ are two linear operators with domains $D(S)$ and $D(T)$ respectively, then $T$ is said to be an extension of $S$, written as $S \subset T$, if $D(S) \subset D(T)$ and $S$ and $T$ coincide on $D(S)$.

An operator $T$ is called closed if its graph is closed in $H \oplus H$. It is called closable if it has a closed extension. The smallest closed extension of it is called its closure and it is denoted by $\bar{T}$ (a standard result states that a densely defined $T$ is closable iff $T^{*}$ has a dense domain, and in which case $\bar{T}=T^{* *}$ ). If $T$ is closable, then

$$
S \subset T \Rightarrow \bar{S} \subset \bar{T}
$$

If $T$ is densely defined, we say that $T$ is self-adjoint when $T=T^{*}$; symmetric if $T \subset T^{*}$; normal if $T$ is closed and $T T^{*}=T^{*} T$.

The product $S T$ and the sum $S+T$ of two operators $S$ and $T$ are defined in the usual fashion on the natural domains:

$$
D(S T)=\{x \in D(T): T x \in D(S)\}
$$

and

$$
D(S+T)=D(S) \cap D(T)
$$

In the event that $S, T$ and $S T$ are densely defined, then

$$
T^{*} S^{*} \subset(S T)^{*},
$$

with the equality occurring when $S \in B(H)$. If $S+T$ is densely defined, then

$$
S^{*}+T^{*} \subset(S+T)^{*}
$$

with the equality occurring when $S \in B(H)$.
Let $T$ be a linear operator (possibly unbounded) with domain $D(T)$ and let $B \in B(H)$. Say that $B$ commutes with $T$ if

$$
B T \subset T B
$$

In other words, this means that $D(T) \subset D(T B)$ and

$$
B T x=T B x, \forall x \in D(T)
$$

Let $A$ be an injective operator (not necessarily bounded) from $D(A)$ into $H$. Then $A^{-1}: \operatorname{ran}(A) \rightarrow H$ is called the inverse of $A$, with $D\left(A^{-1}\right)=\operatorname{ran}(A)$.

If the inverse of an unbounded operator is bounded and everywhere defined (e.g. if $A: D(A) \rightarrow H$ is closed and bijective), then $A$ is said to be boundedly invertible. In other words, such is the case if there is a $B \in B(H)$ such that

$$
A B=I \text { and } B A \subset I
$$

If $A$ is boundedly invertible, then it is closed.
The resolvent set of $A$, denoted by $\rho(A)$, is defined by

$$
\rho(A)=\left\{\lambda \in \mathbb{C}: \lambda I-A \text { is bijective and }(\lambda I-A)^{-1} \in B(H)\right\} .
$$

The complement of $\rho(A)$, denoted by $\sigma(A)$,

$$
\sigma(A)=\mathbb{C} \backslash \rho(A)
$$

is called the spectrum of $A$.

## 2. Introduction

The aim of this paper is to obtain some generalizations of the Fuglede-Putnam theorem involving unbounded operators.

Recall that the original version of the Fuglede-Putnam theorem reads:

THEOREM 2.1 ([6, 20]). If $A \in B(H)$ and if $M$ and $N$ are normal (nonnecessarily bounded) operators, then

$$
A N \subset M A \Longrightarrow A N^{*} \subset M^{*} A
$$

The problem leading to the above theorem was first mooted by J. von Neumann in [18] who had already established it in a finite-dimensional setting. B. Fuglede was the first one to prove this theorem in [6] in the case $N=M$, and where $\operatorname{dim} H=\infty$ was allowed. It is important to tell readers that P. R. Halmos obtained in [8] almost simultaneously as B. Fuglede a quite different proof of the theorem above. More precisely, at the end of August 1949, B. Fuglede communicated his proof to P. R. Halmos at the Boulder meeting of the American Mathematical Society. Halmos' proof dealt with the all bounded version, however, P. R. Halmos indicated that only minor modifications were needed to adapt his proof to the more general case of unbounded operators.

Then, C. R. Putnam [20] proved the above version. S. K. Berberian [3] amazingly noted that the two versions were equivalent.

There are different proofs of the Fuglede-Putnam theorem. The most elegant proof perhaps is the one due to M. Rosenblum [23]. For other proofs, see e.g. [21] and [22].

There have been many generalizations of the Fuglede-Putnam theorem since Fuglede's paper. However, most generalizations were devoted to relaxing the normality assumption (see e.g. [12], and the references therein). Apparently, the first generalization of the Fuglede theorem to an unbounded $A$ was established in [19]. Then, the first generalization involving unbounded operators of the Fuglede-Putnam theorem is:

Theorem 2.2. Let $A$ be a closed symmetric operator and let $N$ be an unbounded normal operator. If $D(N) \subset D(A)$, then

$$
A N \subset N^{*} A \Longrightarrow A N^{*} \subset N A
$$

In fact, the previous result was established in [10] under the assumption of the self-adjointness of $A$. However, and by scrutinizing the proof in [10] or [11], it is seen that only the closedness and the symmetricity of $A$ were needed. Other unbounded generalizations may be consulted in [1], [2], and [13], as well as some of the references therein. In the end, readers may wish to consult the survey [16] exclusively devoted to the Fuglede-Putnam theorem and its applications.

## 3. Generalizations of the Fuglede-Putnam theorem

If a densely defined operator $N$ is normal, then so is its adjoint. However, if $N^{*}$ is normal, then $N^{* *}$ does not have to be normal (unless $N$ itself is closed). A
simple counterexample is to take the identity operator $I_{D}$ restricted to some unclosed dense domain $D \subset H$. Then $I_{D}$ cannot be normal for it is not closed. But, $\left(I_{D}\right)^{*}=I$ which is the full identity on the entire $H$, is obviously normal. Notice in the end that if $N$ is a densely defined closable operator, then $N^{*}$ is normal if and only if $\bar{N}$ is.

The first improvement is that in the very first version by B. Fuglede, the normality of the operator is not needed as only the normality of its closure will do. This observation has already appeared in [4], but we reproduce the proof here.

Theorem 3.1. Let $B \in B(H)$ and let $A$ be a densely defined and closable operator such that $\bar{A}$ is normal. If $B A \subset A B$, then

$$
B A^{*} \subset A^{*} B
$$

Proof. Since $\bar{A}$ is normal, $\bar{A}^{*}=A^{*}$ remains normal. Now,

$$
\begin{aligned}
B A \subset A B & \Longrightarrow B^{*} A^{*} \subset A^{*} B^{*}(\text { by taking adjoints }) \\
& \Longrightarrow B^{*} \bar{A} \subset \bar{A} B^{*}(\text { by using the classical Fuglede theorem }) \\
& \Longrightarrow B A^{*} \subset A^{*} B \text { (by taking adjoints again) }
\end{aligned}
$$

establishing the result.
Remark 3.2. Notice that $B A^{*} \subset A^{*} B$ does not yield $B A \subset A B$ even in the event of the normality of $A^{*}$ (see [15]).

Let us now turn to the extension of the Fuglede-Putnam version. A similar argument to the above one could be applied.

Theorem 3.3. Let $B \in B(H)$ and let $N, M$ be densely defined closable operators such that $\bar{N}$ and $\bar{M}$ are normal. If $B N \subset M B$, then

$$
B N^{*} \subset M^{*} B
$$

Proof. Since $B N \subset M B$, it ensues that $B^{*} M^{*} \subset N^{*} B^{*}$. Taking adjoints again gives $B \bar{N} \subset \bar{M} B$. Now, apply the Fuglede-Putnam theorem to the normal $\bar{N}$ and $\bar{M}$ to get the desired conclusion $B N^{*} \subset M^{*} B$.

Jabłoński et al. obtained in [9] the following version.
Theorem 3.4. If $N$ is a normal (bounded) operator and if $A$ is a closed densely defined operator with $\sigma(A) \neq \mathbb{C}$, then:

$$
N A \subset A N \Longrightarrow g(N) A \subset A g(N)
$$

for any bounded complex Borel function $g$ on $\sigma(N)$. In particular, we have $N^{*} A \subset A N^{*}$.

Remark 3.5. It is worth noticing that B. Fuglede obtained, long ago, in [7] a unitary $U \in B(H)$ and a closed and symmetric $T$ with domain $D(T) \subset H$ such that $U T \subset T U$ but $U^{*} T \not \subset T U^{*}$.

Next, we give a generalization of Theorem 3.4 to an unbounded $N$, and as above, only the normality of $\bar{N}$ is needed.

Theorem 3.6. Let $p$ be a one variable complex polynomial. If $N$ is a densely defined closable operator such that $\bar{N}$ is normal and if $A$ is a densely defined operator with $\sigma[p(A)] \neq \mathbb{C}$, then

$$
N A \subset A N \Longrightarrow N^{*} A \subset A N^{*}
$$

whenever $D(A) \subset D(N)$.
Remark 3.7. This is indeed a generalization of the bounded version of the Fuglede theorem. Observe that when $A, N \in B(H)$, then $\bar{N}=N, D(A)=$ $D(N)=H$, and $\sigma[p(A)]$ is a compact set.

Proof of Theorem 3.6. First, we claim that $\sigma(A) \neq \mathbb{C}$, whereby $A$ is closed. Let $\lambda$ be in $\mathbb{C} \backslash \sigma[p(A)$ ]. Then, and as in [5], we obtain

$$
p(A)-\lambda I=\left(A-\mu_{1} I\right)\left(A-\mu_{2} I\right) \cdots\left(A-\mu_{n} I\right)
$$

for some complex numbers $\mu_{1}, \mu_{2}, \cdots, \mu_{n}$. By consulting again [5], readers see that $\sigma(A) \neq \mathbb{C}$.

Now, let $\lambda \in \rho(A)$. Then

$$
N A \subset A N \Longrightarrow N A-\lambda N \subset A N-\lambda N=(A-\lambda I) N
$$

Since $D(A) \subset D(N)$, it is seen that $N A-\lambda N=N(A-\lambda I)$. So

$$
N(A-\lambda I) \subset(A-\lambda I) N \Longrightarrow(A-\lambda I)^{-1} N \subset N(A-\lambda I)^{-1}
$$

Since $\bar{N}$ is normal, we may now apply Theorem 3.1 to get

$$
(A-\lambda I)^{-1} N^{*} \subset N^{*}(A-\lambda I)^{-1}
$$

because $(A-\lambda I)^{-1} \in B(H)$. Hence

$$
N^{*} A-\lambda N^{*} \subset N^{*}(A-\lambda I) \subset(A-\lambda I) N^{*}=A N^{*}-\lambda N^{*}
$$

But

$$
D\left(A N^{*}\right) \subset D\left(N^{*}\right) \text { and } D\left(N^{*} A\right) \subset D(A) \subset D(N) \subset D(\bar{N})=D\left(N^{*}\right)
$$

Thus, $D\left(N^{*} A\right) \subset D\left(A N^{*}\right)$, and so

$$
N^{*} A \subset A N^{*}
$$

as needed.

Now, we present a few consequences of the preceding result. The first one is given without proof.

Corollary 3.8. If $N$ is a densely defined closable operator such that $\bar{N}$ is normal and if $A$ is an unbounded self-adjoint operator with $D(A) \subset D(N)$, then

$$
N A \subset A N \Longrightarrow N^{*} A \subset A N^{*}
$$

Corollary 3.9. If $N$ is a densely defined closable operator such that $\bar{N}$ is normal and if $A$ is a boundedly invertible operator, then

$$
N A \subset A N \Longrightarrow N^{*} A \subset A N^{*}
$$

Proof. We may write

$$
N A \subset A N \Longrightarrow N A A^{-1} \subset A N A^{-1} \Longrightarrow A^{-1} N \subset N A^{-1}
$$

Since $A^{-1} \in B(H)$ and $\bar{N}$ is normal, Theorem 3.1 gives

$$
A^{-1} N^{*} \subset N^{*} A^{-1} \text { and so } N^{*} A \subset A N^{*}
$$

as needed.
A Putnam's version seems impossible to obtain unless strong conditions are imposed. However, the following special case of a possible Putnam's version is worth stating and proving. Besides, it is somewhat linked to the important notion of anti-commutativity.
Proposition 3.10. If $N$ is a densely defined closable operator such that $\bar{N}$ is normal and if $A$ is a densely defined operator with $\sigma(A) \neq \mathbb{C}$, then

$$
N A \subset-A N \Longrightarrow N^{*} A \subset-A N^{*}
$$

whenever $D(A) \subset D(N)$.
Proof. Consider

$$
\widetilde{N}=\left(\begin{array}{cc}
N & 0 \\
0 & -N
\end{array}\right) \quad \text { and } \quad \widetilde{A}=\left(\begin{array}{cc}
0 & A \\
A & 0
\end{array}\right)
$$

where $D(\widetilde{N})=D(N) \oplus D(\underset{\sim}{N})$ and $D(\widetilde{A})=D(A) \oplus D(A)$. Then $\widetilde{N}$ is normal and $\widetilde{A}$ is closed. Besides $\sigma(\widetilde{A}) \neq \mathbb{C}$. Now

$$
\widetilde{N} \widetilde{A}=\left(\begin{array}{cc}
0 & N A \\
-N A & 0
\end{array}\right) \subset\left(\begin{array}{cc}
0 & -A N \\
A N & 0
\end{array}\right)=\widetilde{A} \widetilde{N}
$$

for $N A \subset-A N$. Since $D(\widetilde{A}) \subset D(\widetilde{N})$, Theorem 3.6 applies, i.e. it gives $\widetilde{N}^{*} \widetilde{A} \subset$ $\widetilde{A} \widetilde{N}^{*}$ which, upon examining their entries, yields the required result.

We finish this section by giving counterexamples to some "generalizations". Example 3.11 ([13]). Consider the unbounded linear operators $A$ and $N$ which are defined by

$$
A f(x)=(1+|x|) f(x) \text { and } N f(x)=-i(1+|x|) f^{\prime}(x)
$$

(with $i^{2}=-1$ ) on the domains

$$
D(A)=\left\{f \in L^{2}(\mathbb{R}):(1+|x|) f \in L^{2}(\mathbb{R})\right\}
$$

and

$$
D(N)=\left\{f \in L^{2}(\mathbb{R}):(1+|x|) f^{\prime} \in L^{2}(\mathbb{R})\right\}
$$

respectively, and where the derivative is taken in the distributional sense. Then $A$ is a boundedly invertible, positive, self-adjoint unbounded operator. As for $N$, it is an unbounded normal operator $N$ (details may consulted in [13]). It was shown that such that

$$
A N^{*}=N A \text { but } A N \not \subset N^{*} A \text { and } N^{*} A \not \subset A N
$$

(in fact $A N f \neq N^{*} A f$ for all $f \neq 0$ ).
So, what this example is telling us is that $N A=A N^{*}$ (and not just an "inclusion"), that $N$ and $N^{*}$ are both normal, $\sigma(A) \neq \mathbb{C}$ (as $A$ is self-adjoint), but $N A \not \subset A N^{*}$.

This example can further be beefed up to refute certain possible generalizations.

Example 3.12 (Cf. [17]). There exist a closed operator $T$ and a normal $M$ such that $T M \subset M T$ but $T M^{*} \not \subset M^{*} T$ and $M^{*} T \not \subset T M^{*}$. Indeed, consider

$$
M=\left(\begin{array}{cc}
N^{*} & 0 \\
0 & N
\end{array}\right) \quad \text { and } \quad T=\left(\begin{array}{cc}
0 & 0 \\
A & 0
\end{array}\right)
$$

where $N$ is normal with domain $D(N)$ and $A$ is closed with domain $D(A)$ and such that $A N^{*}=N A$ but $A N \not \subset N^{*} A$ and $N^{*} A \not \subset A N$ (as defined above). Clearly, $M$ is normal and $T$ is closed. Observe that $D(M)=D\left(N^{*}\right) \oplus D(N)$ and $D(T)=D(A) \oplus L^{2}(\mathbb{R})$. Now,

$$
T M=\left(\begin{array}{cc}
0 & 0 \\
A & 0
\end{array}\right)\left(\begin{array}{cc}
N^{*} & 0 \\
0 & N
\end{array}\right)=\left(\begin{array}{cc}
0_{D\left(N^{*}\right)} & 0_{D(N)} \\
A N^{*} & 0
\end{array}\right)=\left(\begin{array}{cc}
0 & 0_{D(N)} \\
A N^{*} & 0
\end{array}\right)
$$

where e.g. $0_{D(N)}$ is the zero operator restricted to $D(N)$. Likewise

$$
M T=\left(\begin{array}{cc}
N^{*} & 0 \\
0 & N
\end{array}\right)\left(\begin{array}{cc}
0 & 0 \\
A & 0
\end{array}\right)=\left(\begin{array}{cc}
0 & 0 \\
N A & 0
\end{array}\right) .
$$

Since $D(T M)=D\left(A N^{*}\right) \oplus D(N) \subset D(N A) \oplus L^{2}(\mathbb{R})=D(M T)$, it ensues that $T M \subset M T$. Now, it is seen that

$$
T M^{*}=\left(\begin{array}{cc}
0 & 0 \\
A & 0
\end{array}\right)\left(\begin{array}{cc}
N & 0 \\
0 & N^{*}
\end{array}\right)=\left(\begin{array}{cc}
0 & 0_{D\left(N^{*}\right)} \\
A N & 0
\end{array}\right)
$$

and

$$
M^{*} T=\left(\begin{array}{cc}
N & 0 \\
0 & N^{*}
\end{array}\right)\left(\begin{array}{cc}
0 & 0 \\
A & 0
\end{array}\right)=\left(\begin{array}{cc}
0 & 0 \\
N^{*} A & 0
\end{array}\right) .
$$

Since $A N f \neq N^{*} A f$ for any $f \neq 0$, we infer that $T M^{*} \not \subset M^{*} T$ and $M^{*} T \not \subset T M^{*}$.

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# Products of sequentially compact spaces with no separability assumption 

Paolo Lipparini


#### Abstract

Let $X$ be a product of topological spaces. We prove that $X$ is sequentially compact if and only if all subproducts by $\leq \mathfrak{s}$ factors are sequentially compact. If $\mathfrak{s}=\mathfrak{h}$, then $X$ is sequentially compact if and only if all factors are sequentially compact and all but at most $<\mathfrak{s}$ factors are ultraconnected. We give a topological proof of the inequality cf $\mathfrak{s} \geq \mathfrak{h}$. Recall that $\mathfrak{s}$ denotes the splitting number and $\mathfrak{h}$ the distributivity number. Some corresponding invariants are introduced, relative to an arbitrary topological property, more generally, relative to a subset of a partial infinitary semigroup.


Keywords: sequential compactness, Tychonoff product, splitting number, distributivity number, partial infinitary semigroup.
MS Classification 2020: 54D30, 54B10, 03E17, 54A20, 20 M 75.

## 1. Introduction

A countable product of sequentially compact spaces is still sequentially compact [7, Theorem 3.10.35]. The problem whether the above assertion generalizes to uncountable products involves the so-called combinatorial cardinal characteristics of the Continuum $[1,2,16]$. These are cardinals which are provably uncountable and less than or equal to the continuum $\mathfrak{c}$, but consistently strictly smaller than $\mathfrak{c}$. In particular, they all equal $\mathfrak{c}$ if the Continuum Hypothesis holds.

A cardinal characteristic has a standard definition which involves infinite combinatorics and frequently many equivalent formulations in different settings. For example, P. Simon [15] proved that one of these characteristics, the distributivity number $\mathfrak{h}$, is the smallest cardinal such that every product of $<\mathfrak{h}$ sequentially compact spaces is still sequentially compact. Thus the problem mentioned at the beginning is dependent on set theory: in some models of set theory $\mathfrak{h}=\omega_{1}$, in which case the classical result cannot be improved, but in other models $\mathfrak{h}=\mathfrak{c}>\omega_{1}$ [2], hence there are uncountable products which are sequentially compact.

As another influence of cardinal characteristics on products, Booth [3] showed that the splitting number $\mathfrak{s}$ is the smallest cardinal such that the product $\mathbf{2}^{\mathfrak{5}}$ is not sequentially compact. Here $\mathbf{2}$ denotes the discrete 2-element space. Since any nontrivial $T_{1}$ space contains a closed subspace isomorphic to $\mathbf{2}$, we get that if $\mathfrak{h}=\mathfrak{s}$ (an identity which is relatively consistent with the usual axioms of set theory [2]), then a product of $T_{1}$ spaces is sequentially compact if and only if all factors are sequentially compact and the set of nontrivial factors has cardinality $<\mathfrak{s}$. On the other hand, we are not aware of any former result of this kind when separation axioms are not assumed, apart from some partial results in S. Brandhorst thesis [4] under the strong assumption of the Continuum Hypothesis.

While many topologists usually deal with Hausdorff spaces-possibly, even with spaces satisfying higher regularity conditions-recently the interest on spaces satisfying lower separability conditions has newly arisen, e. g. [8, 9, 19]. In particular, see [13] for an interesting recent manifesto in support of the study of spaces satisfying lower separation axioms from a purely topological point of view.

In this note we show that a product of topological spaces is sequentially compact if and only if all subproducts by $\leq \mathfrak{s}$ factors are sequentially compact. If $\mathfrak{h}=\mathfrak{s}$, then a product is sequentially compact if and only if all factors are sequentially compact, and all but at most $<\mathfrak{s}$ factors are ultraconnected. While the proofs are elementary and do not rely on set theory, apart from the mentioned known characterizations of the cardinals $\mathfrak{h}$ and $\mathfrak{s}$, we believe that the results deserve to be explicitly presented with the details of the proofs.

Finally, a longstanding open problem has been recently solved by Dow and Shelah [6] who showed that it is consistent that $\mathfrak{s}$ is singular. Here we present a simple topological proof that the cofinality of $\mathfrak{s}$ is $\geq \mathfrak{h}$. The argument has a general flavor and suggests the idea of attaching similar invariants to an arbitrary property $P$ of topological spaces. At the end of Section 3 we argue that the right framework for the argument is the context of partial infinitary semigroups with a specified subclass. While the ideas are simple, there is the possibility that the arguments and the general framework turn out to be a useful paradigm for many disparate situations. We exemplify the methods in the case of chain compactness.

## 2. Products of sequentially compact spaces

For the sake of simplicity, all topological spaces are assumed to be nonempty.
Recall that a space $X$ is called ultraconnected if no pair of nonempty closed sets of $X$ is disjoint.

Definition 2.1. The splitting number $\mathfrak{s}$ is the least cardinal such that $\mathbf{2}^{\mathfrak{s}}$ is not sequentially compact, where $\mathbf{2}$ is the two-element discrete topological space. Usually the definition of $\mathfrak{s}$ is given in equivalent forms, but the present one is the most suitable for our purposes. See Booth [3, Theorem 2] or van Douwen [16, Theorem 6.1] for a proof of the equivalences. See [2, 16, 18] for further information about $\mathfrak{s}$.

A proof of the next lemma can be found in [12, Lemmata 4.1 and 4.2].
Lemma 2.2. (i) A topological space $X$ is both ultraconnected and sequentially compact if and only if every sequence in $X$ converges.
(ii) A product of $\geq \mathfrak{s}$ spaces which are not ultraconnected is not sequentially compact.

Proposition 2.3. If a product is sequentially compact, then the set of factors with a nonconverging sequence has cardinality $<\mathfrak{s}$.

Proof. Suppose by contradiction that there are $\geq \mathfrak{s}$ factors with a nonconverging sequence. Since each factor is sequentially compact, then, by Lemma 2.2(i), there are $\geq \mathfrak{s}$ factors which are not ultraconnected, and Lemma 2.2(ii) gives a contradiction.

Theorem 2.4. A product of topological spaces is sequentially compact if and only if all subproducts by $\leq \mathfrak{s}$ factors are sequentially compact.

Proof. Necessity is trivial, since we assume that all the spaces are nonempty and sequential compactness is preserved by taking images of surjective continuous functions. For the other direction, suppose that each subproduct of $X=\prod_{j \in J} X_{j}$ by $\leq \mathfrak{s}$ factors is sequentially compact, and let $J^{\prime}=\{j \in$ $J \mid X_{j}$ has a nonconverging sequence $\}$. If $\left|J^{\prime}\right| \geq \mathfrak{s}$, choose $J^{\prime \prime} \subseteq J^{\prime}$ with $\left|J^{\prime \prime}\right|=\mathfrak{s}$. By assumption, $\prod_{j \in J^{\prime \prime}} X_{j}$ is sequentially compact, and we get a contradiction from Proposition 2.3. Thus $\left|J^{\prime}\right|<\mathfrak{s}$. Now $X$ is homeomorphic to $\prod_{j \in J^{\prime}} X_{j} \times \prod_{j \in J \backslash J^{\prime}} X_{j}$. The first factor is sequentially compact by assumption, since we have proved that $\left|J^{\prime}\right|<\mathfrak{s}$. For each $j \in J \backslash J^{\prime}$, we have that every sequence on $X_{j}$ converges, thus in $\prod_{j \in J \backslash J^{\prime}} X_{j}$, too, every sequence converges; a fortiori, $\prod_{j \in J \backslash J^{\prime}} X_{j}$ is sequentially compact. Then $X$ is sequentially compact, being the product of two sequentially compact spaces.

In the context of $T_{1}$ spaces, Theorem 2.4 is an immediate consequence of Definition 2.1, since any nontrivial $T_{1}$ space contains a closed subspace isomorphic to 2 . Thus if a product of $T_{1}$ spaces is sequentially compact, then all but $<\mathfrak{s}$ factors are one-element spaces. Then Theorem 2.4, restricted to $T_{1}$ spaces, follows, since if all subproducts of $\leq \mathfrak{s}$ factors are sequentially compact, then all but $<\mathfrak{s}$ factors are one-element spaces and the product of the nontrivial factors
is sequentially compact by hypothesis. Thus the main point of Theorem 2.4 is the case of spaces satisfying few separation axioms.

The value $\mathfrak{s}$ in Theorem 2.4 is the best possible value: by Definition 2.1, all subproducts of $\mathbf{2}^{\mathfrak{s}}$ by $<\mathfrak{s}$ factors are sequentially compact, but $\mathbf{2}^{\mathfrak{s}}$ is not.

We now show that, under a relatively weak cardinality assumption, we can replace "subproducts" with "factors" in Theorem 2.4.

Definition 2.5. The distributivity number $\mathfrak{h}$ is the smallest cardinal such that there are $\mathfrak{h}$ sequentially compact spaces whose product is not sequentially compact. Usually, the definition of $\mathfrak{h}$ is given in some equivalent form: see Simon [15] for the proof of the equivalence, and [2, 18], for further information. Obviously, $\mathfrak{h} \leq \mathfrak{s}$. It is known that $\mathfrak{h}<\mathfrak{s}$ is relatively consistent [2].

Theorem 2.6. Assume that $\mathfrak{h}=\mathfrak{s}$. If $X$ is a product of topological spaces, then the following conditions are equivalent.
(i) $X$ is sequentially compact.
(ii) All factors of $X$ are sequentially compact, and the set of factors with a nonconverging sequence has cardinality $<\mathfrak{s}$.
(iii) All factors of $X$ are sequentially compact, and all but at most $<\mathfrak{s}$ factors are ultraconnected.

Proof. Conditions (ii) and (iii) are equivalent by Lemma 2.2(i).
Condition (i) implies Condition (ii) by Proposition 2.3 .
The proof that (ii) implies (i) is similar to the proof of Theorem 2.4. Suppose that (ii) holds, and that $X=\prod_{j \in J} X_{j}$. Split $X$ as $\prod_{j \in J^{\prime}} X_{j} \times \prod_{j \in J \backslash J^{\prime}} X_{j}$, where $J^{\prime}=\left\{j \in J \mid X_{j}\right.$ has a nonconverging sequence $\}$. By (ii) and the assumption, $\left|J^{\prime}\right|<\mathfrak{s}=\mathfrak{h}$, hence, by the very definition of $\mathfrak{h}$ (the one we have presented), $\prod_{j \in J^{\prime}} X_{j}$ is sequentially compact. Moreover $\prod_{j \in J \backslash J^{\prime}} X_{j}$ is sequentially compact, since in it every sequence converges, hence also $X$ is sequentially compact.

Under the stronger assumption of the Continuum Hypothesis, we have learned of the equivalence of (i) and (ii) in Corollary 2.6 from Brandhorst [4]. See also Brandhorst and Erné [5]. As mentioned in the introduction, when restricted to $T_{1}$ spaces, Theorem 2.6 follows immediately from Definitions 2.1 and 2.5 (for a $T_{1}$ space $X$ the following are equivalent: all sequences converge; $X$ is ultraconnected; $X$ is trivial, that is, a one-point space). On the other hand, we are not aware of any former result of this kind when no separation axiom is assumed, apart from the mentioned partial result in [4].

Notice that the assumption $\mathfrak{h}=\mathfrak{s}$ is necessary in Theorem 2.6. Indeed, it is now almost immediate to show that Conditions (i) and (ii) in Theorem 2.6 are equivalent if and only if $\mathfrak{h}=\mathfrak{s}$.

Corollary 2.7. The following conditions are equivalent.
(i) $\mathfrak{h}=\mathfrak{s}$
(ii) For every product $X$ of topological spaces, condition (i) in Theorem 2.6 holds if and only if condition (ii) there holds.
(iii) For every product $X$ with $\mathfrak{h}$ factors, condition (ii) in Theorem 2.6 implies condition (i) there.

Proof. (i) $\Rightarrow$ (ii) is given by Theorem 2.6 itself, and (ii) $\Rightarrow$ (iii) is trivial.
To prove (iii) $\Rightarrow$ (i) we shall prove the contrapositive. Suppose that (i) fails. By the definition of $\mathfrak{h}$ there is a not sequentially compact product $X$ by $\mathfrak{h}$ sequentially compact factors. If $\mathfrak{h}<\mathfrak{s}$, then condition (ii) in Theorem 2.6 trivially holds for such an $X$, while condition (i) there fails. Thus condition (iii) in the present corollary fails.

## 3. A topological proof that $\operatorname{cf} \mathfrak{s} \geq \mathfrak{h}$ and a generalization

We begin this section by giving a curious and purely topological proof of the inequality $\operatorname{cf} \mathfrak{s} \geq \mathfrak{h}$. The proof does not use any of the results proved before, but relies heavily on the characterizations of the cardinals $\mathfrak{s}$ and $\mathfrak{h}$ that we have presented as Definitions 2.1 and 2.5. See Blass [1, Corollary 2.2] for another proof of cf $\mathfrak{s} \geq \mathfrak{h}$. Andreas R. Blass (personal communication, June 2014) has kindly communicated us a direct simple proof which uses the combinatorial definitions of $\mathfrak{s}$ and $\mathfrak{h}$.

By the way, Dow and Shelah [6] have recently showed that it is consistent that $\mathfrak{s}$ is $\mathfrak{s i n g}$ ular, solving a longstanding problem.

Proposition 3.1. $\operatorname{cf} \mathfrak{s} \geq \mathfrak{h}$.
Proof. Suppose by contradiction that $\operatorname{cf} \mathfrak{s}=\lambda<\mathfrak{h}$, hence we can express $\mathfrak{s}$ as $\bigcup_{\alpha \in \lambda} s_{\alpha}$, with $\left|s_{\alpha}\right|<\mathfrak{s}$, for $\alpha \in \lambda$; moreover, without loss of generality, we can take the $s_{\alpha}$ 's to be pairwise disjoint. Thus $\mathbf{2}^{\mathfrak{s}}$ is (homeomorphic to) $\prod_{\alpha \in \lambda} \mathbf{2}^{s_{\alpha}}$. By the definition of $\mathfrak{s}$ (the one we have given) and since $\left|s_{\alpha}\right|<\mathfrak{s}$, for $\alpha \in \lambda$, then each $\mathbf{2}^{s_{\alpha}}$ is sequentially compact. By the definition of $\mathfrak{h}$, and since $\lambda<\mathfrak{h}$, we have that $\prod_{\alpha \in \lambda} \mathbf{2}^{s_{\alpha}}$ is sequentially compact. But then $\mathbf{2}^{\mathfrak{s}} \cong \prod_{\alpha \in \lambda} \mathbf{2}^{s_{\alpha}}$ would be sequentially compact, contradicting the definition of $\mathfrak{s}$.

As we mentioned in the introduction, the arguments in the proofs of Proposition 3.1 have a general form and work for every property $P$ of topological
spaces. We could work as well with some property (= a subclass) of objects in a category in which some infinite products or coproducts are defined. However, the right ambient in which the results can be stated in their full generality appears to be the context of partial infinitary semigroups. We shall sketch here a basic result. For more details and further invariants, see Section 7 in the unpublished manuscript [11] from which the present note has been extracted.

Definition 3.2. A partial infinitary semigroup is a $\Sigma$-algebra satisfying properties $(U)$ and $(P)$, in the terminology from [10].

For short, in a partial infinitary semigroup we have a partially defined infinitary operation $\sum_{i \in I} a_{i}$, for every index set I. Property ( $U$ ) asserts that if $|I|=1$, then $\sum_{i \in I} a_{i}$ is defined and its outcome is the only element $a_{i}$ of the sequence.

Property ( $P$ ) asserts that if $\sum_{i \in I} a_{i}$ is defined, then, for every partition $\left(J_{k}\right)_{k \in K}$ of $I$, all the sums in the following equation are defined, and equality actually holds: $\sum_{i \in I} a_{i}=\sum_{k \in K} \sum_{i \in J_{k}} a_{i}$.

With the customary foundational caution, classes of topological spaces modulo homeomorphism and with the Tychonoff product form a partial infinitary semigroups.

Definition 3.3. If $S$ is a partial infinitary semigroup and $P \subseteq S$, let $\mathfrak{H}(P)$ be the class of all cardinals $\kappa \geq 2$ such that the following holds. There are some $I$ of cardinality $\kappa$ and some sum $\sum_{i \in I} a_{i}$ which is defined, whose outcome is not in $P$, while $\sum_{i \in J} a_{i} \in P$, for every $J \subseteq I$ with $|J|<\kappa$, $J \neq \emptyset$.

Notice that property $(P)$ implies that if $\sum_{i \in I} a_{i}$ is defined, then $\sum_{i \in J} a_{i}$ is defined, for every nonempty $J \subseteq I$.

Let $\mathfrak{H}^{*}(P)$ be the class of all cardinals $\kappa \geq 2$ such that there are some $I$ of cardinality $\kappa$ and some sum $\sum_{i \in I} a_{i}$ which is defined, whose outcome is not in $P$, while $a_{i} \in P$, for every $i \in I$. In most examples, if $\kappa \in \mathfrak{H}^{*}(P)$, then $\lambda \in \mathfrak{H}^{*}(P)$, for every $\lambda \geq \kappa$. In this case $\mathfrak{H}^{*}(P)$, if nonempty, is determined by $\mathfrak{h}(P)=\inf \mathfrak{H}^{*}(P)$. However, we shall not need to assume this further property of $\mathfrak{H}^{*}(P)$ in what follows.

Proposition 3.4. Suppose that $S$ is a partial infinitary semigroup and $P \subseteq S$. Then
(i) $\mathfrak{H}(P) \subseteq \mathfrak{H}^{*}(P)$.
(ii) If $\kappa \in \mathfrak{H}(P)$, then $1+\operatorname{cf} \kappa \in \mathfrak{H}^{*}(P)$.
(iii) If $\mathfrak{H}^{*}(P)$ is not empty, then $\inf \mathfrak{H}^{*}(P) \in \mathfrak{H}(P)$, thus $\mathfrak{H}(P) \neq \emptyset$, inf $\mathfrak{H}^{*}(P)=$ $\inf \mathfrak{H}(P)$, and $\inf \mathfrak{H}^{*}(P)$ is a regular cardinal.

Proof. (i) follows from the definitions and Property (U).
(ii) If $\kappa$ is an infinite regular cardinal, then $\kappa=\operatorname{cf} \kappa=1+\operatorname{cf} \kappa$, hence (ii) follows from (i).

If $\kappa$ is finite, say, $\kappa=n \geq 2$ and $\sum_{i<n} a_{i}$ witnesses $\kappa \in \mathfrak{H}(P)$, then $a_{n-1}+$ $\sum_{i<n-1} a_{i}$ witnesses $1+\operatorname{cf} n=1+1=2 \in \mathfrak{H}^{*}(P)$.

The remaining case is similar to Proposition 3.1. Suppose that $\kappa$ is $\sin -$ gular, thus $\kappa=\bigcup_{k \in K} J_{k}$, for some sets $K$ and pairwise disjoint $J_{k}$ such that $|K|,\left|J_{k}\right|<\kappa$, for $k \in K$. Let $c=\sum_{\gamma \in \kappa} a_{\gamma}$ witness $\kappa \in \mathfrak{H}(P)$. For $k \in K$, let $b_{k}=\sum_{\gamma \in J_{k}} a_{\gamma}$. Since $\left|J_{k}\right|<\kappa$, for $k \in K$, then, by the definition of $\mathfrak{H}(P)$, each $b_{k}$ is in $P$. By Property (P), $c=\sum_{k \in K} b_{k}$ and this sum witnesses cf $\kappa \in \mathfrak{H}^{*}(P)$.
(iii) Let $\kappa=\inf \mathfrak{H}^{*}(P)$ and let $\sum_{\gamma \in \kappa} a_{\gamma}$ witness $\kappa \in \mathfrak{H}^{*}(P)$. By assumption, $\kappa \geq 2$ and each $a_{\gamma}$ is in $P$. If there is $J \subseteq \kappa$ such that $2 \leq|J|<\kappa$ and $\sum_{j \in J} a_{j} \notin$ $P$, then $\sum_{j \in J} a_{j}$ witnesses $|J| \in \mathfrak{H}^{*}(P)$, contradicting the minimality of $\kappa$. Thus, by (U), for every $J \subseteq \kappa$ with $1 \leq|J|<\kappa$, we have $\sum_{j \in J} a_{j} \in P$. This means that $\sum_{\gamma \in \kappa} a_{\gamma}$ witnesses $\kappa \in \mathfrak{H}(P)$. The rest follows from (i) and (ii).

If $S$ is the class of topological spaces modulo homeomorphism with Tychonoff products and $P$ is the class of sequentially compact spaces, then $\mathfrak{h}=$ $\inf \mathfrak{H}^{*}(P)$, by Definition 2.5. Moreover, $\mathfrak{s} \in \mathfrak{H}(P)$, by Definition 2.1. Thus Proposition 3.4(ii) generalizes Proposition 3.1. Moreover, the last assertion in Proposition 3.4(iii) generalizes the known fact that $\mathfrak{h}$ is a regular cardinal.

By Theorem 2.4, $\mathfrak{s}=\sup \mathfrak{H}(P)$, hence $\mathfrak{H}(P) \subseteq[\mathfrak{h}, \mathfrak{s}]$, where $[\mathfrak{h}, \mathfrak{s}]$ is the set of those cardinals $\lambda$ such that $\mathfrak{h} \leq \lambda \leq \mathfrak{s}$. It is an open problem whether the inclusion $\mathfrak{H}(P) \subseteq[\mathfrak{h}, \mathfrak{s}]$ may be strict (of course, this is a nontrivial problem only when $\mathfrak{h}<\mathfrak{s}$ ).

As an application of Proposition 3.4, one can consider chain compactness. If $\lambda \leq \mu$ are infinite cardinals, a topological space $X$ is $[\lambda, \mu]$-chain compact [17] if, for every cardinal $\nu$ such that $\lambda \leq \nu \leq \mu$, every $\nu$-indexed sequence of elements of $X$ has a converging cofinal subsequence. Thus $[\omega, \omega]$-chain compactness is the same as sequential compactness.

A product of countably many $[\lambda, \mu]$-chain compact spaces is still $[\lambda, \mu]$-chain compact [17]. Thus if $P_{[\lambda, \mu]-c}$ is the property of being $[\lambda, \mu]$-chain compact, then $\mathfrak{h}\left(P_{[\lambda, \mu]-c}\right)=\inf \mathfrak{H}^{*}\left(P_{[\lambda, \mu]-c}\right)>\omega$. By Proposition 3.4, $\mathfrak{h}\left(P_{[\lambda, \mu]-c}\right)$ is a regular cardinal, and if $\kappa \in \mathfrak{H}\left(P_{[\lambda, \mu]-c}\right)$, then $\operatorname{cf} \kappa \geq \mathfrak{h}\left(P_{[\lambda, \mu]-c}\right)$. To the best of our knowledge, it is an open problem to explicitly characterize the cardinal $\mathfrak{h}\left(P_{[\lambda, \mu]-c}\right)$ and the class $\mathfrak{H}\left(P_{[\lambda, \mu]-c}\right)$. Some results about products of $[\omega, \mu]$-chain compact spaces can be found in [14]. If follows from [11, Theorem 3.1] that $\sup \mathfrak{H}\left(P_{[\lambda, \mu]-c}\right) \leq 2^{\mu}$.

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# Analysis of existence and non-existence of limit cycles for a family of Kolmogorov systems 

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#### Abstract

The main objective of this paper is to study existence and non existence of limit cycles by using the idea of Green's theorem and inverse integrating factor method respectively, for some a significant family of Kolmogorov differential systems.


Keywords: Sixteenth problem of Hilbert, planar differential system, Kolmogorov System, invariant curve, hyperbolic limit cycle, first integral.
MS Classification 2020: 34C25, 34C05, 34C07.

## 1. Introduction

We consider the following Kolmogorov system

$$
\left\{\begin{array}{l}
\dot{x}=x P(x, y),  \tag{1}\\
\dot{y}=y Q(x, y),
\end{array}\right.
$$

where $P(x, y)$ and $Q(x, y)$ are polynomials, the dot denotes derivative with respect to the time $t$, and the coefficients are real numbers.

Generally, Kolmogorov system is introduced as the structure of many natural phenomena models. Their applications can appear in several fields such as, physics, biology, chemical reactions, hydrodynamics, fluid dynamics, economics, etc. for more detail see $[1,7,16,17]$.

One of the most important topics in qualitative theory of planar dynamical systems is related to the second part of the unsolved Hilbert 16th problem which consisted to study the maximum number of limit cycles and their relative distributions of the real planar polynomial system of degree $n$, see [12].

Many different methods have been used for proving the existence and nonexistence of limit cycles in simply connected region, for instance see [3, 11, 18]. In recent years, existence and nonexistence of limit cycle for some class of Kolmogorov system has been studied, see for instance $[2,4,5,6,8,13,14]$. In this paper we will give a unifying characterization on the invariant algebraic curves and first integrals to investigate existence and non existence of limit cycle for system (1).

Firstly, we need to give some necessary definitions. We define a vector field associated to the system (1) as follows

$$
\mathcal{X}=x P(x, y) \frac{\partial}{\partial x}+y Q(x, y) \frac{\partial}{\partial y}
$$

Let $\mathcal{W} \subset \mathbb{R}^{2}$ be an open subset such that $\mathbb{R}^{2} \backslash \mathcal{W}$ has zero Lebesgue measure. We say that a non-constant real function $\mathcal{H}=\mathcal{H}(x, y): \mathbb{R}^{2} \rightarrow \mathbb{R}$, is a first integral if $\mathcal{H}(x(t), y(t))$ is constant on all solutions $(x(t), y(t))$ of $\mathcal{X}$ contained in $\mathcal{W}$, i.e. $\left.\mathcal{X H}\right|_{\mathcal{W}}=0$.

A polynomial $\mathcal{V}(x, y) \in \mathbb{R}[x, y]$, the ring of the real coefficient polynomials in $x, y$ is called a Invariant algebraic curve for the system (1) if

$$
\begin{equation*}
\mathcal{X} \mathcal{V}=\mathcal{K} \mathcal{V} \tag{2}
\end{equation*}
$$

for some real polynomial $\mathcal{K}(x, y)$, which is called cofactor of $\mathcal{V}$.
The curve $\Gamma=\left\{(x, y) \in \mathbb{R}^{2} ; \mathcal{V}(x, y)=0\right\}$, is non-singular of system (1) if the equilibrium points of the system that satisfy the following system

$$
\left\{\begin{array}{l}
x P(x, y)=0  \tag{3}\\
y Q(x, y)=0
\end{array}\right.
$$

are not contained on the curve $\Gamma$.
A solution $(x(t), y(t))$ for a differential system (1) is said to be T-periodic solution, if its satisfies

$$
(x(t), y(t))=(x(t+T), y(t+T))
$$

for all $t$, and for some $T>0$.
A limit cycle is an isolated periodic solution of a differential equation, or is a T-periodic solution of system (1), isolated with respect to all other possible periodic solutions of the system and defined as

$$
\gamma=\{(x(t), y(t)), t \in[0, T]\} .
$$

Let $\gamma$ be periodic orbit of system (1) of period $T$, then $\gamma$ is an hyperbolic limit cycle if

$$
\int_{0}^{T} \operatorname{div}(\mathcal{X})(\gamma(t)) d t \neq 0
$$

for more detail see [18].
Let $\mathcal{W} \subset \mathbb{R}$ and $\Psi: \mathcal{W} \rightarrow \mathbb{R}$ be a function, $\Psi$ is said to be an inverse integrating factor of (1) if it is not locally null and satisfies the partial differential equation

$$
\begin{equation*}
\mathcal{X} \Psi=\operatorname{div}(\mathcal{X}) \Psi \tag{4}
\end{equation*}
$$

where $\operatorname{div}(\mathcal{X})=\frac{\partial(x P(x, y))}{\partial x}+\frac{\partial(y Q(x, y))}{\partial y}$.

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## 2. Main Results

As a main result, we have the following theorem,
Theorem 2.1. We consider Kolmogorov system of degree $m(m \geq 5)$

$$
\left\{\begin{array}{l}
\dot{x}=x\left(\mathcal{V}\left(a x^{2 n-1} y^{2 k-1}+b\right)+\alpha x^{2 n-1} y^{2 n} \mathcal{V}_{y}\right)  \tag{5}\\
\dot{y}=y\left(\mathcal{V}\left(c y^{2 n-1} x^{2 k-1}+d\right)-\alpha x^{2 n} y^{2 n-1} \mathcal{V}_{x}\right)
\end{array}\right.
$$

where $\mathcal{V}=\mathcal{V}(x, y)$, is a polynomial function, and $\mathcal{V}_{x}$ and $\mathcal{V}_{y}$ denotes the partial derivative of variables $x$ and $y$ respectively. The coefficients $a, b, c, d, \alpha$ are non zero reals, and the degree $n$ and $k$ are positive integers. Then the following statements hold.
(1) Let $\Gamma=\left\{(x, y) \in \mathbb{R}^{2}, \mathcal{V}(x, y)=0\right\}$, be a degree $l \geq 2$ invariant and nonsingular curve of the differential system (5). If $b+d \neq 0$ and the bounded components of $\Gamma$ do not intersect the axes $(x=0, y=0)$, then the system (5) admits all bounded components of $\Gamma$ as hyperbolic limit cycles.
(2) If $b+d=0$ the system is integrable with first integral

$$
\mathcal{H}=\left\{\begin{array}{lr}
\exp \left(\frac{(-2 c n+c) x^{-2 n+2 k}+(2 a n-a) y^{-2 n+2 k}-2 y^{-2 n+1} b x^{-2 n+1}(k-n)}{2 \alpha(k-n)(2 n-1)}\right) \mathcal{V}  \tag{6}\\
\frac{y^{\frac{\alpha}{\alpha}}}{x^{\frac{c}{\alpha}}} \exp \left(\frac{-b}{(2 n-1) \alpha x^{2 n-1} y^{2 n-1}}\right) \mathcal{V} & \text { if } k=n
\end{array}\right.
$$

moreover the system has no limit cycle.
Proof of statement 1. Let $\Gamma=\left\{(x, y) \in \mathbb{R}^{2}, \mathcal{V}(x, y)=0\right\}$ with degree $l(l \geq 2)$, be a non-singular of system (5) and the bounded components of $\Gamma$ do not intersect the lines $(x=0, y=0)$. To show that all the bounded components of $\Gamma$ are hyperbolic limit cycles of system (5), we will prove that $\Gamma$ is an invariant curve of the system (5), and

$$
\int_{0}^{T} \operatorname{div}(\mathcal{X})(\gamma(t)) d t \neq 0
$$

see for instance Perko[15, Pages 216-217].
Its clearly $\mathcal{V}$ is an invariant curve of system (5), because

$$
\begin{aligned}
\frac{\partial \mathcal{V}}{\partial x} \dot{x}+\frac{\partial U}{\partial y} \dot{y}= & \mathcal{V}_{x} x\left(\mathcal{V}\left(a x^{2 n-1} y^{2 k-1}+b\right)+\alpha x^{2 n-1} y^{2 n} \mathcal{V}_{y}\right) \\
& +\mathcal{V}_{y} y\left(U\left(c y^{2 n-1} x^{2 k-1}+d\right)-\alpha x^{2 n} y^{2 n-1} \mathcal{V}_{x}\right) \\
= & \mathcal{V}\left(b \mathcal{V}_{x} x+d y \mathcal{V}_{y}+a x^{2 n} y^{2 k-1} \mathcal{V}_{x}+c y^{2 n} x^{2 k-1} \mathcal{V}_{y}\right)
\end{aligned}
$$

where the cofactor is

$$
\mathcal{K}=\left(a x^{2 n-1} y^{2 k-1}+b\right) x \mathcal{V}_{x}+\left(c y^{2 n-1} x^{2 k-1}-d\right) y \mathcal{V}_{y}
$$

To see $\int_{0}^{T} \operatorname{div}(\mathcal{X})(\gamma(t)) d t$ is nonzero, we have show that

$$
\begin{equation*}
\int_{0}^{T} \operatorname{div}(\mathcal{X})(\gamma(t)) d t=\int_{0}^{T} \mathcal{K}(x(t), y(t)) d t \tag{7}
\end{equation*}
$$

is non zero (see for instance Giacomini \& Grau [10, theo 2]).

$$
\begin{array}{rl}
\int_{0}^{T} & \mathcal{K}(x(t), y(t)) d t \\
& =\oint_{\Gamma} \frac{\left(a x^{2 n-1} y^{2 k-1}+b\right) x \mathcal{V}_{x}}{-\alpha x^{2 n} y^{2 n} \mathcal{V}_{x}} d y+\oint_{\Gamma} \frac{\left(c y^{2 n-1} x^{2 k-1}+d\right) y \mathcal{V}_{y}}{\alpha x^{2 n} y^{2 n} \mathcal{V}_{y}} d x \\
& =-\oint_{\Gamma} \frac{\left(a x^{2 n-1} y^{2 k-1}+b\right)}{\alpha x^{2 n-1} y^{2 n}} d y+\oint_{\Gamma} \frac{\left(c y^{2 n-1} x^{2 k-1}+d\right)}{\alpha x^{2 n} y^{2 n-1}} d x
\end{array}
$$

by applying the Green formula,

$$
\begin{aligned}
& \oint_{\Gamma} \frac{\left(c y^{2 n-1} x^{2 k-1}+d\right)}{\alpha x^{2 n} y^{2 n-1}} d x-\oint_{\Gamma} \frac{\left(a x^{2 n-1} y^{2 k-1}+b\right)}{\alpha x^{2 n-1} y^{2 n}} d y \\
&=\frac{1}{\alpha} \iint_{\operatorname{Int}(\Gamma)}\left(\frac{\partial\left(\frac{\left(a x^{2 n-1} y^{2 k-1}+b\right)}{x^{2 n-1} y^{2 n}}\right)}{\partial x}+\frac{\partial\left(\frac{\left(c y^{2 n-1} x^{2 k-1}+d\right)}{x^{2 n} y^{2 n-1}}\right)}{\partial y}\right) d x d y \\
& \quad=-\frac{2 n-1}{\alpha}(b+d) \iint_{\operatorname{Int}(\Gamma)} \frac{1}{y^{2 n} x^{2 n}} d x d y
\end{aligned}
$$

where $\operatorname{Int}(\Gamma)$ denotes the interior of $\Gamma$. As $\alpha \neq 0, b+d \neq 0$ and the bounded components of $\Gamma$ do not intersect the lines $(x=0, y=0)$ then $\int_{0}^{T} \mathcal{K}(x(t), y(t)) d t$ is non zero.

To prove the second statement of Theorem 2.1, we will use the following Theorem.

Theorem 2.2 ([11, Theorem 9]). Let $\Psi: \Omega \rightarrow \mathbb{R}$ be an inverse integrating factor of system(1), if $\Gamma \subset \Omega$ is a limit cycle of (1) then $\Gamma$ is contained in the set $\Psi^{-1}(0)=\{(x, y) \in \Omega, \Psi(x, y)=0\}$.
Proof of statement 2. For $d=-b$, we have a first integral in the form of equation (6). We separate the proof in two different cases . Firstly, if $k \neq n$

$$
\mathcal{H}(x, y)=\exp \left(\frac{(-2 c n+c) x^{-2 n+2 k}+(2 a n-a) y^{-2 n+2 k}-2 y^{-2 n+1} b x^{-2 n+1}(k-n)}{2 \alpha(k-n)(2 n-1)}\right) \mathcal{V}
$$

$$
\begin{aligned}
& \frac{\partial \mathcal{H}}{\partial x} \dot{x}+\frac{\partial \mathcal{H}}{\partial y} \dot{y} \\
& =\frac{1}{\alpha}\left(\left(b y^{-2 n+1} x^{-2 n}-c x^{-2 n+2 k-1}\right) \mathcal{V}+\alpha \mathcal{V}_{x}\right) \\
& \quad \exp \left(\frac{(-2 c n+c) x^{-2 n+2 k}+(2 a n-a) y^{-2 n+2 k}-2 y^{-2 n+1} b x^{-2 n+1}(k-n)}{2 \alpha(k-n)(2 n-1)}\right) \\
& \quad\left(x\left(\mathcal{V}\left(a x^{2 n-1} y^{2 k-1}+b\right)+\alpha x^{2 n-1} y^{2 n} \mathcal{V}_{y}\right)\right) \\
& +\frac{1}{\alpha}\left(\left(a y^{-2 n+2 k-1}+b y^{-2 n} x^{-2 n+1}\right) \mathcal{V}+\alpha \mathcal{V}_{y}\right) \\
& \quad \exp \left(\frac{(-2 c n+c) x^{-2 n+2 k}+(2 a n-a) y^{-2 n+2 k}-2 y^{-2 n+1} b x^{-2 n+1}(k-n)}{2 \alpha(k-n)(2 n-1)}\right) \\
& \quad\left(y\left(\mathcal{V}\left(c y^{2 n-1} x^{2 k-1}-b\right)-\alpha x^{2 n} y^{2 n-1} \mathcal{V}_{x}\right)\right)=0
\end{aligned}
$$

Therefore

$$
\dot{x} \frac{\partial \mathcal{H}}{\partial x}+\dot{y} \frac{\partial \mathcal{H}}{\partial y}=0, \quad \text { then } \quad \frac{\dot{y}}{\frac{\partial \mathcal{H}}{\partial x}}=\frac{\dot{x}}{-\frac{\partial \mathcal{H}}{\partial y}}=\Psi
$$

where $\Psi$ is an inverse integrating factor. Thus

$$
\frac{\dot{y}}{\frac{\partial \mathcal{H}}{\partial x}}=\frac{\dot{x}}{-\frac{\partial \mathcal{H}}{\partial y}}=-\alpha x^{2 n} y^{2 n} \exp (\mathcal{U}(x, y))
$$

where

$$
\mathcal{U}(x, y)=\frac{(2 n-1)\left(-a y^{2 k} x^{2 n}+c y^{2 n} x^{2 k}\right)+2 b x y(k-n)}{2 \alpha x^{2 n} y^{2 n}(2 n-1)(k-n)}
$$

According to Theorem 2.2, the system has no limit cycle because the set

$$
\begin{aligned}
& \Psi^{-1}(0)=\left\{(x, y) \in \mathbb{R}^{2} \mid\right. \\
& \left.\quad-\alpha x^{2 n} y^{2 n} \exp \left(\frac{(2 n-1)\left(-a y^{2 k} x^{2 n}+c y^{2 n} x^{2 k}\right)+2 b x y(k-n)}{2 x^{2 n} y^{2 n} \alpha(2 n-1)(k-n)}\right)=0\right\}
\end{aligned}
$$

contains no closed curve.
Secondly, if $k=n$, then

$$
\mathcal{H}(x, y)=\frac{y^{\frac{\alpha}{\alpha}}}{x^{\frac{c}{\alpha}}} \exp \left(\frac{-b}{(2 n-1) \alpha x^{2 n-1} y^{2 n-1}}\right) \mathcal{V}(x, y)
$$

is first integral and satisfies the following equation

$$
\begin{aligned}
& \frac{\partial \mathcal{H}}{\partial x} \dot{x}+\frac{\partial \mathcal{H}}{\partial y} \dot{y}=\left(\left(-\frac{1}{x^{\frac{1}{\alpha}(c+\alpha+2 n \alpha)}} \frac{y^{\frac{1}{\alpha}(a-2 n \alpha)}}{\alpha}\left(c x^{2 n} y^{2 n}-b x y\right)\right) \mathcal{V}+\frac{y^{\frac{a}{\alpha}}}{x^{\frac{c}{\alpha}}} \mathcal{V}_{x}\right) \\
& \exp \left(\frac{-b}{(2 n-1) \alpha x^{2 n-1} y^{2 n-1}}\right)\left(x\left(\mathcal{V}\left(a x^{2 n-1} y^{2 n-1}+b\right)+\alpha x^{2 n-1} y^{2 n} \mathcal{V}_{y}\right)\right) \\
& +\left(\left(\frac{1}{x^{\frac{1}{\alpha}(c+2 n \alpha)} y^{\frac{1}{\alpha}(\alpha-a+2 n \alpha)} \alpha}\left(a x^{2 n} y^{2 n}+b x y\right)\right) \mathcal{V}+\frac{y^{\frac{a}{\alpha}}}{x^{\frac{c}{\alpha}}} \mathcal{V}_{y}\right) \\
& \exp \left(\frac{-b}{(2 n-1) \alpha x^{2 n-1} y^{2 n-1}}\right)\left(y\left(\mathcal{V}\left(c y^{2 n-1} x^{2 n-1}-b\right)-\alpha x^{2 n} y^{2 n-1} \mathcal{V}_{x}\right)\right)=0
\end{aligned}
$$

Thus

$$
\frac{\dot{y}}{\frac{\partial \mathcal{H}}{\partial x}}=\frac{\dot{x}}{-\frac{\partial \mathcal{H}}{\partial y}}=-\frac{x^{\frac{1}{\alpha}(c+2 n \alpha)}}{y^{\frac{1}{\alpha}(a-2 n \alpha)}} \alpha \exp \left(b x^{1-2 n} \frac{y^{1-2 n}}{\alpha(2 n-1)}\right)
$$

By using Theorem 2.2, the system has no limit cycle because the set

$$
\Psi^{-1}(0)=\left\{(x, y) \in \mathcal{R}^{2} \left\lvert\,-\frac{x^{\frac{1}{\alpha}(c+2 n \alpha)}}{y^{\frac{1}{\alpha}(a-2 n \alpha)}} \alpha \exp \left(b x^{1-2 n} \frac{y^{1-2 n}}{\alpha(2 n-1)}\right)=0\right.\right\}
$$

contains no closed curve.

Now, we present two examples for illustrating the result.
Example 2.3. Let $a=b=c=d=\alpha=n=1, \mathcal{V}(x, y)=2\left(x^{2}+y^{2}-2\right)^{2}-$ $4 x^{2} y^{2}+2 x y+1$. The system (5) reduced to

$$
\left\{\begin{align*}
\dot{x}=x & \left(\left(2\left(x^{2}+y^{2}-2\right)^{2}-4 x^{2} y^{2}+2 x y+1\right)(x y+1)\right.  \tag{8}\\
& \left.+y^{2} x\left(8\left(x^{2}+y^{2}-2\right) y-8 x^{2} y+2 x\right)\right) \\
\dot{y}=y & \left(\left(2\left(x^{2}+y^{2}-2\right)^{2}-4 x^{2} y^{2}+2 x y+1\right)(y x+1)\right. \\
& \left.-x^{2} y\left(8\left(x^{2}+y^{2}-2\right) x-8 x y^{2}+2 y\right)\right)
\end{align*}\right.
$$

$\Gamma=\left\{(x, y) \in \mathbb{R}^{2}, 2\left(x^{2}+y^{2}-2\right)^{2}-4 x^{2} y^{2}+2 x y+1=0\right\}$, does not intersect the axes $(x=0, y=0)$, and $b+d \neq 0$, then the system (8) admits all bounded components of $\Gamma$ as hyperbolic limit cycles. So the system (8) admits four limit cycles represented by the curve $2\left(x^{2}+y^{2}-2\right)^{2}-4 x^{2} y^{2}+2 x y+1=0$ and nine singular points where $(0,0)$ is an unstable node, $(0.16868,-1.4441)$ ) is saddle point, $(-0.16868,1.4441)$ is a saddle point, $(1.1170,1.4373)$ is a strong

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unstable focus, $(-1.1170,-1.4373)$ is a strong unstable focus, $(1.3788,-1.5843)$ is a strong unstable focus, $(-1.3788,1.5843)$ is a strong unstable focus, $(1.4235$, $0.46345)$ is a saddle point, $(-1.4235,-0.46345)$ is a saddle point. See Figure 1.


Figure 1: Limit cycles and singular points of system(8).
EXAMPLE 2.4. Let $a=b=c=k=1, d=-1, n=2, \alpha=\frac{1}{2}$, and $\mathcal{V}(x, y)=$ $(x-2)^{2}+(y-2)^{2}-1$. Then system (5) becomes as follows

$$
\left\{\begin{array}{l}
\dot{x}=x\left(\left((x-2)^{2}+(y-2)^{2}-1\right)\left(x^{3} y+1\right)+x^{3} y^{4}(y-2)\right),  \tag{9}\\
\dot{y}=y\left(\left((x-2)^{2}+(y-2)^{2}-1\right)\left(y^{3} x-1\right)-x^{4} y^{3}(x-2)\right),
\end{array}\right.
$$

it has a first integral as follows

$$
\mathcal{H}(x, y)=\exp \left(-\frac{1}{3 x^{3} y^{3}}\left(3 x^{3} y-3 x y^{3}+2\right)\right)\left((x-2)^{2}+(y-2)^{2}-1\right)
$$

It's clearly aforementioned $\mathcal{H}(x, y)$ satisfies the definition of first integral. Then

$$
\frac{\dot{y}}{\frac{\partial \mathcal{H}}{\partial x}}=\frac{\dot{x}}{-\frac{\partial \mathcal{H}}{\partial y}}=-\frac{1}{2} x^{4} y^{4} \exp \left(\frac{1}{3 x^{3} y^{3}}\left(3 x^{3} y-3 x y^{3}+2\right)\right)
$$

and the set

$$
\Psi^{-1}(0)=\left\{(x, y) \in \mathbb{R}^{2} \left\lvert\,-\frac{1}{2} x^{4} y^{4} \exp \left(\frac{1}{3 x^{3} y^{3}}\left(3 x^{3} y-3 x y^{3}+2\right)\right)=0\right.\right\}
$$

contains no closed curve. The system (9) admits three singular points, where $(0,0)$ is a saddle point, $(1.85773,2.110525)$ is a strong stable focus and (1.99589, $0.777487)$ is a saddle point. The circle $(x-2)^{2}+(y-2)^{2}-1=0$ is an invariant curve for system, but the system has not a limit cycle. See Figure 2.


Figure 2: Phase portraits of system(9) in Poincaré disk.

## 3. Conclusion

In this paper, we investigate existence and non nonexistence of limit cycle for a class of Kolmogorov system (1). We characterized all conditions for the suggested system in order to find hyperbolic limit cycle. In addition, for investigating non existence limit cycle the general form of the first integral for system (1) has been found under suitable conditions of the coefficients.

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Remark 3.1. All figures are plotted on the Poincaré disc by using polynomial planar phase portraits program, see for instance [9, pages 233-257].

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## Monotonicity theorems and inequalities for certain sine sums

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Abstract. Inspired by the work of Askey-Steinig, Szegö, and Schweitzer, we provide several monotonicity theorems and inequalities for certain sine sums. Among others, we prove that for $n \geq 1$ and $x \in(0, \pi / 2)$, we have

$$
\frac{d}{d x} \frac{C_{n}(x)}{1-\cos (x)}<0 \quad \text { and } \quad \frac{d}{d x}(1-\cos (x)) C_{n}(x)>0
$$

where

$$
C_{n}(x)=\sum_{k=1}^{n} \frac{\sin ((2 k-1) x)}{2 k-1}
$$

denotes Carslaw's sine polynomial. Another result states that the inequality

$$
\sum_{k=1}^{n}(n-k+a)(n-k+b) k \sin (k x)>0 \quad(a, b \in \mathbb{R})
$$

holds for all $n \geq 1$ and $x \in(0, \pi)$ if and only if $a=b=1$.
Many corollaries and applications of these results are given. Among them, we present a two-parameter class of absolutely monotonic rational functions.

Keywords: Sine sum, inequality, absolutely monotonic, rational function, subadditive. MS Classification 2020: 26A48, 26C15, 26D05.

## 1. Introduction and statement of the results

I. A classical result in the theory of trigonometric polynomials states that

$$
\begin{equation*}
F_{n}(x)=\sum_{k=1}^{n} \frac{\sin (k x)}{k}>0 \quad(n \geq 1 ; 0<x<\pi) . \tag{1}
\end{equation*}
$$

Fejér conjectured the validity of (1) in 1910. The first proof was published by Jackson [21] one year later. Since then, more than 20 proofs of the Fejér-Jackson
inequality were discovered. A remarkable stronger result than (1) was given by Askey and Steinig [13] in 1976. They proved the monotonicity property

$$
\begin{equation*}
\frac{d}{d x} \frac{F_{n}(x)}{\sin (x / 2)}<0 \quad(n \geq 1 ; 0<x<\pi) \tag{2}
\end{equation*}
$$

Some related theorems were published by Gasper [18] and Alzer and Koumandos [2].

The inequality

$$
\begin{equation*}
C_{n}(x)=\sum_{k=1}^{n} \frac{\sin ((2 k-1) x)}{2 k-1}>0 \quad(n \geq 1 ; 0<x<\pi) \tag{3}
\end{equation*}
$$

is an elegant counterpart of (1). It is due to Carslaw [15]. We note that (3) is equivalent to the functional inequality

$$
F_{n}(2 x)<2 F_{2 n}(x) \quad(n \geq 1 ; 0<x<\pi)
$$

Extensions and refinements of (3) as well as various similar results can be found in Alzer and Koumandos [1], Alzer and Kwong [4, 5, 7], Koschmieder [23], Meynieux and Tudor [25], Ruscheweyh and Salinas [29]; see also Milovanović et al. [26, p. 317].

In view of (2) it is natural to ask: do there exist monotonicity properties of functions which are defined in terms of $C_{n}(x)$ ? Our first theorem gives an affirmative answer to this question.
Theorem 1.1. Let $n \geq 1$ be an integer. Then, for $x \in(0, \pi / 2)$,

$$
\begin{equation*}
\frac{d}{d x} \frac{C_{n}(x)}{1-\cos (x)}<0 \quad \text { and } \quad \frac{d}{d x}(1-\cos (x)) C_{n}(x)>0 \tag{4}
\end{equation*}
$$

For $x \in(\pi / 2, \pi)$, we have

$$
\frac{d}{d x} \frac{C_{n}(x)}{1+\cos (x)}>0 \quad \text { and } \quad \frac{d}{d x}(1+\cos (x)) C_{n}(x)<0
$$

Remark 1.2. (i) It follows from the formula $C_{n}(\pi-x)=C_{n}(x)$ that each of the two different sets of inequalities in Theorem 1.1 can be derived from the other.
(ii) As an immediate consequence of the monotonicity results we obtain the estimates

$$
(1-\cos (x)) L_{n}<C_{n}(x)<\frac{L_{n}}{1-\cos (x)} \quad(n \geq 1 ; 0<x<\pi / 2)
$$

where

$$
L_{n}=\sum_{k=1}^{n} \frac{(-1)^{k-1}}{2 k-1}
$$

denotes the $n$-th partial sum of the classical Leibniz series for $\pi / 4$.

A new lower bound for $C_{n}(x)$, given in the next theorem, plays a crucial role in the proof of (4).
Theorem 1.3. Let $n \geq 1$ be an integer. For $x \in(0, \pi)$, we have

$$
\begin{equation*}
|\sin (2 n x)| \frac{1-|\cos (x)|}{1-\cos (2 x)}<C_{n}(x) \tag{5}
\end{equation*}
$$

II. The inequality

$$
\begin{equation*}
\sum_{k=1}^{n}(n-k+1) \sin (k x)>0 \quad(n \geq 1 ; 0<x<\pi) \tag{6}
\end{equation*}
$$

was first published by Fejér [17] in 1928. It is due to Lukács. Fejér offered a proof of (6) by using properties of power series. An elegant short proof and an extension involving a binomial coefficient were given by Turán [32]; see also Alzer and Kwong [3]. Askey and Gasper [12] pointed out that (6) is a special case of an inequality for the sum of Jacobi polynomials. We define

$$
S_{n}(x)=\sum_{k=1}^{n}(n-k+1)^{2} k \sin (k x)
$$

Here, we present a companion to (6).
Theorem 1.4. Let $n \geq 1$ be an integer. For $x \in(0, \pi)$, we have $S_{n}(x)>0$.
The following representation for $S_{n}(x)$ plays a key role in our proof of Theorem 1.4.

Theorem 1.5. Let $n \geq 1$ be an integer. For $x \in \mathbb{R}$, we have

$$
\begin{align*}
16 \sin ^{4}(x / 2) S_{n}(x)=4( & n+1) \sin (x)-(n+2) \sin (n x) \\
& -4 \sin ((n+1) x)+n \sin ((n+2) x) \tag{7}
\end{align*}
$$

Remark 1.6. Integrating $S_{n}(t)$ from $t=x$ to $t=y$ yields, from Theorem 1.4, an inequality involving the cosine function,

$$
\begin{equation*}
\sum_{k=1}^{n}(n-k+1)^{2}(\cos (k x)-\cos (k y))>0 \quad(n \geq 1 ; 0 \leq x<y \leq \pi) \tag{8}
\end{equation*}
$$

Remark 1.7. From Theorem 1.4 we conclude that the function

$$
M_{n}(x)=\sum_{k=1}^{n}(n-k+1)^{2} \frac{\sin (k x)}{k}
$$

is strictly concave on $[0, \pi]$. Applying the Petrović inequality (see Mitrinović [27, section 1.4.7]) gives that $M_{n}$ satisfies the subadditive property

$$
M_{n}(x+y)<M_{n}(x)+M_{n}(y) \quad(n \geq 1 ; x, y>0, x+y \leq \pi)
$$

Robertson [28] proved the inequality: For $n \geq 2$ and $0<x<\pi$,

$$
(n+1) \frac{\sin ((n-1) x)}{\sin (x)}-(n-1) \frac{\sin ((n+1) x)}{\sin (x)} \leq 4\left(n-\frac{\sin (n x)}{\sin (x)}\right)
$$

and used it to deduce properties of certain analytic functions. Askey and Gasper [11] refined this inequality by showing that the factor 4 can be replaced by $3+\cos (x)$. The inequality

$$
\begin{equation*}
\frac{\sin (n x)}{n \sin (x)} \leq \frac{\sqrt{6}}{9} \quad(n \geq 2 ; \pi / n \leq x \leq \pi-\pi / n) \tag{9}
\end{equation*}
$$

is due to Askey; see Jagers [22]. It plays a role in the proof of Theorem 1.4. An application of Theorem 1.4 leads to the following related result.

Corollary 1.8. Let $\lambda \in \mathbb{R}$ with $\lambda \geq 1$. The inequality

$$
\begin{equation*}
\frac{\sin (n x)}{n \sin (x)}<\frac{\lambda+\cos (n x)}{\lambda+\cos (x)} \tag{10}
\end{equation*}
$$

holds for all integers $n \geq 2$ and $x \in(0, \pi)$ if and only if $\lambda \geq 2$.
A function $f: I \rightarrow \mathbb{R}$ (where $I \subset \mathbb{R}$ is an interval) is called absolutely monotonic if $f$ has derivatives of all orders and satisfies

$$
f^{(n)}(x) \geq 0 \quad(n=0,1,2, \ldots ; x \in I)
$$

These functions have applications in probability theory and the theory of analytic functions. We refer to Widder [33, chapter IV] and Boas [14] for more information on this subject. An additional application of Theorem 1.4 provides a two-parameter class of absolutely monotonic rational functions.

Corollary 1.9. Let $a, b \in \mathbb{R}$ with $-1<a, b<1$. The function

$$
\begin{equation*}
R_{a, b}(x)=\left(\frac{1+x}{1-x}\right)^{2} \frac{x}{\left(x^{2}+2 a x+1\right)\left(x^{2}+2 b x+1\right)} \tag{11}
\end{equation*}
$$

is absolutely monotonic on $[0,1)$.
We discovered Theorem 1.4 when studying a remarkable paper published by Szegö [31] in 1941. His work on univalent functions led Szegö to the inequality

$$
\begin{equation*}
\sum_{k=1}^{n}(n-k+1)(n-k+2) k \sin (k x)>0 \quad(n \geq 1 ; 0<x \leq \tau) \tag{12}
\end{equation*}
$$

where $\tau=1.98 \ldots$ is defined by the equation $\sin ^{2}(\tau / 2)=7 / 10$. Schweitzer [30] improved this result. He showed that (12) is valid for all $n \geq 1, x \in(0,2 \pi / 3)$ and that $2 \pi / 3$ cannot be replaced by a larger constant. Applications and counterparts of (12) can be found in Askey and Fitch [10] and Alzer and Kwong [6]. The following companion to (12) is valid.

Theorem 1.10. Let $a, b \in \mathbb{R}$. The inequality

$$
\begin{equation*}
\sum_{k=1}^{n}(n-k+a)(n-k+b) k \sin (k x)>0 \tag{13}
\end{equation*}
$$

holds for all integers $n \geq 1$ and $x \in(0, \pi)$ if and only if $a=b=1$.
III. In the literature, we can find numerous papers on inequalities for trigonometric sums. The main reason for the great interest is that these results have applications in various fields, like, for instance, geometric function theory, numerical analysis, and number theory. Detailed information on this subject with interesting historical comments and many references are given in Askey [8], Askey and Gasper [12], Milovanović et al. [26, chapter 4]; see also Askey [9], Dimitrov and Merlo [16], Gluchoff and Hartmann [19], and Koumandos [24].
IV. Our proofs of the stated theorems and corollaries are given in Sections 2-8. The algebraic and numerical computations have been carried out by using the computer program MAPLE 13.

## 2. Proof of Theorem 1.3

Let $n \geq 1, x \in(0, \pi)$ and

$$
B_{n}(x)=C_{n}(x)-|\sin (2 n x)| \frac{1-|\cos (x)|}{1-\cos (2 x)} .
$$

Since $B_{n}(\pi-x)=B_{n}(x)$, it suffices to prove that $B_{n}$ is positive on $(0, \pi / 2]$.
Let $x \in(0, \pi / 2]$. Then,

$$
\begin{equation*}
B_{n}(x)=C_{n}(x)-\frac{|\sin (2 n x)|}{2(1+\cos (x))} \tag{14}
\end{equation*}
$$

We obtain

$$
B_{1}(x)=\frac{\sin (x)}{1+\cos (x)}>0
$$

Let $t=\cos (x)$. If $x \in(0, \pi / 4]$, then

$$
B_{2}(x)=\frac{2 \sin (x)(1+2 \cos (x))}{3(1+\cos (x))} p(t)
$$

and if $x \in(\pi / 4, \pi / 2]$, then

$$
B_{2}(x)=\frac{2 \sin (x)}{3(1+\cos (x))} q(t)
$$

with

$$
p(t)=-2 t^{2}+2 t+1 \quad \text { and } \quad q(t)=8 t^{3}+2 t^{2}-2 t+1
$$

Since $p$ is positive on $[\sqrt{2} / 2,1]$ and $q$ is positive on $[0, \sqrt{2} / 2]$, we conclude that $B_{2}(x)>0$ for $x \in(0, \pi / 2]$.

Next, let $n \geq 3$. We consider two cases.
Case 1. $x \in(0, \pi /(2 n))$.
We have

$$
B_{n}(x)=C_{n}(x)-\frac{\sin (2 n x)}{2(1+\cos (x))}
$$

Using

$$
\begin{equation*}
C_{n}^{\prime}(x)=\sum_{k=1}^{n} \cos ((2 k-1) x)=\frac{\sin (2 n x)}{2 \sin (x)} \tag{15}
\end{equation*}
$$

gives

$$
2 \sin (x) B_{n}^{\prime}(x)=\sin (2 n x) \eta(x)-2 n \tan (x / 2) \cos (2 n x)
$$

with

$$
\eta(x)=1-\left(\frac{\sin (x)}{1+\cos (x)}\right)^{2}
$$

Since $\eta$ is decreasing on $(0, \pi)$, we conclude from $0<x<\pi /(2 n) \leq \pi / 6$ that

$$
\eta(x) \geq \eta(\pi / 6)>0.92
$$

It follows that

$$
\begin{equation*}
2 \sin (x) B_{n}^{\prime}(x)>0.92 \sin (2 n x)-2 n \tan (x / 2) \cos (2 n x) \tag{16}
\end{equation*}
$$

Case 1.1. $x \in(0, \pi /(4 n))$.
$\overline{\text { From (16) }}$ we obtain

$$
2 \sin (x) B_{n}^{\prime}(x)>\cos (2 n x) \sigma_{n}(x)
$$

with

$$
\sigma_{n}(x)=0.92 \tan (2 n x)-2 n \tan (x / 2)
$$

Since

$$
\frac{1}{n} \sigma_{n}^{\prime}(x)=\frac{1.84}{\cos ^{2}(2 n x)}-\frac{1}{\cos ^{2}(x / 2)}>\frac{1}{\cos ^{2}(2 n x)}-\frac{1}{\cos ^{2}(x / 2)}>0
$$

we get $\sigma_{n}(x)>\sigma_{n}(0)=0$. Thus, $B_{n}^{\prime}(x)>0$.
Case 1.2. $x \in[\pi /(4 n), \pi /(2 n))$.
Since $\sin (2 n x)>0 \geq \cos (2 n x)$, we get from (16) that $B_{n}^{\prime}(x)>0$.
From Case 1.1 and Case 1.2 we conclude that $B_{n}^{\prime}$ is positive on $(0, \pi /(2 n)]$. This leads to $B_{n}(x)>B_{n}(0)=0$.

Case 2. $x \in[\pi /(2 n), \pi / 2]$.
From (15) we obtain the integral representation

$$
C_{n}(x)=\int_{0}^{x} \frac{\sin (2 n s)}{2 \sin (s)} d s
$$

Carslaw [15] proved that in $[\pi /(2 n), \pi / 2], C_{n}$ attains its global minimum at $x=\pi / n$. Thus,

$$
\begin{equation*}
C_{n}(x) \geq C_{n}(\pi / n)=Y_{n}+Z_{n} \tag{17}
\end{equation*}
$$

where

$$
Y_{n}=\int_{0}^{\pi /(2 n)} \frac{\sin (2 n s)}{2 \sin (s)} d s=\frac{1}{4 n} \int_{0}^{\pi} \frac{\sin (t)}{\sin (t /(2 n))} d t
$$

and

$$
Z_{n}=\int_{\pi /(2 n)}^{\pi / n} \frac{\sin (2 n s)}{2 \sin (s)} d s=\frac{1}{4 n} \int_{\pi}^{2 \pi} \frac{\sin (t)}{\sin (t /(2 n))} d t
$$

Using the estimate

$$
\frac{2 n}{t} \leq \frac{1}{\sin (t /(2 n))} \quad(0<t<\pi)
$$

gives

$$
Y_{n} \geq \frac{1}{4 n} \int_{0}^{\pi} \frac{2 n}{t} \sin (t) d t>0.92
$$

Since $t \mapsto \sin (t) / t$ is decreasing on $(0, \pi)$, we obtain for $t \in(\pi, 2 \pi)$,

$$
\frac{2 n}{t} \sin \left(\frac{t}{2 n}\right) \geq \frac{3}{\pi} \sin \left(\frac{\pi}{3}\right)
$$

Thus,

$$
\frac{1}{\sin (t /(2 n))} \leq \frac{4 \pi n}{3 \sqrt{3} t} \quad(\pi<t<2 \pi)
$$

This leads to

$$
Z_{n} \geq \frac{\pi}{3 \sqrt{3}} \int_{\pi}^{2 \pi} \frac{\sin (t)}{t} d t>-0.27
$$

It follows that

$$
\begin{equation*}
Y_{n}+Z_{n}>\frac{1}{2} \tag{18}
\end{equation*}
$$

Moreover, we have

$$
\begin{equation*}
\frac{|\sin (2 n x)|}{2(1+\cos (x))} \leq \frac{1}{2(1+\cos (x))} \leq \frac{1}{2} \tag{19}
\end{equation*}
$$

From (14), (17), (18) and (19) we conclude that $B_{n}(x)>0$. The proof of Theorem 1.3 is complete.

## 3. Proof of Theorem 1.1

Let $n \geq 1$. We define

$$
G_{n}(x)=\frac{C_{n}(x)}{1-\cos (x)}, \quad H_{n}(x)=(1-\cos (x)) C_{n}(x)
$$

Using (15) and (5) gives, for $x \in(0, \pi / 2)$,

$$
\begin{aligned}
\frac{(1-\cos (x))^{2}}{\sin (x)} \frac{d}{d x} G_{n}(x) & =\frac{1}{\sin (x)}\left[(1-\cos (x)) C_{n}^{\prime}(x)-\sin (x) C_{n}(x)\right] \\
& =\sin (2 n x) \frac{1-|\cos (x)|}{1-\cos (2 x)}-C_{n}(x)<0
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\sin (x)} \frac{d}{d x} H_{n}(x) & =\frac{1}{\sin (x)}\left[(1-\cos (x)) C_{n}^{\prime}(x)+\sin (x) C_{n}(x)\right] \\
& =C_{n}(x)+\sin (2 n x) \frac{1-|\cos (x)|}{1-\cos (2 x)}>0
\end{aligned}
$$

We define

$$
G_{n}^{*}(x)=\frac{C_{n}(x)}{1+\cos (x)}, \quad H_{n}^{*}(x)=(1+\cos (x)) C_{n}(x)
$$

Since $G_{n}^{*}(x)=G_{n}(\pi-x)$ and $H_{n}^{*}(x)=H_{n}(\pi-x)$, we obtain, for $x \in(\pi / 2, \pi)$,

$$
\frac{d}{d x} G_{n}^{*}(x)=-G_{n}^{\prime}(\pi-x)>0 \quad \text { and } \quad \frac{d}{d x} H_{n}^{*}(x)=-H_{n}^{\prime}(\pi-x)<0
$$

## 4. Proof of Theorem 1.5

We have

$$
\begin{equation*}
\sum_{k=1}^{n} k \sin (k x)=\frac{\sin ((n+1) x)}{4 \sin ^{2}(x / 2)}-(n+1) \frac{\cos ((n+1 / 2) x)}{2 \sin (x / 2)} \tag{20}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{k=1}^{n} k \cos (k x)=(n+1) \frac{\sin ((n+1 / 2) x)}{2 \sin (x / 2)}-\frac{1-\cos ((n+1) x)}{4 \sin ^{2}(x / 2)} \tag{21}
\end{equation*}
$$

see Gradshteyn and Ryzhik [20, p. 38]. Next, we set

$$
s(k)=\sin (k x) \quad \text { and } \quad T(k)=(2 \sin (x / 2))^{k}
$$

By differentiation, we obtain from (20) and (21),

$$
\begin{equation*}
\sum_{k=1}^{n} k^{2} s(k)=\frac{A_{n}^{*}}{T(4)} \tag{22}
\end{equation*}
$$

where

$$
\begin{aligned}
A_{n}^{*}=- & 2 s(1)-(n+1)^{2} s(n-1)+n(3 n+4) s(n) \\
& -(n+1)(3 n-1) s(n+1)+n^{2} s(n+2)
\end{aligned}
$$

and

$$
\begin{equation*}
\sum_{k=1}^{n} k^{3} s(k)=\frac{B_{n}^{*}}{T(4)} \tag{23}
\end{equation*}
$$

where

$$
\begin{aligned}
B_{n}^{*}=- & (n+1)^{3} s(n-1)+\left(3 n^{3}+6 n^{2}-4\right) s(n) \\
& -\left(3 n^{3}+3 n^{2}-3 n+1\right) s(n+1)+n^{3} s(n+2) .
\end{aligned}
$$

Moreover, (20) can be written as

$$
\begin{equation*}
\sum_{k=1}^{n} k s(k)=\frac{C_{n}^{*}}{T(4)} \tag{24}
\end{equation*}
$$

where

$$
C_{n}^{*}=-(n+1) s(n-1)+(3 n+2) s(n)-(3 n+1) s(n+1)+n s(n+2)
$$

Applying (22), (23), (24) and the representation

$$
S_{n}(x)=(n+1)^{2} \sum_{k=1}^{n} k s(k)-2(n+1) \sum_{k=1}^{n} k^{2} s(k)+\sum_{k=1}^{n} k^{3} s(k)
$$

we conclude that (7) holds.

## 5. Proof of Theorem 1.4

Using (7) we obtain

$$
2 \frac{(1-\cos (x))^{2}}{\sin (x)} S_{n}(x)=A_{n}(x)
$$

where

$$
A_{n}(x)=2(n+1)-\frac{\sin (n x)}{\sin (x)}-2 \frac{\sin ((n+1) x)}{\sin (x)}+n \cos ((n+1) x)
$$

We show that $A_{n}(x)>0$ for $n \geq 1$ and $x \in(0, \pi)$. First, we consider the cases $n=1,2,3,4,5,6$. We set $t=\cos (x) \in(-1,1)$. Then,

$$
A_{1}(x)=2(1-t)^{2}>0 \quad \text { and } \quad A_{2}(x)=8(1+t)(1-t)^{2}>0 .
$$

Moreover,

$$
\begin{array}{ll}
A_{3}(x)=4(1-t)^{2} p_{3}(t), & A_{4}(x)=8(1+t)(1-t)^{2} p_{4}(t) \\
A_{5}(x)=2(1-t)^{2} p_{5}(t), & A_{6}(x)=16(1+t)(1-t)^{2} p_{6}(t)
\end{array}
$$

with

$$
\begin{array}{ll}
p_{3}(t)=6 t^{2}+8 t+3, & p_{4}(t)=8 t^{2}+4 t+1, \\
p_{5}(t)=80 t^{4}+128 t^{3}+48 t^{2}+3, & p_{6}(t)=24 t^{4}+16 t^{3}-4 t^{2}-2 t+1 .
\end{array}
$$

A short calculation yields that the polynomials $p_{3}, p_{4}, p_{5}$ and $p_{6}$ are positive on $(-1,1)$. It follows that $A_{3}, A_{4}, A_{5}$ and $A_{6}$ are positive on $(0, \pi)$.

Let $n \geq 7$. We consider five cases.
Case 1. $x \in(0, \pi / n]$.
We set $x=s /(n+1)$ with $s \in(0,(n+1) \pi / n]$ and define

$$
\begin{align*}
J_{n}(s) & =\sin \left(\frac{s}{n+1}\right) A_{n}\left(\frac{s}{n+1}\right) \\
& =(2 n+2+n \cos (s)) \sin \left(\frac{s}{n+1}\right)-\sin \left(\frac{n s}{n+1}\right)-2 \sin (s) . \tag{25}
\end{align*}
$$

Case 1.1. $s \in(0,3 \pi / 4]$.
Using

$$
1-\frac{1}{2} \theta^{2}+\frac{1}{24} \theta^{4}-\frac{1}{720} \theta^{6} \leq \cos (\theta) \quad(\theta \geq 0)
$$

and

$$
\theta-\frac{1}{6} \theta^{3} \leq \sin (\theta) \leq \theta-\frac{1}{6} \theta^{3}+\frac{1}{120} \theta^{5} \quad(\theta \geq 0)
$$

we obtain the estimates

$$
\begin{aligned}
\sin \left(\frac{s}{n+1}\right) & \geq \frac{s}{n+1}-\frac{s^{3}}{6(n+1)^{3}} \\
-\sin \left(\frac{n s}{n+1}\right) & \geq-\frac{n s}{n+1}+\frac{n^{3} s^{3}}{6(n+1)^{3}}-\frac{n^{5} s^{5}}{120(n+1)^{5}}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
J_{n}(s) \geq & \left(2 n+2+n\left(1-\frac{1}{2} s^{2}+\frac{1}{24} s^{4}-\frac{1}{720} s^{6}\right)\right)\left(\frac{s}{n+1}-\frac{s^{3}}{6(n+1)^{3}}\right) \\
& \quad-\frac{n s}{n+1}+\frac{n^{3} s^{3}}{6(n+1)^{3}}-\frac{n^{5} s^{5}}{120(n+1)^{5}}-2 s+\frac{1}{3} s^{3}-\frac{1}{60} s^{5} \\
= & \frac{s^{5} P_{n}(s)}{4320(n+1)^{5}}
\end{aligned}
$$

with

$$
\begin{gathered}
P_{n}(s)=n(n+1)^{2} s^{4}-6 n(n+1)^{2}\left(n^{2}+2 n+6\right) s^{2}+72 n^{5} \\
+360 n^{4}+720 n^{3}+720 n^{2}+180 n-72 .
\end{gathered}
$$

It remains to show that $P_{n}(s)>0$, or, equivalently, after replacing $s^{2}$ by $t \in\left(0,(3 \pi / 4)^{2}\right) \subset(0,6)$,

$$
Q_{n}(t)=t^{2}-6\left(n^{2}+2 n+6\right) t+\frac{72 n^{5}+360 n^{4}+720 n^{3}+720 n^{2}+180 n-72}{n(n+1)^{2}}>0
$$

Since

$$
Q_{n}^{\prime}(t)=2 t-6\left(n^{2}+2 n+6\right)<0 \quad(0<t<6)
$$

we obtain

$$
Q_{n}(t)>Q_{n}(6)=\frac{36\left(n^{5}+6 n^{4}+10 n^{3}+8 n^{2}-2\right)}{n(n+1)^{2}}>0
$$

Case 1.2. $s \in[3 \pi / 4,(n+1) \pi / n]$.

## Applying

$$
\sin \left(\frac{n s}{n+1}\right) \leq \sin \left(\frac{3 n \pi}{4(n+1)}\right) \leq \sin \left(\frac{21 \pi}{32}\right)<0.882
$$

and

$$
2 \sin (s) \leq 2 \sin \left(\frac{3 \pi}{4}\right)<1.415
$$

leads to

$$
\begin{equation*}
-\sin \left(\frac{n s}{n+1}\right)-2 \sin (s)>-2.297 \tag{26}
\end{equation*}
$$

Using the monotonicity of $x \mapsto \sin (x) / x$ we obtain

$$
\begin{align*}
(n+2) \sin \left(\frac{s}{n+1}\right) & \geq(n+2) \sin \left(\frac{3 \pi}{4(n+1)}\right) \\
& \geq \frac{8(n+2)}{n+1} \sin \left(\frac{3 \pi}{32}\right)>2.321 \tag{27}
\end{align*}
$$

From (25), (26) and (27) we get

$$
J_{n}(s) \geq(n+2) \sin \left(\frac{s}{n+1}\right)-\sin \left(\frac{n s}{n+1}\right)-2 \sin (s)>0 .
$$

Case 2. $x \in[\pi / n, \pi-\pi / n]$.
Using (9) we obtain

$$
\begin{aligned}
A_{n}(x) & \geq 2(n+1)-\frac{\sqrt{6}}{9} n-\frac{2 \sqrt{6}}{9}(n+1)-n \\
& =\left(1-\frac{\sqrt{6}}{3}\right) n+2\left(1-\frac{\sqrt{6}}{9}\right)>0 .
\end{aligned}
$$

Case 3. $n$ is odd and $x \in[\pi-\pi / n, \pi-\pi /(n+1)]$.
Since $x \mapsto-\sin (n x)$ is decreasing on $I=[\pi-\pi / n, \pi-\pi /(n+1)]$, we obtain

$$
-\sin (n x) \geq-\sin \left(n \pi-\frac{n \pi}{n+1}\right)=-\sin \left(\frac{\pi}{n+1}\right)
$$

Moreover, we have

$$
\begin{equation*}
\sin (x) \geq \sin \left(\pi-\frac{\pi}{n+1}\right)=\sin \left(\frac{\pi}{n+1}\right) . \tag{28}
\end{equation*}
$$

This gives

$$
\begin{equation*}
-\frac{\sin (n x)}{\sin (x)} \geq-1 \tag{29}
\end{equation*}
$$

The function $x \mapsto-\sin ((n+1) x)$ is increasing on $I$. Thus,

$$
\begin{equation*}
-2 \sin ((n+1) x) \geq-2 \sin \left((n+1) \pi-\frac{n+1}{n} \pi\right)=-2 \sin \left(\frac{\pi}{n}\right) . \tag{30}
\end{equation*}
$$

Using (28) and (30) gives

$$
\begin{align*}
-2 \frac{\sin ((n+1) x)}{\sin (x)} \geq-2 \frac{\sin (\pi / n)}{\sin (x)} & \geq-2 \frac{\sin (\pi / n)}{\sin (\pi /(n+1))} \\
& \geq-2 \frac{n+1}{n} \geq-\frac{16}{7} \tag{31}
\end{align*}
$$

From (29) and (31) we conclude that

$$
A_{n}(x) \geq 2(n+1)-1-\frac{16}{7}-n=n-\frac{9}{7}>0
$$

Case 4. $n$ is odd and $x \in(\pi-\pi /(n+1), \pi)$.

We have $\sin ((n+1) x)<0$, and since $0<\pi-x<\pi /(n+1)<\pi / n$, we conclude from Case 1 that $A_{n}(\pi-x)>0$. It follows that

$$
A_{n}(x)=A_{n}(\pi-x)-4 \frac{\sin ((n+1) x)}{\sin (x)}>0
$$

Case 5. $n$ is even and $x \in(\pi-\pi / n, \pi)$.
We have

$$
\begin{equation*}
A_{n}(x)=A_{n}(\pi-x)+\frac{n(n+2)}{\sin (x)} \omega_{n}(x) \tag{32}
\end{equation*}
$$

where

$$
\omega_{n}(x)=\frac{\sin ((n+2) x)}{n+2}-\frac{\sin (n x)}{n}
$$

Then,

$$
\omega_{n}^{\prime}(x)=-2 \sin (x) \sin ((n+1) x)
$$

It follows that $\omega_{n}^{\prime}$ is positive on $(\pi-\pi / n, n \pi /(n+1))$ and negative on $(n \pi /(n+$ $1), \pi)$. This leads to

$$
\omega_{n}(x) \geq \min \left(\omega_{n}(\pi-\pi / n), \omega_{n}(\pi)\right)=0
$$

Moreover, from Case 1 we obtain that $A_{n}(\pi-x)>0$. Applying (32) gives $A_{n}(x)>0$. This completes the proof of Theorem 1.4.

## 6. Proof of Corollary 1.8

Let $n \geq 2$ and $x \in(0, \pi)$. We define for $\lambda \geq 2$,

$$
D(\lambda)=D_{n}(\lambda, x)=\lambda+\cos (n x)-(\lambda+\cos (x)) \frac{\sin (n x)}{n \sin (x)}
$$

Applying Theorem 1.4 and Theorem 1.5 gives

$$
\begin{aligned}
D(2) & =2+\cos (n x)-(2+\cos (x)) \frac{\sin (n x)}{n \sin (x)} \\
& =\frac{2 \sin (x)(1-\cos (x))}{n(1+\cos (x))} S_{n-1}(x)>0 .
\end{aligned}
$$

Since

$$
D^{\prime}(\lambda)=1-\frac{\sin (n x)}{n \sin (x)}>0
$$

we obtain $D(\lambda) \geq D(2)>0$. This leads to (10). Next, we assume that (10) is valid for all $n \geq 2$ and $x \in(0, \pi)$. We define

$$
E(x)=E_{n}(\lambda, x)=n(\lambda+\cos (n x)) \sin (x)-(\lambda+\cos (x)) \sin (n x)
$$

Then, $E(x)>0$. Since $E(0)=E^{\prime}(0)=E^{\prime \prime}(0)=0$, we get

$$
E^{\prime \prime \prime}(0)=n\left(n^{2}-1\right)(\lambda-2) \geq 0
$$

This yields $\lambda \geq 2$.

## 7. Proof of Corollary 1.9

Let $x \in(-1,1)$ and $t \in(0, \pi)$. We define

$$
\begin{aligned}
U(x) & =\sum_{k=0}^{\infty}(k+1)^{2} x^{k}=\frac{1+x}{(1-x)^{3}} \\
\text { and } \quad V_{t}(x) & =\sum_{k=1}^{\infty} \cos (k t) x^{k}=\frac{x(\cos (t)-x)}{x^{2}-2 x \cos (t)+1} .
\end{aligned}
$$

Let $0<\alpha<\beta<\pi$ and $W_{\alpha, \beta}(x)=U(x)\left(V_{\alpha}(x)-V_{\beta}(x)\right)$. Then,

$$
\begin{align*}
W_{\alpha, \beta}(x) & =\sum_{k=0}^{\infty}(k+1)^{2} x^{k} \sum_{k=1}^{\infty}(\cos (k \alpha)-\cos (k \beta)) x^{k} \\
& =\sum_{n=1}^{\infty} \sum_{k=1}^{n}(n-k+1)^{2}(\cos (k \alpha)-\cos (k \beta)) x^{n} \\
& =(\cos (\alpha)-\cos (\beta))\left(\frac{1+x}{1-x}\right)^{2} \frac{x}{\phi(x)}, \tag{33}
\end{align*}
$$

where

$$
\phi(x)=\left(x^{2}-2 x \cos (\alpha)+1\right)\left(x^{2}-2 x \cos (\beta)+1\right) .
$$

Moreover, we obtain

$$
\begin{align*}
\sum_{n=1}^{\infty} \sum_{k=1}^{n}(n-k+1)^{2} k \frac{\sin (k \beta)}{\sin (\beta)} x^{n} & =\lim _{\alpha \rightarrow \beta} \frac{W_{\alpha, \beta}(x)}{\cos (\alpha)-\cos (\beta)} \\
& =\left(\frac{1+x}{1-x}\right)^{2} \frac{x}{\left(x^{2}-2 x \cos (\beta)+1\right)^{2}} \tag{34}
\end{align*}
$$

Applying (8) and Theorem 1.4 we conclude that the power series in (33) and (34) have positive coefficients. We set $a=-\cos (\alpha) \in(-1,1)$ and $b=$ $-\cos (\beta) \in(-1,1)$. It follows that the function $R_{a, b}$, defined in (11), is absolutely monotonic on $[0,1)$.

## 8. Proof of Theorem 1.10

We denote the sum in (13) by $K_{n}(a, b ; x)$. From Theorem 1.4 we conclude that $K_{n}(1,1 ; x)>0$ for $n \geq 1$ and $x \in(0, \pi)$. Next, we assume that (13) is valid for all $n \geq 1$ and $x \in(0, \pi)$. From $K_{1}(a, b ; x)=a b \sin (x)>0$ we conclude that $a b>0$. For $n=2$ we obtain

$$
K_{2}(a, b ; x)=(1+a+b+a b(1+4 \cos (x))) \sin (x)>0 .
$$

This gives $1+a+b-3 a b \geq 0$. Thus,

$$
\begin{equation*}
0<3 a b \leq 1+a+b \tag{35}
\end{equation*}
$$

Since $K_{n}(a, b ; \pi)=0$, we obtain for $n \geq 1$,

$$
\left.\frac{d}{d x} K_{n}(a, b ; x)\right|_{x=\pi}=\sum_{k=1}^{n}(-1)^{k}(n-k+a)(n-k+b) k^{2} \leq 0 .
$$

We consider two cases.
Case 1. $n=2 N$.
We obtain

$$
\left.\frac{d}{d x} K_{2 N}(a, b ; x)\right|_{x=\pi}=N^{2}(2 a b-a-b)+N(a b-1) \leq 0 .
$$

This gives

$$
\begin{equation*}
2 a b-a-b \leq 0 \tag{36}
\end{equation*}
$$

Case 2. $n=2 N+1$.
Then,

$$
\left.\frac{d}{d x} K_{2 N+1}(a, b ; x)\right|_{x=\pi}=N^{2}(a+b-2 a b)+N(a+b-3 a b)-a b \leq 0
$$

It follows that

$$
\begin{equation*}
a+b-2 a b \leq 0 \tag{37}
\end{equation*}
$$

From (36) and (37) we get

$$
\begin{equation*}
a+b=2 a b . \tag{38}
\end{equation*}
$$

Using (35) and (38) we conclude that $a, b>0$. This gives

$$
a b=\frac{a+b}{2} \geq \sqrt{a b} .
$$

Thus, $a b \geq 1$. Using (35) and (38) yields

$$
1+2 a b=1+a+b \geq 3 a b
$$

Hence, $a b \leq 1$. It follows that $a b=1$. Applying this result and (38) leads to

$$
0=a+b-2 a b=\frac{1}{a}(a-1)^{2} .
$$

Thus, $a=1$. It follows that $b=1$.

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## Section 2

Proceedings of TAGSS School 2021
Hyperkähler and Prym varieties:
Classical and New Results

## Preface

TAGSS (Trieste Algebraic Geometry Summer School) is an ongoing series of schools taught by outstanding women mathematicians, with the aim of bringing doctoral students, postdocs, and anyone interested, from a review of the basic constructions to current, state-of-the art research themes.

The previous successful editions featured programs on enumerative geometry (TAGSS I - Summer School in Enumerative Geometry, held at SISSA in 2017), geometry of moduli spaces of curves (TAGSS II - Summer School on Geometry of Moduli Spaces of Curves, held at ICTP in 2018), and applications of geometry to biochemical networks and data clouds (TAGSS III - Algebraic Geometry towards Applications, again at ICTP in 2019).

After a break in 2020 due to the pandemic situation, we reprised the series in July 2021 with the online event TAGSS 2021 - Hyperkähler and Prym Varieties: Classical and New Results, sponsored by ICTP (activity smr 3609, available at the page https://indico.ictp.it/event/9610/overview). It featured courses by Elham Izadi (Hyperkähler manifolds, an overview and some open problems) and Angela Ortega (Prym varieties). Following the same format as the previous editions, the Summer School lasted one week and included exercise sessions that complemented the lectures, as well as contributed talks delivered by young participants.

The topics of the school concerned the connection between Abelian varieties and algebraic curves, which has inspired algebraic geometers for more than a century, with each field helping to shed light on the other. With each (smooth projective) curve one can associate its Jacobian; however, most principally polarized abelian varieties cannot be obtained in this way. A more general construction associates with a finite morphism of curves its Prym variety; this construction leads to the concept of Prym-Tyurin variety. These associations work well in families, leading to the Torelli map from the moduli of curves (and the Prym map from the moduli of covers) to the moduli of principally polarized abelian varieties, and have led to the proof of numerous results on all these important moduli spaces. Other applications include Hodge theory, and in particular primal cohomology of the theta divisor, highlighting its connections to root lattices.

The main goal of the school was to provide a stimulating intellectual environment where all the participants could learn about some of these important
aspects of algebraic geometry, as well as the basic notions required for working in this field. In particular, we encouraged female students and researchers at the beginning of their career.

This volume contains the lecture notes of the two courses given at the school, as well as a contribution by one of the young speakers and her collaborator.

The lecture notes Hyperkähler manifolds contain contributions by Elham Izadi and some of her students and postdocs, who helped with the exercise sessions, namely Samir Canning, Yajnaseni Dutta, and David Stapleton. They give an elementary introduction to Hyperkähler manifolds, survey some of their interesting properties and some open problems.

The lecture notes Prym varieties and Prym maps have been co-authored by Angela Ortega and Paweł Borówka, who also led the exercise sessions of Ortega's course. They contain an introduction to the theory of Prym varieties, and a detailed analysis of the fibres of the Prym map for étale double coverings over genus 6 curves.

Finally, the article by Gian Paolo Grosselli and Irene Spelta concerns positive dimensional fibres of the Prym map $\mathcal{P}_{g, r}$. The authors present a direct procedure to investigate infinitely many examples of positive dimensional fibres.

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The guest Editors
Valentina Beorchia
Ada Boralevi
Barbara Fantechi

# Explicit analysis of positive dimensional fibres of $\mathcal{P}_{g, r}$ and Xiao conjecture 

Gian Paolo Grosselli and Irene Spelta


#### Abstract

We focus on the positive dimensional fibres of the Prym map $\mathcal{P}_{g, r}$. We present a direct procedure to investigate infinitely many examples of positive dimensional fibres. Such procedure uses families of Galois coverings of the line admitting a 2-sheeted Galois intermediate quotient. Then we generalize to families of Galois coverings of the line admitting a Galois intermediate quotient of higher degree and we show that the higher degree analogue of the aforementioned procedure gives all the known counterexamples to a conjecture by Xiao on the relative irregularity of a fibration.


Keywords: Families of Galois covers, Prym maps, Xiao conjecture. MS Classification 2020: $14 \mathrm{H} 10,14 \mathrm{H} 30,14 \mathrm{H} 40$.

## 1. Introduction

Let $C$ be a curve of genus $g \geq 1$ and let $f: \tilde{C} \rightarrow C$ be a 2-sheeted covering of $C$ ramified at $r \geq 0$ points. The Prym variety $P:=P(\tilde{C}, C)$ is a polarized abelian variety of dimension $g-1+\frac{r}{2}$ associated with $f$. It is defined as the identity component of the kernel of the Norm map $\mathrm{Nm}_{f}: J \tilde{C} \rightarrow J C$. The theta divisor of $J \tilde{C}$ induces a polarization on the Prym variety $P$ of type $\delta:=(1, \ldots, 1,2, \ldots, 2)$ with 2 repeated $g$ times if $r>0$ and $g-1$ times if $r=0$.

Let us denote by $\mathcal{R}_{g, r}$ the coarse moduli space of isomorphism classes of coverings $f$ and by $\mathcal{A}_{g-1+\frac{r}{2}}^{\delta}$ the one of abelian varieties of dimension $g-1+\frac{r}{2}$ with polarization of type $\delta$. The theory of double coverings provides an alternative description of $\mathcal{R}_{g, r}$. Indeed there is a 1-1 correspondence between double covers $f: \tilde{C} \rightarrow C$ and triples $(C, \eta, B)$ in

$$
\mathcal{R}_{g, r}=\left\{(C, \eta, B): C \in \mathcal{M}_{g}, \eta \in \operatorname{Pic}^{\frac{r}{2}}(C), B \text { reduced in }\left|\eta^{\otimes 2}\right|\right\} / \cong .
$$

When $r=0$, the branch divisor $B$ is empty, hence we can identify $f$ with a pair $(C, \eta)$, with $\eta \in \operatorname{Pic}^{0}(C) \backslash\left\{\mathcal{O}_{C}\right\}$ such that $\eta^{\otimes 2} \cong \mathcal{O}_{C}$.

The Prym map is the morphism

$$
\mathcal{P}_{g, r}: \mathcal{R}_{g, r} \rightarrow \mathcal{A}_{g-1+\frac{r}{2}}^{\delta}
$$

which sends triples $[(C, \eta, B)$ ] to the corresponding Prym variety $P$.
The case $r=0$ is very classical. Indeed unramified Prym varieties are principally polarized abelian varieties and they have been studied for over one hundred years. Our (algebraic) point of view has been presented for the first time by Mumford in [24] in 1974. Then many papers investigated this case and nowadays we have a lot of information on $\mathcal{P}_{g, 0}$. Donagi very well discusses it in [6]. Good surveys are [8] and [29].

The case $r>0$ has become of interest only more recently. Indeed ramified Prym varieties are abelian varieties no longer principally polarized (except in the case of $r=2$ ). As such, they started to be studied quite late. Even if some cases with $r=4$ were already considered by [25] and by [2], the seminal paper is the one by Marcucci and Pirola ([22]), which came out only in 2012. From this work, many other authors have investigated the ramified Prym maps $\mathcal{P}_{g, r}$. At this moment it is a very active area of research. For instance, very recently, it has been proved that if $r \geq 6$ then $\mathcal{P}_{g, r}$ is injective ( $[18,28]$ ).

When the genus $g$ and the number $r$ are low, more precisely when $g<6$ for $r=0, g \leq 4$ for $r=2$ and $g \leq 2$ for $r=4$, the fibres of the Prym maps $\mathcal{P}_{g, r}$ are positive dimensional and they carry plenty of geometry which is wellunderstood (see [6] and [13]). The structure of the generic positive dimensional fibre is so peculiar that one needs to find an ad-hoc procedure to describe each of them.

In [26], Naranjo stated that, for $g$ large enough, the étale Prym map $\mathcal{P}_{g, 0}$ has positive dimensional fibres only on the locus of coverings of hyperelliptic curves and on some components of the locus of coverings of bielliptic curves. Naranjo and Ortega (see [28, Theorem 1.2]) showed that the ramified Prym map $\mathcal{P}_{g, r}$, with $r=2,4$ and any value for $g$, has positive dimensional fibres when restricted to covers of hyperelliptic curves. In the same paper, the authors also proved that the Prym map $\mathcal{P}_{5,2}$ carries positive dimensional fibres when restricted to the locus of trigonal curves. Finally, Casalaina-Martin and Zhang [4, Theorem 5.1] produced some positive dimensional fibres for $\mathcal{P}_{3,4}$ under the assumption $\eta$ effective.

Thus it turns out that, except for a few isolated cases, the hyperelliptic locus represents a good place to look for positive dimensional fibres for the Prym map. Nevertheless, when $r>0$ it is unknown if we should expect the existence of other examples. Indeed we only have the following:
Proposition 1.1 ([28]). Assume $r>0$. If the differential $d \mathcal{P}_{g, r}$ is not injective then we are in one among these cases.
$r=2: C$ is hyperelliptic or trigonal or plane quintic or $\eta=\mathcal{O}_{C}(x+y-z)$, for $x, y, z \in C$.
$r=4: C$ is hyperelliptic or $h^{0}(C, \eta)>0$.
This Proposition doesn't conclude the classification. Apart from the hyperelliptic locus and the isolated examples mentioned in the previous paragraph (i.e. the trigonal locus in genus 5 and $\eta$ effective in genus 3 ), we still do not known the behaviour of the differential $d \mathcal{P}_{g, r}$ on the remaining cases.

The goal of this paper is to analyse infinitely many examples of positive dimensional fibres of the Prym maps, both in the étale or in the ramified case. We use positive dimensional families of Galois covers $\tilde{C} \rightarrow \tilde{C} / \tilde{G} \cong \mathbb{P}^{1}$ of the line where the genus $\tilde{g}:=g(\tilde{C})$, the number of ramification points $s$ and the monodromy are fixed. Then we look for which among these families admit as intermediate quotient $\tilde{C} \rightarrow C$ a 2:1 map ramified in $r=0,2,4$. Hence we select the ones with associated $d \mathcal{P}_{g, r}$ non-injective. This request corresponds to a simple numerical condition as follows.

Proposition 1.2. If $\operatorname{dim}\left(S^{2} H^{0}\left(\omega_{\tilde{C}}\right)\right)^{\tilde{G}}-\operatorname{dim}\left(S^{2} H^{0}\left(\omega_{C}\right)\right)^{\tilde{G}}<\operatorname{dim} H^{0}\left(\omega_{\tilde{C}}^{\otimes 2}\right)^{\tilde{G}}$ then the differential $d \mathcal{P}_{g, r}$ along the family is not injective. Hence the Prym map has positive dimensional fibres along the family.

In particular, when $\operatorname{dim}\left(S^{2} H^{0}\left(\omega_{\tilde{C}}\right)\right)^{\tilde{G}}=\operatorname{dim}\left(S^{2} H^{0}\left(\omega_{C}\right)\right)^{\tilde{G}}$ the family is all contained in a fibre of $\mathcal{P}_{g, r}$. This is always the situation in case of $H^{0}\left(\omega_{\tilde{C}}^{\otimes 2}\right)^{\tilde{G}}=$ 1, i.e. of 1-dimensional families, under the assumption of Proposition 1.2.

We show the following:
Theorem 1.3. For any $N \in \mathbb{N}$ there exist 1-dimensional families of Galois covers $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^{1}$ contained in the fibres of $\mathcal{P}_{2 N, 0}, \mathcal{P}_{2 N, 2}, \mathcal{P}_{2 N-1,0}, \mathcal{P}_{2 N-1,4}$.

Easily we show that (unfortunately) all such families arise as coverings of curves $C$ lying in the hyperelliptic locus.

In general, starting from a family of curves, one can construct fibrations that have the curves of the family as fibres. In particular, we focus on those obtained from families under the assumption of Proposition 1.2. At the same time, we generalize to families of Galois coverings $\tilde{C} \rightarrow \tilde{C} / \tilde{G} \cong \mathbb{P}^{1}$ (with fixed genus $\tilde{g}:=g(\tilde{C})$ and monodromy) admitting Galois intermediate quotient $\tilde{C} \rightarrow C$ of degree $d \geq 2$. Accordingly, we give the definitions of Prym variety $P(\tilde{C}, C)$ and of higher degree Prym map $\mathcal{P}_{g, r}(d)$.

First, we show that a higher-degree analogue of Proposition 1.2 holds. Indeed we prove the following:

Proposition 1.4. If $\operatorname{dim}\left(S^{2} H^{0}\left(\omega_{\tilde{C}}\right)\right)^{\tilde{G}}-\operatorname{dim}\left(S^{2} H^{0}\left(\omega_{C}\right)\right)^{\tilde{G}}<\operatorname{dim} H^{0}\left(\omega_{\tilde{C}}^{\otimes 2}\right)^{\tilde{G}}$ then the differential of the Prym map $\mathcal{P}_{g, r}(d)$ along the family is not injective. Hence the Prym map has positive dimensional fibres along the family.

It appears that fibrations $h: S \rightarrow B$ obtained from families under the assumption of Proposition 1.4 are quite interesting. Indeed, using them, we produce all the known counterexamples to a conjecture of Xiao.

In [35], Xiao proved that a non trivial fibration with base curve $B \cong \mathbb{P}^{1}$, general fibre of genus $\tilde{g}$ and irregularity $q$, satisfies $q \leq \frac{\tilde{g}+1}{2}$. Furthermore, for $g(B)>0$, he conjectured that the relative irregularity of the fibration $q_{h}$ satisfies $q_{h} \leq \frac{\tilde{q}+1}{2}$. The four known counterexamples have been constructed in [30] and in [1] as fibrations associated with families of cyclic prime odd étale covers of hyperelliptic curves (elliptic curves in case of [30]) carrying a non-injective differential $d \mathcal{P}_{g, r}(d)$. In particular, the three examples of [1] fit perfectly in the structure of our families: a Theorem by Ries ([31]) guarantees that they yield dihedral Galois cover of $\mathbb{P}^{1}$, i.e. they provide towers $\tilde{C} \xrightarrow{d: 1} C \rightarrow$ $\tilde{C} / D_{d} \cong \mathbb{P}^{1}$.

In light of Proposition 1.4, it seemed natural to us to check if there exist other families of Galois coverings $\tilde{C} \xrightarrow{d: 1} C \rightarrow \mathbb{P}^{1}$ with non-injective differential and which disprove the conjecture. Notice that we do not require $C$ to be hyperelliptic and we consider any Galois (cyclic or not) intermediate quotient $\tilde{C} \xrightarrow{d: 1} C$ of any degree. By means of computer calculations (our MAGMA script is available at http://mate.unipv.it/grosselli/publ/), we show the following

Proposition 1.5. $U p$ to $\tilde{g}=12, s=14$ (i.e. dimension 11), the only positive dimensional families of Galois towers $\tilde{C} \xrightarrow{\text { d:1 }} C \rightarrow \mathbb{P}^{1}$ carrying non-injective differential and disproving Xiao's conjecture are the one of [30] and the ones of [1].

The third example of [1, Theorem 1.2] is obtained via a degeneration argument. Therefore one of the four examples that we find with Proposition 1.5 is presented in a slightly different way from the original, although it is clearly the same. For this reason, we think it may be useful to give a very brief description of all the examples.

The paper is organized as follows: in Section 2 we give an overview on Prym maps while in Section 3 we explain our constructive method. Finally, in Section 4, we recall something on Xiao fibrations and we describe the counterexamples.

## 2. The state of the art

In this section we would like to overview the literature on positive dimensional fibres of the Prym maps $\mathcal{P}_{g, r}: \mathcal{R}_{g, r} \rightarrow \mathcal{A}_{g-1+\frac{r}{2}}^{\delta}$. We recall that

$$
\begin{equation*}
\operatorname{dim} \mathcal{R}_{g, r}=3 g-3+r \quad \text { and } \quad \operatorname{dim} \mathcal{A}_{g-1+\frac{r}{2}}^{\delta}=\frac{1}{2}\left(g-1+\frac{r}{2}\right)\left(g+\frac{r}{2}\right) \tag{2.1}
\end{equation*}
$$

First, let us briefly recall the classical case, that is $r=0$ (the standard notation refers to $\mathcal{R}_{g, 0}$ as to $\mathcal{R}_{g}$, the same for $\mathcal{P}_{g, 0}$ ). Easily from (2.1) we see that the generic fibre of $\mathcal{P}_{g}$ is positive dimensional when $g \leq 5$. A detailed study of its geometric structure is provided by the works of Verra for $g=3$ ([34]), Recillas for $g=4$ ([32]) and Donagi for $g=5$ ([6]). Cases with $g=1, g=2$ are summarized in [6, Section 6]. When $g=6$ the fibre is generically finite of degree 27 ([7]). On the other hand, when $g \geq 7$, the Prym map is generically injective but never injective (see [6] and references therein).

The positive dimensional fibres of $\mathcal{P}_{g}$ are characterized as follows:
Theorem 2.1 (Mumford [24], Naranjo [26]). Assume $g \geq 13$. Then $\mathcal{P}_{g}$ has positive dimensional fibres at $(\tilde{C}, C)$ if and only if $C$ is either hyperelliptic or it belongs to one among the components of the bielliptic locus where $\tilde{C}$ carries $\mathbb{Z} / 2 \times \mathbb{Z} / 2 \subseteq \operatorname{Aut}(\tilde{C})$.

Now we focus on the ramified cases, i.e. $r>0$. The inequality $\operatorname{dim} \mathcal{R}_{g, r}>$ $\operatorname{dim} \mathcal{A}_{g-1+\frac{r}{2}}^{\delta}$ is satisfied only in six cases, that is $r=2$ with $1 \leq g \leq 4$ and $r=4$ with $1 \leq g \leq 2$. All of them are considered in [13]. Indeed it is shown the following:

Proposition 2.2. ([13, Proposition 1.2 and Corollary 1.3]) Under the assumptions

$$
(g, r) \in\{(1,2),(1,4),(2,2),(2,4),(3,2),(4,2)\}
$$

the ramified Prym map $\mathcal{P}_{g, r}$ is dominant. Therefore the generic fibre $F_{g, r}$ of $\mathcal{P}_{g, r}$ has $\operatorname{dim} F_{1,2}=1, \operatorname{dim} F_{2,2}=2, \operatorname{dim} F_{3,2}=2, \operatorname{dim} F_{4,2}=1, \operatorname{dim} F_{1,4}=$ $1, \operatorname{dim} F_{2,4}=1$.

Hence the paper gives a detailed description of the generic fibre for all the six cases (see [13, Theorem 0.1]).

Let us now focus on $\operatorname{dim} \mathcal{R}_{g, r} \leq \operatorname{dim} \mathcal{A}_{g-1+\frac{r}{2}}^{\delta}$. The first result is the following:

Theorem 2.3 (Marcucci-Pirola [22], Marcucci-Naranjo [21], Naranjo-Ortega [27]). The ramified Prym map is generically injective as far as the dimension of $\mathcal{R}_{g, r}$ is less than or equal to the dimension of $\mathcal{A}_{g-1+\frac{r}{2}}^{\delta}$.

Actually, the equality between the dimensions is reached only in the case of $g=3$ and $r=4$ where more it is known:

Theorem 2.4 (Nagaraj-Ramanan [25]). Let $g \geq 3$. The Prym map $\mathcal{P}_{g, 4}$ restricted to the locus of tetragonal curves has generically degree 3.

Quite recently Theorem 2.3 has been improved:
Theorem 2.5 (Ikeda [18] for $g=1$, Naranjo-Ortega [28] for all $g$ ). The Prym map $\mathcal{P}_{g, r}$ is injective with injective differential for all $r \geq 6$ and $g>0$.

Thus we will look for the positive dimensional fibres of $\mathcal{P}_{g, 2}$ and $\mathcal{P}_{g, 4}$. Since $\mathcal{P}_{1,2}$ and $\mathcal{P}_{1,4}$ have positive dimensional generic fibre, we will assume $g \geq 2$.

The codifferential of $\mathcal{P}_{g, r}$ at a point $[(C, \eta, B)] \in \mathcal{R}_{g, r}$ is given by the multiplication map ([22])

$$
\begin{equation*}
d \mathcal{P}_{g, r}^{*}(C, \eta, B): S^{2} H^{0}\left(C, \omega_{C} \otimes \eta\right) \rightarrow H^{0}\left(C, \omega_{C}^{2} \otimes \mathcal{O}(B)\right) \tag{2.2}
\end{equation*}
$$

We have the following:
Proposition 2.6 ([28]). Let $L:=\omega_{C} \otimes \eta$. If $d \mathcal{P}_{g, r}$ is not injective at $[(C, \eta, B)]$ then one of the following holds:

1. $L$ is not very ample or
2. L very ample and
(a) $r=2$ and $\operatorname{Cliff}(C) \leq 1$ or
(b) $r=4$ and $\operatorname{Cliff}(C)=0$.

Proof. The proof is a straightforward application of Green-Lazarsfeld Theorem for the surjectivity of a multiplication map (see [16, Theorem 1]): the map $d \mathcal{P}_{g, r}^{*}: S^{2} H^{0}(C, L) \rightarrow H^{0}\left(C, L^{2}\right)$ is surjective if $L$ is very ample and $\operatorname{deg} L \geq$ $2 g+1-2 h^{1}(C, L)-\operatorname{Cliff}(C)$. Since

$$
\operatorname{deg} L=2 g-2+\frac{r}{2} \quad \text { and } \quad h^{1}(C, L)=h^{0}\left(C, \omega_{C} \otimes L^{-1}\right)=0
$$

we conclude.
As already observed in [28, Remark 2.2], the above Proposition can be rephrased as follows:
Proposition 2.7. If the differential $d \mathcal{P}_{g, r}$ is not injective at $[(C, \eta, B)]$ then:

1. $r=2$ and $\eta=\mathcal{O}(x+y-z)$ for $x, y, z \in C$ or $r=4$ and $h^{0}(C, \eta)>0$. Otherwise
2. $r=2$ and $C$ is hyperelliptic, trigonal or a quintic plane curve or $r=4$ and $C$ is hyperelliptic.

Proof. (1) is borrowed from [19, Lemma 2.1] while (2) follows from the definition of the Clifford Index.

Now we list evidence of positive dimensional fibres that we find in the literature. For a proof of these results, we refer to the cited papers.

Proposition 2.8. (Naranjo-Ortega, [28, Theorem 1.2]) Let $\mathcal{P}_{g, r}^{h}$ be the restriction of $\mathcal{P}_{g, r}$ to the locus of coverings of hyperelliptic curves of genus $g$ ramified in $r$ points $(r=2,4)$. Then the generic fibre of $\mathcal{P}_{g, 2}^{h}$, respectively of $\mathcal{P}_{g, 4}^{h}$, is birational to a projective plane, respectively to an elliptic curve.

Remark 2.9. When $g=2$ the restriction $\mathcal{P}_{2, r}^{h}$ coincides with $\mathcal{P}_{2, r}$. Indeed the paper [13] studies $\mathcal{P}_{2,2}$, respectively $\mathcal{P}_{2,4}$, and it shows that the generic fibre is isomorphic to a plane minus 15 lines, respectively to an elliptic curve minus 15 points.

Proposition 2.10. (Naranjo-Ortega, [28, Proposition 2.4]) Let $\mathcal{P}_{5,2}^{t r}$ be the restriction of $\mathcal{P}_{5,2}$ to the locus of coverings of trigonal curves of genus 5 ramified in 2 points. Then the fibres are all positive dimensional.

Proposition 2.11. (Casalaina Martin-Zhang, [4, Theorem 5.1]) Let $\mathcal{R}_{3,4}^{E c k} \subset$ $\mathcal{R}_{3,4}$ be the subset of triples $(C, \eta, B)$ such that $C$ is a smooth quartic plane curve canonically embedded in a plane where $B$ is reduced and cut by a line $l$ and $\eta$ is of degree 2 and cut by a bitangent. Then the generic fibre of $\mathcal{P}_{3,4}$ restricted to $\mathcal{R}_{3,4}^{E c k}$ is isomorphic to the elliptic curve described as the covering of $l$ ramified on $B$.

Remark 2.12. Notice that here the positive dimensional fibres are realized under the assumption $\eta$ effective.

## 3. The procedure

In this section, we describe our strategy to investigate infinitely many positive dimensional fibres of the Prym maps. In particular, we study families of towers of Galois covers $\tilde{C} \xrightarrow{2: 1} C \rightarrow \mathbb{P}^{1}$. Our procedure does not bound the genus of the curves occurring in such towers. For this reason, we are able to describe infinitely many examples.

In order to do this, we introduce the Prym datum as given in [5, 11]. We recall the definition.

Definition 3.1. A Prym datum of type $(s, r)$ is a triple $\Xi=(\tilde{G}, \tilde{\theta}, \sigma): \tilde{G}$ is a finite group, $\tilde{\theta}: \Gamma_{s} \rightarrow \tilde{G}$ is an epimorphism, $\sigma \in \tilde{G}$ is a central involution and

$$
r=\sum_{i: \sigma \in\left\langle\tilde{\theta}\left(\gamma_{i}\right)\right\rangle} \frac{|\tilde{G}|}{\operatorname{ord}\left(\tilde{\theta}\left(\gamma_{i}\right)\right)},
$$

where $\Gamma_{s}:=\left\langle\gamma_{1}, \ldots, \gamma_{s}: \gamma_{1} \ldots \gamma_{s}=1\right\rangle$.
This datum corresponds to a family of Galois coverings and the generic point of the family fits in the following diagram

$$
\begin{equation*}
\tilde{C} \underset{\mathbb{P}^{1}}{\text { }} \underset{\sim}{f} \longleftrightarrow C=\tilde{C} /\langle\sigma\rangle \tag{3.1}
\end{equation*}
$$

where $\tilde{C}$ and $C$ are curves of genus $\tilde{g}$ and $g$ respectively. The cover $\tilde{C} \rightarrow \mathbb{P}^{1}$ is branched on $s$ points. The cover $f$ is 2 -sheeted and branched on a degree $r$ divisor $B$. Let $\eta \in \operatorname{Pic}^{\frac{r}{2}}(C)$ be the corresponding line bundle such that $\eta^{2}=\mathcal{O}_{C}(B) . G:=\tilde{G} /\langle\sigma\rangle$ is the quotient group acting on $C$, the composition of $\tilde{\theta}$ with the projection to the quotient is an epimorphism $\theta: \Gamma_{s} \rightarrow G$. The maps $\tilde{\theta}$ and $\theta$ are respectively the monodromies of the two Galois covers $\tilde{C} \rightarrow$ $\mathbb{P}^{1}=\tilde{C} / \tilde{G}$ and $C \rightarrow \mathbb{P}^{1}=C / G$. Moreover, in all the examples we consider, $s=4$ and $C \rightarrow \mathbb{P}^{1}$ is branched on the same points of $\tilde{C} \rightarrow \mathbb{P}^{1}$.

There is a natural identification between the tangent space to the family at the generic point and the space of the infinitesimal deformations of $\tilde{C}$ that preserve the action of $\tilde{G}$. The latter is isomorphic to $H^{1}\left(\tilde{C}, T_{\tilde{C}}\right)^{\tilde{G}}$ $\left(\left(\cong H^{0}\left(\tilde{C}, \omega_{\tilde{C}}^{2}\right)^{\tilde{G}}\right)^{*}\right)$. Thus $\operatorname{dim} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}^{2}\right)^{\tilde{G}}=s-3$ equals the dimension of the family.

Recall that $\sigma$ gives a decomposition of $V:=H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)$ in $\pm 1$-eigenspaces, resp. $V_{+}$and $V_{-}$, where $V_{+} \cong H^{0}\left(C, \omega_{C}\right)$ and $V_{-} \cong H^{0}\left(C, \omega_{C} \otimes \eta\right)$. Similarly we can define $W:=H^{0}\left(\tilde{C}, \omega_{\tilde{C}}^{2}\right)$ and get a decomposition $W=W_{+} \oplus W_{-}$with $W_{+} \cong H^{0}\left(C, \omega_{C}^{2} \otimes \eta^{2}\right)=H^{0}\left(C, \omega_{C}^{2} \otimes \mathcal{O}(B)\right)$. Let us denote

$$
\begin{equation*}
\tilde{N}:=\operatorname{dim}\left(S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)\right)^{\tilde{G}} \quad \text { and } \quad N:=\operatorname{dim}\left(S^{2} H^{0}\left(C, \omega_{C}\right)\right)^{G} \tag{3.2}
\end{equation*}
$$

Immediately, we have $\tilde{N}-N=\operatorname{dim}\left(S^{2} V_{-}\right)^{\tilde{G}}$.
We are interested in families that lie in positive dimensional fibres of the Prym map, so we look for a condition that makes the codifferential of the Prym map not surjective. We have the following

Proposition 3.2. If $\tilde{N}-N<s-3$ the differential of the Prym map $d \mathcal{P}_{g, r}$ along the family is not injective, hence the Prym map has positive dimensional fibres along the family.

Proof. As in (2.2), the codifferential of the Prym map at the generic element of the family is the multiplication map

$$
m=\left(d \mathcal{P}_{g, r}\right)^{*}:\left(S^{2} V_{-}\right)^{\tilde{G}} \rightarrow W_{+}^{\tilde{G}}
$$

Since by assumption $\operatorname{dim}\left(S^{2} V_{-}\right)^{\tilde{G}}=\tilde{N}-N<s-3=\operatorname{dim} W_{+}^{\tilde{G}}$ the codifferential cannot be surjective hence its dual has a non-trivial kernel.

Corollary 3.3. If $\tilde{N}=N$ the family is contained in a fibre of the Prym map.
Now we present our construction. It produces infinitely many Prym data. They yield 1-dimensional families of Galois towers of type (3.1) that carry constant Prym variety. Thus they lie in positive dimensional fibres of the Prym map. We organize these data into 5 classes as follows.

Theorem 3.4. For any $N \in \mathbb{N}$ there are 1-dimensional families of Galois covers $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^{1}$ in the fibres of the Prym maps $\mathcal{P}_{2 N, 2}, \mathcal{P}_{2 N-1,0}$ (2 families), $\mathcal{P}_{2 N, 0}$ and $\mathcal{P}_{2 N-1,4}$. All the families lie in their respective hyperelliptic locus.

All the families carry an abelian Galois group, so we can refer to [12, Section 4]. We describe in detail the first case.

Fix a positive integer $N$, set the odd number $k=2 N+1$ and denote $C_{n} \cong \mathbb{Z} / n \mathbb{Z}$. The Prym datum is defined by the group $\tilde{G}=C_{2} \times C_{2 k} \subset C_{2 k}^{2}$ under the inclusion $\binom{1}{0} \mapsto\binom{k}{0},\binom{0}{1} \mapsto\binom{0}{1}$, the monodromy $\tilde{\theta}: \Gamma_{4} \rightarrow \tilde{G}$ is represented by the matrix

$$
A=\left(\begin{array}{cccc}
k & 0 & 0 & k \\
0 & k & 2 & k-2
\end{array}\right)
$$

where the $i$-th column is $\tilde{\theta}\left(\gamma_{i}\right)$ under the inclusion in $C_{2 k}^{2}$ and the involution $\sigma=(k, k)^{t}$.

In case of abelian groups, the character group $\tilde{G}^{*}=\operatorname{Hom}(\tilde{G}, \mathbb{C})$ is isomorphic to $\tilde{G}$. In our situation, a character in $\tilde{G}^{*}$ can be identified with an element $n=\left(n_{1}, n_{2}\right)$ where $n_{1} \in C_{2}$ and $n_{2} \in C_{2 k}$. Set $V=H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)$ and let $V=V_{+} \oplus V_{-}$be the eigenspace decomposition induced by the action of $\sigma$. As before, $V_{+} \cong H^{0}\left(C, \omega_{C}\right)$.

Our target is to compute the dimension of $\left(S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{-}\right)^{\tilde{G}}$ and to show that it is zero. In this way, the codifferential of the Prym map would be trivial on the family and thus the Prym map would be constant.

Let $H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{n}$ be the subspace of $H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)$ where $\tilde{G}$ acts via the character $n$, and denote by $d_{n}$ its dimension. By [23, Prop. 2.8], the dimension for a non trivial character $n$ is given by

$$
d_{n}=-1+\sum_{i=1}^{4}\left\langle-\frac{\alpha_{i}}{2 k}\right\rangle
$$

where $\langle x\rangle$ denotes the fractional part of a real number $x$ and

$$
\alpha=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}\right):=\left(n_{1}, n_{2}\right) \cdot A=\left(k n_{1}, \quad k n_{2}, \quad 2 n_{2}, \quad k n_{1}+(k-2) n_{2}\right) .
$$

We remind that if $x \in \mathbb{R} \backslash \mathbb{Z}$ then $\langle x\rangle+\langle-x\rangle=1$. It is straightforward that for any $n$ such that $2 n=0$ we have $d_{n}=0$. Indeed

$$
d_{n}=-1+\left\langle\frac{-n_{1}}{2}\right\rangle+\left\langle\frac{-n_{2}}{2}\right\rangle+\left\langle\frac{-n_{2}}{k}\right\rangle+\left\langle-\frac{k n_{1}+(k-2) n_{2}}{2 k}\right\rangle,
$$

therefore when $n_{1}=0, n_{2}=k$

$$
d_{n}=-1+0+\left\langle-\frac{n_{2}}{2}\right\rangle+0+\left\langle-\frac{n_{2}}{2}\right\rangle=0
$$

and when $n_{1}=1, n_{2}=0, k$

$$
d_{n}=-1+\frac{1}{2}+\left\langle\frac{-n_{2}}{2}\right\rangle+0+\left\langle-\frac{1+n_{2}}{2}\right\rangle=0
$$

So now we suppose $-n \neq n$, i.e. $-n_{2} \neq n_{2}$, so $n_{2} \neq 0, k$.

- If $n_{1}=0$ and $n_{2}$ is even, then $d_{n}=-1+\left\langle-\frac{n_{2}}{k}\right\rangle+\left\langle-\frac{(k-2) n_{2}}{2 k}\right\rangle=0$.
- If $n_{1}=0$ and $n_{2}$ is odd, then $d_{n}=-1+\frac{1}{2}+\left\langle-\frac{n_{2}}{k}\right\rangle+\left\langle-\frac{(k-2) n_{2}}{2 k}\right\rangle$. Thus $d_{n}+d_{-n}=1$, hence exactly one between $d_{n}$ and $d_{-n}$ is 1 and the other is 0 . In both cases the product $d_{n} d_{-n}$ is zero. Moreover the sum of all $d_{n}$ of this kind is $\frac{k-1}{2}$.
- If $n_{1}=1$ and $n_{2}$ is even, then $d_{n}=-1+\frac{1}{2}+\left\langle-\frac{n_{2}}{k}\right\rangle+\left\langle-\frac{k+(k-2) n_{2}}{2 k}\right\rangle$. As in the previous case $d_{n}+d_{-n}=1$ and $d_{n} d_{-n}$ is always 0 . Therefore the sum of all $d_{n}$ of this kind equals $\frac{k-1}{2}$.
- If $n_{1}=1$ and $n_{2}$ is odd, then $d_{n}=-1+\frac{1}{2}+\frac{1}{2}+\left\langle-\frac{n_{2}}{k}\right\rangle+\left\langle-\frac{k+(k-2) n_{2}}{2 k}\right\rangle=1$. So all $k-1$ terms of this form are 1 , hence there are $\frac{k-1}{2}$ couples such that $d_{n} d_{-n}=1$.

The decomposition $H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)=\bigoplus_{n} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{n}$ gives us

$$
\tilde{g}=g(\tilde{C})=\operatorname{dim} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)=\sum_{n} d_{n}=2 k-2
$$

Moreover we have

$$
\operatorname{dim}\left(S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)\right)^{\tilde{G}}=\frac{1}{2} \sum_{2 n \neq 0} d_{n} d_{-n}=\frac{k-1}{2}=N
$$

As explained in [12, Lemma 4.3] the terms of $H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)$ which are invariant for the action of $\sigma$ are those such that $n_{1}+n_{2}$ is even. Hence the genus $g$ of $C$ is

$$
g=\operatorname{dim} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{+}=\sum_{n_{1}+n_{2} \text { even }} d_{n}=k-1
$$

and by the Riemann-Hurwitz formula we obtain that the degree of the ramification divisor is

$$
r=2 \tilde{g}-2-2(2 g-2)=2(2 k-2)-4(k-1)+2=2 .
$$

Finally we compute the dimension of $\left(S^{2} V_{+}\right)^{\tilde{G}}$ :

$$
\operatorname{dim}\left(S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{+}\right)^{\tilde{G}}=\frac{1}{2} \sum_{n_{1}+n_{2} \text { even }} d_{n} d_{-n}=\frac{k-1}{2}=N
$$

This gives $\left(S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{-}\right)^{\tilde{G}}=0$ and thus it allows to conclude.
It only remains to observe that the unique element of $G=\tilde{G} /\langle\sigma\rangle \cong C_{2 k}$ of order 2 gives the hyperelliptic involution of $C$.

In the following table, we summarize all the examples outlined in Theorem 3.4. For any integer $N$ we define $k$ and consequently, we give the data. The first line corresponds to the example described above. As seen $\left(S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{+}\right)^{\tilde{G}}=N$. The same holds for the remaining families. Since computations are almost identical, except case (3) which is slightly more tricky, we do not repeat them.

| $n$ | $k$ | $\tilde{g}$ | $g$ | $r$ | $\tilde{G}$ | $A$ |  |  | $\sigma$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $(1)$ | $2 N+1$ | $2 k-2$ | $k-1$ | 2 | $C_{2} \times C_{2 k}$ | $\left(\begin{array}{cccc}k & 0 & 0 & k \\ 0 & k & 2 & k-2\end{array}\right)$ | $\binom{k}{k}$ |  |  |
| $(2)$ | $2 N-1$ | $2 k-1$ | $k$ | 0 | $C_{2} \times C_{2 k}$ | $\left(\begin{array}{cccc}k & 0 & 0 & k \\ 0 & k & 1 & k-1\end{array}\right)$ | $\binom{k}{k}$ |  |  |
| $(3)$ | $N$ | $4 k-3$ | $2 k-1$ | 0 | $C_{2} \times C_{2 k}$ | $\left(\begin{array}{cccc}k & k & 0 & 0 \\ k-1 & k-1 & 1 & 1\end{array}\right)$ | $\binom{k}{k}$ |  |  |
| $(4)$ | $2 N$ | $2 k$ | $k$ | 0 | $C_{2} \times C_{2 k}$ | $\left(\begin{array}{cccc}k & 0 & 0 & k \\ 0 & k & 1 & k-1\end{array}\right)$ | $\binom{k}{k}$ |  |  |
| $(5)$ | $2 N$ | $2 k-1$ | $k-1$ | 4 | $C_{2 k}$ | $\left(\begin{array}{llll}1 & 1 & k-1 & k-1\end{array}\right)$ | $(k)$ |  |  |

In order to find other (possibly higher dimensional) families contained in the fibres of $\mathcal{P}_{g, r}$ we use a MAGMA script similar to the one described in [11] and [12]. In these papers, the authors look for Shimura subvarieties generically contained in the Prym locus. For this reason, they require the differential of the Prym map to be an isomorphism. Here we want exactly the converse, so we look for data that satisfy Proposition 3.2. In this way, MAGMA produces various examples of families of different dimensions and genus. While in the higher dimensional cases we only find some sporadic examples, in the 1-dimensional case we realized that the examples behaved cyclically in the same way. This motivated our choice to organize them into five classes as in the above table.

## 4. The Xiao conjecture

In this section we look at the higher-degree analogue of the Prym maps $\mathcal{P}_{g, r}$. Indeed nothing prevents us to consider Galois covers of curves $C \in \mathcal{M}_{g}$ of
degree greater than 2. The theory of Prym varieties easily extends to these cases (see [19] for cyclic covers). In general, to any Galois covering $f: \tilde{C} \xrightarrow{d: 1} C$ one can associate an abelian variety $P:=P(\tilde{C}, C)$ defined as the connected component to the origin of the kernel of the Norm map $\mathrm{Nm}: J \tilde{C} \rightarrow J C$. Letting $\tilde{g}$, resp. $g$, be the genus of the curves $\tilde{C}$ and $C$, then $P$ has dimension $\tilde{g}-g$ and inherits a (non-principal) polarization $L$ from the theta divisor associated with $J \tilde{C}$.

We denote by $\mathcal{R}(K, g, r)$ the Hurwitz scheme parametrizing coverings $f: K$ is the Galois group acting on $\tilde{C}, g$ is the genus of the quotient curve and $r$ the number of branch points. Thus we can define the Prym map:

$$
\begin{align*}
\mathcal{P}_{g, r}(d): \mathcal{R}(K, g, r) & \longrightarrow \mathcal{A}_{\tilde{g}-g}^{\delta}  \tag{4.1}\\
{[f] \longmapsto } & \longmapsto P, L]
\end{align*}
$$

Let us fix a group $\tilde{G} \subseteq \operatorname{Aut}(\tilde{C})$ with $K \triangleleft \tilde{G}$ normal subgroup. For computational reasons, we assume that $\tilde{C}$ has Galois quotient $\tilde{C} / \tilde{G}$ isomorphic to $\mathbb{P}^{1}$ and we let $s$ be the number of branch points of $\tilde{C} \rightarrow \tilde{C} / \tilde{G} \cong \mathbb{P}^{1}$. This means that we focus on towers $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^{1}$. By moving the branch points in $\mathbb{P}^{1}$, we get a family of dimension $s-3$. This family naturally gives rise to a family of the same dimension contained in $\mathcal{R}(K, g, r)$. For more details, we refer the reader to Section 3, where such construction is considered in case of covers $f$ of degree 2 .

If we set $V:=H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)$, we decompose $V=V_{+} \oplus V_{-}$, where we identify $V_{+}:=H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)^{K}\left(\cong H^{0}\left(C, \omega_{C}\right)\right)$ and $V_{-}:=H^{1,0}(P)$. Similarly we define $H_{1}(\tilde{C}, \mathbb{Z})_{+}$and $H_{1}(\tilde{C}, \mathbb{Z})_{-}$. As already seen, the tangent space to the family at the generic point is $H^{1}\left(\tilde{C}, T_{\tilde{C}}\right)^{\tilde{G}}$. The space of the infinitesimal deformations of $(P, L)$ that preserve the action of $\tilde{G}$ is $S^{2} H^{0,1}(P)^{\tilde{G}}$. Indeed $P=V_{-}^{*} / H_{1}(\tilde{C}, \mathbb{Z})_{-}$, hence the tangent space of $P$ at the origin is $V_{-}^{*}$ and thus $T_{P} \mathcal{A}_{\tilde{g}-g}=S^{2} V_{-}^{*}$. Since we have an inclusion of $\tilde{G}$ in Aut $P$, we restrict to the $\tilde{G}$-invariant part. Thus the differential of the Prym map $d \mathcal{P}_{g, r}(d)$ yields the map

$$
d \mathcal{P}_{g, r}(d): H^{1}\left(\tilde{C}, T_{\tilde{C}}\right)^{\tilde{G}} \rightarrow S^{2} H^{0,1}(P)^{\tilde{G}}
$$

Dualizing we get

$$
d \mathcal{P}_{g, r}(d)^{*}=\left.m\right|_{S^{2} H^{1,0}(P)^{\tilde{G}}}: S^{2} H^{1,0}(P)^{\tilde{G}} \rightarrow H^{0}\left(\tilde{C}, \omega_{\tilde{C}}^{\otimes 2}\right)^{\tilde{G}}
$$

where, as usual, the multiplication map $m: S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right) \rightarrow H^{0}\left(\tilde{C}, \omega_{\tilde{C}}^{\otimes 2}\right)$ is the codifferential of the Torelli map.

As in (3.2), we define $\tilde{N}, N$. We have a higher degree analogue of Proposition 3.2:

Proposition 4.1. If $\tilde{N}-N<s-3$, the differential of the Prym map $d \mathcal{P}_{g, r}(d)$ along the family is not injective, hence the Prym map has positive dimensional fibres along the family.

Proof. The proof works exactly like the one of Proposition 3.2.
Starting from our families of curves, we can construct fibrations $h: S \rightarrow B$ that carry $\tilde{C}$ as fibres. Indeed the closure of the image of the modular map $t \mapsto\left[\tilde{C}_{t}\right]$ gives a curve in $\mathcal{M}_{\tilde{g}}$. Then, up to resolving singularities and taking pull-backs, we get a fibration $h: S \rightarrow B$, as claimed.

The irregularity of the surface $S$ is $q:=\operatorname{dim} H^{1}\left(S, \mathcal{O}_{S}\right)$, and $q_{h}=q-g(B)$ is called relative irregularity of the fibration. It is quite famous that Xiao in [35] proved the following

Theorem 4.2 (Xiao). If $h$ is not isotrivial and $B=\mathbb{P}^{1}$ then

$$
q \leq \frac{\tilde{g}+1}{2}
$$

Furthermore he conjectured that, for a base $B$ of positive genus, the relative irregularity of the fibration should satisfy

$$
\begin{equation*}
q_{h} \leq \frac{\tilde{g}+1}{2} \tag{4.2}
\end{equation*}
$$

It is known that the inequality (4.2) is false: Pirola in [30], resp. Albano and Pirola in [1], explicitly constructed 1 fibration, resp. 3 fibrations, that do not satisfy (4.2). Indeed, a modified version of the conjecture supposes $q_{h} \leq\left\lceil\frac{\tilde{g}+1}{2}\right\rceil$.

All the counterexamples are constructed considering families of covers in $\mathcal{R}(K, g, r)$. They have data:

- $K=\mathbb{Z} / 3 \mathbb{Z}, g=1, r=3 ;$
- $K=\mathbb{Z} / 5 \mathbb{Z}, g=2, r=0$;
- $K=\mathbb{Z} / 3 \mathbb{Z}, g=4, r=0$;
- $K=\mathbb{Z} / 3 \mathbb{Z}, g=3, r=0$;

They all have constant Prym variety: indeed they have been found by considering families of coverings of hyperelliptic curves (elliptic curves in the example of Pirola [30]) which lie in positive dimensional fibres of Prym maps.
Remark 4.3. The family studied by Pirola, i.e. the first one listed above, turns out to be interesting also from another point of view. Indeed, in [14], the authors show that the locus described by $J \tilde{C}$, for $\tilde{C}$ varying in the family, yields a Shimura subvariety of $\mathcal{A}_{4}$ generically contained in the Torelli locus. Indeed it satisfies the condition $(*)$ studied in the same paper which is sufficient to
produce Shimura subvarieties generically contained in the Torelli locus. Moreover, in [15] and also in [10], it is proven that, via its Prym map, it is fibred in totally geodesic curves, countably many of which are Shimura. One of these Shimura fibres is the family (12) of [9]. This is exactly the family we use to study the family of Pirola.
Remark 4.4. The example with data $K=\mathbb{Z} / 5 \mathbb{Z}, g=2, r=0$ is curious in the same spirit of the previous Remark. Indeed, by [31, Theorem 3.1], it involves curves whose Jacobians have a non-trivial endomorphism algebra and the endomorphisms are not induced by the automorphisms of the curves. In [33], the second author shows that the Jacobians of such curves yield a new explicit Shimura subvariety of $\mathcal{A}_{2}$ generically contained in the Torelli locus.

In order to find new counterexamples to Xiao's conjecture, it seems quite natural to generalize the idea of [30] and [1] considering families of towers $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^{1}$ whose Prym map is constant but without requiring $C$ to be hyperelliptic and considering any Galois covering $\tilde{C} \rightarrow C$ of any degree. We have the following:

Proposition 4.5. Up to $\tilde{g}=12, s=14$ (i.e. dimension 11), the only positive dimensional families of Galois towers $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^{1}$ that have $\tilde{N}-N<s-3$ and disprove (4.2) are the family (12) of [9] and the three examples of [1].

Proof. Using Proposition 4.1, we know that when $\tilde{N}-N<s-3$ the differential of the Prym map associated with the family is not injective. Under this assumption, computer calculations in MAGMA that impose $q_{h}>\frac{\tilde{q}+1}{2}$ find only the four examples of the statement.

Now we would like to explicitly describe the four examples as families of Galois towers $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^{1}$.

The first example we treat is the family (12) of [9]. Indeed the family of Pirola ([30]) cannot be realized as a Galois cover of $\mathbb{P}^{1}$. For this reason, we will study the family (12) of [9] which is contained in the family of Pirola and which describes Galois towers $\tilde{C} \rightarrow C \rightarrow \mathbb{P}^{1}$, as already mentioned in Remark 4.3.

Example 1.
$d=3, s=4, \tilde{g}=4, g=1, r=3$.
$\tilde{G}_{\tilde{\theta}}=G(6,2)_{\tilde{\theta}}=\mathbb{Z} / 6 \mathbb{Z}=\left\langle g: g^{6}=1\right\rangle$.
$\left(\tilde{\theta}\left(\gamma_{1}\right), \ldots, \tilde{\theta}\left(\gamma_{4}\right)\right)=\left(g^{3}, g^{5}, g^{5}, g^{5}\right), \quad K=\left\langle\sigma=g^{2}\right\rangle \cong \mathbb{Z} / 3 \mathbb{Z}$. The monodromy matrix is $A=(3,5,5,5)$.
The action of $\tilde{G}$ gives the decomposition $H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)=W_{3} \oplus W_{4} \oplus W_{5}$, where $W_{n}$ is the subspace of $H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)$ where $\tilde{G}$ acts via the character $n$. We have $\operatorname{dim} W_{3}=\operatorname{dim} W_{4}=1$ and $\operatorname{dim} W_{5}=2$. Moreover $H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{+}=W_{3}$. Therefore $\left(S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)\right)^{\tilde{G}}=\left(S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{+}\right)^{\tilde{G}}=S^{2} W_{3}$ and so the multiplication map $m:\left(S^{2} V_{-}\right)^{\tilde{G}} \rightarrow H^{0}\left(\omega_{\tilde{C}}^{\otimes 2}\right)^{\tilde{G}}$ is trivial. Hence the family is a curve that
lies in a fibre of the Prym map $\mathcal{P}_{1,3}(3)$. The relative irregularity of the fibred surface is $q_{h}=\tilde{g}-g=3>\frac{5}{2}=\frac{\tilde{g}+1}{2}$, hence it violates the inequality (4.2).

Next example is the first one that appears in [1, Section 4], given by an étale 5:1 cover.

Example 2.
$d=5, s=6, \tilde{g}=6, g=2, r=0$.
$\tilde{G}=G(10,1)=D_{5}=\left\langle g_{1}, g_{2}: g_{1}^{2}=g_{2}^{5}=1, g_{1} g_{2} g_{1}=g_{2}^{-1}\right\rangle$.
$\left(\tilde{\theta}\left(\gamma_{1}\right), \ldots, \tilde{\theta}\left(\gamma_{6}\right)\right)=\left(g_{1} g_{2}^{2}, g_{1} g_{2}^{4}, g_{1} g_{2}, g_{1} g_{2}^{4}, g_{1} g_{2}, g_{1} g_{2}\right)$,
$K=\left\langle\sigma=g_{2}\right\rangle \cong \mathbb{Z} / 5 \mathbb{Z}$.
Using the notation of MAGMA, we get $H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)=2 V_{2} \oplus V_{3} \oplus V_{4}$, where $V_{i}$ are irreducible representations of $\tilde{G}$ such that $\operatorname{dim} V_{2}=1$ and $\operatorname{dim} V_{3}=\operatorname{dim} V_{4}=2$. We have that $\left(S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)\right)^{\tilde{G}}=3 S^{2} V_{2} \oplus\left(S^{2} V_{3}\right)^{\tilde{G}} \oplus\left(S^{2} V_{4}\right)^{\tilde{G}}$ has dimension 5 and that $H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{+}=2 V_{2}$. Therefore $\operatorname{dim}\left(S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{+}\right)^{\tilde{G}}=3$. Since $\operatorname{dim}\left(S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{-}\right)^{\tilde{G}}=2<s-3=3$, the Prym map is not injective on the family. Again inequality (4.2) does not hold: $\tilde{g}-g=4>\frac{7}{2}=\frac{\tilde{q}+1}{2}$.

Here we have the example of [1, Section 5].

## Example 3.

$d=3, s=10, \tilde{g}=10, g=4, r=0$.
$\tilde{G}=G(6,1)_{\tilde{\theta}}=D_{3}=\left\langle g_{1}, g_{2}: g_{1}^{2}=g_{2}^{3}=1, g_{1} g_{2} g_{1}=g_{2}^{-1}\right\rangle$.
$\left(\tilde{\theta}\left(\gamma_{1}\right), \ldots, \tilde{\theta}\left(\gamma_{10}\right)\right)=\left(g_{1}, g_{1}, g_{1} g_{2}^{2}, g_{1} g_{2}^{2}, g_{1} g_{2}, g_{1}, g_{1} g_{2}^{2}, g_{1} g_{2}, g_{1} g_{2}, g_{1}\right)$,
$K=\left\langle\sigma=g_{2}\right\rangle \cong \mathbb{Z} / 3 \mathbb{Z}$.
In this case $H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)=4 V_{2} \oplus 3 V_{3}$, where $\operatorname{dim} V_{2}=1$ and $\operatorname{dim} V_{3}=2$, $H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{+}=4 V_{2}$. Then $\left(S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)\right)^{\tilde{G}}=10 S^{2} V_{2} \oplus 6\left(S^{2} V_{3}\right)^{\tilde{G}}$ has dimension 16 and $\left(S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{+}\right)^{\tilde{G}}=10 S^{2} V_{2}$ has dimension 10. Hence $\tilde{N}-N=$ $6<7$ guarantees $d \mathcal{P}_{4,0}(3)$ not injective. Moreover $\tilde{g}-g=6>\frac{11}{2}=\frac{\tilde{g}+1}{2}$ violates (4.2).

Finally we have the example [1, Section 6].

## Example 4.

$d=3, s=7, \tilde{g}=6, g=2, r=2$.
$\tilde{G}=G(6,1)_{\tilde{\theta}}=D_{3}=\left\langle g_{1}, g_{2}: g_{1}^{2}=g_{2}^{3}=1, g_{1} g_{2} g_{1}=g_{2}^{-1}\right\rangle$.
$\left(\tilde{\theta}\left(\gamma_{1}\right), \ldots, \tilde{\theta}\left(\gamma_{7}\right)\right)=\left(g_{1} g_{2}, g_{1} g_{2}, g_{1}, g_{1} g_{2}^{2}, g_{1}, g_{1}, g_{2}\right)$,
$K=\left\langle\sigma=g_{2}\right\rangle \cong \mathbb{Z} / 3 \mathbb{Z}$.
MAGMA gives $H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)=2 V_{2} \oplus 2 V_{3}$, where $\operatorname{dim} V_{2}=1$ and $\operatorname{dim} V_{3}=2$ and $H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{+}=2 V_{2}$. We have that $\left(S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)\right)^{\tilde{G}}=3 S^{2} V_{2} \oplus 3\left(S^{2} V_{3}\right)^{\tilde{G}}$ has dimension 6 and that $\left(S^{2} H^{0}\left(\tilde{C}, \omega_{\tilde{C}}\right)_{+}\right)^{\tilde{G}}=3 S^{2} V_{2}$ has dimension 3. Once again the differential of the Prym map along the family is not injective and $\tilde{g}-g=$ $4>\frac{7}{2}=\frac{\tilde{g}+1}{2}$ violates (4.2).

As already said, our data give a slightly different presentation of the last example presented by [1]. We easily observe that they are the same. Indeed our family yields the following diagram:

the curve $D$ is obtained as the quotient of $\tilde{C}$ by a lift of the hyperelliptic involution of $C$. We have 7 branch points $z_{1}, \ldots, z_{7}$ in $\mathbb{P}^{1}$ and the map $\tilde{C} \rightarrow \mathbb{P}^{1}$ has three ramification points of order 2 over $z_{1}, \ldots, z_{6}$. Let us call them $p_{i j}, i=$ $1, \ldots, 6, j=1,2,3$. Moreover $\psi$ has 2 ramification points of order three over $z_{7}$. Let us call them $q_{1}, q_{2}$. The $2: 1$ map $\tilde{C} \rightarrow D$ ramifies on one among the $p_{i j}$ for every $i$, while it is étale over the remaining two and it is étale over $q_{1}, q_{2}$. If we denote $p=\pi\left(q_{1}\right)=\pi\left(q_{2}\right)$, then we get the map $D \xrightarrow{3: 1} \mathbb{P}^{1}$ associated with the linear system $|3 p|$. This is the starting point of the construction of Albano and Pirola. Indeed the map $\tilde{C} \rightarrow C$ is étale over all the $p_{i j}$ 's while it is completely ramified over $q_{1}$ and $q_{2}$. When these two points are glued we get the singular curve $C_{p}$ of [1]. The fibration of $[1$, Section 6$]$ is constructed desingularizing such curves $C_{p}$, i.e. considering the curves $\tilde{C}$ provided by our example. Therefore this example is clearly another presentation of the one discussed in [1, Section 6].

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# Prym varieties and Prym map 

## PaweŁ Borówka and Angela Ortega


#### Abstract

These are introductory notes to the theory of Prym varieties. Subsequently, we focus on the description and geometry of the fibres of the Prym map for étale double coverings over genus 6 curves.


Keywords: Prym varieties, Prym map. MS Classification 2020: 14H40, 14H30.

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P. BORÓWKA AND A. ORTEGA

## 1. Introduction

Given a finite morphism between smooth curves one can associate to it a polarized abelian variety (not necessarily principally polarized), called Prym variety. This construction induces a map from the moduli space of coverings to the moduli space of polarized abelian varieties, know as Prym map, depending on the genus of the base curve, the degree of the map and its ramification pattern. The classical Prym varieties revisited by Mumford in [25] are principally polarized obtained from double coverings (étale or ramified in two points). Since then they have been studied not only as a way of understanding abelian varieties and their moduli space, but also as interesting objects on their own, see for instance the recent work in $[1,9,20,21,30,31]$. Prym maps in low genera often display very rich geometry and interesting structure. These notes summarize the early developments in the theory of Prym varieties which continue to inspire recent work in algebraic geometry. As an introduction to the subject, we chose to focus on the structure of the fibres of the Prym $\mathcal{P}_{6}$ for étale double coverings over a genus 6 curve, which is generically finite of degree 27 . The computation of the degree of $\mathcal{P}_{6}$ is the ideal occasion to encounter classical algebraic objects (cubic surfaces and threefolds, plane quintics, Fano surface of lines, etc.), geometric constructions (tetragonal and trigonal constructions, conic bundles), as well as moduli spaces (of coverings, abelian varieties, intermediate Jacobians). We tried to put together the main ingredients for a good understanding of the geometric structure of the fibres of $\mathcal{P}_{6}$.

These notes cover the material presented in the course "Prym Varieties" of the Trieste Algebraic Summer School (TAGSS) 2021 given by the second author. The series of lectures included exercise sessions run by the first author. Some of the exercises can be found here. The main references are Beauville, Donagi and Donagi-Smith papers [4, 5, 14, 15].

## 2. Basics on abelian varieties

Through this notes we work over $\mathbb{C}$.
Definition 2.1. A complex torus $A$ is a quotient $V / \Lambda$, with $V \simeq \mathbb{C}^{g} a \mathbb{C}$ vector space and $\Lambda \simeq \mathbb{Z}^{2 g}$ a full rank lattice inside $V$. A polarization on $A$ is an ample line bundle ${ }^{1} L$ on $A$. An abelian variety is a complex torus admitting a polarization, so $(A, L)$ is polarized abelian variety.

REmARK 2.2. In particular, with the addition operation inherited from $V$, an abelian variety is an abelian group.

[^0]By definition of ampleness, given a line bundle $L$ on $A$ we have that the map

$$
\begin{aligned}
\varphi_{L^{\otimes k}}: & A \hookrightarrow \mathbb{P} H^{0}\left(A, L^{\otimes k}\right)^{*} \\
& x \mapsto\left[s_{0}(x): s_{1}(x): \cdots: s_{N}(x)\right],
\end{aligned}
$$

defined by the sections of $L^{\otimes k}$ is an embedding for some $k>1$. In fact, in the case of polarized abelian varieties it suffices to take $k=3$. Then an abelian variety is also a projective variety.

Different incarnations of a polarization on $A$. The following data are equivalent:

- A first Chern class $c_{1}(L) \in H^{2}(A, \mathbb{Z})$ of an ample line bundle $L$ on $A$.
- A non degenerated alternating form $E: V \times V \rightarrow \mathbb{R}$ such that $E(\Lambda, \Lambda) \subset \mathbb{Z}$ and $E(i v, i w)=E(v, w)$.
- A positive definite Hermitian form $H: V \times V \rightarrow \mathbb{C}$ with $H(\Lambda, \Lambda) \subset \mathbb{Z}$.
- An isogeny $\phi_{L}: A \rightarrow \widehat{A}:=\operatorname{Pic}^{0}(A)$ that satisfies 'positivity' properties.
- An effective Weil divisor $\Theta \subset A$ such that the subgroup $\left\{x \in A \mid t_{x}^{*} \Theta \sim\right.$ $\Theta\}$ is finite.
Let $E$ be an alternating form representing a polarization on $A=V / \Lambda$. There exists a basis $\lambda_{1}, \ldots, \lambda_{g}, \mu_{1}, \ldots \mu_{g}$ of $\Lambda$ with respect to which $E$ is given by the matrix $\left(\begin{array}{cc}0 & D \\ -D & 0\end{array}\right)$, where $D$ is the diagonal matrix with positive integer entries $d_{1}, \ldots, d_{g}$ satisfying $d_{i} \mid d_{i+1}$ for $i=1, \ldots, g-1$.

Definition 2.3. The vector $\left(d_{1}, \ldots, d_{g}\right)$ is called the type of the polarization of $L$ and when it is of the form $(1, \ldots, 1)$ the polarization is principal and the variety is called ppav.

Let $C$ be a smooth curve and let $H_{1}(C, \mathbb{Z}) \simeq \mathbb{Z}^{2 g}$ be the group of closed paths in $C$ (which does not depend on the starting point) modulo homology. This group can be seen as a full rank lattice inside of $H^{0}\left(C, \omega_{C}\right)^{*}$, via the injective map

$$
\gamma \mapsto\left\{\omega \mapsto \int_{\gamma} \omega\right\}
$$

assigning to a path $\gamma$ the functional which integrates the holomorphic differentials along $\gamma$.

Definition 2.4. The Jacobian of a smooth algebraic curve $C$ (or a compact Riemann surface) is the complex torus

$$
J C=H^{0}\left(C, \omega_{C}\right)^{*} / H_{1}(C, \mathbb{Z})
$$

The intersection product on $H_{1}(C, \mathbb{Z})$ induces an alternating form $E$ on $V:=H^{0}\left(C, \omega_{C}\right)^{*}$. More precisely, if we choose a basis over $\mathbb{Z}, \gamma_{1}, \ldots, \gamma_{2 g}$ of $H_{1}(C, \mathbb{Z})$ as in the Figure 1, the intersection product has a matrix of the form $\left(\begin{array}{cc}0 & \mathbf{1}_{g} \\ -\mathbf{1}_{g} & 0\end{array}\right)$. As $H_{1}(C, \mathbb{Z})$ is a full rank lattice in $V$, the $\left\{\gamma_{i}\right\}$ also form a basis of $V$ as an $\mathbb{R}$-vector space. One verifies then that, with respect to this basis, the intersection matrix gives an alternating form $E$ on $V$ defining a principal polarization $\Theta$.


Figure 1: Curve of genus $g$

A one-dimensional abelian variety also is an algebraic curve of genus one (with a distinguished point), that is, an elliptic curve. The Jacobian of a genus one curve is then isomorphic to the curve itself.

### 2.1. Abel-Jacobi map

Let $\operatorname{Pic}^{0}(C)$ be the group of line bundles of degree 0 on $C$, it is the quotient of the group of divisors $\operatorname{Div}^{0}(C)$ of degree 0 modulo principal divisors. Define the Abel-Jacobi map

$$
\operatorname{Div}^{0}(C) \rightarrow \operatorname{Pic}^{0}(C), \quad D=\sum\left(p_{\nu}-q_{\nu}\right) \mapsto\left\{\omega \mapsto \sum \int_{q_{\nu}}^{p_{\nu}} \omega\right\} \bmod H_{1}(C, \mathbb{Z})
$$

Theorem 2.5. The Abel-Jacobi map induces an isomorphism $\operatorname{Pic}^{0}(C) \simeq J C$.
A variation of this Abel-Jacobi map is given by

$$
\alpha_{D_{n}}: C^{(n)} \rightarrow J C, \quad \sum n_{\nu} p_{\nu} \mapsto\left\{\omega \mapsto \sum \int_{c}^{p_{\nu}} \omega\right\} \bmod H_{1}(C, \mathbb{Z})
$$

where $D_{n}=n c$ for a fixed point $c \in C$ and $C^{(n)}$ denotes the cartesian product $C^{n}$ of the curve modulo the symmetric group $S_{n}$, so its elements can be seen
as effective divisors of degree $n$ on $C$. For $n=1$, we denote the map by $\alpha_{c}$. Let $\beta: C^{(n)} \rightarrow \operatorname{Pic}^{n}(C)$ be the map $D \mapsto \mathcal{O}_{C}(D)$, so for a line bundle $L$ of degree $n$ the fibre $\beta^{-1}(L)$ consists of all divisors in the linear system $|L|$. We have the following commutative diagram


Proposition 2.6. The projectivized differential of the Abel-Jacobi map $\alpha_{c}$ : $C \rightarrow J C$ is the canonical map $\varphi_{\omega_{C}}: C \rightarrow \mathbb{P}^{g-1}$.

Corollary 2.7. For any $g \geq 1$ and $c \in C$ the Abel-Jacobi map $\alpha_{c}: C \rightarrow J C$ is an embedding.

Remark 2.8. Note that for any $c, c^{\prime} \in C$ we have $\alpha_{c}=t_{c^{\prime}-c}^{*} \alpha_{c^{\prime}}$, where $t_{D}$ : $J C \rightarrow J C$ is the translation map $t_{D}\left(D^{\prime}\right)=D^{\prime}+D$, so we sometimes omit a base point $c$ of the Abel-Jacobi map.

Algebraic geometers typically gather their objects of study in families to investigate a general property or single out interesting elements. Ideally, the set of all the objects forms itself an algebraic variety where one can apply known tools. This leads to the notion of moduli space, which is the variety parametrizing the objects. Fortunately, there exists a nice parameter space for all principally polarized abelian varieties (ppav) of fixed dimension $g$ (up to isomorphism classes). Let $\mathfrak{h}_{g}$ be the Siegel upper half plane

$$
\mathfrak{h}_{g}:=\left\{\tau \in M_{g \times g}(\mathbb{C}) \mid \tau^{t}=\tau, \operatorname{Im} \tau>0\right\}
$$

(where $\operatorname{Im} \tau>0$ means that the imaginary part is a positive definite 2-form) and

$$
S p_{2 g}(\mathbb{Z})=\left\{M \in G L_{2 g}(\mathbb{Z}): M\left(\begin{array}{cc}
0 & \mathbf{1}_{\mathbf{g}} \\
-\mathbf{1}_{\mathbf{g}} & 0
\end{array}\right)^{t} M=\left(\begin{array}{cc}
0 & \mathbf{1}_{\mathbf{g}} \\
-\mathbf{1}_{\mathbf{g}} & 0
\end{array}\right)\right\}
$$

the symplectic group, which acts on $\mathfrak{h}_{g}$ by

$$
M=\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \in S p_{2 g}(\mathbb{Z}), \quad M \cdot \tau=(a+b \tau)(c+d \tau)^{-1} .
$$

Thus, every point in the quotient $\mathfrak{h}_{g} / S p_{2 g}(\mathbb{Z})$ represents an isomorphism class of a principally polarized abelian variety of dimension $g$ : for each $\tau \in \mathfrak{h}_{g}$ set $A_{\tau}=\mathbb{C}^{g} / \tau \mathbb{Z}^{g} \oplus \mathbb{Z}^{g}$, then

$$
A_{\tau} \simeq A_{\tau^{\prime}} \text { as ppav } \Leftrightarrow \exists M \in S p_{2 g}(\mathbb{Z}) \quad \text { s.t. } \quad \tau^{\prime}=M \cdot \tau
$$

In the sequel, we denote by $\mathcal{A}_{g}$ the moduli space of principally polarized abelian varieties of dimension $g$. Observe that the dimension of this space is the same as the dimension of the space of symmetric matrices of size $g$, thus $\operatorname{dim} \mathcal{A}_{g}=$ $\frac{g(g+1)}{2}$.

Let $\mathcal{M}_{g}$ be the moduli space of smooth projective curves of genus $g>1$, it is an irreducible algebraic variety of dimension $3 g-3$. By associating to each smooth curve $[C] \in \mathcal{M}_{g}$ its Jacobian we get the Torelli map:

$$
\mathfrak{t}: \mathcal{M}_{g} \rightarrow \mathcal{A}_{g}, \quad[C] \mapsto(J C, \Theta)
$$

Theorem 2.9. The Torelli map $\mathfrak{t}$ is injective.
Comparing the dimensions of both spaces, one deduces that a general principally polarized abelian variety of dimension 2 and 3 is the Jacobian of some curve.

### 2.2. The theta divisor

Let $W_{n}:=\beta\left(C^{(n)}\right) \subset \operatorname{Pic}^{n}(C)$ for $n \geq 1$; it consists of the line bundles of degree $n$ with non-empty linear system. According to Riemann-Roch Theorem $W_{n}=\operatorname{Pic}^{n}(C)$ for $n \geq g$. For a general divisor $D$ of degree $1 \leq n \leq g$, $h^{0}\left(C, \mathcal{O}_{C}(D)\right)=1$, that is, in this range $\beta$ is birational onto $W_{n}$. Since $\beta$ is proper $W_{n}$ is an irreducible closed subvariety of $\operatorname{Pic}^{n}(C)$ of dimension $n$, so in particular $W_{g-1}$ is a divisor in $\mathrm{Pic}^{g-1}(C)$. For a fixed point $c \in C$ we set $\widetilde{W}_{n}=\alpha_{\mathcal{O}_{C}(n c)}\left(P^{n} c^{n}(C)\right) \subset J C$. Recall that the fundamental class $[Y]$ of an $n$-dimensional subvariety $Y$ of a variety $X, \operatorname{dim} X=g$ is the element in $H^{2 g-2 n}(X, \mathbb{Z})$, Poincaré dual to the homology class of $Y$ in $H_{2 n}(X, \mathbb{Z})$.
Theorem 2.10 (Poincaré's Formula). $\left[\widetilde{W}_{n}\right]=\frac{1}{(g-n)!} \bigwedge^{g-n}[\Theta]$ for any $1 \leq n \leq g$.
Corollary 2.11. There is a line bundle $\eta \in \operatorname{Pic}^{g-1}(C)$ such that $W_{g-1}=\alpha_{\eta}^{*} \Theta$.
Proof. By Poincaré Formula $\left[\widetilde{W}_{g-1}\right]=[\Theta]$ so $c_{1}\left(\mathcal{O}_{J C}\left(\widetilde{W}_{g-1}\right)=c_{1}\left(\mathcal{O}_{J C}(\Theta)\right)\right.$. There exists $x \in J C \simeq \operatorname{Pic}^{0}(C)$ such that $\widetilde{W}_{g-1}=t_{x}^{*} \Theta$. Hence

$$
W_{g-1}=\alpha_{\mathcal{O}_{C}((g-1) c)}^{*} \widetilde{W}_{g-1}=\alpha_{\eta}^{*} \Theta
$$

with $\eta=\mathcal{O}_{C}((g-1) c) \otimes x^{-1}$.
We recall that a theta characteristic on $C$ is a line bundle $\kappa$ such that $\kappa^{\otimes 2} \simeq \omega_{C}$. A divisor $D$ is called symmetric if $(-1)^{*} D \sim D$.

Theorem 2.12. Riemann's Theorem] For any symmetric theta divisor $\Theta$ there is a theta characteristic $\kappa$ on $C$ such that

$$
W_{g-1}=\alpha_{\kappa}^{*} \Theta
$$

The divisor $W_{g-1}$ is called the canonical theta divisor.
Given a theta characteristic $\kappa$, the map $\alpha_{\kappa}: \mathrm{Pic}^{g-1} \rightarrow J C$ induces a bijection between the set of theta characteristics on $C$ and the subgroup $J C[2]=$ $\{a \in J C[2] \mid 2 a=0\}$

Theorem 2.13 (Riemann's Singularity Theorem). For every $L \in \operatorname{Pic}^{(g-1)}(C)$

$$
\text { mult }_{L} W_{g-1}=h^{0}(C, L)
$$

## 3. Prym varieties

Consider a finite covering $\pi: \widetilde{C} \rightarrow C$ of degree $d$ between two smooth projective curves and let $g$ and $\tilde{g}$ denote the genera of $C$ and $\widetilde{C}$ respectively. By the Hurwitz formula these genera are related by

$$
\begin{equation*}
\tilde{g}=d(g-1)+\frac{\operatorname{deg} R}{2}+1 \tag{1}
\end{equation*}
$$

where $R$ denotes the ramification divisor of $f$, that is the set of points in $\widetilde{C}$ (counted with multiplicities) where the map is not locally a homeomorphism. The map $\pi$ induces a map between the Jacobians of the curves, the norm map. As a group, the Jacobian $J C$ is generated by the points of the curve $\alpha(C)$, and in fact, by Theorem 2.5, JC parametrizes classes of linear equivalence of divisors of degree zero. With this in mind, one can simply define the norm map as the push forward of divisors from $\widetilde{C}$ to $C$ :

$$
\mathrm{Nm}_{\pi}: J \widetilde{C} \rightarrow J C, \quad\left[\sum_{i} n_{i} p_{i}\right] \mapsto\left[\sum_{i} n_{i} \pi\left(p_{i}\right)\right]
$$

where the sum is finite, $\sum n_{i}=0$ with $n_{i} \in \mathbb{Z}$ and the bracket denotes the class of linear equivalence. The kernel of $\mathrm{Nm}_{\pi}$ is not necessarily connected but since $\mathrm{Nm}_{\pi}$ is a group homomorphism the connected component containing the zero is naturally a subgroup of $J \widetilde{C}$. This subgroup is the Prym variety of $f$ denoted by

$$
\begin{equation*}
P(\pi):=\left(\operatorname{Ker} \mathrm{Nm}_{\pi}\right)^{0} \subset J \widetilde{C} \tag{2}
\end{equation*}
$$

Moreover, the restriction $\Xi$ of the principal polarization $\Theta$ on $J \widetilde{C}$ to $P(\pi)$, defines a polarization so $(P(\pi), \Xi)$ is an abelian subvariety of the Jacobian $J \widetilde{C}$ of dimension

$$
\operatorname{dim} P(\pi)=\operatorname{dim} J \widetilde{C}-\operatorname{dim} J C=\tilde{g}-g
$$

The Prym variety can be regarded as the complementary abelian subvariety to the image of $\pi^{*}: J C \rightarrow J \widetilde{C}$ inside $J \widetilde{C}$, see [8, p. 125].

Theorem 3.1. Let $\pi: \widetilde{C} \rightarrow C$ be of degree $d \geq 2$ with $g \geq 1$. Then $\Xi$ defines $a$ principal polarization if and only if one of the following cases occur:
(a) $\pi$ is étale of degree 2, in this case $\Theta_{\mid P} \equiv 2 \Xi$, with $\Xi$ principal;
(b) $\pi$ is a double covering ramified in exactly 2 points, so $\Theta_{\mid P} \equiv 2 \Xi$;
(c) $g(\widetilde{C})=2, g=1$ (any degree);
(d) $g=2, d=3, \pi$ is non-cyclic.

Proof. Uses that $\left(\pi^{*}\right)^{*} \tilde{\Theta} \equiv n \Theta$ and that $P$ and $\pi^{*} J C$ are complementary subvarieties of a ppav. The cases (a),(b),(c) can be found in [8, Thm 12.3.3], where the case (d) is omitted by mistake. The case (d) is considered in [19].

From now on, we assume that the covering $\pi: \widetilde{C} \rightarrow C$ is étale of degree 2. Then the dimension of the corresponding Prym variety is $\operatorname{dim} P(\pi)=2 g-$ $1-g=g-1$. If $\iota$ denotes the involution on $\widetilde{C}$ exchanging the sheets of the covering $f$, it induces an automorphism $\iota^{*}$ on $J \widetilde{C}$. We can also describe the Prym variety of $\pi$ as

$$
P=\operatorname{Im}\left(1-\iota^{*}\right) \subset J \widetilde{C} .
$$

So $P$ is the $\iota^{*}$-anti-invariant part of $J \widetilde{C}$ orthogonal to $\pi^{*} J C$. Further, the addition map defines an isogeny

$$
\pi^{*} J C \times P \rightarrow J \widetilde{C}
$$

Let

$$
\mathcal{R}_{g}:=\left\{[C, \eta] \mid[C] \in \mathcal{M}_{g}, \eta \in \operatorname{Pic}^{0}(C) \backslash\left\{\mathcal{O}_{C}\right\}, \eta^{\otimes 2} \simeq \mathcal{O}_{C}\right\}
$$

be the moduli space parametrizing all étale double coverings over curves of genus $g$ up to isomorphism. Given a pair $[C, \eta] \in \mathcal{R}_{g}$ the isomorphism $\eta^{\otimes 2} \simeq$ $\mathcal{O}_{C}$ endows $\mathcal{O}_{C} \oplus \eta$ with a ring structure (actually with a structure of $\mathcal{O}_{C^{-}}$ algebra). Thus, the corresponding double covering is given by taking the spectrum $\widetilde{C}:=\operatorname{Spec}\left(\mathcal{O}_{C} \oplus \eta\right)$ and the map $\pi$ is just the natural projection $\operatorname{Spec}\left(\mathcal{O}_{C} \oplus \eta\right) \rightarrow C=\operatorname{Spec} \mathcal{O}_{C}$, induced by the inclusion $\mathcal{O}_{C} \hookrightarrow \mathcal{O}_{C} \oplus \eta$. There are finitely many "square roots" of $\mathcal{O}_{C}$, that is, line bundles $\eta$ with $\eta^{\otimes 2} \simeq \mathcal{O}_{C}$. In other words, the forgetful map

$$
\mathcal{R}_{g} \rightarrow \mathcal{M}_{g}, \quad[C, \eta] \mapsto \eta
$$

is finite of degree $2^{2 g}-1$ and hence $\operatorname{dim} \mathcal{R}_{g}=\operatorname{dim} M_{g}=3 g-3$. The Prym map is then defined as

$$
\mathcal{P}_{g}: \mathcal{R}_{g} \rightarrow \mathcal{A}_{g-1} \quad[C, \eta] \mapsto(P(\pi), \Xi)
$$

By comparing the dimensions on both sides, one sees that for $g \leq 6$, we have $\operatorname{dim} \mathcal{R}_{g} \geq \operatorname{dim} \mathcal{A}_{g-1}=\frac{g(g-1)}{2}$ so it makes sense to ask if for low values of $g$ the Prym map is dominant, i.e. if we can realize a (general) principally polarized abelian varieties of dimension $\leq 6$ as the Prym variety of some covering.

In order to investigate when the map $\mathcal{P}_{g}$ is generically finite one has to check if the differential map is injective at a generic point of $\mathcal{R}_{g}$ or equivalently, when the codifferential map $d^{*} \mathcal{P}_{g}$ is surjective. On one side, the tangent space at $0 \in P(\pi)$ to the Prym variety can be identified with

$$
T_{0} P \simeq H^{0}\left(C, \omega_{C} \otimes \eta\right)^{*}
$$

which is the $(-1)$-eigenspace for the action of $\iota$ on $H^{0}\left(\widetilde{C}, \omega_{\widetilde{C}}\right)^{*}$. Further, we have the identification $T_{[P, \Xi]}^{*} \mathcal{A}_{g-1} \simeq \operatorname{Sym}^{2}\left(T_{0} P\right)^{*}$ of the cotangent space to $[P, \Xi]$. On the other hand, notice that the forgetful map $[C, \eta] \mapsto[C]$ is finite over the moduli space $\mathcal{M}_{g}$ of smooth curves of genus $g$. Therefore the cotangent space to a generic point $[C, \eta] \in \mathcal{R}_{g}$ can be identified to the cotangent space to $\mathcal{M}_{g}$ at [C], that is,

$$
T_{[C, \eta]}^{*} \mathcal{R}_{g} \simeq T_{[C]}^{*} \mathcal{M}_{g} \simeq H^{0}\left(C, \omega_{C}^{2}\right)
$$

Via these identifications one obtains that the codifferential of $\mathcal{P}_{g}$ at a generic point $[C, \eta]$ is given by the multiplication of sections

$$
d^{*} \mathcal{P}_{g}: \operatorname{Sym}^{2}\left(T_{0} P\right)^{*} \rightarrow H^{0}\left(C, \omega_{C}^{2} \otimes \mathcal{O}\right)
$$

which is surjective for $g \geq 6$ at a generic point $[(C, \eta)]$. More precisely, the following theorem summarizes the situation for the classical Prym map:
Theorem 3.2. (a) The Prym map is dominant if $g \leq 6$.
(b) The Prym map is generically injective if $g \geq 7$.
(c) The Prym map is never injective.

Proof. Let $\mathcal{B}_{g-1}$ denote the image of $\mathcal{P}_{g}$. Wirtinger showed [34] that the closure $\overline{\mathcal{B}}_{g-1}$ is an irreducible subvariety in $\mathcal{A}_{g-1}$ of dimension $3 g-3$, so $\overline{\mathcal{B}}_{g-1}=\mathcal{A}_{g-1}$ for $g \leq 6$, which implies part (a). Moreover, he also proved that the Jacobian locus in $\mathcal{A}_{g-1}$ (i.e. the image of the Torelli map $\mathfrak{t}$ ) is contained in $\overline{\mathcal{B}}_{g-1}$. In this sense, Pryms are a generalization of Jacobians. Part (b) was first proved by R. Friedman and R. Smith [16] and for $g \geq 8$, by V. Kanev [18] by using degeneration methods. More geometric proofs were given by G. Welters [33] and later by O. Debarre [12], in the spirit of the proof of Torelli's theorem.

The fact that the Prym map is non-injective was first observed by Beauville [5]. Donagi's tetragonal construction [13] provides examples for the non-injectivity in any genus.

Open question. What is exactly the non-injectivity locus of the Prym map $\mathcal{P}_{g}$ ?

## 4. Allowable covers

We will focus now on the rich geometry of the fibres of the Prym map $\mathcal{P}_{6}$ : $\mathcal{R}_{6} \rightarrow \mathcal{A}_{5}$. The following theorem is proved in [15].
Theorem 4.1. The degree of $\mathcal{P}_{6}: \mathcal{R}_{6} \rightarrow \mathcal{A}_{5}$ is 27 .
This number encodes the geometry of the fibres, which have the structure of the 27 lines in a smooth cubic surfaces. Although the degree is the number of étale double coverings mapping to a general element in $\mathcal{A}_{5}$, which in this case is a Prym variety, the count is done over loci with positive dimensional fibers, involving a very precise description of the blow ups. In order to compute the degree we shall

1. extend the $\mathcal{P}_{g}$ to a proper map (Theorem 4.7),
2. study the Prym varieties on the boundary,
3. compute the local degree along the different loci.

Parts (2) and (3) will be treated in Sections 6,7 and 8.
Beauville introduced the notion of generalized Prym varieties in [4] to denote Prym varieties associated with double coverings of stable curves, that is, lying on the boundary of $\mathcal{R}_{g}$. Since the objects in the compactification of $\mathcal{R}_{g}$ are known as admissible covers, we use the terminology in [15] and denote those covers in the boundary that give rise to a generalized Prym variety, allowable covers.

Let $\overline{\mathcal{M}}_{g}$ be the compactification of $\mathcal{M}_{g}$ by stable curves of genus $g$. Recall that a (complete) curve $C$ is stable if it is connected, the only singularities are ordinary double points and $|A u t(C)|<\infty$. In particular, $\rho_{a}(C)=g \neq 1$ (arithmetic genus) and every non singular rational component meet other components in at least 3 points. Let $C \in \overline{\mathcal{M}}_{g}, \widetilde{C} \in \overline{\mathcal{M}}_{2 g-1}$ and $\pi: \widetilde{C} \rightarrow C$ be a (possibly branched) double covering with an involution $\iota: \widetilde{C} \rightarrow \widetilde{C}$. In order to analyse the "good" coverings for the Prym map, we make the following assumption:
(*) The fixed points of $\widetilde{C}$ under the involution $\iota$ are exactly the singular points and at a singular point the two branches are not exchanged under $\iota$.
The reason for this assumption is that in this case the quotient $C:=\widetilde{C} / \iota$ has only ordinary double points as singularities. We have the following commutative diagram

where $f$ and $\tilde{f}$ are the normalization maps. Thus, $\pi^{\prime}$ is ramified at the points $x_{i}, y_{i} \in \widetilde{N}$ lying over a singular point $z_{i} \in \widetilde{C}$. One can also show that $\pi^{*} \omega_{C} \simeq$ $\omega_{\widetilde{C}}$; as a consequence,

$$
\rho_{a}(\widetilde{C})=2 \rho_{a}(C)-1
$$

Let $\widetilde{K}$ (resp. $K$ ) be the ring of rational functions on $\widetilde{C}$ (resp. $C$ ). The group of Cartier divisors on $\widetilde{C}$ can be described as

$$
\operatorname{Div} \widetilde{C}=\bigoplus_{x \in \widetilde{C}_{s m}} \mathbb{Z} x \oplus \bigoplus_{s \in \widetilde{C}_{s i n g}} \widetilde{K}_{s}^{*} / \mathcal{O}_{s}^{*}
$$

Let $s_{1}, s_{2} \in \widetilde{N}$ be the points over a singular point $s \in \widetilde{C}$. By choosing parameters $t_{1}, t_{2}$ around $s_{1}$ and $s_{2}$, we have the following isomorphism

$$
\widetilde{K}_{s}^{*} / \mathcal{O}_{s}^{*} \xrightarrow{\sim} \mathbb{C}^{*} \times \mathbb{Z} \times \mathbb{Z}, \quad a \mapsto\left(\frac{u}{v}, m, n\right)
$$

where $a=u t_{1}^{m}$ and $a=v t_{2}^{n}$ are the local descriptions of $a$ around $s_{1}$, resp. $s_{2}$. Assuming that $\iota^{*} t_{1}=-t_{1}$ and $\iota^{*} t_{2}=-t_{2}$, the action of $\iota$ on $\widetilde{K}_{s}^{*} / \mathcal{O}_{s}^{*}$ is

$$
\iota^{*}(z, m, n)_{s}=\left((-1)^{m+n} z, m, n\right)_{s}
$$

which yields the commutative diagram

where $\pi_{*}\left(\sum_{i} x_{i}\right)=\sum_{i} \pi\left(x_{i}\right)$ for $x_{i} \in \widetilde{C}_{s m}$ and for singularities we have $\pi_{*}\left((z, m, n)_{s}\right)=\left((-1)^{m+n} z^{2}, m, n\right)_{\pi(s)}$. The norm map Pic $(\widetilde{C}) \rightarrow \operatorname{Pic}(C)$ restricts to a norm map between the generalized Jacobians $\mathrm{Nm}: J \widetilde{C} \rightarrow J C$. We want to consider coverings such that the kernel of this map is an abelian variety.
Example 4.2. Let $X$ be a smooth genus $g$ curve with two marked points $p, q$ and $X_{1}=X_{2}=X$ be two copies with marked points $p_{i}, q_{i} \in X_{i}$. Define the Wirtinger cover $\pi: \widetilde{C} \rightarrow C$ (see Figure 2) by

$$
\widetilde{C}:=X_{1} \cup X_{2} / p_{1} \sim q_{2}, p_{2} \sim q_{1}, \quad C=X / p \sim q
$$

Let $\nu: X \rightarrow C$ denote the normalization map and $s \in C$ be the node. To specify a line bundle $L$ on $C$ one has to specify $\tilde{L}:=\nu^{*} L$ and a descent data, i.e. when a section of $\nu^{*} L$ is the pullback of a section of $L$. In this case it
suffices to give the identification of the fibers $\varphi_{s}: \tilde{L}_{p} \xrightarrow{\sim} \tilde{L}_{q}$ over $p$ and $q$, so $\varphi_{s} \in \mathbb{C}^{*}$. More generally, we have a short exact sequence

$$
0 \rightarrow\left(\mathbb{C}^{*}\right)^{b} \rightarrow J C \xrightarrow{\nu^{*}} J N \rightarrow 0
$$

where $b$ is the first Betti number of the dual graph of $C$. As will see, this is an example of an allowable cover.
Lemma 4.3. If $L$ is a line bundle on $\widetilde{C}$ such that $\operatorname{Nm}_{\pi} L=\mathcal{O}_{C}$ then $L=$ $M \otimes \iota^{*} L^{-1}$ for some line bundle $M$ on $\widetilde{C}$ which can be chosen of multidegree $\underline{\operatorname{deg} M}=(0, \ldots, 0)$ or $(1,0, \ldots, 0)$.

We define Prym variety of the covering $\pi: \widetilde{C} \rightarrow C$ as the connected algebraic subgroup

$$
P:=\left\{M \otimes \iota^{*} L^{-1} \quad \mid \underline{\operatorname{deg}} M=(0, \ldots, 0)\right\} .
$$

Proposition 4.4. The variety $P$ is an abelian variety of dimension $\rho_{a}(C)-1$.
Proof. We have a commutative diagram


Notice that $\mathrm{Nm} \circ \pi^{*}$ is the multiplication by 2 and since $\pi^{*}: T \rightarrow \widetilde{T}$ is an isomorphism the left vertical arrow Nm is surjective and its kernel $\widetilde{T}_{2}$ corresponds to the points of order 2 in $\widetilde{T}$. The Kernel $R$ is a complete subvariety of $J \widetilde{N}$, so $P$ is also complete.

Choose a line bundle $L \in \operatorname{Pic}(C)$ with multidegree satisfying $2 \underline{\operatorname{deg}} L=$ $\operatorname{deg} \omega_{\widetilde{C}}$ and define

$$
\Theta_{L}:=\left\{M \in J \widetilde{C} \mid H^{0}(\widetilde{C}, L \otimes M) \neq 0\right\}
$$

It turns out that, as in the smooth case, $\Theta_{L \mid P} \equiv 2 \Xi$, with $\Xi \in \operatorname{NS}(P)$ a principal polarization. Thus $(P, \Xi)$ is a ppav associated to $(\widetilde{C}, \iota)$.

Definition 4.5. A covering $(\widetilde{C}, \iota)$ with $\widetilde{C} \in \overline{\mathcal{M}}_{2 g-1}$ and $\iota$ an involution, is allowable if its associated Prym variety $P$ is an abelian variety and $\rho_{a}(\widetilde{C} / \iota)=g$.

This definition is equivalent to any of these properties:
(a) The only fixed points of $\iota$ are the nodes where the two branches are not exchanged and the number of nodes exchanged under $\iota$ equals the number of irreducible components exchanged under $\iota$.
(b) The components of $\widetilde{C}$ can be grouped as $\widetilde{C}=A \cup A^{\prime} \cup \widetilde{B}$ where $\iota$ interchanges $A$ and $A^{\prime}$ and fixes $\widetilde{B}$, each $A$ is tree-like and either

- $\widetilde{B}=\emptyset, A$ connected and $\left|A \cap A^{\prime}\right|=2$, or
$-A \cap A^{\prime}=\emptyset,\left|\widetilde{B} \cap A_{i}\right|=1$ for each connected component $A_{i}$ of $A$ the fixed points of $\iota$ in $\widetilde{B}$ are precisely the nodes and the two branches there are never exchanged (so that $\widetilde{B} / \iota$ ) also has nodes at the corresponding points.

Remark 4.6. The condition $\left(^{*}\right)$ is equivalent to $(a)$ and $(b)$ if there is no exchanged components under $\iota$.

Let us denote

$$
\overline{\mathcal{R}}_{g}:=\left\{[\pi: \widetilde{C} \rightarrow C] \mid[\widetilde{C}] \in \overline{\mathcal{M}}_{2 g-1},[C] \in \overline{\mathcal{M}}_{g}, \pi \text { is an allowable cover }\right\}
$$

which is an open subspace in the compactification by admissible coverings of the moduli space $\mathcal{R}_{g}$.

Theorem 4.7. The Prym map $\mathcal{P}_{g}: \mathcal{R}_{g} \rightarrow \mathcal{A}_{g-1}$, extends to a proper map

$$
\overline{\mathcal{P}}_{g}: \overline{\mathcal{R}}_{g} \rightarrow \mathcal{A}_{g-1}
$$

For the proof of this theorem we refer [15, Theorem 1.1]. Now the aim is to compute the local degree of $\overline{\mathcal{P}}_{g}$ along the relevant divisors (those which are not contracted under the Prym map).

## 5. Computation of the local degree

Let $f: Y \rightarrow X$ be a proper dominant map between two varieties, with $\operatorname{dim} X=$ $\operatorname{dim} Y=n$, so $f$ is generically finite. Set $d=\operatorname{deg} f$. Let $W \subset Y$ be an irreducible closed subvariety of codimension $k$, thus $f^{-1}(W)$ consists of finitely many irreducible components $Z_{i}$ of codimension $l_{i}$ in $X$. The local degree of $d_{i}$ of $f$ along $Z_{i}$ is the degree of the map obtained from $f$ by localizing $X$ at $Z_{i}$; thus $d=\sum_{i} d_{i}$. Let $Z \subset X$ be one of these components, $\widetilde{X}=B l_{Z} X$ (resp.
$\widetilde{Y}=B l_{Z} Y$ ) the blow up fo $X$ (resp. $Y$ ) along $Z$. Consider the following commutative diagram


The map $\tilde{f}$ induces a map between the exceptional divisors $f_{*}: \widetilde{Z} \rightarrow \widetilde{W}$, described as follows. Recall that $\widetilde{Z}=\mathbb{P}\left(\mathcal{N}_{Z \backslash X}\right)$ and $\widetilde{W}=\mathbb{P}\left(\mathcal{N}_{W \backslash Y}\right)$ are the projectivized normal bundles. Let $z \in Z$ and $w=f(z) \in W$. The differential $d f_{z}: T_{z} X \rightarrow T_{w} W$ at $z$ maps $T_{z} Z$ to $T_{w} W$. therefore, this induces a map

$$
f_{*, z}: \mathcal{N}_{Z \backslash X} \rightarrow \mathcal{N}_{W \backslash Y}
$$

This lemma follows from the universal property of blow ups.
Lemma 5.1.
(a) The map $\tilde{f}$ is regular at a generic $z \in \widetilde{Z}$ if and only if $f_{*, z}$ is not identically zero at a generic $z \in Z$.
(b) The map $\tilde{f}$ is regular for all $\tilde{z}$ in the fiber over $z \in \widetilde{Z}$ if and only if $f_{*, z}$ is injective on the normal space $\mathcal{N}_{Z \backslash X, z}$ to $Z$ at $z$. In this case $\tilde{f}_{\mid \text {fiber over } z}$ is the projectivization of the linear map $f_{*, z}$.

Lemma 5.2. Assume $f_{*, z}$ is injective on $\mathcal{N}_{Z \backslash X, z}$ at each $z \in Z$. Then the local degree of $f$ along $Z$ equals the degree of the $\operatorname{map} f_{*}: \widetilde{Z} \rightarrow \widetilde{W}$ on the exceptional divisors.

Remark 5.3 (Warning). The lemma requires the injectivity for all $z \in Z$. Otherwise $\tilde{f}$ is not regular on a neighbourhood of $\widetilde{Z}$ and could involve a blow up of some small dimensional subvariety onto $\widetilde{W}$, implying that $\widetilde{Z}$ is only one of several components of the graph $\tilde{f}$ over $Z$. In this case the degree of $f_{*}$ is possibly smaller than the degree of $\tilde{f}$ restricted to $f^{-1}($ neighbourhood of $z)$, that is, smaller than the local degree of $f$ on $Z$.

Consider the locus $\mathcal{J}_{5}$ of Jacobians of smooth curves of genus 5 , which is of codimension 3 in $\mathcal{A}_{5}$. Given a generic curve $X \in \mathcal{M}_{5}$, Mumford [25] provided a list of smooth double covers such that their Prym variety $P(\widetilde{C} / C) \simeq J X$. Later Donagi and Smith [15] extended this list to the allowable covers with the same Jacobian. From this list only four cases are relevant for the computation of the degree, since the rest of the cases involve covers mapping to smaller loci in $\mathcal{J}_{5}$, that is whose image is of dimension $<12=\operatorname{dim} \mathcal{J}_{5}$, hence these loci do not contain the generic Jacobian. These are the four relevant loci in $\overline{\mathcal{R}}_{g}$ :
(a) $[C] \in \mathcal{M}_{6}$ is a smooth plane quintic and $\pi: \widetilde{C} \rightarrow C$ is an even double cover, that is, with the property $h^{0}\left(\widetilde{C}, \pi^{*} \mathcal{O}_{C}(1)\right)=0 \bmod 2^{2}$. We will denote this locus by $\mathcal{R}_{Q}{ }^{3}$.
(b) Double covers over trigonal curves, denoted $\mathcal{R}_{\mathcal{T}}$.
(c) Wirtinger covers, later denoted by $\mathcal{R}_{S}$.
(d) Elliptic tails, $\mathcal{R}_{E}$, given by : $\pi: \widetilde{C} \rightarrow C$ where

$$
\widetilde{C}:=X_{1} \cup \tilde{E} \cup X_{2} / p_{1} \sim 0, p_{2} \sim a, \quad C:=X \cup E / p \sim 0
$$

with $[X] \in \mathcal{M}_{5}, X_{1} \simeq X_{2}$ copies of $X$ and $\widetilde{E}$ an étale double cover of an elliptic curve $E$. Here $p_{i} \in X_{i}, i=1,2$ map to $p \in X$ and intersection points $0, a \in \widetilde{E}$ map to $0 \in E$.
The loci (a) and (b) are of dimension 12, whereas (c) and (d) are of dimension 14. In the next section we shall apply Lemma 5.2 to the computation of the local degree along these loci mapping onto the Jacobi locus. We will prove that their contributions to the degree are $1,10,16$ and 0 respectively.

## 6. Plane quintics

Recall that a theta characteristic on a curve $C$ of genus $g$ is a line bundle $\kappa \in \operatorname{Pic}^{g-1}(C)$ such that $\kappa^{\otimes 2} \simeq \Omega_{C} ; \kappa$ is even or odd according to the parity of $h^{0}(C, \kappa)$. Let $[C] \in \mathcal{M}_{6}$ be a smooth plane quintic. There is a natural odd theta characteristic $\kappa$ given by the pullback of the hyperplane class $\ell$ under the embedding $C \hookrightarrow \mathbb{P}^{2}$. Define

$$
\mathcal{R}_{Q}^{\prime}=\left\{[C, \eta] \in \mathcal{R}_{6} \mid C \text { is a plane quintic }\right\} .
$$

We distinguish two types of coverings in $\mathcal{R}_{Q}^{\prime}$, if $h^{0}(\eta \otimes \kappa)=0 \bmod 2$ (respectively $=1 \bmod 2$ ) we say that the cover $[C, \eta]$ is even (respectively odd). This gives a decomposition of $\mathcal{R}_{Q}^{\prime}$ into two irreducible components

$$
\mathcal{R}_{Q}^{\prime}=\mathcal{R}_{Q} \sqcup \mathcal{R}_{\mathcal{C}}
$$

We will show that the locus of even coverings, $\mathcal{R}_{Q}$, maps onto $\mathcal{J}_{5}$ and odd coverings, $\mathcal{R}_{\mathcal{C}}$, maps to the locus of intermediate Jacobians of cubic threefolds.

Assume now, that $J X=\mathcal{P}_{6}([C, \eta]) \in \mathcal{J}_{5}$, with $[X] \in \mathcal{M}_{5}$ generic (non hyperelliptic, nor trigonal), so $(J X, \Theta) \in \mathcal{A}_{5}$. According to the Riemann Singularity Theorem

$$
\Theta_{\text {Sing }}=\left\{L \in \operatorname{Pic}^{4}(X) \mid h^{0}(X, L) \geq 2\right\}
$$

[^1]On the other hand, the genericity of $X$ implies that the image of the canonical embedding $X \hookrightarrow \mathbb{P}^{4}$ is given by the intersections of three smooth quadrics

$$
X=Q_{0} \cap Q_{1} \cap Q_{2}
$$

Any $g_{4}^{1}$ on $X$ is cut out by a 1-parameter family of 2-planes sweeping out a quadric (of rank 3 or 4 ) in $\mathbb{P}^{4}$ containing $X$. Set

$$
\Pi=\left\langle Q_{0}, Q_{1}, Q_{2}\right\rangle=\left\{\lambda_{0} Q_{0}+\lambda_{1} Q_{1}+\lambda_{2} Q_{2} \mid \lambda=\left(\lambda_{0}, \lambda_{1}, \lambda_{2}\right) \in \mathbb{P}^{2}\right\}
$$

the net parametrizing all the quadrics containing $X$. The discriminant locus

$$
\left\{\lambda \in \Pi \mid Q_{\lambda} \text { is a singular quadric }\right\}
$$

is a plane quintic $C \subset \mathbb{P}^{2}$ defined by the vanishing of $5 \times 5$ linear determinant. For a given $\lambda \in C$, the quadric $Q_{\lambda}$ possesses two 1-parameter families of planes cutting a $g_{4}^{1}$. Let $\widetilde{C}$ be the curve parametrizing the $g_{4}^{1}$ 's. By construction, this defines an étale double cover $\widetilde{C} \rightarrow C$. Actually, $\widetilde{C} \simeq \Theta_{\text {Sing }}$.

In conclusion, one can recover uniquely a double covering $[C, \eta] \in \mathcal{R}_{Q}$ from the Jacobian of a generic genus 5 curve.
Remark 6.1. For $X$ generic, $\widetilde{C}$ is smooth and $\pi$ is étale. This fails when $X$ possesses a vanishing thetanull, i.e., an even theta characteristic $\kappa$ with $h^{0}(C, \kappa) \geq 2$. In this case $\widetilde{C}$ is singular. Masiewicki showed [24] that the corresponding cover is allowable, extending the result to all the curves in $\mathcal{M}_{5}$.

In order to show that the local degree of $\mathcal{P}_{6}$ on $\mathcal{R}_{Q}$ equals 1 , we have to show that $\mathcal{P}_{6}$ is not ramified on $\mathcal{R}_{Q}$. This is equivalent to showing that the codifferential map

$$
d \mathcal{P}_{6}^{*}: \operatorname{Sym}^{2} H^{0}\left(C, \omega_{C} \otimes \eta\right) \rightarrow H^{0}\left(C, \omega_{C}^{\otimes 2}\right)
$$

is injective on the generic element of $\mathcal{R}_{Q}$. Using the identification $\left(T_{0} P\right)^{*} \simeq$ $H^{0}\left(C, \omega_{C} \otimes \eta\right)$ one can show that the projectivized of the Abel-Prym map $\widetilde{C} \rightarrow P$ is the composition [8, Prop. 12.5.3]

$$
\widetilde{C} \xrightarrow{\pi} C \rightarrow \mathbb{P}\left(H^{0}\left(C, \omega_{C} \otimes \eta\right)\right) \simeq \mathbb{P}^{5}
$$

Therefore, the injectivity of the map follows from:
Proposition 6.2. The Prym-canonical image $\Psi(C) \subset \mathbb{P}^{5}$ for a generic $[C, \eta] \in$ $\mathcal{R}_{Q}$ is contained in no quadrics.

Proof. Beauville proved [5, Prop. 7.10] that for a non-hyperelliptic curve $X \in$ $\mathcal{M}_{5}$, the corresponding plane quintic $C$ is contained in no quadrics. Since $\mathcal{R}_{Q}$ is irreducible [15, II.3.3], this suffices to to prove the proposition.

## 7. Trigonal curves

Let us recall Recillas construction [32] that shows that the Jacobian of a tetragonal curve is the Prym variety of a covering of a trigonal curve.

Let $\left(X, g_{4}^{1}\right)$ be a tetragonal curve of genus $g-1$. Consider

$$
\widetilde{C}:=\left\{p_{1}+p_{2} \in X^{(2)} \mid \exists p_{3}, p_{4} \in X, p_{1}+p_{2}+p_{3}+p_{4} \in g_{4}^{1}\right\}
$$

Note that there exists a natural involution $\sigma: \widetilde{C} \rightarrow \widetilde{C}, \sigma\left(p_{1}+p_{2}\right)=p_{3}+p_{4}$, so we can define $C=\widetilde{C} / \sigma$. For the construction, we assume that $X$ is general tetragonal, i.e. a map $f: X \rightarrow \mathbb{P}^{1}$ induced by the $g_{4}^{1}$ has in any fibre at least three points. The assumption implies that $\sigma$ is a fixed point free involution. A technical lemma shows that $\widetilde{C}$ is smooth [8, Lemma 12.7.1].

Note that $C$ is trigonal. This is because 4 points can be divided to pairs in 3 different ways, as in the following Diagram.


This gives a $g_{3}^{1}$ and a map $h: C \rightarrow \mathbb{P}^{1}$. The map $h$ is ramified in exactly the same locus as $f$, so one can apply Hurwitz formula to get $g(C)=g-1+1=g$. Then $g(\widetilde{C})=2 g-1$ and the aim will be to show that $P(\widetilde{C} / C)=J X$.

Now, we will show an inverse construction that will be drawn in Diagram (3). Let $\pi: \widetilde{C} \rightarrow C$ be a double covering of a trigonal curve $C$ of genus $g$. Let $\pi^{(3)}: \widetilde{C}^{(3)} \rightarrow C^{(3)}$ be an induced $8: 1$ covering. Note that one can embed $\mathbb{P}^{1}=g_{3}^{1} \ni p_{1}+p_{2}+p_{3} \hookrightarrow\left[p_{1}+p_{2}+p_{3}\right] \in C^{(3)}$ and restrict $\pi^{(3)}$ to the preimage of $\mathbb{P}^{1}$, called $\tilde{X}$. An involution $\sigma$ acts on $\widetilde{C}$, hence on $\widetilde{C}^{(3)}$ and $\tilde{X}$ is $\sigma$-invariant. Hence, $\left.\pi^{(3)}\right|_{\tilde{X}}$ factorises via $X=\tilde{X} / \sigma$ which is a tetragonal curve of genus $g-1$.


Trigonal construction allows us to define a map:

$$
\begin{aligned}
& \tau: \mathcal{J}_{4, g-1}^{1} \longrightarrow \overline{\mathcal{R}}_{g} \\
& \quad\left(X, g_{4}^{1}\right) \longmapsto[\widetilde{C} \rightarrow C]
\end{aligned}
$$

that gives us an allowable covering. We have the following proposition.
Proposition 7.1. Recall that $\widetilde{C} \subseteq X^{(2)}$ and let $\alpha=\alpha_{g_{4}^{1}}: X \longrightarrow J X$ be the Abel map chosen such that $\alpha(x)=4 x-g_{4}^{1}$. We get:

1. $P_{g}\left(\tau\left(X, g_{4}^{1}\right)\right)=J X$.
2. The map $\Psi: \widetilde{C} \longrightarrow J X=P_{g}(\widetilde{C} \rightarrow C)$ given by $(a, b) \mapsto \alpha(a)+\alpha(b)$ is the Abel-Prym map of the covering $\widetilde{C} \rightarrow C$.

Proof. Fix $\tilde{c} \in \widetilde{C}$. We will use the universal property of Prym varieties [8, Thm 12.5.1] to get the bottom row of the diagram:

and to show the $\tilde{\Psi}$ is an isomorphism.
Firstly, in order to get the diagram, we need to show that $\Psi \circ \iota=-\Psi$. This is satisfied since

$$
\Psi \circ \iota(a, b)=\Psi(c, d)=\alpha(c)+\alpha(d)=-\alpha(a)-\alpha(b),
$$

for $a+b+c+d \in g_{4}^{1}$.
Now, by Matsusaka's criterion [8, Rmk 12.2.5] it is enough to show that

$$
\psi(\widetilde{C})=\frac{2}{(g-2)!} \bigwedge^{(g-2)} \Theta_{J X} \in H^{2 g-4}(J X, \mathbb{Z})
$$

(note that $g(X)=g-1$, both $J X$ and $P$ are of the same dimension and the polarisation on $P$ is twice the principal one).

To prove it we will use a degeneration method. Let $X_{t}$ degenerate to $X_{0} \cup$ $\mathbb{P}^{1}$ with $X_{0}$ being trigonal curve and $g_{4}^{1}$ degenerates to $g_{3}^{1}$ on $X_{0}$ and the intersection point $X_{0} \cap \mathbb{P}^{1}=p_{0}$. Let $p_{0}+p_{1}+p_{2} \in g_{3}^{1}$ and consider $C=X_{0} / p_{1} \sim$ $p_{2}$ with the Wirtinger cover $\widetilde{C}$ where $q_{1}=p_{2}, q_{2}=p_{1}$ as in Example 4.2. Note that the class $[\Psi(\widetilde{C})]$ does not change in the degeneration. We compute

$$
[\Psi(\widetilde{C})]=\left[\alpha\left(X_{0}\right)+\alpha\left(X_{0}\right)\right]=2\left[\alpha\left(X_{0}\right)\right]=\frac{2}{(g-2)!} \bigwedge^{(g-2)} \Theta_{J X_{0}}
$$

Denote by

$$
\mathcal{R}_{T, g}=\left\{[\widetilde{C} \rightarrow C] \in \mathcal{R}_{g}: C \text { trigonal }\right\} .
$$

Then $\overline{\mathcal{R}}_{T, g} \subset \overline{\mathcal{R}}_{g}$ and $\operatorname{Im}(\tau)=\overline{\mathcal{R}}_{T, g}$.
Remark 7.2. By Brill-Noether theory, every curve of genus 5 has got a $g_{4}^{1}$.
We have the following diagram:

where $\widetilde{\mathcal{R}}_{T}$ and $\tilde{J}_{5}$ are exceptional divisors of the blow ups.
Recall that for $X$ of genus $g-1$

$$
\Phi: X \longrightarrow \mathbb{P} H^{0}\left(X, \omega_{X}\right)^{*} \simeq \mathbb{P}^{g-2}
$$

is the canonical map and for $(C, \eta) \in \mathcal{R}_{g}$ of genus $g$

$$
\Psi: C \longrightarrow \mathbb{P} H^{0}\left(C, \omega_{C} \otimes \eta\right)^{*} \simeq \mathbb{P}^{g-2}
$$

is called the Prym-canonical map. Consider again the map

$$
d \mathcal{P}_{g}^{*}: \operatorname{Sym}^{2} H^{0}\left(C, \omega_{C} \otimes \eta\right) \longrightarrow H^{0}\left(C, \omega_{C}^{2}\right)
$$

LEmma 7.3. Let $\widetilde{C} \rightarrow C$ be a double covering of a trigonal curve $C$ of genus $g$, and $X$ its corresponding tetragonal curve of genus $g-1$. Then
(i) The image in $\mathbb{P}^{g-2}$ of the 4 points $a, b, c, d \in X$ under the canonical embedding of each divisor $D \in g_{4}^{1}$ are coplanar.
(ii) On each of these planes the 3 points of intersection of opposite lines, that is, $\overline{a b} \cap \overline{c d}, \overline{a c} \cap \overline{b d}, \overline{a d} \cap \overline{b c}$ are on $\Psi(C)$ and as $D$ varies in $g_{4}^{1}$, they trace $\Psi(C)$ once, giving the $g_{3}^{1}$ on $C$.

Lemma 7.4. The intersection of $\Phi(X) \cap \Psi(C)$ in $\mathbb{P}^{g-2}$ consists of $2 g+4$ points, corresponding to the ramification points of the $g_{4}^{1}$ and $g_{3}^{1}$.

Proposition 7.5. Let $[C, \eta] \in \widetilde{\mathcal{R}}_{T}$, then $\operatorname{ker}\left(d \mathcal{P}_{6}^{*}\right)$ is the one-dimensional subspace corresponding to the unique quadric in $\mathbb{P}^{4}$ containing $\Phi(X)$ and the family of planes cutting the given $g_{4}^{1}$ on $X$.

Proof. Recall that $\operatorname{Ker}\left(d \mathcal{P}_{6}^{*}\right)=\left\{\right.$ quadrics in $\mathbb{P}^{4}$ containing $\left.\Psi(C)\right\}$. A $g_{4}^{1}$ is given by cutting out $X$ with 1-dimensional family of plane of a quadric $Q \subset \mathbb{P}^{4}$. By Lemma $7.3, Q$ contains $\Psi(C)$ since $\Psi(C)$ is contained in the union of these planes. Moreover, every quadric in $\operatorname{Ker}\left(d \mathcal{P}_{6}^{*}\right)$ also contains $\Phi(X)$. Suppose that $\Psi(C)$ is contained in another quadric $Q^{\prime}$, so $\Psi(C) \subset Q \cap Q^{\prime}$ and let $Q^{\prime \prime}$ be the quadric so that

$$
\Phi(X)=Q \cap Q^{\prime} \cap Q^{\prime \prime}
$$

In the smooth case $\Psi(C)$ has degree $2 g-2=10$. Hence $\Psi(C) \cap \Phi(X)=$ $\Psi(C) \cap Q^{\prime \prime}$ has degree 20, but this contradicts Lemma 7.4, since the trigonal map has $2 g+4=16$ ramification points.

Theorem 7.6. The local degree of $\overline{\mathcal{P}}_{6}$ at $\overline{\mathcal{R}}_{T}$ equals 10.
Proof. Identifying $\mathcal{J}_{5}$ to $\mathcal{M}_{5}$ via the Torelli map, we denote by $\mathcal{N}\left(\mathcal{M}_{5} \backslash \mathcal{A}_{5}\right)$ the normal subbundle to $\mathcal{J}_{5}$ in $\mathcal{A}_{5}$, and $\mathcal{N}\left(\widetilde{\mathcal{R}}_{T} \backslash \widetilde{\mathcal{R}}_{6}\right)$ denotes the normal subbundle to the exceptional divisor $\widetilde{\mathcal{R}}_{T}$ in the blowup $\widetilde{\mathcal{R}}_{6}$. Consider the codifferential map on the conormal subbundles

$$
\mathcal{N}^{*}\left(\mathcal{M}_{5} \backslash \mathcal{A}_{5}\right) \rightarrow \mathcal{N}^{*}\left(\widetilde{\mathcal{R}}_{T} \backslash \widetilde{\mathcal{R}}_{6}\right)
$$

Notice that the source bundle is of rank 3 and the target one is of rank 2. By Proposition 7.5 , the kernel of this map is at most of rank one, therefore the map is surjective. According to Lemma 5.2 the local degree equals the degree of the

$$
\widetilde{P}_{e}: \widetilde{\mathcal{R}}_{T} \rightarrow \widetilde{\mathcal{M}}_{5}
$$

where $\widetilde{P}_{e}$ denotes the projectivization of the conormal map on the exceptional divisors. Let $[X] \in \mathcal{M}_{5}$ be a generic curve and $\mathbb{P}^{2}$ the fiber over $[X]$ in $\widetilde{\mathcal{M}}_{5}$ and $\mathcal{R}=\widetilde{P}_{e}^{-1}([X])$ in $\widetilde{\mathcal{R}}_{T}$. We shall describe the map

$$
\widetilde{P}_{e}: \mathcal{R} \rightarrow \mathbb{P}^{2}=\mathbb{P}\left(\mathcal{N}_{X}\left(\mathcal{M}_{5} \backslash \mathcal{A}_{5}\right)\right)
$$

The target plane is dual to the plane $\Pi$ containing the discriminant plane quintic $F$ parametrizing the singular quadrics containing the canonical embedding of $X$. Thus, a point of $\mathbb{P}^{2}$ corresponds to a line in $\Pi$, that is a pencil of quadrics. And viceversa, a line in $\mathbb{P}^{2}$ corresponds to a point in $\Pi$, that is a quadric $Q$ containing $\Phi(X)$. The quadric $Q$ is singular if and only if, $p \in F \subset \Pi$. We have that $\mathcal{R}$ is a $\mathbb{P}^{1}$-bundle over $[\widetilde{F} \rightarrow F] \in \mathcal{R}_{T}$. Moreover, $\widetilde{F}=\operatorname{Sing} \Theta_{X}$ parametrizes the fiber of the Prym map over $J X$. Indeed, an element $L \in \operatorname{Sing} \Theta_{X}$ corresponds to a $g_{1}^{4}$ on $X$ and the data $\left(X, g_{4}^{1}\right)$ produces, via the trigonal construction, a double covering $[\widetilde{C} \rightarrow C] \in \mathcal{R}_{T}$ over a trigonal curve whose Prym variety is isomorphic to $J X$. The map

$$
\widetilde{P}_{e}: \mathbb{P}^{1} \rightarrow \mathbb{P}^{2}
$$

on the fibers is injective and its image is a line in $\mathbb{P}^{2}$, that is a point in $\Pi$ corresponding to a singular quadric in $\operatorname{ker} d P_{6}^{*}{ }_{\mid[\widetilde{C} \rightarrow C]}$.

Now, let $p \in \mathbb{P}^{2}$ be a generic point, then

$$
\begin{aligned}
\operatorname{deg} \widetilde{P}_{e} & =\left|\left\{\widetilde{P}_{e}\right\}\right| \\
& =\left|\left\{[\widetilde{C} \rightarrow C] \in \widetilde{F} \quad \mid p \in \widetilde{P}_{e}\left(\mathbb{P}_{C}^{1}\right)\right\}\right| \\
& =|\{[\widetilde{C} \rightarrow C] \in \widetilde{F} \mid \pi(C) \in \ell(p)\}|,
\end{aligned}
$$

where $\ell(p)$ is the dual line to $p$ in $\Pi$ and $\pi: \widetilde{F} \rightarrow F$ is the double covering. Hence,

$$
\operatorname{deg} \widetilde{P}_{e}=\operatorname{deg}(\widetilde{F} \rightarrow \Pi)=\operatorname{deg}(\pi) \cdot \operatorname{deg} F=2 \cdot 5=10
$$

## 8. Boundary components

In this section we will describe the Prym map on the boundary $\overline{\mathcal{R}}_{6} \backslash \mathcal{R}_{6}$ mapping onto the Jacobian locus $\mathcal{J}_{5}$. Denote by:

$$
\begin{array}{r}
\mathcal{R}_{S}=\left\{[\widetilde{C} \rightarrow C] \in \overline{\mathcal{R}}_{6}:[C] \in \overline{\mathcal{M}}_{g}\right. \text { is irreducible with one node and its } \\
\text { degenerations }\}
\end{array}
$$

$$
\begin{aligned}
\mathcal{R}_{E}=\left\{[\widetilde{C} \rightarrow C] \in \overline{\mathcal{R}}_{6}:\right. & {[C] \in \overline{\mathcal{M}}_{g} \text { has an irreducible component of } } \\
& \text { genus } g-1 \text { and an elliptic tail and its degenerations }\}
\end{aligned}
$$

The general element of $\mathcal{R}_{S}$ is a Wirtinger cover and in fact $\mathcal{R}_{S}$ is a boundary component over $\mathcal{M}_{5}$ (see Figure 2). Let $\mathcal{R}_{E, S}$ denote the intersection of $\mathcal{R}_{S}$ and $\mathcal{R}_{E}$.
LEmma 8.1. The only irreducible components of $\overline{\mathcal{R}}_{6} \backslash \mathcal{R}_{6}$ whose image contains $\mathcal{J}_{5}$ are $\mathcal{R}_{S}, \mathcal{R}_{E}$ and $\overline{\mathcal{R}}_{T} \backslash \mathcal{R}_{T}$.

A fine analysis is required to study the Prym map near the singular curves (for instance one needs to distinguish the dualising sheaf $\omega_{C}$ from the Kähler differentials $\Omega_{C}$ ). Donagi and Smith avoid the difficulties arising along the locus $\mathcal{R}_{E}$ and $\mathcal{R}_{E, S}$ by constructing a new compactification $\mathcal{M}_{6}^{\prime}$ of $\mathcal{M}_{6}$ and a space $\mathcal{R}^{\prime}$ over $\mathcal{M}_{6}^{\prime}$, such that $\overline{\mathcal{P}}_{6}$ factorises through $\mathcal{R}^{\prime}$ :

such that the map $\beta$ blow downs the component $\mathcal{R}_{E}$. This shows that $\mathcal{R}_{E}$ has no contribution to the degree of $\overline{\mathcal{P}}_{6}$. The details of these constructions can be found in [15, IV.§2,§4]

## General elements of the boundary components



Figure 2: Stable covers

In this section we will use the genericity of $X \in \mathcal{M}_{5}$ (1) to ignore families in $\overline{\mathcal{R}}_{6}$ of dimension smaller than $12\left(=\operatorname{dim} \mathcal{M}_{5}\right)$ and (2) to assume that $[X] \in \mathcal{M}_{5}$ is smooth (in particular has no automorphisms).

Note that $\left.\overline{\mathcal{P}}_{6}\right|_{\mathcal{R}_{S}}: \mathcal{R}_{S} \longrightarrow \mathcal{J}_{5}$ is proper and surjective. The fibre of a general $J X$ is naturally isomorphic to $S^{2}(X)$, since all you need is to choose $p, q \in X$ where you glue.

Lemma 8.2. For an element $[C, \eta] \in \mathcal{R}_{S} \backslash \mathcal{R}_{E, S}$ over a generic $[X] \in \mathcal{M}_{g-1}$ we have

Ker $d P_{g-1}^{*}=\{Q u a d r i c s Q$ containing $\Phi(X)$ and the chord $\overline{\Phi(p) \Phi(q)}\}$.
For $[C, \eta] \in \mathcal{R}_{S}$ with $p=q$
$\operatorname{Ker} d P_{g-1}^{*}=\{$ Quadrics $Q$ containing $\Phi(X)$ and its tangent line at the normalisation of the cusp $\}$.

Proposition 8.3. For $C=X / p \sim q$ we have that ker $d P^{*}$ is two dimensional. Proof. Recall that for the canonical model $X \subset \mathbb{P}^{4}$ we have $X=Q_{1} \cap Q_{2} \cap$ $Q_{3}$, for some quadrics $Q_{1}, Q_{2}, Q_{3}$. The secant $\overline{p q}$ (or the tangent line if $p=$ $q$ ) imposes 1 linear condition on the quadrics. The proposition follows from Lemma 8.2.

Theorem 8.4. The local degree of the Prym map $P_{6}$ at the boundary of $\overline{\mathcal{R}}_{6}$ equals 16.

Proof. The degree can be computed after blowing up $\mathcal{M}_{5} \hookrightarrow \mathcal{A}_{5}$ and restricting the map to the exceptional divisor. On the fiber over a fixed generic $[X] \in \mathcal{M}_{5}$ the map becomes

$$
f: S^{2} X \rightarrow \mathbb{P}^{2}=\mathbb{P}\left(\mathcal{N}_{X}\left(\mathcal{M}_{5} \backslash \mathcal{A}_{5}\right)\right)
$$

sending the $(p, q) \in S^{2} X$ to the pencil of quadrics trough the line $\overline{\Phi(p) \Phi(q)}$. Therefore the degree is computed by the number of chords of $\Phi(X)$ contained in the intersection of two quadrics in general position. The theorem follows from Lemma 8.5 and 8.6.

Lemma 8.5. The intersection of two quadrics in general position in $\mathbb{P}^{4}$ contains 16 lines.

This is the number of lines on a del Pezzo surface of degree 4 obtained as the blow up of 5 points in general position on $\mathbb{P}^{2}$.

Lemma 8.6. The canonical curve $\Phi(X) \subset \mathbb{P}^{4}$ meets each of the 16 lines twice.
Proof. Recall that $\Phi(X)$ is a complete intersection of $Q_{0} \cap Q_{1} \cap Q_{2} \in \mathbb{P}^{4}$ of three quadrics. Let $\ell$ be a line in $Q_{1} \cap Q_{2}$. The result follows from

$$
\left|(\Phi(X) \cap \ell)_{\mid Q_{1} \cap Q_{2}}\right|=\left|\left(Q_{0} \cap \ell\right)_{\mid \mathbb{P}^{4}}\right|=2 .
$$

## 9. Cubic threefolds and their intermediate Jacobians

In this section we study the fiber of the Prym map on the locus of intermediate Jacobians. Although the preimage of an intermediate Jacobian is 2-dimensional, after blow up the Prym map displays the structure of the finite fiber. The tetragonal construction provides a beautiful geometric way of recovering the fiber starting from one element in the preimage identifying it with the structure of the 27 lines on a smooth cubic surface. The original references for the theory of cubic threefolds are [11, 27, 28].

Let $X \subset \mathbb{P}^{4}$ be a smooth cubic hypersurface. Since a generic hyperplane section intersects $X$ in 27 lines, there is a 2-dimensional family of lines lying in $X$ parmetrized by the Fano surface $F(X)$. The intermediate Jacobian

$$
J X=H^{1,2}(X, \mathbb{C}) / H^{3}(X, \mathbb{Z})
$$

of $X$ is isomorphic as a ppav to the Albanese variety $\operatorname{Alb}(F(X))$. The theta divisor $\Theta$ in $J X$ is the image of the map

$$
F(X) \times F(X) \rightarrow J X, \quad\left(\ell, \ell^{\prime}\right) \mapsto[\ell]-\left[\ell^{\prime}\right],
$$

which collapses the diagonal to $0 \in J X$ giving the only singularity in $\Theta$ (a triple point). One can identify the projectivized tangent space $\mathbb{P}\left(T_{0}(J X)\right)$ with the ambient $\mathbb{P}^{4}$.

Let $\mathcal{C}$ be the 10 -dimensional moduli space parametrizing the smooth cubic threefolds. The construction of the intermediate Jacobian yields a map $\mathcal{C} \rightarrow$ $\mathcal{A}_{5}$. Since one can recover $X$ from its tangent cone to $\Theta$ at 0 , this map is an embedding (Torelli Theorem for cubic threefolds [11], see [6] for a proof involving Prym varieties).

Let $\left(X, l_{0}\right)$ be a pair consisting of a cubic threefold $X$ and a generic line $l_{0} \subset X \subset \mathbb{P}^{4}$. Let $\pi_{l_{0}}$ be a projection from $l_{0}$ to $\mathbb{P}^{2}$ and $b l$ the blowing up map of $X$ along $\ell_{0}$. We have the following diagram

where $(\widetilde{X}, p r)$ can be seen as a conic bundle over $\mathbb{P}^{2}$. It is because a point $p \in \mathbb{P}^{2}$ is the image of a plane that contain $l_{0}$, so its intersection with $X$ (that is of degree 3 ) is the union of $l_{0}$ and a conic, either smooth, or degenerated to two lines.

We define the discriminant locus and denote it as

$$
C:=\left\{p \in \mathbb{P}^{2}: p r^{-1}(p) \text { contains } 2 \text { lines }\right\}
$$

Note that $C$ is a plane quintic, smooth for generic $l_{0}$, since a generic hyperplane section of $X$ contains 5 pairs of lines coplanar with $l$. We also have a natural line bundle $L=\left.\mathcal{O}_{\mathbb{P}^{2}}(1)\right|_{C}$ of degree 5 .

Let

$$
\widetilde{C}=\left\{l \in F(X): l \cap l_{0} \neq \emptyset\right\}
$$

be the curve of lines intersecting $l_{0}$. The plane generated by $l_{0}, l$ intersects $X$ in the third line $l^{\prime}$ and hence there is a natural $2: 1$ map $\pi=p r_{\mid \widetilde{C}}: \widetilde{C} \rightarrow C$ that sends a line $l$ to $l \cap l^{\prime} \in C$. One checks that $\widetilde{C}$ is also smooth and the covering $\pi$ is unramified. For any line $l_{0}$ one obtains an allowable cover.
Proposition 9.1. The Prym variety of the covering $\widetilde{C} \rightarrow C$ is isomorphic to $P(\widetilde{C} / C) \simeq J X$. Moreover the fibre $\mathcal{P}_{6}^{-1}(J X) \simeq F(X)$ is 2-dimensional.

Proof. We give a sketch of the proof (a complete proof is available in [5]). Since $\widetilde{C}$ parametrizes a family of lines on $X$, there exists a map (an Abel-Prym map)

$$
\psi: \widetilde{C} \rightarrow J X, \quad l \mapsto[l]-\left[l_{0}\right]
$$

defined up to translation. This induces a homomorphism $a: J \widetilde{C} \rightarrow J X$. The family of fibres of $p r$ is parametrized by $\mathbb{P}^{2}$, hence its corresponding Abel-Jacobi map is constant. Therefore, $a$ is zero on $\pi^{*} J C$ and it gets factorized through a map $u$ :

were $\sigma$ is the involution exchanging the lines on the fiber of $\pi$. One shows that $u$ is an isomorphism by means of cohomology properties and that $u$ pulls back the principal polarization of $J X$ to the principal polarization of the Prym variety [5, §2.6].

Since $\widetilde{C}$ is defined via a line $l_{0} \subset X$ one can get that $\mathcal{P}_{6}^{-1}(J X) \simeq F(X)$.
For a generic $p \in C, p r^{-1}(p)$ is a conic in $X$ meeting $l_{0}$ in two points. For $p \in C \subset \mathbb{P}^{2}$, we denote

$$
p r^{-1}(p)=l_{1}(p) \cup l_{2}(p)
$$

with $l_{1}(p) \cup l_{2}(p)$ coplanar to $l_{0}$.
Proposition 9.2. The map $C \rightarrow X \subset \mathbb{P}^{4}$ sending

$$
p \mapsto l_{1}(p) \cap l_{2}(p)
$$

is the Prym-canonical map of $(C, \eta)$
Proof. The Abel-Prym map $\psi: \widetilde{C} \rightarrow P(\widetilde{C}, C) \simeq J X \simeq \operatorname{Alb}(F(X))$ is just the restriction to $\widetilde{C}$ of the map

$$
F(X) \rightarrow J X, \quad l \mapsto[l]-\left[l_{0}\right]
$$

The Prym-canonical image of $l_{1}(p) \in \widetilde{C}$ is the projectivized of the derivative of $\psi$ at $l_{1}(p)$ and corresponds to a point of $l_{1}(p) \subset \mathbb{P}^{4}$ and similarly for $l_{2}(p)$. Hence this point should be the intersection of both lines.

Proposition 9.3. The 2 -torsion point $\eta \in J C$ defining the covering $\pi: \widetilde{C} \rightarrow C$ satisfies $h^{0}(C, \eta \otimes L)=1$, so $(C, \eta) \in \mathcal{R}_{C}$.

Proof. The Prym-canonical map $\Psi$ of $(C, \eta)$ is given by the line bundle

$$
\omega_{C} \otimes \eta \simeq \mathcal{O}_{\mathbb{P}^{2}}(2)_{\mid C} \otimes \eta
$$

and after projecting from $l, C$ is mapped to $\mathbb{P}^{2}$ by $\mathcal{O}_{\mathbb{P}^{2}}(1)_{\mid C}=L$. Hence, the line bundle $\mathcal{O}_{\mathbb{P}^{2}}(1)_{\mid C} \otimes \eta$ has a unique effective divisor given by the 5 -points intersection of the image of $l$ with $\Psi(C)$.

Let $\mathcal{R}_{X}:=\mathcal{P}_{6}^{-1}(J X) \cap \mathcal{R}_{\mathcal{C}}$. We will see that the codifferential is of maximal rank so $\mathcal{R}_{X}$ is isolated in $\mathcal{P}_{6}^{-1}(J X)$, that is, it is a connected and irreducible component. Therefore $\mathcal{R}_{X}=\mathcal{P}_{6}^{-1}(J X)$.

As we have seen the choice of a line in the Fano variety $F(X)$ produces an étale double covering whose Prym variety is isomorphic to the intermediate Jacobian $J X$. Thus $F(X)$ parametrizes a subvariety $\mathcal{R}_{X}^{\prime} \subset \mathcal{R}_{X}$. It can be shown that the closure of the union of $\cup_{X} \mathcal{R}_{X}^{\prime}$ for all the smooth cubic threefolds $X$ equals the locus $\mathcal{R}_{\mathcal{C}}$ of pairs $(C, \eta)$ with $\eta$ odd. Let $\mathcal{A}_{\mathcal{C}} \subset \mathcal{A}_{5}$ be the closure of the locus of intermediate Jacobians of cubic threefolds. We have the following blow up diagram:

where $\widetilde{\mathcal{R}}_{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ are the exceptional divisors, $H=\left.\mathcal{O}_{\mathbb{P}^{4}}(1)\right|_{X}$ is a hyperplane section and $L \in F(X)$. We will see that $\widetilde{\mathcal{P}}_{e}^{-1}(X, H)=\{l \in F(X): l \in X \cap H\}$. We can find the following dimensions of spaces and general fibres of maps that appears in Diagram (6)


The cotangent space

$$
T_{J X}^{*} \mathcal{A}_{5}=\operatorname{Sym}^{2} T_{0}^{*}(J X)
$$

consists of all quadrics in $\mathbb{P}^{4}=\mathbb{P}\left(T_{0}^{*}(J X)\right)$. The quadrics corresponding to the conormal space $\mathcal{N}_{J X}^{*}\left(\mathcal{A}_{\mathcal{C}} \backslash \mathcal{A}_{5}\right)$ are those $X_{p}$ polar to points $p \in \mathbb{P}^{4}$ with respect to $X$ (see [17]). Thus

$$
\pi_{2}^{-1}(J X) \simeq \mathbb{P}(\mathcal{N}) \simeq\left(\mathbb{P}^{4}\right)^{*}
$$

Since $\mathcal{R}_{\mathcal{C}}^{\prime}$ is an unramified cover over the moduli space of plane quintics, we can identify the fiber of $\pi_{1}$ over $(C, \eta) \in \mathcal{R}_{\mathcal{C}}^{\prime}$ with the dual of the ambient $\mathbb{P}^{2}$ of $C$. In terms of given pair $(X, l)$ with $l \in F(X)$ this $\mathbb{P}_{l}^{2}$ is the space of planes through $l$ in $\mathbb{P}^{4}$ and $\left(\mathbb{P}_{l}^{2}\right)^{*}$ is the subspace of $\left(\mathbb{P}^{4}\right)^{*}$ dual to $l$.
Lemma 9.4. Let $\widetilde{\mathcal{R}}_{X}=\pi^{-1}\left(\mathcal{R}_{X}^{\prime}\right)$ and $\mathcal{P}_{X}$ be the restricted map. Then

$$
\mathcal{P}_{X}: \widetilde{\mathcal{R}}_{X}=\cup_{l \in F(X)}\left(\mathbb{P}_{l}^{2}\right)^{*} \rightarrow\left(\mathbb{P}^{4}\right)^{*}
$$

is the natural injection on each $\left(\mathbb{P}_{l}^{2}\right)^{*}$.

This lemma shows that $\widetilde{\mathcal{P}}_{e}$ is of maximal rank and it is of degree 27 , since a generic hyperplane section of $X$ contains 27 lines. Hence, an element in $\left(\mathbb{P}^{4}\right)^{*}$ has 27 planes in its preimage.

## 10. Tetragonal construction

As we have seen, the fiber of $\widetilde{\mathcal{P}}_{6}$ over an intermediate Jacobian corresponds to the 27 lines on a smooth cubic surface, so it carries also a structure of the incidence correspondence of the lines. The tetragonal construction on elements $(C, \eta) \in \mathcal{R}_{\mathcal{C}}$ on the fiber reflects this correspondence.

Let $C$ denote a tetragonal curve of genus $g$ (with $f: C \rightarrow \mathbb{P}^{1}$ given by a $g_{4}^{1}$ ) and let $\pi: \widetilde{C} \rightarrow C$ be an étale double covering. As usual, we have the following construction.


Note that for $D, D^{\prime} \in \operatorname{Pic}(\widetilde{X})$ we have $D \sim D^{\prime}$ if and only if they push down to the same divisor on $C^{(4)}$ and they share an even number of points in each orbit. This shows in particular that $\widetilde{X}$ has two connected components $\widetilde{C}_{0}$ and $\widetilde{C}_{1}$.

We have so called triality $(\widetilde{C}, C, f)$, $\left(\widetilde{C}_{0}, C_{0}, f_{0}\right)$, ( $\left.\widetilde{C}_{1}, C_{1}, f_{1}\right)$ because the construction does not depend from which curve we have started. This phenomenon can be explained by the monodromy representation $\pi\left(\mathbb{P}^{1} \backslash\{\right.$ branch points $\left.\}\right) \longrightarrow W D_{4}$, where $W\left(D_{4}\right)$ is the Weyl group of $D_{4}$. Note that $S_{3}$ acts on $W\left(D_{4}\right)$ as the group of outer automorphisms. The outer automorphism of order 3 is responsible for the appearance of the three tetragonally related double covers.

Theorem 10.1. The tetragonal construction commutes with the Prym map, that is,

$$
\mathcal{P}_{g}(\widetilde{C}, C) \simeq \mathcal{P}_{g}\left(\widetilde{C}_{0}, C_{0}\right) \simeq \mathcal{P}_{g}\left(\widetilde{C}_{1}, C_{1}\right)
$$

are isomorphic as ppav (for any genus $g \geq 5$ ).
Proof. One can use Masiewicki's criterion to prove the isomorphisms. Instead, we sketch here the degeneration argument given in [14]. Consider $\mathcal{R}_{g}^{\text {Tet }} \subset$


Figure 3: Stable cover
$\mathcal{R}_{g}$ the space parametrizing pairs $(\widetilde{C}, C)$ of étale double coverings with $C$ a tetragonal curve of genus $g$. This is an irreducible space and the construction varies continuously with $(\widetilde{C}, C)$, so we can make the computation for a single pair. Consider the allowable covering

$$
\widetilde{C}:=\mathbb{P}^{1} \cup_{q^{\prime}} \widetilde{T} \cup_{q^{\prime \prime}} \mathbb{P}^{1}, \quad C:=T \cup_{q} \mathbb{P}^{1}
$$

with $\widetilde{T} \rightarrow T$ an étale double cover over a trigonal curve $T$ as in Figure 3.
The tetragonal construction applied to the cover produces other two Wirtinger covers $\widetilde{C}_{i}, C_{i}, i=0,1$, such that the normalization of $C_{i}$ is the tetragonal curve $N$ associated to $(\widetilde{T}, T)$ via the trigonal construction. In this sense, the trigonal is a degeneration of the tetragonal construction. We have then isomorphisms of ppav

$$
J N \simeq \mathcal{P}_{g}(\widetilde{T}, T) \simeq \mathcal{P}_{g}(\widetilde{C}, C)
$$

such that image of the Abel-Prym map $\alpha_{i}: C_{i} \rightarrow \mathcal{P}_{g}(\widetilde{C}, C)$ consists of the image of the Abel-Jacobi map $\varphi: N \rightarrow J N$ and its involution. Thus the fundamental class is twice that of $\varphi(N)$.

Curves of genus 6 are tetragonal and the generic one possess $5 g_{4}^{1}$ 's. Let $\mathcal{M}_{6}^{T e t}$ denote the moduli space parametrizing pairs of genus- 6 curves with a $g_{4}^{1}$. So the forgetful map $\mathcal{M}_{6}^{T e t} \rightarrow \mathcal{M}_{6}$ is generically finite of degree 5. By base change we get the following diagram


The tetragonal construction induces a $(2,2)$ correspondence on $\mathcal{R}_{6}^{T e t}$ whose image in $\mathcal{R}_{6}$ is a $(10,10)$ correspondence $\operatorname{Tet} \subset \mathcal{R}_{6} \times \mathcal{R}_{6}$.
Theorem 10.2. The correspondence Tet on the fiber $\mathcal{P}_{6}^{-1}(A)$ for a generic $A \in \mathcal{A}_{5}$ is isomorphic the the incidence correspondence of the lines on a smooth cubic surface. Moreover, the Galois group of the Galois closure of $\mathcal{R}_{6} \rightarrow \mathcal{A}_{5}$ is the Weyl group $W\left(E_{6}\right)$, the symmetry group of the incidence of the 27 lines on the cubic surface.

Proof. The generically finite map $\mathcal{R}_{6}^{T e t} \rightarrow \mathcal{R}_{6}$ has 1-dimesional fibers over the locus of double coverings $\widetilde{C} \rightarrow C$ with $C$ trigonal or a plane quintic. After blowing up and normalizing one gets generically finite fibers over the corresponding exceptional loci. One checks that the tetragonal correspondence lifts to a generically finite $(10,10)$ correspondence

$$
\widetilde{\operatorname{Tet}} \subset \widetilde{\mathcal{R}}_{6} \times \widetilde{\mathcal{R}}_{6}
$$

It suffices to identify the structure over a point over which $\widetilde{\mathcal{P}}_{6}$ and $\widetilde{\text { Tet }}$ are étale. For instance over a generic $(X, H) \in \widetilde{\mathcal{C}}$, where the group $W\left(E_{6}\right)$ acts on the line of the cubic surface $X \cap H$. So the monodromy is contained in $W\left(E_{6}\right)$.

For instance, for an element $(C, \eta, l) \in \widetilde{\mathcal{R}}_{\mathcal{C}}$ (that is a plane quintic $C$ with an odd 2-torsion point $\eta$ and $l$ a line in $\mathbb{P}^{2}$ ), the $5 g_{4}^{1}$ 's correspond to the projections of the plane quintic $C$ from one of the 5 points of the intersection $C \cap l$. We have the identification of $\widetilde{\mathcal{P}}_{6}(X, H)$ with the set of lines of the cubic surface $X \cap H$, which for generic $X$ and $H$ there are 27 lines. For each $l$ of these lines the conic bundle construction (blow up of the projection from $l$ ) gives a double cover $\pi: \widetilde{C} \rightarrow C$, with $L=\pi(H) \subset \mathbb{P}^{2}$. In order to corroborate Theorem 10.2, we need to check that for two given lines $l, l^{\prime} \in F(X)$ they intersect each other if and only if the corresponding objects $(C, \eta, l),\left(C^{\prime} \eta^{\prime}, l^{\prime}\right)$ are tetragonally related, that is, the pair belongs to Tet. If $l \cap l^{\prime} \neq \emptyset$, let $A \subset \mathbb{P}^{4}$ be the plane containing $l, l^{\prime}$ and $l^{\prime \prime}$ the line such that $A \cap X=l \cup l^{\prime} \cup l^{\prime \prime}$. The conic bundle construction gives then 3 plane quintics $C, C^{\prime}, C^{\prime \prime}$ with their respective double covers $\widetilde{C}, \widetilde{C}^{\prime}, \widetilde{C}^{\prime \prime}$. Note that the $l, l^{\prime}$ map to a point $p \in C$ and this point determines a $4: 1$ map $f: C \rightarrow \mathbb{P}^{1}$ by projecting from it. Similarly, for $C^{\prime}, C^{\prime \prime}$ we obtain tetragonal maps $f^{\prime}, f^{\prime \prime}$. These 3 maps can be realised simultaneously via the pencil of hyperplanes $S_{\lambda} \subset \mathbb{P}^{4}$ containing $A$. For a generic $\lambda, S_{\lambda} \cap X=: Y_{\lambda}$ is a smooth cubic surface. A line $m \in Y_{\lambda}$, with $m \notin A$, $m \cap l^{\prime} \neq \emptyset$ also meets 4 of the 8 lines in $Y_{\lambda} \backslash A$ meeting $l$. This gives the injection

$$
\widetilde{C}^{\prime} \hookrightarrow \widetilde{C}^{(4)}, \quad m \mapsto\left\{m^{\prime}: m^{\prime} \cap l \neq \emptyset, m^{\prime} \cap m \neq \emptyset\right\}
$$

This shows that the three covers are tetragonally related, hence

$$
(\widetilde{C}, C, f),\left(\widetilde{C}^{\prime}, C^{\prime}, f^{\prime}\right) \in \widetilde{\text { Tet. }}
$$

Since both, the line incidence and the tetragonal correspondence are of bidegree $(10,10)$ and we have the inclusion, they must be equal.

## 11. Exercises

The course has been supplemented with the exercise sessions. The idea was to compute some examples in low genera and show that Prym theory combines constructions from curve theory and theory abelian varieties. We would like to show some ideas of what was covered.

Exercise 11.1. Show that if $C$ is a genus 2 curve and $f: C \rightarrow E$ is an $n: 1$ covering of an elliptic curve, then there is another $n: 1$ covering $g: C \rightarrow E^{\prime}$.

Proof. By dimension count, one gets that the Prym variety $P(f)$ is of dimension 1 , hence an elliptic curve, say $E^{\prime}$. Since we have an inclusion $j: E^{\prime} \rightarrow J C$, we can dualize it to get $\widehat{j}: J C \rightarrow E^{\prime}$ and restrict to an image of an Abel Jacobi map to get a map $g=\widehat{j} \circ \alpha_{C}: C \rightarrow E^{\prime}$. Here, we have used the fact that both $J C$ and $E^{\prime}$ are principally polarized, hence isomorphic to their duals. It is also worth noting that a change of the base point of an Abel Jacobi map results in a map that differs by a translation on $E^{\prime}$, so the map $g$ is (up to translation on $E^{\prime}$ ) unique. Since $f$ is $n: 1$, we have that $E^{\prime}$ has restricted polarization being $n$ times the principal one, so $g$ is of order $n$.

Remark 11.2. The locus of Jacobians of curves mentioned in Exercise 11.1 coincides with the locus of abelian surfaces that are polarized isogenous to a product of elliptic curves of exponent $n$ and is called the Humbert surface of degree $n^{2}$.

From the proof of Exercise 11.1, we get an immediate corollary.
Corollary 11.3. An elliptic curve $E$ can be embedded in a Jacobian JC with exponent $n$ if and only if there exists an $n: 1$ covering $C \rightarrow E$ (that does not factorize via $C \rightarrow E^{\prime} \rightarrow E$ with $E^{\prime} \rightarrow E$ an isogeny).

Before showing next exercise, we need to recall result from [7] that deals with curves on surfaces.

Lemma 11.4 ([7, Prop 4.3]). Let $C$ be a smooth curve and $(J C, \Theta)$ its Jacobian. Let $(A, H)$ be a polarised abelian surface and suppose $f_{C}: C \rightarrow A$ is a morphism and $f: J C \rightarrow A$ is the canonical homomorphism defined by the universal property of Jacobians. Then the following are equivalent:

- $\hat{f}^{*}(\Theta) \equiv \hat{H}$;
- $\left(f_{C}\right)_{*}[C]=H$ in $H^{2}(A, \mathbb{Z})$.

If $C$ is of genus 3 , we can use the fact that if $J C$ contains an abelian subvariety, then it contains an elliptic curve and therefore $C$ is a covering of an elliptic curve. We will recall Barth's result [3] in the following exercise.
Exercise 11.5. Show that a smooth genus 3 curve $C$ can be embedded in a $(1,2)$ polarized abelian surface if and only if it is a double covering of an elliptic curve. In such a case, the curve $C$ is hyperelliptic if and only if $C$ is an étale double covering of a genus 2 curve.

Proof. Note that by [2], a general section of a polarization of type $(1, d)$ is a smooth curve and by Riemann-Roch, it is of genus $d+1$. Hence, if $f_{C}: C \rightarrow A$ is an embedding of a genus 3 curve, then $f_{C}(C)$ has to generate $A$ as a group and hence $\mathcal{O}\left(f_{C}(C)\right)$ is a $(1,2)$ polarization. Now, by Universal Property of Jacobians, we can extend $f_{C}$ to a map $f: J C \rightarrow A$ which will be surjective and hence $\operatorname{Ker}(f)=E$ is an elliptic curve. Since $E$ is complementary to $\hat{A}$ in $J C$, its exponent equals 2 and so there exists a double covering $C \rightarrow E$.

On the other hand, if $g: C \rightarrow E$ is a double covering then $\mathrm{Nm}_{g}: J C \rightarrow E$ has kernel $\operatorname{Ker}\left(\mathrm{Nm}_{g}\right)=A$ that is a $(1,2)$ polarized abelian surface. If we take the dual map to the inclusion, we get a map $J C \rightarrow \hat{A}$ and by composing with an Abel-Jacobi map, using Lemma 11.4 we get that the image is of arithmetic genus 3 and hence it is a desired embedding of $C$.

As for the second part, it is well known that an étale double covering of a genus 2 curve is bielliptic, i.e. hyperelliptic and a double cover of an elliptic curve (see [25]). On the other hand, if $C$ is hyperelliptic, we can use the hyperelliptic involution $\iota$ to show that for any degree 0 divisor $D$, we have that $D+\iota^{*} D$ is a principal divisor. In particular $\iota$ extends to -1 on the Jacobian $J C$. Now, if $\tau$ is the involution defining the covering $f: C \rightarrow E$, then $\iota \tau$ defines another double covering $\pi: C \rightarrow C^{\prime}$ and one can compute

$$
E=\operatorname{Im}(\operatorname{Nm}(f))=\operatorname{Im}(1+\tau)=\operatorname{Im}(1-(-\tau))=P\left(C / C^{\prime}\right)
$$

and in particular $C^{\prime}$ is of genus 2 and $\pi$ an étale double covering.
REMARK 11.6. A trick of composing an involution with the hyperelliptic involution used in the proof of Exercise 11.5 can be generalised to any genus. If $C$ is hyperelliptic and $f^{\prime}: C \rightarrow C^{\prime}$ and $f^{\prime \prime}: C \rightarrow C^{\prime \prime}$ are double coverings given by involutions $\tau$ and $\iota \tau$ respectively then the Prym varieties equals $P\left(f^{\prime}\right)=\left(f^{\prime \prime}\right)^{*}\left(J C^{\prime \prime}\right)$ and $P\left(f^{\prime \prime}\right)=\left(f^{\prime}\right)^{*}\left(J C^{\prime}\right)$.

One may suppose that if a Jacobian contains an abelian subvariety, then there is a covering of curves involved. The aim of the last exercise is to show that it may not be the case.
Exercise 11.7. Show that there exists a Jacobian of a curve that contains abelian subvarieties but does not come from the Prym construction (i.e. the curve is not a covering of a positive genus curve).

Proof. Here, we will show a heuristic argument. Let $C$ be a smooth genus 4 curve embedded in a $(1,3)$ polarised abelian surface $A$. Then we can construct the exact sequence $0 \rightarrow K \rightarrow J C \rightarrow A \rightarrow 0$ and its dual sequence $0 \rightarrow \hat{A} \rightarrow$ $J C \rightarrow \hat{K} \rightarrow 0$. Since $K$ and $\hat{A}$ are complementary to each other in $J C$ and therefore of the same type $(1,3)$ we get that $C$ is also embedded in $\hat{K}$. The moduli of abelian surfaces is three dimensional and on a fixed surface there is a two dimensional family of genus 4 curves (since $h^{0}(A, \mathcal{L})=3$ ), hence there is a five dimensional family of such curves (locally). Note that any curve can be embedded in only finitely many surfaces, since for a fixed abelian variety (in this case a Jacobian) there is only finitely many abelian subvarieties of a fixed exponent. Because of that, we can assume that all $A, \hat{A}, K, \hat{K}$ do not contain elliptic curves. In such a case, $C$ is not a covering of an elliptic curve. Now, the only possible covering is a triple étale covering of a genus 2 curve but there are only finitely many such curves on a fixed $A$.

An explicit example when we additionally assume that $C$ is hyperelliptic and get that $K=A$ and a precise proof that $C$ is not a covering of a positive genus curve can be found in [10].

To show that the inverse to the Torelli map is mysterious even in dimensions 2 and 3, we have finished the exercises with two open questions:

Exercise 11.8. Can you find an explicit example (i.e. a hyperelliptic equation) of a smooth genus 2 curve that is a $2021: 1$ covering of an elliptic curve? Can you find an explicit example (i.e. a bivariate quartic equation) of a smooth genus 3 curve that is a 2021:1 covering of an elliptic curve?

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# Hyperkähler manifolds 

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#### Abstract

We give an elementary introduction to hyperkähler manifolds, survey some of their interesting properties and some open problems.


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## 1. Introduction

The cohomology of a compact Kähler manifold has remarkable properties, abstractified in the modern notion of a (polarized) Hodge structure. While the
datum of a Hodge structure of weight 1 is equivalent to the datum of a compact complex torus, this is no longer the case in higher weights. In weight 2 there are remarkable examples of compact Kähler manifolds which are, mostly, determined by the polarized Hodge structure on their second cohomology. These are the hyperkähler manifolds: higher dimensional analogues of K3 surfaces. In these lecture notes, we give an elementary introduction to hyperkähler manifolds and survey some of their interesting properties.

We start by reviewing the notions of tensors, connections, the curvature tensor, Ricci curvature and some of their properties. We define parallel transport, holonomy and the Levi-Civita connection. We also describe the constraints posed by the holonomy on the curvature tensor. We define (locally) symmetric spaces and state the main structure theorem for them. We then state De Rham's decomposition theorem for simply connected complete Riemannian manifolds and Berger's classification of the holonomy groups of nonsymmetric, complete, connected, irreducible Riemannian manifolds. Berger's classification shows that hyperkähler manifolds are the nonsymmetric complete connected irreducible Riemannian manifolds with holonomy group contained in $S p(r)$ : the group of automorphisms of the quaternions $\mathbb{H}^{r}$ preserving a quaternionic hermitian form. It follows that they are Ricci flat. In fact, it follows from the theorems of De Rham and Berger, the Calabi-Yau theorem and results of Cheeger-Gromoll and Bochner that, after possibly taking a finite étale cover, Ricci-flat compact Riemannian manifolds are products of complex tori, CalabiYau manifolds and hyperkähler manifolds (see Paragraph 4.5).

Constructing examples of compact hyperkähler manifolds has proven particularly challenging. Two infinite series were constructed by Beauville [2], using an idea of Fujiki [19]. Two sporadic families of hyperkählers of dimensions 6 and 10 were constructed by O'Grady ( $[37,38]$ ) via desingularization of certain singular moduli spaces of sheaves on K3 surfaces and complex tori of dimension 2. We give an overview of Beauville's constructions of the two infinite series.

It is the content of the Torelli theorem that hyperkähler manifolds are essentially determined by their second cohomology. This is consistent with the fact that all constructions to date of hyperkähler manifolds involve surfaces.

We briefly describe the moduli spaces of compact hyperkähler manifolds, their period domains and some of their properties. By a result of Tian-Todorov and Bogomolov, the deformations of hyperkähler manifolds are unobstructed. This essentially means that the moduli spaces of compact hyperkähler manifolds are smooth analytic spaces. It is known however, that they are not Hausdorff.

The period domain of a given family of hyperkähler manifolds is constructed from the lattice abstractly isometric to the second integral cohomology of the hyperkähler together with a natural non-degenerate quadratic form called the

Beauville-Bogomolov form. This form generalizes the intersection form in the case of dimension 2 and the natural form on the second cohomology of the Fano variety of lines of a smooth cubic fourfold. In the case of the Fano variety of lines, the form is induced by the intersection form on the fourth cohomology of the cubic fourfold, via the Abel-Jacobi isomorphism between the second cohomology of the Fano variety if lines and the fourth cohomology of the cubic fourfold.

For a fixed compact hyperkähler $X$, we describe the local and the global period domains with their respective maps from the local and global deformation spaces of $X$. We explain the local Torelli theorem and Verbitsky's weaker version of global Torelli which holds in the hyperkähler case.

We conclude with a brief discussion of twistor conics and twistor families, the proof of the global Torelli theorem by Verbitsky and the relation between twistor families and hyperholomorphic bundles.

Some good general references for the material that we present here are: $[2,5,6,18,22,45]$.

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## 2. $C^{\infty}$ manifolds

### 2.1. Tangent and cotangent bundles

For a $C^{\infty}$ manifold $M$, we denote by $T_{M}$ the tangent bundle of $M$ and $T_{M}^{*}$ the cotangent bundle.

For any non-negative integers $(k, l)$, the sections of the bundle $T_{M}^{\otimes k} \otimes\left(T_{M}^{*}\right)^{\otimes l}$ are called $(k, l)$-tensors. Section of $T_{M}$ are vector fields and sections of $\Lambda^{p} T_{M}^{*}$ differential $p$-forms. Alternatively, vector fields can be defined as first order differential operators on $C^{\infty}$ functions.

In a local coordinate chart with local coordinates $\left(x^{1}, \ldots, x^{n}\right)$, the (local) vector fields $\partial / \partial x^{1}, \ldots, \partial / \partial x^{n}$ form a basis of vector fields and the (local) 1forms $d x^{1}, \ldots, d x^{n}$ form a basis of differential 1-forms. A local ( $k, l$ )-tensor can
be written as

$$
T=\sum T_{j_{1}, \ldots, j_{l}}^{i_{1}, \ldots, i_{k}} \frac{\partial}{\partial x^{i_{1}}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_{k}}} \otimes d x^{j_{1}} \otimes \ldots \otimes d x^{j_{l}}
$$

### 2.2. The Lie bracket

Given a vector field $v=\sum v^{i} \frac{\partial}{\partial x^{i}}$ and a $C^{\infty}$ function $f$ on $M$,

$$
v(f)=\sum_{i=1}^{n} v^{i} \frac{\partial f}{\partial x^{i}}
$$

Given two vector fields $v=\sum v^{i} \frac{\partial}{\partial x^{i}}, w=\sum w^{i} \frac{\partial}{\partial x^{i}}$, the Lie bracket of $v$ and $w$ is given by

$$
[v, w]=\sum_{j=1}^{n}\left(\sum_{i=1}^{n} v^{i} \frac{\partial w^{j}}{\partial x^{i}}-w^{i} \frac{\partial v^{j}}{\partial x^{i}}\right) \frac{\partial}{\partial x^{j}} .
$$

Alternatively, the Lie bracket can be defined via its action on $C^{\infty}$ functions on $M$ :

$$
[v, w](f)=v(w(f))-w(v(f))
$$

### 2.3. Connections

Tangent vectors allow us to take derivatives of $C^{\infty}$ functions. Connections allow us to take derivatives of sections of arbitrary vector bundles.

For a $C^{\infty}$ vector bundle $E$ on $M$, a connection is a linear map

$$
\nabla: C^{\infty}(E) \longrightarrow C^{\infty}\left(E \otimes T_{M}^{*}\right)
$$

satisfying the Leibnitz rule

$$
\nabla(f e)=f \nabla(e)+e \otimes d f
$$

for all $C^{\infty}$ sections $e$ of $E$ and $C^{\infty}$ functions $f$ on $M$. For any vector field $v$ on $M$, the connection $\nabla$ defines a linear map $\nabla_{v}: C^{\infty}(E) \rightarrow C^{\infty}(E)$ via

$$
\nabla_{v}(e):=\nabla(e)(v)
$$

We call $\nabla_{v}$ the covariant derivative in the direction of $v$.
We may thus also think of $\nabla$ as a linear map

$$
\nabla: C^{\infty}\left(E \otimes T_{M}\right) \longrightarrow C^{\infty}(E)
$$

When $E=T_{M}$, the torsion of a connection $\nabla: C^{\infty}\left(T_{M} \otimes T_{M}\right) \rightarrow C^{\infty}\left(T_{M}\right)$ is the linear map

$$
T: C^{\infty}\left(\Lambda^{2} T_{M}\right) \longrightarrow C^{\infty}\left(T_{M}\right)
$$

defined as

$$
T(v \wedge w):=\nabla_{v}(w)-\nabla_{w}(v)-[v, w] .
$$

We say $\nabla$ is torsion-free or symmetric when $T=0$.

### 2.4. Curvature

Euclidean space is "flat". What this means is that when we take second partial derivatives of vector fields, the order of differentiation does not affect the final result. Roughly speaking, the curvature of a connection measures the difference between the second partials of a section of a vector bundle taken in different orders.

For general vector fields $v, w$, the curvature measures the difference between $\nabla_{v} \nabla_{w}-\nabla_{w} \nabla_{v}$ and the derivative in the direction of the bracket $[v, w]$. On the tangent bundle of Euclidean space this difference is 0 .

Precisely, the curvature of a connection $\nabla$ is a linear map

$$
R: C^{\infty}(E) \longrightarrow C^{\infty}\left(E \otimes \Lambda^{2} T_{M}^{*}\right)
$$

or, equivalently,

$$
R: C^{\infty}\left(E \otimes \Lambda^{2} T_{M}\right) \longrightarrow C^{\infty}(E)
$$

or a global section

$$
R \in C^{\infty}\left(\operatorname{End}(E) \otimes \Lambda^{2} T_{M}^{*}\right)
$$

It can be defined via its action on sections $e$ of $E$ and vector fields $v, w$ as

$$
R(e \otimes(v \wedge w))=\nabla_{v}\left(\nabla_{w}(e)\right)-\nabla_{w}\left(\nabla_{v}(e)\right)-\nabla_{[v, w]}(e)
$$

We say that the connection $\nabla$ (or sometimes the bundle $E$ ) is flat if $R=0$.
In a coordinate chart with coordinates $\left(x^{1}, \ldots, x^{n}\right)$, the partial derivatives commute, i.e.,

$$
\left[\frac{\partial}{\partial x^{i}}, \frac{\partial}{\partial x^{j}}\right]=0
$$

for all $i, j$. Hence

$$
R\left(e \otimes\left(\frac{\partial}{\partial x^{i}} \wedge \frac{\partial}{\partial x^{j}}\right)\right)=\nabla_{\frac{\partial}{\partial x^{i}}}\left(\nabla_{\frac{\partial}{\partial x^{j}}}(e)\right)-\nabla_{\frac{\partial}{\partial x^{j}}}\left(\nabla_{\frac{\partial}{\partial x^{i}}}(e)\right)
$$

and the connection is flat if and only if its partial (covariant) derivatives commute.

### 2.5. Parallel transport

Suppose given a $C^{\infty}$ vector bundle $E$ on $M$ with a connection

$$
\nabla: E \longrightarrow E \otimes T_{M}^{*}
$$

and a smooth curve $\gamma:[0,1] \rightarrow M$. Parallel transport along $\gamma$ produces sections of the pull-back $\gamma^{*} E$ that are 'constant' or 'horizontal' along $\gamma$. As we
see below, such sections exist and are determined by their values at one point of $\gamma$.

The pull-back $\gamma^{*} E$ is a $C^{\infty}$ vector bundle on $[0,1]$ with fiber $E_{\gamma(t)}$ at $t \in$ $[0,1]$. The connection $\nabla$ defines the connection $\gamma^{*} \nabla$ on $\gamma^{*} E$ as the composition

$$
\gamma^{*} \nabla: \gamma^{*} E \longrightarrow \gamma^{*} E \otimes \gamma^{*} T_{M}^{*} \longrightarrow \gamma^{*} E \otimes T_{[0,1]}^{*}
$$

where the second map is induced by the projection $T_{M}^{*} \rightarrow T_{[0,1]}^{*}$.
In local coordinates $\left(x^{1}, \ldots, x^{n}\right)$ on $M$, with $\gamma(t)=\left(x^{1}(t), \ldots, x^{n}(t)\right)$,

$$
\dot{\gamma}(t)=\left(\dot{x}^{1}(t), \ldots, \dot{x}^{n}(t)\right)=\sum_{i=1}^{n} \dot{x}^{i}(t) \frac{\partial}{\partial x^{i}}
$$

and, for all (local) sections $e$ of $E$,

$$
\nabla_{\dot{\gamma}(t)}(e):=\nabla_{\sum_{i=1}^{n} \dot{x}^{i}(t) \frac{\partial}{\partial x^{i}}}(e)=\sum_{i=1}^{n} \dot{x}^{i}(t) \nabla_{\frac{\partial}{\partial x^{i}}}(e) .
$$

Definition and Proposition 2.1. Put $x:=\gamma(0), y:=\gamma(1)$. Then, for all $e \in E_{x}=\left(\gamma^{*} E\right)_{0}$, there exists a unique smooth section s of $\gamma^{*} E$ such that $s(0)=e$ and $\gamma^{*} \nabla(s)=0$, i.e., $\nabla_{\dot{\gamma}(t)}(s)=0$.

The parallel transport of e along $\gamma$ to $y$ is $P_{\gamma}(e):=s(1) \in E_{y}=\left(\gamma^{*} E\right)_{1}$. The map $P_{\gamma}: E_{x} \longrightarrow E_{y}$ is a linear isomorphism.

### 2.6. Holonomy

As we saw above, parallel transport defines linear isomorphisms between fibers of $E$ at points of $M$. In particular, for a given point $x$ of $M$, it defines linear automorphisms of the fiber $E_{x}$. The holonomy of $\nabla$ is the group generated by these automorphisms. It acts on all tensors of $E$ and its invariants are the covariantly constant tensors:

Definition and Proposition 2.2. If $\gamma$ is a loop (i.e. $x=y$ ), then $P_{\gamma} \in$ $G L\left(E_{x}\right)$. The holonomy group $\operatorname{Hol}_{x}(\nabla)$ at $x$ is

$$
\operatorname{Hol}_{x}(\nabla):=\left\{P_{\gamma} \mid \gamma \text { is a loop based at } x\right\} .
$$

It has the following properties.

1. $\operatorname{Hol}_{x}(\nabla)$ is a Lie subgroup of $G L\left(E_{x}\right)$ :

$$
\begin{aligned}
& \gamma \delta(t)= \begin{cases}\delta(2 t) & \text { if } t \in\left[0, \frac{1}{2}\right] \\
\gamma(2 t-1) & \text { if } t \in\left[\frac{1}{2}, 1\right]\end{cases} \\
& \gamma^{-1}(t)=\gamma(1-t), \\
& P_{\gamma \delta}=P_{\gamma} \circ P_{\delta}, \quad P_{\gamma^{-1}}=P_{\gamma}^{-1} .
\end{aligned}
$$

2. If $\gamma$ is a path from $x$ to $y$, then

$$
\operatorname{Hol}_{y}(\nabla)=P_{\gamma} \operatorname{Hol}_{x}(\nabla) P_{\gamma}^{-1}
$$

Hence, up to conjugation, $\operatorname{Hol}_{x}(\nabla)$ only depends on the connected component of $M$ containing $x$.
3. if $M$ is simply connected, then $\operatorname{Hol}_{x}(\nabla)$ is connected. Any loop can be shrunk to a point:

$$
\gamma:[0,1] \times[0,1] \longrightarrow M ; \quad \gamma_{s}(t):=\gamma(s, t) ; \gamma_{1}(t)=x \text { for all } t
$$

Then $\left\{P_{s}:=P_{\gamma_{s}} \mid s \in[0,1]\right\}$ is a path in $\operatorname{Hol}_{x}(\nabla)$ from $P_{0}=P_{\gamma_{0}}$ to $P_{1}=P_{\gamma_{1}}=\mathrm{Id}$.
4. Let $\mathfrak{h o l}_{x}(\nabla) \subset \mathfrak{g l}\left(E_{x}\right)=\operatorname{End}\left(E_{x}\right)$ be the Lie algebra of $\operatorname{Hol}_{x}(\nabla)$. Recall that the curvature operator $R(\nabla)$ belongs to $C^{\infty}\left(E^{*} \otimes E \otimes \Lambda^{2} T_{M}^{*}\right)=$ $C^{\infty}\left(\operatorname{End}(E) \otimes \Lambda^{2} T_{M}^{*}\right)$. At a point $x$, the fiber $R(\nabla)_{x}$ of $R(\nabla)$ belongs to $\operatorname{End}\left(E_{x}\right) \otimes \Lambda^{2} T_{x}^{*} M$. We have

$$
R(\nabla)_{x} \in \mathfrak{h o l}_{x}(\nabla) \otimes \Lambda^{2} T_{x}^{*} M
$$

As we shall see below, Riemannian holonomy plays a central role in the structure theory of Riemannian manifolds.

The connection $\nabla$ induces connections on all tensor powers $E^{\otimes k} \otimes\left(E^{*}\right)^{\otimes l}$, and all exterior and symmetric powers of $E$ and $E^{*}$ and their tensor products. We shall denote these induced connections by $\nabla$ as well.

Definition 2.3. A tensor $S$ is called (covariantly) constant if $\nabla(S)=0$.
Theorem 2.4. For a tensor $S, \nabla(S)=0$ if and only if $S$ is fixed by $\operatorname{Hol}_{x}(\nabla)$, if and only if $P_{\gamma}(S(x))=S(y)$ for all $x, y \in M$ and all paths $\gamma$ from $x$ to $y$.

## 3. Riemannian manifolds

A $C^{\infty}$ manifold is called Riemannian if it has a Riemannian metric, i.e., a (2,0)-tensor $g \in C^{\infty}\left(\left(T_{M}^{*}\right)^{2}\right.$ which is symmetric:

$$
g \in C^{\infty}\left(\operatorname{Sym}^{2} T_{M}^{*}\right),
$$

and defines a positive definite quadratic form on the tangent space $T_{M, x}$ for all $x \in M$. It is a fundamental result in differential geometry that every smooth manifold can be endowed with a Riemannian metric.

Riemannian manifolds have canonical connections on their tangent bundles: the Levi-Civita connection. The holonomy of the Levi-Civita connection is called Riemannian holonomy and the classification of Riemannian manifolds is based on the classification of Riemannian holonomy groups.

### 3.1. Levi-Civita connection

Suppose $(M, g)$ is a Riemannian manifold. The fundamental theorem of Riemannian geometry is the following.

Theorem 3.1. There exists a unique torsion free (or symmetric) connection $\nabla$ on $T_{M}$ such that $\nabla g=0$. This unique connection is called the Levi-Civita or Riemannian connection of $(M, g)$.

One can verify that the condition $\nabla g=0$ is equivalent to the following compatibility property: For all vector fields $u, v, w$ on $M$,

$$
u(g(v, w))=g\left(\nabla_{u} v, w\right)+g\left(v, \nabla_{u} w\right)
$$

The Levi-Civita connection $\nabla$ can be explicitly defined via

$$
\begin{aligned}
& 2 g\left(\nabla_{u} v, w\right)=u(g(v, w))+v(g(u, w))-w(g(u, v)) \\
& +g([u, v], w)-g([v, w], u)-g([u, w], v) .
\end{aligned}
$$

The curvature $R(\nabla)$ is a $(1,3)$ tensor:

$$
R(\nabla): T_{M} \longrightarrow T_{M} \otimes \Lambda^{2} T_{M}^{*}
$$

More symmetries of $R(\nabla)$ can be exhibited by defining the $(0,4)$ tensor $\widetilde{R}(\nabla)$ as the compostion

$$
\widetilde{R}(\nabla): T_{M} \xrightarrow{R(\nabla)} T_{M} \otimes \Lambda^{2} T_{M}^{*} \xrightarrow{g \otimes \mathrm{Id}} T_{M}^{*} \otimes \Lambda^{2} T_{M}^{*} .
$$

While a priori $\widetilde{R}(\nabla) \in C^{\infty}\left(\left(T_{M}^{*}\right)^{\otimes 2} \otimes \Lambda^{2} T_{M}^{*}\right)$, one can show that in fact

$$
\widetilde{R}(\nabla) \in C^{\infty}\left(\operatorname{Sym}^{2}\left(\Lambda^{2} T_{M}^{*}\right)\right)
$$

The Bianchi identities can be written in the form

$$
\begin{aligned}
& R(u, v) w+R(v, w) u+R(w, u) v=0 \\
& \nabla_{u} R(u, v)+\nabla_{v} R(w, u)+\nabla_{w} R(u, v)=0
\end{aligned}
$$

In a basis of local coordinates $x^{1}, \ldots, x^{n}$, we can write $\widetilde{R}(\nabla)$ as

$$
\widetilde{R}(\nabla)=\sum_{a, b, c, d} \widetilde{R}_{a b c d} \quad\left(d x^{a} \wedge d x^{b} \odot d x^{c} \wedge d x^{d}\right)
$$

where $\alpha \odot \beta:=\alpha \otimes \beta+\beta \otimes \alpha$ is the symmetric tensor. The Bianchi identities then can be written as

$$
\widetilde{R}_{a b c d}+\widetilde{R}_{a c d b}+\widetilde{R}_{a d b c}=0, \quad \frac{\partial}{\partial x^{e}} \widetilde{R}_{a b c d}+\frac{\partial}{\partial x^{c}} \widetilde{R}_{a b d e}+\frac{\partial}{\partial x^{d}} \widetilde{R}_{a b e c}=0
$$

### 3.2. Ricci curvature

The Ricci curvature is a $(0,2)$ tensor, obtained by contracting $R(\nabla)$ :
At each point $x \in M$, the curvature tensor $R$ defines a multilinear map

$$
\begin{array}{ccc}
R_{x}: T_{x} M \times T_{x} M \times T_{x} M & \longrightarrow & T_{x} M \\
(u, v, w) & \longmapsto & R(u, v) w
\end{array}
$$

The Ricci curvature is the $(0,2)$ tensor defined as

$$
\begin{array}{ccc}
\operatorname{Ric}_{x}: T_{x} M \times T_{x} M & \longrightarrow & \mathbb{R} \\
(u, v) & \longmapsto \operatorname{tr}\left(w \mapsto R_{x}(u, w) v\right)
\end{array}
$$

where tr is the trace of a linear map. It follows from the symmetries of the curvature tensor that the Ricci curvature is symmetric. In local coordinates, if we write the curvature tensor as

$$
R(\nabla)=\sum_{a, b, c, d} R_{b c d}^{a} \frac{\partial}{\partial x^{a}} \otimes d x^{b} \otimes d x^{c} \wedge d x^{d}
$$

then the coordinates of the Ricci tensor are

$$
\operatorname{Ric}_{a b}=\sum_{c} R_{a c b}^{c} .
$$

Definition 3.2. We say $g$ is an Einstein metric if the Ricci curvature is a constant multiple of the metric. We say $g$ is Ricci flat if the Ricci curvature is 0 .

### 3.3. Riemannian holonomy

For a Riemannian manifold $(M, g)$, the holonomy of the Levi-Civita connection $\nabla$ is called Riemannian holonomy. For $x \in M$, we write

$$
\begin{aligned}
& \operatorname{Hol}_{x}(g):=\operatorname{Hol}_{x}(\nabla) \subset \operatorname{GL}\left(T_{x} M\right), \\
& \mathfrak{h o l}_{x}(g):=\mathfrak{h o l}_{x}(\nabla) \subset \mathfrak{g l}\left(T_{x} M\right)=\operatorname{End}\left(T_{x} M\right)=T_{x} M \otimes T_{x}^{*} M .
\end{aligned}
$$

A first symmetry property of Riemannian holonomy is seen using the isomorphism $g: T_{M} \rightarrow T_{M}^{*}$.

Proposition 3.3. We have

$$
\left(g_{x} \otimes \operatorname{Id}_{x}\right)\left(\mathfrak{h o l}_{x}(g) \subset \Lambda^{2} T_{x}^{*} M\right.
$$

We saw that the curvature tensor $\widetilde{R} \in\left(\mathfrak{h o l}_{x}(g) \otimes \Lambda^{2} T_{x}^{*} M\right) \cap \operatorname{Sym}^{2}\left(\Lambda^{2} T_{x}^{*} M\right)$. Hence

Theorem 3.4.

$$
\widetilde{R} \in \operatorname{Sym}^{2} \mathfrak{h o l}_{x}(g) \subset \operatorname{Sym}^{2}\left(\Lambda^{2} T_{x}^{*} M\right)
$$

### 3.4. Reducibility

The first step in the classification of Riemannian manifolds is to decompose them into their 'irreducible' factors. As we see below, these correspond to the irreducible summands in the representation of the Riemannian holonomy group on the tangent space of $M$.

Definition 3.5. A Riemannian manifold is called (locally) reducible if every point has a neighborhood isometric to a product. It is called irreducible if it is not locally reducible. We have

Proposition 3.6. Suppose a neighborhood of $x \in M$ is isometric to the product $\left(M_{1}, g_{1}\right) \times\left(M_{2}, g_{2}\right)$. Then

$$
\operatorname{Hol}_{x}\left(g_{1} \times g_{2}\right)=\operatorname{Hol}_{x}\left(g_{1}\right) \times \operatorname{Hol}_{x}\left(g_{2}\right)
$$

Theorem 3.7. If $(M, g)$ is irreducible at $x$, then $\mathbb{R}^{n}=T_{x} M$ is an irreducible representation of $\operatorname{Hol}_{x}(g)$.

### 3.5. Symmetric and locally symmetric spaces

A large and relatively well understood class of irreducible Riemannian manifolds is that of locally symmetric spaces.

Definition 3.8. A Riemannian manifold is called symmetric if, for all $p \in M$, there exists an isometry $s_{p}: M \rightarrow M$ such that $s_{p}^{2}=\operatorname{Id}_{M}$ and $p$ is an isolated fixed point for $s_{p}$.

Definition 3.9. A Riemannian manifold is called locally symmetric if every point has an open neighborhood isometric to an open subset of a symmetric space. It is called nonsymmetric if it is not locally symmetric.

Theorem 3.10. $(M, g)$ is locally symmetric if and only if $\nabla R=0$.

### 3.6. Geodesics and completeness

To better understand locally symmetric spaces, we use 'geodesics'. Geodesics allow us to define a notion of 'completeness' (often called geodesic completeness) for Riemannian manifolds. Among other things, these notions allow us to describe all symmetric spaces in terms of Lie groups.

Definition 3.11. A geodesic is a parametrized smooth curve $\gamma:(a, b) \rightarrow M$ such that, for all $t \in(a, b), \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t)=0$.

Intuitively, a geodesic is the trajectory of a particle moving with constant velocity on the manifold: the equation $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t)=0$ means that the acceleration of the particle is 0 with respect to the Levi-Civita connection.

The Riemannian metric defines a norm in the tangent space at each point of $M$. By integrating the length of the velocity vector of a parametrized (piecewise) smooth curve, we define the length of such a curve. One can show that geodesics are locally the 'shortest' curves on $M$ for the Riemannian length. It can happen however that there are many geodesics of different lengths between two given points on a manifold. The simplest example of this is the cylinder with Riemannian metric induced from $\mathbb{R}^{3}$ : Consider a vertical cylinder, obtained by revolving the vertical line $L$ at $x=1, y=0$ around the $z$-axis. Then the line $L$ and all helixes on the cylinder are geodesics. Such curves give infinitely many geodesics between any pair of distinct points on $L$.

The Riemannian distance is defined as the infimum of the lengths of the (piecewise) smooth curves between two points on $M$. We have the following useful existence and uniqueness theorem for geodesics.
Theorem 3.12. For all $p \in M, v \in T_{p} M$, there exists a unique geodesic $\gamma$ : $(a, b) \rightarrow M$ such that $\gamma(0)=p, \dot{\gamma}(0)=v$.
Definition 3.13. A manifold $(M, g)$ is (geodesically) complete if every geodesic $(a, b) \rightarrow M$ can be defined on all of $\mathbb{R} \supset(a, b)$.

All compact Riemannian manifolds and all symmetric spaces are complete. Every path connected Riemannian manifold which is also a complete metric space with respect to the Riemannian distance is geodesically complete.

We can now give a description of symmetric spaces in terms of Lie groups.
Proposition 3.14. Suppose $(M, g)$ is a connected, simply connected symmetric space. Then $(M, g)$ is complete. Put

$$
G:=\left\{s_{p} \circ s_{q} \mid p, q \in M\right\} \subset \operatorname{Isom}(M)
$$

Then $G$ is a connected Lie group. Choose $p \in M$ and let $H$ be the stabilizer subgroup of $p$ in $G$. Then $H$ is a closed connected Lie subgroup of $G$ and the map

$$
\begin{array}{clc}
G / H & \longrightarrow & M \\
g & \longmapsto & g(p)
\end{array}
$$

is a diffeomorphism.

### 3.7. De Rham's theorem

De Rham's theorem describes the decomposition of a Riemannian manifold into the product of its irreducible factors.

Theorem 3.15. Suppose $(M, g)$ is Riemannian, complete, simply connected. Then $M$ is isometric to a product $M_{0} \times M_{1} \times \ldots \times M_{k}$ where $M_{0}$ is a Euclidean space and $M_{1}, \ldots, M_{k}$ are irreducible. The decomposition is unique up to reordering $M_{1}, \ldots, M_{k}$. The holonomy group of $M$ is the product of the holonomies of $M_{1}, \ldots, M_{k}$.

### 3.8. Berger's theorem

Suppose $(M, G)$ is connected. Then, $\operatorname{Hol}(g):=\operatorname{Hol}_{x}(g)$ is independent of the choice of $x$ up to conjugation in $\mathrm{GL}_{n}(\mathbb{R})$.

Definition 3.16. The restricted holonomy group $\operatorname{Hol}(g)^{0}$ is the connected component of the identity of $\operatorname{Hol}(g)$.

Berger's theorem classifies the possibilities for the restricted holonomy group $\operatorname{Hol}(g)^{0}$ and describes the corresponding manifolds.

Theorem 3.17. Suppose $(M, g)$ is Riemannian, complete, connected, nonsymmetric, irreducible. Then the restricted holonomy group $\operatorname{Hol}(g)^{0}$ is one of the following:

1. $\operatorname{Hol}(g)^{0} \cong S O(n)$ (automorphisms of $\mathbb{R}^{n}$ preserving the metric, generic metric),
2. $n=2 m \geq 4, \operatorname{Hol}(g)^{0}=U(m) \subset S O(n)$ (automorphisms of $\mathbb{C}^{m}$ perserving a hermitian form, Kähler),
3. $n=2 m \geq 4, \operatorname{Hol}(g)^{0}=S U(m) \subset S O(n)$ (automorphisms of $\mathbb{C}^{m}$, CalabiYau, Ricci-flat, Kähler),
4. $n=4 r \geq 4, \operatorname{Hol}(g)^{0}=S p(r) \subset S O(n)$ ( $\mathbb{R}$-linear automorphisms of $\mathbb{H}^{r}$ preserving a quaternionic hermitian form, hyperkähler, Ricci-flat, Kähler), (when $r=1$, the group $S p(1)$ is abstractly isomorphic to the group $S U(2)=S^{3}$ of unit quaternions)
5. $n=4 r \geq 8, \operatorname{Hol}(g)^{0}=S p(r) S p(1) \subset S O(n)(\mathbb{R}$-linear automorphisms of $\mathbb{H}^{r}$, quaternionic-Kähler, Einstein, not Ricci-flat, not Kähler), (the group $S p(1)=S U(2)=S^{3}$ of unit length quaternions acts on $\mathbb{H}^{r}$ by right scalar multiplication and commutes with $\operatorname{Sp}(r)$, however, this action is different from the action of $S p(1)$ on $\mathbb{H}$ preserving a quaternionic hermitian form; the Lie group $S p(r) S p(1)$ generated by combining this action with that of $S p(r)$ is abstractly isomorphic to $(S p(r) \times S p(1)) /(\mathbb{Z} / 2 \mathbb{Z})$; when $r=1$, $S p(1) S p(1)=S O(4))$,
6. $n=7, \operatorname{Hol}(g)^{0}=G_{2} \subset S O(7)$ (automorphisms of $\operatorname{Im} \mathbb{O} \cong \mathbb{R}^{7}$, exceptional, Ricci-flat),
7. $n=8, \operatorname{Hol}(g)^{0}=\operatorname{Spin}(7) \subset S O(8)$ (automorphisms of $\mathbb{O} \cong \mathbb{R}^{8}$, exceptional, Ricci-flat).

## 4. Kähler manifolds

For a complex manifold $M$, multiplication by $i$ defines an endomorphism $I$ : $T_{M} \rightarrow T_{M}$ satisfying $I^{2}=-$ Id. This is called the complex structure (operator) of $M$. A metric $g$ on $M$ is called Hermitian if

$$
g(v, w)=g(I v, I w), \quad \text { for all vector fields } \quad v, w
$$

The $(1,1)$ form associated to $g$ and $I$ is

$$
\omega(v, w):=g(I v, w), \quad \text { for all vector fields } \quad v, w
$$

Equivalently, $\omega$ is the composition

$$
\omega: T_{M} \xrightarrow{I} T_{M} \xrightarrow{g} T_{M}^{*} .
$$

The fact that $\omega$ is a $(1,1)$ form means $\omega(I v, I w)=\omega(v, w)$. One also checks that $\omega$ is anti-symmetric.

It is easy to check that any two of $\{I, g, \omega\}$ determine the third.
Definition and Proposition 4.1. The metric $g$ is Kähler with respect to $I$ if one of the following equivalent conditions hold:

1. $d \omega=0$,
2. $\nabla \omega=0$,
3. $\nabla I=0$.

In such a case, $\omega$ is called the Kähler form of $g$.
So $g$ is Kähler if and only if $\omega$ and $I$ are constant. Equivalently $\operatorname{Hol}(g)$ preserves $\omega$ and $I$. The subgroup of $S O(n)$ preserving $I$ is $U(m)(n=2 m)$. Therefore, $M$ is Kähler if and only if $\operatorname{Hol}(g) \subset U(m)$.

### 4.1. Ricci form

Given a Kähler manifold $(M, g, I)$, its Ricci form $\rho$ is the differential form associated to the Ricci curvature via $I$ :

$$
\rho(v, w):=\operatorname{Ric}(I v, w), \quad \text { for all vector fields } \quad v, w .
$$

Equivalently, $\rho$ is the composition

$$
\rho: T_{M} \xrightarrow{I} T_{M} \xrightarrow{\mathrm{Ric}} T_{M}^{*} .
$$

As in the case of $\omega$ and $g$, one has $\rho \in C^{\infty}\left(\Lambda^{2} T_{M}^{*}\right)$. The Ricci form is also the curvature of the connection induced on $K_{M}:=\Omega_{M}^{m}$ by the Levi-Civita connection. We have the following
Proposition 4.2. The Ricci form $\rho$ is a closed $(1,1)$ form. Its cohomology class in $H^{2}(M, \mathbb{R})$ is $[\rho]=2 \pi c_{1}\left(K_{M}\right)=2 \pi c_{1}\left(T_{M}^{*}\right)$.

### 4.2. Ricci flatness (the Calabi-Yau case)

Since the Ricci form $\rho$ is the curvature of the connection induced on $K_{M}$ by the Levi-Civita connection, if $\rho=0$, then $K_{M}$ is a flat bundle.

Assume now that $M$ is Ricci-flat and simply connected. The flat bundle $K_{M}$ admits locally flat, i.e., covariantly constant, sections. Since $M$ is simply connected, $K_{M}$ has a global flat section. Such a section is hence invariant under Riemannian holonomy and, by the following lemma which is a consequence of Bochner's principle, holomorphic.

Lemma 4.3. Suppose $(M, I, g)$ is a compact Kähler, simply connected, Ricciflat manifold with holonomy group $H$. For all $x \in M$ and all positive integers $p$, the natural evaluation map

$$
\begin{array}{ccc}
H^{0}\left(M, \Omega_{M}^{p}\right) & \longrightarrow & \left(\Omega_{M, x}^{p}\right)^{H} \\
w & \longmapsto & w_{x}
\end{array}
$$

is an isomorphism.
Hence $K_{M}$ has a nowhere vanishing holomorphic section, which implies that $K_{M}$ is trivial, i.e., $M$ is Calabi-Yau. Furthermore, on the tangent space $T_{p} M$ at a point $p \in M$, a nonvanishing differential $m$-form is a multiple of the determinant. Hence $\operatorname{Hol}(g)$ preserves the determinant. Since we already know that $\operatorname{Hol}(g) \subset U(m)$, this implies that $\operatorname{Hol}(g) \subset S U(m)$.

Conversely, if $\operatorname{Hol}(g) \subset S U(m)$, then $M$ admits a nowhere vanishing differential $m$-form, $K_{M}$ is trivial and $\rho=0$.

### 4.3. The hyperkähler case

Recall that the quaternions have bases of the form

$$
\mathbb{H}=\mathbb{R} 1 \oplus \mathbb{R} i \oplus \mathbb{R} j \oplus \mathbb{R} k, \quad \text { with } \quad i^{2}=j^{2}=k^{2}=i j k=-1
$$

A triple $(i, j, k)$ as above is called a quaternionic triple. The Lie group $S p(r)$ is the group of $\mathbb{R}$-linear endomorphisms of $\mathbb{H}^{r}$ preserving a quaternionic Hermitian form $q$. Recall that $q$ is quaternionic Hermitian if

$$
q(a v, b w)=\bar{a} b q(v, w), \quad \text { for all } a, b \in \mathbb{H}, v, w \in \mathbb{H}^{r}
$$

where, if $a=\lambda+\mu i+\nu j+\rho k$, then $\bar{a}=\lambda-\mu i-\nu j-\rho k$. Such a $q$ can be represented by an $r \times r$ matrix $A$ with entries in $\mathbb{H}$ such that $A \bar{A}^{t}=\operatorname{Id}$ is the identity of $\mathbb{H}^{r}$.

We can embed $S p(r)$ in $S U(2 r)$ each time we choose $i \in \mathbb{H}$ with $i^{2}=-1$ as follows.

Complete $i$ to a quaternionic triple $(i, j, k)$ and write

$$
q=h+\omega j
$$

where $h$ is Hermitian with respect to $i$ and $\omega$ is alternating $\mathbb{C}$-bilinear with respect to the complex structure on $\mathbb{H}^{r}$ given by $i$. Then $S p(r)$ can be identified with the group of $\mathbb{R}$-linear automorphisms of $\mathbb{H}$ preserving $h$ and $\omega$. Hence, thinking of $U(2 r)$ as the group of transformations of $\mathbb{H}^{r}=\mathbb{C} \oplus \mathbb{C} i$ preserving $h$, we can identify $S p(r)$ as the subgroup of $U(2 r)$ of transformations preserving $\omega$. In particular, they preserve $\wedge^{r} \omega$, which means they belong to $S U(2 r)$.

Given a Riemannian manifold $M$ with $\operatorname{Hol}_{p}(g) \subset S p(r)$, we can identify $T_{p} M$ with $\mathbb{H}^{r}$. The form $\omega$ obtained as above by decomposing the form $q$ is invariant under the holonomy group of $M$, hence globalizes to an alternating flat, i.e., holomorphic, 2 -form on $M$ which is non-degenerate everywhere. Furthermore, the quaternionic triple $(i, j, k)$ gives three complex structures $I, J, K$ on $M$ satisfying the quaternionic relations and with respect to which $g$ is Kähler ( $I, J, K$ are invariant under $\operatorname{Hol}_{p}(g)$, hence flat). We then obtain a sphere of complex structures $\lambda=a I+b J+c K$ with $a, b, c \in \mathbb{R}, a^{2}+b^{2}+c^{2}=1$ such that $\nabla \lambda=0$. The metric $g$ is therefore Kähler with respect to all these complex structures.

Note that if $\operatorname{Hol}(g)=U(m)$ or $S U(m)$, then $M$ has a unique complex structure with respect to which $g$ is Kähler because the only complex endomorphisms commuting with $U(m)$ or $S U(m)$ are multiplication by scalars. So Calabi-Yaus have only one Kähler complex structure.

If $\operatorname{Hol}(g)=S p(r)$, then $M$ has exactly an $S^{2}$ of Kähler complex structures because the only quaternionic endomorphisms commuting with $S p(r)$ are multiplication by quaternionic scalars.

If $M$ is a complex torus, then $\operatorname{Hol}(g)=0$. Any complex structure is then Kähler.

Definition 4.4. We say that a simply connected Ricci-Flat manifold $M$ is irreducible hyperkähler if $\operatorname{Hol}(g)=S p(r)$, i.e., $M$ has exactly an $S^{2}$ of Kähler complex structures.

### 4.4. The Calabi conjecture and its consequence

Theorem 4.5. Calabi's conjecture, Yau's theorem:
Let $(M, I)$ be a compact complex manifold and $g$ a metric Kähler with respect to I with Kähler form $\omega$ and Ricci form $\rho$. Let $\rho^{\prime}$ be a real closed $(1,1)$ form on $M$ with cohomology class $\left[\rho^{\prime}\right]=[\rho]=2 \pi c_{1}\left(K_{M}\right)$. There exists a unique Kähler metric $g^{\prime}$ on $M$ whose Ricci form is $\rho^{\prime}$ and whose Kähler form $\omega^{\prime}$ satisfies $\left[\omega^{\prime}\right]=[\omega]$.

For Ricci-flat manifolds this has the following useful consequence.

Corollary 4.6. Suppose $(M, I, g)$ is compact Kähler with $c_{1}\left(K_{M}\right)=0$. There exists a unique Ricci-flat Kähler metric in each Kähler class on M. The Ricciflat Kähler metrics on $M$ form a smooth family of dimension $h^{1,1}(M)$, isomorphic to the Kähler cone of $M$.

### 4.5. The decomposition theorem

The following decomposition theorem for Ricci-flat manifolds, usually referred to as the Beauville-Bogomolov decomposition theorem, is a consequence of De Rham's decomposition theorem, the Berger classification theorem and results of Cheeger-Gromoll and Bochner. (see [2, Théorème 1]).

Theorem 4.7. Let $(M, I, g)$ be a compact Kähler, complete, Ricci-flat manifold. Then

1. the universal cover of $M$ is isomorphic to $\mathbb{C}^{k} \times \prod_{i} V_{i} \times \prod_{j} X_{j}$ where $\mathbb{C}^{k}$ has the standard Kähler metric, and, for all $i, V_{i}$ is compact simply connected with holonomy $S U\left(m_{i}\right)$ and, for all $j, X_{j}$ is compact simply connected with holonomy $\operatorname{Sp}\left(r_{j}\right)$,
2. there exists a finite étale cover of $M$ isomorphic to $T \times \prod_{i} V_{i} \times \prod_{j} X_{j}$ where $T$ is a complex torus of complex dimension $k$.

The proof uses
Lemma 4.8. Suppose $(M, I, g)$ is a compact Kähler, simply connected, Ricci-flat manifold. The group of automorphisms of $(M, I)$ is discrete. In particular, the group of automorphisms of $(M, I, g)$ is finite (because it is contained in $S O(n)$ which is compact).

## 5. Holomorphic symplectic manifolds

We now present the infinite series of examples of compact hyperkähler manifolds constructed by Beauville [2]. For this, the point of view of holomorphic symplectic geometry is more convenient. We begin with the following.

Proposition 5.1. Suppose $(M, I, g)$ is a compact Kähler, simply connected, Ricci-flat manifold of complex dimension $2 r$ with holonomy group $\operatorname{Sp}(r)$. Then

1. there exists a holomorphic 2 -form $\varphi$ on $M$ which is nondegenerate everywhere (represented by the form $\omega$ in the decomposition of the quaternionic Hermitian form $q=h+\omega j$ ),
2. for all $0 \leq p \leq r$,

$$
H^{0}\left(M, \Omega_{M}^{2 p+1}\right)=0, \quad H^{0}\left(M, \Omega_{M}^{2 p}\right)=\mathbb{C} \varphi^{p}
$$

Definition and Proposition 5.2. A compact Kähler manifold $X$ is called holomorphic symplectic if there exists an everywhere non-degenerate holomorphic 2-form on $X$. This is equivalent to: $X$ is compact hyperkähler or $X$ is Kähler and $\operatorname{Hol}_{g}(X) \subset S p(r)$.

A compact Kähler manifold $X$ is called irreducible holomorphic symplectic if $X$ is simply connected and $H^{0}\left(X, \Omega_{X}^{2}\right)$ is generated by an everywhere nondegenerate holomorphic 2 -form. This is equivalent to: $X$ is irreducible compact hyperkähler $X$ is Kähler and $\operatorname{Hol}_{g}(X)=S p(r)$.

### 5.1. The case of surfaces

In dimension 2, $S p(1)=S U(2)$, so Calabi-Yau and hyperkähler are the same: these are K3 surfaces and complex tori.

DEFINITION 5.3. A K3 surface is a compact complex manifold of dimension 2 such that $\Omega_{X}^{2} \cong \mathcal{O}_{X}$ and $H^{1}\left(X, \mathcal{O}_{X}\right)=0$.

One can prove that K3 surfaces are simply connected and their integral cohomology is torsion free.

It is a deep theorem of Siu that a K3 surface admits a unique Kähler metric. Examples of algebraic K3 surfaces (see, e.g., [3]):

1. Double covers of $\mathbb{P}^{2}$ branched along smooth sextics.
2. Smooth quartics in $\mathbb{P}^{3}$.

3 . $(2,3)$ complete intersections in $\mathbb{P}^{4}$.
4. $(2,2,2)$ complete intersections in $\mathbb{P}^{5}$.

### 5.2. Hilbert schemes of points

Both infinite series of examples are constructed using the Hilbert schemes of points, the first uses the Hilbert schemes of points of K3 surfaces, and the second uses the Hilbert schemes of points of complex tori of dimension 2. The construction begins by showing that these Hilbert schemes have natural holomorphic symplectic structures.

Suppose $S$ is a compact complex manifold of dimension 2. Denote $S^{r}$ the $r$-th Cartesian power of $S$ and

$$
\pi: S^{r} \rightarrow S^{(r)}:=S^{r} / \mathfrak{S}_{r}
$$

its quotient by the action of $\mathfrak{S}_{r}$ permuting the factors. Let $\Delta_{i j} \subset S^{r}$ be the diagonal where the $i$-th and $j$-th components are equal. The action of $\mathfrak{S}_{r}$ is not free on the diagonals $\Delta_{i j}$. The stabilizer of a generic point of $\Delta_{i j}$ is the
subgroup $\{1,(i j)\} \subset \mathfrak{S}_{r}$ where $(i j)$ is the transposition exchanging $i$ and $j$. The quotient morphism $\pi$ is étale away from $\cup_{i, j} \Delta_{i j}$. For any $i, j$, the diagonal $\Delta_{i j}$ has codimension 2 in $S^{r}$. Hence, by the theorem on the purity of the ramification locus of a morphism of smooth varieties, the symmetric power $S^{(r)}$ is singular along the diagonal $D:=\pi\left(\Delta_{i j}\right)=\pi\left(\cup_{i, j} \Delta_{i j}\right)$. Note that $D$ is irreducible.

The symmetric power $S^{(r)}$ has a natural desingularization: the Hilbert scheme $S^{[r]}$ of length $r$ Artinian subschemes of $S$. The natural map $\epsilon: S^{[r]} \rightarrow$ $S^{(r)}$ sends a subscheme $Z$ of length $r$ to its underlying 0 -cycle. Since, for any $r$ distinct points $x_{1}, \ldots, x_{r} \in S$, there exists a unique Artinian subscheme supported on $\left\{x_{1}, \ldots, x_{r}\right\}$, the map $\epsilon: S^{[r]} \backslash \epsilon^{-1}(D) \rightarrow S^{(r)} \backslash D$ is an isomorphism.

Let $D_{*} \subset D$ be the open subset where exactly two points of the K3 surface are equal. Given a fixed $2 x_{1}+x_{2}+\ldots+x_{r-1} \in D_{*}$, the datum of an Artinian subscheme of length $r$ supported on $2 x_{1}+x_{2}+\ldots+x_{r-1}$ is equivalent to the datum of a tangent line to $S$ at $x_{1}$. So the set of Artinian subschemes of length $r$ supported on $2 x_{1}+x_{2}+\ldots+x_{r-1}$ is naturally identified with $\mathbb{P} T_{x_{1}} S$.

Let $S_{*}^{(r)}$, respectively $S_{*}^{r}$, be the open subset where at most two of the coordinates coincide and let $S_{*}^{[r]}$ be the inverse image of $S_{*}^{(r)}$ in $S^{[r]}$. The fiber of $\epsilon: S_{*}^{[r]} \rightarrow S_{*}^{(r)}$ at $x=2 x_{1}+x_{2}+\ldots+x_{r-1} \in D_{*}$ is naturally identified with $\mathbb{P} T_{x_{1}} S$. One can prove:

THEOREM 5.4. 1. The complex analytic pair $\left(S_{*}^{(r)}, D_{*}\right)$ is locally isomorphic to $(B \times C, B \times\{O\})$, where $B$ is a ball, $C$ is a cone with vertex $O$ over a smooth conic in $\mathbb{P}^{2}$.
2. The complex manifold $S_{*}^{[r]}$ is the blow up of $S_{*}^{(r)}$ along $D_{*}$.
3. If we denote $B l_{\Delta}\left(S_{*}^{r}\right)$ the blow up of $S_{*}^{r}$ along the union of its diagonals, then the action of $\mathfrak{S}_{r}$ lifts to $B l_{\Delta}\left(S_{*}^{r}\right)$ and

$$
S_{*}^{[r]}=B l_{\Delta}\left(S_{*}^{r}\right) / \mathfrak{S}_{r}
$$

So we have the Cartesian diagram


Note that when $r=2$, we have $D_{*}=D, S_{*}^{2}=S^{2}, S_{*}^{(2)}=S^{(2)}, S_{*}^{[2]}=S^{[2]}$, and $S^{[2]}$ is the blow up of $S^{(2)}$ along the diagonal.

Next we construct differential forms on $S^{[r]}$, starting from differential forms on $S$.

Given a holomorphic differential form $\omega$ on $S$, the form $\psi:=p r_{1}^{*} \omega+\ldots+$ $\operatorname{pr}_{r}^{*} \omega$ and its pull-back $\eta^{*} \psi$ to $B l_{\Delta}\left(S_{*}^{r}\right)$ are invariant under the action of $\mathfrak{S}_{r}$. Hence there exists a holomorphic differential form $\varphi$ on $S_{*}^{[r]}$ such that

$$
\eta^{*} \psi=\rho^{*} \varphi .
$$

Proposition 5.5. If $K_{S}$ is trivial, then $S^{[r]}$ admits a holomorphic symplectic form.

Proof. Let $\omega$ be a generator of $K_{S}$. Defining $\psi$ and $\varphi$ as above, we show that $\varphi$ extends to $S^{[r]}$ as an everywhere non-degenerate form.

The form $\varphi$ extends to all of $S^{[r]}$ because $S^{[r]} \backslash S_{*}^{[r]}$ has codimension $\geq 2$ in $S^{[r]}$. The fact that $\varphi$ is everywhere non-degenerate means that $\wedge^{r} \varphi$ does not vanish anywhere.

The form $\wedge^{r} \varphi$ is a section of $K_{S^{[r]}}$, so the locus where it vanishes is a canonical divisor on $S^{[r]}$.

Denote $E_{i j}:=\eta^{*} \Delta_{i j}$. Then the divisors $E_{i j}$ are the exceptional divisors of the blow up $\eta: B l_{\Delta}\left(S_{*}^{r}\right) \rightarrow S_{*}^{r}$ and the ramification divisors of the morphism $\rho: B l_{\Delta}\left(S_{*}^{r}\right) \rightarrow S_{*}^{[r]}$. Hence

$$
K_{B l_{\Delta}\left(S_{*}^{r}\right)}=\rho^{*} K_{S_{*}^{[r]}}+\sum_{i<j} E_{i j},
$$

and the divisor of zeros of $\rho^{*} \wedge^{r} \varphi$ is

$$
\operatorname{Div}\left(\rho^{*} \wedge^{r} \varphi\right)=\rho^{*} \operatorname{Div}\left(\wedge^{r} \varphi\right)+\sum_{i<j} E_{i j}
$$

However,

$$
\operatorname{Div}\left(\rho^{*} \wedge^{r} \varphi\right)=\operatorname{Div}\left(\eta^{*} \wedge^{r} \psi\right)=\operatorname{Div}\left(\wedge^{r} \eta^{*} \psi\right)=\sum_{i<j} E_{i j} .
$$

Indeed, choose $z=\left(x_{1}, \ldots, x_{r}\right) \in S^{r}$, then

$$
T_{z} S^{r}=T_{x_{1}} S \oplus \ldots \oplus T_{x_{r}} S
$$

The differential form $\psi$ is a bilinear form on $T_{z} S^{r}$, the decomposition $T_{z} S^{r}=$ $T_{x_{1}} S \oplus \ldots \oplus T_{x_{r}} S$ is orthogonal with respect to $\psi$ and $\psi$ is non-degenerate at any $z$. Hence $\operatorname{Div}\left(\wedge^{r} \psi\right)=0$ on $S^{r}$. However, the differential of the blow up $\eta: B l_{\Delta}\left(S_{*}^{r}\right) \rightarrow S_{*}^{r}$ has image of dimension $2 r-1$ along the union of the diagonals, so $\eta^{*} \psi$ is degenerate of rank $2 r-2$ along $\cup_{i<j} E_{i j}$. It follows that $\operatorname{Div}\left(\wedge^{r} \eta^{*} \psi\right)=\sum_{i<j} E_{i j}$.

So $\rho^{*} \operatorname{Div}\left(\wedge^{r} \varphi\right)=0$ and $\operatorname{Div}\left(\wedge^{r} \varphi\right)=0$.

To determine the type of $S^{[r]}$, we compute its fundamental group. The map $S^{r} \rightarrow S^{(r)}$ is a Galois cover with Galois group $\mathfrak{S}_{r}$. So we have the exact sequence of fundamental groups

$$
1 \longrightarrow \mathfrak{S}_{r} \longrightarrow \pi_{1}\left(S^{r}\right) \longrightarrow \pi_{1}\left(S^{(r)}\right) \longrightarrow 1
$$

We have

$$
\pi_{1}\left(S^{r}\right)=\pi_{1}\left(S_{*}^{r}\right)=\pi_{1}\left(B l_{\Delta}\left(S_{*}^{r}\right)\right), \quad \pi_{1}\left(S^{(r)}\right)=\pi_{1}\left(S_{*}^{(r)}\right), \quad \pi_{1}\left(S^{[r]}\right)=\pi_{1}\left(S_{*}^{[r]}\right)
$$

The map $B l_{\Delta}\left(S_{*}^{r}\right) \rightarrow S_{*}^{[r]}$ is also a Galois cover with Galois group $\mathfrak{S}_{r}$. So we have the commutative diagram of exact sequences


Therefore, we also have $\pi_{1}\left(S_{*}^{[r]}\right) \xlongequal{\cong} \pi_{1}\left(S^{(r)}\right)$.
It is a fact from algebraic topology and group theory that $\pi_{1}\left(S^{(r)}\right)$ is the largest commutative quotient of $\pi_{1}(S)$, hence it is isomorphic to $H_{1}(S, \mathbb{Z})$.

Lemma 5.6. 1. $H^{i}\left(S^{(r)}, \mathbb{Q}\right)=H^{i}\left(S^{r}, \mathbb{Q}\right)^{\mathfrak{G}_{r}}$,
2. $H^{2}\left(S^{[r]}, \mathbb{Q}\right)=H^{2}\left(S^{(r)}, \mathbb{Q}\right) \oplus \mathbb{Q}[E]$,
3. $H^{2}\left(S^{(r)}, \mathbb{Q}\right)=H^{2}(S, \mathbb{Q}) \oplus \Lambda^{2} H^{1}(S, \mathbb{Q})$.

Proof. 1. Standard.
2. Replace $S^{r}$ by $S_{*}^{r}, S^{(r)}$ by $S_{*}^{(r)}$ and $S^{[r]}$ by $S_{*}^{[r]}$ : the second cohomology does not change. We compute

$$
\begin{aligned}
& H^{2}\left(S_{*}^{[r]}, \mathbb{Q}\right)=H^{2}\left(B l_{\Delta}\left(S_{*}^{r}\right), \mathbb{Q}\right)^{\mathfrak{S}_{r}} \\
& \quad=\left(H^{2}\left(S_{*}^{r}, \mathbb{Q}\right) \oplus\left(\oplus_{1 \leq i<j \leq r} \mathbb{Q}\left[E_{i j}\right]\right)\right)^{\mathfrak{G}_{r}}=H^{2}\left(S_{*}^{r}, \mathbb{Q}\right)^{\mathfrak{G}_{r}} \oplus \mathbb{Q}\left[\rho^{*} E\right]
\end{aligned}
$$

3. We compute, using part (1),

$$
\begin{aligned}
H^{2}\left(S^{(r)}, \mathbb{Q}\right) & =H^{2}\left(S^{r}, \mathbb{Q}\right)^{\mathfrak{G}_{r}} \cong\left(H^{2}(S, \mathbb{Q})^{\oplus r} \oplus\left(H^{1}(S, \mathbb{Q})^{\otimes 2}\right)^{\oplus\binom{r}{2}}\right)^{\mathfrak{G}_{r}} \\
& =H^{2}(S, \mathbb{Q}) \oplus\left(H^{1}(S, \mathbb{Q})^{\otimes 2}\right)^{\tau} \cong H^{2}(S, \mathbb{Q}) \oplus \Lambda^{2} H^{1}(S, \mathbb{Q})
\end{aligned}
$$

where, by skew-symmetry, $\tau$ sends a tensor $v \otimes w$ to $-w \otimes v$.

We immediately obtain.
Corollary 5.7. If $S$ is a K3 surface, then $S^{[r]}$ is an irreducible holomorphic symplectic manifold and

$$
H^{2}\left(S^{[r]}, \mathbb{Q}\right)=H^{2}(S, \mathbb{Q}) \oplus \mathbb{Q}[E]
$$

$S^{[r]}$ is Kähler by results of Varouchas.

### 5.3. Generalized Kummers

Now take $S=A$ a complex torus of dimension 2. Then $A^{[r+1]}$ is a holomorphic symplectic manifold. As in the case of K3 surfaces, it is Kähler. By the previous results,

$$
\begin{aligned}
& \pi_{1}\left(A^{[r+1]}\right)=H_{1}(A, \mathbb{Z})=\pi_{1}(A) \neq\{1\} \\
& H^{2}\left(A^{[r+1]}, \mathbb{Q}\right)=H^{2}(A, \mathbb{Q}) \oplus \Lambda^{2} H^{1}(A, \mathbb{Q}) \oplus \mathbb{Q}[E]
\end{aligned}
$$

So in this case, the Hilbert scheme is not irreducible holomorphic symplectic. We determine its factors according to the decomposition theorem.

Consider the addition map $s: A^{(r+1)} \rightarrow A$ and its composition

$$
\zeta: A^{[r+1]} \xrightarrow{\rho} A^{(r+1)} \xrightarrow{s} A .
$$

Definition 5.8. The $(r+1)$-st generalized Kummer manifold of $A$ is

$$
K_{r}:=\zeta^{-1}(0)
$$

One can see that $K_{r}$ is a manifold as follows.
The complex torus $A$ acts on itself by translation, hence also on $A^{[r+1]}$ by pull-back:

If $Z \subset A$ is an analytic subspace of length $r+1$, then $a \in A$ acts as $Z \mapsto t_{a}^{*} Z$ on $A^{[r+1]}$. The map $\zeta$ is equivariant for this action on $A^{[r+1]}$ and the action of $A$ on itself via $x \mapsto t_{(r+1) a}^{*} x$. In other words we have the Cartesian diagram

which induces the Cartesian diagram


It follows that $\zeta$ is a smooth map and all its fibers are isomorphic to $K_{r}$ which is therefore also smooth.

Proposition 5.9. The holomorphic symplectic structure of $A^{[r+1]}$ restricts to a holomorphic symplectic structure on $K_{r}$.

Proof. Since $K_{r}$ is a fiber of a smooth morphism, its normal bundle is trivial: the normal space at every point of $K_{r}$ maps isomorphically onto $T_{0} A$, so that we have $N_{K_{r} \mid A^{[r+1]}} \cong T_{0} A \otimes \mathcal{O}_{K_{r}}$. From the normal bundle sequence

$$
\left.0 \longrightarrow T_{K_{r}} \longrightarrow T_{A^{[r+1]}}\right|_{K_{r}} \longrightarrow N_{K_{r} \mid A^{[r+1]}} \longrightarrow 0
$$

we obtain $\left.K_{K_{r}} \cong K_{A^{[r+1]}}\right|_{K_{r}} \cong \mathcal{O}_{K_{r}}$.
Recall the differential forms $\psi=p r_{1}^{*} \omega \oplus \ldots \oplus p r_{r+1}^{*} \omega$ and $\varphi$ with $\eta^{*} \psi=\rho^{*} \varphi$. The form $\wedge^{r}\left(\left.\varphi\right|_{K_{r}}\right)$ is a section of $K_{K_{r}} \cong \mathcal{O}_{K_{r}}$. We show that it remains everywhere non-degenerate. As before, this means that $\wedge^{r}\left(\left.\varphi\right|_{K_{r}}\right)$ does not vanish anywhere. Since $K_{K_{r}}$ is trivial, either $\wedge^{r}\left(\left.\varphi\right|_{K_{r}}\right)$ is zero everywhere or it does not vanish anywhere. We prove that it is nonzero at one point.

Let $Z=x_{1}+\ldots+x_{r+1} \in K_{r}$ be such that the $x_{i}$ are all distinct. Then
$\left.T_{Z} A^{[r+1]} \cong T_{( } x_{1}, \ldots, x_{r+1}\right) A^{r+1} \cong T_{x_{1}} A \oplus \ldots \oplus T_{x_{r+1}} A \cong\left(T_{0} A\right)^{\oplus(r+1)}$.
We can choose the isomorphism above in such a way that the differential $d \zeta$ : $T_{Z} A^{[r+1]} \rightarrow T_{0} A$ of $\zeta$ is the sum map. The form $\varphi$ acts as $\omega$ on each summand $T_{0} A$ of $T_{Z} A^{[r+1]}$ and the summands are orthogonal to each for $\varphi$. It is then an exercise in linear algebra to check that $\left.\varphi\right|_{\operatorname{Ker~d\zeta }}$ is non-degenerate, i.e., $\wedge^{r}\left(\left.\varphi\right|_{K_{r}}\right)$ is not 0 .

Proposition 5.10. The manifold $K_{r}$ is simply connected. For $r \geq 2$, we have

$$
H^{2}\left(K_{r}, \mathbb{Q}\right) \cong H^{2}(A, \mathbb{Q}) \oplus \mathbb{Q}[E]
$$

where $E$ is the intersection of the exceptional divisor of $A^{[r+1]}$ with $K_{r}$.
Proof. Immediate from the definition of $K_{r}$ and the description of the cohomology and fundamental group of $A^{[r+1]}$.

It now follows that the factors of $A^{[r+1]}$ in the decomposition theorem are $K_{r}$ and $A$ itself.

Note that $S^{[r]}$ (for K3 surfaces $S$ ) and $K_{r}$ have different Betti numbers, hence are not deformation equivalent. These provide two infinite series of families of hyperkähler manifolds.

There are two known examples of families of hyperkähler manifolds due to O'Grady that are not deformation equivalent to Hilbert schemes of K3s or generalized Kummers: these are hyperkählers of dimensions 6 and 10.

Question 5.11. Are there other families of compact irreducible hyperkählers?

## 6. Moduli of hyperkählers, the Beauville-Bogomolov form, the period domain and the period map

### 6.1. Moduli of complex structures and Teichmüller space

Given a differentiable manifold $X$, there can be many different complex structures on $X$. We define the Teichmüller space of $X$ as

$$
\operatorname{Teich}(X):=\{\text { complex structures on } X\} / \sim^{0}
$$

where two complex structures $I, J$ on $X$ satisfy $I \sim^{0} J$ if there exists a diffeomorphism $\varphi: X \rightarrow X$ isotopic (or homotopic) to the identity $\mathrm{Id}_{X}$ such that $\varphi^{*} I=J$. The moduli space of complex structures on $X$ is, by definition,

$$
\mathcal{M}_{c x}(X):=\{\text { complex structures on } X\} / \sim
$$

where two complex structures $I, J$ on $X$ satisfy $I \sim J$ if there exists a diffeomorphism $\varphi: X \rightarrow X$ such that $\varphi^{*} I=J$. If we denote $\operatorname{Diff}(X)$ the group of diffeomorphisms of $X$ and $\operatorname{Diff}^{0}(X)$ its connected component of the identity, then $G:=\operatorname{Diff}(X) / \operatorname{Diff}^{0}(X)$ is the discrete group of components of $\operatorname{Diff}(X)$, and

$$
\mathcal{M}_{c x}(X)=\operatorname{Teich}(X) / G
$$

A priori, $\mathcal{M}_{c x}(X)$ is the space that we are interested in. However, it usually does not have many good properties while $\operatorname{Teich}(X)$ does. So we will, most of the time, work with small open sets of $\operatorname{Teich}(X)$ which describe small deformations of given complex structures.

### 6.2. Universal families and Kuranishi's theorem

Suppose given a complex manifold $(X, I)$.
Definition 6.1. A family of complex manifolds is a smooth proper morphism of complex spaces

$$
\pi: \mathcal{X} \rightarrow S
$$

A deformation of $(X, I)$ is a family of complex manifolds with a point $s_{0} \in S$ and an isomorphism $\mathcal{X}_{0}:=\pi^{-1}\left(s_{0}\right) \cong X$.

A deformation is called universal if, for any deformation $\mathcal{X}^{\prime} \rightarrow S^{\prime}$, there exists a unique morphism $\varphi: S^{\prime} \rightarrow S$ such that $\varphi\left(s_{0}^{\prime}\right)=s_{0}$ and $\mathcal{X}^{\prime} \rightarrow S^{\prime}$ is the pull-back of $\mathcal{X} \rightarrow S$ under $\varphi$. In other words, we have the Cartesian diagram


It immediately follows from its definition that the universal deformation is unique up to unique isomorphism and we denote it

$$
\mathcal{X} \rightarrow \operatorname{Def}(X)
$$

Kuranishi's theorem is the following.
Theorem 6.2. Suppose $(X, I)$ is a compact complex manifold with $H^{0}\left(X, T_{X}\right)=$ 0 . Then a local universal deformation of $(X, I)$ exists and it is universal for all of its fibers.

Under the conditions of the theorem, the local universal deformation $\mathcal{X} \rightarrow$ $\operatorname{Def}(X)$ is sometimes called the Kuranishi family.

Note that the condition $H^{0}\left(X, T_{X}\right)=0$ means that there are no global holomorphic vector fields on $X$ or $X$ has no infinitesimal automorphisms: given two complex manifolds $X, Y$ and a holomorphic map $f: X \rightarrow Y$, the tangent space to the space of holomorphic maps $\operatorname{Hom}(X, Y)$ at $f$ can be identified with $H^{0}\left(X, f^{*} T_{Y}\right)$. This can be deduced from general results in deformation theory, applied to the deformations of the graph of $f$ in $X \times Y$.

### 6.3. Unobstructedness for $K$-trivial Kähler manifolds

For any compact complex manifold $X$, if $H^{0}\left(X, T_{X}\right)=0$, then $X$ has a local or small universal deformation denoted $\mathcal{X} \rightarrow \operatorname{Def}(X)$. By this we mean a germ of a deformation, i.e., whose base is suitably small. Such a deformation is universal for all its fibers, its base $\operatorname{Def}(X)$ is a "Kuranishi slice" $\subset H^{1}\left(X, T_{X}\right)$. For $t \in \operatorname{Def}(X)$ small, we have

$$
T_{t} \operatorname{Def}(X)=H^{1}\left(X_{t}, T_{X_{t}}\right)
$$

The obstructions to deformations (to various orders) provide local analytic equations for $\operatorname{Def}(X)$ in a neighborhood of $0 \in H^{1}\left(X, T_{X}\right)$. We say that the deformations of $X$ are unobstructed if all the obstructions to deformations are 0 . If the deformations of $X$ are unobstructed (i.e., $\operatorname{dim} T_{0} \operatorname{Def}(X)=\operatorname{dim} \operatorname{Def}(X)$ ), then the base $\operatorname{Def}(X)$ is a small open neighborhood of the origin in $H^{1}\left(X, T_{X}\right)$. The following theorem is due to Bogomolov in the hyperkähler case and to Tian-Todorov in the general case.

Theorem 6.3. If the canonical bundle $K_{X}$ is trivial (we say $X$ is $K$-trivial), then the deformations of $X$ are unobstructed.

We have the following facts.

- If $X$ is Kähler, then so is any small deformation of $X$.
- If $X$ is Kähler and $K$-trivial, then small deformations $X_{t}$ of $X$ are also Kähler and $K$-trivial and $h^{1}\left(T_{X_{t}}\right)$ is constant.
- If $X$ is holomorphic symplectic, then small deformations of $X$ are also holomorphic symplectic. If $X$ is irreducible holomorphic symplectic, then all fibers of any deformation of $X$ are irreducible holomorphic symplectic.


### 6.4. The Beauville-Bogomolov form

The key to understanding the deformations of hyperkähler manifolds is the period domain. Small open subsets of the period domain are isomorphic to $\operatorname{Def}(X)$. We define the period domain using the second cohomology of hyperkähler manifolds, together with a non-degenerate quadratic form: the Beauville-Bogomolov form.

Suppose $X$ is irreducible holomorphic symplectic (irreducible hyperkähler) of dimension $2 n$ and choose $\sigma \in H^{0}\left(\Omega_{X}^{2}\right)$ such that

$$
\int_{X}(\sigma \bar{\sigma})^{n}=1
$$

For $\alpha \in H^{2}(X, \mathbb{C})$, define

$$
q_{X}(\alpha):=\frac{n}{2} \int_{X} \alpha^{2}(\sigma \bar{\sigma})^{n-1}+(1-n) \int_{X} \sigma^{n-1} \bar{\sigma}^{n} \alpha \int_{X} \sigma^{n} \bar{\sigma}^{n-1} \bar{\alpha} .
$$

One can show this is equal to

$$
q_{X}(\alpha)=\lambda \mu+\frac{n}{2} \int_{X} \beta^{2}(\sigma \bar{\sigma})^{n-1}
$$

where $\alpha=\lambda \sigma+\beta+\mu \bar{\sigma}$ with $\beta \in H^{1,1}(X)$.
Beauville showed that there exists $d_{X} \in \mathbb{N}$ such that

$$
\int_{X} \alpha^{2 n}=d_{X}\left(q_{X}(\alpha)\right)^{n}
$$

In fact $d_{X}=\binom{2 n}{n}$ by $[22,23.4]$. Therefore, if $r_{X}$ is the positive real root of $d_{X}$, then $\widetilde{q}_{X}:=r_{X} q_{X}$ is an $n$-th root of the $n$-th power cup-product on $H^{2}(X, \mathbb{C})$.

Beauville [2] and Fujiki [20] proved that the quadratic form $\widetilde{q}_{X}$ is nondegenerate of signature $\left(3, b_{2}-3\right)$ on $H^{2}(X, \mathbb{R})$. Furthermore,

$$
\widetilde{q}_{X}(\sigma)=0, \quad \widetilde{q}_{X}(\sigma+\bar{\sigma})>0
$$

and

$$
\widetilde{q}_{X}\left(\sigma_{t}\right)=0, \quad \widetilde{q}_{X}\left(\sigma_{t}+\bar{\sigma}_{t}\right)>0
$$

for $t$ close to 0 in any deformation of $X$.
The form $\widetilde{q}$ is called the Beauville-Bogomolov form of the hyperkähler manifold. The inspiration for the Beauville-Bogomolov form came from the study
of the Fano variety of lines of a cubic fourfold. There, it naturally appears as the intersection form on the fourth cohomology of the cubic threefold which is isomorphic to the second cohomology of its Fano variety of lines which is a hyperkähler manifold.

Note that for $n=1, \widetilde{q}_{X}=2 q_{X}$ is the usual intersection form on $H^{2}(X, \mathbb{Z})$.
Beauville and Fujiki, loc. cit., also proved that $\widetilde{q}_{X}$ has a positive real multiple, say $q_{X}^{\prime}$, which is integer valued and indivisible on $H^{2}(X, \mathbb{Z})$. The Fujiki constant is the positive rational number $c_{X}$ such that $\int_{X} \alpha^{2 n}=c_{X} \frac{(2 n)!}{2^{n} n!}\left(q_{X}^{\prime}(\alpha)\right)^{n}$ for all $\alpha \in H^{2}(X, \mathbb{R})$. The reason for the factor $\frac{(2 n)!}{2^{n} n!}$ is that, in all known examples of compact hyperkählers, the Fujiki constant as defined is in fact an integer, see, e.g., [40] and [8].

### 6.5. The local period domain and the local Torelli theorem

Define

$$
Q_{X}:=\left\{\alpha \mid q_{X}(\alpha)=0, q_{X}(\alpha+\bar{\alpha})>0\right\} \subset \bar{Q}_{X} \subset \mathbb{P} H^{2}(X, \mathbb{C})
$$

We saw that for $t \in \operatorname{Def}(X)$ close to $0, q_{X}\left(\sigma_{t}\right)=0, q_{X}\left(\sigma_{t}+\bar{\sigma}_{t}\right)>0$. Hence we can define the local period map

$$
\begin{aligned}
P_{X}: \operatorname{Def}(X) & \longrightarrow Q_{X} \\
t & \longmapsto\left[\sigma_{t}\right] .
\end{aligned}
$$

This is holomorphic because $\left\langle\sigma_{t}\right\rangle=H^{2,0}\left(X_{t}\right)=H^{0}\left(\Omega_{X_{t}}^{2}\right)$ varies holomorphically with $t: H^{0}\left(\Omega_{X_{t}}^{2}\right)$ is the fiber of the holomorphic line bundle $\pi_{*} \Omega_{\mathcal{X} / \operatorname{Def}(X)}^{2}$ on $\operatorname{Def}(X)$.

We have the local Torelli theorem [2]:
Theorem 6.4. The local Torelli map $P_{X}$ is a local isomorphism, i.e., $d P_{X}$ is an isomorphism at 0 .

### 6.6. The period domain

We now construct the global period domain for hyperkähler manifolds. For this we first fix the discrete data of a lattice which will usually be abstractly isomorphic to the second integral cohomology of a hyperkähler manifold with its Beauville-Bogomolov form.

Definition 6.5. A lattice is the data of a free $\mathbb{Z}$-module $\Gamma$ of finite rank with an integral non-degenerate quadratic form $q_{\Gamma}$.
Definition 6.6. Given a lattice $\left(\Gamma, q_{\Gamma}\right)$, the period domain $Q_{\Gamma}$ is

$$
Q_{\Gamma}:=\left\{\alpha \mid q_{X}(\alpha)=0, q_{X}(\alpha+\bar{\alpha})>0\right\} \subset \bar{Q}_{\Gamma} \subset \mathbb{P}\left(\Gamma \otimes_{\mathbb{Z}} \mathbb{C}\right)
$$

### 6.7. The moduli space of marked holomorphic symplectic manifolds and local period maps

We will construct a moduli space of marked holomorphic symplectic manifolds and a global period map on it which is, roughly speaking, a glueing of local period maps.

Definition 6.7. 1. A marking of an irreducible holomorphic symplectic manifold is a lattice isomorphism

$$
\varphi:\left(H^{2}(X, \mathbb{Z}), \widetilde{q}_{X}\right) \xrightarrow{\cong}\left(\Gamma, q_{\Gamma}\right)
$$

2. The pair $(X, \varphi)$ is called a marked manifold.
3. Two marked manifolds $(X, \varphi),\left(X^{\prime}, \varphi^{\prime}\right)$ are isomorphic if there exists $f$ : $X \rightarrow X^{\prime}$ such that $\varphi^{\prime}=\varphi \circ f^{*}$. We write $(X, \varphi) \cong\left(X^{\prime}, \varphi^{\prime}\right)$.
4. The moduli space of marked irreducible holomorphic symplectic manifolds is the set

$$
\mathcal{M}_{\Gamma}:=\{(X, \varphi)\} / \cong .
$$

We use the local period map to show that $\mathcal{M}_{\Gamma}$ is a smooth (non-Hausdorff) complex analytic space.

Given an irreducible holomorphic manifold $X$, we choose a marking $\varphi: H^{2}(X, \mathbb{Z}) \rightarrow \Gamma$. The Kuranishi family $\mathcal{X} \rightarrow \operatorname{Def}(X)$ is locally isomorphic to the period domain $Q_{\Gamma}$, and the marking $\varphi: H^{2}(X, \mathbb{Z}) \rightarrow \Gamma$ induces isomorphisms forming the commutative diagram


Hence an open ball in the Kuranishi space $\operatorname{Def}(X)$ is isomorphic to an open ball in $Q_{\Gamma}$. Such open balls cover $\mathcal{M}_{\Gamma}$ and the analytic structures on intersections coincide because the Kuranishi family is the local universal deformation of all of its fibers. Hence we obtain a well-defined smooth complex analytic structure on $\mathcal{M}_{\Gamma}$.

### 6.8. The global period map and Verbitsky's global Torelli theorem

Definition 6.8. The global period map is

$$
\begin{array}{cccc}
P: & \mathcal{M}_{\Gamma} & \longrightarrow & Q_{\Gamma} \subset \bar{Q}_{\Gamma} \subset \mathbb{P}\left(\Gamma \otimes_{\mathbb{Z}} \mathbb{C}\right) \\
(X, \varphi) & \longmapsto & {[\varphi(\sigma)] .}
\end{array}
$$

Verbitsky's global Torelli theorem [44] (also see [25] and [32]) for compact hyperkähler manifolds is the following.

Theorem 6.9. The map $P$ is generically injective on each connected component of $\mathcal{M}_{\Gamma}$.

Note that the datum of the line $H^{2,0}(X) \subset H^{2}(X, \mathbb{C})$ determines the Hodge structure on $H^{2}(X, \mathbb{Z}): H^{0,2}(X)=\overline{H^{2,0}(X)}$ (complex conjugate), $H^{2,0}(X)^{\perp}=$ $H^{2,0}(X) \oplus H^{1,1}(X), H^{1,1}(X)=\left(H^{2,0}(X) \oplus H^{1,1}(X)\right) \cap \overline{\left(H^{2,0}(X) \oplus H^{1,1}(X)\right)}$.

We say that the global Torelli theorem holds for a class of manifolds, if a manifold is determined by its Hodge structure, possibly together with the data of a polarization (such as the form $\widetilde{q}_{X}$ in the hyperkähler case). For instance, two complex tori are isomorphic if and only if their first cohomologies are isomorphic as Hodge structures. Two Riemann surfaces are isomorphic if and only if their first cohomologies are Hodge isometric, i.e., they are isomorphic as Hodge structures and, under the given Hodge isomorphism, the intersection forms for the two curves coincide. Similarly, two K3 surfaces are isomorphic if their second cohomologies are Hodge isometric.

In fact we have stronger Torelli theorems in the above cases: for complex tori, any Hodge isomorphism between the first cohomologies of two tori is induced by an isomorphism of the tori. For curves, any Hodge isometry between their first cohomologies is induced by an isomorphism between the curves up to a change of sign. For generic K3 surfaces, any Hodge isometry between the second cohomologies is induced by an isomorphism of the surfaces up to a sign.

For hyperkähler manifolds of dimension $>4$, none of the above stronger versions of Torelli hold. There are examples of

1. non-isomorphic (but bimeromorphic) compact hyperkähler manifolds with Hodge isometric second cohomologies [16],
2. non-birational projective hyperkähler manifolds of dimension 4 with Hodge isometric second cohomologies, [36].

QUESTION 6.10. Is there a good characterization of irreducible holomorphic symplectic manifolds that are Hodge isometric but not isomorphic?

We have the following maps of moduli spaces

and the period map
Teich $(X) \xrightarrow{\text { local isom. }} \mathcal{M}_{\Gamma}(X) \xrightarrow{P_{\Gamma}} Q_{\Gamma} \subset \bar{Q}_{\Gamma} \subset \mathbb{P}(\Gamma \otimes \mathbb{C})$.
The spaces Teich $(X)$ and $\mathcal{M}_{\Gamma}(X)$ are non Hausdorff smooth analytic spaces and $Q_{\Gamma}$ is a (Hausdorff) simply connected complex manifold. Verbitsky constructed a new (Hausdorff) complex manifold $\mathcal{M}_{\Gamma}^{s}(X)$ which is obtained by identifying all non-separated points of $\mathcal{M}_{\Gamma}(X)$. In other words

$$
\mathcal{M}_{\Gamma}^{s}(X)=\mathcal{M}_{\Gamma}(X) / \equiv
$$

where, for two points $p, q \in \mathcal{M}_{\Gamma}(X), p \equiv q$ when every neighborhood of $p$ contains $q$ and every neighborhood of $q$ contains $p$. The period map then factors through $\mathcal{M}_{\Gamma}^{s}(X)$ :

$$
P_{\Gamma}: \quad \mathcal{M}_{\Gamma}(X) \xrightarrow{\text { local isom. }} \mathcal{M}_{\Gamma}^{s}(X) \xrightarrow{P_{\Gamma}^{s}} Q_{\Gamma}
$$

Verbitsky proved
THEOREM 6.11. The map $P_{\Gamma}^{s}$ is surjective from any connected component of $\mathcal{M}_{\Gamma}^{s}(X)$ to $Q_{\Gamma}$.

Combined with the facts that $P_{\Gamma}^{s}$ is a local isomorphism and $Q_{\Gamma}$ is simply connected, this implies

Corollary 6.12. The map $P_{\Gamma}^{s}$ induces an isomorphism from any connected component of $\mathcal{M}_{\Gamma}^{s}(X)$ to $Q_{\Gamma}$.

Verbitsky's proof uses twistor conics which we will describe in the next section.

The following results of Huybrechts help us understand the difference between $\mathcal{M}_{\Gamma}(X)$ and $\mathcal{M}_{\Gamma}^{s}(X)$.

Proposition 6.13. If two marked hyperkähler manifolds $(X, \varphi)$ and $\left(X^{\prime}, \varphi^{\prime}\right)$ correspond to two non-separated points of $\mathcal{M}_{\Gamma}(X)$, then $X$ and $Y$ are bimeromorphic and their period $P_{\Gamma}(X, \varphi)=P_{\Gamma}\left(X^{\prime}, \varphi\right)$ is contained in the hyperplane $Q_{\Gamma} \cap \alpha^{\perp}$ for some $\alpha \in \Gamma$.

Proposition 6.14. Suppose given a bimeromorphism $f: X \rightarrow X^{\prime}$ between compact, hyperkähler manifolds. Then there exist families of compact hyperkähler manifolds

$$
\mathcal{X} \longrightarrow D, \quad \mathcal{X}^{\prime} \longrightarrow D
$$

over a complex disc $D$ such that

$$
\text { 1. } \mathcal{X}_{0} \cong X \text { and } \mathcal{X}_{0}^{\prime} \cong X^{\prime}
$$

2. there exists a bimeromorphism $F: \mathcal{X} \rightarrow \mathcal{X}^{\prime}$ commuting with the projections to $D$ which is an isomorphism over $D \backslash\{0\}$ and induces $f$ on $\mathcal{X}_{0} \cong X \rightarrow \mathcal{X}_{0}^{\prime} \cong X^{\prime}$.

Proposition 6.15. For any $x \in Q_{\Gamma}$, the set of hyperkähler complex structures on a differentiable manifold $X$ with period $x \in Q_{\Gamma}$ consists of a finite number of bimeromorphic equivalence classes.

## 7. Twistor spaces and twistor conics

### 7.1. Hyperkähler structures

Given $X$ hyperkähler, let $g$ be the hyperkähler metric of $X$. We saw that there exist complex structures $I, J, K$ such that $g$ is Kähler with respect to $I, J, K$ and $I J K=-1$. In fact $g$ is Kähler with respect to any linear combination $\lambda=a I+b J+c K$ such that $a^{2}+b^{2}+c^{2}=1$. The Kähler form associated to $\lambda$ is $\omega_{\lambda}(\cdot, \cdot):=g(\lambda \cdot, \cdot)$. So we have a family $\left\{(X, \lambda) \mid \lambda \in S^{2}\right\}$ of compact Kähler manifolds.

### 7.2. Twistor spaces

With the notation above, the twistor space $\mathcal{X} \rightarrow \mathbb{P}^{1}$ of $(X, g)$ is the product $X \times \mathbb{P}^{1}$ (as a real manifold) endowed with the almost complex structure

$$
\begin{array}{cccc}
I_{X \times \mathbb{P}^{1}}: & T_{x} X \oplus T_{\lambda} \mathbb{P}^{1} & \longrightarrow & T_{x} X \oplus T_{\lambda} \mathbb{P}^{1} \\
(v, w) & \longmapsto & \left(\lambda(v), I_{\mathbb{P}^{1}}(w)\right)
\end{array}
$$

which is integrable by a result of Hitchin, Karlhede, Lindström, Roček.

### 7.3. Twistor conics

Fix a lattice $\left(\Gamma, q_{\Gamma}\right)$, isometric to $\left(H^{2}(X, \mathbb{Z}), \widetilde{q}_{X}\right)$. Recall that the signature of $q_{\Gamma} \otimes \mathbb{R}$ is $\left(3, b_{2}-3\right)$ where $b_{2}$ is the second Betti number of $X$. Since $\mathbb{P}^{1}$ is simply connected, we can choose consistent markings on all the fibers of $\mathcal{X} \rightarrow \mathbb{P}^{1}$ to obtain the period map

$$
\begin{array}{cccc}
P_{g}: & \mathbb{P}^{1} & \longrightarrow & Q_{\Gamma} \\
\lambda & \longmapsto & {\left[\sigma_{(X, \lambda)}\right]}
\end{array}
$$

whose image is a twistor conic.
One can show that it is the intersection of a linearly embedded $P=\mathbb{P}^{2}$ with $Q_{\Gamma}$ in $\mathbb{P}(\Gamma \otimes \mathbb{C})$. Furthermore $P=\mathbb{P}(W \otimes \mathbb{C})$ where $W$ is a three dimensional real subspace of $\Gamma \otimes \mathbb{R}$ totally positive for the intersection form $q_{\Gamma}$.

Conversely, one can show that each choice of a 3-dimensional real space $W \subset \Gamma \otimes \mathbb{R}$ positive for $q_{\Gamma}$ gives a twistor conic:

$$
C:=\mathbb{P}(W \otimes \mathbb{C}) \cap Q_{\Gamma} \subset Q_{\Gamma}
$$

Recall the following
Definition 7.1. A Kähler class is the cohomology class of a $(1,1)$ form which is Kähler with respect to some metric. The Kähler cone is the cone generated by all Kähler classes.

By Corollary 4.6, given the family $\left\{(X, \lambda) \mid \lambda \in S^{2}\right\}$ as in 7.1, for every Kähler class $\alpha \in H^{1,1}(M)$, there exists a unique hyperkähler metric $g_{\lambda}$, Kähler with respect to $\lambda$, such that $\left[\omega_{g_{\lambda}}\right]=\alpha$.

For each such metric $g_{\lambda}$, we can construct a twistor family $\mathcal{X}_{\lambda} \rightarrow \mathbb{P}^{1}$. In other words, through each point $[(X, I)]$ of the twistor conic there passes another twistor conic.

One can show [22, §25.4]
Proposition 7.2. $Q_{\Gamma}$ is twistor path connected, i.e., any two points of $Q_{\Gamma}$ can be joined by a connected sequence of twistor conics.

From which it follows (again see Huybrechts' lecture notes [22, §25.4])
Corollary 7.3. The period map $P_{\Gamma}: \mathcal{M}_{\Gamma} \rightarrow Q_{\Gamma}$ is surjective on any connected component of $\mathcal{M}_{\Gamma}$.

### 7.4. Hyperholomorphic bundles

Verbitsky studied conditions under which a holomorphic bundle which is stable for a particular Kähler class $\lambda$ is hyperholomorphic, i.e., extends to a holomorphic bundle on the twistor family $\mathcal{X}_{\lambda} \rightarrow \mathbb{P}^{1}$. Verbitsky's results form the basis for Markman's proof of the Hodge conjecture for abelian fourfolds of Weil type with discriminant 1 [33]. We start with the precise definition of hyperholomorphic bundles.

Definition 7.4. Given a hermitian vector bundle $B$ on $X$, with hermitian connection $\theta$, we say $(B, \theta)$ is hyperholomorphic if it is compatible with all the complex structures $\lambda \in S^{2}=\mathbb{P}^{1}$.

Definition 7.5. A $C^{\infty}$ vector bundle $B$ on $X$ is hermitian if it has a hermitian metric (denoted $\langle$,$\rangle ). A connection$

$$
\theta: B \longrightarrow B \otimes T_{X}^{*}
$$

is hermitian if the metric is (covariantly) constant with respect to $\theta$. If we are given a complex structure $I$ on $B$, we say that $\theta$ and $I$ are compatible if the curvature form

$$
\Theta: B \longrightarrow B \otimes \Lambda^{2} T_{M}^{*}
$$

is a $(1,1)$-form with respect to $I$.
Intuitively, considering the twistor family

the $C^{\infty}$ vector bundle $B \times \mathbb{P}^{1}$ on $\mathcal{X}$ has a structure of complex vector bundle holomorphic on each fiber $(X, \lambda)$ of $\mathcal{X} \rightarrow \mathbb{P}^{1}$.

Stability conditions allow us to construct moduli spaces of bundles.
Definition 7.6. Fix a Kähler form $\omega$ on $X$. For a coherent sheaf $F$ on $X$, put

$$
\operatorname{deg}(F):=\frac{1}{\operatorname{vol}(X)} \int_{X} c_{1}(F) \wedge \omega^{n-1}
$$

where $n$ is the complex dimension of $X$ and $\operatorname{vol}(X):=\int_{X} \omega^{n}$. Define

$$
\operatorname{slope}(F):=\frac{\operatorname{deg}(F)}{\operatorname{rank}(F)}
$$

where $\operatorname{rank}(F)$ is the complex rank of $F$. We say $F$ is stable with respect to $\omega$ if for all subsheaves $F^{\prime} \subset F$ with $\operatorname{rank}\left(F^{\prime}\right)<\operatorname{rank}(F)$, we have

$$
\operatorname{slope}\left(F^{\prime}\right)<\operatorname{slope}(F)
$$

We say $F$ is semi-stable with respect to $\omega$ if for all subsheaves $F^{\prime} \subset F$, we have

$$
\operatorname{slope}\left(F^{\prime}\right) \leq \operatorname{slope}(F)
$$

Verbitsky (see [45]) proved that, given a stable vector bundle $B$ on $(X, I)$, if $c_{1}(B)$ and $c_{2}(B)$ are of type $(1,1)$ and $(2,2)$ with respect to all complex structures $\lambda \in S^{2}=\mathbb{P}^{1}$ on $X$, then $B$ is hyperholomorphic. In particular, the class $c_{2}(B)$ is analytic on each $(X, \lambda)$.

A useful characterization of stable bundles is given by the HitchinKobayashi correspondence. To state it, we first need the following definition.
Definition 7.7. Let $\omega$ be the Kähler form of $M$ and denote by $\Lambda: \Omega_{M}^{1,1} \otimes B \rightarrow B$ the adjoint of cup-product with $\omega$. A hermitian metric with curvature form $\Theta: B \rightarrow B \otimes \Omega_{M}^{1,1}$ is Hermitian-Einstein if the composition $\Lambda \Theta: B \rightarrow B$ is a multiple of the identity.

The Hitchin-Kobayashi correspondence, proved by Donaldson, Uhlenbeck and Yau is the following theorem.

Theorem 7.8. Suppose $B$ is an indecomposable bundle on a compact Kähler manifold $M$. Then $B$ is stable if and only if $B$ has a Hermitian-Einstein metric.

## 8. Examples of hyperkählers in dimension 2 and beyond, by Samir Canning

### 8.1. Betti and Hodge numbers of $K 3$ surfaces

The purpose of this exercise is to compute the Betti and Hodge numbers of a complex $K 3$ surface $X$, which is the simplest example of a hyperkähler manifold. Feel free to add the additional assumption that $X$ is algebraic if you are more comfortable in that setting.
Problem 8.1. Show that $H^{0}(X, \mathbb{Z})=H^{4}(X, \mathbb{Z})=\mathbb{Z}, H^{1}(X, \mathbb{Z})=0$, and $H^{3}(X, \mathbb{Z})$ is torsion. (Hint: use the exponential exact sequence.)
Problem 8.2. Show that $H^{2}(X, \mathbb{Z})$ is torsion free. Conclude that $H^{3}(X, \mathbb{Z})=$ 0 . (Hint: continue analyzing the exponential exact sequence, using that $\operatorname{Pic}(X)$ is torsion free. Prove this if you know about Riemann-Roch. For the second statement, use the universal coefficient theorem for cohomology.)

Recall the Hirzebruch-Riemann-Roch Theorem.
Theorem 8.3 (Hirzebruch-Riemann-Roch). Let $F$ be a (holomorphic) vector bundle on a compact complex manifold $X$. Then,

$$
\chi(X, F)=\int_{X} \operatorname{ch}(F) \operatorname{td}(X)
$$

When we write $c_{i}(X)$, we mean $c_{i}\left(T_{X}\right)$, where $T_{X}$ is the tangent bundle. Here are the first few terms of the Chern character and Todd class for reference:

$$
\operatorname{ch}(F)=\operatorname{rank}(F)+c_{1}(F)+\frac{1}{2}\left(c_{1}(F)^{2}-2 c_{2}(F)\right)+\cdots
$$

and

$$
\operatorname{td}(F)=1+\frac{1}{2} c_{1}(F)^{2}+\frac{1}{12}\left(c_{1}(F)^{2}+c_{2}(F)\right)+\cdots
$$

Problem 8.4. Compute $c_{2}(X)$ for $X$ a $K 3$ surface. (Hint: set $F=\mathcal{O}_{X}$.)
Problem 8.5. Compute $H^{2}(X, \mathbb{Z})$. (Hint: take $F=\Omega_{X}$.)
You have now computed all of the Betti numbers. Next, we will compute the Hodge numbers.

Definition 8.6. Let $X$ be a compact Kähler manifold. The Hodge numbers of $X$ are

$$
h^{p, q}=\operatorname{dim} H^{q}\left(X, \Omega_{X}^{p}\right) .
$$

Theorem 8.7 (The Hodge Decomposition). Let $X$ be a compact Kähler manifold. There is a direct sum decomposition

$$
H^{i}(X, \mathbb{Z}) \otimes \mathbb{C}=H^{i}(X, \mathbb{C})=\bigoplus_{p+q=i} H^{q}\left(X, \Omega_{X}^{p}\right)
$$

Moreover $h^{p, q}=h^{q, p}$.
Problem 8.8. Compute all of the Hodge numbers of a compact complex $K 3$ surface $X$.

Further remarks 8.9. The same ideas, especially the use of the Hirzebruch-Riemann-Roch Theorem, can be used to give restrictions on the Betti and Hodge numbers of higher dimensional hyperkähler manifolds. For more in this direction, see the paper of Salamon [41] and Debarre's exposition thereof [15]. For even further restrictions on the Betti numbers of hyperkähler fourfolds, see the paper of Guan [23]. For sixfolds, see the paper by Sawon [42].

### 8.2. Identifying hyperkähler manifolds

One of the most interesting areas of research in hyperkähler geometry is the construction of examples. This exercise will focus on identifying examples. We begin with some basic problems.
Problem 8.10. Convince yourself that any holomorphic two-form $\sigma$ on a complex manifold $X$ induces a morphism of bundles

$$
\sigma: T_{X} \rightarrow \Omega_{X}^{1}
$$

where $T_{X}$ is the tangent bundle and $\Omega_{X}^{1}$ is the cotangent bundle.
We call $\sigma$ non-degenerate if the morphism above is an isomorphism.
Problem 8.11. Can you convince yourself that K3 surfaces are irreducible hyperkähler? (Hint: the tricky part is probably the simply connectedness. It may require some extra background knowledge.)
Problem 8.12. Show that $h^{2,0}=h^{0,2}=1, K_{X} \cong \mathcal{O}_{X}$, and that $\operatorname{dim}(X)$ is even for any irreducible compact hyperkähler manifold $X$.

Now that we know that $K_{X}$ is trivial for compact hyperkähler manifolds $X$, a natural question is: given a $K_{X}$-trivial manifold, how can we show that it is hyperkähler, if it is? We will focus on a real-life example due to DebarreVoisin [17]. The same type of argument works for another famous example of Beauville-Donagi [7] (the Fano variety of lines on a cubic fourfold.)

Let $V_{10}$ be a 10 -dimensional complex vector space. Let $\omega \in \wedge^{3} V_{10}^{\vee}$ be a 3 -form on $V_{10}$. We define a subvariety of $G\left(6, V_{10}\right)$ :

$$
X_{\omega}:=\left\{[W] \in G\left(6, V_{10}\right):\left.\omega\right|_{W \times W \times W} \equiv 0\right\}
$$

Problem 8.13. Show that for a general choice of $\omega, X_{\omega}$ is a smooth fourfold. (Hint: show that $X_{\omega}$ is given by the vanishing of a section of a certain globally generated vector bundle.)
Problem 8.14. Show that $K_{X_{\omega}} \cong \mathcal{O}_{X_{\omega}}$. (Hint: use adjunction.)
Now that we know we have a $K_{X}$-trivial variety, we want to show it's hyperkähler. Using something called the Koszul resolution, one can compute the Euler characteristic of the structure sheaf:

$$
\chi\left(X_{\omega}, \mathcal{O}_{X_{\omega}}\right)=3
$$

Definition 8.15. A strict Calabi-Yau manifold is a simply connected projective manifold $X$ such that $H^{0}\left(X, \Omega_{X}^{p}\right)=0$ for $0<p<\operatorname{dim}(X)$.

Problem 8.16. Show that any simply connected smooth $K_{X}$-trivial compact Kähler fourfold with $\chi\left(X, \mathcal{O}_{X}\right)=3$ is irreducible compact hyperkähler. (Hint: use the nice multiplicative properties of $\chi\left(X, \mathcal{O}_{X}\right)$.)

Further remarks 8.17. The proof that $X_{\omega}$ above is hyperkähler is done differently (more geometrically) in [17]. I also highly recommend the classic paper [7]. It turns out in both cases, the resulting hyperkähler is deformation equivalent to the Hilbert scheme of 2 points on a K3 surface.

## 9. Basic properties of Lagrangian fibrations of Hyperkählers, by Yajnaseni Dutta

The following exercises are based on a couple of fundamental results from [34] and [35]. Given a Lagrangian fibration $f: X \rightarrow B$ of a Hyperkähler manifold $X$, the geometry and topology of $B$ are heavily influenced by $X$. In fact, Matsushita conjectured that $B \simeq \mathbb{P}^{n}$. It is known by work of Hwang [28] that if $B$ is smooth then $B \simeq \mathbb{P}^{n}$. The conjecture is known to be true if $\operatorname{dim} B=2$ by recent results of $[9,27,39]$

### 9.1. Lagrangian fibrations

Let $S$ be a K3 surface and $f: S \rightarrow C$ a proper surjective morphism onto a smooth irreducible curve with connected fibres ${ }^{1}$.

[^2]Problem 9.1. Show that $C \simeq \mathbb{P}^{1}$. (Hint: Use that $S$ is simply connected.)
Problem 9.2. Show that the general fibres of $f$ are elliptic curves. (Hint: Use Adjunction.)
Problem 9.3. Find an explicit fibration of the Fermat quartic $\left(x^{4}+y^{4}+z^{4}+\right.$ $\left.w^{4}=0\right) \subset \mathbb{P}^{3}$. (Hint: rewrite as an equality of two fractions.)

Let $X$ be a hyperkähler manifold of dimension $2 n$. The following exercises show how similar the situation is in higher dimensions. The quadratic space $\left(H^{2}(X, \mathbb{R}), q_{X}\right)$ controls much of the geometry of $X$ and is a central gadget in the study of hyperkähler manifolds.

Recall that $q_{X}$ is a priori dependent on the symplectic form $\sigma \in H^{0}\left(X, \Omega_{X}^{2}\right)$, however, up to scaling, it is independent of $\sigma$. Here are some key properties of $q_{X}$ (we denote the associated bilinear form again by $q_{X}$ ).

- The symplectic form $\sigma$, upto a normalization, satisfies $q_{X}(\sigma)=0$ and $q_{X}(\sigma+\bar{\sigma})=1$.
- More generally, for $\alpha_{i} \in H^{2}(X)$, we have

$$
\int_{X} \alpha_{1} \cdots \alpha_{2 n}=c_{X} \sum_{s \in S_{n}} q_{X}\left(\alpha_{s(1)}, \alpha_{s(2)}\right) \ldots q_{X}\left(\alpha_{s(2 n-1)}, \alpha_{s(2 n-2)}\right)
$$

for some constant $c_{X}$ depending only on $X$. As a consequence, we obtain $\int_{X} \sigma \bar{\sigma} \omega^{2 n-2}=c^{\prime} q_{X}(\omega)^{n-1}$.

- If a line bundle $L$ is ample, then $q_{X}\left(c_{1}(L)\right)>0$. The Kähler cone is contained in a connected component of $\left\{\alpha \in H^{1,1}(X, \mathbb{R}) \mid q_{X}(\alpha)>0\right\}$. Partial converses to these statements exist. For instance, if $L$ is a line bundle with $q_{X}(L)>0$ then $X$ is projective [22, Prop. 26.13]. Furthermore, if $q_{X}(\alpha)>0$ and, for every rational curve $C \subset X, \int_{C} \alpha>0$, then $\alpha$ is a Kähler class [11, Théorème 1.2].
- $H^{1,1}(X, \mathbb{C})$ is orthogonal to $H^{2,0}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C})$ with respect to $q_{X}$.
- By $[10,43]$ whenever there exists $0 \neq \beta \in H^{2}(X, \mathbb{C})$ that satisfies $q_{X}(\beta)=$ 0 , we have $\beta^{n} \neq 0$ and $\beta^{n+1}=0$

We begin with a Hodge index type theoerem.
Problem 9.4. Given a divisor $E$ on $X$, show that if $E$ satisfies $E^{2 n}=0$ and $E \cdot A^{2 n-1}=0$ for some ample bundle $A$, then $E \sim 0$. (Hint: Use $q_{X}(t E+A)=$ $t^{2} q(E)+2 t q(E, A)+q(A)$ for any variable $t$ and that $(t E+A)^{2 n}=c_{X} q_{X}(t E+$ $A)^{n}$.)

Problem 9.5. Given a divisor $E$ on $X$, show that if $E$ satisfies $E^{2 n}=0$ and $E \cdot A^{2 n-1}>0$ for some ample line bundle $A$, then $q_{X}(E, A)>0$ and the following are true

$$
\begin{aligned}
& E^{m} \cdot A^{2 n-m}=0 ; \text { for } m>n \\
& E^{m} \cdot A^{2 n-m}>0 ; \text { for } m \leq n
\end{aligned}
$$

(Hint: Expand $q_{X}(t E+A)$ as in the previous exercise.)
Problem 9.6. Let $f: X \rightarrow B$ be a fibration of a hyperkähler manifold $X^{2}$. Using the previous exercise show that $\operatorname{dim} B=n$. (Hint: Apply the previous exercise to the pull-back of an ample class $H$ on $B$.)
Problem 9.7. Show that $\operatorname{Pic}(B)$ is of rank 1. (Hint: Show that any divisor $E$ on $X$ that satisfies $E^{2 n}=0$ and $E^{n} \cdot\left(f^{*} H\right)^{n}=0$ is in fact a rational multiple of $f^{*} H$.)
For the next exercise we need the definition of a Lagrangian (possibly singular) subvariety. Recall that
Definition 9.8. A subvariety $Y \subset X$ is said to be a Lagrangian subvariety if $\operatorname{dim} Y=\frac{1}{2} \operatorname{dim} X$ and there exists a resolution of singularities $\mu: Y^{\prime} \rightarrow Y$ such that $\left.\mu^{*} \sigma\right|_{Y^{\prime}}=0$.

Problem 9.9. Show that a general fibre of $f$ is Lagrangian. By a classical theorem, the general fibres of $f$ are then complex tori. A more recent result of Voisin [12, Prop. 2.1] or, more generally, [31, Theorem 1.1], shows that even if $X$ is not projective, a Lagrangian subvariety of a hyperkähler manifold is always projective. Thus, a general fibre $F$ is isomorphic to an abelian variety. (Hint: Let $A$ be an ample class on $X$.)
Problem 9.10. Show that every fibre of $f$ is Lagrangian and hence $f$ is equidimensional. (Hint: Use the map $H^{2}\left(X, \mathcal{O}_{X}\right) \rightarrow H^{0}\left(B, R^{2} f_{*} \mathcal{O}_{X}\right)$ induced by the Leray spectral sequence and that $R^{2} f_{*} \mathcal{O}_{X}$ is torsion free.)
Problem 9.11. Show that $B$ is $\mathbb{Q}$-factorial with at worst Kawamata log terminal singularities. (Hint: see [26, Prop. 5.10] or use that f is equidimensional and [30, Lemma 5.16] which states that if the source of a finite surjective map between normal varieties is $\mathbb{Q}$-factorial and klt then so is the target.)

For the next exercise, recall and use the following
Definition 9.12 (Kodaira Dimension). Let $X$ be a $\mathbb{Q}$-factorial variety. Then

$$
\kappa(X)=\sup _{m} \operatorname{dim} \overline{\phi_{m}(X)}
$$

[^3]where $\phi_{m}: X \longrightarrow \mathbb{P}^{P_{m}}$ is the rational map defined by the global sections of $\omega_{X}^{\otimes m}$ and $P_{m}=\operatorname{dim} H^{0}\left(X, \omega_{X}^{\otimes m}\right)$. Another way to interpret this is
$$
\kappa(X):=\operatorname{trdeg}_{k}\left(\bigoplus_{m} H^{0}\left(X, \omega_{X}^{\otimes m}\right)\right)-1
$$
where the algebra structure on the right side is given by the multiplication map.
Iitaka's $C_{n, m}$ conjecture then states that
Conjecture 9.13. Let $f: X \rightarrow B$ be a fibration of smooth projective varieties, and let $F$ be a general fibre of $f$. Then,
$$
\kappa(X) \geq \kappa(F)+\kappa(B)
$$

By a result of Kawamata [29, Theorem 1.1(2)], the conjecture is known when $F$ is a minimal variety.
Problem 9.14. Assume $B$ is smooth, show that $B$ is Fano, i.e., the inverse of the canonical bundle of $B$ is ample. (Hint: use that the Picard rank of $B$ is 1 and Kawamata's result above.)
Problem 9.15. Assume $B$ is smooth. Let $B^{0}$ be the open set where $f$ is smooth. Let $X^{0}:=f^{-1}\left(B^{0}\right)$. Show that $R^{i} f_{*}^{0} \mathcal{O}_{X^{0}}=\Omega_{B^{0}}^{i}$. (Hint: Use $\Omega_{X^{0}}^{1} \simeq$ $\mathcal{T}_{X^{0}}$ to conclude that $\left.f^{*} \mathcal{T}_{B^{0}} \simeq \Omega_{X^{0} / B^{0}}^{1}.\right)$

Matsushita [35] (also see [34]) extends this equality to the big open set $U$ which includes the smooth points of the discriminant divisor $D_{f}$, using Deligne's canonical extension. Then, using the reflexivity of $R^{i} f_{*} \mathcal{O}_{X}$ and the isomorphism $R^{n} f_{*} \mathcal{O}_{X} \simeq \omega_{B}$, he shows that $R^{i} f_{*} \mathcal{O}_{X} \simeq \Omega_{B}^{i}$.

## 10. Rational curves on K3 surfaces and Euler characteristics of Moduli spaces, by David Stapleton

We follow a paper of Beauville [4], inspired by work of Yau and Zaslow [46], which uses hyperkähler geometry to count the number of rational curves in a very general K3 surface of degree 2d.
Problem 10.1. Assume that a K3 surface $X$ admits an elliptic pencil - that is a map

$$
\pi: X \rightarrow \mathbb{P}^{1}
$$

so that the general fibers are smooth genus 1 curves. Assume that all the fibers that do not have geometric genus 1 are irreducible rational curves with a single
node. Count the number of rational fibers. (Hint: If $R=\sqcup_{i=1}^{n} R_{i}$ is the union of rational curves, compute the topological Euler characteristic using the formula:

$$
e(X)=e(R)+e(X \backslash R)
$$

and compute $e\left(R_{i}\right)$.)

### 10.1. Hyperkählers as moduli spaces of sheaves on K3 surfaces.

Let $X$ be a very general K3 surface of degree 2d with primitive line bundle $L$ (with $L^{2}=2 d$ ) and let $\Pi=\mathbb{P}\left(H^{0}(X, L)\right) \cong \mathbb{P}^{d-1}$. Moduli spaces of sheaves on $X$ are frequently hyperkähler manifolds. Here are two examples:

1. Hilbert schemes of $n$ points on $X$ - denoted $X^{[n]}$, this space compactifies the space of unordered distinct points on $X$ by considering length $n$ subschemes as their limits.
2. Compactified Jacobians - denoted $\overline{\mathcal{J}}^{d}(X)$ - parametrizing coherent sheaves supported on curves $C \in \Pi$, which when thought of as sheaves on $C$ are line bundles (or torsion-free sheaves of rank 1 when $C$ is singular) of degree $d$.

Problem 10.2. Show that if $X$ is a K3 surface, then $\Pi$ contains only finitely many rational curves (curves with geometric genus 0 ).
Problem 10.3. Compute the dimension of $X^{[n]}$ and $\overline{\mathcal{J}}^{d}(X)$.
Problem 10.4. Show that the hyperkählers $X^{[g]}$ and $\overline{\mathcal{J}}^{g}(X)$ are birational.
There is a natural map

$$
\pi: \overline{\mathfrak{J}}^{g}(X) \rightarrow \Pi
$$

which sends a coherent sheaf $\mathcal{F}$ to the curve in $\Pi$ that it is supported on.
Problem 10.5. Show that the general fiber of $\pi$ is an Abelian variety. Describe the fibers over a general point $C \in \Pi$.

Problem 10.6. (this is [4, Prop. 2.2]) Let $C$ be an integral curve such that the normalization $\widehat{C}$ has genus $\geq 1$. We show that $e\left(\overline{\mathcal{J}}^{d}(C)\right)=0$ as follows.

1. Find a line bundle $\mathcal{M}$ on $C$ which is torsion of order $m$ (for any $m>0$ ). (This uses the comparison between the Jacobian of $C$ and of $\widehat{C}$.)
2. Show that tensoring by $\mathcal{M}$ is a free action of $\mathbb{Z} / m \mathbb{Z}$ on $\overline{\mathcal{J}}^{d}(\widehat{C})$.
3. Conclude that $m$ divides $e\left(\overline{\mathcal{J}}^{d}(C)\right)$ for all $m>0$.

It follows by the scissor property of Euler characteristics that

$$
e\left(\overline{\mathcal{J}}^{g}(X)\right)=\sum_{R_{i} \in \Pi} e\left(\overline{\mathcal{J}}^{g}\left(R_{i}\right)\right)
$$

where $R_{i} \in \Pi$ is a rational curve and $\pi^{-1}\left(R_{i}\right)$ is the fiber over $R_{i}$ (i.e., the set of torsion free sheaves of rank 1 and degree $g$ supported on $\left.R_{i}\right)$.
Problem 10.7. Show that

$$
e\left(\overline{\mathcal{J}}^{g}\left(R_{i}\right)\right)=1
$$

if $R_{i}$ is a nodal, irreducible rational curve. Thus by a result of Xi Chen [13], if $X$ is very general then

$$
e\left(\overline{\mathcal{J}}^{g}(X)\right)=\#\left\{R_{i} \in \Pi\right\}
$$

Hint: Locally at a node $p \in R_{i}$ there are only 2 types of rank 1 torsion free sheaves (1) line bundles and (2) the ideal sheaf of a point. Show that if $p_{1}, \cdots, p_{g} \in R_{i}$ are the nodes then $\overline{\mathcal{J}}^{g}\left(R_{i}\right)$ is stratified into loci $\overline{\mathcal{J}}_{S}^{g} \subset \overline{\mathcal{J}}^{g}\left(R_{i}\right)$ consisting of torsion-free sheaves that are not locally free exactly at the points in a subset $S \subset\left\{p_{1}, \cdots, p_{g}\right\}$. Conclude that the only stratum where $e\left(\overline{\mathcal{J}}_{S}^{g}\right) \neq 0$ is when $S=\left\{p_{1}, \cdots, p_{g}\right\}$ (a single point). See also [4, §3].

It remains to actually calculate the Euler characteristic of $\overline{\mathcal{J}}^{g}(X)$. This relies on

1. The birational invariance of Euler characteristic for hyperkählers (see [24] or use the birational invariance of betti numbers of Calabi-Yaus [1]).
2. The computation of the Euler characteristic of $X^{[n]}$ by Göttsche [21] (see [14] for a nice explanation of these results).

In particular, for a K3 surface, by (1) and (2) we have:
$\sum(\#$ rational curves on a K3 of genus $g) q^{g}=\sum_{g \geq 0} e\left(\overline{\mathcal{J}}^{g}(X)\right) q^{g}$

$$
=\sum_{g \geq 0} e\left(X^{[g]}\right) q^{g}=\Pi_{k=1}^{\infty}\left(\frac{1}{1-q^{k}}\right)^{e(X)}
$$

where the sum over $g \geq 0$ is understood to take a very general K3 surface of genus $g$.
Problem 10.8. Compute the Euler characteristic of $X^{[2]}$ for any complex surface using that

1. there is a birational map

$$
h: X^{[2]} \rightarrow X^{(2)}
$$

to the symmetric product $X^{(2)}:=X^{2} / \Sigma_{2}$ which is given by blowing up the diagonal locus and
2. the exceptional divisor of $h$ is a $\mathbb{P}^{1}$-bundle over $X$.

Problem 10.9. Find the number of bitangents to a very general plane sextic curve $C \subset \mathbb{P}^{2}$ using that a very general K3 surface of genus 2 is a double cover of $\mathbb{P}^{2}$ branched at such a sextic.

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## Section 3

> Proceedings of the Conference GO60 Pure and Applied Algebraic Geometry celebrating Giorgio Ottaviani's 60 th birthday

## Preface

Section 3 of Volume 54 of Rendiconti dell'Istituto di Matematica dell'Università di Trieste is dedicated to the proceedings of the conference GO60-Pure and Applied Algebraic Geometry celebrating Giorgio Ottaviani's 60th birthday. This workshop, originally planned in Levico (Trento, Italia), June 20-26, 2020 and held on-line, June 21-25, 2021 due to Covid-19 restrictions, was organized on the occasion of the sixtieth birthday of our friend Giorgio Ottaviani, in appreciation of his many contributions to Algebraic Geometry and of his role as teacher and mentor.

Giorgio Ottaviani, Full Professor at University of Florence since November 1, 1997, is a world-wide recognized mathematician, who is able to convey enthusiasm and passion to open new perspectives to other researchers. His scientific interests cover both the pure and applied sides of Algebraic Geometry. Giorgio's main results on pure Algebraic Geometry concern the geometry of projective subvarieties in projective space, in particular those of small codimension, vector bundles and their moduli space, instantons, Lefschetz properties, homogeneous vector bundles on rational homogeneous varieties and classical chapters of Algebraic Geometry such as Lüroth quartics, Hessians etc. His interests encompass also topics which lie close to applications, for instance higher secant varieties and tensor decomposition, Euclidean distance degree, tensor rank and identifiability of tensors, complexity of matrix multiplication algorithm, entanglement and Quantum Information and algorithms developed by current symbolic software.

This section collects articles by many of the speakers of the conference, together with contributions by some of Giorgio's many students, now professional researchers, and from other distinguished mathematicians who have actively collaborated with him and are still very close to him. Many of the papers we are presenting are stricly related to the topics that fascinate Giorgio and witness how much he is respected by his colleagues, both from the human and scientific point of view.

We would like to thank Giorgio, for everything he has taught us, in his passionate and informal style. For his friendship and the support that he gave us all over the years. For showing us how one can be open minded, humble and brilliant at the same time. And finally for giving us the opportunity to organize this conference, whose interesting talks were very useful to outline the
state of the art of the mathematical research in our field.
We heartly thank the authors of the articles published in this section for their highly appreciated contributions. We are indebted with Emilia Mezzetti, Editor-in-Chief of RIMUT, for accepting our proposal of this section and for her suggestions and technical support during its achievement. A special thank to the referees of all the papers, for their careful readings and interesting remarks.

Finally we gratefully acknowledge CIRM-FBK, Università Politecnica delle Marche, GNSAGA-INdAM, and Università degli Studi di Firenze for the support in the organization of the conference.

The guest Editors<br>Elena Angelini<br>Maria Chiara Brambilla<br>Daniele Faenzi

and the other members of the Organizing Committee
Ada Boralevi
Simone Naldi
Elena Rubei


# Examples of non-effective rays at the boundary of the Mori cone of blow-ups of the plane 

Ciro Ciliberto and Rick Miranda

We dedicate this paper to Giorgio Ottaviani on his 60th birthday


#### Abstract

In this paper we prove that no multiple of the linear system of plane curves of degree $d \geqslant 4$ with a point of multiplicity $d-m$ (with $2 \leqslant m \leqslant d)$ and $m(2 d-m)$ simple general points is effective.


Keywords: Linear systems, Mori cone, Nagata's conjecture, nef rays. MS Classification 2020: 14E07, 14E30, 14J26.

## 1. Introduction

Alex Massarenti and Massimiliano Mella asked us the following question. Consider 13 general points $p_{0}, \ldots, p_{12}$ in the projective plane and consider the class of a quartic curve with a singular point at $p_{0}$ and passing through $p_{1}, \ldots, p_{12}$. Is it the case that no multiple of this class is effective?

In trying to answer this question we got aware of the fact that we are able to prove the following more general result.

Theorem 1.1. Let $d$ be any integer and $p_{0}, \ldots, p_{m(2 d-m)}$ general points in the plane with $m \leqslant d$. Consider the class (or system) $\xi_{d, m}$ of plane curves of degree $d$ with a point of multiplicity at least $d-m$ at $p_{0}$ and passing through $p_{1}, \ldots, p_{m(2 d-m)}$. Fix $k \geqslant 1$.
(a) For any $d \geqslant 4$ and any $m$ with $2 \leqslant m \leqslant d$, the class $k \xi_{d, m}$ is not effective.
(b) For any d, the system $\xi_{d, 1}$ is a pencil of rational curves and $\xi_{d, 0}$ is composed with the pencil of lines through the point $p_{0}$ and has dimension $d$. The multiple linear systems $k \xi_{d, 1}$ are composed with the corresponding pencil and have dimension $k$. There is no member of these systems that contains an irreducible curve which is not a component of a member of this pencil. The same is true for the system $\xi_{2,2}$.
(c) For $d=3$, the systems $\xi_{3,3}$ and $\xi_{3,2}$ coincide with the system of cubics through 9 general points, which consists of a unique cubic $C$. The systems $k \xi_{3,3}$ and $k \xi_{3,2}$ consist of the unique curve $k C$.

A few remarks are in order. First statements (b) and (c) in the theorem are trivial (we stated them for completeness and we will inductively use them in the proof of (a) which is the core of the theorem. Secondly, if $m=d$ or $m=d-1$ the statement is Nagata's theorem for $d^{2}$ general points in the plane (see [4]), hence the theorem can be viewed as a generalization of Nagata's theorem. So for the proof of (a) we may and will assume $2 \leqslant m \leqslant d-2$.

Let $X_{n}$ be the blow-up of the projective plane at $n$ general points. Let $\mathcal{L}_{d}\left(m_{1}, \ldots, m_{n}\right)$ be the linear system on $X_{n}$ corresponding to plane curves of degree $d$ with general points of multiplicities at least $m_{1}, \ldots, m_{n}$.

If we blow-up $p_{0}, \ldots, p_{m(2 d-m)}$ we get the surface $X_{m(2 d-m)+1}$ and $\xi_{d, m}$ can be interpreted as an element in $\operatorname{Pic}\left(X_{m(2 d-m)+1}\right)$; note that $\xi_{d, m}^{2}=0$. Moreover $\xi_{d, m}$ is nef. Indeed we consider a general plane curve $C$ of degree $d$ with a point $p_{0}$ of multiplicity $d-m$ and we can fix $m(2 d-m)$ general points on $C$. If we blow up $p_{0}$ and the $m(2 d-m)$ chosen points, the proper transform of $C$ is an irreducible curve with 0 self-intersection, and therefore it is nef on the blow-up. Since nefness is an open condition, this is true for the general class $\xi_{d, m}$.

Our result says that there is no positive number $k$ such that $\mathcal{L}_{k d}(k(d-$ $m$ ), $k^{m(2 d-m)}$ ) is non-empty (the exponential notation for repeated multiplicities is clear). If we set $N_{1}\left(X_{m(2 d-m)+1}\right)=\operatorname{Pic}\left(X_{m(2 d-m)+1}\right) \otimes_{\mathbb{Z}} \mathbb{R}$, then $\xi_{d, m}$ generates a rational ray in $N_{1}\left(X_{m(2 d-m)+1}\right)$ that is not effective (see [1, §3.1]) and therefore it sits in the boundary of the Mori cone of $X_{m(2 d-m)+1}$. Such a ray, if rational in $N_{1}\left(X_{m(2 d-m)+1}\right)$, is called a good ray in [1, §3.2] whereas, if irrational, it is called a wonderful ray. So far no wonderful ray has been discovered ${ }^{1}$. However, proving that a given ray is good is in general not easy, and in [1] the authors were able to exhibit some examples. Therefore it is interesting to find good rays, and in this paper we make a new contribution in this direction.

Our proof uses the degeneration technique we introduced for analyzing the dimension of such linear system (see, e.g., [2]). We briefly recall this in Section 2 . The proof is by induction on $m$, the case $m=2$ being the critical one. We prove the $m=2$ case in Section 5. This particular example relies on a subtlety that requires us to analyze, more deeply than what we did in [3], the case in which there are multiple $(-1)$-curves splitting off a linear system in the limit. This we describe in Section 3. Finally in Section 5 we finish the proof of Theorem 1.1.

We notice that the surprising phenomenon that allows us to make the final analysis of the limit linear systems in the case $m=2$ is that we eventually end up with curves of a certain degree $t n$ in the plane with $n^{2}$ points of multiplicity $t$,

[^4]which is currently the only case in which Nagata's conjecture is proven (see [4]). Indeed we use here an argument inspired by the original one of Nagata (see l.c.) to deal with these cases.

The ideas in this note can be generalized to prove more general similar results about general linear systems with zero self-intersection and we will do this in a forthcoming paper.

## 2. The degeneration method

In this section we briefly recall the degeneration technique that we use to analyse planar linear systems (see [2]). We want to study a linear system $\mathcal{L}_{d}\left(m_{1}, \ldots, m_{n}\right)$. To do this we consider a trivial family $\mathbb{P}^{2} \times \mathbb{D} \rightarrow \mathbb{D}$ over a disc $\mathbb{D}$. In the central fibre over $0 \in \mathbb{D}$ we blow-up a line $R$ producing a new family $\mathcal{X} \rightarrow \mathbb{D}$ with an exceptional divisor $F \cong \mathbb{F}_{1}$ and the proper transform $P \cong \mathbb{P}^{2}$ of the original central fibre. The new central fibre consists now of $F \cup P$, with $F, P$ transversely intersecting along the line $R$, which is the ( -1 )-curve in $F$.

Next we fix $a$ general points on $P$ and $b$ general points of $F$, so that $a+b=n$. Consider sections of the family $\mathcal{X} \rightarrow \mathbb{D}$ extending these $n$ points to general points on the general fibre. Blowing up these sections, we have a degeneration of $X_{n}$ to the union of an $X_{a}$ (the blow-up of $P$ at the $a$ general points) and of an $X_{b+1}$ (the blow-up of $F$ at the $b$ general points).

Since there is an obvious map $\pi: \mathcal{X} \rightarrow \mathbb{P}^{2}$, we have the bundle $\mathcal{O}_{\mathcal{X}}(d)=$ $\pi^{*}\left(\mathcal{O}_{\mathbb{P}^{2}}(d)\right)$. This bundle restricts to the general fibre to $\mathcal{O}_{\mathbb{P}^{2}}(d)$. On the central fibre it restricts to the bundle $\mathcal{O}_{\mathbb{P}^{2}}(d)$ on $P$ and to $\mathcal{O}_{F}(d f)$, where $f$ is the class of a fibre of the ruling of $F$ over $\mathbb{P}^{1}$. This is a limit of the line bundle on the general fibre; there are other limits obtained by twisting by $\mathcal{O}_{\mathcal{X}}(-l P)$, i.e., by tensoring the above limit bundle by $\mathcal{O}_{\mathcal{X}}(-l P)$, with $l$ an integer. This restricts to $\mathcal{O}_{\mathbb{P}^{2}}(d+l)$ on $P$ and to $\mathcal{O}_{F}(d f-l R)$ on $F$. So we have a discrete set of limits of $\mathcal{L}_{d}\left(m_{1}, \ldots, m_{n}\right)$, depending on all choices for $a, b, l$ and distribution of the multiplicities among the $a+b$ points on the central fibre.

A section of a limit line bundle is given by a pair of sections on $P$ and $F$, that restrict equally to the double curve $R$. We will call this the naive matching condition. Such a section could be identically zero on one of the components of the central fibre, and in this case a matching section on the other component corresponds to a section of the linear system (called a kernel linear system) of curves on the other component containing the double curve $R$. One way to prove emptiness of the system on the general fibre is to find $a, b$, a distribution of the multiplicities and a twisting parameter $l$ such that there is no section of the limit line bundle on the central fibre that verifies the naive matching condition and that is non-zero on at least one of the two components.

An alternative approach to proving the emptiness of the linear system on the general fibre is the following. Suppose that the system is non-empty on the general fibre. Then for every choice of $a, b$ and a distribution of the multiplicities, there will be a limit curve which must be the zero of a section of a limit line bundle given by some particular twist parameter $l$, this section being not identically zero on both $P$ and $F$. As we said, naively the matching condition means that the two curves restrict equally to $R$. However we will see in the next section that when the curves are non-reduced the matching conditions are more subtle. We will call these conditions refined matching conditions. Hence, to prove that the system on the general fibre is empty, it suffices to find $a, b$ and a distribution of the multiplicities so that for no $l$ there is a limit curve as above, i.e., a pair of curves on $P$ and $F$ satisfying the refined matching. This will be the approach we will use in the proof of the case $m=2$.

Clearly the former approach is easier than the latter, which however could be necessary if the naive approach fails for every twist, which could be the case.

## 3. Refined matching conditions

In this section we will perform an analysis, needed later, which is a generalization of the concepts of 1 -throws and 2 -throws considered in [3].

Suppose that a ( -1 )-curve $C$ lives on a component $P$ in a degeneration with two components $P$ and $F$ in the central fibre of a family $\mathcal{X} \rightarrow \mathbb{D}$, intersecting transversely along a curve $R$, and suppose we are given a line bundle $\mathcal{L}$ on $\mathcal{X}$. Suppose that the intersection number of $C$ with the restriction of $\mathcal{L}$ to $P$ is $-s$. Suppose in addition that $C$ meets transversely at $m$ points the double curve $R$.

For $m=1$ we have the 1 -throw considered in [3], which reveals that the appropriate matching conditions for a curve on $F$ to be a limit is that it must have a point of multiplicity $s$ at the intersection point of $C$ with $R$, not simply an intersection multiplicity $s$.

Now suppose that $m>1$. We blow up $C$ in the threefold $\mathcal{X} m$ times, thus obtaining a new threefold $\mathcal{X}^{\prime}$ and a new family $\mathcal{X}^{\prime} \rightarrow \mathbb{D}$. This blows up $F$ $m$ times at each of the $m$ intersection points of $C$ with $R$, for a total of $m^{2}$ blow-ups. We denote by $\bar{F}$ the resulting surface.

These blow-ups create $m$ ruled surfaces $Q_{m-1}, Q_{m-2}, \ldots, Q_{1}, Q_{0}$ which are stacked one on the other. In the central fiber of $\mathcal{X}^{\prime}, Q_{i}$ appears with multiplicity $m-i$, for $i=0, \ldots, m-1$. One checks that $Q_{i} \cong \mathbb{F}_{i}$, with non-positive section $B_{i}$ and disjoint non-negative section $S_{i}$; on $Q_{i}$ we have $B_{i}^{2}=-i, S_{i}^{2}=i$, and $S_{i} \sim B_{i}+i f$, where $f$ denotes as usual the fibre class and $\sim$ is the linear equivalence. $Q_{0}$ meets the surface $P$ in a section $B_{0}$ (equal to $C$ on $P$ ), with $B_{0}^{2}=0$. Each $Q_{i}$ meets $Q_{i+1}$ so that $S_{i}$ (on $Q_{i}$ ) is identified with $B_{i+1}$ (on $\left.Q_{i+1}\right)$. Each $Q_{i}$ also meets the other component $\bar{F}$ in $m$ fibers of the ruling, corresponding to the $m$ points where $C$ meets $R$.

The normal bundle of $Q_{0}$ in $\mathcal{X}^{\prime}$ is $(-1 / m)\left(B_{0}+(m-1) S_{0}+m f\right)=-B_{0}-f$. For $1<i<m-1$, the normal bundle of $Q_{i}$ in $\mathcal{X}^{\prime}$ is $(-1 /(m-i))((m-i+$ 1) $\left.B_{i}+(m-i-1) S_{i}+m f\right)=-2 B_{i}-(i+1) f$. For $i=m-1$, the normal bundle of $Q_{m-1}$ in $\mathcal{X}^{\prime}$ is $(-1)\left(2 B_{m-1}+m f\right)=-2 B_{m-1}-m f$.

When we pull back the bundle $\mathcal{L}$ to $\mathcal{X}^{\prime}$, this pull back $\mathcal{L}^{\prime}$ restricts to -sf on each $Q_{i}$. At this point we make the additional assumption that $s$ is divisible by $m$ : write $s=h m$. Twist $\mathcal{L}^{\prime}$ by $\mathcal{O}_{\mathcal{X}}\left(-h\left(\sum_{i=0}^{m-1}(m-i) Q_{i}\right)\right)$. Let us analyze the restriction of this new bundle on each component of the central fibre.

First we consider the surface $P$, on which the original curve $C$ sits. Since the only exceptional surface that meets $P$ is $Q_{0}$, we are twisting the restriction of the bundle on $P$ by $-h m Q_{0}=-s Q_{0}$, and since $Q_{0}$ restricts to $C$ on $P$ this removes $s C$ from the restriction of the bundle on $P$, and then this restriction is trivial on $C$.

The restriction to $Q_{0}$ is

$$
-s f-\left.h m Q_{0}\right|_{Q_{0}}-\left.h(m-1) Q_{1}\right|_{Q_{0}}=-s f-s\left(-B_{0}-f\right)-h(m-1) S_{0}=h B_{0}
$$

For $1<i<m-1$, the restriction to $Q_{i}$ is

$$
\begin{aligned}
-s f-\left.h(m-i+1) Q_{i-1}\right|_{Q_{i}}-\left.h(m-i) Q_{i}\right|_{Q_{i}}-\left.h(m-i-1) Q_{i+1}\right|_{Q_{i}} & = \\
-s f-h(m-i+1) B_{i}-h(m-i)\left(-2 B_{i}-(i+1) f\right)-h(m-i-1) S_{i} & =0
\end{aligned}
$$

Finally for $i=m-1$ the restriction to $Q_{m-1}$ is

$$
-s f-\left.2 h Q_{m-2}\right|_{Q_{m-1}}-\left.h Q_{m-1}\right|_{Q_{m-1}}=0
$$

The above analysis shows that the bundle is now trivial on $Q_{m-1}, Q_{m-2}, \ldots, Q_{1}$, and non-trivial only on $Q_{0}$, where it consists of $h B_{0}$, i.e., $h$ horizontal sections. Therefore the matching divisor on $\bar{F}$ does not meet any of the exceptional divisors of the first $m-1$ blow-ups, and meets only the last ones $h$ times at each of the $m$ points. Moreover, there is a correspondence on the divisors on the final exceptional curves, namely they must all agree with $h$ horizontal sections. In other words, any one of these intersections determines all the other $m-1$ ones. This behaviour of the curves on $\bar{F}$ means that the curve on $F$ must have at each of the $m$ points of the intersection of $C$ and $R, m$ infinitely near points of multiplicity $h$ along $R$. We denote this phenomenon by $\left[h^{m}\right]_{R}$. Hence the matching conditions for the curves on $F$ can be written as $\left(\left[h^{m}\right]_{R}\right)^{m}$, plus the correspondence.

We can summarize what we proved in this section in the following statement:
Proposition 3.1. Suppose we have a semistable degeneration of surfaces $\pi$ : $\mathcal{X} \rightarrow \mathbb{D}$ over a disc $\mathbb{D}$ (i.e., $\mathcal{X}$ is smooth, all fibres of $\pi$ are smooth except perhaps for the one over 0 , that has normal crossings) and a line bundle $\mathcal{L}$ on $\mathcal{X}$ which restricts to line bundles on every component of the central fibre. Let $P$ be $a$
component of the central fibre containing a (-1)-curve $C$ which is not a double curve and intersects the double curve $R$ transversely at $m$ points $p_{1}, \ldots, p_{m}$ that are not triple points of the central fibre. Suppose that $C \cdot \mathcal{L}=-h m$, with $h>0$. Then any curve on the central fibre that is a limit of a curve in the general fibre in the linear system determined by the restriction of $\mathcal{L}$, must satisfy the following conditions: for every point $p_{i}$ the curve on the component different from $P$ has the singularity of type $\left[h^{m}\right]_{R}$ at $p_{i}$ and the final $h$ infinitely near points to the $p_{i}$ 's of order $m$ correspond in the sense described above.

## 4. The proof of the case $m=2$

We focus in this section on the case $m=2$. We will consider the degeneration described in Section 2 and first we want to describe the distribution of the multiplicities and the limit linear systems on $P$ and $F$. For convenience we set $n=d-1$ and note that there are $4 n$ simple points in the case $m=2$, which we will distribute evenly among $P$ and $F$. So the limit linear systems will be

$$
\mathcal{L}_{P}:=\mathcal{L}_{k n+t}\left(k(n-1), k^{2 n}\right), \quad \mathcal{L}_{F}:=\mathcal{L}_{k(n+1)}\left(k n+t, k^{2 n}\right)
$$

with $t$ the twisting parameter.
In order to prove Theorem 1.1(a), for $m=2$, we will use the refined matching approach. This requires that we prove that for any twisting parameter $t$ there is no limit curve satisfying the refined matching conditions stated in Proposition 3.1.

Consider the curve class (useful on both $P$ and $F$ ) equal to $\mathcal{L}_{n}\left(n-1,1^{2 n}\right)$. We note that this linear system is of dimension 0 and consists of a unique (-1)-curve $C$.

The linear system $\mathcal{L}_{k n}\left(k(n-1), k^{2 n}\right)$ is equal to $|k C|$, and has dimension 0 . Therefore if $t<0$, then $\mathcal{L}_{P}$ is empty. Hence we may assume $t \geqslant 0$.

Let us analyze $\mathcal{L}_{F}$. The lines through the first point and through any one of the other $2 n$ points split off with multiplicity $t$. The residual system has the form $\mathcal{L}_{k(n+1)-2 n t}\left(k n+t-2 t n,(k-t)^{2 n}\right)$. Now we intersect this system with $C$ and get $-t(n-1)$. So $C$ splits $t(n-1)$ times. The further residual system is $\mathcal{L}_{F}^{\prime}=\mathcal{L}_{(n+1)(k-t n)}\left(n(k-t n),(k-t n)^{2 n}\right)$. For $\mathcal{L}_{F}^{\prime}$ to be effective, one needs $t \leqslant k / n$, which we will assume from now on. A sequence of $n$ quadratic transformations (each based at the first point and at two of the $2 n$ points) brings $\mathcal{L}_{F}^{\prime}$ to the complete linear system $\mathcal{L}_{k-t n}$.

As for $\mathcal{L}_{P}$, one sees that $C$ splits off with multiplicity $k-t n$ and the residual system is $\mathcal{L}_{P}^{\prime}=\mathcal{L}_{t\left(n^{2}+1\right)}\left(\operatorname{tn}(n-1),(t n)^{2 n}\right)$. A sequence of $n$ quadratic transformations (each based at the first point and at two of the $2 n$ points) brings this system to $\mathcal{L}_{t(n+1)}\left(t^{2 n}\right)$.

Let us see what the refined matching implies on $\mathcal{L}_{P}^{\prime}$ or rather on its Cremona transform $\mathcal{L}_{t(n+1)}\left(t^{2 n}\right)$.

Each of the $2 n$ lines splitting $t$ times from $\mathcal{L}_{F}$ are $(-1)$-curves meeting the double curve $R$ once. Hence in the notation of the previous section $m=1$ and $s=h=t$ and therefore we are imposing $2 n$ points of multiplicity $t$ to the linear system $\mathcal{L}_{t(n+1)}\left(t^{2 n}\right)$. These points are located along the curve $T$, the Cremona image of $R$ on $P$, which is easy to see to be equal to a curve of degree $n+1$, with a point of multiplicity $n$.

Also the curve $C$ splits $t(n-1)$ times from $\mathcal{L}_{F}$ and meets the double curve $R$ transversely at $n-1$ points. In the notation of the previous section $m=n-1$, $s=t(n-1)=t m$ hence $h=t$. Therefore we are imposing to $\mathcal{L}_{t(n+1)}\left(t^{2 n}\right)$ also the multiple points $\left(\left[t^{n-1}\right]_{T}\right)^{n-1}$, plus the correspondence.

Eventually the resulting system on $P$ is Cremona equivalent to the system $\mathcal{L}$ of plane curves of degree $t(n+1)$, with $(n+1)^{2}$ points of multiplicity $t$, plus the correspondence. These $(n+1)^{2}$ points are distributed in $2 n$ general points, $2 n$ general points on $T$, and $n-1$ general points of type $\left[t^{n-1}\right]_{T}$.

Assume now $t>0$. We want to prove that $\mathcal{L}$ is empty and therefore $\mathcal{L}_{P}$ is empty. To prove this, we need the following:

Lemma 4.1. For any $t>0$ the linear system of plane curves of degree $t(n+1)$ with $n-1$ general points of type $\left[t^{n-1}\right]_{T}$, with $2 n$ general points of multiplicity $t$ on $T$ and $2 n$ additional general points of multiplicity $t$ consists of at most one element.

Proof. We specialize the configuration of the imposed multiple points to $n-1$ general points of type $\left[t^{n-1}\right]_{T}$, and with $4 n$ more general points of multiplicity $t$ on $T$. This is a total $(n+1)^{2}$ points (some of them are infinitely near) forming a divisor $D$ on $T$ supported on the smooth locus of $T$. By generality, for no positive integer $t, t D$ belongs to $\left|\mathcal{O}_{T}(t(n+1))\right|$. So any curve of degree $t(n+1)$ with the above multiple points on $T$ must contain $T$. To the residual curve we may apply the same argument, so $T$ recursively splits; by induction we conclude that the only possible member of the system is $t T$. This implies the assertion.

To prove that $\mathcal{L}$ is empty, we notice that the possible unique curve satisfying the multiplicity conditions imposed on $\mathcal{L}$ (see Lemma 4.1) will not satisfy the required correspondence as soon as $m=n-1 \geqslant 2$, i.e., $n \geqslant 3$, hence $\mathcal{L}_{P}$ is empty.

Finally we have to deal with the case $t=0$. In this case we take $k=$ $h n$ and $\mathcal{L}_{P}$ consists of the unique curve $h n C$. Now $C$ is a $(-1)$-curve that intersects $R$ transversely at $n$ points. Therefore, in the notation of Section 3, we have $m=n, s=k$. So the refined matching implies that we eventually have to impose to $\mathcal{L}_{F}$, or rather to its Cremona transform $\mathcal{L}_{h n}, n$ points of type $\left[h^{n}\right]_{T^{\prime}}$, where $T^{\prime}$ is the Cremona image of the double curve $R$, plus the correspondence. Note that $T^{\prime}$ is a curve of degree $n$ with a point of multiplicity $n-1$.

By the same argument as in Lemma 4.1, we see that the only curve verifying all the above conditions is $h T^{\prime}$, so in the original analysis it is $h R$ plus some exceptional curves which appear in the Cremona transformation. However, as we saw in the refined matching analysis, on $P$ the bundle is now trivial, so the corresponding section on $P$ has to vanish identically on $P$. This shows that there is no limit curve in the case $t=0$ either.

Eventually we have seen that for any twisting parameter $t$ there is no limiting curve verifying the refined matching conditions, finishing the proof of Theorem 1.1(a) for $m=2$ and $d \geqslant 4$.

## 5. The proof for $m>2$

In this section we will complete the proof of Theorem 1.1(a) in the case $m \geqslant 3$, arguing by induction on $m$ (the case $m=2$ for all $d \geqslant 4$ is the starting case of the induction). For this we will again use the degeneration as in Section 2 and the naive matching approach will be sufficient.

Let us describe the limit linear systems we will use, i.e.,

$$
\begin{gathered}
L_{P}=\mathcal{L}_{k(d-2)}\left(k(d-m), k^{(m-2)(2 d-m-2)}\right)=k \xi_{d-2, m-2} \\
L_{F}=\mathcal{L}_{k d}\left(k(d-2), k^{4 d-4}\right)=k \xi_{d, 2}
\end{gathered}
$$

By the $m=2$ case, $L_{F}$ is empty, and therefore also the kernel system is empty. Hence it suffices to show that the kernel system on $P$ is also empty.

First consider the case $m=3$. Then, by Theorem 1.1(b), $L_{P}$ is composed with a pencil of rational curves, and the kernel system is empty because it consists of the members of $L_{P}$ that vanish along the double curve $R$, which is a general line on $P$. This proves the $m=3$ case for all $d \geqslant 5$ (remember that $m \leqslant d-2$ ).

Next assume $m \geqslant 4$, and therefore, since $m \leqslant d-2$, we have $d \geqslant 6$. Then, by induction, $L_{P}$ is empty and hence also the kernel system is empty, finishing the proof of Theorem 1.1.

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# Geometry of dependency equilibria 

Irem Portakal and Bernd Sturmfels


#### Abstract

An n-person game is specified by $n$ tensors of the same format. We view its equilibria as points in that tensor space. Dependency equilibria are defined by linear constraints on conditional probabilities, and thus by determinantal quadrics in the tensor entries. These equations cut out the Spohn variety, named after the philosopher who introduced dependency equilibria. The Nash equilibria among these are the tensors of rank one. We study the real algebraic geometry of the Spohn variety. This variety is rational, except for $2 \times 2$ games, when it is an elliptic curve. For $3 \times 2$ games, it is a del Pezzo surface of degree two. We characterize the payoff regions and their boundaries using oriented matroids, and we develop the connection to Bayesian networks in statistics.


Keywords: Nash equilibria, dependency equilibria, Spohn variety, conditional independence models.
MS Classification 2020: 14A10, 14Q30, 62R01, 91A06, 91A80.

## 1. Introduction

The geometry of Nash equilibria has been a topic of considerable interest in economics, mathematics and computer science. It is known, thanks to Datta's Universality Theorem [6], that the set of Nash equilibria can be an essentially arbitrary semialgebraic set. Yet, a game with generic payoff tables has only finitely many Nash equilibria, with tight bounds known for their number [11]. They can be found with the tools of computational algebraic geometry.

For many games one encounters Nash equilibria with undesirable or counterintuitive properties. This issue has been a concern not just in the economics literature, but also in philosophy. Several authors proposed more inclusive notions of equilibria. One of these is the concept of correlated equilibria, due to Aumann [1]. In this concept, one augments the original game with a coordination device which allows players to coordinate actions (i.e. a joint probability distribution). These equilibria form a convex polytope in tensor space, studied in $[4,14]$, with Nash equilibria being precisely the rank one tensors.

In this article, we examine another inclusive notion of equilibria, introduced by a prominent philosopher, Wolfgang Spohn, in his articles [19, 20]. Spohn's notion of dependency equilibria leads to interesting structures in non-
linear algebra [12]. Unlike the polyhedral setting of correlated equilibria, the characterization of dependency equilibria requires nonlinear polynomials, even for two-player games. This is the reason why they are interesting for us.

Spohn offers the following warning about the nonlinear algebra that arises in his approach: The computation of dependency equilibria seems to be a messy business. Obviously it requires one to solve quadratic equations in two-person games, and the more persons, the higher the order of the polynomials we become entangled with. All linear ease is lost. Therefore, I cannot offer a well developed theory of dependency equilibria [20, page 779, Section 3].

This paper lays the foundations for the desired theory, by introducing novel algebraic varieties in tensor spaces. The bemoaned loss of linear ease is our journey's point of departure.

It is useful to think of this article as a case study in algebraic statistics [3, 22]. In that field one examines statistical models for $n$ discrete random variables. Such a model is a semialgebraic set whose points are positive tensors whose entries sum to one. These represent joint probability distributions, and the statistical task is to identify points that best explain some given data set. To address such an optimization problem, it is advantageous to relax the constraint that tensors are real and positive. Thus, one replaces the model by its Zariski closure in a complex projective space, and one studies algebro-geometric features - such as dimension, degree, equations, decomposition, and singularities - of these varieties.

The statistical model in this article is the set of dependency equilibria of an $n$-person game in normal form. These equilibria are real positive tensors whose entries sum to one. Relaxing the reality constraints yields an algebraic variety in complex projective space. This is called the Spohn variety of the game, in recognition of the fundamental work in [19, 20].

Our presentation is organized as follows. In Section 2 we review the basics on $n$-player games in normal form, and we present the equations that define dependency equilibria. After clearing denominators, these are expressed as the $2 \times 2$ minors of $n$ matrices whose entries are linear forms in the entries of $P$. Small cases are worked out in Examples 2.1, 2.2, 2.3 and 2.4.

The Spohn variety $\mathcal{V}_{X}$ of a normal form game $X$ is formally introduced in Section 3. We determine its dimension and degree in Theorem 3.2. The intersection of $\mathcal{V}_{X}$ with the Segre variety recovers the Nash equilibria. Theorem 3.4 shows that $\mathcal{V}_{X}$ is generally rational, with an explicit rational parametrization. Example 3.6 covers Del Pezzo surfaces of degree two.

Section 4 offers a detailed study of the dependency equilibria for $2 \times 2$ matrix games. This case is an exceptional case because the Spohn variety $\mathcal{V}_{X}$ is not rational. It is the intersection of two quadrics in $\mathbb{P}^{3}$, hence an elliptic curve, when the payoff matrices are generic. A formula for the j -invariant is given in Proposition 4.2. The real picture is determined in Theorem 4.4.

Section 5 concerns the payoff region $\mathcal{P}_{X}$. This is a semialgebraic subset of $\mathbb{R}^{n}$, visualized in Figures 2 and 3 . The points of $\mathcal{P}_{X}$ are the expected utilities of positive points on $\mathcal{V}_{X}$. Theorem 5.5 identifies that region in the oriented matroid stratification given by the Konstanz matrix $K_{X}(x)$. Its algebraic boundaries are determinantal hypersurfaces, such as the K3 surfaces in Example 5.7. These offer an algebraic representation for Pareto optimal equilibria.

Section 6 develops a perspective that offers dimensionality reduction and a connection to data analysis. Namely, we consider conditional independence models, in the sense of algebraic statistics [3, 22]. These models are represented by projective varieties. We focus on the case of Bayesian networks [8]. Their importance for dependency equilibria was already envisioned by Spohn in [19, Section 3]. This offers many opportunities for future research.

## 2. Games, Tensors and Equilibria

We work in the setting of normal form games, using the notation fixed in [21, Section 6.3]. Our game has $n$ players, labeled as $1,2, \ldots, n$. The $i$ th player can select from $d_{i}$ pure strategies. This set of pure strategies is taken to be $\left[d_{i}\right]=$ $\left\{1,2, \ldots, d_{i}\right\}$. The game is specified by $n$ payoff tables $X^{(1)}, X^{(2)}, \ldots, X^{(n)}$. Each payoff table is a tensor of format $d_{1} \times d_{2} \times \cdots \times d_{n}$ whose entries are arbitrary real numbers. The entry $X_{j_{1} j_{2} \cdots j_{n}}^{(i)} \in \mathbb{R}$ represents the payoff for player $i$ if player 1 chooses pure strategy $j_{1}$, player 2 chooses pure strategy $j_{2}$, etc. These choices are to be understood probabilistically. Think of the $n$ players as random variables. The $i$ th random variable has the state space $\left[d_{i}\right]$. The players collectively choose a mixed strategy, which is a joint probability distribution $P$. More precisely, $P$ is a tensor of format $d_{1} \times d_{2} \times \cdots \times d_{n}$ whose entries are positive reals that sum to 1 . The entry $p_{j_{1} j_{2} \cdots j_{n}}$ is the probability that player 1 chooses pure strategy $j_{1}$, player 2 chooses pure strategy $j_{2}$, etc.

We write $V=\mathbb{R}^{d_{1} \times d_{2} \times \cdots \times d_{n}}$ for the real vector space of all tensors. Let $\mathbb{P}(V)$ denote the corresponding projective space, and let $\Delta$ be the open simplex of positive real points in $\mathbb{P}(V)$. The set of equilibria of our game is a subset of $\Delta$, and we are interested in its Zariski closure in $\mathbb{P}(V)$. The classical theory of Nash equilibria arises through the Segre variety $\mathbb{P}^{d_{1}-1} \times \mathbb{P}^{d_{2}-1} \times \cdots \times \mathbb{P}^{d_{n}-1}$ whose points are the tensors of rank one in $\mathbb{P}(V)$. Namely, the entries of a rank one tensor $P$ factor into the decision variables of [21, Section 6.3] as follows:

$$
p_{j_{1} j_{2} \cdots j_{n}}=\pi_{j_{1}}^{(1)} \cdot \pi_{j_{2}}^{(2)} \cdot \ldots \cdot \pi_{j_{n}}^{(n)}
$$

Here $\pi_{j_{i}}^{(i)}$ represents the probability that player $i$ unilaterally selects pure strategy $j_{i}$. In the study of totally mixed Nash equilibria, these quantities are
positive reals, and they satisfy

$$
\begin{equation*}
\pi_{1}^{(i)}+\pi_{2}^{(i)}+\cdots+\pi_{d_{i}}^{(i)}=1 \tag{1}
\end{equation*}
$$

However, in what follows the $n$ players are not independent. We view them as acting together. Their collective choice of a mixed strategy is thus a tensor $P$ which need not have rank 1 .

Consider two players with binary choices, so $n=d_{1}=d_{2}=2$. Here $V=\mathbb{R}^{2 \times 2}$ is a four-dimensional vector space, and $P(V)=\mathbb{P}^{3}$ is the projective space whose points are $2 \times 2$ matrices up to scaling. A game is specified by two matrices $X^{(1)}$ and $X^{(2)}$ in $V$. The two players collectively choose a joint probability distribution $P$ for two binary random variables. Thus, they choose a positive matrix $P=\left[\begin{array}{ll}p_{11} & p_{12} \\ p_{21} & p_{22}\end{array}\right]$ whose entries sum to one, i.e. $P \in \Delta$.
Example 2.1 (Bach or Stravinsky). A couple decides which of two concerts to attend. The payoff matrices indicate their preferences among composers, Bach $=1$ or Stravinsky $=2$ :

$$
X^{(1)}=\left[\begin{array}{ll}
3 & 0  \tag{2}\\
0 & 2
\end{array}\right] \quad \text { and } \quad X^{(2)}=\left[\begin{array}{ll}
2 & 0 \\
0 & 3
\end{array}\right] .
$$

In texts on game theory, this is called a bimatrix game. The two payoff matrices are often written in a combined table. For the game (2), the combined table looks as follows:

Player 2


Different entries are used in [20, Section 3]. We refer to that source for further examples. The four pure choices $\mathrm{BB}, \mathrm{BS}, \mathrm{SB}$ and SS label the vertices of the tetrahedron in Figure 1. In the game, the couple selects a mixed strategy $P$, which is a point in that tetrahedron.

Returning to our general set-up, we consider the expected payoff for the $i$ th player. By definition, this is the dot product of the tensors $X^{(i)}$ and $P$. In symbols, the expected payoff is

$$
\begin{equation*}
P X^{(i)}=\sum_{j_{1}=1}^{d_{1}} \sum_{j_{2}=1}^{d_{2}} \cdots \sum_{j_{n}=1}^{d_{n}} p_{j_{1} j_{2} \cdots j_{n}} X_{j_{1} j_{2} \cdots j_{n}}^{(i)} \tag{3}
\end{equation*}
$$

Player $i$ desires this quantity to be as large of possible. Aumann's correlated equilibria [1] are choices of $P$ where no player can raise their expected payoff
by changing their strategy or breaking their part of the agreed joint probability distribution while assuming that the other players adhere to their own recommendations. The set of correlated equilibria is a convex polytope inside the simplex $\Delta$. Its combinatorial structure is studied in [4, 14].

In Spohn's theory [20], expected payoff is replaced by conditional expected payoff. We focus on the payoff expected by player $i$ conditioned on player $i$ having fixed pure strategy $k \in\left[d_{i}\right]$. In precise mathematical terms, the conditional expected payoff is the ratio of two linear forms in the entries of $P$, each of which has $d_{1} \cdots d_{i-1} d_{i+1} \cdots d_{n}$ summands. The numerator is the subsum of (3) given by all summands with $j_{i}=k$. The denominator is the sum of all probabilities $p_{j_{1} j_{2} \cdots j_{n}}$ where $j_{i}=k$. In algebraic statistics texts, this is denoted $p_{+\cdots+k+\cdots+}$.

Here is now the key definition due to Spohn [19, 20]. Consider the game given by the tuple $X=\left(X^{(1)}, X^{(2)}, \ldots, X^{(n)}\right)$. A tensor $P$ in $\Delta$ is a dependency equilibrium for $X$ if the conditional expected payoff of each player $i$ does not depend on player $i$ 's choice $k$. In symbols, this definition says that the following equations hold, for all $i \in[n]$ and all $k, k^{\prime} \in\left[d_{i}\right]$ :

$$
\begin{align*}
\sum_{j_{1}=1}^{d_{1}} \cdots \widehat{\sum_{i}=1} \widehat{d_{i}} & \sum_{j_{n}=1}^{d_{n}} X_{j_{1} \cdots k \cdots j_{n}}^{(i)} \frac{p_{j_{1} \cdots k \cdots j_{n}}}{p_{+\cdots+k+\cdots+}} \\
& =\sum_{j_{1}=1}^{d_{1}} \cdots \widehat{\sum_{j_{i}=1}^{d_{i}}} \cdots \sum_{j_{n}=1}^{d_{n}} X_{j_{1} \cdots k^{\prime} \cdots j_{n}}^{(i)} \frac{p_{j_{1} \cdots k^{\prime} \cdots j_{n}}}{p_{+\cdots+k^{\prime}+\cdots+}} \tag{4}
\end{align*}
$$

Thus, dependency equilibria are defined by certain equalities among ratios of linear forms.

One issue with this definition is that $p_{+\cdots+k+\cdots+}$ might be zero. Spohn calls this a "technical flaw" [20, Section 2], and he suggests a fix by taking limits to the boundary of $\Delta$. From the algebraic statistics perspective, this is not a flaw but a feature. Many models are defined by constraints on strictly positive probabilities. Possible extensions to the boundary are studied using the technique of primary decomposition [22, Section 4.3]. We here disregard boundary phenomena since $\Delta$ is the open simplex. This allows us to divide by $p_{+\cdots+k+\cdots+}$.

We have argued that clearing denominators in (4) does not change the solution sets of interest. Thus we can write our equations as $2 \times 2$ determinants of linear forms in the entries of $P$. We define a matrix $M_{i}=M_{i}(P)$ with $d_{i}$ rows and two columns as follows. The $k$ th row of $M_{i}$ consists of the denominator
and the numerator of the ratio on the left of (4):

$$
M_{i}=M_{i}(P):=\left[\begin{array}{cc}
\vdots & \vdots  \tag{5}\\
p_{+\cdots+k+\cdots+} & \widehat{\sum_{j_{1}=1}^{d_{1}} \cdots} \begin{array}{c}
\widehat{d_{i}}
\end{array} \cdots \sum_{j_{i}=1}^{d_{n}} X_{j_{n}=1}^{(i)}{ }_{j_{1} \cdots k \cdots j_{n}} p_{j_{1} \cdots k \cdots j_{n}} \\
\vdots & \vdots
\end{array}\right]
$$

Dependency equilibria for $X$ are the points $P \in \Delta$ for which each $M_{i}$ has rank one. If $n$ is small then we simplify our notation by using letters $a, b, c$ for the tensors $X^{(1)}, X^{(2)}, X^{(3)}$.

Example $2.2(2 \times 2$ games $)$. Let $n=d_{1}=d_{2}=2$ and $a_{i j}, b_{i j} \in \mathbb{R}$. The matrices in (5) are

$$
\begin{aligned}
M_{1} & =\left[\begin{array}{ll}
p_{11}+p_{12} & a_{11} p_{11}+a_{12} p_{12} \\
p_{21}+p_{22} & a_{21} p_{21}+a_{22} p_{22}
\end{array}\right], \\
M_{2} & =\left[\begin{array}{ll}
p_{11}+p_{21} & b_{11} p_{11}+b_{21} p_{21} \\
p_{12}+p_{22} & b_{12} p_{12}+b_{22} p_{22}
\end{array}\right] .
\end{aligned}
$$

The dependency equilibria are solutions in $\Delta$ to the equations $\operatorname{det}\left(M_{1}\right)=$ $\operatorname{det}\left(M_{2}\right)=0$.
Example $2.3(2 \times 2 \times 2$ games). Consider a game with three players who have binary choices, i.e. $n=3$ and $d_{1}=d_{2}=d_{3}=2$. In [21, Section 6.1] the players are called Adam, Bob and Carl, and their payoff tables are $X^{(1)}=\left(a_{i j k}\right)$, $X^{(2)}=\left(b_{i j k}\right)$ and $X^{(3)}=\left(c_{i j k}\right)$. Dependency equilibria are $2 \times 2 \times 2$ tensors $P=\left(p_{i j k}\right)$ such that these three $2 \times 2$ matrices have rank $\leq 1$ :

$$
\begin{aligned}
M_{1} & =\left[\begin{array}{ll}
p_{111}+p_{112}+p_{121}+p_{122} & a_{111} p_{111}+a_{112} p_{112}+a_{121} p_{121}+a_{122} p_{122} \\
p_{211}+p_{212}+p_{221}+p_{222} & a_{211} p_{211}+a_{212} p_{212}+a_{221} p_{221}+a_{222} p_{222}
\end{array}\right], \\
M_{2} & =\left[\begin{array}{ll}
p_{111}+p_{112}+p_{211}+p_{212} & b_{111} p_{111}+b_{112} p_{112}+b_{211} p_{211}+b_{212} p_{212} \\
p_{121}+p_{122}+p_{221}+p_{222} & b_{121} p_{121}+b_{122} p_{122}+b_{221} p_{221}+b_{222} p_{222}
\end{array}\right], \\
M_{3} & =\left[\begin{array}{ll}
p_{111}+p_{121}+p_{211}+p_{221} & c_{111} p_{111}+c_{121} p_{121}+c_{211} p_{211}+c_{221} p_{221} \\
p_{112}+p_{122}+p_{212}+p_{222} & c_{112} p_{112}+c_{122} p_{122}+c_{212} p_{212}+c_{222} p_{222}
\end{array}\right] .
\end{aligned}
$$

If $X=(A, B, C)$ is generic then their determinants are quadrics that intersect transversally. This defines an irreducible variety $\mathcal{V}_{X}$ of dimension 4 and degree 8 in the tensor space $\mathbb{P}^{7}$. We now intersect $\mathcal{V}_{X}$ with the Segre variety $\mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{1}$ of rank one tensors in $\mathbb{P}^{7}$. Setting $\alpha=\pi_{1}^{(1)}, \beta=\pi_{1}^{(2)}$, and $\gamma=\pi_{1}^{(3)}$, we use the
following parametrization for the Segre variety:

$$
\begin{array}{ll}
p_{111}=\alpha \beta \gamma, & p_{211}=(1-\alpha) \beta \gamma, \\
p_{112}=\alpha \beta(1-\gamma), & p_{212}=(1-\alpha) \beta(1-\gamma), \\
p_{121}=\alpha(1-\beta) \gamma, & p_{221}=(1-\alpha)(1-\beta) \gamma, \\
p_{122}=\alpha(1-\beta)(1-\gamma), & p_{222}=(1-\alpha)(1-\beta)(1-\gamma) .
\end{array}
$$

After this substitution, and after removing extraneous factors, the three $2 \times 2$ determinants are precisely the three bilinear polynomials exhibited in [21, Corollary 6.3]. These equations have two solutions in $\mathbb{P}(V)$, so there can be two distinct totally mixed Nash equilibria.

For any game $X$, the set of dependency equilibria contains the set of Nash equilibria. The latter is usually finite. It is instructive to compare these objects for some examples from game theory text books. Some of these games are not presented in normal form, but in extensive form. It takes practise to derive the payoff tensors $X^{(i)}$ from extensive forms.
Example 2.4 (Centipede Game). This is a famous class of two-person games due to Robert Rosenthal [17]. They are presented in extensive form, by graphs that looks like a centipede. We discuss an instance with $d_{1}=3, d_{2}=2$. Our game is presented by the following graph:


The two players chose sequentially between going right $r$ or down $d$. A down choice ends the game. In our instance, the game also ends after three right choices. The payoffs for the four outcomes $d, r d, r r d$ or $r r r d$ are the labels of the leaves. This translates into a $3 \times 2$-game:


This table gives the $3 \times 2$ payoff matrices $X^{(1)}$ and $X^{(2)}$. Similarly to the Prisoner's Dilemma, the Nash equilibrium of the centipede game is not Pareto efficient. To compute the dependency equilibria, we consider four quadrics in
six unknowns, namely the $2 \times 2$ minors of the matrices $M_{1}$ and $M_{2}$. The ideal they generate is the intersection of two prime ideals:

$$
\begin{aligned}
\left\langle p_{31}-p_{32}, p_{21}-2 p_{22},\right. \\
\left.p_{11} p_{22}-4 p_{12} p_{22}-2 p_{22}^{2}+4 p_{11} p_{32}-2 p_{12} p_{32}+3 p_{22} p_{32}+2 p_{32}^{2}\right\rangle \\
\cap \quad\left\langle p_{11}+p_{12}, 3 p_{22} p_{31}-2 p_{21} p_{32}+p_{22} p_{32}\right. \\
\left.6 p_{12} p_{21}+3 p_{12} p_{22}+3 p_{21} p_{22}+6 p_{12} p_{31}+12 p_{12} p_{32}-4 p_{21} p_{32}-p_{22} p_{32}-6 p_{31} p_{32}\right\rangle .
\end{aligned}
$$

The second component, a singular quartic surface in a hyperplane in $\mathbb{P}^{5}$, is disjoint from $\Delta$. The first component is a hyperboloid in a 3 -space $\mathbb{P}^{3}$ which intersects the open simplex $\Delta$. That intersection is the set of dependency equilibria. There are no Nash equilbria in $\Delta$.

## 3. The Spohn variety

In this section we work in the complex projective space $\mathbb{P}(V)$ of $d_{1} \times \cdots \times d_{n}$ tensors. We write $\mathcal{V}_{X}$ for the subvariety of $\mathbb{P}(V)$ that is given by requiring $M_{1}, \ldots, M_{n}$ to have rank one. We call $\mathcal{V}_{X}$ the Spohn variety of the game $X$. Thus $\mathcal{V}_{X}$ is defined by $\sum_{i=1}^{n}\binom{d_{i}}{2}$ quadratic forms in $\prod_{i=1}^{n} d_{i}$ unknowns $p_{j_{1} \cdots j_{n}}$, namely the $2 \times 2$ minors of the $n$ matrices $M_{i}$ in (5).

We already saw several examples in the previous section. For three-person games with binary choices (Example 2.3), the Spohn variety $\mathcal{V}_{X}$ is a fourfold in $\mathbb{P}^{7}$. For the centipede game (Example 2.4), the Spohn variety $\mathcal{V}_{X}$ is a surface in $\mathbb{P}^{8}$. We next consider a $2 \times 2$ game .
Example 3.1 (Bach or Stravinsky). For the game in Example 2.1, we consider the matrices

$$
M_{1}=\left[\begin{array}{ll}
p_{11}+p_{12} & 3 p_{11} \\
p_{21}+p_{22} & 2 p_{22}
\end{array}\right] \quad \text { and } \quad M_{2}=\left[\begin{array}{ll}
p_{11}+p_{21} & 2 p_{11} \\
p_{12}+p_{22} & 3 p_{22}
\end{array}\right]
$$

The ideal generated by $\operatorname{det}\left(M_{1}\right)$ and $\operatorname{det}\left(M_{2}\right)$ is the intersection of three prime ideals:
$\left\langle p_{11}, p_{22}\right\rangle \cap\left\langle 2 p_{12}+3 p_{21}, 3 p_{11} p_{21}+p_{11} p_{22}+3 p_{21} p_{22}\right\rangle \cap\left\langle 2 p_{12}-3 p_{21}-p_{22}, p_{11}-p_{22}\right\rangle$.
This shows that the Spohn variety $\mathcal{V}_{X}$ is the reduced union of three curves, two lines and one conic, shown in Figure 1. Only one component, namely a line, intersects the open tetrahedron $\Delta$. This game has two pure Nash equilibria $(1,0,0,0),(0,0,0,1)$ and one totally mixed Nash equilibrium $\left(\frac{6}{25}, \frac{9}{25}, \frac{4}{25}, \frac{6}{25}\right)$. The latter is the positive point of rank one on the curve $\mathcal{V}_{X}$.

The curve in Figure 1 has multiple components because the payoff matrices in (2) are very special. If we perturb the matrix entries, then the resulting


Figure 1: The Spohn variety is a reducible curve of degree four in $\mathbb{P}^{3}$. It has three components but only one passes through the tetrahedron. The figure also shows the Segre surface in the tetrahedron. The curve and the surface meet in one point, namely the totally mixed Nash equilibrium.
curve $\mathcal{V}_{X}$ will be smooth and irreducible in $\mathbb{P}^{3}$. As we shall see, the analogous result holds for games of arbitrary size.

We now present our first result in this section. It summarizes the essential geometric features of Spohn varieties, and it shows how these varieties are related to the Nash equilibria.

Theorem 3.2. If the payoff tables $X$ are generic then the Spohn variety $\mathcal{V}_{X}$ is irreducible of codimension $d_{1}+d_{2}+\cdots+d_{n}-n$ and degree $d_{1} d_{2} \ldots d_{n}$. The intersection of $\mathcal{V}_{X}$ with the Segre variety in the open simplex $\Delta$ is precisely the set of totally mixed Nash equilibria for $X$.

Proof. Consider a generalized column of the $d_{i} \times 2$ matrix $M_{i}$, i.e. a linear combination of the columns of $M_{i}$ with coefficients $\lambda_{1}, \lambda_{2} \in \mathbb{R}$ that are not both zero. Since the payoff table $X^{(i)}$ is generic, every generalized column of $M_{i}$ consists of linearly independent linear forms. We know from [7, Theorem 6.4] that the ideal generated by the $2 \times 2$ minors of $M_{i}$ is prime of codimension $d_{i}-1$. Moreover, by [5, Proposition 2.15], the degree of this linear determinantal variety is $\binom{2+d_{i}-1-1}{d_{i}-1}=d_{i}$. Since the tensor $X^{(i)}$ is generic and its entries
occur only in $M_{i}$, the intersection of the $n$ varieties is transversal. Now, [18, Theorem 1.24] and Bézout's Theorem for dimensionally transverse intersections yield the first assertion.

The second assertion says that the totally mixed Nash equilibria are the dependency equilibria of rank one. Set $p_{j_{1} \ldots j_{n}}=\pi_{j_{1}}^{(1)} \cdots \pi_{j_{n}}^{(n)}$ with $\pi_{k}^{(i)}>0$ for $k \in\left[d_{i}\right]$ and $i \in[n]$. Suppose (1) holds. The dependency equilibria of rank one are defined by the $2 \times 2$ minors of the matrix

$$
\left[\begin{array}{ccc}
1 & \sum_{j_{1}=1}^{d_{1}} \cdots \widehat{\sum_{j_{i}=1}^{d_{i}}} \cdots \sum_{j_{n}=1}^{d_{n}} X_{j_{1} \cdots 1 \cdots j_{n}}^{(i)} \pi_{j_{1}}^{(1)} \cdots \pi_{j_{i-1}}^{(i-1)} \pi_{j_{i+1}}^{(i+1)} \cdots \pi_{j_{n}}^{(n)} \\
\vdots & \vdots \\
1 & \sum_{j_{1}=1}^{d_{1}} \cdots \widehat{\sum_{j_{i}=1}^{d_{i}}} \cdots \sum_{j_{n}=1}^{d_{n}} X_{j_{1} \cdots d_{i} \cdots j_{n}}^{(i)} \pi_{j_{1}}^{(1)} \cdots \pi_{j_{i-1}}^{(i-1)} \pi_{j_{i+1}}^{(i+1)} \cdots \pi_{j_{n}}^{(n)}
\end{array}\right] .
$$

We subtract the first row from the $k$ th row for all $k \in\left\{2, \ldots, d_{i}\right\}$. The $2 \times 2$ minors of the resulting matrix are the pairwise differences of the entries in the second column. These differences are precisely the $d_{i}-1$ multilinear equations exhibited in [21, Theorem 6.6].

The Spohn variety $\mathcal{V}_{X}$ is a high-dimensional projective variety associated with a game $X$. Each point $P$ on $\mathcal{V}_{X}$ is a tensor. We say that $P$ is a Nash point if that tensor has rank one. The positive Nash points in $\mathcal{V}_{X} \cap \Delta$ are the totally mixed Nash equilibria. Their number is given by the formula in [21, Section 6.4], namely it expressed as the mixed volume of certain products of simplices. That mixed volume is zero when the tensor format is too unbalanced.
Remark 3.3. A generic game $X$ has no Nash points unless

$$
\begin{equation*}
d_{i} \leq d_{1}+\cdots+d_{i-1}+d_{i+1}+\cdots+d_{n}-n+2 \quad \text { for } \quad i=1,2, \ldots, n \tag{6}
\end{equation*}
$$

Experts on tensor geometry recognize these inequalities from a result by Gel'fand, Kapranov and Zelevinsky on hyperdeterminants [9, Theorem 14.I.1.3]. Namely, the existence of Nash points for a given tensor format is equivalent to the hyperdeterminant being a hypersurface. In particular, two-person games have Nash points if and only if the matrix is square $\left(d_{1}=d_{2}\right)$.

We continue to assume that the payoff tables are generic. Then the following result holds.

Theorem 3.4. If $n=d_{1}=d_{2}=2$ then the Spohn variety $\mathcal{V}_{X}$ is an elliptic curve. In all other cases, the Spohn variety $\mathcal{V}_{X}$ is rational, represented by a map onto $\left(\mathbb{P}^{1}\right)^{n}$ with linear fibers.

Proof. We shall provide a parametrization of $\mathcal{V}_{X}$. Along the way, we shall see why the case $n=d_{1}=d_{2}=2$ is special. The entries of these $n$ matrices $M_{i}$ in (5) are linear forms in the entries $p_{j_{1} \cdots j_{n}}$ of the tensor $P$. Their coefficients depend linearly on the entries of $X$.

Consider the affine line whose coordinate $x_{i}=P X^{(i)}$ is the expected payoff (3) for player $i$. We embed this into a projective line $\mathbb{P}^{1}$ by setting $z_{i}=\left(x_{i}:-1\right)$. We call $\left(\mathbb{P}^{1}\right)^{n}$ the algebraic payoff space. Its homogeneous coordinates are $z=\left(z_{1}, z_{2}, \ldots, z_{n}\right)$. The algebraic payoff map is the following rational map from the Spohn variety to the algebraic payoff space:

$$
\begin{equation*}
\pi_{X}: \mathcal{V}_{X} \rightarrow\left(\mathbb{P}^{1}\right)^{n}, P \mapsto\left(\operatorname{ker}\left(M_{1}(P)\right), \operatorname{ker}\left(M_{2}(P)\right), \ldots, \operatorname{ker}\left(M_{n}(P)\right)\right) \tag{7}
\end{equation*}
$$

The name "payoff map" is justified as follows. Suppose that $P$ is a dependency equilibrium, so $P$ is a point in the set $\mathcal{V}_{X} \cap \Delta$. The expected payoff $x_{i}$ for the $i$ th player satisfies

$$
M_{i}(P) \cdot\left[\begin{array}{r}
x_{i}  \tag{8}\\
-1
\end{array}\right]=0 \quad \text { for } \quad i=1,2, \ldots, n
$$

To see this, augment the rank one matrix $M_{i}(P)$ by its row of column sums, like in (18). Equation (8) implies $\pi_{X}(P)=\left(\left(x_{1}:-1\right), \ldots,\left(x_{n}:-1\right)\right)$. We now write (8) on $\left(\mathbb{P}^{1}\right)^{n}$ as follows:

$$
\begin{equation*}
M_{i}(P) \cdot z_{i}^{T}=K_{X, i}\left(z_{i}\right) \cdot P \tag{9}
\end{equation*}
$$

where the tensor $P$ is vectorized as column. The matrix $K_{X, i}\left(z_{i}\right)$ has $d_{i}$ rows and $d_{1} d_{2} \cdots d_{n}$ columns. Its entries are binary forms in $z_{i}$ whose coefficients depend on the entries of $X^{(i)}$.
Definition 3.5. The Konstanz matrix $K_{X}(z)$ of the game $X$ is a matrix with $\sum_{i=1}^{n} d_{i}$ rows and $d_{1} d_{2} \cdots d_{n}$ columns. It is obtained by stacking the matrices $K_{X, 1}\left(z_{1}\right), \ldots, K_{X, n}\left(z_{n}\right)$ on top of each other. When working on the affine chart $z_{i}=\left(x_{i}:-1\right)$, we write $K_{X}(x)$.

The Konstanz matrix $K_{X}(z)$ has linearly independent rows when $z$ is generic. Therefore, its kernel is a vector space of dimension $D=\prod_{i=1}^{n} d_{i}-\sum_{i=1}^{n} d_{i}$. We regard $\operatorname{ker}\left(K_{X}(z)\right)$ as a linear subspace of dimension $D-1$ in the projective space $\mathbb{P}(V)$. Our construction implies that the Spohn variety is the union of these linear spaces for $z \in\left(\mathbb{P}^{1}\right)^{n}$ :

$$
\begin{equation*}
\mathcal{V}_{X}=\left\{P \in \mathbb{P}(V): K_{X}(z) \cdot P=0 \text { for some } z \in\left(\mathbb{P}^{1}\right)^{n}\right\} \tag{10}
\end{equation*}
$$

At this point we must distinguish the cases $D \geq 1$ and $D=0$. First, let $D \geq 1$. Then the map $\pi_{X}$ is dominant, and its generic fiber is a linear space $\mathbb{P}^{D-1}$. This map furnishes an explicit birational isomorphism between the Spohn variety $\mathcal{V}_{X}$ and $\mathbb{P}^{D-1} \times\left(\mathbb{P}^{1}\right)^{n}$. The representation (10) gives the inverse, hence the desired rational parametrization of $\mathcal{V}_{X}$. This confirms the dimension formula in Theorem 3.2, which is here rewritten as $\operatorname{dim}\left(\mathcal{V}_{X}\right)=D-1+n$.

Finally, let $D=0$. This implies $n=d_{1}=d_{2}=2$, so the Konstanz matrix has format $4 \times 4$. It is shown in (19). The determinant of $K_{X}(z)$ is a curve of
degree $(2,2)$ in $\mathbb{P}^{1} \times \mathbb{P}^{1}$, so it is an elliptic curve. The map $\pi_{X}$ gives a birational isomorphism from $\mathcal{V}_{X}$ onto this curve. This elliptic curve is studied in detail in Section 4, and we will revisit it in Example 5.2.

The case $D=1$ is also of special interest, because here $\pi_{X}$ is a birational isomorphism.
Example 3.6 (Del Pezzo surfaces of degree two). Let $n=2, d_{1}=3, d_{2}=2$. Up to relabelling, this is the only case satisfying $D=1$. The Konstanz matrix equals

$$
K_{X}(x)=\left[\begin{array}{cccccc}
x_{1}-a_{11} & x_{1}-a_{12} & 0 & 0 & 0 & 0  \tag{11}\\
0 & 0 & x_{1}-a_{21} & x_{1}-a_{22} & 0 & 0 \\
0 & 0 & 0 & 0 & x_{1}-a_{31} & x_{1}-a_{32} \\
x_{2}-b_{11} & 0 & x_{2}-b_{21} & 0 & x_{2}-b_{31} & 0 \\
0 & x_{2}-b_{12} & 0 & x_{2}-b_{22} & 0 & x_{2}-b_{32}
\end{array}\right]
$$

Here $\left(x_{1}, x_{2}\right)$ are coordinates on an affine chart $\mathbb{C}^{2}$ of $\mathbb{P}^{1} \times \mathbb{P}^{1}$. The rank of (11) drops from 5 to 4 at precisely six points in $\mathbb{P}^{1} \times \mathbb{P}^{1}$. Five of these lie in $\mathbb{C}^{2}$. We obtain a rational map

$$
\mathbb{P}^{1} \times \mathbb{P}^{1} \longrightarrow \mathbb{P}^{5},\left(x_{1}, x_{2}\right) \mapsto \operatorname{ker}\left(K_{X}(x)\right)
$$

This blows up six points, and its image is the Spohn surface $\mathcal{V}_{X}$. The inverse map is $\pi_{X}$. We conclude that $\mathcal{V}_{X}$ is the blow-up of $\mathbb{P}^{1} \times \mathbb{P}^{1}$ at six general points. When seen through the lens of algebraic geometry [16, Example 1.9], this is a del Pezzo surface of degree two.

Konstanz matrices for three other tensor formats are shown in Examples 5.2, 5.6 and 5.7.

## 4. Elliptic Curves

In this section we take a closer look at $2 \times 2$ games, with payoff matrices $X^{(1)}=\left(a_{i j}\right)$ and $X^{(2)}=\left(b_{i j}\right)$. The Spohn variety $\mathcal{V}_{X}$ is the elliptic curve in $\mathbb{P}^{3}$ defined by the two quadrics

$$
\begin{aligned}
f_{1}=\operatorname{det}\left(M_{1}\right)= & \left(a_{21}-a_{11}\right) p_{11} p_{21}+\left(a_{22}-a_{11}\right) p_{11} p_{22} \\
& +\left(a_{21}-a_{12}\right) p_{12} p_{21}+\left(a_{22}-a_{12}\right) p_{12} p_{22} \\
f_{2}=\operatorname{det}\left(M_{2}\right)= & \left(b_{12}-b_{11}\right) p_{11} p_{12}+\left(b_{22}-b_{11}\right) p_{11} p_{22} \\
& +\left(b_{12}-b_{21}\right) p_{12} p_{21}+\left(b_{22}-b_{21}\right) p_{21} p_{22}
\end{aligned}
$$

This curve passes through the coordinate points $E_{11}, E_{12}, E_{21}, E_{22}$ in $\mathbb{P}^{3}$. It is smooth and irreducible when $a_{i j}$ and $b_{i j}$ are generic. A planar model of
this elliptic curve is obtained by eliminating $p_{22}$ from $f_{1}$ and $f_{2}$. Setting $p_{11}=x, p_{12}=y, p_{21}=z$, we find the ternary cubic

$$
\begin{align*}
& \left(a_{11}-a_{22}\right)\left(b_{11}-b_{12}\right) x^{2} y+\left(a_{11}-a_{21}\right)\left(b_{22}-b_{11}\right) x^{2} z \\
+ & \left(a_{12}-a_{22}\right)\left(b_{11}-b_{12}\right) x y^{2}+\left(a_{11}-a_{21}\right)\left(b_{22}-b_{21}\right) x z^{2}  \tag{12}\\
+ & \left(a_{12}-a_{22}\right)\left(b_{21}-b_{12}\right) y^{2} z+\left(a_{12}-a_{21}\right)\left(b_{22}-b_{21}\right) y z^{2} \\
+ & \left(\left(a_{12}-a_{21}\right)\left(b_{22}-b_{11}\right)+\left(a_{11}-a_{22}\right)\left(b_{21}-b_{12}\right)\right) x y z .
\end{align*}
$$

A ternary cubic of the form (12) is called a Spohn cubic. This passes through the three coordinate points in $\mathbb{P}^{2}$. But there are other restrictions. To see this, we consider all cubics

$$
\begin{equation*}
c_{1} x^{2} y+c_{2} x^{2} z+c_{3} x y^{2}+c_{4} x z^{2}+c_{5} y^{2} z+c_{6} y z^{2}+c_{7} x y z \tag{13}
\end{equation*}
$$

The set of such cubics is a projective space $\mathbb{P}^{6}$ with homogeneous coordinates $c_{1}, \ldots, c_{7}$.
Proposition 4.1. The Spohn cubics (12) form the 4-dimensional variety in $\mathbb{P}^{6}$ given by $c_{1}+c_{2}-c_{3}-c_{4}+c_{5}+c_{6}-c_{7}=c_{2} c_{4} c_{5}-c_{3} c_{4} c_{5}-c_{2} c_{3} c_{6}+c_{4} c_{5} c_{6}+c_{3} c_{4} c_{7}-$ $c_{4} c_{5} c_{7}-c_{4}^{2} c_{5}+c_{4} c_{5}^{2}=0$. This is a cubic hypersurface inside a hyperplane $\mathbb{P}^{5}$. Its singular locus consists of nine points.
Proof. This is obtained by a direct computation using the software Macaulay2 [10].

While the general Spohn cubic is smooth, it can be singular for special payoff matrices. To identify these, we compute the discriminant $\mathcal{D}$ of the ternary cubic (13). This discriminant is an irreducible polynomial of degree 12 in seven unknowns. It is a sum of 127 terms:

$$
\mathcal{D}=16 c_{1}^{5} c_{4}^{2} c_{5}^{2} c_{6}^{3}+16 c_{1}^{4} c_{2}^{2} c_{5}^{2} c_{6}^{4}-24 c_{1}^{4} c_{2} c_{4}^{2} c_{5}^{3} c_{6}^{2}+\cdots+c_{2}^{2} c_{3}^{2} c_{4}^{2} c_{5}^{2} c_{7}^{4}-c_{2}^{2} c_{3}^{2} c_{4} c_{5} c_{6} c_{7}^{5}
$$

We now plug in the Spohn cubic (12). The resulting discriminant is a polynomial of degree 24 in the eight unknowns $a_{i j}, b_{i j}$. It factors into nine irreducible factors, namely

$$
\begin{aligned}
& \mathcal{D}(a, b)=\left(a_{11}-a_{12}\right)^{2}\left(a_{11}-a_{21}\right)^{2}\left(a_{12}-a_{22}\right)^{2}\left(a_{21}-a_{22}\right)^{2} \\
& \cdot\left(b_{11}-b_{12}\right)^{2}\left(b_{11}-b_{21}\right)^{2}\left(b_{12}-b_{22}\right)^{2}\left(b_{21}-b_{22}\right)^{2} \mathcal{E}(a, b) .
\end{aligned}
$$

The last factor $\mathcal{E}(a, b)$ has 587 terms of degree 8 . Nonvanishing of the discriminant $\mathcal{D}(a, b)$ ensures that the Spohn cubic (12) is smooth in $\mathbb{P}^{2}$, and hence so is the curve $\mathcal{V}_{X}$ in $\mathbb{P}^{3}$.

We have argued that the general Spohn curve $\mathcal{V}_{X}$ is an elliptic curve. It is thus natural to express its $j$-invariant, which identifies the isomorphism type, in terms of the payoff matrices.

Proposition 4.2. The j-invariant of the Spohn cubic equals $\mathcal{I}(a, b)^{3} / \mathcal{D}(a, b)$, where $\mathcal{I}(a, b)$ is an irreducible polynomial of degree 8 with 633 terms in the entries of the two payoff tables.

Proof. For any ternary cubic, the j-invariant is the cube of the Aronhold invariant divided by the discriminant; see [12, Example 11.12]. Here, $\mathcal{I}(a, b)$ is the Aronhold invariant of (12).

The dependency equilibria of our game are the points in $\mathcal{V}_{X} \cap \Delta$. To better understand this semialgebraic set, we identify some landmarks on the curve $\mathcal{V}_{X}$. The first such landmark is the Nash point, which is the unique rank one matrix in $\mathbb{P}^{3}$ lying on $\mathcal{V}_{X}$ :

$$
N=\left[\begin{array}{l}
b_{22}-b_{21}  \tag{14}\\
b_{11}-b_{12}
\end{array}\right]\left[\begin{array}{ll}
a_{22}-a_{12} & a_{11}-a_{21}
\end{array}\right]
$$

Suppose that the following holds and the two signs are non-zero:

$$
\begin{equation*}
\operatorname{sign}\left(a_{11}-a_{21}\right)=\operatorname{sign}\left(a_{22}-a_{12}\right) \quad \text { and } \quad \operatorname{sign}\left(b_{11}-b_{12}\right)=\operatorname{sign}\left(b_{22}-b_{21}\right) \tag{15}
\end{equation*}
$$

Then we can scale the matrix $N$ in (14) by $\left(\left(a_{11}-a_{21}+a_{22}-a_{12}\right)\left(b_{11}-b_{12}+\right.\right.$ $\left.\left.b_{22}-b_{21}\right)\right)^{-1}$ to land in $\Delta$, and the result is the unique totally mixed Nash equilibrium of the game.

Next recall that the four coordinate points $E_{i j}$ lie on the curve $\mathcal{V}_{X}$. Their tangent lines $\operatorname{span}\left(D_{i j}, E_{i j}\right)$ are specified by their intersection points with the opposite coordinate planes:

$$
\begin{aligned}
D_{11} & =\left[\begin{array}{cc}
0 & \left(a_{11}-a_{21}\right)\left(b_{22}-b_{11}\right) \\
\left(a_{22}-a_{11}\right)\left(b_{11}-b_{12}\right) & \left(a_{11}-a_{21}\right)\left(b_{11}-b_{12}\right)
\end{array}\right], \\
D_{12} & =\left[\begin{array}{cc}
\left(a_{22}-a_{12}\right)\left(b_{12}-b_{21}\right) & 0 \\
\left(a_{22}-a_{12}\right)\left(b_{11}-b_{12}\right) & \left(a_{12}-a_{21}\right)\left(b_{11}-b_{12}\right)
\end{array}\right], \\
D_{21} & =\left[\begin{array}{cc}
\left(a_{21}-a_{12}\right)\left(b_{21}-b_{22}\right) & \left(a_{11}-a_{21}\right)\left(b_{21}-b_{22}\right) \\
0 & \left(a_{11}-a_{21}\right)\left(b_{12}-b_{21}\right)
\end{array}\right], \\
D_{22} & =\left[\begin{array}{cc}
\left(a_{22}-a_{12}\right)\left(b_{21}-b_{22}\right) & \left(a_{11}-a_{22}\right)\left(b_{21}-b_{22}\right) \\
\left(a_{12}-a_{22}\right)\left(b_{11}-b_{22}\right) & 0
\end{array}\right],
\end{aligned}
$$

And, finally, our curve intersects each coordinate plane in a unique non-coor-
dinate point:

$$
\begin{aligned}
& F_{11}=\left[\begin{array}{cc}
0 & \left(a_{12}-a_{21}\right)\left(b_{21}-b_{22}\right) \\
\left(a_{12}-a_{22}\right)\left(b_{21}-b_{12}\right) & \left(a_{12}-a_{21}\right)\left(b_{12}-b_{21}\right)
\end{array}\right], \\
& F_{12}=\left[\begin{array}{cc}
\left(a_{11}-a_{22}\right)\left(b_{21}-b_{22}\right) & 0 \\
\left(a_{11}-a_{22}\right)\left(b_{22}-b_{11}\right) & \left(a_{11}-a_{21}\right)\left(b_{11}-b_{22}\right)
\end{array}\right], \\
& F_{21}=\left[\begin{array}{cc}
\left(a_{12}-a_{22}\right)\left(b_{11}-b_{22}\right) & \left(a_{11}-a_{22}\right)\left(b_{22}-b_{11}\right) \\
0 & \left(a_{11}-a_{22}\right)\left(b_{11}-b_{12}\right)
\end{array}\right], \\
& F_{22}=\left[\begin{array}{cc}
\left(a_{12}-a_{21}\right)\left(b_{12}-b_{21}\right) & \left(a_{11}-a_{21}\right)\left(b_{21}-b_{12}\right) \\
\left(a_{12}-a_{21}\right)\left(b_{11}-b_{12}\right) & 0
\end{array}\right] .
\end{aligned}
$$

We now show that dependency equilibria may exist even if there are no Nash equilibria in $\Delta$ :
Example 4.3 (Disconnected equilibria). Consider the game $X$ given by the payoff matrices

$$
\begin{array}{r}
{\left[\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right]=\left[\begin{array}{ll}
2 & 0 \\
4 & 1
\end{array}\right] \text { and }\left[\begin{array}{ll}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{array}\right]=\left[\begin{array}{ll}
2 & 1 \\
4 & 3
\end{array}\right],} \\
\text { with Nash point } N=\left[\begin{array}{rr}
-1 & 2 \\
1 & -2
\end{array}\right] .
\end{array}
$$

Here, $\mathcal{V}_{X}$ is smooth and irreducible.
This elliptic curve has j-invariant $-\left(7^{3} 103^{3}\right) /\left(2^{8} 3^{2} 47\right)$. The real curve $\mathcal{V}_{X} \cap \Delta$ has two connected components, both disjoint from the Segre surface $\left\langle p_{11} p_{22}-p_{12} p_{21}\right\rangle$. One arc connects $E_{11}$ and $F_{21}$, and the other arc connects $E_{22}$ and $F_{12}$.

The combinatorics of the curve $\mathcal{V}_{X} \cap \Delta$ is given by the signs of the entries in the nine matrices $N, D_{i j}$ and $F_{i j}$. These signs are determined by the respective orderings of $a_{11}, a_{12}, a_{21}, a_{22}$ and $b_{11}, b_{12}, b_{21}, b_{22}$, assuming that these are quadruples of distinct numbers. We derive the following theorem by analyzing all $(4!)^{2}=576$ possibilities for these pairs of orderings.
Theorem 4.4. For a generic $2 \times 2$ game $X$, the curve of dependency equilibria $\mathcal{V}_{X} \cap \Delta$ has either 0,1 or 2 connected components, each of which is an arc between two boundary points. If (15) holds then there is exactly one $E E, E F$ or $F F$ arc. If (15) does not hold then all components are $E F$ arcs, and their number can be 0,1 or 2 .

## 5. The Payoff Region

The $n$ payoff tensors $X^{(i)}$ define a canonical linear map from tensor space to payoff space:

$$
\begin{equation*}
\pi_{X}: V \rightarrow \mathbb{R}^{n}, P \mapsto\left(P X^{(1)}, P X^{(2)}, \ldots, P X^{(n)}\right) \tag{16}
\end{equation*}
$$

The $i$ th coordinate $P X^{(i)}$ is the expected payoff for player $i$, given by the formula in (3). We call $\pi_{X}$ the payoff map. By (8), this is the lifting to $V$ of the algebraic payoff map in (7).

The image of the probability simplex $\Delta$ is a convex polytope $\pi_{X}(\Delta)$ that is usually full-dimensional in $\mathbb{R}^{n}$. This polytope is known as the cooperative payoff region of the game $X$. Its points are all possible expected payoff vectors for the game in question. Tu and Jiang [23] investigate the semialgebraic subset that is obtained by projecting all rank one tensors in $\Delta$. This is a nonconvex subset of $\pi_{X}(\Delta)$, known as the noncooperative payoff region.

For $2 \times 2$ games, this region is the image of the Segre surface under a linear projection into the plane. Our readers might like to compare [23, Figure 1] with the surface shown in Figure 1.

We are interested in the subset of payoff vectors that arise from dependency equilibria:

$$
\mathcal{P}_{X}:=\pi_{X}\left(\mathcal{V}_{X} \cap \Delta\right) \subset \pi_{X}(\Delta) \subset \mathbb{R}^{n}
$$

The set $\mathcal{P}_{X}$ is semialgebraic, by Tarski's Theorem on Quantifier Elimination. The authors of [23] would probably call $\mathcal{P}_{X}$ the dependency payoff region of the game $X$. In the present paper, we just use the term payoff region for $\mathcal{P}_{X}$, since our focus is on dependency equilibria.

We begin by noting that, at every dependency equilibrium of $X$, the expected payoffs agree with the various conditional expected payoffs. We can thus use conditional expectations in (16) to define the payoff region $\mathcal{P}_{X}$. This is the content of the following lemma.
Lemma 5.1. Let $P$ be a tensor in $V$ with $p_{++\cdots+}=1$ that represents a point in $\mathcal{V}_{X}$. Then

$$
\begin{align*}
& P X^{(i)}=\sum_{j_{1}=1}^{d_{1}} \cdots \widehat{\sum_{j_{i}=1}^{d_{i}}} \cdots \sum_{j_{n}=1}^{d_{n}} X_{j_{1} \cdots k \cdots j_{n}}^{(i)} \frac{p_{j_{1} \cdots k \cdots j_{n}}}{p_{+\cdots+k+\cdots+}} \\
& \quad \text { for all } i \in[n] \text { and } k \in\left[d_{i}\right] . \tag{17}
\end{align*}
$$

Proof. The $d_{i} \times 2$ matrix $M_{i}$ in (5) has rank one, by definition of $\mathcal{V}_{X}$. We replace the first row by the sum of all rows. This transforms $M_{i}$ into the following matrix whose rank is one:

$$
\left[\begin{array}{cc}
1 & P X^{(i)}  \tag{18}\\
p_{+\cdots+2+\cdots+} & \sum_{j_{1}=1}^{d_{1}} \cdots \widehat{\sum_{j_{i}=1}^{d_{i}} \cdots} \sum_{j_{n}=1}^{d_{n}} X_{j_{1} \cdots 2 \cdots j_{n}}^{(i)} p_{j_{1} \cdots 2 \cdots j_{n}} \\
\vdots & \vdots \\
p_{+\cdots+d_{i}+\cdots+} & \sum_{j_{1}=1}^{d_{1}} \cdots \widehat{\sum_{j_{i}=1}^{d_{i}}} \cdots \sum_{j_{n}=1}^{d_{n}} X_{j_{1} \cdots d_{i} \cdots j_{n}}^{(i)} p_{j_{1} \cdots d_{i} \cdots j_{n}}
\end{array}\right]
$$

The $2 \times 2$ minor given by the first row and the $k$ th row is zero; see also (8). This implies the desired identity (17) for $k \geq 2$. The case $k=1$ is obtained by swapping rows in $M_{i}$.


Figure 2: The payoff region for each of these $2 \times 2$ games is the blue arc in the yellow triangle.

Example $5.2(2 \times 2$ games $)$. The polygon $\pi_{X}(\Delta)$ is the convex hull in $\mathbb{R}^{2}$ of the points $\left(a_{11}, b_{11}\right),\left(a_{12}, b_{12}\right),\left(a_{21}, b_{21}\right)$ and $\left(a_{22}, b_{22}\right)$, so it is typically a triangle or a quadrilateral. This polygon contains the payoff curve $\mathcal{P}_{X}$, which is the image of the curve $\mathcal{V}_{X} \cap \Delta$ under the payoff map $\pi_{X}$. This is a plane cubic, defined by the determinant of the Konstanz matrix

$$
K_{X}(x)=\left[\begin{array}{cccc}
x_{1}-a_{11} & x_{1}-a_{12} & 0 & 0  \tag{19}\\
0 & 0 & x_{1}-a_{21} & x_{1}-a_{22} \\
x_{2}-b_{11} & 0 & x_{2}-b_{21} & 0 \\
0 & x_{2}-b_{12} & 0 & x_{2}-b_{22}
\end{array}\right] .
$$

For each point $x$ on this curve, the kernel of (19) gives the unique matrix $P$ satisfying $\pi_{X}(P)=x$. The payoff region $\mathcal{P}_{X}$ is the subset of points $x$ on the curve for which $P>0$.

Figure 2a shows the payoff region for the Bach or Stravinsky game in Example 2.1. It is the blue arc inside the yellow triangle $\pi_{X}(\Delta)=\operatorname{conv}\{(0,0),(2,3)$, $(3,2)\}$. This picture is the image of Figure 1 under the payoff map $\pi_{X}$. Figure 2 b shows a perturbed version, with $a_{11}=3.3$ and $b_{22}=3.2$, where the Spohn curve is irreducible.

We now consider cases other than $2 \times 2$ games, so that $\operatorname{dim}\left(\mathcal{V}_{X}\right) \geq n$ holds. We further assume that $X$ is generic and that $\mathcal{V}_{X} \cap \Delta$ is non-empty. Since the algebraic payoff map $\pi_{X}$ in (7) is dominant, the payoff region $\mathcal{P}_{X}$ is a full-dimensional semialgebraic subset of $\mathbb{R}^{n}$.
Example $5.3(3 \times 2$ games). The following two payoff matrices exhibit the


Figure 3: The payoff region $\mathcal{P}_{X}$ for the $3 \times 2$ game in Example 5.3 consists of two curvy triangles, inside the pentagon $\pi_{X}(\Delta)$. Its boundary is given by two lines and two cubics.
generic behavior:

$$
X^{(1)}=\left[\begin{array}{ll}
a_{11} & a_{12}  \tag{20}\\
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{array}\right]=\left[\begin{array}{cc}
0 & 30 \\
5 & 25 \\
13 & 24
\end{array}\right] \quad \text { and } \quad X^{(2)}=\left[\begin{array}{ll}
b_{11} & b_{12} \\
b_{21} & b_{22} \\
b_{31} & b_{32}
\end{array}\right]=\left[\begin{array}{cc}
6 & 42 \\
21 & 12 \\
36 & 0
\end{array}\right]
$$

The polygon $\pi_{X}(\Delta)$ is the pentagon whose vertices are $\left(a_{i j}, b_{i j}\right)$ with $\{i, j\} \neq$ $\{2,2\}$. The payoff region $\mathcal{P}_{X}=\pi_{X}\left(\mathcal{V}_{X} \cap \Delta\right)$ is shaded in blue in Figure 3. The algebraic boundary of $\mathcal{P}_{X}$ is given by the two cubics $9 x_{1}^{2} x_{2}-2 x_{1} x_{2}^{2}-162 x_{1}^{2}-$ $189 x_{1} x_{2}+30 x_{2}^{2}+3906 x_{1}-540 x_{2}+2160$ and $72 x_{1}^{2} x_{2}-19 x_{1} x_{2}^{2}-1512 x_{1}^{2}-$ $1614 x_{1} x_{2}+390 x_{2}^{2}+36288 x_{1}-2340 x_{2}$, plus the two vertical lines $x_{1}-13$ and $x_{1}-24$. The two curvy triangles that form $\mathcal{P}_{X}$ meet at the special point

$$
\begin{equation*}
(22.9902299164,16.2987107576) \tag{21}
\end{equation*}
$$

Figure 3 illustrates the general behavior for $3 \times 2$ games. We can understand
this via the del Pezzo geometry in Example 3.6. The Spohn surface $\mathcal{V}_{X}$ is the blow-up of $\mathbb{P}^{1} \times \mathbb{P}^{1}$ at six points. One of these six is the special point (21). The Konstanz matrix $K_{X}(x)$ in (11) has rank four at this point, so there is a line segment in $\mathcal{V}_{X} \cap \Delta$ that maps to (21) under $\pi_{X}$. At all nearby points $x \in \mathbb{R}^{2}$, the rank of $K_{X}(x)$ is five. Here, $\pi_{X}$ gives a bijection between $\mathcal{V}_{X} \cap \Delta$ and the payoff region $\mathcal{P}_{X}$. The boundary curves of $\mathcal{P}_{X}$ are defined by maximal minors of $K_{X}(x)$. Each minor is a $5 \times 5$-determinant, but it has degree four as a polynomial in $x=\left(x_{1}, x_{2}\right)$. That quartic factors into a linear factor $x_{1}-a_{i j}$ times a cubic in $\left(x_{1}, x_{2}\right)$.

We now work towards the main result of this section, generalizing Example 5.3 to arbitrary tensor formats. The key players are the maximal minors of the Konstanz matrix $K_{X}(x)$.

Lemma 5.4. Given any game $X$, each of the $\binom{d_{1} d_{2} \cdots d_{n}}{d_{1}+d_{2}+\cdots+d_{n}}$ maximal minors of the Konstanz matrix $K_{X}(x)$ is a polynomial of degree at most $\sum_{i=1}^{n} d_{i}-n+1$ in the unknowns $x_{1}, \ldots, x_{n}$.

Proof. The highest degree seen in the maximal minors is the rank of $K_{X}(x)$ after setting all entries in the payoff tables $X^{(i)}$ to zero. After rescaling the rows, the columns of this matrix are homogeneous coordinates for the vertices of the product of standard simplices $\Delta_{d_{1}-1} \times \cdots \times \Delta_{d_{n}-1}$. The dimension of this polytope is one less than the matrix rank.

Suppose now that $X$ is fixed and generic. We consider the stratification of the payoff space $\mathbb{R}^{n}$ defined by the signs taken on by the maximal minors of $K_{X}(x)$. We call this the oriented matroid stratification of the game $X$. Indeed, it is the restriction to $\mathbb{R}^{n}$ of the usual oriented matroid stratification (cf. [13]) of the space of matrices with $\sum_{i=1}^{n} d_{i}$ rows and $\prod_{i=1}^{n} d_{i}$ columns. The maximal minors of $K_{X}(x)$ that are nonzero polynomials give the bases of a matroid. The full-dimensional strata correspond to orientations of that matroid. The open stratum containing a given point $x \in \mathbb{R}^{n}$ consists of all points $x^{\prime} \in \mathbb{R}^{n}$ such that corresponding nonzero maximal minors of $K_{X}(x)$ and $K_{X}\left(x^{\prime}\right)$ have the same sign +1 or -1 .

The oriented matroid strata in $\mathbb{R}^{n}$ are semialgebraic. Their boundaries are delineated by the maximal minors of $K_{X}(x)$. These minors are the polynomials in Lemma 5.4. The oriented matroid strata can be disconnected (cf. [13]). This happens in Examples 4.3 and 5.3. Note that the union of the two open curvy triangles in Figure 3 is a single chamber (open stratum) for the game $X$ given in (20). It is given by prescribing a fixed sign +1 or -1 for each of the six maximal minors of (11). Interestingly, $\mathcal{P}_{X}$ itself is connected in this case. The point (21) lies in $\mathcal{P}_{X}$ because its fiber under $\pi_{X}$ is a line that meets the interior of $\Delta$.

We now present our characterization of the payoff region $\mathcal{P}_{X}$ of a generic game $X$. By the algebraic boundary of $\mathcal{P}_{X}$ we mean the Zariski closure of its topological boundary.

Theorem 5.5. The payoff region $\mathcal{P}_{X}$ for a generic game $X$ is a union of oriented matroid strata in $\mathbb{R}^{n}$ that are given by the signs of the maximal minors of the Konstanz matrix $K_{X}(x)$. Its algebraic boundary is a union of irreducible hypersurfaces of degree at most $\sum_{i=1}^{n} d_{i}-n+1$.
Proof. For fixed $x \in \mathbb{R}^{n}$, the set of probability tensors $P$ with expected payoffs $x$ is equal to

$$
\begin{equation*}
\operatorname{kernel}\left(K_{X}(x)\right) \cap \Delta . \tag{22}
\end{equation*}
$$

This is a convex polytope which is either empty or has the full dimension $\prod_{i=1}^{n} d_{i}-\sum_{i=1}^{n} d_{i}-1$. The payoff region $\mathcal{P}_{X}$ is the set of all $x \in \mathbb{R}^{n}$ such that this polytope is nonempty. We know from oriented matroid theory [2, Chapter 9] that the combinatorial type of the polytope (22) is determined by the oriented matroid of the matrix $K_{X}(x)$. Therefore, the combinatorial type is constant as $x$ ranges over a fixed oriented matroid stratum in $\mathbb{R}^{n}$. In particular, whether or not (22) is empty depends only on the oriented matroid of $K_{X}(x)$. Namely, it is non-empty if and only if every column index lies in a positive covector of that oriented matroid. This proves the first sentence. The second sentence follows from Lemma 5.4.

One of the reasons for our interest in the algebraic boundary is that it helps in characterizing dependency equilibria $P$ that are Pareto optimal. We thus address a question raised in [20, Section 4]. Recall that $P$ is Pareto optimal if its image $x=\pi_{X}(P)$ in $\mathcal{P}_{X}$ satisfies $\left(x+\mathbb{R}_{>0}^{n}\right) \cap \overline{\mathcal{P}_{X}}=\{x\}$. This condition implies that $x$ lies in the boundary of $\mathcal{P}_{X}$, hence one of the maximal minors of $K_{X}(x)$ must vanish. For instance, for the $3 \times 2$ game in Example 5.3, the Pareto optimal equilibria correspond to the points on the upper-right boundaries of the two curvy triangles in Figure 3. At such points $x$, the product of our two cubics vanishes.

We close this section by discussing Theorem 5.5 for two cases larger than Example 5.3.
Example $5.6(3 \times 3$ games $)$. Let $n=2$ and $d_{1}=d_{2}=3$. The Konstanz matrix $K_{X}(x)$ equals
$\left[\begin{array}{ccccccccc}x_{1}-a_{11} & x_{1}-a_{12} & x_{1}-a_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{1}-a_{21} & x_{1}-a_{22} & x_{1}-a_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_{1}-a_{31} & x_{1}-a_{32} & x_{1}-a_{33} \\ x_{2}-b_{11} & 0 & 0 & x_{2}-b_{21} & 0 & 0 & x_{2}-b_{31} & 0 & 0 \\ 0 & x_{2}-b_{12} & 0 & 0 & x_{2}-b_{22} & 0 & 0 & x_{2}-b_{32} & 0 \\ 0 & 0 & x_{2}-b_{13} & 0 & 0 & x_{2}-b_{23} & 0 & 0 & x_{2}-b_{33}\end{array}\right]$.
Among the $\binom{9}{6}=84$ maximal minors of this $6 \times 9$ matrix, six are identically zero. Six others are irreducible polynomials of degree five in $x=\left(x_{1}, x_{2}\right)$. Each of the
remaining 72 minors is an irreducible cubic times a product $\left(x_{1}-a_{i j}\right)\left(x_{2}-b_{k l}\right)$. The resulting arrangement of lines, cubics and quintics divides the plane $\mathbb{R}^{2}$ into open chambers. We examine the chambers that lie inside the polygon $\pi_{X}(\Delta)$. The rank 6 oriented matroid of $K_{X}(x)$, given by 78 signed bases, is constant on each chamber. The payoff region is a union of some of them.
Example $5.7(2 \times 2 \times 2$ games $)$. The game played by Adam, Bob and Carl in Example 2.3 has the Konstanz matrix $K_{X}(x)$ as:
$\left[\begin{array}{cccccccc}x_{1}-a_{111} & x_{1}-a_{112} & x_{1}-a_{121} & x_{1}-a_{122} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{1}-a_{211} & x_{1}-a_{212} & x_{1}-a_{221} & x_{1}-a_{222} \\ x_{2}-b_{111} & x_{2}-b_{112} & 0 & 0 & x_{2}-b_{211} & x_{2}-b_{212} & 0 & 0 \\ 0 & 0 & x_{2}-b_{121} & x_{2}-b_{122} & 0 & 0 & x_{2}-b_{221} & x_{2}-b_{222} \\ x_{3}-c_{111} & 0 & x_{3}-c_{121} & 0 & x_{3}-c_{211} & 0 & x_{3}-c_{221} & 0 \\ 0 & x_{3}-c_{112} & 0 & x_{3}-c_{122} & 0 & x_{3}-c_{212} & 0 & x_{3}-c_{222}\end{array}\right]$.

All $\binom{8}{6}=28$ maximal minors are irreducible polynomials of degree four in $x=\left(x_{1}, x_{2}, x_{3}\right)$. Each of them defines a smooth quartic surface in $\mathbb{C}^{3}$ that has three isolated singularities at infinity in $\mathbb{P}^{3}$. This data specifies an arrangement of 28 K 3 surfaces in $\mathbb{P}^{3}$. We examine its chambers inside the polytope $\pi_{X}(\Delta)$, which has $\leq 8$ vertices. The payoff region $\mathcal{P}_{X}$ is the union of a subset of these chambers, so its algebraic boundary consists of quartic surfaces.

## 6. Conditional Independence and Bayesian Networks

One drawback of dependency equilibria is that they are abundant. Indeed, if the Spohn variety $\mathcal{V}_{X}$ intersects the open simplex $\Delta$, then the semialgebraic set $\mathcal{V}_{X} \cap \Delta$ of all dependency equilibria has dimension $\prod_{i=1}^{n} d_{i}-\sum_{j=1}^{n} d_{j}+n-1$. This follows from Theorem 3.2. To mitigate this drawback, we restrict to intersections of $\mathcal{V}_{X}$ with statistical models in $\Delta$. Natural candidates are the conditional independence models in [21, Section 8.1] and [22, Section 4.1].

We view the $n$ players as random variables with state spaces $\left[d_{1}\right], \ldots,\left[d_{n}\right]$. A point $P$ in $\Delta$ is a joint probability distribution. Let $\mathcal{C}$ be any collection of conditional independence (CI) statements on $[n]$. These statements have the form $A \Perp B \mid C$, where $A, B, C$ are pairwise disjoint subsets of $[n]$. Each CI statement translates into a system of homogeneous quadratic constraints in the tensor entries $p_{j_{1} j_{2} \cdots j_{n}}$. This translation is explained in [21, Proposition 8.1] and [22, Proposition 4.1.6]. We write $\mathcal{M}_{\mathcal{C}}$ for the projective variety in $\mathbb{P}(V)$ that is defined by these quadrics, arising from all statements $A \Perp B \mid C$ in $\mathcal{C}$. Here we assume that components lying in the hyperplanes $\left\{p_{j_{1} j_{2} \cdots j_{n}}=0\right\}$ and $\left\{p_{++\cdots+}=0\right\}$ have been removed.

Suppose $X$ is any game in normal form, and $\mathcal{C}$ is any collection of CI statements. We define the Spohn CI variety to be the intersection of the Spohn variety with the CI model:

$$
\begin{equation*}
\mathcal{V}_{X, \mathcal{C}}=\mathcal{V}_{X} \cap \mathcal{M}_{\mathcal{C}} \tag{23}
\end{equation*}
$$

We again assume that components lying in the special hyperplanes above have been removed. The intersection $\mathcal{V}_{X, \mathcal{C}} \cap \Delta$ with the simplex $\Delta$ is the set of all $C I$ equilibria of the game $X$. This is a semialgebraic set which is a natural extension of the set of Nash equilibria of $X$. In what follows we assume that all random variables are binary, i.e. $d_{1}=d_{2}=\cdots=d_{n}=2$.
Example 6.1 (Nash points). Let $\mathcal{C}$ be the set of all CI statements on [ $n$ ]. The model $\mathcal{M}_{\mathcal{C}}$ is the Segre variety of rank one tensors, and the Spohn CI variety (23) is the set of all Nash points in the Spohn variety $\mathcal{V}_{X}$. By [21, Corollary 6.9], this variety is finite, and its cardinality is the number of derangements of $[n]$, which is $1,2,9,44,265, \ldots$ for $n=1,2,3,4,5, \ldots$

For $n \geq 3$, the Nash points span a linear subspace of codimension $2 n$ in $\mathbb{P}(V) \simeq \mathbb{P}^{2^{n}-1}$. To see this, we note that the $i$ th multilinear equation in $[21$, Theorem 6.6] has degree $n-1$ and it misses the $i$ th unknown $\pi^{(i)}$. Multiplying that equation by $\pi^{(i)}$ and by $1-\pi^{(i)}$ gives two linear constraints on $\mathbb{P}(V)$ for each $i$. These $2 n$ linear forms are linearly independent.
Example $6.2\left(n=3, d_{1}=d_{2}=d_{3}=2\right)$. Consider games $X$ for three players with binary choices. The Spohn variety $\mathcal{V}_{X}$ is a complete intersection of dimension 4 and degree 8 in $\mathbb{P}^{7}$. It is defined by imposing rank one constraints on the three matrices $M_{i}$ in Example 2.3. It is parametrized by the lines $\operatorname{ker}\left(K_{X}(x)\right)$ where $x \in \mathbb{C}^{3}$ and $K_{X}(x)$ is the matrix in Example 5.7.

We examine the Spohn CI varieties given by three models $\mathcal{M}_{\mathcal{C}}$ in [21, Section 8.1]. In each case, the intersection (23) is transversal in $\Delta$, and we find that $\mathcal{V}_{X, \mathcal{C}}$ is irreducible in $\mathbb{P}^{7}$.
(a) Let $\mathcal{C}=\{1 \Perp 2 \mid 3\}$ as in [21, eqn (8.3)]. The CI model $\mathcal{M}_{\mathcal{C}}$ has codimension 2 and degree 4, and the Spohn CI variety $\mathcal{V}_{X, \mathcal{C}}$ is a surface of degree 28 in $\mathbb{P}^{7}$. We find that the prime ideal of $\mathcal{V}_{X, \mathcal{C}}$ is minimally generated by five quadrics and three quartics.
(b) Let $\mathcal{C}=\{2 \Perp 3\}$ as in [21, eqn (8.4)], so here $C=\emptyset$. The CI model $\mathcal{M}_{\mathcal{C}}$ is the hypersurface, defined by the quadric $p_{+11} p_{+22}-p_{+12} p_{+21}$. The Spohn CI variety $\mathcal{V}_{X, \mathcal{C}}$ is a threefold of degree 10 in $\mathbb{P}^{7}$. Its prime ideal is minimally generated by six quadrics.
(c) Let $\mathcal{C}=\{1 \Perp 23\}$ as in [21, eqn (8.5)]. Here $\mathcal{M}_{\mathcal{C}} \simeq \mathbb{P}^{1} \times \mathbb{P}^{3}$ is defined by the $2 \times 2$ minors of a $2 \times 4$ matrix obtained by flattening the tensor $P$. The Spohn CI variety $\mathcal{V}_{X, \mathcal{C}}$ is a curve of degree 8 and genus 3 . It lies in a $\mathbb{P}^{5}$ inside $\mathbb{P}^{7}$. Its prime ideal is generated by two linear forms and seven quadrics. These will be explained after Example 6.5.

The computation of the prime ideals is non-trivial. One starts with the ideal generated by the natural quadrics defining (23), and one then saturates that ideal by $p_{+++} \cdot \prod_{i, j, k=1}^{2} p_{i j k}$. We performed these computations with the computer algebra system Macaulay2 [10].

Of special interest are graphical models, such as Markov random fields and Bayesian networks. These allow us to describe the nature of the desired equilibria by means of a graph whose nodes are the $n$ players. This is different from the setting of graphical games in [21, Section 6.5], where the graph structure imposes zero patterns in the payoff tables $X^{(i)}$.

Inspired by [19, Section 3], we now focus on Bayesian networks, where the CI statements $\mathcal{C}$ describe the global Markov property of an acyclic directed graph with vertex set $[n]$. These CI statements and their ideals are explained in [8, Section 3]. In Macaulay2, they can be computed using the commands globalMarkov and conditionalIndependenceIdeal in the GraphicalModels package. Sometimes, it is preferable to work with the prime ideal $\operatorname{ker}(\Phi)$ in $[8$, Theorem 8]. From this we obtain the ideal of the Spohn CI variety $\mathcal{V}_{X, \mathcal{C}}$ by saturation, as described at the end of Example 6.2. For all the models we were able to compute, this ideal turned out to be of the expected codimension. In each case, except for the network with no edges, the variety $\mathcal{V}_{X, \mathcal{C}}$ is irreducible. We conjecture that these facts hold in general.

Conjecture 6.3. For every Bayesian network $\mathcal{C}$ on $n$ binary random variables, the Spohn CI variety $\mathcal{V}_{X, \mathcal{C}}$ has the expected codimension $n$ inside the model $\mathcal{M}_{\mathcal{C}}$ in $\mathbb{P}^{2^{n}-1}$. The variety $\mathcal{V}_{X, \mathcal{C}}$ is positive-dimensional and irreducible whenever the network has at least one edge.

Proposition 6.4. Conjecture 6.3 holds for $n \leq 3$.
Proof. For the network with no edges, $\mathcal{M}_{\mathcal{C}}$ is the Segre variety $\left(\mathbb{P}^{1}\right)^{n}$. The dimension statement holds, but the Spohn CI variety is reducible, as seen in Example 6.1. We thus examine all Bayesian networks with at least one edge. These satisfy $\operatorname{dim}\left(\mathcal{M}_{\mathcal{C}}\right) \geq n+1$. The case $n \leq 2$ being trivial, we assume that $n=3$. If the network is a complete directed acyclic graph, then the ideal of $\mathcal{M}_{\mathcal{C}}$ is the zero ideal and $\mathcal{V}_{X, \mathcal{C}}=\mathcal{V}_{X}$. There are four networks left to be considered. By [8, Proposition 5], they are precisely the three models in Example 6.2:
(a) $1 \leftarrow 3 \rightarrow 2$ or $2 \rightarrow 3 \rightarrow 1$
(b) $3 \rightarrow 1 \leftarrow 2$
(c) $3 \rightarrow 2 \quad 1$.

This means that the proof was already given by our analysis in Example 6.2.
Consider the next case $n=4$. Up to relabeling, there are 29 Bayesian networks $\mathcal{C}$ with at least one edge. They are listed in [8, Theorem 11], along with a detailed analysis of the variety $\mathcal{M}_{\mathcal{C}}$ in each case. We embarked towards a proof of Conjecture 6.3 , by examining all 29 models. But the computations are quite challenging, and we leave them for the future.
Example 6.5. Consider the network $\# 15$ in [8, Table 1]. The variety $\mathcal{M}_{\mathcal{C}}$ has dimension 9 and degree 48. An explicit parametrization $\phi$ is shown in [21, page 109]. We can represent $\mathcal{V}_{X, \mathcal{C}}$ by substituting this parametrization into the equations $\operatorname{det}\left(M_{i}\right)=0$ for $i=1,2,3,4$.

The smallest irreducible variety in Conjecture 6.3 arises from the Bayesian network $\mathcal{C}$ with only one edge, here taken to be $n \rightarrow n-1$. The Spohn CI variety $\mathcal{V}_{X, \mathcal{C}}$ contains all the Nash points in Example 6.1. The rest of this paper is dedicated to this scenario. It is important for applications of dependency equilibria because of its proximity to Nash equilibria.

For our one-edge network, $\mathcal{M}_{\mathcal{C}}$ is the Segre variety $\left(\mathbb{P}^{1}\right)^{n-2} \times \mathbb{P}^{3}$ embedded into $\mathbb{P}^{2^{n}-1}$. Hence $\mathcal{M}_{\mathcal{C}}$ has dimension $n+1$. The Spohn CI variety $\mathcal{V}_{X, \mathcal{C}}$ is a curve. This curve lies in a linear subspace of codimension $2 n-4$ in $\mathbb{P}^{2^{n}-1}$. In addition to the quadrics that define the Segre variety $\mathcal{M}_{\mathcal{C}}$, the ideal of $\mathcal{V}_{X, \mathcal{C}}$ contains $2 n-4$ linear forms and $2^{n-1}$ quadrics that depend on the game $X$. The determinants of the matrices $M_{1}, M_{2}, \ldots, M_{n-2}$ give rise to two linear forms each. The determinants of the matrices $M_{n-1}$ or $M_{n}$ give rise to $2^{n-2}$ quadrics.

For example, if $n=3$ then the variety $\mathcal{M}_{\mathcal{C}} \simeq \mathbb{P}^{1} \times \mathbb{P}^{3}$ has the parametric representation

$$
p_{i j k}=\sigma_{i} \tau_{j k} \quad \text { for } 1 \leq i, j, k \leq 2
$$

The prime ideal of $\mathcal{M}_{\mathcal{C}}$ is generated by the six $2 \times 2$ minors of the matrix

$$
\left[\begin{array}{llll}
p_{111} & p_{112} & p_{121} & p_{122}  \tag{24}\\
p_{211} & p_{212} & p_{221} & p_{222}
\end{array}\right] .
$$

After removing common factors from rows and columns, the three matrices in Example 2.3 are

$$
\begin{aligned}
M_{1} & =\left[\begin{array}{ll}
1 & a_{111} \tau_{11}+a_{112} \tau_{12}+a_{121} \tau_{21}+a_{122} \tau_{22} \\
1 & a_{211} \tau_{11}+a_{212} \tau_{12}+a_{221} \tau_{21}+a_{222} \tau_{22}
\end{array}\right], \\
M_{2} & =\left[\begin{array}{ll}
\tau_{11}+\tau_{12} & b_{111} \sigma_{1} \tau_{11}+b_{112} \sigma_{1} \tau_{12}+b_{211} \sigma_{2} \tau_{11}+b_{212} \sigma_{2} \tau_{12} \\
\tau_{21}+\tau_{22} & b_{121} \sigma_{1} \tau_{21}+b_{122} \sigma_{1} \tau_{22}+b_{221} \sigma_{2} \tau_{21}+b_{222} \sigma_{2} \tau_{22}
\end{array}\right], \\
M_{3} & =\left[\begin{array}{ll}
\tau_{11}+\tau_{21} & c_{111} \sigma_{1} \tau_{11}+c_{121} \sigma_{1} \tau_{21}+c_{211} \sigma_{2} \tau_{11}+c_{221} \sigma_{2} \tau_{21} \\
\tau_{12}+\tau_{22} & c_{112} \sigma_{1} \tau_{12}+c_{122} \sigma_{1} \tau_{22}+c_{212} \sigma_{2} \tau_{12}+c_{222} \sigma_{2} \tau_{22}
\end{array}\right] .
\end{aligned}
$$

By multiplying $\operatorname{det}\left(M_{1}\right)$ with $\sigma_{1}$ and with $\sigma_{2}$, we obtain two linear forms in $p_{111}, p_{112}, \ldots, p_{222}$ that vanish on $\mathcal{V}_{X}$. Likewise, by multiplying $\operatorname{det}\left(M_{2}\right)$ and $\operatorname{det}\left(M_{3}\right)$ with $\sigma_{1}$ and with $\sigma_{2}$, we obtain four quadratic forms in $p_{111}, p_{112}, \ldots$, $p_{222}$ that vanish on $\mathcal{V}_{X}$. Three of the six minors of (24) are linearly independent modulo the linear forms. This explains the $2+7$ generators of the prime ideal of the curve $\mathcal{V}_{X, \mathcal{C}}$, which has genus 3 and degree 8 in $\mathbb{P}^{5} \subset \mathbb{P}^{7}$.

Let now $n=4$. The one-edge model $\mathcal{M}_{\mathcal{C}}$ is the Segre variety $\mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{3}$ in $\mathbb{P}^{15}$. Its prime ideal is generated by 46 binomial quadrics. Of these, 32 are linearly independent modulo the four linear forms that arise from the matrices $M_{1}$ and $M_{2}$ as above. Similarly, $M_{3}$ and $M_{4}$ contribute eight quadrics. We conclude that $\mathcal{V}_{X, \mathcal{C}}$ is an curve of genus 23 and degree 30 in $\mathbb{P}^{11} \subset \mathbb{P}^{15}$, and its prime ideal is minimally generated by 4 linear forms and 40 quadrics.

In the recent work [15] it is proven, for generic games, that the Spohn CI curve for the one-edge model is an irreducible complete intersection curve in the Segre variety $\left(\mathbb{P}^{1}\right)^{n-2} \times \mathbb{P}^{3}$. Moreover the authors give an explicit formula for its degree and genus. In the spirit of Datta's universality theorem for Nash equilibria, they show that any affine real algebraic variety $S \subseteq \mathbb{R}^{m}$ defined by $k$ polynomials with $k<m$ can be represented as the Spohn CI variety of an $n$-person game for one-edge Bayesian networks on $n$ binary random variables.

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# Ramification and discriminants of vector bundles and a quick proof of Bogomolov's theorem 

Hirotachi Abo, Robert Lazarsfeld, and Gregory G. Smith

Dedicated to Giorgio Ottaviani on the occasion of his sixtieth birthday.


#### Abstract

By analyzing degeneracy loci over projectivized vector bundles, we recompute the degree of the discriminant locus of a vector bundle and provide a new proof of the Bogomolov instability theorem.


Keywords: Vector bundles, degeneracy loci, discriminant, instability.
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## 1. Introduction

Let $X$ be an $n$-dimensional smooth complex projective variety and let $E$ be a globally generated vector bundle on $X$ of rank $e \leqslant n$. The projective space $\mathbb{P}^{r}=\mathbb{P}\left(H^{0}(X, E)^{*}\right)$ parameterizes sections of $E$ up to scalars. The discriminant of $E$ is the locus in $\mathbb{P}^{r}$, typically a hypersurface, defined by

$$
\Delta(E):=\left\{s \in \mathbb{P}^{r} \mid \text { the zero scheme } \operatorname{Zeroes}(s) \text { of } s \text { is singular }\right\} .
$$

The closed algebraic set $\operatorname{Zeroes}(s)$ is understood to have its natural scheme structure: when $e=n, \Delta(E)$ consists of those sections that vanish at something other than $\int c_{n}(E)$ distinct points. There are various situations where it is of interest to calculate the degree of $\Delta(E)$. This comes up, for instance, in connection with eigenvalues of tensors [2]. In [1], the first author derives a formula for the degree when $e=n$ and $X=\mathbb{P}^{n}$.

The first purpose of this note is to give a very quick derivation of a formula for the (virtual) degree of $\Delta(E)$ reproving some results from [10]. For example, when $e=n$, we show that the expected degree of $\Delta(E)$ is given by

$$
\delta(E)=\int_{X}\left(K_{X}+c_{1}(E)\right) c_{n-1}(E)+n c_{n}(E) .
$$

If each section $s$ in $\Delta(E)$ is singular at several points, then the actual degree of the discriminant hypersurface is smaller than its postulated one. However,
when $E$ is very ample and 1-jet spanned, we also show that $\Delta(E)$ is irreducible of the expected degree.

As one might expect, the basic idea is to compute the class of the singular locus of the universal zero-locus over $\mathbb{P}^{r}$. It turns out that a somewhat related computation leads to an extremely quick proof of the Bogomolov instability theorem for vector bundles of rank 2 on an algebraic surface, reducing the statement in effect to the Riemann-Hurwitz formula. The existence of a proof along these lines seems to have been known to the experts, but as far as we can tell it is not generally familiar. We therefore take this occasion to present the argument. Some time ago, Langer [9, Appendix] gave an even quicker, but related proof, using the fact that stability is preserved under pulling back by generically finite morphisms.

The formula for the ramification locus is derived in Section 1. In Section 2, we show that, when $E$ is very ample and 1-jet spanned, the discriminant locus is irreducible of the expected degree. The proof of the Bogomolov instability theorem occupies Section 3.

## Conventions

We work throughout over the complex numbers $\mathbb{C}$. For any vector space $V$ or vector bundle $E, \mathbb{P}(V)$ or $\mathbb{P}(E)$ denotes the projective space of one-dimensional quotients. Given a smooth variety $X$, the Chow ring of $X$ is $A^{\bullet}(X)$ (or, if the reader prefers, this is the even cohomology ring $\left.H^{2 \bullet}(X, \mathbb{Z})\right)$. We write $c_{i}(E)$ and $s_{i}(E)$ for the $i$-th Chern and Segre classes of a vector bundle $E$ whereas $c(E)$ and $s(E)$ are the corresponding total Chern and Segre classes. Following [6, Example 3.2.7], we use the notation $c(E-F):=c(E) / c(F)=c(E) s(F)$ for the "difference" of the total Chern classes of two bundles. Finally, given a class $\alpha$ in $A^{\bullet}(X)$, the component of $\alpha$ in codimension $k$ is $\alpha_{k} \in A^{k}(X)$.

## 2. Ramification Locus

In this section, we derive a formula for ramification class of certain morphisms from projectivized vector bundles. To be more explicit, fix an $n$-dimensional smooth complex projective variety $X$ and consider a globally-generated vector bundle $E$ on $X$ of rank $e$ such that $e \leqslant n$.

Let $V_{E}:=H^{0}(X, E)$ be the $\mathbb{C}$-vector space of global sections of $E$ and set $r:=\operatorname{dim}_{\mathbb{C}} V_{E}-1$. The trivial vector bundle on $X$ with fibre $V_{E}$ is $V_{E} \otimes_{\mathbb{C}} \mathscr{O}_{X}$ and the kernel of the evaluation map ev $E_{E}: V_{E} \otimes_{\mathbb{C}} \mathscr{O}_{X} \rightarrow E$ is $M_{E}:=\operatorname{Ker}\left(\operatorname{ev}_{E}\right)$. It follows that $M_{E}$ is a vector bundle of rank $r-e+1$ sitting in the short exact sequence

$$
0 \longrightarrow M_{E} \longrightarrow V_{E} \otimes_{\mathbb{C}} \mathbb{O}_{X} \xrightarrow{\mathrm{ev}_{E}} E \longrightarrow 0
$$

Applying the duality functor $(-)^{*}:=\mathscr{H o m}\left(-, \mathscr{O}_{X}\right)$, we obtain the short exact sequence

$$
0 \longrightarrow E^{*} \longrightarrow\left(V_{E} \otimes_{\mathbb{C}} \theta_{X}\right)^{*} \longrightarrow M_{E}^{*} \longrightarrow 0
$$

The surjective map onto $M_{E}^{*}$ identifies the projectivization $\mathbb{P}\left(M_{E}^{*}\right)$ with a closed subscheme in the product $\mathbb{P}\left(\left(V_{E} \otimes_{\mathbb{C}} \mathscr{O}_{X}\right)^{*}\right)=X \times \mathbb{P}\left(V_{E}^{*}\right)$ where $V_{E}^{*}$ is the dual vector space of $V_{E}$. Thus, we have $\mathbb{P}\left(M_{E}^{*}\right)=\left\{(x,[s]) \in X \times \mathbb{P}\left(V_{E}^{*}\right) \mid s(x)=0\right\}$. Let $p_{E}: X \times \mathbb{P}\left(V_{E}^{*}\right) \rightarrow X$ be the projection onto the first factor. We also use $p_{E}$ for the restriction to $\mathbb{P}\left(M_{E}^{*}\right)$. Let $q_{E}: \mathbb{P}\left(M_{E}^{*}\right) \rightarrow \mathbb{P}\left(V_{E}^{*}\right)$ be the restriction of the projection from $X \times \mathbb{P}\left(V_{E}^{*}\right)$ onto the second factor $\mathbb{P}\left(V_{E}^{*}\right)$. When the vector bundle $E$ is unnecessary, we omit the subscripts on $V, M, p$, and $q$.

Guided by Example 14.4.8 in [6], the ramification locus $R(q)$ of the map $q: \mathbb{P}\left(M^{*}\right) \rightarrow \mathbb{P}\left(V^{*}\right)$ is the $(r-1)$-st degeneracy locus of the induced differential $d q: q^{*} \Omega_{\mathbb{P}\left(V^{*}\right)} \rightarrow \Omega_{\mathbb{P}\left(M^{*}\right)} ;$

$$
\begin{aligned}
R(q) & :=\left\{x \in \mathbb{P}\left(M^{*}\right) \mid \text { rank of map } d q \text { at the point } x \text { is at most } r-1\right\} \\
& =\operatorname{Zeroes}\left(\bigwedge^{r} d q\right)
\end{aligned}
$$

Since $\mathbb{P}\left(V^{*}\right)$ and $\mathbb{P}\left(M^{*}\right)$ have dimension $r$ and $n+r-e$, the subscheme $R(q)$ has codimension at most $(r-(r-1))(n+r-e-(r-1))=n-e+1$; see [6, p. 242]. The next proposition provides a formula for the ramification class $[R(q)]$ in the Chow ring $A \bullet\left(\mathbb{P}\left(M^{*}\right)\right)$.

Proposition 2.1. When the ramification locus $R(q)$ has codimension $n-e+1$, its class in $A^{\bullet}\left(\mathbb{P}\left(M^{*}\right)\right)$ is $[R(q)]=\left\{c\left(p^{*} \Omega_{X}\right) s\left(p^{*} E^{*} \otimes \mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(-1)\right)\right\}_{n-e+1}$ and the degree of its pushforward is

$$
\operatorname{deg} q_{*}[R(q)]=\int_{X} p_{*}\left([R(q)] c_{1}\left(\mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(1)\right)^{r-1}\right)
$$

Proof. Since $R(q)$ has codimension $n-e+1$, the Thom-Porteous formula [6, Theorem 14.4] establishes that $[R(q)]=c_{n-e+1}\left(\Omega_{\mathbb{P}\left(M^{*}\right)}-q^{*} \Omega_{\mathbb{P}\left(V^{*}\right)}\right)$. Hence, it suffices to prove that

$$
c_{n-e+1}\left(\Omega_{\mathbb{P}\left(M^{*}\right)}-q^{*} \Omega_{\mathbb{P}\left(V^{*}\right)}\right)=c_{n-e+1}\left(p^{*} \Omega_{X}-p^{*} E^{*} \otimes \mathbb{O}_{\mathbb{P}\left(M^{*}\right)}(-1)\right)
$$

By combining the two short exact sequences

$$
\begin{aligned}
& 0 \longrightarrow \mathscr{I}_{\mathbb{P}\left(M^{*}\right)} /\left.\mathscr{I}_{\mathbb{P}\left(M^{*}\right)}^{2} \xrightarrow{\delta} \Omega_{X \times \mathbb{P}\left(V^{*}\right)}\right|_{\mathbb{P}\left(M^{*}\right)} \longrightarrow \Omega_{\mathbb{P}\left(M^{*}\right)} \longrightarrow 0 \\
& \left.0 \longrightarrow q^{*} \Omega_{\mathbb{P}\left(V^{*}\right)} \longrightarrow \Omega_{X \times \mathbb{P}\left(V^{*}\right)}\right|_{\mathbb{P}\left(M^{*}\right)} \longrightarrow \theta
\end{aligned}
$$

we obtain the commutative diagram:


The snake lemma shows that $\operatorname{Coker}(d q) \cong \operatorname{Coker}(\theta \circ \delta)$, so we deduce that

$$
c_{n-e+1}\left(\Omega_{\mathbb{P}\left(M^{*}\right)}-q^{*} \Omega_{\mathbb{P}\left(V^{*}\right)}\right)=c_{n-e+1}\left(p^{*} \Omega_{X}-\mathscr{S}_{\mathbb{P}\left(M^{*}\right)} / \mathscr{S}_{\mathbb{P}\left(M^{*}\right)}^{2}\right) .
$$

It remains to show that the conormal bundle $\mathscr{I}_{\mathbb{P}\left(M^{*}\right)} / \mathscr{I}_{\mathbb{P}\left(M^{*}\right)}^{2}$ on $\mathbb{P}\left(M^{*}\right)$ is isomorphic to the vector bundle $p^{*} E^{*} \otimes \mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(-1)$. As a closed subscheme of $X \times \mathbb{P}\left(V^{*}\right)$, the projectivization $\mathbb{P}\left(M^{*}\right)$ is the zero scheme of a regular section of $p^{*} E \otimes \mathscr{O}_{X \times \mathbb{P}\left(V^{*}\right)}(1)$; see [6, Appendix B.5.6]. Tensoring the Koszul complex associated to this regular section with $\mathscr{O}_{\mathbb{P}\left(M^{*}\right)}$ produces the desired isomorphism $p^{*} E^{*} \otimes \mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(-1) \cong \mathscr{S}_{\mathbb{P}\left(M^{*}\right)} \otimes \mathscr{O}_{\mathbb{P}\left(M^{*}\right)} \cong \mathscr{I}_{\mathbb{P}\left(M^{*}\right)} / \mathscr{J}_{\mathbb{P}\left(M^{*}\right)}^{2}$.

To prove the second part, observe that $\mathscr{Q}_{\mathbb{P}\left(M^{*}\right)}(1)=q^{*} \mathscr{O}_{\mathbb{P}\left(V^{*}\right)}(1)$; see [11, Example 6.1.5]. It follows from the projection formula that the degree of pushforward is

$$
\begin{aligned}
\operatorname{deg} q_{*}[R(q)] & =\int_{\mathbb{P}\left(V^{*}\right)} q_{*}[R(q)] c_{1}\left(\mathscr{O}_{\mathbb{P}\left(V^{*}\right)}(1)\right)^{r-1} \\
& =\int_{\mathbb{P}\left(M^{*}\right)} q^{*}\left(q_{*}[R(q)] c_{1}\left(\mathscr{O}_{\mathbb{P}\left(V^{*}\right)}(1)\right)^{r-1}\right) \\
& =\int_{\mathbb{P}\left(M^{*}\right)}[R(q)] c_{1}\left(\mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(1)\right)^{r-1} \\
& =\int_{X} p_{*}\left([R(q)] c_{1}\left(\mathbb{O}_{\mathbb{P}\left(M^{*}\right)}(1)\right)^{r-1}\right)
\end{aligned}
$$

In the following examples, we examine three special cases that express ramification class as a polynomial in the Chern classes for $E$ and $\Omega_{X}$. For all nonnegative integers $i$, the defining short exact sequence of the kernel bundle $M_{E}$ shows that $p_{*} c_{1}\left(\mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(1)\right)^{r-e+i}=s_{i}(M)=c_{i}(E)$.

Example $2.2(e=1)$. Suppose that the vector bundle $E$ has rank 1. When ramification locus $R(q)$ has codimension $n$, Proposition 2.1 implies that

$$
\begin{aligned}
{[R(q)] } & =\left\{c\left(p^{*} \Omega_{X}\right) s\left(p^{*} E^{*} \otimes \mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(-1)\right)\right\}_{n} \\
& =\sum_{i=0}^{n} c_{n-i}\left(p^{*} \Omega_{X}\right)(-1)^{i} c_{1}\left(p^{*} E^{*} \otimes \mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(-1)\right)^{i} \\
& =\sum_{i=0}^{n} c_{n-i}\left(p^{*} \Omega_{X}\right) \sum_{j=0}^{i}\binom{i}{j} c_{1}\left(p^{*} E\right)^{j} c_{1}\left(\mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(1)\right)^{i-j},
\end{aligned}
$$

and $\operatorname{deg} q_{*}[R(q)]=\sum_{i=0}^{n}(i+1) \int_{X} c_{n-i}\left(\Omega_{X}\right) c_{1}(E)^{i}$.
Example $2.3(n=e)$. Suppose that the rank of the vector bundle $E$ equals the dimension of its underlying variety $X$. When $R(q)$ has codimension 1, Proposition 2.1 implies that

$$
\begin{aligned}
{[R(q)] } & =\left\{c\left(p^{*} \Omega_{X}\right) s\left(p^{*} E^{*} \otimes \mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(-1)\right)\right\}_{1} \\
& =c_{1}\left(p^{*} \Omega_{X}\right)-c_{1}\left(p^{*} E^{*} \otimes \mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(-1)\right) \\
& =c_{1}\left(p^{*} \Omega_{X}\right)+c_{1}\left(p^{*} E\right)+n c_{1}\left(\mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(1)\right)
\end{aligned}
$$

and $\operatorname{deg} q_{*}[R(q)]=\int_{X}\left(c_{1}\left(\Omega_{X}\right)+c_{1}(E)\right) c_{n-1}(E)+n c_{n}(E)$.
Example $2.4(e=n-1)$. Suppose that the rank of $E$ is the dimension of $X$ minus 1. Observe that $s_{2}\left(p^{*} E\right)=s_{1}\left(p^{*} E\right)^{2}-c_{2}\left(p^{*} E^{*}\right)=c_{1}\left(p^{*} E\right)^{2}-c_{2}\left(p^{*} E\right)$ and

$$
\begin{aligned}
& s_{2}\left(p^{*} E^{*} \otimes \mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(-1)\right) \\
= & \binom{n}{n-2} c_{1}\left(\mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(1)\right)^{2}-n s_{1}\left(p^{*} E^{*}\right) c_{1}\left(\mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(1)\right)+s_{2}\left(p^{*} E^{*}\right) \\
= & \binom{n}{2} c_{1}\left(\mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(1)\right)^{2}-n c_{1}\left(p^{*} E\right) c_{1}\left(\mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(1)\right)+c_{1}\left(p^{*} E\right)^{2}-c_{2}\left(p^{*} E\right)
\end{aligned}
$$

see [6, p. 50 and Example 3.1.1]. When $R(q)$ codimension 2, Proposition 2.1 implies that

$$
\begin{aligned}
& {[R(q)] } \\
= & \left\{c\left(p^{*} \Omega_{X}\right) s\left(p^{*} E^{*} \otimes \mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(-1)\right)\right\}_{2} \\
= & c_{2}\left(p^{*} \Omega_{X}\right)+c_{1}\left(p^{*} \Omega_{X}\right) s_{1}\left(p^{*} E^{*} \otimes \mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(-1)\right)+s_{2}\left(p^{*} E^{*} \otimes \mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(-1)\right) \\
= & c_{2}\left(p^{*} \Omega_{X}\right)+c_{1}\left(p^{*} \Omega_{X}\right)\left(c_{1}\left(p^{*} E\right)+(n-1) c_{1}\left(\mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(1)\right)\right) \\
& \quad+\binom{n}{2} c_{1}\left(\mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(1)\right)^{2}-n c_{1}\left(p^{*} E\right) c_{1}\left(\mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(1)\right)+c_{1}\left(p^{*} E\right)^{2}-c_{2}\left(p^{*} E\right)
\end{aligned}
$$

and

$$
\begin{gathered}
\operatorname{deg} q_{*}[R(q)]=\int_{X}\left(c_{2}\left(\Omega_{X}\right)+c_{1}\left(\Omega_{X}\right) c_{1}(E)+c_{1}(E)^{2}-c_{2}(E)\right) c_{n-2}(E) \\
+\left((n-1) c_{1}\left(\Omega_{X}\right)+n c_{1}(E)\right) c_{n-1}(E)
\end{gathered}
$$

## 3. Discriminant Locus of a Vector Bundle

This section determines the degree of the discriminant of a vector bundle. As in the first section, $X$ is an $n$-dimensional smooth complex projective variety $X$ and $E$ is a globally-generated vector bundle on $X$ of rank $e \leqslant n$. Set $V_{E}:=H^{0}(X, E)$, let $M_{E}$ be the kernel of $\mathrm{ev}_{E}: V_{E} \otimes_{\mathbb{C}} \mathscr{O}_{X} \rightarrow E$, and write $q_{E}: \mathbb{P}\left(M_{E}^{*}\right) \rightarrow \mathbb{P}\left(V_{E}^{*}\right)$ for composition of the inclusion $\mathbb{P}\left(M_{E}^{*}\right) \rightarrow X \times \mathbb{P}\left(V_{E}^{*}\right)$ and the projection $X \times \mathbb{P}\left(V_{E}^{*}\right) \rightarrow \mathbb{P}\left(V_{E}^{*}\right)$ onto the second factor.

The discriminant locus $\Delta(E)$ of the vector bundle $E$ is the reduced scheme structure on the image of the ramification locus $R\left(q_{E}\right)$ under the map $q_{E}$. A section $s$ in $V_{E}^{*}$ is nonsingular if its zero scheme $\operatorname{Zeroes}(s)$ is nonsingular and has codimension $e$ in $\mathbb{P}\left(V_{E}^{*}\right)$; otherwise it is singular. With this terminology, one verifies that

$$
\Delta(E):=\left\{[s] \in \mathbb{P}\left(V_{E}^{*}\right) \mid \text { the section } s \text { is singular }\right\} .
$$

The defect of the vector bundle $E$ is the integer $\operatorname{def}(E):=\operatorname{codim} \Delta(E)-1$, the expected degree of the discriminant locus $\Delta(E)$ is $\delta(E):=\operatorname{deg}\left(q_{E}\right)_{*}\left[R\left(q_{E}\right)\right]$, and the coefficient of $R\left(q_{E}\right)$ in $\left[R\left(q_{E}\right)_{\text {red }}\right]$ is the unique positive integer $m_{E}$ such that $\left[R\left(q_{E}\right)\right]=m_{E}\left[R\left(q_{E}\right)_{\text {red }}\right]$ in the Chow ring $A^{\bullet}\left(\mathbb{P}\left(M_{E}^{*}\right)\right)$.

The significance of these numerical invariants becomes clear with an additional hypothesis.
Remark 3.1. Assume that the ramification locus $R\left(q_{E}\right)$ is irreducible and has dimension $r-1$ (or equivalently codimension $n-e+1$ ). It follows that the discriminant locus $\Delta(E)$ is also irreducible. For the function fields $\mathbb{C}\left(R\left(q_{E}\right)\right)$ and $\mathbb{C}(\Delta(E))$ of the reduced schemes $R\left(q_{E}\right)_{\text {red }}$ and $\Delta(E)$, the degree of the field extension is $\left[\mathbb{C}\left(R\left(q_{E}\right)\right): \mathbb{C}(\Delta(E))\right]$ and the degree of $R\left(q_{E}\right)$ over $\Delta(E)$ is

$$
\operatorname{deg} R\left(q_{E}\right) / \Delta(E):= \begin{cases}{\left[\mathbb{C}\left(R\left(q_{E}\right)\right): \mathbb{C}(\Delta(E))\right]} & \text { if } \operatorname{dim} \Delta(E)=r-1 \\ 0 & \text { if } \operatorname{dim} \Delta(E)<r-1\end{cases}
$$

The definition of the pushforward of a cycle gives

$$
\left(q_{E}\right)_{*}\left[R\left(q_{E}\right)\right]=m_{E}\left(\operatorname{deg} R\left(q_{E}\right) / \Delta(E)\right)[\Delta(E)] ;
$$

see [6, Section 1.4]. Hence, we have $\operatorname{def}(X)>0$ if and only if $\delta(E)=0$. When $R\left(q_{E}\right)$ is integral and birational to $\Delta(E)$, we also have $\operatorname{deg} \Delta(E)=\delta(E)$.

Although the next result is likely known to experts, we could not find an adequate reference.

Theorem 3.2. Assume that $X$ an n-dimensional smooth projective variety $X$ and let $E$ be a very ample vector bundle on $X$ of rank $e \leqslant n$. Let $\pi: \mathbb{P}(E) \rightarrow X$ be the projective bundle associated to $E$ and let $L:=\mathscr{O}_{\mathbb{P}(E)}(1)$ be the tautological line bundle on the projectivization $\mathbb{P}(E)$.

- The discriminant locus $\Delta(E)$ of the vector bundle $E$ is isomorphic to the discriminant locus $\Delta(L)$ of the line bundle L. In particular, the discriminant locus $\Delta(E)$ is irreducible.
- When the discriminant locus $\Delta(E)$ is a hypersurface, the reduced scheme $R\left(q_{E}\right)_{\text {red }}$ is birational to $\Delta(E)$ and

$$
\operatorname{deg} \Delta(E)=m_{E}\left\{c\left(p^{*} \Omega_{X}\right) s\left(p^{*} E^{*} \otimes \mathbb{O}_{\mathbb{P}\left(M_{E}^{*}\right)}(-1)\right)\right\}_{n-e+1} .
$$

Proof. The canonical isomorphism $V_{E}=H^{0}(X, E) \xrightarrow{\cong} H^{0}(\mathbb{P}(E), L)=V_{L}$ induces an isomorphism $\varphi: \mathbb{P}\left(V_{L}^{*}\right) \rightarrow \mathbb{P}\left(V_{E}^{*}\right)$. It is enough to show that the restriction of $\varphi$ to the discriminant locus $\Delta(L)$ yields an isomorphism from $\Delta(L)$ to $\Delta(E)$. To accomplish this, it suffices to prove that a section $s$ in $V_{E}^{*}$ is singular if and only if the corresponding section $\tilde{s}$ in $V_{L}^{*}$ is singular. As this assertion is local, we may assume that $X$ is affine and $E \cong \bigoplus_{i=1}^{e} 0_{X}$. Hence, there exist $f_{1}, f_{2}, \ldots, f_{e} \in H^{0}\left(X, \Theta_{X}\right)$ such that $s=\left(f_{1}, f_{2}, \ldots, f_{e}\right)$ and $\tilde{s}=f_{1} x_{1}+f_{2} x_{2}+\cdots+f_{e} x_{e}$ where $x_{1}, x_{2}, \ldots, x_{e}$ are homogeneous coordinates of $\mathbb{P}^{e-1}=\mathbb{P}\left(V_{E}^{*}\right)$. The assertion now follows from a local calculation of derivatives as appears in [1, Subsection 3.2].

The same calculation shows that restriction of the map

$$
\pi \times \varphi: \mathbb{P}(E) \times \mathbb{P}\left(V_{L}^{*}\right) \rightarrow X \times \mathbb{P}\left(V_{E}^{*}\right)
$$

to $R\left(q_{L}\right)_{\text {red }}$ is a birational map from $R\left(q_{L}\right)_{\text {red }}$ to $R\left(q_{E}\right)_{\text {red }}$. When $\Delta(L)$ is a hypersurface, Proposition 3.2 in [7] demonstrates that reduced scheme $R\left(q_{L}\right)_{\text {red }}$ is birational to discriminant locus $\Delta(L)$. It follows that the reduced scheme $R\left(q_{E}\right)_{\text {red }}$ is birational to discriminant locus $\Delta(E)$. Finally, the degree formula is an immediate consequence of Remark 3.1.

To prove that the ramification locus is reduced, we first record a general observation about degeneracy loci. Consider three vector bundles $A, B$, and $C$ on a smooth projective variety $X$ together with an injective vector bundle morphism $\mu: A \otimes B^{*} \rightarrow C$. Let $\varpi: \mathbb{P}(C) \rightarrow X$ be the projective bundle associated to $C$, let $\eta: \varpi^{*} C \rightarrow \mathscr{O}_{\mathbb{P}(C)}(1)$ be the natural surjective morphism, and let $\tilde{\mu}: \varpi^{*}\left(A \otimes B^{*}\right) \rightarrow \mathscr{O}_{\mathbb{P}(C)}(1)$ be the composition of $\mu$ with $\eta$. The map $\tilde{\mu}$ corresponds to the morphism $\mu^{\prime}: \varpi^{*} A \rightarrow \varpi^{*} B \otimes \mathscr{O}_{\mathbb{P}(C)}(1)$ via tensor-hom adjunction.

Lemma 3.3. For any nonnegative integer $k$, the $k$-th degeneracy locus

$$
D_{k}\left(\mu^{\prime}\right):=\operatorname{Zeroes}\left(\bigwedge^{k+1} \mu^{\prime}\right)
$$

is reduced and Cohen-Macaulay of codimension $(\operatorname{rank}(A)-k)(\operatorname{rank}(B)-k)$.
Proof. As the assertion is local, we may assume that $X$ is affine and the three vector bundles are trivial. Let $U, V$, and $W$ be complex vector spaces such that $A=U \otimes_{\mathbb{C}} \mathscr{O}_{X}, B=V \otimes_{\mathbb{C}} \mathscr{O}_{X}$, and $C=W \otimes_{\mathbb{C}} \mathscr{O}_{X}$. For each nonnegative integer $k$, let $D_{k}(U, V)$ be the locus of points in $\mathbb{P}\left(U \otimes_{\mathbb{C}} V^{*}\right)=\mathbb{P}\left(\operatorname{Hom}_{\mathbb{C}}(U, V)\right)$ whose corresponding linear transformations from $U$ to $V$ have rank at most $k$.

Consider the projective bundle $\rho: \mathbb{P}\left(A \otimes B^{*}\right) \rightarrow X$ associated to $A \otimes B^{*}$. On $Y:=\mathbb{P}\left(A \otimes B^{*}\right)$, the surjective morphism $\theta: \rho^{*}\left(A \otimes B^{*}\right) \rightarrow \mathscr{O}_{Y}(1)$ corresponds to the morphism $\theta^{\prime}: \rho^{*} A \rightarrow \rho^{*} B \otimes \mathscr{O}_{Y}(1)$ whose $k$-th degeneracy locus $D_{k}\left(\theta^{\prime}\right)$ is $X \times D_{k}(U, V)$. In particular, $D_{k}\left(\theta^{\prime}\right)$ is reduced and Cohen-Macaulay of codimension $(\operatorname{rank}(A)-k)(\operatorname{rank}(B)-k)$.

Let $Q$ be the cokernel of the map $\mu: A \otimes B^{*} \rightarrow C$. It follows that $\mathbb{P}(Q)$ is a subbundle of $\mathbb{P}(C)$. Let $\psi: \mathbb{P}(C)-\mathbb{P}(Q) \rightarrow Y$ be the associated trivial affine bundle over $X$. Since the map $\mu^{\prime}: \varpi^{*} A \rightarrow \varpi^{*} B \otimes \mathscr{O}_{\mathbb{P}(C)}(1)$ is nonzero away from $\mathbb{P}(Q)$, we have the commutative diagram

with the property that $\mu^{\prime}=\psi^{*}\left(\theta^{\prime}\right)$. Hence, the $k$-th degeneracy locus $D_{k}\left(\mu^{\prime}\right)$ is the "cone" over $D_{k}\left(\theta^{\prime}\right)$ in $\mathbb{P}(C)$ with vertex $\mathbb{P}(Q)$; it is the product of $X$ and the cone over $D_{k}(U, V)$ in $\mathbb{P}(W)$ with vertex $\mathbb{P}\left(W /\left(U \otimes V^{*}\right)\right)$. We conclude that the $k$-th degeneracy locus $D_{k}\left(\mu^{\prime}\right)$ is also reduced and Cohen-Macaulay of codimension $(\operatorname{rank}(A)-k)(\operatorname{rank}(B)-k)$.

To ensure that the ramification locus $R\left(q_{E}\right)$ is reduced, we rely on a stronger hypothesis than $E$ being very ample. To define this condition, we use the first jet bundle $J_{1}(E)$ that parametrizes the first-order Taylor expansions of the sections of $E$. More precisely, let $\mathscr{F}$ be the ideal sheaf defining the diagonal embedding $X \hookrightarrow X \times X$ and let $\operatorname{pr}_{1}, \operatorname{pr}_{2}$ : Zeroes $\left(\mathscr{g}^{2}\right) \rightarrow X$ be the restrictions of the projections $X \times X \rightarrow X$ to the closed subscheme $\operatorname{Zeroes}\left(\mathscr{F}^{2}\right) \subset X \times X$. The first jet bundle is $J_{1}(E):=\left(\mathrm{pr}_{1}\right)_{*} \mathrm{pr}_{2}^{*} E$; this is also called the bundle of principal parts in [6, Example 2.5.6]. The vector bundle $J_{1}(E)$ has rank $n+1$ and sits in the short exact sequence

$$
0 \longrightarrow \Omega_{X} \otimes E \longrightarrow J_{1}(E) \longrightarrow E \longrightarrow 0
$$

The vector bundle $E$ is 1-jet spanned if the evaluation map $V_{E} \otimes_{\mathbb{C}} \mathcal{O}_{X} \rightarrow J_{1}(E)$ is surjective; see [3, Subsection 1.3]. With this concept, we have the following corollary.

Corollary 3.4. Let $X$ be an n-dimensional smooth projective variety and let $E$ be a very ample vector bundle of rank $e \leqslant n$. Assuming that the vector bundle $E$ is 1-jet spanned, the ramification locus $R\left(q_{E}\right)$ is reduced and CohenMacaulay of codimension $n-e+1$, so $\Delta(E)=\left(q_{E}\right)_{*}\left[R\left(q_{E}\right)\right]$. Furthermore, the discriminant locus $\Delta(E)$ is a hypersurface if and only if we have $\delta(E)>0$. When $\Delta(E)$ is a hypersurface, the degree of discriminant locus is

$$
\operatorname{deg} \Delta(E)=\left\{c\left(p^{*} \Omega_{X}\right) s\left(p^{*} E^{*} \otimes \mathscr{O}_{\mathbb{P}\left(M^{*}\right)}(-1)\right)\right\}_{n-e+1}
$$

Proof. By Theorem 3.2 and Lemma 3.3, it suffices to show the existence of an injective vector bundle morphism from $E^{*} \otimes\left(\Omega_{X}\right)^{*}$ to $M_{E}^{*}$ or equivalently a surjective map from $M_{E}$ to $E \otimes \Omega_{X}$. To establish this, we combine the defining short exact sequence for $M_{E}$ with the canonical short exact sequence for $J_{1}(E)$ to obtain the following commutative diagram with exact rows:


Since $E$ is 1-jet spanned, the second vertical map is surjective. Hence, the snake lemma implies that the first vertical map is also surjective.

Remark 3.5. Remark 0.3.2 in [4] establishes that, for any very ample line bundle $L$ on an $m$-dimensional smooth projective variety $Y$, we have $\operatorname{def}(L)>0$ if and only if $c_{m}\left(J_{1}(L)\right)=0$. When the discriminant locus $\Delta(L)$ is a hypersurface, this remark also shows that $\operatorname{deg} \Delta(L)=\int_{Y} c_{m}\left(J_{1}(L)\right)$.

Given an $n$-dimensional smooth projective variety $X$ and a very ample vector bundle $E$ on $X$ of rank $e \leqslant n$, Lanteri and Muñoz compute the top Chern class of the first jet bundle of the line bundle $L:=\mathscr{O}_{\mathbb{P}(E)}(1)$. More precisely, when $Y=\mathbb{P}(E)$, Proposition 1.1 in [10] expresses $c_{n+e-1}\left(J_{1}(L)\right)$ as a polynomial in the Chern classes of $E$ and the tangent bundle $T_{X}$. Under the assumption that the vector bundle $E$ is 1 -jet spanned, Corollary 3.4 provides a different formula for the degree of $\Delta(E)$.
Example 3.6. Let $L$ be a very ample line bundle on a smooth projective variety $X$. The line bundle $L$ is 1 -jet spanned; see [3, Subsection 1.3]. When the discriminant locus $\Delta(L)$ is a hypersurface, Example 2.2 and Corollary 3.4 show that

$$
\operatorname{deg} \Delta(L)=\sum_{i=0}^{n}(i+1) \int_{X} c_{n-i}\left(\Omega_{X}\right) c_{1}(L)^{i} .
$$

Thus, we recover the degree of the classical discriminant; see [7, Example 3.12].

Our second corollary focuses on vector bundles whose rank equals the dimension of their underlying variety. Part of this result provides an alternative proof for Proposition 2.2 in [10].

Corollary 3.7. Let $X$ be a n-dimensional smooth complex projective variety. For any very ample vector bundle $E$ of rank $n$ on $X$, the discriminant locus $\Delta(E)$ is irreducible and $\operatorname{def}(E)>0$ if and only if $X=\mathbb{P}^{n}$ and $E=\bigoplus_{i=1}^{n} \widehat{O}_{\mathbb{P}^{n}}(1)$. Assuming that the vector bundle $E$ is 1 -jet spanned and $(X, E) \neq\left(\mathbb{P}^{n}, \bigoplus_{i=1}^{n} \mathscr{O}_{\mathbb{P}^{n}}(1)\right)$, the discriminant locus $\Delta(E)$ is an irreducible hypersurface of degree

$$
\int_{X}\left(c_{1}\left(\Omega_{X}\right)+c_{1}(E)\right) c_{n-1}(E)+n c_{n}(E)
$$

Proof. Theorem 3.2 and Example 2.3 show that the discriminant locus $\Delta(E)$ is irreducible and $\operatorname{def}(E)>0$ if and only if

$$
\delta(E)=\int_{X}\left(c_{1}\left(\Omega_{X}\right)+c_{1}(E)\right) c_{n-1}(E)+n c_{n}(E)=0
$$

When $(X, E)=\left(\mathbb{P}^{n}, \bigoplus_{i=1}^{n} \mathscr{P}_{\mathbb{P}^{n}}(1)\right)$, we have $\delta(E)=((-n-1)+n) n+n=0$ and $\operatorname{def}(E)>0$. Hence, it suffices to show that, for any very ample $E$ excluding $\bigoplus_{i=1}^{n} \mathscr{O}_{\mathbb{P}^{n}}(1)$, we have $\delta(E)>0$. If $E$ is 1 -jet spanned as well as very ample, then Corollary 3.4 shows that $\operatorname{deg} \Delta(E)=\delta(E)$.

Since $E$ is very ample, we have $\int_{X} c_{n}(E)>0$; see [5, Proposition 2.2]. Thus, it is enough to prove that $\int_{X}\left(c_{1}\left(\Omega_{X}\right)+c_{1}(E)\right) c_{n-1}(E) \geqslant 0$. Let $K_{X}$ be the canonical divisor on $X$ and let $D$ be the Cartier divisor associated to $\operatorname{det}(E)$. Since $E$ is very ample, $D$ is also. Moreover, Theorem 2 in [13] establishes that the adjoint divisor $K_{X}+D$ is nef unless $(X, E)=\left(\mathbb{P}^{n}, \bigoplus_{i=1}^{n} \mathscr{O}_{\mathbb{P}^{n}}(1)\right)$. The very ampleness of the vector bundle $E$ implies that $c_{n-1}(E) \neq 0$; again see [5, Proposition 2.2]. We deduce that $c_{n-1}(E)$ is the class of a curve $C$ by a Bertini-type argument; see [8, Theorem B]. It follows that

$$
\int_{X}\left(c_{1}\left(\Omega_{X}\right)+c_{1}(E)\right) c_{n-1}(E)=\left(K_{X}+D\right) \cdot C \geqslant 0
$$

To illustrate this corollary, we recompute the degree of the discriminant locus for nonnegative twists of the tangent bundle on $\mathbb{P}^{n}$; see [2, Corollary 4.2] and [1, Example 4.9].
EXAMPLE 3.8. Let $d$ be a nonnegative integer and let $T_{\mathbb{P}^{n}}$ be the tangent bundle on $\mathbb{P}^{n}$. We have $c_{1}\left(\Omega_{\mathbb{P}^{n}}\right)=c_{1}\left(\mathcal{O}_{\mathbb{P}^{n}}(-n-1)\right)$. From the Euler sequence

$$
0 \longrightarrow \mathscr{O}_{\mathbb{P}^{n}} \longrightarrow \bigoplus_{i=1}^{n} \mathscr{O}_{\mathbb{P}^{n}}(1) \longrightarrow T_{\mathbb{P}^{n}} \longrightarrow 0
$$

we deduce that

$$
\int_{\mathbb{P}^{n}} c_{i}\left(T_{\mathbb{P}^{n}}(d)\right)=\sum_{j=0}^{i}\binom{n-j}{i-j} d^{i-j}\binom{n+1}{j}
$$

for all nonnegative integers $i$. Combining Propositions 2.1-2.3 in [3], the Euler sequence also shows that vector bundle $T_{\mathbb{P}^{n}}(d)$ is very ample and 1 -jet spanned. Thus, Corollary 3.7 establishes that the discriminant locus $\Delta\left(T_{\mathbb{P}^{n}}(d)\right)$ is an irreducible hypersurface and

$$
\begin{aligned}
\operatorname{deg} \Delta\left(T_{\mathbb{P}^{n}}(d)\right) & =n d \sum_{j=0}^{n-1}(n-j) d^{n-1-j}\binom{n+1}{j}+n \sum_{j=0}^{n} d^{n-j}\binom{n+1}{j} \\
& =n \sum_{j=0}^{n} d^{n-j}(n+1-j)\binom{n+1}{n+1-j} \\
& =n(n+1) \sum_{j=0}^{n} d^{n-j}\binom{n}{j}=n(n+1)(d+1)^{n}
\end{aligned}
$$

## 4. Bogomolov Instability Theorem

In this section, we use calculations involving the discriminant divisor of a multisection to give a simple proof of the Bogomolov instability theorem for vector bundles having rank 2 on an algebraic surface. At the very least, it was known to experts that one could give an argument along these lines. However, since it fits well with the themes of this note and is not widely known, we felt it worthwhile to include it here. We refer the reader to [9] for another approach having several points of contact with the present proof.

Let $X$ be a smooth complex projective surface. We consider a vector bundle $E$ of rank 2 on $X$, and denote by $D$ a Cartier divisor associated to $\operatorname{det}(E)$. The vector bundle $E$ is Bogomolov unstable if there exist a divisor $A$ and a finite scheme $W \subset X$ (possibly empty) such that the sequence

$$
0 \longrightarrow \mathscr{O}_{X}(A) \longrightarrow E \longrightarrow \mathscr{O}_{X}(D-A) \otimes \mathscr{I}_{W} \longrightarrow 0,
$$

is exact, $(2 A-D)^{2}>4$ length $(W)$, and $(2 A-D) \cdot H>0$ for some (or any) ample divisor $H$ on $X$. Roughly speaking, being Bogomolov unstable means that the vector bundle $E$ contains an unexpectedly positive subsheaf.

Bogomolov's theorem asserts that instability is detected numerically.
Theorem 4.1. The vector bundle $E$ is Bogomolov unstable if and only if

$$
\int_{X} c_{1}(E)^{2}-4 c_{2}(E)>0
$$

The defining exact sequence for a Bogomolov unstable vector bundle gives

$$
\int_{X} c_{2}(E)=\operatorname{length}(W)+A \cdot(D-A)
$$

so the inequality holds. Thus, the essential content of the Theorem 4.1 is the converse statement: the inequality implies the existence of a destabilizing subsheaf $\mathscr{O}_{X}(A)$.

For our proof of this implication, suppose that $\int_{X}\left(c_{1}(E)^{2}-4 c_{2}(E)\right)>0$. Let $\pi: \mathbb{P}(E) \rightarrow X$ the projectivization of $E$, so $\operatorname{dim} \mathbb{P}(E)=3$. The starting point, as in other arguments, is the next lemma.

Lemma 4.2. When the vector bundle E satisfies the inequality in Theorem 4.1, the line bundle $\mathscr{O}_{\mathbb{P}(E)}(2) \otimes \pi^{*} \mathbb{O}_{X}(-D)$ on $\mathbb{P}(E)$ is big. In other words, there is a positive number $C>0$ such that, for all sufficiently large integers $m$, we have
$h^{0}\left(\mathbb{P}(E), \mathscr{O}_{\mathbb{P}(E)}(2 m) \otimes \pi^{*} \mathscr{O}_{X}(-m D)\right)=h^{0}\left(X, \operatorname{Sym}^{2 m}(E) \otimes \mathscr{O}_{X}(-m D)\right) \geqslant C m^{3}$.
Idea of proof. The asymptotic Riemann-Roch theorem [11, Theorem 1.1.24] shows that

$$
\chi\left(X, \operatorname{Sym}^{2 m}(E) \otimes \mathscr{O}_{X}(-m D)\right)=\frac{1}{3}\left(c_{1}^{2}(E)-4 c_{2}(E)\right) m^{3}+O\left(m^{2}\right)
$$

The assertion follows via Serre duality and the fact that the vector bundle $\operatorname{Sym}^{2 m}(E) \otimes \mathscr{O}_{X}(-m D)$ has trivial determinant; see [12, Proposition 2].

Now let $H$ be an ample divisor on $X$. By an argument of Kodaira [11, Proposition 2.2.6], it follows from the lemma that, for all sufficiently large integers $m$, we have $H^{0}\left(\mathbb{P}(E), \mathscr{O}_{\mathbb{P}(E)}(2 m) \otimes \pi^{*} \mathscr{O}_{X}(-m D-H)\right) \neq 0$. Fix one such integer $m$ and choose nonzero section

$$
s \in H^{0}\left(\mathbb{P}(E), \mathscr{O}_{\mathbb{P}}(E)(2 m) \otimes \pi^{*} \mathscr{O}_{X}(-m D-H)\right)
$$

Let $Z:=\operatorname{Zeroes}(s)$ be the zero locus of the global section $s$. The subscheme $Z$ is a divisor on $\mathbb{P}(E)$ of relative degree $2 m$ over $X$.

We study the irreducible components of $Z$ with the aim of singling out a particularly interesting one. To begin, let $Z_{0} \subset \mathbb{P}(E)$ denote the union of any "vertical" components of $Z: Z_{0}$ is the preimage under $\pi$ of the zeroes of a section of $\mathscr{O}_{X}\left(-A_{0}\right)$ for some anti-effective divisor $A_{0}$ on $X$. Write $Z_{1}, Z_{2}, \ldots, Z_{t}$ in $\mathbb{P}(E)$ for the remaining irreducible components of $Z$ allowing repetitions to account for multiplicities. In other words, each $Z_{i} \subset \mathbb{P}(E)$ is a reduced and irreducible divisor that is defined by a section of $\mathscr{O}_{\mathbb{P}(E)}\left(d_{i}\right) \otimes \pi^{*} \mathscr{O}_{X}\left(-A_{i}\right)$ for some divisor $A_{i}$ on $X$ and positive integer $d_{i}$. By construction, the divisor $A_{0}+A_{1}+\cdots+A_{t}$ is linearly equivalent to $m D+H$ and $d_{1}+d_{2}+\cdots+d_{t}=2 m$, so the divisor $\sum_{i \geqslant 1}\left(A_{i}-\frac{d_{i}}{2} D\right)$ is numerically equivalent to $H-A_{0}$. Since $-A_{0}$
is an effective divisor, it follows that $\left(\sum_{i \geqslant 1} 2 A_{i}-d_{i} D\right) \cdot H>0$. By reindexing the components if necessary, we may assume that $\left(2 A_{1}-d_{1} D\right) \cdot H>0$.

The idea is to consider the discriminant divisor $\Delta \subseteq X$ over which the fibre of the map $Z_{1} \rightarrow X$ is not $d_{1}$ distinct points. Specifically, Proposition 4.3 shows that the class of $\Delta$ is given by $\delta=d_{1}\left(d_{1}-1\right) D-2\left(d_{1}-1\right) A_{1}$ and $\delta$ is either effective or zero, so $\delta \cdot H \geqslant 0$. However, if $d_{1}>1$, then this would contradict the assumption that $\left(2 A_{1}-d_{1} D\right) \cdot H>0$. Thus, we have $d_{1}=1$ and $Z_{1}$ is defined by a (necessarily saturated) section in $H^{0}\left(\mathbb{P}(E), \mathscr{O}_{\mathbb{P}(E)}(1) \otimes \pi^{*} \mathscr{O}_{X}\left(-A_{1}\right)\right)$. The corresponding section in $H^{0}\left(X, E \otimes \mathcal{O}_{X}(-A)\right)$ defines a closed subscheme $W$ of $X$ and gives rise to a short exact sequence

$$
0 \longrightarrow \mathscr{O}_{X}\left(A_{1}\right) \longrightarrow E \longrightarrow \mathscr{O}_{X}\left(D-A_{1}\right) \otimes \mathscr{I}_{W} \longrightarrow 0
$$

The inequality $\int_{X} c_{1}(E)^{2}-4 c_{2}(E)>0$ implies that $(2 A-D)^{2}>4$ length $(W)$ and $(2 A-D) \cdot H>0$. Therefore, we have established that the vector bundle $E$ is unstable.

It remains to prove the following proposition.
Proposition 4.3. Let $E$ be a vector bundle on $X$ having rank 2 and satisfying $\operatorname{det}(E)=\mathscr{O}_{X}(D)$, let $\pi: \mathbb{P}(E) \rightarrow X$ be the projectivization of $E$, and consider $a$ reduced and irreducible divisor

defined by a section of $\mathscr{O}_{\mathbb{P}(E)}(d) \otimes \pi^{*} \mathscr{O}_{X}(-A)$ for some positive integer $d$. The locus $\Delta(f) \subseteq X$ of points $x \in X$ over which the fibre $f^{-1}(x)$ fails to consist of $d$ distinct points supports an effective divisor in the class

$$
\delta=d(d-1) D+2(d-1) A
$$

In particular, this class is effective or zero.
Proof. Consider the set $\Gamma:=\{y \in Y \mid f$ is not étale at $y\}$. The map $f$ is generically étale because $Y$ is reduced. It follows that $\Gamma$ has dimension 1 (or is empty) and $\Delta(f)=f(\Gamma)$. We claim that, viewed as a cycle of codimension 2 on $\mathbb{P}(E), \Gamma$ supports the effective class

$$
\begin{equation*}
\gamma:=\left((d-2) c_{1}\left(\mathscr{O}_{\mathbb{P}(E)}(1)\right)+\pi^{*}(D-A)\right) \cdot\left(d c_{1}\left(\mathscr{O}_{\mathbb{P}(E)}(1)\right)-\pi^{*} A\right) . \tag{*}
\end{equation*}
$$

There are at least two ways to confirm this claim.

- As a cycle on $Y, \gamma$ is the class of the first degeneracy locus of the induced differential $d f: f^{*} \Omega_{X} \rightarrow \Omega_{Y}$, so

$$
\gamma=c_{1}\left(\Omega_{Y}-f^{*} \Omega_{X}\right)=c_{1}\left(\mathscr{O}_{Y}\left(K_{Y}-f^{*} K_{X}\right)\right)
$$

which is the class of the relative canonical divisor $K_{Y / X}:=K_{Y}-f^{*} K_{X}$. The adjunction formula shows that $K_{Y / X}=\left.\left(K_{\mathbb{P}(E) / X}+Y\right)\right|_{Y}$. Thus, as a cycle on $\mathbb{P}(E)$, we have $\gamma=\left[\left.\left(K_{\mathbb{P}(E) / X}+Y\right)\right|_{Y}\right] \cap[Y]$. Since we also have $[Y]=c_{1}\left(\mathscr{O}_{\mathbb{P}}(E)(d) \otimes \pi^{*} \mathscr{O}_{X}(-A)\right)$, the equation $(*)$ follows from the equality $\left[K_{\mathbb{P}(E) / X}\right]=c_{1}\left(\mathscr{O}_{\mathbb{P}(E)}(-2) \otimes \pi^{*} \mathscr{O}_{X}(D)\right)$; see [11, Section 7.3.A].

- The section $s$ in $H^{0}\left(X, \mathscr{O}_{\mathbb{P}(E)}(d) \otimes \pi^{*} \mathscr{O}_{X}(-A)\right)$ defining $Y$ lifts to a section of the first relative jet bundle of $d s \in H^{0}\left(\mathbb{P}(E), J_{1}^{\pi}\left(\mathscr{O}_{\mathbb{P}(E)}(d) \otimes \pi^{*} \mathscr{O}_{X}(-A)\right)\right.$ ), and $\Gamma=\operatorname{Zeroes}(d s)$. From the canonical short exact sequence

we see that $\gamma=c_{2}\left(J_{1}^{\pi}\left(\mathscr{O}_{\mathbb{P}}(E)(d) \otimes \pi^{*} \mathscr{O}_{X}(-A)\right)\right)$, which again establishes the equation (*).
It remains to check that $\pi_{*}(\gamma)=\delta$. This follows from the Grothendieck relation

$$
c_{1}\left(\mathscr{O}_{\mathbb{P}(E)}(1)\right)^{2}-\pi^{*}\left(c_{1}(E)\right) \cdot c_{1}\left(\mathscr{O}_{\mathbb{P}(E)}(1)\right)+\pi^{*}\left(c_{2}(E)\right)=0,
$$

$\pi_{*}\left(\pi^{*}(\alpha) \cdot c_{1}\left(\mathscr{O}_{\mathbb{P}}(E)(1)\right)\right)=\alpha$, and $\pi_{*}\left(\pi^{*}(\beta)\right)=0$ for any classes $\alpha \in A^{1}(X)$ and $\beta \in A^{2}(X)$.

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# Birational geometry and the canonical ring of a family of determinantal 3 -folds 

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#### Abstract

Few explicit families of 3-folds are known for which the computation of the canonical ring is accessible and the birational geometry non-trivial. In this note we investigate a family of determinantal 3 -folds in $\mathbb{P}^{2} \times \mathbb{P}^{3}$ where this is the case.


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## 1. Introduction

There has been substantial progress in higher dimensional birational geometry over $\mathbb{C}$ in the past decade. For instance, we currently know that for every smooth projective variety $X$, the canonical ring

$$
R\left(X, K_{X}\right)=\bigoplus_{m \in \mathbb{N}} H^{0}\left(X, m K_{X}\right)
$$

is finitely generated and that varieties with mild singularities and of log general type have good minimal models [1, 2, 3]. Numerous other results have also recently been obtained when $X$ is not necessarily of general type, but the existence of minimal models and the Abundance conjecture remain unproven in general.

Lack of examples in higher dimensional geometry is one of the problems in the field for two reasons: (a) ultimately, one wants to apply the general theory in concrete examples, preferably described by concrete equations, and (b) without examples, it is often difficult to decide if a certain conjecture is plausible or to devise a route to a possible proof of a conjecture.

Recall that some of the main examples of higher dimensional constructions are the following: projective bundles (this is probably the most common class of examples, see [11, §2.3.B]); toric bundles, see [16, Chapter IV]; deformations. Recently, blowups of $\mathbb{P}^{3}$ along a very general configuration of points were used in [14] to give counterexample to a conjecture of Kawamata, and a relatively simple example from [17] (a complete intersection of general hypersurfaces of
bi-degrees $(1,1),(1,1)$ and $(2,2)$ in $\left.\mathbb{P}^{3} \times \mathbb{P}^{3}\right)$ was used in $[15]$ to disprove a widely believed claim from $[6,13,16]$ about an expected behaviour of the numerical dimension.

The last two examples above should illustrate that more examples are needed in order to speed up progress in the field. We provide a general class of new examples in this note, and investigate the birational geometry of a particular subclass of examples in detail.

The class of examples we study in this paper are a particular case of determinantal varieties. The situation in general is explained in detail in Section 2. In particular, denote $\mathbb{P}=\mathbb{P}^{2} \times \mathbb{P}^{3}$ and $\mathcal{F}=\mathcal{O}_{\mathbb{P}}^{\oplus 2}$, and for each integer $b \geq 1$ define the sheaf

$$
\mathcal{G}_{b}=\mathcal{O}_{\mathbb{P}}(1, b) \oplus \operatorname{ker}\left(H^{0}\left(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1,0)\right) \otimes \mathcal{O}_{\mathbb{P}}(1,1) \rightarrow \mathcal{O}_{\mathbb{P}}(2,1)\right)
$$

Pick $\varphi \in \operatorname{Hom}\left(\mathcal{F}, \mathcal{G}_{b}\right)$ general, and $X_{b}$ let be the 3 -fold given as

$$
X_{b}=\{p \in \mathbb{P} \mid \operatorname{rank} \varphi(p) \leq 1\} .
$$

Our main result is:
Theorem 1.1. The variety $X_{b}$ is birational to a hypersurface $Y_{b}$ of degree $2 b+2$ in the weighted projective space $\mathbb{P}(1,1,1,1, b+1)$. In particular, we have

$$
\kappa\left(X_{b}\right)= \begin{cases}-\infty & \text { if } b=1 \text { or } 2 \\ 0 & \text { if } b=3 \\ 3 & \text { if } b \geq 4\end{cases}
$$

The image $X_{b}^{1}$ of $X$ in $\mathbb{P}^{1} \times \mathbb{P}^{3}$ is a small resolution of $Y_{b}$ in $(b+1)^{3} A_{1}$-singularities. The morphism $X \rightarrow X_{b}^{1}$ is the blowup of one of the two components of the preimage of a twisted $C \subseteq \mathbb{P}^{3}$ which intersects the branch divisor of $Y_{b} \rightarrow \mathbb{P}^{3}$ tangentially. The variety $X_{b}^{1}$ has precisely two minimal models and one nontrivial birational automorphism $\iota$ of order two. The automorphism $\iota$ interchanges the two models.

Thus for $b \geq 4$ the 3 -fold $X_{b}^{1}$ is a minimal model of $X$ and $Y_{b}$ is the canonical model. In particular, this family of examples has an unexpectedly rich birational geometry.

## 2. Determinantal varieties

In this section we describe a general construction of determinantal varieties in products of projective spaces, and specialise to a particular case which is the main object of this paper.

### 2.1. A general construction

Let $\mathbb{P}$ be a product of projective spaces, let $\mathcal{F}$ and $\mathcal{G}$ be vector bundles on $\mathbb{P}$ of rank $f$ and $g \geq f$ respectively, and let $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ be a general homomorphism. Define an algebraic set $X \subseteq \mathbb{P}$ by

$$
X=\{p \in \mathbb{P} \mid \varphi(p) \text { does not have maximal rank } f\} .
$$

For example, if the sheaf $\mathcal{H o m}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{F}, \mathcal{G}) \simeq \mathcal{F}^{*} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{G}$ is ample, then $X$ is nonempty, connected and has codimension $g-f+1$ by [9], and is smooth outside a sublocus of codimension $2(g-f+2)$ by [10], which is empty if $\operatorname{dim} \mathbb{P}<2(g-f+2)$. Moreover, in this case the sheaf

$$
\begin{equation*}
\mathcal{L}=\operatorname{coker}\left(\varphi^{t}: \mathcal{G}^{*} \rightarrow \mathcal{F}^{*}\right) \tag{1}
\end{equation*}
$$

is a line bundle on $X$.
If $f=1$, then $X$ is a zero loci of a section of a vector bundle on $\mathbb{P}$. If additionally $\mathcal{G}$ is a direct sum of line bundles, then $X$ is a complete intersection.

Perhaps the simplest case beyond the one above is when $f=g-1$. In that case, $X$ is a codimension 2 subvariety in $\mathbb{P}$ and, if $\mathcal{J}_{X}$ is the ideal sheaf of $X$ in $\mathbb{P}$, then we have the resolution

$$
\begin{equation*}
0 \rightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \rightarrow \mathcal{J}_{X} \otimes \mathcal{O}_{\mathbb{P}}\left(c_{1}(\mathcal{G})-c_{1}(\mathcal{F})\right) \rightarrow 0 \tag{2}
\end{equation*}
$$

see $[4,5]$. By above, we expect these $X$ to be a smooth variety only when $\operatorname{dim} \mathbb{P} \leq 5$.

### 2.2. Examples

Thus, from now on we choose $\mathbb{P}=\mathbb{P}^{2} \times \mathbb{P}^{3}$, and we let $f=2$ and $g=3$. Specifying further $\mathcal{F}:=\mathcal{O}_{\mathbb{P}}^{\oplus 2}$, then $X$ is a 3 -fold and the linear system $|\mathcal{L}|$, where $\mathcal{L}$ is defined as in (1), defines a morphism $\mathbb{P} \rightarrow \mathbb{P}^{1}$. Since we also have the projections from $\mathbb{P}$ to its two factors, we obtain three maps

$$
\begin{equation*}
\pi_{1}: X \rightarrow \mathbb{P}^{1}, \quad \pi_{2}: X \rightarrow \mathbb{P}^{2}, \quad \pi_{3}: X \rightarrow \mathbb{P}^{3} \tag{3}
\end{equation*}
$$

which we use to study $X$.
At first sight, the case $\mathcal{G}=\mathcal{O}_{\mathbb{P}}(1,1)^{\oplus 3}$ might look like the simplest possible case. In this case, the morphism $\pi_{2}: X \rightarrow \mathbb{P}^{2}$ is a fibration into twisted cubic curves, $\pi_{3}: X \rightarrow \mathbb{P}^{3}$ is generically finite of degree $3: 1$, and $\pi_{1}: X \rightarrow \mathbb{P}^{1}$ is a fibration into cubic surfaces.

Now, let $\theta: \mathcal{O}_{\mathbb{P}}(1,1)^{\oplus 4} \rightarrow \mathcal{O}_{\mathbb{P}}(2,1)$ be a general morphism and consider the case $\mathcal{G}=\operatorname{ker} \theta$. In suitable coordinates on $\mathbb{P}^{2}$ we have

$$
\mathcal{G}=\mathcal{O}_{\mathbb{P}}(1,1) \oplus \operatorname{ker}\left(H^{0}\left(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1,0)\right) \otimes \mathcal{O}_{\mathbb{P}}(1,1) \rightarrow \mathcal{O}_{\mathbb{P}}(2,1)\right)
$$

where the map is the evaluation morphism. This case is even simpler, in the sense that $\pi_{3}: X \rightarrow \mathbb{P}^{3}$ is generically finite of degree $2: 1$. Indeed, let $F$ be a general fiber of the second projection $\mathbb{P} \rightarrow \mathbb{P}^{3}$. Then the sheaf

$$
\left.\mathcal{G}\right|_{F} \simeq \operatorname{ker}\left(\mathcal{O}_{\mathbb{P}^{2}}(1)^{\oplus 4} \rightarrow \mathcal{O}_{\mathbb{P}^{2}}(2)\right)
$$

has the Chern polynomial

$$
c_{t}\left(\left.\mathcal{G}\right|_{F}\right)=\frac{(1+t)^{4}}{1+2 t}=1+2 t+2 t^{2}
$$

and thus $c_{2}\left(\left.\mathcal{G}\right|_{F}\right)=2$ implies that $\pi_{3}$ is generically $2: 1$.

## 3. Cohomological properties

### 3.1. The main example

Our main example is a generalisation of this last construction. As announced in the introduction, for each integer $b \geq 1$ we consider 3 -folds $X_{b}$ constructed as follows: we set $\mathbb{P}=\mathbb{P}^{2} \times \mathbb{P}^{3}, \mathcal{F}=\mathcal{O}_{\mathbb{P}}^{\oplus 2}$, and

$$
\mathcal{G}_{b}=\mathcal{O}_{\mathbb{P}}(1, b) \oplus \operatorname{ker}\left(H^{0}\left(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1,0)\right) \otimes \mathcal{O}_{\mathbb{P}}(1,1) \rightarrow \mathcal{O}_{\mathbb{P}}(2,1)\right)
$$

where the morphism is the evaluation morphism in suitable coordinates ( $x_{0}$ : $\left.x_{1}: x_{2}\right)$ on $\mathbb{P}^{2}$. Then for a general $\varphi \in \operatorname{Hom}\left(\mathcal{F}, \mathcal{G}_{g}\right)$ we define

$$
X_{b}=\left\{p \in \mathbb{P}^{2} \times \mathbb{P}^{3} \mid \operatorname{rank} \varphi(p) \leq 1\right\} .
$$

This is the main object of this paper.
By (2), there exists a locally free resolution

$$
\begin{equation*}
0 \rightarrow \mathcal{O}_{\mathbb{P}}^{\oplus^{2}} \rightarrow \mathcal{G}_{b} \rightarrow \mathcal{J}_{X_{b}} \otimes \mathcal{O}_{\mathbb{P}}(2, b+2) \rightarrow 0, \tag{4}
\end{equation*}
$$

and $\pi_{3}: X_{b} \rightarrow \mathbb{P}^{3}$ is generically $2: 1$ similarly as in Section 2. Dualizing (4) we obtain a resolution of $\mathcal{L}$ :

$$
\begin{align*}
0 \leftarrow \mathcal{L} \leftarrow \mathcal{O}_{\mathbb{P}}^{\oplus 2} & \leftarrow \mathcal{O}_{\mathbb{P}}(-1,-1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}}(-1,-b)  \tag{5}\\
& \leftarrow \mathcal{O}_{\mathbb{P}}(-2,-1) \oplus \mathcal{O}_{\mathbb{P}}(-2,-2-b) \leftarrow 0
\end{align*}
$$

and thus

$$
\begin{align*}
\omega_{X_{b}} & \simeq \mathcal{E} x t_{\mathcal{O}_{\mathbb{P}}}^{2}\left(\mathcal{O}_{X_{b}}, \mathcal{O}_{\mathbb{P}}(-3,-4)\right)  \tag{6}\\
& \simeq \mathcal{E} x t_{\mathcal{O}_{\mathbb{P}}}^{2}\left(\mathcal{O}_{X_{b}}(2, b+2), \mathcal{O}_{\mathbb{P}}(-1, b-2)\right) \simeq \mathcal{L}(-1, b-2) .
\end{align*}
$$

Some of the computationally accessible information in explicit examples are the dimensions of the cohomology groups $H^{i}\left(X_{b}, \mathcal{O}_{X_{b}}(\alpha, \beta)\right)$. It is useful to arrange this data in cohomology polynomials

$$
p_{\alpha, \beta}=\sum_{i=0}^{3} h^{i}\left(X_{b}, \mathcal{O}_{X_{b}}(\alpha, \beta)\right) \cdot h^{i} \in \mathbb{Z}[h] .
$$

We also consider the ring

$$
R=\bigoplus_{\beta \geq 0} H^{0}\left(X_{b}, \mathcal{O}_{X_{b}}(0, \beta)\right)
$$

### 3.2. Cohomology groups of $X_{3}$

Using the theory of Tate resolutions for product of projective spaces [7] we can calculate the dimensions of these groups. In this subsection, we concentrate on the case $b=3$. Fix the range

$$
-3 \leq \alpha \leq 3, \quad-7 \leq \beta \leq 7 .
$$

Then we can summarize the result in matrix of cohomology polynomials $p_{\alpha, \beta}$ as below.

| $88 h$ | $56 h$ | 20 | 140 | 304 | 512 | 764 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $53 h$ | $41 h$ | 8 | 94 | 217 | 377 | 574 |
| $24 h$ | $26 h$ | 2 | 60 | 148 | 266 | 414 |
| $5 h^{2}+8 h$ | $13 h$ | 0 | 36 | 95 | 177 | 282 |
| $10 h^{2}+2 h$ | $4 h$ | 0 | 20 | 56 | 108 | 176 |
| $7 h^{2}$ | $h$ | 0 | 10 | 29 | 57 | 94 |
| $12 h^{3}+4 h^{2}$ | $6 h^{3}$ | $2 h^{3}$ | 4 | 12 | $2 h+24$ | $6 h+40$ |
| $40 h^{3}+h^{2}$ | $21 h^{3}$ | $8 h^{3}$ | $h^{3}+1$ | 3 | $5 h+6$ | $16 h+10$ |
| $88 h^{3}$ | $48 h^{3}$ | $20 h^{3}$ | $4 h^{3}$ | 0 | $8 h$ | $28 h$ |
| $157 h^{3}$ | $89 h^{3}$ | $40 h^{3}$ | $10 h^{3}$ | $h^{2}$ | $h^{2}+8 h$ | $34 h$ |
| $248 h^{3}$ | $146 h^{3}$ | $70 h^{3}$ | $20 h^{3}$ | $4 h^{2}$ | $4 h^{2}+2 h$ | $2 h^{2}+28 h$ |
| $363 h^{3}$ | $221 h^{3}$ | $112 h^{3}$ | $36 h^{3}$ | $7 h^{2}$ | $17 h^{2}$ | $8 h^{2}+14 h$ |
| $504 h^{3}$ | $316 h^{3}$ | $168 h^{3}$ | $60 h^{3}$ | $2 h^{3}+10 h^{2}$ | $36 h^{2}$ | $24 h^{2}$ |
| $673 h^{3}$ | $433 h^{3}$ | $240 h^{3}$ | $94 h^{3}$ | $8 h^{3}+13 h^{2}$ | $57 h^{2}$ | $62 h^{2}$ |
| $872 h^{3}$ | $574 h^{3}$ | $330 h^{3}$ | $140 h^{3}$ | $20 h^{3}+16 h^{2}$ | $78 h^{2}$ | $106 h^{2}$ |

Let us point out a few interesting values: we have

$$
h^{1}\left(X_{b}, \mathcal{O}_{X_{b}}\right)=h^{2}\left(X_{b}, \mathcal{O}_{X_{b}}\right)=0 \text { and } h^{3}\left(X_{b}, \mathcal{O}_{X_{b}}\right)=h^{0}\left(X_{b}, \omega_{X_{b}}\right)=1
$$

from the center entry. Moreover, we see that $h^{0}\left(X_{b}, \mathcal{O}_{X_{b}}(0,4)\right)=36>35$, so the ring $R$ has a further generator in degree 4.

Another interesting sequence of values are the dimensions of the $H^{2}$-cohomology in the first vertical strand (that is, for $\alpha=1$ ):

$$
\ldots, 16,13,10,7,4,1
$$

This looks like the Hilbert function of the twisted cubic in $\mathbb{P}^{3}$.

### 3.3. Cohomology groups of $X_{b}$

The tables for other values of $b$ have a lot of similarity with the table above.
Recall from (6) that $\mathcal{L} \cong \omega_{X_{b}}(1,-b+2)$. Dualising the resolution (5) we obtain

$$
\begin{aligned}
0 & \rightarrow \mathcal{O}_{\mathbb{P}}^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}}(1,1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}}(1, b) \\
& \rightarrow \mathcal{O}_{\mathbb{P}}(2,1) \oplus \mathcal{O}_{\mathbb{P}}(2, b+2) \rightarrow \mathcal{O}_{X_{b}}(2, b+2) \rightarrow 0
\end{aligned}
$$

Twisting back by $\mathcal{O}_{\mathbb{P}}(-2,-b-2)$ we deduce

$$
R \pi_{3, *} \mathcal{O}_{X_{b}}=\pi_{3, *} \mathcal{O}_{X_{b}}=\mathcal{O}_{\mathbb{P}^{3}} \oplus \mathcal{O}_{\mathbb{P}^{3}}(-b-1),
$$

and twisting by back by $\mathcal{O}_{\mathbb{P}}(-3,-b-2)$ gives

$$
R \pi_{3, *} \mathcal{O}_{X_{b}}(-1,0)=\mathcal{O}_{\mathbb{P}^{3}}(-b-2)^{\oplus 2}
$$

Since $R \pi_{3, *} \mathcal{O}_{X_{b}}(\alpha, 0)$ is computed with the vertical strands in the Tate resolution, this explains the values in the 0 -th and $(-1)$-st vertical strand in the cohomology table. In particular, we see that

$$
h^{0}\left(X_{b}, \mathcal{O}_{X_{b}}(-1, b+2)\right)=2
$$

### 3.4. A twisted cubic

As suggested in $\S 3.2$, we can find a twisted cubic on $\mathbb{P}^{3}$ in our construction.
Recall that we fixed coordinates $\left(x_{0}: x_{1}: x_{2}\right)$ on $\mathbb{P}^{2}$. We may write

$$
\mathcal{G}_{b}=\mathcal{O}_{\mathbb{P}}(1, b) \oplus \operatorname{ker}\left(\mathcal{O}_{\mathbb{P}}(1,1)^{\otimes 3} \xrightarrow{\left(x_{0}, x_{1}, x_{2}\right)} \mathcal{O}_{\mathbb{P}}(2,1)\right)
$$

so that we have two projections

$$
\mathcal{G}_{b} \rightarrow \mathcal{O}_{\mathbb{P}}(1, b) \quad \text { and } \quad \mathcal{G}_{b} \rightarrow \mathcal{O}_{\mathbb{P}}(1,1)^{\otimes 3}
$$

The composition

$$
\mathcal{O}_{\mathbb{P}}^{\oplus 2} \xrightarrow{\varphi} \mathcal{G}_{b} \rightarrow \mathcal{O}_{\mathbb{P}}(1,1)^{\oplus 3}
$$

factors over

$$
\mathcal{O}_{\mathbb{P}}^{\oplus 2} \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}}(0,1)^{\oplus 3} \xrightarrow{K_{2}} \mathcal{O}_{\mathbb{P}}(1,1)^{\oplus 3},
$$

where

$$
K_{2}=\left(\begin{array}{ccc}
0 & -x_{2} & x_{1} \\
x_{2} & 0 & -x_{0} \\
-x_{1} & x_{0} & 0
\end{array}\right)
$$

is the Koszul matrix, and in suitable coordinates $\left(y_{0}: y_{1}: y_{2}: y_{3}\right)$ of $\mathbb{P}^{3}$ we have

$$
\psi=\left(\begin{array}{lll}
y_{0} & y_{1} & y_{2} \\
y_{1} & y_{2} & y_{3}
\end{array}\right)
$$

We denote by $C \subseteq \mathbb{P}^{3}$ the twisted cubic curve defined by the $2 \times 2$ minors of $\psi$.
The remaining part $\mathcal{O}_{\mathbb{P}}^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}}(1, b)$ of $\varphi$ can be factored as $B \cdot\left(\begin{array}{lll}x_{0} & x_{1} & x_{2}\end{array}\right)^{t}$, with

$$
B=\left(\begin{array}{lll}
b_{00} & b_{01} & b_{02}  \tag{7}\\
b_{10} & b_{11} & b_{12}
\end{array}\right)
$$

where $b_{i j} \in \mathbb{C}\left[y_{0}, y_{1}, y_{2}, y_{3}\right]$ are forms of degree $b$. To this matrix we associate the matrix

$$
M=\left(\begin{array}{cc}
2 \sum_{i=0}^{2} y_{i} b_{0 i} & \sum_{i=0}^{2} y_{i} b_{1 i}+\sum_{i=0}^{2} y_{i+1} b_{0 i}  \tag{8}\\
\sum_{i=0}^{2} y_{i} b_{1 i}+\sum_{i=0}^{2} y_{i+1} b_{0 i} & 2 \sum_{i=0}^{2} y_{i+1} b_{1 i}
\end{array}\right)
$$

this matrix will be important in $\S 4.2$ below.
Proposition 3.1. In the notation as above, we have:
(a) $\pi_{3}^{-1}(C) \subseteq X_{b}$ decomposes into two components: $C_{1}$ of dimension 1 and $E$ of dimension 2,
(b) $C_{1}$ is defined by the $2 \times 2$ minors of

$$
\left(\begin{array}{lllll}
y_{0} & y_{1} & y_{2} & x_{0} & x_{1} \\
y_{1} & y_{2} & y_{3} & x_{1} & x_{2}
\end{array}\right)
$$

(c) $E$ is defined by the minors of $\psi$ and the entries of

$$
\left(\begin{array}{lll}
x_{0} & x_{1} & x_{2}
\end{array}\right) \cdot B^{t} \cdot\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right) \cdot \psi
$$

(d) $C_{1} \rightarrow C$ is an isomorphism while $E \rightarrow C$ is a $\mathbb{P}^{1}$-bundle. In particular, $C_{1}$ and $E$ are smooth.

Proof. Parts (b) and (c) follow from direct calculations [8] or [12]. Note that

$$
\left\{p \in \mathbb{P}^{3} \mid \operatorname{rank} B(p) \leq 1 \text { and } \operatorname{rank} \psi(p) \leq 1\right\}=\varnothing
$$

for a general choice of $B$. Therefore, $\operatorname{rank} B(p)=2$ for $p \in C$, so $B^{t} \cdot\left(\begin{array}{cc}0 & 1 \\ -1 & 1\end{array}\right) \cdot \psi$ has rank 1 over the points of $C$. Hence, $E$ is a $\mathbb{P}^{1}$-bundle. We have $C_{1} \cong C$
and the projection $\pi_{2}$ maps $C_{1}$ isomorphically to the conic $V\left(x_{0} x_{2}-x_{1}^{2}\right) \subseteq \mathbb{P}^{2}$. This shows (d).

Finally, consider the matrix $\varphi^{t}$ as a $2 \times 4$ matrix with entries in

$$
\mathbb{Q}\left[x_{0}, x_{1}, x_{2}, y_{0}, y_{1}, y_{2}, y_{3}, b_{00}, \ldots, b_{12}\right]
$$

The defining ideal of $X_{b}$ is the annihilator of the $\operatorname{coker} \varphi$, once we substitute the actual values for the $b_{i j}$ in $H^{0}\left(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(0, b)\right)$. Adding the defining equations of $C$, a primary decomposition gives the two components in this generic setting. Since $C_{1}$ and $E$ are smooth, specialising $b_{i j}$ gives the actual components.

## 4. Two minimal models

In this section we describe the birational geometry of $X_{b}$.

### 4.1. An overview

We introduce several new varieties. Denote

$$
X_{b}^{1}:=\left(\pi_{1} \times \pi_{3}\right)\left(X_{b}\right) \subseteq \mathbb{P}^{1} \times \mathbb{P}^{3} .
$$

Moreover, let

$$
R=\mathbb{C}\left[y_{0}, y_{1}, y_{2}, y_{3}, w\right] /\left\langle w^{2}+\operatorname{det} M\right\rangle,
$$

where $w$ has degree $b+1$ and $M$ is defined as in (8), and denote

$$
Y_{b}=\operatorname{Proj} R \subseteq \mathbb{P}(1,1,1,1, b+1) .
$$

An easy argument with an exact sequence in $\S 4.2$ shows the existence of a rational map $\rho: X_{b}^{1} \rightarrow \mathbb{P}^{1}$, and we denote

$$
X_{b}^{2}:=\left(\rho \times \pi_{3}\right)\left(X_{b}^{1}\right) \subseteq \mathbb{P}^{1} \times \mathbb{P}^{3}
$$

We will show that these varieties fit into the diagram

such that the following holds:
(a) $X_{b}^{1} \subseteq \mathbb{P}^{1} \times \mathbb{P}^{3}$ is a hypersurface of bi-degree $(2, b+1)$,
(b) $X_{b}^{1}$ and $X_{b}^{2}$ are small resolutions of $Y_{b}$.

This then implies our main result.

### 4.2. The geometry of $X_{b}^{1}$

Our first goal is to compute $X_{b}^{1}$.
By $\S 3.4$, the defining ideal of $X_{b} \subseteq \mathbb{P}^{1} \times \mathbb{P}^{2} \times \mathbb{P}^{3}$ is given by the four entries of the matrix

$$
\left(\begin{array}{ll}
z_{0} & z_{1}
\end{array}\right) \cdot\left[\begin{array}{lll}
\left.\psi \cdot K_{2} \left\lvert\, B \cdot\left(\begin{array}{lll}
x_{0} & x_{1} & x_{2}
\end{array}\right)^{t}\right.\right] . ~ \tag{10}
\end{array}\right.
$$

The saturation of this ideal with respect to $\left\langle x_{0}, x_{1}, x_{2}\right\rangle$ gives the hypersurface $X_{b}^{1}$.

Proposition 4.1. With notation as in §4.1, we have:
(a) The variety $X_{b}^{1}$ is a smooth hypersurface of bi-degree $(2, b+1)$ in $\mathbb{P}^{1} \times \mathbb{P}^{3}$ defined by

$$
f=\left(\begin{array}{ll}
z_{0} & z_{1}
\end{array}\right) \cdot M \cdot\binom{z_{0}}{z_{1}}
$$

with matrix $M$ given as in (8).
(b) The map $\alpha_{1}: X_{b} \rightarrow X_{b}^{1}$ is birational: it is the blow down of the $\mathbb{P}^{1}$-bundle E from Proposition 3.1 to the rational curve $C^{1} \subseteq X_{b}^{1}$ defined by the $2 \times 2$ minors of the matrix

$$
\left(\begin{array}{cccc}
y_{0} & y_{1} & y_{2} & -z_{1} \\
y_{1} & y_{2} & y_{3} & z_{0}
\end{array}\right)
$$

Proof. We rewrite the equation (10) of $X_{b}$ as

$$
\left(\begin{array}{lll}
x_{0} & x_{1} & x_{2}
\end{array}\right) \cdot N=0,
$$

where

$$
N=\left(\begin{array}{cccc}
0 & z_{0} y_{2}+z_{1} y_{3} & -z_{0} y_{1}-z_{1} y_{2} & z_{0} b_{00}+z_{1} b_{10} \\
-z_{0} y_{2}-z_{1} y_{3} & 0 & z_{0} y_{0}+z_{1} y_{1} & z_{0} b_{01}+z_{1} b_{11} \\
z_{0} y_{1}+z_{1} y_{2} & -z_{0} y_{0}-z_{1} y_{1} & 0 & z_{0} b_{02}+z_{1} b_{12}
\end{array}\right) .
$$

We conclude that $X_{b}^{1} \subseteq \mathbb{P}^{1} \times \mathbb{P}^{3}$ coincides with the variety defined by the radical of the $3 \times 3$ minors of $N$. This radical coincides with the form $f$ in the statement of the proposition; the details of the calculations are in [8] or [12]. Moreover, the map $\alpha_{1}$ is birational outside the preimage of the ideal defined by the $2 \times 2$ minors of $N$ : this is the curve $C^{1}$. Since $\alpha_{1}$ blows down a smooth $\mathbb{P}^{1}$-bundle $E$, the variety $X_{b}^{1}$ is smooth.

With this information, one can calculate the cohomology table of $X_{b}^{1}$ in the test case $b=3$, using the Macaulay2 package TateOnProducts:

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| $148 h$ | $96 h$ | $44 h$ | 8 | 60 | 112 | 164 | 216 | 268 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $100 h$ | $66 h$ | $32 h$ | 2 | 36 | 70 | 104 | 138 | 172 |
| $60 h$ | $40 h$ | $20 h$ | 0 | 20 | 40 | 60 | 80 | 100 |
| $30 h$ | $20 h$ | $10 h$ | 0 | 10 | 20 | 30 | 40 | 50 |
| $12 h$ | $8 h$ | $4 h$ | 0 | 4 | 8 | 12 | 16 | 20 |
| $5 h^{3}+3 h$ | $4 h^{3}+2 h$ | $3 h^{3}+h$ | $2 h^{3}$ | $h^{3}+1$ | 2 | $h^{2}+3$ | $2 h^{2}+4$ | $3 h^{2}+5$ |
| $20 h^{3}$ | $16 h^{3}$ | $12 h^{3}$ | $8 h^{3}$ | $4 h^{3}$ | 0 | $4 h^{2}$ | $8 h^{2}$ | $12 h^{2}$ |
| $50 h^{3}$ | $40 h^{3}$ | $30 h^{3}$ | $20 h^{3}$ | $10 h^{3}$ | 0 | $10 h^{2}$ | $20 h^{2}$ | $30 h^{2}$ |
| $100 h^{3}$ | $80 h^{3}$ | $60 h^{3}$ | $40 h^{3}$ | $20 h^{3}$ | 0 | $20 h^{2}$ | $40 h^{2}$ | $60 h^{2}$ |
| $172 h^{3}$ | $138 h^{3}$ | $104 h^{3}$ | $70 h^{3}$ | $36 h^{3}$ | $2 h^{3}$ | $32 h^{2}$ | $66 h^{2}$ | $100 h^{2}$ |
| $268 h^{3}$ | $216 h^{3}$ | $164 h^{3}$ | $112 h^{3}$ | $60 h^{3}$ | $8 h^{3}$ | $44 h^{2}$ | $96 h^{2}$ | $148 h^{2}$ |

From the table, we see that $h^{0}\left(X_{b}^{1}, \mathcal{O}_{X_{b}^{1}}(-1,4)\right)=2$. In fact, for every $b$ we have

$$
\begin{equation*}
h^{0}\left(X_{b}^{1}, \mathcal{O}_{X_{b}^{1}}(-1, b+1)=2\right. \tag{11}
\end{equation*}
$$

This follows from the exact sequence

$$
0 \rightarrow \mathcal{O}_{\mathbb{P}^{1} \times \mathbb{P}^{3}}(-3,0) \rightarrow \mathcal{O}_{\mathbb{P}^{1} \times \mathbb{P}^{3}}(-1, b+1) \rightarrow \mathcal{O}_{X_{b}^{1}}(-1, b+1) \rightarrow 0
$$

and the fact that $h^{1}\left(\mathbb{P}^{1} \times \mathbb{P}^{3}, \mathcal{O}_{\mathbb{P}^{1} \times \mathbb{P}^{3}}(-3,0)\right)=2$. Therefore, as announced in $\S 4.1$, by (11) we obtain a rational map

$$
\begin{equation*}
\rho: X_{b}^{1} \rightarrow \mathbb{P}^{1} \tag{12}
\end{equation*}
$$

### 4.3. The first small resolution

Next we show that $X_{b}^{1}$ is a small resolution of $Y_{b}$ and analyse in detail the geometry of $Y_{b}$. Recall that by the definition of $Y_{b}$ in $\S 4.1$, there exists a double cover

$$
\begin{equation*}
\delta: Y_{b} \rightarrow \mathbb{P}^{3} . \tag{13}
\end{equation*}
$$

Proposition 4.2. For a general choice of $b_{i j}$ in (7) we have:
(a) the double cover $\delta$ has $A_{1}$-singularities above the $(b+1)^{3}$ distinct points defined by the zero loci of entries of $M$, and is otherwise smooth,
(b) $X_{b}^{1}$ is a small resolution of $Y_{b}$.

Proof. Recall that the variety $X_{b}^{1}$ comes with a projection to $\mathbb{P}^{3}$. By the description in Proposition 4.1, the fibre of the map $X_{b}^{1} \rightarrow \mathbb{P}^{3}$ over a point $p \in \mathbb{P}^{3}$ consist either of two points, of one point or is isomorphic to $\mathbb{P}^{1}$, depending on whether $M(p)$ has rank 2,1 or 0 respectively. For general $b_{i j}$, the three entries of the matrix $M$ form a regular sequence, which intersect in $(b+1)^{3}$ distinct points. Since this is an open condition for the values of $b_{i j}$, it suffices to construct an example.

To this end, pick $\lambda_{0}, \ldots, \lambda_{b}, \mu_{0}, \ldots \mu_{b} \in \mathbb{C}$ which are algebraically independent over $\mathbb{Q}$. Define forms

$$
\widetilde{b}_{01} \in \mathbb{Q}\left[\lambda_{0}, \ldots, \lambda_{b}\right]\left[y_{0}, y_{1}\right], \quad \widetilde{b}_{11} \in \mathbb{Q}\left[\mu_{0}, \ldots \mu_{b}\right]\left[y_{2}, y_{3}\right]
$$

of degree $b$ by the relations

$$
\prod_{i=0}^{b}\left(y_{0}-\lambda_{i} y_{1}\right)=y_{0}^{b+1}+y_{1} \widetilde{b}_{01}, \quad \prod_{j=0}^{b}\left(y_{3}-\mu_{j} y_{2}\right)=y_{3}^{b+1}+y_{2} \widetilde{b}_{11},
$$

and define the matrix

$$
B^{\circ}=\left(\begin{array}{ccc}
y_{0}^{b} & \widetilde{b}_{01} & 0 \\
0 & \widetilde{b}_{11} & y_{3}^{b}
\end{array}\right)
$$

We consider $B^{\circ}$ as the matrix $B$ from (7) for special values of $b_{i j}$. For these values, the corresponding matrix $M$ from (8) turns into

$$
M^{\circ}=\left(\begin{array}{cc}
2\left(y_{0}^{b+1}+y_{1} \widetilde{b}_{01}\right) & y_{0}^{b} y_{1}+y_{1} \widetilde{b}_{11}+y_{2} y_{3}^{b}+y_{2} \widetilde{b}_{01} \\
y_{0}^{b} y_{1}+y_{1} \widetilde{b}_{11}+y_{2} y_{3}^{b}+y_{2} \widetilde{b}_{01} & 2\left(y_{3}^{b+1}+y_{2} \widetilde{b}_{11}\right)
\end{array}\right) .
$$

Fix $0 \leq i, j \leq b$. The diagonal entries of $M^{\circ}$ have solutions $y_{0}=\lambda_{i} y_{1}$ and $y_{3}=\mu_{j} y_{2}$. Substituting these values for $y_{0}$ and $y_{3}$ into the off diagonal entry of $M^{\circ}$ yields non-zero polynomials

$$
\begin{aligned}
P_{i j} & =\lambda_{i}^{b} y_{1}^{b+1}+y_{1} \widetilde{b}_{11}\left(y_{2}, \mu_{j} y_{2}\right)+\mu_{j}^{b} y_{2}^{b+1}+y_{2} \widetilde{b}_{01}\left(\lambda_{i} y_{1}, y_{1}\right) \\
& =\lambda_{i}^{b} y_{1}^{b+1}-\left(\mu_{j}^{b+1}+\ldots\right) y_{1} y_{2}^{b}+\mu_{j}^{b} y_{2}^{b+1}-\left(\lambda_{i}^{b+1}+\ldots\right) y_{2} y_{1}^{b} \\
& \in \mathbb{Q}\left[\lambda_{0}, \ldots, \lambda_{b}, \mu_{1}, \ldots, \mu_{b}\right]\left[y_{1}, y_{2}\right] .
\end{aligned}
$$

The highest exponent of $\lambda_{i}$ and $\mu_{j}$ in the Sylvester matrix for the resultant

$$
R\left(\frac{\partial P_{i j}}{\partial y_{1}}, \frac{\partial P_{i j}}{\partial y_{2}}\right)
$$

is $b+1$ and the coefficient of $\left(\lambda_{i} \mu_{j}\right)^{b(b+1)}$ is $\pm 1$ obtained from the coefficient of $y_{2}^{b}$ in $\frac{\partial P_{i j}}{\partial y_{1}}$ and the coefficient of $y_{1}^{b}$ in $\frac{\partial P_{i j}}{\partial y_{2}}$. Hence, the discriminant of $P_{i j}$ in $\mathbb{Q}\left[\lambda_{0}, \ldots, \lambda_{b}, \mu_{1}, \ldots, \mu_{b}\right]$ is not identically zero. Since $\lambda_{0}, \ldots, \lambda_{b}, \mu_{1}, \ldots, \mu_{b}$ are algebraically independent over $\mathbb{Q}$, each $P_{i j}$ factors into $b+1$ distinct linear forms in $\mathbb{C}\left[y_{1}, y_{2}\right]$. Hence, the entries of $M^{\circ}$ vanish in precisely $(b+1)^{3}$ distinct points, as desired.

Now, write

$$
M=\left(\begin{array}{ll}
a_{0} & a_{1} \\
a_{1} & a_{2}
\end{array}\right)
$$

for forms $a_{i}$ of degree $b+1$ on $\mathbb{P}^{3}$ as in (8). For any $B$ leading to $(b+1)^{3}$ distinct points in $\mathbb{P}^{3}$, the entries $a_{0}, a_{1}, a_{2}$ generate locally at each point its

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maximal ideal, so the branch divisor $\operatorname{det} M=0$ has $A_{1}$-singularities at these points. Since $X_{b}^{1}$ is smooth by Proposition 4.1, the branch divisor $\operatorname{det} M=0$ is smooth outside the $A_{1}$-singularities.

Consider the subvariety of $\mathbb{P}^{1} \times \mathbb{P}(1,1,1,1, b+1)$ defined by the $2 \times 2$ minors of the matrix

$$
\left(\begin{array}{ccc}
a_{0} & a_{1}-w & z_{1}  \tag{14}\\
a_{1}+w & a_{2} & -z_{0}
\end{array}\right)
$$

This is a small resolution of $Y_{b}$, and it is easy to see that it is isomorphic to $X_{b}^{1}$, as defined in Proposition 4.1(a).

Proposition 4.3. Let $C \subseteq \mathbb{P}^{3}$ be the twisted cubic defined in $\S 3.4$ and let $\delta$ be the double cover from (13). Then $C$ intersects the branch divisor of $\delta$ tangentially. We have $\left(\delta \circ \xi_{1}\right)^{-1}(C)=C^{1} \cup C^{2} \subseteq X_{b}^{1}$, where $C^{1}$ is the curve from Proposition 4.1, and $C^{2}$ is defined by the $4 \times 4$ Pfaffians of the matrix

$$
\left(\begin{array}{ccccc}
0 & 0 & y_{1} & y_{2} & y_{3} \\
0 & 0 & y_{0} & y_{1} & y_{2} \\
-y_{0} & -y_{1} & 0 & z_{0} b_{02}+z_{1} b_{12} & -z_{0} b_{01}-z_{1} b_{11} \\
-y_{1} & -y_{2} & -z_{0} b_{02}-z_{1} b_{12} & 0 & z_{0} b_{00}+z_{1} b_{10} \\
-y_{2} & -y_{3} & z_{0} b_{01}+z_{1} b_{11} & -z_{0} b_{00}-z_{1} b_{10} & 0
\end{array}\right) .
$$

The projection $\pi_{1}$ induces a map $C^{2} \rightarrow \mathbb{P}^{1}$ which is a covering of degree $3 b+2$.
Proof. Let $I_{C}=\left\langle y_{1}^{2}-y_{0} y_{2}, y_{1} y_{2}-y_{0} y_{3}, y_{2}^{2}-y_{1} y_{3}\right\rangle$ denote the homogeneous ideal of $C \subseteq \mathbb{P}^{3}$. Since

$$
\operatorname{det} M \equiv-\left(y_{1} b_{00}+y_{2} b_{01}+y_{3} b_{02}-y_{0} b_{10}-y_{1} b_{11}-y_{2} b_{12}\right)^{2} \bmod I_{C},
$$

the curve $C$ intersects the branch divisor of $\delta$ tangentially in $3(b+1)$ distinct points for general choices of $b_{i j}$ and the preimage of $C$ in $\mathbb{P}(1,1,1,1, b+1)$ has two components defined by $I_{C}$ and

$$
w \pm\left(y_{1} b_{00}+y_{2} b_{01}+y_{3} b_{02}-y_{0} b_{10}-y_{1} b_{11}-y_{2} b_{12}\right)=0 .
$$

The second statement follows by computing a primary decomposition of $I_{C}+\langle f\rangle \subseteq \mathbb{Q}\left[z_{0}, z_{1}, y_{0}, y_{1}, y_{2}, y_{3}, b_{00}, \ldots, b_{12}\right]$, where $f$ is given as in Proposition 4.1(a).

### 4.4. The second small resolution

Finally, we show that the variety $X_{b}^{2}$ defined in $\S 4.1$ is another small resolution of $Y_{b}$, and we finish the proof of the main theorem.

Proposition 4.4. The variety $X_{b}^{2}$ is another small resolution of $Y_{b}$.

Proof. As in the proof of Proposition 4.2, write

$$
M=\left(\begin{array}{ll}
a_{0} & a_{1} \\
a_{1} & a_{2}
\end{array}\right)
$$

for forms $a_{i}$ of degree $b+1$ on $\mathbb{P}^{3}$ as in (8). Let ( $u_{0}: u_{1}$ ) be the coordinates on $\mathbb{P}^{1}$. Consider the subvariety of $\mathbb{P}^{1} \times \mathbb{P}(1,1,1,1, b+1)$ defined by the $2 \times 2$ minors of the matrix

$$
\left(\begin{array}{cc}
a_{0} & a_{1}-w \\
a_{1}+w & a_{2} \\
u_{1} & -u_{0}
\end{array}\right)
$$

compare to (14). This is another small resolution of $Y_{b}$, and we will show that it is isomorphic to $X_{b}^{2}$, as defined in $\S 4.1$. To this end, it suffices to show that the base locus of the linear system $\left|\mathcal{O}_{X_{b}^{1}}(-1, b+1)\right|$ is precisely the collection of the $(b+1)^{3}$ exceptional curves of the small resolution $\xi_{1}: X_{b}^{1} \rightarrow Y_{b}$, see Proposition 4.2.

We have $\left\{u_{1}=0\right\}=V\left(a_{0}, a_{1}+w, w^{2}+\operatorname{det} M\right)$. In $X_{b}^{1}$ this fiber is contained in $V\left(a_{0}, f\right)$. Since

$$
f \equiv z_{1}\left(2 z_{0} a_{1}+z_{1} a_{2}\right) \quad \bmod a_{0}
$$

is reducible, the locus $V\left(a_{0}\right)$ cuts $X_{b}^{1}$ in two components: $V\left(z_{1}\right) \in\left|\mathcal{O}_{X_{b}^{1}}(1,0)\right|$ and

$$
V\left(a_{0}, 2 z_{0} a_{1}+z_{1} a_{2}\right) \in\left|\mathcal{O}_{X_{b}^{1}}(-1, b+1)\right| .
$$

By analysing $\left\{u_{0}=0\right\}$, we get that another divisor in this linear system is $V\left(a_{2}, z_{0} a_{0}+2 z_{1} a_{1}\right)$. Hence, the base locus of $\left|\mathcal{O}_{X_{b}^{1}}(-1, b+1)\right|$ is the zero locus $V\left(a_{0}, a_{2}, 2 z_{0} a_{1}, 2 z_{1} a_{1}\right)=V\left(a_{0}, a_{2}, a_{1}\right)$, which is precisely the collection of the $(b+1)^{3}$ exceptional curves of $\xi_{1}$.

Finally, our main result follows from combining all these results with the following theorem.

Theorem 4.5. The Picard group of $X_{b}^{1}$ is $\operatorname{Pic}\left(X_{b}^{1}\right) \simeq \operatorname{Pic}\left(\mathbb{P}^{1} \times \mathbb{P}^{3}\right)$. The nef, effective and movable cones of $X_{b}^{1}$ are

$$
\begin{aligned}
& \operatorname{Nef}\left(X_{b}^{1}\right)=\langle(1,0),(0,1)\rangle \quad \text { and } \\
& \operatorname{Eff}\left(X_{b}^{1}\right)=\operatorname{Mov}\left(X_{b}^{1}\right)=\langle(1,0),(-1, b+1)\rangle
\end{aligned}
$$

The variety $X_{b}^{1}$ has precisely two minimal models and one nontrivial birational automorphism $\iota$ of order two. The automorphism $\iota$ interchanges the two models.
Proof. The isomorphism $\operatorname{Pic}\left(X_{b}^{1}\right) \simeq \operatorname{Pic}\left(\mathbb{P}^{1} \times \mathbb{P}^{3}\right)$ follows from $H^{2}\left(X_{b}^{\prime}, \mathcal{O}_{X_{b}^{\prime}}\right)=0$ and from $H^{2}\left(\mathbb{P}^{1} \times \mathbb{P}^{3}, \mathbb{Z}\right) \simeq H^{2}\left(X_{b}^{1}, \mathbb{Z}\right)$, see [11, §3.2. A].

We first prove that $\operatorname{Nef}\left(X_{b}^{1}\right)=\langle(1,0),(0,1)\rangle$. Indeed, the fibres $\mathbb{P}^{1}$ of the small resolution $\xi_{1}: X_{b}^{1} \rightarrow Y_{b}$ have intersection number 0 with $\mathcal{O}_{X_{b}^{1}}(0,1)$ and 1
with $\mathcal{O}_{X_{b}^{1}}(1,0)$. Thus, $\mathcal{O}_{X_{b}^{1}}(\alpha, \beta)$ with $\alpha<0$ has negative intersection number with these curves. On the other hand, the curves which arise as the intersection of a fiber of $\pi_{1}: X_{b}^{1} \rightarrow \mathbb{P}^{1}$ with $\pi_{3}^{-1}(H)$, where $H \in\left|\mathcal{O}_{\mathbb{P}^{3}}(1)\right|$, have positive intersection number with $\mathcal{O}_{X_{b}^{1}}(0,1)$ and intersection number 0 with $\mathcal{O}_{X_{b}^{1}}(1,0)$. Since these curves form a covering family, the line bundles $\mathcal{O}_{X_{b}^{1}}(\alpha, \beta)$ with $\beta<0$ are neither nef nor effective.

Next we compute the effective and movable cone. Since $\mathcal{O}_{X_{b}^{1}}(-1, b+1)$ has no fixed component by the proof of Proposition 4.4, we have $\langle(1,0),(-1, b+1)\rangle \subseteq$ $\operatorname{Mov}\left(X_{b}^{1}\right)$. To see that this coincides with $\operatorname{Eff}\left(X_{b}^{1}\right)$ we note that the two small resolutions $X_{b}^{1}$ and $X_{b}^{2}$ of $Y_{b}$ coincide in codimension 1 and are isomorphic as abstract varieties. Thus, we have

$$
h^{0}\left(X_{b}^{1}, \mathcal{O}_{X_{b}^{1}}(\alpha, \beta)\right)=h^{0}\left(X_{b}^{2}, \mathcal{O}_{X_{b}^{2}}(\alpha, \beta)\right)=h^{0}\left(X_{b}^{1}, \mathcal{O}_{X_{b}^{1}}(-\alpha, \alpha(b+1)+\beta)\right)
$$

In particular, these groups are zero for $\alpha>0$ and $\beta<0$ and

$$
\operatorname{Eff}\left(X_{b}^{1}\right)=\operatorname{Mov}\left(X_{b}^{1}\right)=\langle(1,0),(0,1)\rangle \cup\langle(0,1),(-1, b+1)\rangle .
$$

The interiors of the two subcones are ample on $X_{b}^{1}$ and $X_{b}^{2}$, respectively.
All computations in Macaulay2 can be found in [12].

## Supporting file

A supporting file for this paper is available on the journal website.

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# On the study of decompositions of forms in four variables 

Luca Chiantini

Al mio amico Giorgio, insieme al quale ho brindato ai recenti successi della Lupa


#### Abstract

In the space of sextic forms in 4 variables with a decomposition of length 18 we determine and describe a closed subvariety which contains all non-identifiable sextics. The description of the subvariety is geometric, but one can derive from that an algorithm which can guarantee that a given form is identifiable.


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## 1. Introduction

The paper describes an application of geometric tools, mainly from the theory of finite sets in projective spaces, to the study of Waring decompositions of forms.

The tools have been introduced and employed, in a series of papers, mainly for forms of degree 4 or for forms in three variables (see [2, 3, 4, 5, 11]). Since quaternary forms of degree 5 are considered in a forthcoming paper [9], we turn now our attention to forms of degree 6 in four variables.

Our starting point is the same starting point of the celebrated Kruskal's criterion for the minimality and uniqueness of a decomposition (to be precise, in its version for symmetric tensors). We assume that we know a (Waring) expression of a form $F$ in terms of powers of linear forms, as the one given in formula (1) below. The problem consists of determining if the expression is minimal, in which case it computes the Waring rank of F. In addition, one would like to know if the expression is unique (up to trivialities).

We attack the problem by considering the linear forms appearing in the expression as a set of points $A$ in a projective space $\mathbb{P}^{3}$, and analyzing the existence of another set $B$, of length smaller or equal than the length of $A$, whose 6 -Veronese image spans $F$.

It turns out that the union $Z=A \cup B$ must satisfy several geometric and algebraic restrictions. This makes it possible to analyze the situation up to rank 18. Indeed, we prove that when the length ( $=$ cardinality) of $A$ is strictly smaller than 18 and $A$ is sufficiently general in a very precise sense (see the statement of Proposition 4.4), then the expression is necessarily unique. The geometric situation in this case is similar to the one treated in Kruskal's criterion (which, by the way, even in its reshaped version described in [12], cannot work for $r>14$ in the case of quaternary sextics).

The case $r=18$ turns out to be different. For $r=18$, even if $A$ is completely general, there are forms in the span of the 6 -Veronese image of $A$ for which a second decomposition $B$ exists. We can be more specific: when $A$ is general, so that cubic surfaces through $A$ define a complete intersection irreducible curve $C$ of degree 9 , then $B$ is forced to be residual to $A$ in a complete intersection of $C$ and a quartic surface. This allows us to parameterize the possible sets $B$, and thus parameterize a (locally closed) subvariety $\Gamma$ of the span of $v_{6}(A)$, which contains the forms $F$ of degree 6 in 4 variables, rank 18 , which are not identifiable. The closure of $\Gamma$ is the image of a map from a subspace of the projective space $\mathbb{P}\left(\left(I_{A}\right)_{4}\right)$ to $\left\langle v_{6}(A)\right\rangle$. We refer to Theorem 5.5 for a more precise description.

In particular, we get that if $F$ is a non-identifiable form, then the second decomposition $B$ is bounded to an invariant curve $C$, defined by $A$. This is a case of confinement for decompositions of forms, as described in general in [1].

Since the generic rank of a form of degree 6 in four variables is 21 , one may wonder what happens for the missing cases $r=19,20,21$. For $r=19$, the same procedure proves that a hypothetical second decomposition $B$ must be bounded to the unique cubic surface defined by $A$, but we are not able to characterize it any more. For $r=20,21$ we have no precise characterization. This is probably due to the fact that the theory of finite sets in $\mathbb{P}^{3}$ is far from being completely understood, and also opens a series of questions on the structure of finite sets in higher dimensional spaces, which could suggest directions to investigators in the field.

## 2. Preliminaries

All polynomials in the paper are defined over the complex field.
We will often, by abuse, use the same letter to indicate a form in a polynomial ring, the projective hypersurface defined by the form, and the point defined by the form in the corresponding projective space.

Given a finite set $A$ in a projective space, we denote by $\ell(A)$ its length (i.e. its cardinality).

Consider a form $F$ of degree 6 in 4 variables, over the complex field.

Assume we know a Waring expression of $F$ (of length $r$ ) as a linear combination of powers of linear forms

$$
\begin{equation*}
F=\sum_{i=1}^{r} a_{i} L_{i}^{6} \tag{1}
\end{equation*}
$$

but we do not know a priori if the expression is minimal or unique (up to trivialities). Thus we do not know if $r$ is the (Waring) rank of $F$, and we do not know whether $F$ is identifiable or not.

On the other hand, we can certainly assume that the expression is nonredundant, in the sense that the powers $L_{i}^{6}$ 's are linearly independent and no coefficient $a_{i}$ is 0 .

Call $A=\left\{L_{1}, \ldots, L_{r}\right\}$ the set of linear forms involved in the expression, considered as points in a projective space $\mathbb{P}^{3}$. If we denote with $v_{d}: \mathbb{P}^{3} \rightarrow \mathbb{P}^{N}$ the $d$-Veronese map, the expression tells us that $F$ (as we said above, identified by abuse with one point of the space $\mathbb{P}^{83}$ of sextic forms in $\mathbb{P}^{3}$ ) belongs to the span of the Veronese image $v_{6}(A)$. The non-redundancy of $A$ is equivalent to saying that, for all proper subsets $A^{\prime} \subset A, F$ is not contained in the span of $v_{6}\left(A^{\prime}\right)$.

We have full control on the set $A$, so we may assume that we know all its invariants. Thus we can assume that

## (*) $\quad A$ is in General Position (GP)

which, in this setting, means that all subsets of $A$ have maximal Hilbert function.

Notice that if $A$ has this property, then all subsets of $A$ also have it.
Remark 2.1. When $r \leq 14$, then the celebrated Kruskal's criterion, in its reshaped version (see [12]) guarantees that $r$ is the $\operatorname{rank}$ of $F$, and the expression is unique (up to trivialities: product by a scalar or reordering).

Namely, if $u=\min \{r, 10\}$ then necessarily

$$
r \leq \frac{u+u+u-2}{2}
$$

thus we can take a partition $6=2+2+2$ and consider $F$ as a tensor of $\operatorname{Sym}^{2}\left(\mathbb{C}^{4}\right) \otimes \operatorname{Sym}^{2}\left(\mathbb{C}^{4}\right) \otimes \operatorname{Sym}^{2}\left(\mathbb{C}^{4}\right)$. Since the second Kruskal's rank of $A$ is $u$ by the genericity assumption, then a direct application of Kruskal's criterion guarantees that (1) is the unique expression of $F$ of length $r$.

When $r>14$, we assume the existence of another expression

$$
\begin{equation*}
F=\sum_{i=1}^{s} a_{i} M_{i}^{6}, \quad s \leq r \tag{2}
\end{equation*}
$$

and call $B=\left\{M_{1}, \ldots, M_{s}\right\}$ the consequent finite set in $\mathbb{P}^{3}$.
Again we may directly assume that also $B$ is non-redundant.
When $r \geq 15$ the Kruskal's criterion cannot provide a proof of the minimality and uniqueness of the expression (1). Indeed in this case new expressions are possible. A finer geometrical analysis is required to understand the situation.

Call $h_{A}, h_{B}, h_{Z}$ the Hilbert functions of $A, B$ and $Z=A \cup B$ respectively.
By assumptions we know that the difference $D h_{A}(i)=h_{A}(i)-h_{A}(i-1)$ is defined by the following table

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $D h_{A}(j)$ | 1 | 3 | 6 | $r-10$ | $\max \{0, r-20\}$ | 0 | $\cdots$ |

From [2, Proposition 2.19], we know that

$$
\operatorname{dim}\left(\left\langle v_{6}(A)\right\rangle \cap\left\langle v_{6}(B)\right\rangle\right)=\ell(A \cap B)-1+h_{Z}^{1}(6)
$$

where $h_{Z}^{1}(i)$ is defined by $h_{Z}^{1}(i)=\ell(Z)-h_{Z}(i)$.
In particular $h_{Z}(6)<\ell(Z)$ when $A, B$ are disjoint.
We recall the Cayley-Bacharach property of $Z$ from [5] and [2, Section 2.4]. Remark 2.2. Since $A, B$ are both non-redundant, if $A \cap B=\emptyset$ then the set $Z$ satisfies the Cayley-Bacharach property. In particular for $j=0,1,2,3$,

$$
\sum_{i=0}^{j} D h_{Z}(i) \leq \sum_{i=0}^{j} D h_{Z}(7-j)
$$

Proposition 2.3. Assume $r \leq 20$. Then $s=\ell(B) \geq r$. If $r=15$ then $A, B$ are disjoint. Moreover, for all $r$ the ideals of $A$ and $Z$ agree up to degree 3 .

Proof. If $A \cap B=\emptyset$, then by Remark 2.2 we must have:

$$
\begin{aligned}
\ell(Z)=\ell(A)+\ell(B) & \geq \sum_{i=0}^{7} D h_{Z}(i) \\
& \geq 2 \sum_{i=0}^{3} D h_{Z}(i) \geq 2 \sum_{i=0}^{3} D h_{A}(i)=2 \ell(A)
\end{aligned}
$$

which proves $s \geq r$. If $r=s$, the inequalities become equalities, and this implies the result on the ideals of $A$ and $Z$.

Assume $A \cap B \neq \emptyset$, i.e. assume $L_{i}=M_{i}$ for $i=1, \ldots, j$, for some $j>0$.

Then

$$
\begin{aligned}
F & =a_{1} L_{1}^{6}+\cdots+a_{j} L_{j}^{6}+a_{j+1} L_{j+1}^{6}+\cdots+a_{r} L_{r}^{6} \\
& =b_{1} L_{1}^{6}+\cdots+b_{j} L_{j}^{6}+b_{j+1} M_{j+1}^{6}+\cdots+b_{r} M_{r}^{6}
\end{aligned}
$$

Define $F^{\prime}$ by

$$
\begin{aligned}
F^{\prime}=\left(a_{1}-b_{1}\right) L_{1}^{6}+\cdots+\left(a_{j}-b_{j}\right) L_{j}^{6} & +a_{j+1} L_{j+1}^{6}+\cdots+a_{r} L_{r}^{6} \\
& =b_{j+1} M_{j+1}^{6}+\cdots+b_{s} M_{s}^{6} .
\end{aligned}
$$

$F^{\prime}$ has two disjoint decomposition. The former can have some vanishing coefficients, but its length in any case is at least $r-j$, while the latter has length $\leq s-j$.

If $r=15$ we obtain a contradiction by the reshaped Kruskal's criterion (Remark 2.1) or by what we concluded above in the disjoint case. Then, arguing by induction on $r$, we get that $s \geq r$.

If $A_{0}, B_{0}$ are the two decompositions of $F^{\prime}$ defined above, then by induction the ideals of $A_{0}$ and $Z_{0}=A_{0} \cup B_{0}$ agree up to degree 3. Since $A, B$ are obtained from $A_{0}, B_{0}$ by adding the same subset $S$, then also the ideals of $A$ and $Z$ agree up to degree 3 .

The minimality of the expression (1) proved in the previous result indeed also follows from [6, Theorem 1.2], or by [14, Theorem 3.1].

## 3. The case $r=15$

We know from Proposition 2.3 and its proof that if $F$ has two decompositions $A, B$, then $A \cap B=\emptyset$.

We show an example in which the second decomposition $B$ exists.
Example 3.1. Assume that $A$ is a general set of 15 points in a general elliptic quintic curve $C$. The 6 -Veronese map maps $C$ to a normal elliptic curve of degree 30 which spans a $\mathbb{P}^{29}$. In $\mathbb{P}^{29}$ a general point has two different decompositions with respect to the elliptic curve $C$ (see [10, Proposition 5.2]). Thus one gets that a general $F$ in the span of $v_{6}(A)$ has exactly two different decompositions.

It is easy indeed to construct examples of forms $F$ with two decompositions of this type. A general set $A$ of 15 points in an elliptic quintic and a general $F$ in the span of $v_{6}(A)$ will do.

On the other hand, it is also simple to realize that a general set $A$ of 15 points in $\mathbb{P}^{3}$ does not lie in an elliptic quintic. This is just a count of parameters: the Hilbert scheme of elliptic quintics has dimension $5 \cdot 4=20$, so the sets
of 15 points in such curves cannot depend on more than $20+15=35$ parameters; on the other hand, the family of sets of 15 points in $\mathbb{P}^{3}$ has dimension 45.

One can easily exclude that a given set $A$ of 15 points in $\mathbb{P}^{3}$ lies in an elliptic quintic by considering the base locus of the system of cubics through $A$ which, by assumption, has dimension 5 .

Proposition 3.2. Assume $r=15$ and assume that the base locus of the system of cubics through $A$ contains no curves. Then $A$ is the unique minimal decomposition of $F$.

Proof. Assume there exists a second decomposition $B$ of length $\leq 15$. Arguing as in the final part of the proof of Proposition 2.3, since we can apply the reshaped Kruskal's criterion for decompositions of length $\leq 14$, we see that $A, B$ must be disjoint. We know that the ideal of $Z=A \cup B$ coincides with the ideal of $A$ in degree 3 . Since the base locus of the system of cubics through $A$ contains no curves, then by Bézout $Z$ has length at most 27 . Thus $\ell(B) \leq 12$, which is excluded by Proposition 2.3.

One checks easily the dimension of the base locus of the system of cubics through $A$, by standard computer algebra packages.

## 4. The cases $r=16,17$

The situation for $r=16,17$ is quite similar to the case $r=15$, except that now an intersection between the two decompositions is allowed.
Example 4.1. Let $A_{0}$ be a general set of 15 points lying in a general elliptic quintic curve $C$. We saw in Example 3.1 that a general form $F_{0}$ in the span of $v_{6}\left(A_{0}\right)$ has a second decomposition $B_{0}$ of length 15 , disjoint from $A_{0}$. If $L_{0}$ is a general linear form, then $\left\{L_{0}\right\} \cup A_{0}$ and $\left\{L_{0}\right\} \cup B_{0}$ are two different, non-disjoint, decompositions of length 16 of $L_{0}^{6}+F_{0}$.

Arguing as in Proposition 2.3, one sees that these two decompositions are minimal, when $A_{0}, B_{0}, L_{0}$ are general.

Also examples with different disjoint decompositions are possible.
Example 4.2. Let $A$ be a general set of 16 points lying in a general rational quintic curve $C$. By Bézout, since $C$ is irreducible, the ideal of $C$ and the ideal of $A$ agree in degree 3. The Veronese map $v_{6}$ maps $C$ to $\mathbb{P}^{30}$. Since no curves are defective, a general point $F$ of $\mathbb{P}^{30}$ has infinitely many (mostly disjoint) decompositions of length 16 with respect to $v_{6}(C)$.

Sets $A$ of this type lie in the Terracini locus, as defined in [7]: the differential of the map from the abstract 16 -secant variety to the space $\mathbb{P}^{83}$ of $v_{6}\left(\mathbb{P}^{3}\right)$ drops rank over a general $F \in\left\langle v_{6}(A)\right\rangle$.

Example 4.3. Starting with forms with two decompositions of length 16 , as e.g. in Example 4.2, and adding one point as in Example 4.1, one finds easily examples of non-disjoint different decompositions of length 17 for some sextics $F$.

As in the case $r=15$, if the system of cubics through $A$ has no curves in the base locus, then the decomposition $A$ of $F$ is unique.

Proposition 4.4. Assume $r=16$ or $r=17$ and assume that the base locus of the system of cubics through $A$ contains no curves. Then $A$ is the unique minimal decomposition of $F$.

Proof. The proof is given only for $r=16$, since the other case is completely analogous.

Assume there exists a second decomposition $B$ of length 16. If $A \cap B=\emptyset$, since the ideal of $Z=A \cup B$ coincides with the ideal of $A$ in degree 3 , by Bézout $Z$ has length at most 27 . Thus $\ell(B) \leq 11$, which is excluded by Proposition 2.3.

If the intersection $A \cap B$ contains $j>0$ points, then as above write

$$
\begin{aligned}
F & =a_{1} L_{1}^{6}+\cdots+a_{j} L_{j}^{6}+a_{j+1} L_{j+1}^{6}+\cdots+a_{16} L_{16}^{6} \\
& =b_{1} L_{1}^{6}+\cdots+b_{j} L_{j}^{6}+b_{j+1} M_{j+1}^{6}+\cdots+b_{16} M_{16}^{6} .
\end{aligned}
$$

Define $F^{\prime}$ by

$$
\begin{aligned}
F^{\prime}=\left(a_{1}-b_{1}\right) L_{1}^{6}+\cdots+\left(a_{j}-b_{j}\right) L_{j}^{6} & +a_{j+1} L_{j+1}^{6}+\cdots+a_{16} L_{16}^{6} \\
& =b_{j+1} M_{j+1}^{6}+\cdots+b_{16} M_{16}^{6}
\end{aligned}
$$

$F^{\prime}$ has two disjoint decompositions, one for which $A^{\prime}$ is contained in $A$. Thus the system of cubics through $A^{\prime}$ has no curves in the base locus. Even if the length of $A^{\prime}$ is 15 , we have a contradiction with Proposition 3.2.

Since for $r \leq 17$ and $A$ very general the system of cubics through $A$ has no curves in the base locus, the previous proposition excludes the existence of a second decomposition, except for sets $A$ contained in a Zariski closed subset of $\left(\mathbb{P}^{3}\right)^{r}$.

## 5. The case $r=18$

For $r=18$ and $A$ general, the base locus of the system of cubics through $A$ is a complete intersection curve $C$ of degree 9 and genus 10 . There is no way to use a strategy similar to the statement of Proposition 4.4 in order to prove the identifiability of $F$.
Remark 5.1. From Proposition 3.2 and Proposition 4.4 it turns out that, when $r=15,16,17$ and the system of cubics through $A$ has no curves in the base
locus, then all forms $F$ in the span of $v_{6}(A)$ are identifiable of (Waring) rank $r$, unless the decomposition $A$ is redundant for $F$, i.e. unless $F$ sits in the span of some strict subset of $v_{6}(A)$.

We can see immediately that the situation changes completely for $r=18$. Example 5.2. Let $A$ be a general set of 18 points in $\mathbb{P}^{3}$. Then $A$ is contained in the complete intersection of two cubics $G_{1}, G_{2}$. Consider the complete intersection curve $C=G_{1} \cap G_{2}$ and let $G$ be a general quartic not containing $C$. The intersection of $C$ with the surface $G$ consists of 36 points $Z=A \cup B . B$ is thus a set of 18 points in the curve $C$, disjoint from $A$. By the Cayley-Bacharach property of complete intersections, one knows that $h_{Z}^{1}(6)>0$. Thus by [2, Proposition 2.19], we know that $\left\langle v_{6}(A)\right\rangle$ and $\left\langle v_{6}(B)\right\rangle$ meet in some point $F$. Such $F \in\left\langle v_{6}(A)\right\rangle$ has a second decomposition $B$ of length 18 .
Remark 5.3. By [4, Proposition 3.9], when $A, B$ are disjoint decompositions of $F$, then the sum of the homogeneous ideals $I_{A}+I_{B}$ does not coincide with the polynomial ring $R$ in degree 6 , and $F$ is dual to $I_{A}+I_{B}$

Consider again the sets $A, B$ described in Example 5.2.
The ideal of $B$ can be found from $G$ and the ideal of $A$ as a result of the mapping cone process (see [13]). By the Minimal Resolution Conjecture, which holds in $\mathbb{P}^{3}$ (see [8]), a resolution of the ideal $I_{A}$ is given by $0 \rightarrow R^{8}(-6) \rightarrow$ $R^{18}(-5) \rightarrow R^{2}(-3) \oplus R^{9}(-4) \rightarrow I_{A} \rightarrow 0$. Combining with the Koszul complex of $G_{1}, G_{2}, G$ one obtains a diagram

$$
\begin{array}{ccc}
0 \rightarrow R(-10) & \xrightarrow{\alpha^{\prime}} R(-6) \oplus R^{2}(-7) & \xrightarrow{\beta^{\prime}} R^{2}(-3) \oplus R(-4)  \tag{3}\\
\gamma \downarrow & \rightarrow I_{Z} \rightarrow 0 \\
0 \rightarrow R^{8}(-6) \xrightarrow{\gamma^{\prime} \downarrow} \quad R^{18}(-5) & \xrightarrow{\beta} R^{2}(-3) \oplus R^{9}(-4) \rightarrow I_{A} \rightarrow 0
\end{array}
$$

where the map $\gamma^{\prime \prime}$ is defined by $G_{1}, G_{2}, G$. From the diagram one obtains a resolution of $I_{B}$ by the dual of the mapping cone:

$$
0 \rightarrow R^{8}(-6) \rightarrow R^{18}(-5) \xrightarrow{\left(\alpha \oplus \gamma^{\prime}\right)^{\vee}} R^{2}(-3) \oplus R^{9}(-4) \xrightarrow{\left(\alpha^{\prime} \oplus \gamma\right)^{\vee}} I_{B} \rightarrow 0
$$

Thus there is a standard way to compute $I_{B}$, hence $I_{A}+I_{B}$, from $I_{A}$ and $G$.
We have then all the ingredients to study the existence of a second decomposition for $F$.

Proposition 5.4. Assume that the decomposition $A$ of length 18 of $F$, satisfying condition (*), also satisfies the following condition: for all subsets $A^{\prime} \subset A$ of length 17, the linear system of cubics through $A^{\prime}$ has base locus of dimension 0 . Then any other decompositions of length 18 of $F$ is disjoint from $A$.

Proof. Assume there exists another decomposition $B$ of length 18 with $\ell(A \cap$ $B)=j>0$. Then arguing as in the proof of Proposition 2.3 one finds another
sextic form $F^{\prime}$ with decompositions

$$
\begin{aligned}
F^{\prime}=\left(a_{1}-b_{1}\right) L_{1}^{6}+\cdots+\left(a_{j}-b_{j}\right) L_{j}^{6} & +a_{j+1} L_{j+1}^{6}+\cdots+a_{18} L_{18}^{6} \\
& =b_{j+1} M_{j+1}^{6}+\cdots+b_{18} M_{18}^{6}
\end{aligned}
$$

where $A=\left\{L_{1}, \ldots, L_{18}\right\}$ and $a_{i}, b_{i} \neq 0$ for all $i$. If some coefficient $a_{i}-$ $b_{i}, i=1, \ldots, j$, is non-zero, then the second decomposition of $F^{\prime}$ has length smaller than the first one, which is contained in $A$. We get a contradiction with Proposition 2.3. Thus $a_{i}=b_{i}$ for all $i=1, \ldots, j$. But then $F^{\prime}$ has two disjoint decompositions of length $18-j$, and one of them $A^{\prime}=\left\{L_{j+1}, \ldots, L_{18}\right\}$ is contained in $A$. By assumption the system of cubics through $A^{\prime}$ has no curves in the base locus. Then we get a contradiction with either the Reshaped Kruskal's Criterion, or Proposition 3.2, or Proposition 4.4.

Theorem 5.5. Let $F$ be a sextic in 4 variables, with a non-redundant decomposition A of length 18. Assume that A satisfies the following properties.
(*) $A$ is in General Position;
$\left(^{* *}\right)$ for all subsets $A^{\prime} \subset A$ of length 17, the linear system of cubics through $A^{\prime}$ has base locus of dimension 0;
(***) the base locus of the pencil of cubics through $A$ is an irreducible curve $C$.
Then $A$ is minimal, and any other decomposition $B$ of length 18 of $F$ (if any) is disjoint from $A$, and $Z=A \cup B$ is a complete intersection of surfaces of degrees $3,3,4$.

Proof. The unique thing that remains to prove is the last assertion, i.e. that $A \cup B$ is the intersection of $C$ with a quartic surface.

If $B$ exists, $Z=A \cup B$ lies in the pencil of cubics containing $A$, by Proposition 2.3. If all the quartics containing $Z$ are composed with the pencil, then $h_{Z}(4)=35-8=27$, so that $D h_{Z}(4)=9$. But then $D h_{Z}(5)+D h_{Z}(6)+$ $D h_{Z}(7) \leq 9<D h_{Z}(2)+D h_{Z}(1)+D h_{Z}(0)$, which contradicts the CayleyBacharach property. hence there is a quartic containing $Z$ and not $C$. The claim follows.

Remark 5.6. For a given form $F$ and a decomposition $A$ of length 18, one can produce a procedure which tests if $A$ is unique i.e. if $F$ is identifiable of rank 18 , as follows.

1. Control if $A$ is in $G P$.
2. Control that the system of cubics through any subset of length 17 of $A$ has 0-dimensional base locus.
3. Control that the system of cubics through $A$ has an irreducible nonic curve as base locus.
4. Consider a linear space $W$ in $\left(I_{A}\right)_{4}$ orthogonal to the 8-dimensional subspace spanned by the cubics through $A$.
5. For all $G \in W$ compute the generators of the residue $B$ of $A$ in $G \cap C$, in terms of coordinates of $G \in W$.
6. Prove that for no choice of the coordinates of $G$ the form $F$ is dual to $I_{A}+I_{B}$.

Notice that the generators of $I_{B}, \bmod$ the cubics containing $C$, are 9 quartics, by the resolution following diagram 3.

One of the most expensive points in the procedure is step (1), which requires to control that none of the $\binom{18}{8}=43,758$ subsets of length 10 of $A$ is contained in quadrics.

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# Secant varieties and the complexity of matrix multiplication 

Joseph M. Landsberg<br>Dedicated to Giorgio Ottaviani on the occasion of his 60th birthday


#### Abstract

This is a survey primarily about determining the border rank of tensors, especially those relevant for the study of the complexity of matrix multiplication. This is a subject that on the one hand is of great significance in theoretical computer science, and on the other hand touches on many beautiful topics in algebraic geometry such as classical and recent results on equations for secant varieties (e.g., via vector bundle and representation-theoretic methods) and the geometry and deformation theory of zero dimensional schemes.


Keywords: Tensor rank, border rank, secant variety, Segre variety, Quot scheme, spaces of commuting matrices, spaces of bounded rank, matrix multiplication complexity, deformation theory.
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## 1. Introduction

This is a survey of uses of secant varieties in the study of the complexity of matrix multiplication, one of many areas in which Giorgio Ottaviani has made significant contributions. I pay special attention to the use of deformation theory because at this writing, deformation theory provides the most promising path to overcoming lower bound barriers. For an introduction to more general uses of algebraic geometry in algebraic complexity theory see [35]. I begin by reviewing some classical results.

### 1.1. Symplectic bundles on the plane, secant varieties, and Lüroth quartics revisited [45]

In the 1860's, Darboux studied degree $n$ curves in $\mathbb{P}^{2}$ that pass through all the $\binom{n+1}{2}$ vertices of a complete $(n+1)$-gon in $\mathbb{P}^{2}$ (i.e., the union of $n+1$ lines in $\mathbb{P}^{2}$ with no points of triple intersection). In 1869 Lüroth studied the $n=4$ case. A naïve dimension count indicates that all quartics should pass through the 10 vertices of some complete pentagon but Lüroth proved it is actually a
codimension one condition.
In 1902 Dixon [24] proved all degree $n$ curves in $\mathbb{P}^{2}$ arise as a $n \times n$ symmetric determinant (also see [23] for the general determinantal case).

In 1977 Barth [6] studied the moduli space of stable (symplectic) vector bundles on $\mathbb{P}^{2}$. In particular he showed that the curve of jumping lines of a rank 2 stable bundle on $\mathbb{P}^{2}$ with Chern classes $\left(c_{1}, c_{2}\right)=(0,4)$ is a Lüroth quartic. Barth also gave a new proof of Lüroth's theorem via vector bundles.

In [45] Giorgio Ottaviani explains these results via the defectivity of secant varieties of $\operatorname{Seg}\left(\mathbb{P}^{2} \times v_{2}\left(\mathbb{P}^{n-1}\right)\right)$, where $\operatorname{Seg}\left(\mathbb{P}^{2} \times v_{2}\left(\mathbb{P}^{n-1}\right)\right) \subset \mathbb{P}\left(\mathbb{C}^{3} \otimes S^{2} \mathbb{C}^{n}\right)$ is the set of points $\left[x \otimes z^{2}\right]$, where $[x] \in \mathbb{P}^{2}$ and $[z] \in \mathbb{P}^{n-1}$. The proof uses the bounded derived category version of Beilinson's monad Theorem [8], see [4] for an excellent introduction.

### 1.2. Secant varieties

Throughout this paper $V, A, B, C$ denote finite dimensional complex vector spaces. Let $X \subset \mathbb{P} V$ be a projective variety, Define its $r$-th secant variety, or variety of secant $\mathbb{P}^{r-1}$ 's, to be

$$
\sigma_{r}(X):=\overline{\bigcup_{x_{1}, \ldots, x_{r} \in X}\left\langle x_{1}, \ldots, x_{r}\right\rangle} .
$$

Here, for a set or subscheme $Z \subset \mathbb{P} V,\langle Z\rangle \subset \mathbb{P} V$ denotes its linear span, and the overline denotes Zariski closure.

In this article I will be particularly interested in the case $X=\operatorname{Seg}(\mathbb{P} A \times$ $\mathbb{P} B \times \mathbb{P} C) \subset \mathbb{P}(A \otimes B \otimes C)$, the variety of rank one (3-way) tensors. Given $T \in A \otimes B \otimes C$, define the border rank of $T, \underline{\mathbf{R}}(T)$ to be the smallest $r$ such that $[T] \in \sigma_{r}(X)$.

Secant varieties have a long history in algebraic geometry dating back to the 1800 's. In the 20th century they were used by J. Alexander and A. Hirschowitz [1] to solve the polynomial interpolation problem, and by F. Zak [51, Chap II, §2] to solve a linearized version of R. Hartshorne's famous conjecture on complete intersections, called Hartshorne's conjecture on linear normality. L. Manivel and I used them to study the geometry of the exceptional groups and their homogeneous varieties, and even to obtain a new proof of the Killing-Cartan classification of complex simple Lie algebras and prove geometric consequences of conjectured categorical generalizations of Lie algebras by Deligne and Vogel, see [37] for a survey. In this article, I discuss their use in the context of algebraic complexity theory, more specifically, in proving lower and upper bounds on the complexity of matrix multiplication.

### 1.3. Matrix multiplication

In 1968, V. Strassen [47] discovered the usual way we multiply $\mathbf{n} \times \mathbf{n}$-matrices, which uses $\mathcal{O}\left(\mathbf{n}^{\mathbf{3}}\right)$ arithmetic operations, is not optimal. After much work, it was generally conjectured that one can in fact multiply matrices using $\mathcal{O}\left(\mathbf{n}^{2+\epsilon}\right)$ arithmetic operations for any $\epsilon>0$. To fix ideas, define the exponent of matrix multiplication $\omega$ to be the infimum over all $\tau$ such that $\mathbf{n} \times \mathbf{n}$ matrices may be multiplied using $\mathcal{O}\left(\mathbf{n}^{\tau}\right)$ arithmetic operations, so the conjecture is that $\omega=2$. The matrix multiplication tensor $M_{\langle\mathbf{n}\rangle}: \mathbb{C}^{\mathbf{n}^{2}} \times \mathbb{C}^{\mathbf{n}^{2}} \rightarrow \mathbb{C}^{\mathbf{n}^{2}}$ executes the bilinear map of multiplying two matrices. Fortunately for algebraic geometry, Bini [9] showed $\underline{\mathbf{R}}\left(M_{\langle\mathbf{n}\rangle}\right)=\mathcal{O}\left(\mathbf{n}^{\omega}\right)$ so we may study the exponent via secant varieties of Segre varieties.

Thus one way to prove complexity lower bounds for matrix multiplication would be to prove lower bounds on the border rank of $M_{\langle\mathbf{n}\rangle}$. I will give a history of such lower bounds. Perhaps more surprising, is that one way of showing upper bounds for the complexity of matrix multiplication would be to prove the border rank of certain auxiliary tensors is small, as I discuss in $\S 4$.

### 1.4. Dimensions of secant varieties

One expects $\operatorname{dim} \sigma_{r}(X)=\min \{r \operatorname{dim} X+r-1, \operatorname{dim} \mathbb{P} V\}$, because one can pick $r$ points on $X$, and then a point on the $\mathbb{P}^{r-1}$ spanned by them. This always gives an upper bound on the dimension.

Strassen [48], motivated by the complexity of matrix multiplication, showed that this expectation fails for $X=S e g\left(\mathbb{P}^{2} \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}\right) \subset \mathbb{P}\left(\mathbb{C}^{3} \otimes \mathbb{C}^{n} \otimes \mathbb{C}^{n}\right)$, $n$ odd, $r=\frac{3 n-1}{2}$.

Previously, E. Toeplitz, in 1877 [50], had already shown it fails for $X=$ $\operatorname{Seg}\left(\mathbb{P}^{2} \times v_{2}\left(\mathbb{P}^{3}\right)\right) \subset \mathbb{P}\left(\mathbb{C}^{3} \otimes S^{2} \mathbb{C}^{4}\right), r=5$.

In 2007 Ottaviani [45] showed that more generally the expectation fails for $X=\operatorname{Seg}\left(\mathbb{P}^{2} \times v_{2}\left(\mathbb{P}^{n-1}\right)\right) \subset \mathbb{P}\left(\mathbb{C}^{3} \otimes S^{2} \mathbb{C}^{n}\right)$ with $n$ even $r=\frac{3 n}{2}-1$, and that this failure implies Lüroth's theorem. In the same paper he also partially recovers Barth's moduli results.

## 2. Koszul flattenings and variants

### 2.1. Idea of Proofs of results in $\S 1.4$

To prove the naïve dimension count for $\operatorname{dim} \sigma_{r}(X)$ is wrong (e.g., in the case of Lüroth's theorem that $\left.\sigma_{r}(X) \neq \mathbb{P} V\right)$, one can show that the ideal of $\sigma_{r}(X)$ is non-empty by exhibiting an explicit polynomial in the ideal.

Strassen did this and his result was revisited by Ottaviani: Consider $\operatorname{Seg}(\mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C) \subset \mathbb{P}(A \otimes B \otimes C), \operatorname{dim} A=3, \operatorname{dim} B=\operatorname{dim} C=m$. Let
$\left\{a_{i}\right\},\left\{b_{j}\right\},\left\{c_{k}\right\}$ be bases of $A, B, C$. Given $T=\sum T^{i j k} a_{i} \otimes b_{j} \otimes c_{k} \in A \otimes B \otimes C$, consider the linear map

$$
\begin{aligned}
T_{A}^{\wedge 1}: A \otimes B^{*} & \rightarrow \Lambda^{2} A \otimes C \\
a \otimes \beta & \mapsto \sum_{i, j, k} T^{i j k} \beta\left(b_{j}\right) a \wedge a_{i} \otimes c_{k}
\end{aligned}
$$

Exercise: If $[T] \in \operatorname{Seg}(\mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C)$, then $\operatorname{rank}\left(T_{A}^{\wedge 1}\right)=2$, and thus, by linearity, if $\operatorname{rank}\left(T_{A}^{\wedge 1}\right)>2 R$, then $[T] \notin \sigma_{R}(S e g(\mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C))$.

Ottaviani states in a remark that these minors are a reformulation of Strassen's equations (however see Remark 2.1 below), which, for tensors $T \in A \otimes B \otimes C=\mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{m} \otimes \mathbb{C}^{m}$ such that there exists $\alpha \in A^{*}$ with $\operatorname{rank}(T(\alpha))=$ $m$, are naturally expressed as follows: consider $T\left(A^{*}\right) \subset B \otimes C$, and for $\alpha \in A^{*}$ with $\operatorname{rank}(T(\alpha))=m$, consider the linear isomorphism $T(\alpha): B^{*} \rightarrow C$. Then $T\left(A^{*}\right) T(\alpha)^{-1} \subset \operatorname{End}(C)$ is a space of endomorphisms. If $T=\sum_{j=1}^{m} e_{j} \otimes b_{j} \otimes c_{j}$ for some $e_{j} \in A$, then one obtains a space of diagonal matrices. In particular, the matrices commute. Since the property of commuting is closed, if $[T] \in \sigma_{m}(\operatorname{Seg}(\mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C))$, then one still obtains a space of commuting endomorphisms. Moreover, the rank of the commutator (a measure of the failure of commutivity) may be computed from the rank of $T_{A}^{\wedge 1}$. Note that in both cases one restricts to a three dimensional subspace of $A$.

To see Strassen's equations as polynomials, for $X \in B \otimes C$, let $X^{\wedge m-1} \in$ $\Lambda^{m-1} B \otimes \Lambda^{m-1} C \simeq B^{*} \otimes C^{*}$ denote the adjucate (cofactor matrix), and recall that $X^{-1}$ is essentially the adjugate times the determinant. Then Strassen's equations for $T$ to have border rank (at most) $m$ [48] become, for all $X, Y, Z \in$ $T\left(A^{*}\right) \subset B \otimes C$,

$$
\begin{equation*}
X Y^{\wedge m-1} Z-Z Y^{\wedge m-1} X=0 \tag{1}
\end{equation*}
$$

These are equations of degree $m+1$.
Using a refinement of these equations that takes into account the rank of the commutator, Strassen proved $\underline{\mathbf{R}}\left(M_{\langle\mathbf{n}\rangle}\right) \geq \frac{3}{2} \mathbf{n}^{2}$, the first non-classical lower bound on the border rank of the matrix multiplication tensor.

Call a tensor $T$ which satisfies the genericity condition that there exists $\alpha \in A^{*}$ with $\operatorname{rank}(T(\alpha))=m, 1_{A}$-generic.

When $T$ is $1_{A}$-generic, taking $Y$ of full rank and changing bases such that it is the identity element, the equations require the space to be abelian. If $T\left(A^{*}\right)$ is of bounded rank $m-1$, for each $X, Y, Z$, the set of $m^{2}$ equations reduces to a single equation. If $T\left(A^{*}\right)$ is of bounded rank $m-2$, then the equations become vacuous.

Lemma 3.1 of [45] states that if $T$ is $1_{A}$-generic, then the condition on $\operatorname{rank}\left(T_{A}^{\wedge 1}\right)$ is indeed a reformulation of Strassen's equations.

Remark 2.1. Recently, with my student Arpan Pal and Joachim Jelisiejew [32], we proved that if $T$ is not $1_{A}$-generic, then the condition on $\operatorname{rank}\left(T_{A}^{\wedge 1}\right)$ is a stronger condition than the $A$-Strassen equations.

### 2.2. Generalizations: Young flattenings [43, 44]

### 2.2.1. Koszul flattenings

Consider $\operatorname{Seg}(\mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C) \subset \mathbb{P}(A \otimes B \otimes C)$, let $\operatorname{dim} A=2 p+1, \operatorname{dim} B=$ $\operatorname{dim} C=m$. (If $\operatorname{dim} A>2 p+1$, restrict to a general $2 p+1$ dimensional subspace.)

Given $T=\sum T^{i j k} a_{i} \otimes b_{j} \otimes c_{k} \in A \otimes B \otimes C$, consider the linear map

$$
\begin{aligned}
T_{A}^{\wedge p}: \Lambda^{p} A \otimes B^{*} & \rightarrow \Lambda^{p+1} A \otimes C \\
a_{i_{1}} \wedge \cdots \wedge a_{i_{p}} \otimes \beta & \mapsto \sum_{i, j, k} T^{i j k} \beta\left(b_{j}\right) a_{i_{1}} \wedge \cdots \wedge a_{i_{p}} \wedge a_{i} \otimes c_{k}
\end{aligned}
$$

Exercise: If $[T] \in \operatorname{Seg}(\mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C)$, then $\operatorname{rank}\left(T_{A}^{\wedge p}\right)=\binom{2 p}{p-1}$. Thus if $\operatorname{rank}\left(T_{A}^{\wedge p}\right)>\binom{2 p}{p-1} R$, then $[T] \notin \sigma_{R}(S e g(\mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C))$. Call these equations the $p$-Koszul flattenings.

When Ottaviani and I found the $p$-Koszul flattenings, we were sure we would get a new lower bound for matrix multiplication. Our first attempts were discouraging, we were attempting to take $2 p+1=\mathbf{n}^{2}$ or nearly so. It turns out that our initial attempts were too greedy, as such values do not give good lower bounds. Only months later, we finally tried taking $p=\mathbf{n}-\mathbf{1}$ which enabled us to prove the first new lower bounds for border rank of matrix multiplication since 1983:

Theorem 2.2 ([44]). $\underline{\mathbf{R}}\left(M_{\langle\mathbf{n}\rangle}\right) \geq 2 \mathbf{n}^{2}-\mathbf{n}$.
It is worth noting that the absolute limit of Koszul flattenings, and any determinantal equations that we found, was $2 \mathbf{n}^{2}-\mathbf{1}$, i.e., for tensors in $\mathbb{C}^{m} \otimes \mathbb{C}^{m} \otimes \mathbb{C}^{m}, 2 m-1$.

### 2.2.2. Young flattenings

We found the $p$-Koszul flattenings as part of a general program to systematically find equations for secant varieties, especially equations of secant varieties of homogeneous varieties, which we call Young flattenings. Giorgio likes to think of these in terms of degeracy loci of maps between vector bundles, and I prefer a more representation-theoretic perspective. The basic idea is for $X \subset \mathbb{P} V$, to find an inclusion of $V$ into a space of matrices. Then if the matrices associated to points of $X$ have rank at most $q$, the size $q r+1$ minors restricted to $V$ give equations for $\sigma_{r}(X)$.

## Vector bundle perspective

Let $E \rightarrow X$ be a vector bundle of rank $e$, write $L=\mathcal{O}_{X}(1)$ so $V=H^{0}(X, L)^{*}=$ $H^{0}(L)^{*}$. Let $v \in V$ and consider the linear map $A_{v}^{E}: H^{0}(E) \rightarrow H^{0}\left(E^{*} \otimes L\right)^{*}$ induced by the multiplication map $H^{0}(E) \otimes H^{0}(L)^{*} \rightarrow H^{0}\left(E^{*} \otimes L\right)^{*}$. Then, assuming all spaces are sufficiently large, the size $(r e+1)$ minors of $A_{v}^{E}$ give equations for $\sigma_{r}(X)$. Here we have an inclusion $V=H^{0}(L)^{*} \subset H^{0}(E)^{*} \otimes H^{0}\left(E^{*} \otimes L\right)^{*}$.

## Representation theory perspective

Let $X=G / P \subset \mathbb{P} V_{\lambda}$ where $V_{\lambda}$ is an irreducible module for the reductive group $G$ of highest weight $\lambda$ and $X$ is the orbit of a highest weight line, i.e., the minimal $G$-orbit in $\mathbb{P} V_{\lambda}$. Look for $G$-module inclusions $V_{\lambda} \subset V_{\mu} \otimes V_{\nu}$, so in coordinates one realizes $V_{\lambda}$ as a space of matrices. Then for $x \in X$ if the associated matrix has rank $k$, the size $r k+1$ minors of $V_{\mu} \otimes V_{\nu}$ restricted to $V_{\lambda}$ give equations in the ideal of $\sigma_{r}(X)$.

We spent some time trying to find more powerful Young flattenings. At first we just thought we were not being clever enough in our search for determinantal equations, but then we came to suspect that there was some limit to the method.

## 3. Beyond Koszul flattenings: steps forward and barriers to future progress

### 3.1. The cactus barrier

Around the same time, in both algebraic complexity theory [25] and algebraic geometry [7, 11] (see [34, Chap. 10] for an overview), it was proven that determinantal methods are subject to an absolute barrier that is at most 6 m for tensors in $\mathbb{C}^{m} \otimes \mathbb{C}^{m} \otimes \mathbb{C}^{m}$.

To explain the barrier from a geometric perspective, rewrite the definition of the secant variety as

$$
\sigma_{r}(X):=\overline{\bigcup\{\langle R\rangle \mid \text { length }(R)=r, R \subset X, R: \text { smoothable }\}} .
$$

Here $R \subset X$ denotes a zero dimensional scheme and the union is taken over all zero dimensional schemes satisfying the conditions in braces. Define the cactus variety [11]:

$$
\kappa_{r}(X):=\overline{\bigcup\{\langle R\rangle \mid \text { length }(R)=r, R \subset X\}} .
$$

It turns out that $\kappa_{6 m-4}\left(\operatorname{Seg}\left(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}\right)\right)=\mathbb{P}\left(\mathbb{C}^{m} \otimes \mathbb{C}^{m} \otimes \mathbb{C}^{m}\right)$, compared with $\sigma_{r}\left(\operatorname{Seg}\left(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \times \mathbb{P}^{m-1}\right)\right)$ which does not fill $\mathbb{P}\left(\mathbb{C}^{m} \otimes \mathbb{C}^{m} \otimes \mathbb{C}^{m}\right)$ until $r \sim \frac{m^{2}}{3}$.

The barrier results from the fact that determinantal equations for $\sigma_{r}(X)$ are also equations for $\kappa_{r}(X)$ !

When I learned this, I became very discouraged.

### 3.2. A Pyrrhic victory

With M. Michałek, we were able to push things a little further for tensors with symmetry. Given $T \in A \otimes B \otimes C, \underline{\mathbf{R}}(T) \leq r$ if and only if there exists a curve $E_{t} \subset G(r, A \otimes B \otimes C)$, the Grassmannian of $r$ planes through the origin in $A \otimes B \otimes C$, such that

- For $t \neq 0, E_{t}$ is spanned by $r$ rank one elements.
- $T \in E_{0}$.

Let $G_{T}:=\{g \in G L(A) \times G L(B) \times G L(C) \mid g \cdot T=T\}$ denote the symmetry group of $T$. Then if we have such a curve $E_{t}$, then for all $g \in G_{T}, g E_{t}$ also gives a border rank decomposition. Thus one can insist on normalized curves, those with $E_{0}$ Borel fixed for a Borel subgroup of $G_{T}$ [40]. Then one can apply a border rank version of the classical substitution method (see, e.g., [2]) to reduce the problem to bounding the border rank of a smaller tensor. Applying this to the matrix multiplication tensor, we proved:

Theorem 3.1 ([41]). $\underline{\mathbf{R}}\left(M_{\langle\mathbf{n}\rangle}\right) \geq 2 \mathbf{n}^{2}-\log _{\mathbf{2}} \mathbf{n}-\mathbf{1}$.
Recall that the limit of lower bounds one can prove with Young flattening is $2 m-1$. We wrote down an explicit sequence of tensors $T_{m} \in \mathbb{C}^{m} \otimes \mathbb{C}^{m} \otimes \mathbb{C}^{m}$ with a one-dimensional symmetry group and proved:

Theorem 3.2 ([42]). $\underline{\mathbf{R}}\left(T_{m}\right) \geq(2.02) m$.
After that, I did not see any path to further lower bounds.

### 3.3. A vast generalization: border apolarity

W. Buczyńska and J. Buczyński [12] had the following idea: Consider not just a curve in the Grassmannian obtained by taking the spans of $r$ moving points $\left\{T_{j, \epsilon}\right\}$, where $T=\lim _{\epsilon \rightarrow 0} \sum_{j=1}^{r} T_{j, \epsilon}$, but the curve of ideals $I_{\epsilon} \in \operatorname{Sym}\left(A^{*} \oplus\right.$ $\left.B^{*} \oplus C^{*}\right)$ that the points give rise to: let $I_{\epsilon}$ be the ideal of polynomials vanishing on $\left[T_{1, \epsilon}\right] \cup \cdots \cup\left[T_{r, \epsilon}\right] \subset \mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C$. Now consider the "limiting ideal". But how should one take limits? If one works in the usual Hilbert scheme the $r$ points limit to some zero dimensional scheme and one could take the span of that scheme. But for secant varieties one is really taking the limit of the spans $\left\langle T_{1, \epsilon}, \ldots, T_{r, \epsilon}\right\rangle$ in the Grassmannian of $r$ planes and the span of the limiting scheme can be strictly smaller than the limit of the spans, so this idea does not work.

The answer is to work in the Haiman-Sturmfels multigraded Hilbert scheme [29], which lives in a product of Grassmannians and keeps track of the entire Hilbert function rather than just the Hilbert polynomial. The price one pays is that now one must allow unsaturated ideals.

As with the border substitution method, one can insist that limiting ideal $I_{0}$ is Borel fixed, which for tensors with a large symmetry group reduces in small multi-degrees to a small search that has been exploited in practice.

Instead of single curve $E_{\epsilon} \subset G(r, A \otimes B \otimes C)$ limiting to a Borel fixed point, for each ( $i, j, k$ ) one gets a curve in each $\operatorname{Gr}\left(r, S^{i} A^{*} \otimes S^{j} B^{*} \otimes S^{k} C^{*}\right)$, each curve limiting to a Borel fixed point and satisfying compatibility conditions. Here $G r(r, V)$ is the Grassmannian of codimension $r$ subspaces in $V$. In this situation, $E_{\epsilon}=\left(I_{\epsilon}\right)_{(111)}^{\perp}$.

The upshot is an algorithm that either produces all normalized candidate $I_{0}$ 's or proves border rank $>r$. The caveat is that once one has a candidate $I_{0}$ one must determine if it actually came from a curve of ideals of $r$ distinct points.

Using border apolarity, in [19] we proved numerous new matrix multiplication border rank lower bounds, as well as determining the border rank of the size three determinant polynomial considered as a tensor $\operatorname{det}_{3} \in \mathbb{C}^{9} \otimes \mathbb{C}^{9} \otimes \mathbb{C}^{9}$, whose importance for complexity is explained below. In particular, our results include the first nontrivial lower bounds for "unbalanced matrix multiplication tensors", something that was untouchable using previous methods.

### 3.4. Border apolarity is subject to the cactus barrier

In practice, one attempts to construct an ideal by building it up from low multi-degrees. The main restrictions one obtains is when one has the ideal constructed up to multi-degrees $(s-1, t, u),(s, t-1, u)$ and $(s, t, u-1)$, and one wants to construct the ideal in degree $(s, t, u)$. In order that the construction may continue, one needs that the natural symmetrization and addition map

$$
\begin{equation*}
I_{s-1, t, u} \otimes A^{*} \oplus I_{s, t-1, u} \otimes B^{*} \oplus I_{s, t, u-1} \otimes C^{*} \rightarrow S^{s} A^{*} \otimes S^{t} B^{*} \otimes S^{u} C^{*} \tag{2}
\end{equation*}
$$

has image of codimension at least $r$. Here $I_{s-1, t, u} \subset S^{s-1} A^{*} \otimes S^{t} B^{*} \otimes S^{u} C^{*}$ denotes the component of the ideal in multi-degree ( $s-1, t, u$ ) etc. (Here and in what follows, the direct sum is the abstract direct sum of vector spaces, so there is no implied assertion that the spaces are disjoint when they live in the same ambient space.)

That is, the minors of the map of appropriate size must vanish. These are determinantal conditions and therefore subject to the cactus barrier.
Remark 3.3. Aside for the experts: J. Buczyński points out that not all components of the usual Hilbert scheme contain ideals with generic Hilbert func-
tion. Thus in those situations, border apolarity may give better lower bounds on border rank than on cactus border rank.

### 3.5. Deformation theory

Although border apolarity alone cannot overcome the cactus barrier, by placing calculations in the Haiman-Sturmfels multigraded Hilbert scheme, it provides a path to overcoming the cactus barrier. Namely one can apply the tools of deformation theory (see, e.g., [30] for an introduction) to determine if a candidate ideal is deformable to the ideal of a smooth scheme. Below, after motivating the problem, I describe a first implementation of this in the special case of tensors of minimal border rank.

## 4. Strassen's laser method for upper-bounding the exponent of matrix multiplication using tensors of minimal or near minimal border rank

The best way to prove an upper bound on matrix multiplication complexity would be to prove an upper bound for matrix multiplication directly. Fortunately for algebraic geometry, Bini [9] showed that this is measured by the border rank of the matrix multiplication tensor. However, there has been little progress in this direction since the early 1980's. Instead, border rank upper bounds for $M_{\langle\mathbf{n}\rangle}$ have been proven using indirect methods, the most important papers are those of Schönhage [46], Strassen [49] and CoppersmithWinograd [22]. The resulting technique is called Strassen's laser method. Remarkably, it begins with a tensor of minimal (or near minimal) border rank, i.e., a concise tensor in $\mathbb{C}^{m} \otimes \mathbb{C}^{m} \otimes \mathbb{C}^{m}$ of border rank $m$ or nearly $m$, and then builds a large tensor from it, using its Kronecker powers defined below, and then, using methods from probability and combinatorics, shows this large tensor admits a degeneration to a large matrix multiplication tensor.

For tensors $T \in A \otimes B \otimes C$ and $T^{\prime} \in A^{\prime} \otimes B^{\prime} \otimes C^{\prime}$, the Kronecker product of $T$ and $T^{\prime}$ is the tensor $T \otimes T^{\prime}:=T \otimes T^{\prime} \in\left(A \otimes A^{\prime}\right) \otimes\left(B \otimes B^{\prime}\right) \otimes\left(C \otimes C^{\prime}\right)$, regarded as 3 -way tensor. Given $T \in A \otimes B \otimes C$, the Kronecker powers of $T$ are $T^{\boxtimes N} \in A^{\otimes N} \otimes B^{\otimes N} \otimes C^{\otimes N}$, defined iteratively. Rank and border rank are submultiplicative under Kronecker product: $\mathbf{R}\left(T \boxtimes T^{\prime}\right) \leq \mathbf{R}(T) \mathbf{R}\left(T^{\prime}\right)$, $\underline{\mathbf{R}}\left(T \boxtimes T^{\prime}\right) \leq \underline{\mathbf{R}}(T) \underline{\mathbf{R}}\left(T^{\prime}\right)$, and both inequalities may be strict.

Given $T, T^{\prime} \in A \otimes B \otimes C$, we say that $T$ degenerates to $T^{\prime}$ if $T^{\prime} \in \overline{G L(A) \times G L(B) \times G L(C) \cdot T}$, the closure of the orbit of $T$, the closures are the same in the Euclidean and Zariski topologies.

Strassen's laser method [49, 21] obtains upper bounds on $\omega$ by showing a certain explicit degeneration of a large Kronecker power of a tensor $T$ satisfying certain combinatorial properties, admits a further degeneration to a large
matrix multiplication tensor. Since border rank is non-increasing under degenerations and one has an upper bound on $\underline{\mathbf{R}}\left(T^{\boxtimes N}\right)$ inherited from the knowledge of $\underline{\mathbf{R}}(T)$, one obtains an upper bound on the border rank of the large matrix multilplication tensor.

An early success of the laser method was with a tensor of border rank $m+1$, now called the small Coppersmith-Winograd tensor $T_{c w, q} \in\left(\mathbb{C}^{q+1}\right)^{\otimes 3}$. Coppersmith and Winograd showed that for all $k$ and $q=m-1,[22]$

$$
\begin{equation*}
\omega \leq \log _{q}\left(\frac{4}{27}\left(\underline{\mathbf{R}}\left(T_{c w, q}^{\boxtimes k}\right)\right)^{\frac{3}{k}}\right) . \tag{3}
\end{equation*}
$$

They used this when $q=8$ and the estimate $\underline{\mathbf{R}}\left(T_{c w, q}^{\boxtimes k}\right) \leq(q+2)^{k}$ to obtain $\omega \leq 2.41$. In particular, one could even potentially prove $\omega$ equaled two were $\lim _{k \rightarrow \infty}\left(\underline{\mathbf{R}}\left(T_{c w, 2}^{\boxtimes k}\right)\right)^{\frac{3}{k}}$ equal to 3 instead of 4. Using an enhancement of border apolarity, with A. Conner and H. Huang, in [20] we solved the longstanding problem of determining $\underline{\mathbf{R}}\left(T_{c w, 2}^{\boxtimes 2}\right)$. Unfortunately for matrix multiplication upper bounds, we proved that $\underline{\mathbf{R}}\left(T_{c w, 2}^{\boxtimes 2}\right)=4^{2}$. Previously, just using Koszul flattenings, analogous (and even higher Kronecker power) results for other small Coppersmith-Wingorad tensors were obtained with A. Conner, F. Gesmundo, and E. Ventura [18].

A more intriguing tensor is the "skew-cousin" of the small CoppersmithWinograd tensor $T_{\text {skewcw, } q}$ occuring in odd dimensions, which similarly satisfies for all $k$ and even $q$, [18]

$$
\begin{equation*}
\omega \leq \log _{q}\left(\frac{4}{27}\left(\underline{\mathbf{R}}\left(T_{s k e w c w, q}^{\boxtimes k}\right)\right)^{\frac{3}{k}}\right) . \tag{4}
\end{equation*}
$$

Again, the $q=2$ case could potentially be used to prove the exponent is two. Here one begins with a handicap, as $\underline{\mathbf{R}}\left(T_{\text {skewcw, } 2}\right)=5>4$, but with A. Conner and A. Harper, using border apolarity for the lower bound and numerical search methods for the upper bound, in [19] we showed $\underline{\mathbf{R}}\left(T_{\text {skewcw, } 2}^{\boxtimes 2}\right)=17<25$. Unfortunately $17>16$. The problem of determining the border rank of the cube remains.

It is worth remarking that $T_{c w, 2}^{\boxtimes 2}$ is isomorphic to the size three permanent polynomial considered as a tensor and $T_{s k e w c w, 2}^{\boxtimes 2}$ is isomorphic to the size three determinant polynomial [18].

The best bounds on the exponent were obtained using the laser method applied to the big Coppersmith-Winograd tensor $T_{C W, q}$, which has minimal border rank. However, barriers to future progress using the laser method applied to this tensor have been discovered, first in [3], and then in numerous follow-up works. In particular, one cannot prove $\omega<2.3$ using $T_{C W, q}$ in the laser method. A geometric interpretation of the barriers is given in [16].

Very recently, at an April 2022 workshop on geometry and complexity theory in Toulouse, France, J. Jelisiejew and M. Michałek announced a path to
improving the laser method. Their observation was that the border rank estimate for the "certain degeneration" of $T^{\boxtimes N}$ in the laser method mentioned above can be improved! The proof exploits properties of the algebra associated to $T^{\boxtimes N}$ (discussed in $\S 7.1$ below) that persist under the degeneration.

Even without that recent developement, other minimal border rank tensors could potentially prove $\omega<2.3$ with the standard laser method. In fact in [31, Cor 4.3] and [17, Cor 7.5] it was observed that among tensors that are $1_{A}, 1_{B}$ and $1_{C}$ generic (such are called 1-generic tensors), $T_{C W, q}$ is the "worst" for the laser method in the sense that any bound one can prove using $T_{C W, q}$ can also be proved using any other minimal border rank 1-generic tensor. This provides strong motivation to understand tensors of minimal border rank. A second motivation is that it can serve as a case study for the implementation of deformation theory to overcome the cactus barrier.

## 5. Classical and neo-classical equations for tensors of minimal border rank

Before discussing recent developments for tensors of minimal border rank, I explain the previous state of the art. I already have discussed Strassen's equations and Koszul flattenings. What follows are additional conditions.

### 5.1. The equations of $[36,38]$

Several modules of equations were found in [36, 38] using representation theory and variants of Strassen's equations. Many of these still lack a geometric interpretation.

### 5.2. The flag condition

If $\underline{\mathbf{R}}(T)=m$ there exists a flag $A_{1} \subset \cdots \subset A_{m-1} \subset A$ such that for all $j$, $T\left(A_{j}^{*}\right) \subset \sigma_{j}(\operatorname{Seg}(\mathbb{P} B \times \mathbb{P} C))$. This has been observed several times, dating back at least to [13, Exercise 15.14]. Note that to convert this condition to polynomial conditions, one would have to use elimination theory, even for the first step that there exists a line $A_{1}^{*}$ such that $\mathbb{P} T\left(A_{1}^{*}\right) \in S e g(\mathbb{P} B \times \mathbb{P} C)$. The flag condition was essential to the results in [20].

### 5.3. The End-closed condition

Gerstenhaber [28] observed the following: Let $\left\langle x_{1}, \ldots, x_{m}\right\rangle \subset \operatorname{End}\left(\mathbb{C}^{m}\right)$ be a limit of spaces of simultaneously diagonalizable matrices. Then $\forall i, j, x_{i} x_{j} \in$ $\left\langle x_{1}, \ldots, x_{m}\right\rangle$. Call this the "End-closed condition". To express the condition as
polynomials, let $\left\{\alpha_{i}\right\}$ be a basis of $A^{*}$, with $\alpha_{1}$ chosen to maximize the rank of $T\left(\alpha_{1}\right)$, then for all $\alpha^{\prime}, \alpha^{\prime \prime} \in A^{*}$, the End-closed condition is

$$
\begin{equation*}
\left(T\left(\alpha^{\prime}\right) T\left(\alpha_{1}\right)^{\wedge m-1} T\left(\alpha^{\prime \prime}\right)\right) \wedge T\left(\alpha_{1}\right) \wedge \cdots \wedge T\left(\alpha_{m}\right)=0 \in \Lambda^{m+1}(\operatorname{End}(C)) \tag{5}
\end{equation*}
$$

These are polynomials of degree $2 m+1$. When $T$ is $1_{A}$-generic and one takes $\alpha_{1}$ such that $\operatorname{rank}\left(T\left(\alpha_{1}\right)\right)=m$, these correspond to $T\left(A^{*}\right) T\left(\alpha_{1}\right)^{-1} \subset \operatorname{End}(C)$ being closed under composition of endomorphisms.

### 5.4. The symmetry Lie algebra condition

Let $\mathfrak{g}=\mathfrak{g l}(A) \oplus \mathfrak{g l}(B) \oplus \mathfrak{g l}(C)$. Let $\hat{\mathfrak{g}}_{T}=\{X \in \mathfrak{g} \mid X . T=0\}$. (This is the pullback of the symmetry Lie algebra of $T$ to $\mathfrak{g l}(A) \oplus \mathfrak{g l}(B) \oplus \mathfrak{g l}(C)$.) With $T$ understood, write $\mathfrak{g}_{A B}=\{X \in \mathfrak{g l}(A) \oplus \mathfrak{g l}(B) \mid X . T=0\}$ and similarly for $\mathfrak{g}_{B C}, \mathfrak{g}_{A C}$.

A concise tensor of rank $m, M_{\langle 1\rangle}^{\oplus m}$, has $\operatorname{dim} \hat{\mathfrak{g}}_{M_{\langle 1\rangle}^{\oplus m}}=2 m$ and $\operatorname{dim} \mathfrak{g}_{A B}=$ $\operatorname{dim} \mathfrak{g}_{A C}=\operatorname{dim} \mathfrak{g}_{B C}=m$. The dimension of the symmetry Lie-algebra is semicontinuous under degenerations, thus if $T$ is of minimal border rank $\operatorname{dim} \hat{\mathfrak{g}}_{T} \geq$ $2 m$ and $\operatorname{dim} \hat{\mathfrak{g}}_{A B} \geq m$ and permuted statements.

Computing these dimensions amounts to determining the dimension of the kernel of a linear map. Precisely to check if $\operatorname{dim} \hat{\mathfrak{g}}_{T} \geq 2 m$ are equations of degree $3 m^{2}-2 m+1$ and $\operatorname{dim} \mathfrak{g}_{A B} \geq m$ are equations of degree $2 m^{2}-m+1$.

## 6. The 111-equations and first consequences

### 6.1. The 111-equations

The 111-equations are the rank conditions on the map (2) when $(s, t, u)=$ $(1,1,1)$ and one is testing for border rank $m$. Note that in this case there are no choices for the ideal in degrees $(110),(101),(011)$, so they are really polynomial equations. These equations first appeared in [12, Thm 1.3].

The 111-equations for concise tensors of minimal border rank may be rephrased as the requirement that

$$
\begin{equation*}
\operatorname{dim}\left(\left(T\left(A^{*}\right) \otimes A\right) \cap\left(T\left(B^{*}\right) \otimes B\right) \cap\left(T\left(C^{*}\right) \otimes C\right)\right) \geq m \tag{6}
\end{equation*}
$$

A special case of the 111-equations are the two-factor 111-equations, which have a natural geometric interpretation and are easier to implement because a pairwise intersection can be computed using inclusion-exclusion: Given subspaces $X_{1}, X_{2}, X_{3}$ of a vector space $V$, by inclusion-exclusion $\operatorname{dim}\left(X_{i} \cap X_{j}\right)=$ $\operatorname{dim}\left(X_{i}\right)+\operatorname{dim}\left(X_{j}\right)-\operatorname{dim}\left\langle X_{i}, X_{j}\right\rangle$.

Thus the two-factor 111-test may be computed by checking if $\operatorname{dim}\left\langle T\left(A^{*}\right) \otimes A, T\left(B^{*}\right) \otimes B\right\rangle \geq 2 m^{2}-m+1$ and permuted statements. These
are equations of degree $2 m^{2}-m+1$ in the $T^{i j k}$. Notice that if $(X, Y) \in \mathfrak{g}_{A B}$, i.e., $X . T=-Y . T$, then $(X,-Y)$ gives rise to an element of $\left(T\left(A^{*}\right) \otimes A\right) \cap$ $\left(T\left(B^{*}\right) \otimes B\right)$, i.e., the two factor 111-tests are equivalent to the dimension requirements on $\mathfrak{g}_{A B}, \mathfrak{g}_{A C}, \mathfrak{g}_{B C}$ for minimal border rank.

More generally, the full 111-equations may also be understood as a generalization of the lower bound on $\operatorname{dim}\left(\hat{\mathfrak{g}}_{T}\right)$, where one not just bounds dimension, but restricts the structure of $\hat{\mathfrak{g}}_{T}$ as well.

To compare the 111-equations with other previously known equations, we have:

Proposition 6.1. [32, Prop. 1.1, Prop. 1.2] The 111-equations imply both Strassen's equations and the End-closed equations. The 111-equations do not always imply the $p=1$ Koszul flattening equations.

Consider the situation of a concise tensor where each of the associated spaces of homomorphisms is of bounded rank $m-1$. Strassen's equations do allow some assertions in this situation. A normal form for such tensors was proved by S. Friedland [26]. This normal form was generalized in [32, Prop. 3.3] by using the 111-equations instead of Strassen's equations. (In fact this generalized normal form allowed the proof that the 111-equations imply Strassen's equations and the End-closed equations.) These normal forms respectively allowed the characterization of concise tensors of minimal border rank when $m=4$ and $m=5$, in fact S . Friedland was even able to resolve the non-concise $m=4$ case using additional equations he developed, solving the set-theoretic "salmon prize problem" posed by E. Allman.

Recall that Strassen's equations and the End-closed equations are trivial when a tensor gives rise to three linear spaces of bounded rank at most $m-2$. The 111-equations do not share this defect. We are currently implementing them for such spaces of tensors. (The $p=1$ Koszul flattenings are not trivial in this setting, we have yet to determine their utility for bounded rank $m-2$ situations.)

## 7. Deformation theory for tensors of minimal border rank

For tensors $T \in \mathbb{C}^{m} \otimes \mathbb{C}^{m} \otimes \mathbb{C}^{m}$ satisfying genericity conditions, one has natural algebraic structures associated to them that can be utilized to help determine if they have minimal border rank.

### 7.1. Binding tensors and algebras

Say $T \in A \otimes B \otimes C$ is $1_{A}$ and $1_{B}$ generic with $T\left(\alpha_{1}\right): B^{*} \rightarrow C$ and $T\left(\beta_{1}\right):$ $A^{*} \rightarrow C$ of full rank. (A tensor that is at least two of $1_{A}, 1_{B}$ or $1_{C}$ generic is called binding.) Use their inverses to obtain a tensor isomorphic to $T$, where

I abuse notation and also denote by $T, T \in C^{*} \otimes C^{*} \otimes C$, i.e., a bilinear map $T: C \times C \rightarrow C$, which gives $C$ the structure of an associative algebra with left identity $\alpha_{1}$ and right identity $\beta_{1}$. (The algebra is associative because matrix multiplication is associative.)

If $T$ satisfies the $A$-Strassen equations then it is isomorphic to a partially symmetric tensor, see Proposition 8.1, and the associated algebra is abelian. Conversely, given such an algebra, one obtains its structure tensor.

Explicitly, let $\mathcal{I} \subset \mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$ be an ideal whose zero set in affine space is finite, more precisely so that $\mathcal{A}_{\mathcal{I}}:=\mathbb{C}\left[x_{1}, \ldots, x_{n}\right] / \mathcal{I}$ is a finite dimensional algebra of dimension $m$. (This will be the case, e.g., if the zero set consists of $m$ distinct points each counted with multiplicity one.) Let $\left\{p_{I}\right\}$ be a basis of $\mathcal{A}_{\mathcal{I}}$ with dual basis $\left\{p_{I}^{*}\right\}$ We can write the structure tensor of $\mathcal{A}_{\mathcal{I}}$ as

$$
T_{\mathcal{A}_{\mathcal{I}}}=\sum_{p_{I}, p_{J} \in \mathcal{A}_{\mathcal{I}}} p_{I}^{*} \otimes p_{J}^{*} \otimes\left(p_{I} p_{J} \bmod \mathcal{I}\right)
$$

Then [10] shows that a binding tensor $T$ that is the structure tensor of a smoothable algebra is of minimal border rank, i.e., the tensor $M_{\langle 1\rangle}^{\oplus m}$ degenerates to $T$, where $M_{\langle 1\rangle}^{\oplus m}$ is the tensor whose associated algebra $\mathcal{A}_{M_{|1\rangle}^{\oplus m}}$ comes from the ideal of $m$ distinct points. The key step is showing that in this situation $T \in$ $\overline{G L(A) \times G L(B) \times G L(C) M_{\langle 1\rangle}^{\oplus m}}$ if and only if (using the above identifications) $T \in \overline{G L(C) M_{\langle 1\rangle}^{\oplus m}}$.

Thus one may utilize deformation theory on the Hilbert scheme of points to determine if a binding tensor satisfying the $A$-Strassen equations has minimal border rank. In particular, such tensors automatically are of minimal border rank when $m \leq 7$ [14].

### 7.2. 1-Generic tensors: Gorenstein algebras

Now say $T$ is $1_{A}, 1_{B}$, and $1_{C}$ generic (such tensors are called 1-generic) and satisfies the $A$-Strassen equations. We have $\gamma_{1} \in C^{*}$ such that $T\left(\gamma_{1}\right) \in \operatorname{End}(C)$ is invertible. What extra structure do we obtain?

Recall that an algebra $\mathcal{A}$ is Gorenstein if there exists $f \in \mathcal{A}^{*}$ such that any of the following equivalent conditions holds:

1) $T_{\mathcal{A}}(f) \in \mathcal{A}^{*} \otimes \mathcal{A}^{*}$ is of full rank,
2) the pairing $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathbb{C}$ given by $(a, b) \mapsto f(a b)$ is non-degenerate,
3) $\mathcal{A} f=\mathcal{A}^{*}$.

Thus $f=\gamma_{1}$ above tells us $\mathcal{A}_{T}$ is Gorenstein by (1). By the second assertion in Proposition 8.1, $T$ is moreover symmetric.

In particular, such $T$ is of minimal border rank when $m \leq 13$ [15]. For an algorithm that resolves the $m=14$ case, see [27].

The additional algebraic structure of being Gorenstein makes the deformation theory easier to implement.
Example 7.1. Consider $\mathcal{A}=\mathbb{C}[x] /\left(x^{2}\right)$, with basis 1 , $x$, so

$$
T_{\mathcal{A}}=1^{*} \otimes 1^{*} \otimes 1+x^{*} \otimes 1^{*} \otimes x+1^{*} \otimes x^{*} \otimes x
$$

Writing $e_{0}=1^{*}, e_{1}=x^{*}$ in the first two factors and $e_{0}=x, e_{1}=1$ in the third,

$$
T_{\mathcal{A}}=e_{0} \otimes e_{0} \otimes e_{1}+e_{1} \otimes e_{0} \otimes e_{0}+e_{0} \otimes e_{1} \otimes e_{0}
$$

That is, $T_{\mathcal{A}}=T_{W \text { State }}$ is a general tangent vector to $\operatorname{Seg}(\mathbb{P} A \times \mathbb{P} B \times \mathbb{P} C)$.
Example 7.2 (The big Coppersmith-Winograd tensor). Consider the algebra

$$
\mathcal{A}_{C W, q}=\mathbb{C}\left[x_{1}, \ldots, x_{q}\right] /\left(x_{i} x_{j}, x_{i}^{2}-x_{j}^{2}, x_{i}^{3}, i \neq j\right)
$$

Let $\left\{1, x_{i},\left[x_{1}^{2}\right]\right\}$ be a basis of $\mathcal{A}$, where $\left[x_{1}^{2}\right]=\left[x_{j}^{2}\right]$ for all $j$. Then

$$
\begin{aligned}
T_{\mathcal{A}_{C W, q}}= & 1^{*} \otimes 1^{*} \otimes 1+\sum_{i=1}^{q}\left(1^{*} \otimes x_{i}^{*} \otimes x_{i}+x_{i}^{*} \otimes 1^{*} \otimes x_{i}+x_{i}^{*} \otimes x_{i}^{*} \otimes\left[x_{1}^{2}\right]\right) \\
& +1^{*} \otimes\left[x_{1}^{2}\right]^{*} \otimes\left[x_{1}^{2}\right]+\left[x_{1}^{2}\right]^{*} \otimes 1^{*} \otimes\left[x_{1}^{2}\right]
\end{aligned}
$$

Set $e_{0}=1^{*}, e_{i}=x_{i}^{*}, e_{q+1}=\left[x_{1}^{2}\right]^{*}$ in the first two factors and $e_{0}=\left[x_{1}^{2}\right], e_{i}=x_{i}$, $e_{q+1}=1$ in the third to obtain

$$
\begin{aligned}
T_{\mathcal{A}_{C W, q}}= & T_{C W, q}=e_{0} \otimes e_{0} \otimes e_{q+1}+\sum_{i=1}^{q}\left(e_{0} \otimes e_{i} \otimes e_{i}+e_{i} \otimes e_{0} \otimes e_{i}+e_{i} \otimes e_{i} \otimes e_{0}\right) \\
& +e_{0} \otimes e_{q+1} \otimes e_{0}+e_{q+1} \otimes e_{0} \otimes e_{0}
\end{aligned}
$$

which is the usual expression for the big Coppersmith-Winograd tensor.

## 7.3. $1_{*}$-generic tensors: modules and the ADHM correspondence

When $\operatorname{dim} A=\operatorname{dim} B=\operatorname{dim} C=m$, one says $T$ is $1_{*}$-generic if it is $1_{A}$ or $1_{B}$ or $1_{C}$ generic.

Consider the case of a tensor that is $1_{A}$-generic but not binding. What structure can we associate to it? Fixing $\alpha_{1}$ as above we obtain $T \in A \otimes C^{*} \otimes C$, i.e., $T\left(A^{*}\right) T\left(\alpha_{1}\right)^{-1} \subset \operatorname{End}(C)$, and if Strassen's equations are satisfied, we have an abelian subspace, and if furthermore the End-closed condition holds, we may think of this space as defining an algebra action on $\operatorname{End}(C)$, which we may lift to an action of the polynomial ring $S:=\mathbb{C}\left[y_{2}, \ldots, y_{m}\right]$ by $y_{s}(c):=T\left(\alpha_{s}\right) T\left(\alpha_{1}\right)^{-1} c$. (The choice of indices is deliberate, as $T\left(\alpha_{1}\right) T\left(\alpha_{1}\right)^{-1}=\mathrm{Id}_{C}$ corresponds to $1 \in S$.)

That is, the vector space $C$ becomes a module over the polynomial ring. This association is called the $A D H M$ correspondence in [33], after [5]. This leads one to deformation theory in the Quot scheme that parametrizes such modules.

This correspondence allowed Jelisiejew, Pal and myself [32] to characterize concise $1_{*}$-generic tensors of border rank $\leq 6$ as the zero set of Strassen's equations and the End-closed equations, and also as the zero set of the 111equations. Strassen's equations, the 111-equations and the End-closed equations fail to characterize minimal border rank tensors when $m \geq 7$.

### 7.4. Concise tensors: the 111-algebra and its modules

Now say we just have a concise tensor. Previously there had not been any algebraic structure available for studying such tensors. Moreover, as remarked above, both Strassen's equations and the End-closed equations are trivially satisfied for such tensors when the three associated spaces of homomorphisms are of rank bounded above by $m-2$. Despite this, the 111-equations still give strong restrictions in these cases. I now explain that they also allow the implementation of deformation theory even in this situation.

For $X \in \operatorname{End}(A)=A^{*} \otimes A$, let $X \circ_{A} T$ denote the corresponding element of $T\left(A^{*}\right) \otimes A$. Explicitly, if $X=\alpha \otimes a$, then $X \circ_{A} T:=T(\alpha) \otimes a$ and the map $(-) \circ_{A} T: \operatorname{End}(A) \rightarrow A \otimes B \otimes C$ is extended linearly.

Definition 7.3 ([32, Def. 1.9]). Let $T$ be a concise tensor. We say that a triple $(X, Y, Z) \in \operatorname{End}(A) \times \operatorname{End}(B) \times \operatorname{End}(C)$ is compatible with $T$ if $X \circ_{A} T=$ $Y \circ_{B} T=Z \circ_{C} T$. The 111-algebra of $T$ is the set of triples compatible with $T$. We denote this set by $\mathcal{A}_{111}^{T}$.

Thus a compatible triple gives a point in the triple intersection (6). The name 111-algebra is justified by the following theorem:

Theorem 7.4 ([32, Thm. 1.10]). The 111-algebra of a concise tensor $T \in$ $A \otimes B \otimes C$ is a commutative unital subalgebra of $\operatorname{End}(A) \times \operatorname{End}(B) \times \operatorname{End}(C)$ and its projection to any factor is injective.

Using the 111-algebra, one obtains four consecutive obstructions for a concise tensor to be of minimal border rank [32]:

1. $\operatorname{dim}\left(\mathcal{A}_{111}^{T}\right) \geq m$. For the next three conditions, assume equality holds.
2. $\mathcal{A}_{111}^{T}$ must be smoothable.
3. Using the 111-algebra, $A, B, C$ become modules for it and the polynomial ring $S$. These three $S$-modules, $\underline{A}, \underline{B}, \underline{C}$ (where the underline is there to emphasize their module structure) must lie in the principal component of the Quot scheme.
4. There exists a surjective module homomorphism $\underline{A} \otimes_{\mathcal{A}_{111}^{T}} \underline{B} \rightarrow \underline{C}$ associated to $T$ and this homomorphism must be a limit of module homomorphisms $\underline{A}_{\epsilon} \otimes_{\mathcal{A}_{\epsilon}} \underline{B}_{\epsilon} \rightarrow \underline{C}_{\epsilon}$ for a choice of smooth algebras $\mathcal{A}_{\epsilon}$ and semisimple modules $\underline{A}_{\epsilon}, \underline{B}_{\epsilon}, \underline{C}_{\epsilon}$.

## 8. New proofs of existing results

In this section I present two significantly simpler proofs than the original that binding tensors satisfying Strassen's equations are partially symmetric and the original, more elementary proof that binding tensors satisfying Strassen's equations automatically satisfy the End-closed condition. These proofs were obtained in conversations with J. Jelisiejew and M. Michałek.

Let $A, B, C \simeq \mathbb{C}^{m}$ and let $T \in A \otimes B \otimes C$ be $1_{A}$-generic. Say $\operatorname{rank}\left(T_{A}\left(\alpha_{0}\right)\right)=m$.
Note the tautological identities: $T(\alpha, \beta)=T_{A}(\alpha) \beta=T_{B}(\beta) \alpha$.
The $A$-Strassen equations for minimal border rank say that for all $\alpha_{1}, \alpha_{2} \in A$,

$$
T_{A}\left(\alpha_{1}\right) T_{A}\left(\alpha_{0}\right)^{-1} T_{A}\left(\alpha_{2}\right)=T_{A}\left(\alpha_{2}\right) T_{A}\left(\alpha_{0}\right)^{-1} T_{A}\left(\alpha_{1}\right)
$$

Proposition 8.1 ([39]). Let $T$ be $1_{A}$ and $1_{B}$-generic and satisfy the $A$-Strassen equations. Then $T$ is isomorphic to a tensor in $S^{2} C^{*} \otimes C$. If $T$ is 1-generic then it is isomorphic to a symmetric tensor.

Here are two proofs:
$\underset{\tilde{T}}{\text { Proof. Assume }} T\left(\alpha_{0}\right) \in B \otimes C$ and $T\left(\beta_{0}\right) \in A \otimes C$ are of full rank. Define $\tilde{T} \in C^{*} \otimes C^{*} \otimes C$ by $\tilde{T}\left(c_{1}, c_{2}\right):=T\left(T_{B}\left(\beta_{0}\right)^{-1} c_{1}, T_{A}\left(\alpha_{0}\right)^{-1} c_{2}\right)$.

Set $\alpha_{1}=T_{B}\left(\beta_{0}\right)^{-1} c_{1}, \alpha_{2}=T_{B}\left(\beta_{0}\right)^{-1} c_{2}$ so

$$
\begin{aligned}
\tilde{T}\left(c_{1}, c_{2}\right) & =T\left(\alpha_{1}, T_{A}\left(\alpha_{0}\right)^{-1} T_{B}\left(\beta_{0}\right) \alpha_{2}\right) & & \text { definition } \\
& =T\left(\alpha_{1}, T_{A}\left(\alpha_{0}\right)^{-1} T_{A}\left(\alpha_{2}\right) \beta_{0}\right) & & \text { taut.id. } \\
& =T_{A}\left(\alpha_{1}\right) T_{A}\left(\alpha_{0}\right)^{-1} T_{A}\left(\alpha_{2}\right) \beta_{0} & & \text { taut.id. } \\
& =T_{A}\left(\alpha_{2}\right) T_{A}\left(\alpha_{0}\right)^{-1} T_{A}\left(\alpha_{1}\right) \beta_{0} & & \text { Strassen } \\
& =\tilde{T}\left(c_{2}, c_{1}\right) & & \text { taut.id. }
\end{aligned}
$$

The second assertion follows as $\mathfrak{S}_{3}$ is generated by the transpositions $(1,2)$ and $(1,3)$.

Proof. Under the hypotheses $T_{A}^{\wedge 1}: A \otimes B^{*} \rightarrow \Lambda^{2} A \otimes C$ has rank $m^{2}-m$ and the $1_{B}$-genericity condition assures that the $m$-dimensional kernel contains an element of full rank, $\psi: A \rightarrow B$, which makes $\left(\psi \otimes \operatorname{Id}_{B \otimes C}\right)(T) \in S^{2} B^{*} \otimes C$. The second assertion follows similarly.

Note that Proposition 8.1 implies the $B$-Strassen equations are satisfied as well.

The following proposition appeared in [32] with a less elementary proof. Below is the original proof.

Proposition 8.2. If $T$ is $1_{A}$ and $1_{B}$ generic and satisfies the $A$-Strassen equations, then $T\left(A^{*}\right) T\left(\alpha_{0}\right)^{-1} \subset \operatorname{End}(C)$ satisfies the End-closed condition.

Proof. We need to show that for all $\alpha_{1}, \alpha_{2}$, that, there exists $\alpha^{\prime}$ such that

$$
T_{A}\left(\alpha_{1}\right) T_{A}\left(\alpha_{0}\right)^{-1} T_{A}\left(\alpha_{2}\right) T_{A}\left(\alpha_{0}\right)^{-1}=T_{A}\left(\alpha^{\prime}\right) T_{A}\left(\alpha_{0}\right)^{-1}
$$

It is sufficient to work with $\tilde{T} \in S^{2} C^{*} \otimes C$. Here, by symmetry $\tilde{T}_{A}(c)=$ $\tilde{T}_{B}(c)=: \tilde{T}_{C^{*}}(c)$. We claim $\tilde{T}_{C^{*}}\left(c_{1}\right) \tilde{T}_{C^{*}}\left(c_{2}\right)=\tilde{T}_{C^{*}}\left(\tilde{T}\left(c_{1}, c_{2}\right)\right)$. This will finish the proof as $c_{1}, c_{2}, \tilde{T}\left(c_{1}, c_{2}\right) \in C \cong A^{*}$ play the role of $\alpha_{1}, \alpha_{2}, \alpha^{\prime}$ above.

To see this

$$
\begin{aligned}
\tilde{T}_{C^{*}}\left(c_{1}\right) \tilde{T}_{C^{*}}\left(c_{2}\right)(c) & =\tilde{T}_{C^{*}}\left(c_{1}\right) \tilde{T}\left(c_{2}, c\right) & & \text { taut. } \\
& \left.=T_{C^{*}}\left(c_{1}\right) \tilde{T}\left(c, c_{2}\right)\right) & & \text { sym. } \\
& =\tilde{T}_{C^{*}}\left(c_{1}\right) \tilde{T}_{C^{*}}(c)\left(c_{2}\right) & & \text { taut. } \\
& =\tilde{T}_{C^{*}}(c) \tilde{T}_{C^{*}}\left(c_{1}\right)\left(c_{2}\right) & & \text { Strassen } \\
& =\tilde{T}\left(c, \tilde{T}\left(c_{1}, c_{2}\right)\right) & & \text { taut. } \\
& =\tilde{T}\left(\tilde{T}\left(c_{1}, c_{2}\right), c\right) & & \text { sym. } \\
& =\tilde{T}_{C^{*}}\left(\tilde{T}\left(c_{1}, c_{2}\right)\right)(c) & & \text { taut. }
\end{aligned}
$$

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# New examples of free projective curves 

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#### Abstract

It is known that a plane projective curve $D$ consisting of a union of degree $n$ curves in the same pencil with a smooth base locus is free if and only if $D$ contains all the singular members of the pencil and its Jacobian ideal is locally a complete intersection. Here we generalize this result to pencils having a singular base locus.


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## 1. Introduction

Let $R=\oplus_{k \geq 0} R_{k}=\boldsymbol{k}[x, y, z]$ be the graded ring in three indeterminates. The partial derivatives in these three variables are denoted $\partial_{x}, \partial_{y}$ and $\partial_{z}$. The $R$ graded-module of derivations is a rank 3 module $\operatorname{Der}_{R}=\oplus_{k \geq 0}\left[R_{k} \partial_{x}+R_{k} \partial_{y}+\right.$ $\left.R_{k} \partial_{z}\right]$. The so-called Euler derivation is $\delta_{E}=x \partial_{x}+y \partial_{y}+z \partial_{z}$.

To a reduced homogeneous polynomial of degree $n \geq 1, f \in R_{n}$, one associates its module of tangent derivations:

$$
\operatorname{Der}(f)=\left\{\delta \in \operatorname{Der}_{R} \mid \delta(f) \in(f)\right\} .
$$

The Euler derivation belongs to $\operatorname{Der}(f)$ and there is a factorization

$$
\operatorname{Der}(f)=R \delta_{E} \oplus \operatorname{Der}_{0}(f),
$$

where

$$
\operatorname{Der}_{0}(f)=\left\{\delta \in \operatorname{Der}_{R} \mid \delta(f)=0\right\} .
$$

Let $\nabla(f)=\left(\partial_{x} f, \partial_{y} f, \partial_{z} f\right)$ be the vector of partial derivatives. Then $\operatorname{Der}_{0}(f)$ is the kernel of the Jacobian map

$$
R^{3} \xrightarrow{\nabla(f)} R[n-1] .
$$

The modules $\operatorname{Der}(f)$ and $\operatorname{Der}_{0}(f)$ could also be defined in higher dimensions where instead of curves, we would have hypersurfaces. One reason to focus on curves is that the module $\operatorname{Der}_{0}(f)$ is locally free (its associated sheaf in $\mathbb{P}^{2}$ is reflexive and then it is a vector bundle for dimensional reasons). In some
very particular cases, these modules can also be free (see the definition below). This was first pointed out in [4] for reduced hypersurfaces and studied in [7] for line arrangements (finite sets of distinct lines in $\mathbb{P}^{2}$ ) presenting a very special combinatorics; for instance, a union of lines invariant under the action of some reflection group or the Hesse arrangement of 12 lines through the 9 inflection points of a smooth cubic curve (see [2] for detailed examples). Actually, in [2], Terao conjectures that freeness of hyperplane arrangements depends only on its combinatorics. This conjecture is still unsolved even for line arrangements; this is certainly because we do not know enough examples of free line arrangements and more generally of free curves to clearly understand what distinguishes a free curve from a non free curve. Although combinatorics is not as relevant for general curves as for line arrangements, understanding why a curve is free, in addition to the interest of this result for itself, could help solve Terao's conjecture. Before going further on this subject, let us recall the definition of freeness for a reduced plane curve.
Definition 1.1. The reduced curve $V(f)$ is free if and only if $\operatorname{Der}_{0}(f)$ (or equivalently $\operatorname{Der}(f))$ is a free module. More precisely $\operatorname{Der}_{0}(f)$ is free with exponents $(a, b)$ if $\operatorname{Der}_{0}(f)=R[-a] \oplus R[-b]$ where $a$ and $b$ are integers verifying $0 \leq a \leq b$ and $a+b+1=\operatorname{deg}(f)($ or $\operatorname{Der}(f)=R[-1] \oplus R[-a] \oplus R[-b])$.
REmark 1.2. A smooth curve of degree $\geq 2$ is not free, an irreducible curve of degree $\geq 3$ with only nodes and cusps as singularities is not free (see [1, Example 4.5]). Few examples of free curves are known and of course very few families of free curves are known. One such family can be found in [6, Prop. 2.2].

One method to produce free curves given in [8] (suggested by E. ArtalBartolo and J. Cogolludo-Agustin in a personal communication), consists in taking the union of all the singular curves in a generic pencil of curves of the same degree ; generic means here that the base locus is smooth. More precisely, it was proved that:
Theorem 1.3. Let $f, g$ two reduced polynomials in $R_{n}$ such that $B=V(f) \cap$ $V(g)$ consists in $n^{2}$ distinct points. Denote by $D_{k}$ the union of $k \geq 2$ curves and by $D^{s}$ the union of all the singular curves of the pencil $\langle f, g\rangle$ of degree $n$ curves generated by $f$ and $g$. Then $D_{k}$ is free with exponents $(2 n-2, n(k-2)+1)$ if and only if $D^{s} \subset D_{k}$ and the singularities of $D^{s}$ are quasihomogeneous.

Let us first give some classical examples.
Example 1.4. The Braid arrangement defined by $x y z(x-y)(x-z)(y-z)=0$ is the union of the three singular curves of the pencil $\langle(x-y) z, y(x-z)\rangle$. It is free with exponents $(2,3)$.
Example 1.5. The Hesse arrangement defined by

$$
\prod_{\epsilon=0,1, j, j^{2}}\left(x^{3}+y^{3}+z^{3}-\epsilon x y z\right)=0
$$

is the union of four triangles, that are all the singular curves of the pencil $\left\langle x^{3}+y^{3}+z^{3}, x y z\right\rangle$. It is free with exponents $(4,7)$.
Example 1.6. The Fermat arrangement defined by

$$
\left(x^{n}-y^{n}\right)\left(y^{n}-z^{n}\right)\left(x^{n}-z^{n}\right)=0
$$

is the union of three sets of $n$ concurent lines that are all the singular curves of the pencil $\left\langle x^{n}-y^{n}, y^{n}-z^{n}\right\rangle$. It is free with exponents $(n+1,2 n-2)$.

As a definition of quasihomogeneous singularity we follow the characterization given in [5]:
Definition 1.7. Let $f \in \mathbb{C}[x, y, z]$ a reduced polynomial. Let $C=V(f)$ its corresponding projective curve. A singular point $p \in V(f)$ is a quasi-homogeneous singularity if and only if $\tau_{p}(C)=\mu_{p}(C)$, where $\tau_{p}(C)$ and $\mu_{p}(C)$ are the Tjurina and Milnor numbers of $C$ at $p$.

Remark 1.8. The definition being local one can assume that $p=(0,0)$ and $\mathbb{C}\{x, y\}$ is the ring of convergent power series ; then $\tau_{p}(C)=\frac{\mathbb{C}\{x, y\}}{\left(\partial_{x} f, \partial_{y} f, f\right)}$ and $\mu_{p}(C)=\frac{\mathbb{C}\{x, y\}}{\left(\partial_{x} f, \partial_{y} f\right)}$. This implies in particular that $\tau_{p}(C) \leq \mu_{p}(C)$.
Remark 1.9. When $p$ is a smooth point of $C$, these numbers vanish.
Remark 1.10. These numbers play a crucial role here. Indeed, denoting by $\mathcal{T}_{f}$ the logarithmic tangent sheaf associated to $V(f)$ which is the sheafification of $\operatorname{Der}_{0}(f)$, and by $\mathcal{J}_{f}$ the sheaf of ideals, called Jacobian ideal, image of the Jacobian map, one has

$$
0 \longrightarrow \mathcal{T}_{f} \longrightarrow \mathscr{O}_{\mathbb{P}^{2}}^{3} \xrightarrow{\nabla(f)} \mathcal{J}_{f}(n-1) \longrightarrow 0
$$

Since the curve $C=V(f)$ is reduced, its singular locus is a finite scheme and the Jacobian ideal defines a finite scheme of length

$$
c_{2}\left(\mathcal{J}_{f}\right)=\sum_{p \in C} \tau_{p}(C)
$$

The sum $\tau(C):=\sum_{p \in C} \tau_{p}(C)$ is called the total Tjurina number of $C$. This gives also the following relation:

$$
c_{2}\left(\mathcal{T}_{f}\right)=(n-1)^{2}-\tau(C)
$$

The proof of Theorem 1.3 was based on the following observations:

1. there exists a canonical derivation $\delta=\operatorname{det}[\nabla f, \nabla g, \nabla]=\langle\nabla f \wedge \nabla g \mid \nabla\rangle$ (where $\langle\mid\rangle$ is the usual scalar product of vectors in $\mathbb{C}^{3}$ ) associated to a pencil $\langle f, g\rangle$ of degree $n$ curves; this canonical derivation induces for any $k \geq 2$ a non zero section $s_{k} \in \mathrm{H}^{0}\left(\mathcal{T}_{D_{k}}(2 n-2)\right)$;
2. the zero locus of this section $s_{k}$ is empty if and only if $D_{k} \subset D^{s}$ and at each singular point $p$ of $D^{s}$ one has $\tau_{p}\left(D^{s}\right)=\mu_{p}\left(D^{s}\right)$.

The smoothness of the base locus $B$ is necessary to certify that its contribution to the length of the Jacobian scheme is

$$
\sum_{i=1}^{n^{2}}(k-1)^{2}=n^{2}(k-1)^{2}
$$

### 1.1. Objectives

We would like to extend this construction of free curves to more general pencils, i.e. pencils with a singular base locus. Here we focus on two cases.

1. The fat case: pencils generated by two powers $\left\langle f^{b}, g^{a}\right\rangle$ where $V(f)$ and $V(g)$ are two curves of degree $a$ and $b$ such that $(a, b)=1$ and $V(f) \cap V(g)$ is a smooth set of $a b$ distinct points. In such pencils any curve is singular along the base locus $B$ when $a>1$ and $b>1$. The interest for this case comes from the celebrated example of the two types of 6 -cusped sextics with non-isomorphic fundamental groups given by Zariski [9]; indeed the six cusps belong to a smooth conic for the first type and do not belong to a conic for the second type. The sextic of the first type is a general curve in a pencil $\left\langle f^{3}, g^{2}\right\rangle$ where $f=0$ is a smooth conic and $g=0$ is a smooth cubic.
2. The tangential case: pencils of degree $n$ curves such that the general one is smooth but with a singular base locus $B$, i.e. $\operatorname{card}(B)<n^{2}$. The complete description of these pencils remains difficult and we will concentrate in this text on the case of pencils generated by conics.

## 2. The fat case

In this section we do not study all the singular pencils but only those defined by two multiple structures on reduced curves with primary degrees meeting along a smooth set. More precisely, we prove:

Theorem 2.1. Let $a, b$ be two positive integers such that $\operatorname{gcd}(a, b)=1, f \in R_{a}$, $g \in R_{b}$ be two reduced polynomials such that the corresponding curves $V(f)$ and $V(g)$ meet along ab distinct points. We consider the pencil $\mathcal{C}_{a b}=\left\langle f^{b}, g^{a}\right\rangle$ of degree ab curves. Then,

1. if $a>1$ and $b>1$ then all curves of $\mathcal{C}_{a b}$ are singular at $B$;
2. there is a finite number of curves in $\mathcal{C}_{a b}$, disjoint from $V\left(f^{b}\right)$ and $V\left(g^{a}\right)$, that are singular outside $B$. We call these curves the + singular curves and their union is denoted by $D^{+s}$; the length of the scheme of all the singular points of these + singular curves, including the singularities of $V(f)$ and $V(g)$ when these generators are not smooth, is

$$
(a-1)^{2}+(b-1)^{2}+(a-1)(b-1)
$$

3. if $V(f)$ and $V(g)$ are smooth, a union $D_{k}$ of $k$ curves of the pencil $\mathcal{C}_{a b}$ is free with exponents $(a+b-2, k a b-(a+b)+1)$ if and only if $D^{+s} \subset D_{k}$ and any singularity of $D^{+s}$ outside $B$ is quasihomogeneous;
4. if $V(f)$ is not smooth (resp. or/and $V(g)$ ), the curve $D_{k} \cup V(f)$ (resp. $D_{k} \cup V(g)$ and $\left.D_{k} \cup V(f) \cup V(g)\right)$ where $D_{k}$ is a union of $k$ curves of the pencil $\mathcal{C}_{a b}$ is free with exponents $(a+b-2, k a b-b+1)$ (resp. $(a+b-2, k a b-a+1),(a+b-2, k a b+1))$ if and only if $D^{+s} \subset D_{k}$ and any singularity of $D^{+s}$ outside $B$ is quasihomogeneous.

Proof. Let us prove each assertion.
(1) If $(x, y)$ is a local system of coordinates at any base point $p \in B$, then any curve of the pencil is contained in the ideal $\left(x^{a}, y^{b}\right)$ then singular at $p$.
(2) Let us consider a curve $H=\lambda f^{b}+\mu g^{a}$ with $\lambda \mu \neq 0$ with a singular point $p \notin B$. Since $p$ is singular we obtain $\nabla H(p)=0$. We have by Liebniz's rule:

$$
\nabla H(p)=b \lambda f^{b-1}(p) \nabla f(p)+a \mu g^{a-1}(p) \nabla g(p)=0
$$

Since $p \notin B, f^{b-1}(p) \neq 0$ and $g^{a-1}(p) \neq 0$. This is equivalent to say that $\nabla f(p)$ and $\nabla g(p)$ are proportional, in other words that the two by two minors of the matrix $[\nabla f, \nabla g]$ vanish simultaneously at $p$. Moreover since $f$ and $g$ meet transversally at $B$, these minors do not vanish at any point in $B$. The scheme $\Gamma$ of singular points outside $B$ is then defined by the following exact sequence
$0 \longrightarrow \mathscr{O}_{\mathbb{P}^{2}}(1-b) \oplus \mathscr{O}_{\mathbb{P}^{2}}(1-a) \xrightarrow{[\nabla f, \nabla g]} \mathscr{O}_{\mathbb{P}^{2}}^{3} \xrightarrow{\nabla f \wedge \nabla g} \mathcal{J}_{\Gamma}(a+b-2) \longrightarrow 0$.
Reciprocally, if $p \in \Gamma$ then one can find easily two non zero constants $\lambda$ and $\mu$ such that $\nabla\left(\lambda f^{b}+\mu g^{a}\right)(p)=0$. The length of $\Gamma$ is the number by

$$
c_{2}\left(\mathcal{J}_{\Gamma}\right)=(a-1)^{2}+(b-1)^{2}+(a-1)(b-1)
$$

(3) Let $D_{k}$, defined by $H_{k}=0$, be a union of $k$ curves in the pencil that contains $D^{+s}$. We consider the canonical derivation

$$
\delta=\operatorname{det}[\nabla f, \nabla g, \nabla]=\langle\nabla f \wedge \nabla g \mid \nabla\rangle
$$

Since by Liebniz's rule, we have $\delta\left(H_{k}\right)=0$ for any $k \geq 2$, this derivation induces a non zero section of $\mathrm{H}^{0}\left(\mathcal{T}_{D_{k}}(a+b-2)\right)$ and gives a commutative diagram:

where the sheaf $\mathcal{F}$ is a rank two sheaf singular along $\Gamma$, the scheme of + singular points defined above. Dualizing the last exact sequence we obtain:

$$
\begin{aligned}
& 0 \longrightarrow \mathscr{O}_{\mathbb{P}^{2}}(1-k a b) \xrightarrow{[U, V]^{t}} \mathscr{O}_{\mathbb{P}^{2}}(1-a) \oplus \mathscr{O}_{\mathbb{P}^{2}}(1-b) \xrightarrow{[-V, U]} \mathscr{O}_{\mathbb{P}^{2}}(1-a-b+k a b) \\
& \longrightarrow \quad \omega_{D_{k}} \longrightarrow \mathscr{O}_{\Gamma} \quad \longrightarrow \mathscr{O}_{Z\left(s_{k}\right)} \quad \longrightarrow 0
\end{aligned}
$$

where $\omega_{D_{k}}$ is the dualizing sheaf of the Jacobian scheme associated to $D_{k}, U$ and $V$ are the polynomials of degree $k a b-a$ and $k a b-b$ such that

$$
\nabla H_{k}=U \nabla f+V \nabla g
$$

Denoting by $T$ the complete intersection defined by $\{U=0\} \cap\{V=0\}$, we find finally a shorter exact sequence:

$$
0 \longrightarrow \mathscr{O}_{T} \longrightarrow \omega_{D_{k}} \longrightarrow \mathscr{O}_{\Gamma} \longrightarrow \mathscr{O}_{Z\left(s_{k}\right)} \longrightarrow 0
$$

Cutting this exact sequence in two short exact sequences we get

$$
\begin{equation*}
0 \longrightarrow \mathscr{O}_{T} \longrightarrow \omega_{D_{k}} \longrightarrow \mathfrak{R} \longrightarrow 0 \tag{s1}
\end{equation*}
$$

and

$$
0 \longrightarrow \mathfrak{R} \longrightarrow \mathscr{O}_{\Gamma} \longrightarrow \mathscr{O}_{Z\left(s_{k}\right)} \longrightarrow 0
$$

The complete intersection $T$ is supported by $B$. Since $\Gamma \cap B=\emptyset$ the exact sequence ( $s 2$ ) proves that the scheme $\Re$ is supported on a subset of $\Gamma$ and does not meet $B$. The exact sequence $(s 1)$ then shows that $\Re$ is supported by all the + singular points appearing in $D_{k}$.

If $D^{+s} \subset D_{k}$ both schemes $\Re$ and $\Gamma$ have the same support ; if the singularities of $D^{+s}$ are quasihomogeneous then these schemes coincide. The curves $V(f)$ and $V(g)$ meeting transversally, the scheme $\Gamma$ is lci (see $[8$, proof of Theorem 2.7]) ; this proves that $\mathfrak{R}=\mathscr{O}_{\Gamma}$ and finally, this implies $Z\left(s_{k}\right)=\emptyset$.

### 2.1. Example

Consider the pencil $\left\langle f^{3}, g^{2}\right\rangle$ of sextic curves where

$$
C_{f}=V(f)=\left\{y^{2}-x z=0\right\} \text { and } C_{g}=V(g)=\left\{x^{3}+y^{3}+z^{3}=0\right\}
$$

The smooth conic $C_{f}$ and the smooth cubic $C_{g}$ meet in six different points $p_{i}=\left(a_{i}^{2}, a_{i}, 1\right)$ where $a_{i}^{6}+a_{i}^{3}+1=0$. All curves of this pencil are singular in the six points $p_{i}$. Let us describe now the + singular curves of this pencil with more details.

Proposition 2.2. In the pencil $\left\langle f^{3}, g^{2}\right\rangle$ there are exactly five curves that are singular in a point not belonging to the $p_{i}$ 's. Two of these five curves $C_{1,0}$ and $C_{0,1}$ are defined respectively by the equation $f^{3}=0 g^{2}=0$, the three other are $C_{1,-1}, C_{4,1}$ and $C_{4,-3}$ defined respectively by the equations $f^{3}-g^{2}=0$, $4 f^{3}+g^{2}$ and $4 f^{3}-3 g^{2}=0$.
The additional singular point of $C_{1,-1}$ is $(0,1,0)$.
The additional singular points of $C_{4,1}$ are $(1,0,1),(1,0, j)$ and $\left(1,0, j^{2}\right)$.
The additional singular points of $C_{4,-3}$ are $\left(\frac{-1}{2}, 1, \frac{-1}{2}\right),\left(\frac{-j^{2}}{2}, 1, \frac{-j}{2}\right)$ and $\left(\frac{-j}{2}, 1, \frac{-j^{2}}{2}\right)$.
The curve $C_{1,-1} \cup C_{4,1} \cup C_{4,-3}$ is free with exponents (3,14).
Proof. The singular points $p=(a, b, c) \neq p_{i}$ of $C_{\lambda, \mu}:=\lambda f^{3}+\mu g^{2}=0$ are those verifying:

$$
\nabla\left(\lambda f^{3}+\mu g^{2}\right)(p)=3 \lambda f^{2}(p) \nabla(f)(p)+2 \mu g(p) \nabla(g)(p)=\overrightarrow{0}
$$

- If $f(p)=0$ then $(\lambda, \mu)=(1,0)$ and the corresponding curve is $f^{3}=0$.
- If $g(p)=0$ then $(\lambda, \mu)=(0,1)$ and the corresponding curve is $g^{2}=0$.
- If $f(p) g(p) \neq 0$ then $\nabla(f)(p)=(-c, 2 b,-a)$ and $\nabla(g)(p)=\left(3 a^{2}, 3 b^{2}, 3 c^{2}\right)$ are proportional. More precisely, $(a, b, c)$ verifies the equations:

$$
\left\{\begin{array}{cl}
3 b\left(b c+2 a^{2}\right) & =0 \\
c^{3}-a^{3} & =0 \\
3 b\left(a b+2 c^{2}\right) & =0
\end{array}\right.
$$

Solving this system by elementary computations, we find the additional singular points and the singular curves associated. According to Theorem 2.1 the curve $C_{1,-1} \cup C_{4,1} \cup C_{4,-3}$ is free with exponents (3,14).

### 2.2. Example

This second example corresponds to the case (4) of the main theorem.
We consider the pencil $\left\langle f^{3}, g^{2}\right\rangle$ of sextic curves where

$$
f(x, y, z)=x^{2}+y^{2}+z^{2} \text { and } g(x, y, z)=x y z
$$

The smooth conic $V(f)$ and the singular cubic $V(g)$ meet in six different points $(1, i, 0),(1,-i, 0),(1,0, i),(1,0,-i),(0,1, i)$ and $(0,1,-i)$. Using the same method than in the previous example, we find that the locus $V(\nabla f \wedge \nabla g)$ consists in 7 points that are the three vertices of the triangle, $(1,0,0),(0,1,0)$, $(0,0,1)$ and the four singular points of $f^{3}-27 g^{2}=0$. Then the curve $x y z\left(f^{3}-27 g^{2}\right)=0$ is free with exponents $(3,5)$.

## 3. The tangential case

The pencil is generated by two curves of degree $n$ that do not meet transversally (i.e. the cardinality of the set $B$ is $<n^{2}$ ). At the point $p \in B$ where $V(f)$ and $V(g)$ share the same tangent line, the canonical derivation $\delta=\operatorname{det}(\nabla f, \nabla g, \nabla)$ verifies $\delta(p)=0$. This is the main difficulty here. Indeed the computation of the length of the Jacobian scheme becomes harder and we could have $\mu_{p}\left(H_{k}\right) \neq$ $\tau_{p}\left(H_{k}\right)$ at such a point $p \in B$ for a union of $k$ curves in the pencil. If $V(f)$ and $V(g)$ are two smooth conics such that $B$ consists in a subscheme of length 3 and a distinct simple point. Then $V(f g(a f+b g))$, where $V(a f+b g)$ is also smooth, is free with exponents $(2,3)$. So it is possible for a union of smooth curves of the same pencil to be free. It is also possible to be free when instead of containing all the singular curves the union contains only some irreducible components of some singular curves. For instance, if $V(f)$ and $V(g)$ are two smooth conics tangent in a point $p$. Then $V(f g(a f+b g) h)$ where $V(a f+b g)$ is also smooth and $V(h)$ is the line passing through the two smooth points in $B$, is free with exponents $(2,4)$.


We will focus on pencil of conics. Our aim is to

1. determine the "smaller" free union of conics for each kind of pencil;
2. compute the Tjurina numbers at the base points for any kind of pencil.

### 3.1. Pencil of conics

There are different regular pencils (the general conic of the pencil is smooth) generated by two conics $C$ and $D$ with no component in common. Let us precise now for any of this different pencils what generators $\langle f, g\rangle$ can be chosen. Recall that the canonical derivation is $\delta=\operatorname{det}[\nabla f, \nabla g, \nabla]$. The pencil is

1. generic when $C \cap D$ consists of 4 distinct points. Then, up to a linear transformation, $C$ and $D$ can be defined by $x^{2}-z^{2}=0$ and $y^{2}-z^{2}=0$. The canonical derivation $\delta$ has degree 2 ; among the intersection points appearing in the picture, the base points are blue and the singular points are red;

2. tangent when $C \cap D$ consists of 3 points, one double and two simple points. Then, up to a linear transformation, $C$ and $D$ can be defined by $x^{2}-z^{2}=0$ and $y z=0$. The canonical derivation $\delta$ has degree 2 ; now base points and singular points are not disjoint;

3. bitangent when $C \cap D$ consists of 2 double points. Then, up to a linear transformation, $C$ and $D$ can be defined by $x^{2}-z^{2}=0$ and $y^{2}=0$. The canonical derivation $\delta$ can be factorized by $y$, i.e. $\delta=y \nu$ where the derivation $\nu$ has degree 1 ;
4. osculating when $C \cap D$ consists of 2 points, one simple and one triple point. Then, up to a linear transformation, $C$ and $D$ can be defined by $x y=0$ and $y^{2}-x z=0$. The canonical derivation $\delta$ has degree 2 ;

5. +osculating when $C \cap D$ consists of one quadruple point. Then, up to a linear transformation, $C$ and $D$ can be defined by $y^{2}-x z=0$ and $x^{2}=0$. The canonical derivation $\delta$ can be factorized by $x$, i.e. $\delta=x \nu$ where the derivation $\nu$ has degree 1 .


### 3.2. A free union of curves remains free by deleting a smooth curve

Proposition 3.1. Assume that $\mathcal{A}$ is a union of curves $V(\lambda f+\mu g)$ of a regular pencil of degree $n$ curves $\langle f, g\rangle$ in $\mathbb{P}^{2}$. Assume also that $\mathcal{A}$ contains a singular member $V\left(h_{1} h_{2}\right)\left(h_{1} h_{2} \in\langle f, g\rangle\right)$ which is a normal crossing divisor at the points $V\left(h_{1}\right) \cap V\left(h_{2}\right)$ and that $V\left(h_{1}\right)$ is smooth. Then if $\mathcal{A}$ is free the arrangement $\mathcal{A} \backslash V\left(h_{1}\right)$ is also free.

Proof. Let $\delta$ be the canonical derivation associated to the pencil $\langle f, g\rangle$. If the pencil does not contain any multiple curve the degree of $\delta$ is $\alpha_{n}=2 n-$ 2. If it contains a multiple curve then one can factorize it to define a new "canonical" derivation (vanishing along any curve of the pencil) with degree $\alpha_{n}<2 n-2$. Since $V\left(h_{1} h_{2}\right)$ belongs to the pencil $\langle f, g\rangle$ one gets $\delta\left(h_{1} h_{2}\right)=$ $\operatorname{det}\left(\nabla(f), \nabla(g), \nabla\left(h_{1} h_{2}\right)\right)=0$. Then $h_{1} \delta\left(h_{2}\right)=-h_{2} \delta\left(h_{1}\right)$. Hence there exists a polynomial $k$ such that $\delta\left(h_{2}\right)=-k h_{2}$ and $\delta\left(h_{1}\right)=k h_{1}$. The derivation $\delta^{\prime}=\delta-\frac{k}{\operatorname{deg}\left(h_{1}\right)} \delta_{E}$ verifies $\delta^{\prime}\left(h_{1}\right)=0$ and it has the same degree than $\delta$. Since $V\left(h_{1} h_{2}\right)$ is a normal crossing divisor at $p \in V\left(h_{1}\right) \cap V\left(h_{2}\right)$ then $k(p) \neq 0$; indeed $h_{1}(p)=k(p)=0$ implies that $\delta\left(h_{1}\right)$ vanishes at $p$ at the order two contradicting the normal crossing at $p$. Then $\delta^{\prime}(p) \neq 0$ and the section induced by $\delta^{\prime}$ does not vanish at $p$. Hence when the component $V\left(h_{1}\right)$ is deleted from $\mathcal{A}, p$ is removed from the scheme defined by the Jacobian ideal $\mathcal{J}_{\nabla \mathcal{A}}$ and also removed from $Z\left(s_{\delta^{\prime}}\right)$ the zero scheme of the section induced by $\delta^{\prime}$. Then $Z\left(s_{\delta^{\prime}}\right)=\emptyset$ and $\mathcal{A} \backslash V\left(h_{1}\right)$ is also free.

Example 3.2. The following arrangement of four lines can be seen as a union of two singular conics, the dashed one and the black one. It is free with exponents $(2,1)$.


The following arrangement of lines is still free by Proposition 3.1 with exponents $(2,0)$.


Example 3.3. Pappus arrangement consists in 9 lines given by the well known configuration $9_{3}$. The 9 lines are the sides of the 3 triangles passing through 9 points. In the pencil generated by two triangles, singular curves are missing. In general three nodal cubics are missing but in the following example there is only one singular cubic missing: it consists in the union of a line union and a
smooth conic; indeed let us consider the pencil generated by one set of three concurrent lines and one triangle $\left[x\left(x^{2}-z^{2}\right),(x+y)(x-2 y+z)(x-2 y-z)\right]$. It still contains another triangle $(x-y)(x+2 y-z)(x+2 y+z)=0$ and a conic+line $y\left(3 x^{2}-4 y^{2}+z^{2}\right)=0$.


The union of all the singular members of the pencil is free with exponents $(4,7)$ (by [8], Theorem 1.3). By Proposition 3.1 we obtain a new arrangement which is free with exponents $(4,6)$ by removing the line from the conic+line member:

$$
x\left(3 x^{2}-4 y^{2}+z^{2}\right)\left(x^{2}-y^{2}\right)\left(x^{2}-z^{2}\right)\left((x+2 y)^{2}-z^{2}\right)\left((x-2 y)^{2}-z^{2}\right)=0 .
$$

By Proposition 3.1 again, we obtain a new arrangement which is free with exponents $(4,5)$ by removing the conic from the conic+line member:

$$
x y\left(x^{2}-y^{2}\right)\left(x^{2}-z^{2}\right)\left((x+2 y)^{2}-z^{2}\right)\left((x-2 y)^{2}-z^{2}\right)=0
$$

### 3.3. A free union of curves remains free by adding a smooth curve

Proposition 3.4. Let $C$ be a smooth curve in a pencil $\langle f, g\rangle$ of degree $n$ curves, $\mathcal{A}$ be an arrangement of curves, or components of curves, of this pencil. Assume
that the section of $\mathcal{T}_{\mathcal{A}}\left(\alpha_{n}\right)$ induced by the canonical derivation $\delta$ (of degree $\alpha_{n}$ ) does not vanish. Then $\mathcal{A}$ is free with exponents $\left(\alpha_{n},-\alpha_{n}-c_{1}\left(\mathcal{T}_{\mathcal{A}}\right)\right)$ and $\mathcal{A} \cup C$ is free with exponents $\left(\alpha_{n},-\alpha_{n}-c_{1}\left(\mathcal{T}_{\mathcal{A} \cup C}\right)\right)$.

Proof. There is a short exact sequence:

$$
0 \longrightarrow \mathcal{T}_{\mathcal{A} \cup C} \longrightarrow \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{L} \longrightarrow 0
$$

where $\mathcal{L}$ is a line bundle over $C$. Indeed on an open affine neighborhood $U \subset \mathbb{P}^{2}$ the first arrow is given by a $2 \times 2$ matrix $\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ where $a, b, c, d \in \mathscr{O}_{U}$ and $C_{\mid U}=\{a d-b c=0\}$. Assuming that the rank of $\mathcal{L}_{p}$ is $>1$ at some $p \in C$ means that $a(p)=c(p)=b(p)=d(p)=0$. But this would imply that $\nabla(a d-b c)(p)=0$ which contradicts the smoothness of $C$.

Since $C$ belongs to the pencil the canonical derivation $\delta$ induces a non zero section of $\mathcal{T}_{\mathcal{A}}\left(\alpha_{n}\right)$ but also a non zero section of $\mathcal{T}_{\mathcal{A} \cup C}\left(\alpha_{n}\right)$. This gives the following commutative diagram:


Then $\mathcal{A}$ is free with exponents $\left.\left(\alpha_{n},-\alpha_{n}-c_{1}\left(\mathcal{T}_{\mathcal{A}}\right)\right)\right)$ and $\left.\mathcal{L}=\mathscr{O}_{C}\left(c_{1}\left(\mathcal{T}_{\mathcal{A}}\right)+\alpha_{n}\right)\right)$. This proves

$$
\left.\mathcal{J}_{Z\left(s_{k+1}\right)}\left(c_{1}\left(\mathcal{T}_{\mathcal{A} \cup C}\right)+\alpha_{n}\right)\right)=\mathscr{O}_{\mathbb{P}^{2}}\left(c_{1}\left(\mathcal{T}_{\mathcal{A} \cup C}\right)+\alpha_{n}\right)
$$

Example 3.5. By Proposition 3.4 the following arrangement (three concurrent lines with one of them tangent to a smooth conic) is free with exponents (2,2). Computing the Chern classes of the logarithmic vector bundle associated, this implies that $\tau_{p}(\mathcal{A} \cup C)=10$. Computing the Milnor number at $p$ we find $\mu_{p}(\mathcal{A} \cup C)=11$ showing that the tangent point $p$ is not a quasihomogeneous singularity.


### 3.4. Tjurina number for pencils of conics

Proposition 3.6. Let $p$ be the double point of a tangent pencil $\langle f, g\rangle$. Let $C_{1}, \ldots, C_{k}$ be $k \geq 3$ smooth conics in the pencil $\langle f, g\rangle$. Then

$$
\tau_{p}\left(\bigcup_{i=1}^{k} C_{i}\right)=2\left((k-1)^{2}+1\right)
$$

Proof. By a direct computation, using for instance Macaulay 2, one can prove that the union of three smooth conics and a line through the two simple points of the base locus $B$ is free with exponents (2,4). Adding smooth conics of the same pencil does not change the freeness and the arrangement $\mathcal{A}$ consisting in $k \geq 3$ smooth conics plus one line through the two simple points is free with exponents ( $2,2 k-2$ ). Then

$$
c_{2}\left(\mathcal{T}_{\mathcal{A}}\right)=4 k-4=(2 k)^{2}-\tau(\mathcal{A})
$$

The total Tjurina number is the sum of the two normal crossing singular points in $B$ counting each of them as $k^{2}$ and the Tjurina number at the double point which is $\tau_{p}\left(\bigcup_{i=1}^{k} C_{i}\right)$. This means

$$
\tau(\mathcal{A})=4 k-4=4 k^{2}-2 k^{2}-\tau_{p}\left(\bigcup_{i=1}^{k} C_{i}\right)
$$

proving the result.
Proposition 3.7. Let $p$ be one of the two double points of a bitangent pencil $\langle f, g\rangle$. Let $C_{1}, \ldots, C_{k}$ be $k \geq 2$ smooth conics in the pencil $\langle f, g\rangle$. Then

$$
\tau_{p}\left(\bigcup_{i=1}^{k} C_{i}\right)=2 k^{2}-3 k+1
$$

Proof. By a direct computation, using for instance Macaulay 2, one can prove that the union of two smooth conics and the tangent lines along $p$ and $q$, the two base points, is free with exponents $(1,4)$ (the degree of the canonical derivation is 1 instead of 2 because of the double line in the pencil). By Proposition 3.4 adding smooth conics of the same pencil does not change the freeness and the arrangement $\mathcal{A}$ consisting in $k \geq 2$ smooth conics plus the two tangent lines is still free with exponents $(1,2 k)$. Then

$$
\begin{aligned}
c_{2}\left(\mathcal{T}_{\mathcal{A}}\right)=2 k=(2 k+1)^{2}-\tau(\mathcal{A})=(2 k+1)^{2}-1 & -\tau_{p}(\mathcal{A})-\tau_{q}(\mathcal{A}) \\
& =(2 k+1)^{2}-1-2 \tau_{q}(\mathcal{A})
\end{aligned}
$$

Then we find $\tau_{q}(\mathcal{A})=k(2 k+1)$. By Proposition 3.1, removing one of these two lines we get a new free arrangement $\mathcal{A}^{\prime}$ with exponents $(1,2 k-1)$. Then

$$
c_{2}\left(\mathcal{T}_{\mathcal{A}^{\prime}}\right)=2 k-1=(2 k)^{2}-\tau\left(\mathcal{A}^{\prime}\right)=(2 k)^{2}-\tau_{p}\left(\mathcal{A}^{\prime}\right)-\tau_{q}\left(\mathcal{A}^{\prime}\right)
$$

Since $\tau_{q}\left(\mathcal{A}^{\prime}\right)=\tau_{q}(\mathcal{A})=k(2 k+1)$, we find $\tau_{p}(\mathcal{A})=2 k^{2}-3 k+1$. At $p$ the Tjurina number of $\mathcal{A}$ coincide with the one of $k$ smooth conics in a bitangent pencil. This proves the assertion.

Proposition 3.8. Let $p$ be the triple point of an osculating pencil $\langle f, g\rangle$. Let $C_{1}, \ldots, C_{k}$ be $k \geq 3$ smooth conics in the pencil $\langle f, g\rangle$. Then

$$
\tau_{p}\left(\bigcup_{i=1}^{k} C_{i}\right)=3\left((k-1)^{2}+1\right)
$$

Proof. The union of three osculating smooth conics is a free divisor with exponents $(2,3)$. This is verified for instance with Macaulay2. Then adding smooth conics remains free, more precisely for $k \geq 3$ smooth osculating conics, this union is free with exponents $(2, n(k-2)+1)$. The second Chern class of the logarithmic bundle associated is $2 \times(n(k-2)+1)$. This number is also computed with the total Tjurina number. There are two points of intersection, $p$ the osculating point and $q$ where the $k$ conics meet transversally. At $q$ the Tjurina number is the Milnor number $(k-1)^{2}$. This gives $\tau_{p}\left(\bigcup_{i=1}^{k} C_{i}\right)$.

Proposition 3.9. Let $p$ be 4-uple point of $a+$ osculating pencil $\langle f, g\rangle$. Let $C_{1}, \ldots, C_{k}$ be $k \geq 2$ smooth conics in the pencil $\langle f, g\rangle$. Then

$$
\tau_{p}\left(\bigcup_{i=1}^{k} C_{i}\right)=4 k^{2}-6 k+3
$$

Proof. The union of two +osculating smooth conics is a free divisor with exponents (1,2). This is verified with Macaulay2. Then adding smooth conics
remains free, more precisely for $k \geq 2$ smooth overosculating conics, this union is free with exponents $(1,2(k-1))$. The second Chern class of the logarithmic bundle associated is $2(k-1)$. This number is also computed with the total Tjurina number. There is only one point of intersection, $p$. This gives $\tau_{p}\left(\bigcup_{i=1}^{k} C_{i}\right)$.

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# Pencils of singular quadrics of constant rank and their orbits 

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#### Abstract

We give a geometric description of singular pencils of quadrics of constant rank, relating them to the splitting type of some naturally associated vector bundles on $\mathbb{P}^{1}$. Then we study their orbits in the Grassmannian of lines, under the natural action of the general linear group.


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## 1. Introduction

A pencil of quadrics in the projective space of dimension $N$ is a two-dimensional linear subspace $\mathcal{L}$ in the space of symmetric matrices of order $N+1$, and it is a widely studied object in algebraic geometry.

A complete classification of pencils of quadrics, based on algebraic considerations, Segre symbols and minimal indices, has been known for a long time: we refer to the classical book by Gantmacher [9] and the expository article by Thompson [14].

There is also an extensive literature on geometric descriptions and interpretations of pencils of quadrics; among the many contributions, let us cite some older works, from [13] to [3], as well as more recent ones, such as [8].

Often, when studying pencils of quadrics in $\mathbb{P}^{N}$, one assumes that they are regular, that is, that they contain quadrics of maximal rank $N+1$. As observed in [8], these pencils form an open subset in the appropriate Grassmannian, that admits a natural stratification by Segre symbols. The pencils in the complementary closed subset, called singular pencils, are less studied, even if in [9] it is shown that their analysis can be traced back to that of regular pencils and of singular pencils of constant rank. The purpose of this article is to give a description of the geometry of such pencils of constant rank, to relate it to the splitting of certain bundles on $\mathbb{P}^{1}$ naturally associated with them, and to give a description of their orbits under the natural action of the general linear
group $\mathrm{GL}(N+1)$.
To be more precise, we set up our notations: we work over an algebraically closed field of characteristic 0 , for simplicity over the complex field $\mathbb{C}$. Let $V$ be a vector space of dimension $N+1$ over $\mathbb{C}$. Denote by $X$ the Veronese variety, that is, the image of the Veronese map $\mathbb{P}(V) \rightarrow \mathbb{P}\left(S^{2} V\right)$. The natural action of the group $\mathrm{GL}(N+1)$ on $\mathbb{P}(V)$ extends to $\mathbb{P}\left(S^{2} V\right)$, and the orbits under this latter action are $X$ and its secant varieties.

Fixing a basis for $V$, the elements of the vector space $S^{2} V$ can be seen as symmetric $(N+1) \times(N+1)$ matrices: then the action of GL $(N+1)$ is the congruence, $X$ corresponds to symmetric matrices of rank 1 , and its $k$-secant variety $\sigma_{k}(X)$ to symmetric matrices of rank at most $k$.

Working in this projective setting, we interpret a pencil of quadrics as a line $\mathbb{P}(\mathcal{L}) \subseteq \mathbb{P}\left(S^{2} V\right)$ : it is singular when it is entirely contained in the determinantal hypersurface $\sigma_{N}(X)$. If a singular pencil is entirely contained in a stratum $\sigma_{k}(X) \backslash \sigma_{k-1}(X)$, we say that the pencil has constant rank $k$. All the quadrics in such a pencil are cones having as vertex a linear space of dimension $N-k$.

In Section 2 we show that a pencil of constant rank $k$ corresponds to a matrix of linear forms in two variables, that naturally defines a map of vector bundles of rank $N+1$ over $\mathbb{P}^{1}$; since the rank is constant, the cokernel $E$ of this map is also a vector bundle over $\mathbb{P}^{1}$, of rank $N+1-k$, and its first Chern class is $\frac{k}{2}$; in particular the constant rank $k$ is an even number that we denote by $2 r$. We prove that the splitting type of $E$ characterizes the orbits, and for each orbit we give two explicit constructions for the canonical form of the representative: one is the expression described in [9], the other one is analogous to the representative given in [7], adapted from the skewsymmetric case. Indeed, several techniques used in articles on spaces of skewsymmetric matrices of constant rank, such as $[12,2,1]$, can be applied to pencils of quadrics.

Analyzing these canonical forms, in Section 3 we describe the geometry of the pencils in the various orbits. If we make the assumption that the bundle $E$ has no trivial direct summand, which is equivalent to the condition that the quadrics in the pencil $\mathcal{L}$ have no common point in their vertices, the pencil is called non-degenerate. In this case, if the splitting type of $E$ is $r_{1}, \ldots, r_{h}$, any two quadrics of $\mathcal{L}$ have a generating space $S$ of (maximal) dimension $N-r$ in common, and are tangent along a rational normal scroll of dimension $r$ and type $r_{1}, \ldots, r_{h}$ contained in $S$.

In Section 4, we prove our main result Theorem 4.1: we find an explicit expression for the dimension of every GL $(N+1)$-orbit of pencils of constant rank. We recall that these pencils are all unstable, nevertheless we are able to find an explicit expression for the matrices in the Lie algebra of the stabilizer of any pencil $\mathcal{L}$. In particular these Lie algebras all have dimension 5 when the corank of the pencil is 1 , i.e. $E$ is a line bundle with $c_{1}=r$. In Proposition 4.6
we prove that they are of the form $\mathfrak{s l}_{2} \ltimes \mathbb{C}^{2}$. We conclude with a table collecting the results for $r \leqslant 6$.

## 2. Classification's details and first results

Recall from the Introduction that, given an $(N+1)$-dimensional vector space $V$, one has the natural Veronese map $\mathbb{P}(V) \rightarrow \mathbb{P}\left(S^{2} V\right)$ sending $[v] \mapsto\left[v^{2}\right]$, whose image is the Veronese variety $X$. Once we fix a basis of $V$, the elements of $S^{2} V$ are identified with symmetric $(N+1) \times(N+1)$ matrices, $X$ corresponds to symmetric matrices of rank 1 , and its $k$-secant variety $\sigma_{k}(X)$ to symmetric matrices of rank at most $k$. The group $\mathrm{GL}(N+1)$ acts by congruence on $\mathbb{P}\left(S^{2} V\right)$, and the orbits are exactly $X$ and its secant varieties.

Now let $\mathbb{P}(\mathcal{L}) \subseteq \sigma_{k}(X) \backslash \sigma_{k-1}(X)$ be a singular pencil of quadrics of constant rank $k$. Notice that $\mathbb{P}(\mathcal{L})$ can be seen as a symmetric matrix whose entries are linear forms in two variables, that is, a vector bundle map on $\mathbb{P}^{1}=\mathbb{P}(\mathcal{L})$ of the form $V^{*} \otimes \mathcal{O}_{\mathbb{P}^{1}}(-1) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^{1}}$, inducing a long exact sequence:

$$
\begin{equation*}
0 \rightarrow E^{*}(-1) \rightarrow V^{*} \otimes \mathcal{O}_{\mathbb{P}^{1}}(-1) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^{1}} \rightarrow E \rightarrow 0 \tag{1}
\end{equation*}
$$

The cokernel is a vector bundle of rank $N+1-k$ on $\mathbb{P}^{1}$, hence it splits as a direct sum of line bundles; we denote it by $E$. The symmetry implies that the kernel is $E^{*}(-1)$.

From a direct computation of invariants (see [10] for details), one finds that the rank $k=2 r$ is even, the bundle $E$ is generated by its global sections, and moreover its first Chern class is $c_{1}(E)=r$.

We start our description of $\mathcal{L}$ generalizing to the symmetric case some results from [7] that refer to the skew-symmetric case. We are of course interested in non-trivial cases: for this, recall that a space of matrices is called nondegenerate if the kernels of its elements intersect in the zero subspace and the images of its elements generate the entire vector space $V$. This is equivalent to saying that the space is not $\mathrm{GL}(N+1)$-equivalent to a space of matrices with a row or a column of zeroes. Therefore the classification of degenerate spaces of matrices can be traced back to that of non-degenerate spaces of matrices of smaller size. From now on, we will only consider non-degenerate spaces of constant rank $2 r$.

Non-degeneracy also implies that as the quadrics vary in the pencil $\mathcal{L}$, their vertices are pairwise disjoint.

An immediate remark is that not all values of $N$ allow a non-degenerate pencil of symmetric matrices of size $N+1$ and fixed constant rank $2 r$.

Proposition 2.1. Let $\mathcal{L} \subset \mathbb{P}\left(S^{2} V\right)$ be a non-degenerate pencil of singular quadrics of constant rank $2 r$. Then $2 r \leqslant N \leqslant 3 r-1$.

Proof. The proof of [7, Proposition 3.6] goes through step by step. Since the cokernel bundle $E$ from (1) is a vector bundle on $\mathbb{P}^{1}$, it is of the form

$$
E=\mathcal{O}_{\mathbb{P}^{1}}^{m_{0}} \oplus \mathcal{O}_{\mathbb{P}^{1}}(1)^{m_{1}} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{1}}(k)^{m_{k}}
$$

where $m_{0}, \ldots, m_{k}$ are non-negative integers such that $m_{1}+2 m_{2}+\ldots+k m_{k}=$ $c_{1}(E)=r$, and $m_{0}+m_{1}+\ldots+m_{k}=\operatorname{rk}(E)=N+1-2 r$.

The assumption that $\mathcal{L}$ is non-degenerate implies $m_{0}=0$.
Obviously $2 r \leqslant N$. For the other inequality, notice that
$r=m_{1}+2 m_{2}+\ldots+k m_{k}=\left(m_{1}+m_{2}+\ldots+m_{k}\right)+\left(m_{2}+2 m_{3}+\ldots+(k-1) m_{k}\right) ;$
since $m_{0}=0, m_{1}+m_{2}+\ldots+m_{k}=N+1-2 r$, while since $k \geqslant 1, m_{2}+2 m_{3}+$ $\left.\ldots+(k-1) m_{k}\right) \geqslant 0$. This means that $r \geqslant N+1-2 r$, and thus $3 r-1 \geqslant N$.

The group $G L(N+1)$ acts by congruence on $\mathbb{P}\left(S^{2} V\right)$, the space of quadrics in $\mathbb{P}(V)$, and thus it acts on pencils of quadrics, that correspond to lines in $\mathbb{P}\left(S^{2} V\right)$ : this induces an action on the Grassmannian $\mathbb{G}\left(1, \mathbb{P}\left(S^{2} V\right)\right)$. Given a non-degenerate pencil of quadrics in $\mathbb{P}(V)$, the splitting type of the vector bundle $E$ determines a partition of the integer $r$ in $h$ parts, where the number of parts $h=N+1-2 r$ is exactly the rank of the bundle $E$. For every choice of constant rank $2 r$ there are exactly $r$ possible sizes $N+1$ for these pencils, namely $N$ can vary from $2 r$ to $3 r-1$. On the other hand, if the rank and the order of the matrix are fixed, the number of parts $h$ of the partition of $r$ is determined.

Our main result in this Section states that, for a fixed $r$, all possible values of $N$ are attained, and that the partitions of $r$ consisting of $h=N+1-2 r$ parts completely characterize the orbits of pencils of quadrics of constant rank.

In our proof we will use the classification of the GL $(N+1)$-orbits given in terms of minimal indices, see [9, Chapter XII, §6].

In fact, even if $\operatorname{GL}(N+1)$ acts on a pencil $\mathcal{L} \subset \mathbb{P}\left(S^{2} V\right)$ by congruence, one can also consider a different natural action of the general linear group on $\mathcal{L}$, namely two pencils of matrices $a A+b B$ and $\lambda L+\mu M$ are called strictly equivalent if there exist two non-singular matrices $P^{\prime}$ and $P^{\prime \prime}$ with the property that $P^{\prime}(a A+b B) P^{\prime \prime}=\lambda L+\mu M$. The latter action implies the former if the matrices are symmetric or skew-symmetric [ 9 , Theorem 6, Chapter XII]: in particular, two pencils of quadrics are strictly equivalent if and only if they are congruent.

Following the same notations as [9] (so slightly different than [7]), our construction is based on the following "building blocks".

DEFINITION 2.2. Let $r \geqslant 1$ be an integer, and $\left(r_{1}, \ldots, r_{h}\right)$ a partition of $r$, with $r_{1} \leqslant \ldots \leqslant r_{h}$. Set $N=2 r+h-1$. Denote by $\mathcal{L}_{\left(r_{1}, \ldots, r_{h}\right)}$ the pencil
of $(N+1) \times(N+1)$ symmetric matrices of constant rank $2 r$ constructed as follows.

First, define the $r_{i} \times\left(r_{i}+1\right)$ matrix

$$
M_{r_{i}}:=\left(\begin{array}{cccccc}
a & b & 0 & 0 & \cdots & 0  \tag{2}\\
0 & a & b & 0 & \cdots & 0 \\
\vdots & & \ddots & \ddots & & \vdots \\
0 & \cdots & & 0 & a & b
\end{array}\right),
$$

and the $\left(2 r_{i}+1\right) \times\left(2 r_{i}+1\right)$ symmetric block matrix

$$
\mathcal{L}_{r_{i}}:=\left(\begin{array}{c:c}
0_{r_{i}, r_{i}} & M_{r_{i}}  \tag{3}\\
\hdashline{ }^{t} M_{r_{i}} & 0_{r_{i}+1, r_{i}+1}
\end{array}\right) .
$$

The pencil of quadrics $\mathcal{L}_{\left(r_{1}, \ldots, r_{h}\right)}$ is the direct sum of the blocks $\mathcal{L}_{r_{i}}$, so

$$
\mathcal{L}_{\left(r_{1}, \ldots, r_{h}\right)}:=\left(\begin{array}{c|c|c|c}
\mathcal{L}_{r_{1}} & & &  \tag{4}\\
\hline & \mathcal{L}_{r_{2}} & & \\
\hline & & \ddots & \\
\hline & & & \mathcal{L}_{r_{h}}
\end{array}\right)
$$

where all off-diagonal blank spaces are blocks of zeros.
By combining the construction of the pencils $\mathcal{L}_{\left(r_{1}, \ldots, r_{h}\right)}$ and the classification contained in Theorem 7 and the subsequent remarks in [9, Chapter XII, §6], we obtain the following Theorem, that achieves a complete description of the $\mathrm{GL}(N+1)$-orbits of singular pencils of quadrics $\mathcal{L} \subset \mathbb{P}\left(S^{2} V\right)$ of constant rank.

Theorem 2.3. Let $V$ be a complex vector space of dimension $N+1$, and let $\mathcal{L} \subseteq \mathbb{P}\left(S^{2} V\right)$ be a singular pencil of quadrics of constant rank $2 r$. If $\mathcal{L}$ is nondegenerate, it is $\mathrm{GL}(N+1)$-equivalent by congruence and strict equivalence to a pencil of type $\mathcal{L}_{\left(r_{1}, \ldots, r_{h}\right)}$ defined in (4) for some partition $\left(r_{1}, \ldots, r_{h}\right)$ of $r$, with $r_{1} \leqslant \ldots \leqslant r_{h}, h=N+1-2 r$, and whose associated vector bundle $E$ has splitting type precisely $\left(r_{1}, \ldots, r_{h}\right)$.

Viceversa, for every integer $r \geqslant 1$ and every partition $\left(r_{1}, \ldots, r_{h}\right)$ of $r$, with $r_{1} \leqslant \ldots \leqslant r_{h}$, there exists a non-degenerate singular pencil of quadrics of constant rank $2 r$ and size $N+1$, for all $2 r \leqslant N \leqslant 3 r-1$.

Remark 2.4. An alternative proof of Theorem 2.3 could be obtained by adapting to the symmetric case the proof of [7, Theorem 3.12], which is based on compression spaces and 1-generic matrices.

REmark 2.5. If one wanted to take into consideration degenerate pencils, it would be enough to consider partitions of $r$ that admit 0 as a summand, with multiplicity corresponding to the number of copies of $\mathcal{O}_{\mathbb{P}^{1}}$ appearing in the splitting of the vector bundle $E$ in (1).

To conclude this Section, we underline the fact that the content of Theorem 2.3 was already known, even though the relation between the classification of the orbits of singular pencils of quadrics of constant rank and the splitting type of the vector bundle has never been explicitly written down. In [3] the Author provides a geometric classification of the orbits, but the relation with the vector bundles is not clarified; on the other hand, in the recent work [6] there is an explicit description of the splitting type of the bundles, but the Authors are interested in different properties than the orbits of pencils in the Grassmannian.

## 3. Geometry of pencils of quadrics and their orbits

We now want to study more in detail the geometry of pencils of quadrics of constant rank and their orbits. To this end, in this Section we use a different canonical form from the one given in Definition 2.2 for the pencils with $h \geqslant 2$. It is analogous to the canonical form described in [7] in the skew-symmetric case, and is more convenient to understand the geometry of our pencils because it highlights that they are compression spaces. Recall that a subspace $\mathcal{L}$ contained in $V \otimes V$ is called a compression space if there exists a subspace $U \subseteq V$ that is "compressed" by the elements of $\mathcal{L}$, that is, $\operatorname{dim}(L(U))<\operatorname{dim}(U)$ for all $L \in \mathcal{L}$. Such a space is $G L(N+1)$-equivalent to a space of matrices having a common block of zeros.

We start by describing some examples, namely the first cases where $r=1,2$ and 3.
Example 3.1. The first (and easiest) example is $r=1$ : then the only possible value for $N$ is 2 , and the only partition of $r$ is (1), so there is a unique orbit, whose representative is the compression space

$$
\mathcal{L}_{(1)}=\left(\begin{array}{c:cc}
0 & a & b  \tag{5}\\
\hdashline a & 0_{2,2} \\
b & &
\end{array}\right) .
$$

The cokernel bundle from the exact sequence (1) is $E=\mathcal{O}_{\mathbb{P}^{1}}(1)$. This is a pencil of conics in $\mathbb{P}^{2}$, generated by $A=\left\{x_{0} x_{1}=0\right\}$ and $B=\left\{x_{0} x_{2}=0\right\}$, that split into a common line $S=\left\{x_{0}=0\right\}$ and a second line that goes through the point $P=[1: 0: 0]$. The base locus of the pencil is exactly the union of the line $S$, and the isolated point $P$. Notice that $S$ is swept by the singular points of the conics of the pencil.

The pencils belonging to the orbit of $\mathcal{L}_{(1)}$ in the Grassmannian $\mathbb{G}\left(1, \mathbb{P}^{5}\right)$ are determined by their base locus, that varies in the open subset of $\mathbb{P}^{2} \times \mathbb{P}^{2^{*}}$ of disjoint pairs point-line. Therefore the orbit in $\mathbb{G}\left(1, \mathbb{P}^{5}\right)$ has dimension 4 .
Example 3.2. When $r=2$, the possible values of $N$ are 4 and 5 , corresponding to the two partitions (2) and $(1,1)$.

The first case gives a $5 \times 5$ symmetric matrix of constant rank 4:

$$
\mathcal{L}_{(2)}=\left(\begin{array}{c:ccc}
0_{2,2} & a & b & 0 \\
\hdashline a & 0 & 0 & a \\
b \\
b & a & 0_{3,3} & \\
0 & b & &
\end{array}\right)
$$

with associated line bundle $\mathcal{O}_{\mathbb{P}^{1}}(2)$. The pencil is generated by the quadrics $A=\left\{x_{0} x_{2}+x_{1} x_{3}=0\right\}$ and $B=\left\{x_{0} x_{3}+x_{1} x_{4}=0\right\}$; its elements are cones over quadrics in $\mathbb{P}^{3}$, having a single point as vertex. As the cones vary, their vertices describe a conic $\Gamma$ in the plane $S=\left\{x_{0}=x_{1}=0\right\}$. The base locus is the union of the plane $S$ and the rational normal scroll of degree 3 in $\mathbb{P}^{4}$ defined by the $2 \times 2$ minors of the matrix

$$
\left(\begin{array}{ccc}
x_{0} & x_{3} & x_{4} \\
-x_{1} & x_{2} & x_{3}
\end{array}\right)
$$

The singular locus of the base locus is the conic $\Gamma$, which coincides with the improper intersection of the 2 irreducible components.

A pencil in this orbit is completely determined by its base locus, that is the union of a rational normal scroll and a plane generated by a unisecant conic. From [5] we learn that the Hilbert scheme of these rational normal scrolls has dimension 12; moreover the linear system of unisecant conics on such a surface has dimension 2 ; it follows that the orbit has dimension 14.

The partition $(1,1)$ of $r=2$ gives a $6 \times 6$ symmetric matrix of constant rank 4 , whose associated bundle is $E=\mathcal{O}_{\mathbb{P}^{1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{1}}(1)$. As we mentioned at the beginning of the Section, we consider the following canonical form (here and in the next examples the blank spaces all represent zeros):

$$
\tilde{\mathcal{L}}_{(1,1)}=\left(\begin{array}{cc|cccc} 
& & a & b & 0 & 0 \\
& & 0 & 0 & a & b \\
\hline a & 0 & & & & \\
b & 0 & & & & \\
0 & a & & & & \\
0 & b & & & &
\end{array}\right)
$$

Of course, $\tilde{\mathcal{L}}_{(1,1)}$ is strictly equivalent to the block construction from Defini-
tion 2.2, namely:

$$
\mathcal{L}_{(1,1)}=\left(\begin{array}{l|l}
\mathcal{L}_{1} & \\
\hline & \mathcal{L}_{1}
\end{array}\right) .
$$

Since the co-rank is 2 , the cones of this pencil have a line as vertex. The generators are $A=\left\{x_{0} x_{2}+x_{1} x_{4}=0\right\}$ and $B=\left\{x_{0} x_{3}+x_{1} x_{5}=0\right\}$, the base locus is reducible, and its components are the 3 -dimensional linear space $S=\left\{x_{0}=x_{1}=0\right\}$ and a rational normal 3-fold scroll of degree 3 in $\mathbb{P}^{5}$, defined by the $2 \times 2$ minors of

$$
\left(\begin{array}{ccc}
x_{0} & x_{4} & x_{5} \\
-x_{1} & x_{2} & x_{3}
\end{array}\right)
$$

The locus swept by vertices is a smooth quadric surface in $S$. By a count of parameters similar to previous case, the dimension of the orbit is 26: indeed, the dimension of the Hilbert scheme of rational normal cubic scrolls in $\mathbb{P}^{5}$ is 24 and the linear system of unisecant quadrics has dimension 2 .

One of the advantages of using the form $\tilde{\mathcal{L}}_{(1,1)}$ lies precisely in the fact that the codimension 2 linear space $S$ contained in the base locus is now apparent, since we are dealing with a compression space. This phenomenon will generalize in the next cases.

Example 3.3. As a last series of examples, aiming to illustrate the general case, we now consider the possible partitions of $r=3$. One has three possible values $6 \leqslant N \leqslant 8$, corresponding to the three partitions (3), (1,2) and ( $1,1,1$ ). By now we know that the representatives of their orbits are, respectively,

$$
\mathcal{L}_{(3)}=\left(\begin{array}{cc:cccc} 
& & a & b & 0 & 0 \\
0_{3,3} & 0 & a & b & 0 \\
\hdashline & 0 & 0 & a & b \\
\hdashline a & 0 & 0 & & & \\
b & a & 0 & & & \\
0 & b & a & & 0_{4,4} & \\
0 & 0 & b & & &
\end{array}\right), \quad \mathcal{L}_{(1,2)}, \quad \text { and } \mathcal{L}_{(1,1,1)}
$$

The base locus of the pencil $\mathcal{L}_{(3)}$ in $\mathbb{P}^{6}$ is an irreducible quartic, complete intersection of the two quadrics $A=\left\{x_{0} x_{3}+x_{1} x_{4}+x_{2} x_{5}=0\right\}$ and $B=$ $\left\{x_{0} x_{4}+x_{1} x_{5}+x_{2} x_{6}=0\right\}$; it is singular along a twisted cubic $C$ swept by the vertices and it contains the 3-dimensional linear space $S=\left\{x_{0}=x_{1}=x_{2}=0\right\}$ spanned by $C$.

To analyze the other two cases, we will again look at representatives that
are strictly equivalent to $\mathcal{L}_{(1,2)}$ and $\mathcal{L}_{(1,1,1)}$, namely:

$$
\tilde{\mathcal{L}}_{(1,2)}=\left(\begin{array}{ccc|ccccc} 
& & & a & b & 0 & 0 & 0 \\
& & & 0 & 0 & a & b & 0 \\
& & & 0 & 0 & a & b \\
\hline a & 0 & 0 & & & & & \\
b & 0 & 0 & & & & & \\
0 & a & 0 & & & & & \\
0 & b & a & & & & & \\
0 & 0 & b & & & & &
\end{array}\right)
$$

and

$$
\tilde{\mathcal{L}}_{(1,1,1)}=\left(\begin{array}{ccc|ccccc} 
& & & a & b & 0 & 0 & 0 \\
0 & 0 \\
& & & 0 & 0 & a & b & 0 \\
0 & 0 & 0 & 0 & a & b \\
\hline a & 0 & 0 & & & & & \\
b & 0 & 0 & & & & & \\
0 & a & 0 & & & & & \\
0 & b & 0 & & & & & \\
0 & 0 & a & & & & & \\
0 & 0 & b & & & & &
\end{array}\right) .
$$

Considering the kernels of these matrices, we see that in both cases the Jacobian locus of the pencil is contained in the linear space $S=\left\{x_{0}=x_{1}=x_{2}=0\right\}$ of codimension 3 (so of dimension 4 and 5 respectively). The base locus is irreducible in both cases and it is singular along the Jacobian locus, that is a rational normal scroll in $S, \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^{1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{1}}(2)\right)$ and $\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^{1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{1}}(1)\right)$ respectively.

We now describe the general case of a pencil $\mathcal{L}=\mathcal{L}_{\left(r_{1}, \ldots, r_{h}\right)}$ of constant rank $2 r$ in $\mathbb{P}^{N}$, corresponding to the partition $\left(r_{1}, \cdots, r_{h}\right)$ of $r, h=N+1-2 r$. Recall that we can write our $\mathcal{L}$ as $\left\{a A+b B \mid[a: b] \in \mathbb{P}^{1}\right\}$. We denote by $B(\mathcal{L})=A \cap B$ the base locus of $\mathcal{L}$. It is a known fact that its singular locus is contained in the Jacobian locus $J(\mathcal{L})$ of $\mathcal{L}$, the union of the vertices of the quadrics in the pencil, and such vertices are linear spaces of dimension $N-2 r$.

As we did in the previous examples, we use a canonical form for the pencils that is slightly different from (4), and instead agrees with the notations used
in [7]: given the $r_{i} \times\left(r_{i}+1\right)$ block $M_{r_{i}}$ defined in (2), we set

where again the blank spaces have blocks of zeros.
From this canonical form, it is immediate to see that all these pencils correspond to compression spaces, because the associated matrices have a block of zeros of dimension $N+1-r$; a direct consequence is that the Jacobian locus $J(\mathcal{L})$ is contained in the linear space $S$ of dimension $N-r$ defined by the equations $x_{0}=x_{1}=\cdots=x_{r-1}=0$.

Moreover, one easily computes that the Jacobian locus coincides with the singular locus of $B(\mathcal{L})$, which is irreducible, and it is exactly a rational normal scroll $\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^{1}}\left(r_{1}\right) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^{1}}\left(r_{h}\right)\right)$. Any element of the pencil is a cone over a smooth quadric of dimension $2 r-2$, so it admits two families of linear spaces of dimension $(r-1)+(N-2 r)+1=N-r$. Two quadrics of the pencil share a maximal linear subspace $S$ of dimension $N-r$ belonging to one of the two families, and are tangent along a rational normal scroll of type $r_{1}, \ldots, r_{h}$ in $S$.

As a last remark ending this Section, we quote the article [13], a continuation and completion of the thesis of Corrado Segre, where he studied the geometry of singular pencils of quadrics in $\mathbb{P}^{N}$ of rank at most $k$, that he calls "coni quadrici di specie $N-k$ ", relating them to rational normal scrolls contained in their Jacobian locus.

## 4. Orbits' dimension

We recalled in Section 2 that the natural action of the group GL $(N+1)$ on $V=\mathbb{C}^{N+1}$ extends to the congruence action on $\mathbb{P}\left(S^{2} V\right)$, and hence on the lines contained in $\mathbb{P}\left(S^{2} V\right)$. Looking at pencils of quadrics as points in the Grassmannian $\mathbb{G}\left(1, \mathbb{P}\left(S^{2} V\right)\right.$ ), we get an action of $\mathrm{GL}(N+1)$ on the Grassmannian. We are interested in the orbits of singular pencils of quadrics $\mathcal{L} \subseteq \mathbb{P}\left(S^{2} V\right)$ of constant rank $2 r$ under this latter action. As we saw in Theorem 2.3 nondegenerate pencils of quadrics in $\mathbb{P}^{N}$ of constant rank $2 r$ exist if and only if
$2 r \leqslant N \leqslant 3 r-1$ and the orbits of these pencils correspond bijectively to the partitions $\left(r_{1}, \ldots, r_{h}\right)$ of $r$, with $1 \leqslant r_{1} \leqslant r_{2} \leqslant \ldots r_{h}$, where $h=N+1-2 r$.

This last Section contains our main result Theorem 4.1, namely we compute the dimension of all the orbits of pencils of singular quadrics of constant rank. More precisely, for every partition $\left(r_{1}, \ldots, r_{h}\right)$ we describe explicitly the Lie algebra of the stabilizer of the pencil $\mathcal{L}_{\left(r_{1}, \ldots, r_{h}\right)}$.
THEOREM 4.1. Let $r \geqslant 1$ be an integer, and $\left(r_{1}, \ldots, r_{h}\right)$ a partition of $r$, with $r_{1} \leqslant \ldots \leqslant r_{h}$. Set $N=2 r+h-1$. Under the natural action of GL $(N+1)$, the dimension of the stabilizer of the singular pencil $\mathcal{L}_{\left(r_{1}, \ldots, r_{h}\right)}$ of symmetric matrices of size $N+1$ and constant rank $2 r$ is

$$
\begin{equation*}
\delta\left(r_{1}, \ldots, r_{h}\right):=h+4+\sum_{i<j}\left(2 r_{j}+1\right)+\#\left\{(i, j) \mid r_{i}=r_{j}\right\} . \tag{7}
\end{equation*}
$$

Corollary 4.2. The GL $(N+1)$-orbits of singular pencils $\mathcal{L}_{\left(r_{1}, \ldots, r_{h}\right)}$ of symmetric matrices of size $N+1$ and constant rank $2 r$ have (affine) dimension $(N+1)^{2}-\delta\left(r_{1}, \ldots, r_{h}\right)$.

The plan of the proof of Theorem 4.1 is the following: we first analyze, in Propositions 4.5 and 4.6 the case of partitions with only one part, i.e. pencils of symmetric matrices of constant corank 1; then, in Proposition 4.7, we consider the case of partitions with two parts, i.e. pencils of constant corank 2. We obtain a complete description of the Lie algebra of the stabilizer in both cases. The key remark is then that, in the general case, due to the particular canonical form of the representatives of the orbits under consideration, a matrix $X$ in the Lie algebra of the stabilizer can be interpreted as a block matrix of the form (16), where the blocks involved already appear and are described in the first two cases.

The next Lemma is probably well known. We report it here for completeness and because it is a fundamental ingredient for computing the Lie algebras of the stabilizers in the two cases $h=1,2$.

Lemma 4.3. Let $\mathcal{L}$ be the pencil generated by the symmetric matrices $A$ and $B$, let $X$ be a $(N+1) \times(N+1)$ matrix with entries in $\mathbb{C}$. Then $X$ belongs to the Lie algebra of the stabilizer of $\mathcal{L}$ for the action of $\mathrm{GL}(N+1)$ on the Grassmannian if and only if the following relations hold:

$$
\begin{equation*}
\left({ }^{t} X A+A X\right) \wedge A \wedge B=\left({ }^{t} X B+B X\right) \wedge A \wedge B=0 \tag{8}
\end{equation*}
$$

Proof. The point in the Grassmannian $\mathbb{G}\left(1, \mathbb{P}\left(S^{2} V\right)\right)$ corresponding to the pencil $\mathcal{L}$ via the Plücker map is $[A \wedge B]$. Its $\mathrm{GL}(N+1)$-orbit is the image of the map $\mathrm{GL}(N+1) \rightarrow \mathbb{G}\left(1, \mathbb{P}\left(S^{2} V\right)\right)$ given by $X \mapsto\left({ }^{t} X A X\right) \wedge\left({ }^{t} X B X\right)$. So the condition for $X$ to belong to the stabilizer of $\mathcal{L}$ is $[A \wedge B]=\left[\left({ }^{t} X A X\right) \wedge\left({ }^{t} X B X\right)\right]$. This is equivalent to the equations $\left({ }^{t} X A X\right) \wedge A \wedge B=\left({ }^{t} X B X\right) \wedge A \wedge B=0$. Differentiating these equations at the origin we get the thesis.

Remark 4.4. In the article [4], the Authors are interested in the same problem of computing the dimensions of orbits of pencils of symmetric matrices. But instead of interpreting them as points in the appropriate Grassmannian, they work with pairs of matrices generating the pencil, thus obtaining a different result from ours.

We start with the partition having only $h=1$ part. We have a pencil of symmetric matrices of size $N+1=2 r+1$ and rank $2 r$, whose cokernel is the line bundle $E=\mathcal{O}_{\mathbb{P}^{1}}(r)$; the orbit representative is $\mathcal{L}_{(r)}$, that we write in the following form, suitable to apply Lemma 4.3:

Proposition 4.5. Let $r \geqslant 1$ be an integer. The GL( $2 r+1$ )-orbit of pencils of singular quadrics of constant rank $2 r$ and order $2 r+1$ has a stabilizer of dimension 5. The Lie algebra of the stabilizer is the vector space of matrices $X$ of the form:

$$
X=\left(\begin{array}{c:c}
X_{1} & 0_{r, r+1}  \tag{10}\\
\hdashline 0_{r+1, r} & X_{2}
\end{array}\right)
$$

where, for $r \geqslant 2$ :

1. $X_{1}$ and $X_{2}$ are square matrices of order $r$ and $r+1$ respectively;
2. both $X_{1}$ and $X_{2}$ are tridiagonal, i.e. all the elements out of the main diagonal, the sub-diagonal (the first diagonal below this), and the supradiagonal (the first diagonal above the main diagonal) are zero;
3. the sub-diagonal, main diagonal, and supradiagonal of $X_{1}$ are respectively:

$$
\begin{gathered}
y(r-1, r-2, \ldots, 1), x_{00}(1,0,-1,-2, \ldots,-(r-2))+x_{11}(0,1,2, \ldots, r-1) \\
z(1,2, \ldots, r-1)
\end{gathered}
$$

4. the sub-diagonal, main diagonal, and supradiagonal of $X_{2}$ are respectively:

$$
-z(1,2, \ldots, r),\left(x_{00}-x_{11}\right)(0,1, \ldots, r)+x_{33}(1,1, \ldots, 1),-y(r, r-1, \ldots, 1)
$$

where $x_{00}, x_{11}, x_{33}, y, z$ are independent parameters.
For instance, if $r=3, X$ is as follows:

Proof. Let $X=\left(x_{i j}\right)_{i, j=0, \ldots, N}$ be a matrix of unknowns. If $A, B$ are the matrices introduced in (9), the elements of indices $i \leqslant j$ in the symmetric matrices ${ }^{t} X A+A X$ and ${ }^{t} X B+B X$ are as described below:

$$
\begin{align*}
& \left({ }^{t} X A+A X\right)_{i j}=\left\{\begin{array}{lll}
x_{j+r, i}+x_{i+r, j} & \text { if } & 0 \leqslant i \leqslant j \leqslant r-1 \\
x_{j-r, i}+x_{i+r, j} & 0 \leqslant i \leqslant r-1, r \leqslant j \leqslant 2 r-1 \\
x_{i+r, 2 r} & 0 \leqslant i \leqslant r-1, j=2 r \\
x_{j-r, i}+x_{i-r, j} & r \leqslant i \leqslant j \leqslant 2 r-1 \\
x_{i-r, 2 r} & r \leqslant i \leqslant 2 r-1, j=2 r \\
0 & i=j=2 r
\end{array}\right.  \tag{11}\\
& \left({ }^{t} X B+B X\right)_{i j}= \begin{cases}x_{j+r+1, i}+x_{i+r+1, j} & \text { if } 0 \leqslant i \leqslant j \leqslant r-1 \\
x_{i+r+1, r} & 0 \leqslant i \leqslant r-1, j=r \\
x_{j-r-1, i}+x_{i+r+1, j} & 0 \leqslant i \leqslant r-1, r+1 \leqslant j \leqslant 2 r \\
0 & i=r=j \\
x_{j-r-1, i}+x_{i-r-1, j} & r+1 \leqslant i \leqslant j \leqslant 2 r \\
x_{j-r-1, r} & i=r, r+1 \leqslant j \leqslant 2 r\end{cases} \tag{12}
\end{align*}
$$

In view of Lemma 4.3, $X$ belongs to the Lie algebra of the stabilizer of the orbit of $\mathcal{L}_{(r)}$ if and only if it satisfies the equations (8), that are equivalent to a series of equations in the entries of each of the two matrices ${ }^{t} X A+A X$ and ${ }^{t} X B+B X$, and precisely:
(i) vanishing of the elements with equal indices;
(ii) vanishing of the elements with indices $0 \leqslant i<j \leqslant r-1, r \leqslant i<j \leqslant 2 r$, $(i, i+r+2), \ldots,(i, 2 r)$ for $i=0, \ldots, r-2$, and $(i, r), \ldots,(i, i+r-1)$ for $i=1, \ldots, r-1$;
(iii) elements with indices $(0, r),(1, r+1), \ldots,(r-1,2 r-1)$ must be two by two equal;
(iv) elements with indices $(0, r+1),(1, r+2), \ldots,(r-1,2 r)$ must be two by two equal.

Now, using (11) and (12) together with (i) we get $x_{0, r}=x_{1, r+1}=\cdots=$ $x_{r-1,2 r-1}=x_{r, 0}=\cdots=x_{2 r-1, r-1}=0$, and also $x_{0, r+1}=x_{1, r+2}=\cdots=$ $x_{r-2,2 r-1}=x_{r+1,0}=\cdots=x_{2 r, r-1}=0$; note that in all these cases the difference of the indices is either $r$ or $r+1$.

From the vanishings just obtained and those in (ii) whose indices differ by 1 , we get $x_{0, r-1}=x_{1, r}=\cdots=x_{r-1,2 r-2}=x_{r-1,0}=\cdots=x_{2 r-2, r-1}=0$, $x_{0, r+2}=x_{1, r+3}=\cdots=x_{r-2,2 r}=x_{r+2,0}=\cdots=x_{2 r, r-2}=0$, and also $x_{2 r, r}=x_{r-1,2 r}=0$.

We continue in this way, considering relations in (ii) whose indices differ by 2 and so on, until we get all the claimed vanishings in matrix (10) and moreover the following $2 r$ equations:

$$
x_{0,1}+x_{r+1, r}=x_{1,2}+x_{r+2, r+1}=\cdots=x_{r-1, r}+x_{2 r, 2 r-1}=0
$$

and the symmetric ones

$$
x_{1,0}+x_{r, r+1}=x_{2,1}+x_{r+1, r+2}=\cdots=x_{r, r-1}+x_{2 r-1,2 r}=0 .
$$

The relations in (iii) and (iv) impose $2 r-2$ conditions on the elements of the main diagonal of $X$, and $2 r-2$ conditions on the elements of the subdiagonal and supradiagonal of $X$, and precisely:

$$
\begin{gathered}
x_{0,0}+x_{r, r}=x_{1,1}+x_{r+1, r+1}=\cdots=x_{r-1, r-1}+x_{2 r-1,2 r-1}, \\
x_{0,0}+x_{r+1, r+1}=x_{1,1}+x_{r+2, r+2}=\cdots=x_{r-1, r-1}+x_{2 r, 2 r}, \\
x_{1,0}+x_{r, r+1}=x_{2,1}+x_{r+1, r+2}=\cdots=x_{2 r-1,2 r}, \\
x_{r+1, r}=x_{0,1}+x_{r+2, r+1}=\cdots=x_{r-2, r-1}+x_{2 r, 2 r-1} .
\end{gathered}
$$

Combining everything, we obtain for $X$ the expression in (10), with $z=x_{0,1}$ and $y=x_{r-1, r-2}$; the Proposition is proved.

Our description of the stabilizer compared with the known classification of Lie algebras of small dimension ([11]) gives the following result.

Proposition 4.6. The Lie algebra of the stabilizer of the GL $(2 r+1)$-orbit of pencils of quadrics of constant rank $2 r$ and order $2 r+1$ described in Proposition 4.5 is isomorphic to $\mathfrak{s l}_{2} \ltimes \mathbb{C}^{2}$.

Proof. From the detailed description of the Lie algebra of the stabilizer given in Proposition 4.5, one sees that its elements depend on 5 independent parameters, namely any element $X$ in this Lie algebra is $X=X\left(x_{00}, x_{11}, x_{33}, y, z\right)$. With obvious notation, let us call

$$
\begin{array}{ll}
\mathcal{C}_{1}=X(1,0,0,0,0), & \mathcal{C}_{2}=X(0,1,0,0,0), \\
\mathcal{X}=X(0,0,0,0,1), & \mathcal{Y}=X(0,0,0,1,0), \quad \mathcal{Z}=X(r-1, r-3,-r, 0,0) .
\end{array}
$$

If we compute the bracket of these elements, we get that $\left[\mathcal{C}_{1}, \mathcal{C}_{2}\right]=0$ and

$$
\left\{\begin{array}{l}
{[\mathcal{X}, \mathcal{Y}]=\mathcal{Z}} \\
{[\mathcal{Z}, \mathcal{X}]=2 \mathcal{X}} \\
{[\mathcal{Z}, \mathcal{Y}]=-2 \mathcal{Y}}
\end{array}\right.
$$

which tells us that $\mathbb{C}^{2}=<\mathcal{C}_{1}, \mathcal{C}_{2}>$ and $\left.\mathfrak{s l}_{2}=<\mathcal{X}, \mathcal{Y}, \mathcal{Z}\right\rangle$. The fact that

$$
\left\{\begin{array}{l}
{\left[\mathcal{C}_{1}, \mathcal{X}\right]=\mathcal{X}=-\left[\mathcal{C}_{2}, \mathcal{X}\right]} \\
{\left[\mathcal{C}_{1}, \mathcal{Y}\right]=-\mathcal{Y}=-\left[\mathcal{C}_{2}, \mathcal{Y}\right]} \\
{\left[\mathcal{C}_{1}, \mathcal{Z}\right]=\left[\mathcal{C}_{2}, \mathcal{Z}\right]=0}
\end{array}\right.
$$

allows us to conclude that our Lie algebra falls into the first case in the classification table appearing in [11, Section 4], namely the semidirect product $\mathfrak{s l}_{2} \ltimes \mathbb{C}^{2}$.

When the partition has $h=2$ parts, the balanced and unbalanced case have two different behaviors, as explained in the following result.

Proposition 4.7. Let $r \geqslant 1$ be an integer. The GL( $2 r+2$ )-orbit of pencils of singular quadrics of constant rank $2 r$ and order $2 r+2$, whose associated bundle is $\mathcal{O}_{\mathbb{P}^{1}}\left(r_{1}\right) \oplus \mathcal{O}_{\mathbb{P}^{1}}\left(r_{2}\right)$, with $r_{1}+r_{2}=r$, and $r_{1} \leqslant r_{2}$, has stabilizer of dimension

1. $2 r_{2}+8=r+8$ when $r$ is even and $r_{1}=r_{2}=\frac{r}{2}$;
2. $2 r_{2}+7$ when $r_{1}<r_{2}$.

Proof. In the notation of Section 2, a representative of the orbit is the matrix

$$
\mathcal{L}_{\left(r_{1}, r_{2}\right)}=a A+b B=\left(\begin{array}{l|l}
\mathcal{L}_{r_{1}} & \\
\hline & \mathcal{L}_{r_{2}}
\end{array}\right) .
$$

We also introduce the notation $A=\left(\begin{array}{l|l}A_{1} & \\ \hline & A_{2}\end{array}\right), B=\left(\begin{array}{l|l}B_{1} & \\ \hline & B_{2}\end{array}\right)$, where $A_{i}, B_{i}$ are matrices of order $2 r_{i}+1$, for $i=1,2$.

Let $X=\left(x_{i j}\right)_{i, j=0, \ldots, N}$ be a matrix of unknowns. We write $X$ as a block matrix as follows:

$$
X=\left(\begin{array}{c|c}
X_{11} & X_{12} \\
\hline X_{21} & X_{22}
\end{array}\right)=\binom{\left(x_{i j}\right)_{i=0, \ldots, 2 r_{1}+1}}{j=0, \ldots, 2 r_{1}+1}\left(\begin{array}{rl}
\left(x_{i j}\right)_{\substack{i=0, \ldots, 2 r_{1}+1 \\
j=2 r_{1}+2, \ldots, N}} \\
\hline\left(x_{i j}\right)_{\substack{i=2 r_{1}+2, \ldots, N \\
j=0, \ldots, 2 r_{1}+1}} & \left(x_{i j}\right)_{i=2 r_{1}+2, \ldots, N}^{j=2 r_{1}+2, \ldots, N}
\end{array}\right)
$$

where $X_{i i}$ are square matrices of order $\left(2 r_{i}+1\right)$, and $X_{12}, X_{21}$ have order $\left(2 r_{1}+1\right) \times\left(2 r_{2}+1\right)$ and $\left(2 r_{2}+1\right) \times\left(2 r_{1}+1\right)$ respectively.

Then ${ }^{t} X A+A X$ and ${ }^{t} X B+B X$ can be written as block matrices as well, and precisely:

$$
{ }^{t} X A+A X=\left(\begin{array}{c|c}
{ }^{t} X_{11} A_{1}+A_{1} X_{11} & { }^{t} X_{21} A_{2}+A_{1} X_{12}  \tag{13}\\
\hline{ }^{t} X_{12} A_{1}+A_{2} X_{21} & { }^{t} X_{22} A_{2}+A_{2} X_{22}
\end{array}\right)
$$

and similarly for $B$. Lemma 8 implies that $X$ belongs to the Lie algebra of the stabilizer if and only if equations (8) are satisfied. We analyze separately what this means for the diagonal blocks $X_{11}, X_{22}$ and for the off-diagonal blocks $X_{12}, X_{21}$ of $X$.

Diagonal blocks. We use Proposition 4.5: $X_{11}, X_{22}$ must belong to the Lie algebras of the stabilizers of the orbits of $\mathcal{L}_{\left(r_{1}\right)}$ and $\mathcal{L}_{\left(r_{2}\right)}$ respectively, therefore each of them depends on 5 parameters and has the form described in Proposition 4.5. But equations (8) imply that the parameters appearing in $X_{11}$ and $X_{22}$ are not independent, and precisely, after fixing the 5 parameters required to describe $X_{11}$, an explicit computation shows that only one new parameter is needed to describe $X_{22}$, therefore the two diagonal blocks depend on a total of 6 parameters.

Off-diagonal blocks. The matrices ${ }^{t} X_{21} A_{2}+A_{1} X_{12}$ and ${ }^{t} X_{12} A_{1}+A_{2} X_{21}$ are the transpose of each other, and they both have to be the zero matrix. The same holds for ${ }^{t} X_{21} B_{2}+B_{1} X_{12}$ and ${ }^{t} X_{12} B_{1}+B_{2} X_{21}$.

From the explicit expressions of their entries, we get the following conditions:

$$
\begin{align*}
& x_{a, b}+x_{i, j}=0 \text { for any } 2 r_{1}+1 \leqslant a, j \leqslant 2 r, 0 \leqslant i, b \leqslant 2 r_{1}-1  \tag{14}\\
& \text { with }|b-i|=r_{1},|a-j|=r_{2}, \\
& x_{a, b}+x_{i, j}=0 \text { for any } 2 r_{1}+1 \leqslant a, j \leqslant 2 r+1,0 \leqslant i, b \leqslant 2 r_{1}  \tag{15}\\
& \text { with }|b-i|=r_{1}+1,|a-j|=r_{2}+1 .
\end{align*}
$$

We also get a first series of four vanishings, referring to the last and the central columns of $X_{12}$ and $X_{21}$ :
(i) the last column of $X_{12}$ except its last element:

$$
x_{0,2 r+1}=x_{1,2 r+1}=\cdots=x_{2 r_{1}-1,2 r+1}=0
$$

(ii) the central column of $X_{12}$, of index $2 r_{1}+r_{2}+1$, except its central element $x_{r_{1}, 2 r_{1}+r_{2}+1}$;
(iii) the last column of $X_{21}$ except its last element:

$$
x_{2 r_{1}+1,2 r_{1}}=x_{2 r_{1}+2,2 r_{1}}=\cdots=x_{2 r, 2 r_{1}}=0
$$

(iv) the central column of $X_{21}$, of index $r_{1}$, with the exception of its central element $x_{2 r_{1}+r_{2}+1, r_{1}}$.

The vanishing of these columns, together with conditions (14) and (15), implies, in order, the following second series of vanishings, referring to the rows of the two matrices:
(i) the row of index $2 r_{1}+r_{2}$ of $X_{21}$, except the element $x_{2 r_{1}+r_{2}, r_{1}-1}$; this is the row above the middle;
(ii) the first row of $X_{21}$ except its first element $x_{2 r_{1}+1,0}$;
(iii) the row of index $r_{1}-1$ of $X_{12}$ except $x_{r_{1}-1,2 r_{1}+r_{2}}$; this is the row above the middle;
(iv) the first row of $X_{12}$ except its first element $x_{0,2 r_{1}+1}$.

We now analyze separately the two cases (1) and (2) in our statement.
Case (1): when $r_{1}=r_{2}, X_{12}, X_{21}$ are square matrices. Going on with the argument above, we deduce that in both $X_{12}$ and $X_{21}$ all the elements above the central row and to the right of the central column are zero, except those of the main diagonal. Moreover, the first $r_{2}$ entries of the main diagonal of $X_{12}$ are equal to each other and also to the last $r_{2}$ elements of the main diagonal of $X_{21}$, and similarly the last $r_{2}$ elements of the main diagonal of $X_{12}$ are equal to each other and also to the first $r_{2}$ elements of the main diagonal of $X_{21}$.

We are left to analyze the two rectangles of order $\left(r_{2}+1\right) \times r_{2}$ in the lower left corner: from conditions (14) and (15) we get that they depend on $2 r_{2}$ parameters, independent of those previously considered. More precisely, we can divide each of the two rectangles into its $2 r_{2}$ anti-diagonals; each of them results to be formed by elements all equal to each other and to those of the same anti-diagonal of the other matrix.

All in all, there are $2+2 r_{2}$ independent parameters for this case (1). For the reader's convenience, we illustrated the case $(2,2)$ in Figure 1.

Case (2): assume now $r_{1}<r_{2}$. We obtain the vanishing of the entire first $r_{1}$ rows of $X_{12}$ and of the last $r_{1}+1$ columns of $X_{21}$. Now we need to look at the last $r_{1}+1$ rows of $X_{12}$ and the first $r_{1}$ columns of $X_{21}$. The former is divided into two blocks $\alpha_{12}$ and $\beta_{12}$ of size $\left(r_{1}+1\right) \times r_{2}$ and $\left(r_{1}+1\right) \times\left(r_{2}+1\right)$ respectively, while the latter is divided into two blocks $\alpha_{21}$ and $\beta_{21}$ of size $r_{2} \times r_{1}$ and $\left(r_{2}+1\right) \times r_{1}$ respectively. All entries in each of the $r_{2}+2$ anti-diagonals of $\alpha_{12}$ are equal to each other, and the same is true for the $r_{2}+2$ anti-diagonals of $\beta_{21}$. Moreover, these diagonals are paired, in the sense that they depend in order exactly on the same $r_{2}+2$ parameters. Finally, the same relations hold for the $r_{2}-1$ principal diagonals of the blocks $\beta_{12}$ and $\alpha_{21}$, with the difference that this time all entries above and below these $r_{2}-1$ principal diagonals are


Figure 1: Structure of the submatrices $X_{12}$ and $X_{21}$ in an element of the Lie algebra of the stabilizer of $\mathcal{L}_{(2,2)}$ : entries that are equal (up to a sign) are highlighted with the same color.
zero. (By "principal diagonal" we mean a maximal length diagonal with $r_{1}+1$ entries in $\beta_{12}$ and $r_{1}$ entries in $\alpha_{21}$.)

All in all, there are $\left(r_{2}+2\right)+\left(r_{2}-1\right)=2 r_{2}+1$ independent parameters for this case (2). Figure 2 illustrates the case (2,3).

Notice that the unknowns appearing in the on and off-diagonal blocks are independent from each other: this means that we only need to add the number of independent parameters coming from the off-diagonal blocks to the 6 ones needed for the diagonal blocks. This concludes the proof in both cases.

Proof of Theorem 4.1. We mimic and generalize the proof of Proposition 4.7. Given a pencil $\mathcal{L}_{\left(r_{1}, \ldots, r_{h}\right)}$ in the canonical form (4) and generated by $A$ and $B$, with obvious notation we write


To describe the matrices $X$ belonging to the Lie algebra of the stabilizer of $\mathcal{L}_{\left(r_{1}, \ldots, r_{h}\right)}$ we use Lemma 4.3. We write a general matrix of unknowns $X=$ $\left(x_{i j}\right)_{i, j=0, \ldots, N}$ as a block matrix with the same type of blocks $X_{i j}$ as above, each of size $\left(2 r_{i}+1\right) \times\left(2 r_{j}+1\right)$ :

$$
X=\left(\begin{array}{c|c|c|c}
X_{11} & X_{12} & \cdots & X_{1 h}  \tag{16}\\
\hline X_{21} & X_{22} & & \vdots \\
\hline \vdots & & \ddots & \vdots \\
\hline X_{1 h} & \ldots & \ldots & X_{h h}
\end{array}\right) .
$$

$$
X_{12}=\left(\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\begin{array}{|ccc|ccc|}
x_{2,5} & x_{3,6} & x_{2,7} & x_{2,8} & x_{2,9} & 0 \\
x_{3,5} & x_{3,6} & x_{3,7} & 0 & x_{3,9} & x_{3,10} \\
x_{4,5} & x_{4,6} & x_{4,7} & 0 & 0 & x_{4,10}
\end{array} & x_{4,11}
\end{array}\right) \quad X_{21}=\left(\begin{array}{ccc}
\begin{array}{|ccc|}
x_{5,0} & 0 \\
0 & 0 & 0 \\
x_{6,0} & x_{6,1} \\
0 & 0 & 0 \\
0 & x_{7,1}
\end{array} \\
0 & 0 & 0 \\
x_{8,0} & x_{8,1} \\
x_{9,0} & x_{9,1} & 0 \\
0 & 0 & 0 \\
x_{10,0} & x_{10,1} \\
x_{11,0} & x_{11,1} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

Figure 2: Structure of the submatrices $X_{12}$ and $X_{21}$ in an element of the Lie algebra of the stabilizer of $\mathcal{L}_{(2,3)}$ : again, the entries that are equal (up to a sign) are highlighted with the same color.

Then ${ }^{t} X A+A X$ can also be written as a block matrix, where the square blocks on the diagonal have the form

$$
{ }^{t} X_{i i} A_{i}+A_{i} X_{i i}
$$

while the off-diagonal ones with $i<j$ are

$$
{ }^{t} X_{j i} A_{j}+A_{i} X_{i j}
$$

and similarly for $B$. As in the proof of Proposition 4.7, the upper left diagonal block $X_{11}$ depends on 5 independent parameters, and each other diagonal block contributes with 1 more degree of freedom. This accounts for $5+(h-1)=4+h$ parameters. The off-diagonal blocks $X_{i j}$ and its symmetric $X_{j i}$ are in the same relation described for $X_{12}$ and $X_{21}$ in the proof of Proposition 4.7, so each pair accounts for $2 r_{j}+2$ if $r_{i}=r_{j}$, and $2 r_{j}+1$ if $r_{i}<r_{j}$.

As anticipated, now we make the key remark that the blocks $X_{i j}$ and $X_{k \ell}$ are independent for $(i, j) \neq(k, \ell)$, meaning that none of the variables $x_{p q}$ appear in two different blocks; therefore, the total number of parameters is

$$
4+h+\sum_{i<j}\left(2 r_{j}+1\right)+\#\left\{(i, j) \mid r_{i}=r_{j}\right\}
$$

and this concludes our proof.
To illustrate our result, we collected in Table 1 all orbits of pencils of quadrics of constant rank $2 r, r \leqslant 6$, their dimension, and the dimension of their stabilizer.

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| $r$ | $h$ | partition | $N=2 r+h-1$ | dim orbit | dim stabilizer |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | $(1)$ | 2 | 4 | 5 |
| 2 | 1 | $(2)$ | 4 | 20 | 5 |
|  | 2 | $(1,1)$ | 5 | 26 | 10 |
| 3 | 1 | $(3)$ | 6 | 44 | 5 |
|  | 2 | $(1,2)$ | 7 | 53 | 11 |
|  | 3 | $(1,1,1)$ | 8 | 62 | 19 |
| 4 | 1 | $(4)$ | 8 | 76 | 5 |
|  | 2 | $(2,2)$ | 9 | 88 | 12 |
|  | 2 | $(1,3)$ | 9 | 87 | 13 |
|  | 3 | $(1,1,2)$ | 10 | 100 | 21 |
|  | 4 | $(1,1,1,1)$ | 11 | 112 | 32 |
| 5 | 1 | $(5)$ | 10 | 116 | 5 |
|  | 2 | $(2,3)$ | 11 | 131 | 13 |
|  | 2 | $(1,4)$ | 11 | 129 | 15 |
|  | 3 | $(1,2,2)$ | 12 | 146 | 23 |
|  | 3 | $(1,1,3)$ | 12 | 144 | 25 |
|  | 4 | $(1,1,1,2)$ | 13 | 161 | 35 |
|  | 5 | $(1,1,1,1,1)$ | 14 | 176 | 49 |
| 6 | 1 | $(5)$ | 12 | 164 | 5 |
|  | 2 | $(3,3)$ | 13 | 182 | 14 |
|  | 2 | $(2,4)$ | 13 | 181 | 15 |
|  | 2 | $(1,5)$ | 13 | 179 | 16 |
|  | 3 | $(2,2,2)$ | 14 | 200 | 25 |
|  | 3 | $(1,2,3)$ | 14 | 199 | 26 |
|  | 3 | $(1,1,4)$ | 14 | 196 | 29 |
| 4 | $(1,1,2,2)$ | 15 | 218 | 38 |  |
|  | 4 | $15,1,1,3)$ | 15 | 215 | 40 |
|  | 5 | $1,1,1,1,2)$ | 16 | 236 | 53 |
|  | 6 | $(1,1,1,1,1,1)$ | 17 | 254 | 70 |
|  |  |  |  |  |  |

Table 1: Dimension of orbits of pencils of quadrics and their stabilizers.

Looking at Table 1, it is interesting to observe the phenomenon occurring when there are two different partitions of $r$ of the same length. As expected from the behaviour of a rational normal scroll $\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^{1}}\left(r_{1}\right) \oplus \mathcal{O}_{\mathbb{P}^{1}}\left(r_{2}\right)\right)$ degenerating to a $\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^{1}}\left(r_{1}-1\right) \oplus \mathcal{O}_{\mathbb{P}^{1}}\left(r_{2}+1\right)\right)$, the dimension of the relative orbit increases.

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# On the quadratic equations for odeco tensors 

Benjamin Biaggi, Jan Draisma, and Tim Seynnaeve

To Giorgio Ottaviani, on the occasion of his 60th birthday.


#### Abstract

Elina Robeva discovered quadratic equations satisfied by orthogonally decomposable ("odeco") tensors. Boralevi-Draisma-Horobef-Robeva then proved that, over the real numbers, these equations characterise odeco tensors. This raises the question to what extent they also characterise the Zariski-closure of the set of odeco tensors over the complex numbers. In the current paper we restrict ourselves to symmetric tensors of order three, i.e., of format $n \times n \times n$. By providing an explicit counterexample to one of Robeva's conjectures, we show that for $n \geq 12$, these equations do not suffice. Furthermore, in the open subset where the linear span of the slices of the tensor contains an invertible matrix, we show that Robeva's equations cut out the limits of odeco tensors for dimension $n \leq 13$, and not for $n \geq 14$. To this end, we show that Robeva's equations essentially capture the Gorenstein locus in the Hilbert scheme of $n$ points and we use work by Casnati-JelisiejewNotari on the (ir)reducibility of this locus.


Keywords: symmetric tensors, orthogonally decomposable tensors, Gorenstein algebras. MS Classification 2020: 15A69 Multilinear algebra, tensor calculus.

## 1. Introduction

In [11], Robeva discovered quadratic equations satisfied by orthogonally decomposable (odeco) tensors. In [1], it is proved that over the real numbers, these quadratic equations in fact characterise odeco tensors.

This raises the question whether Robeva's equations also define (the Zariski closure of) the set of complex odeco tensors. Indeed, Robeva conjectured that they might even generate the prime ideal of this Zariski closure, at least in the case of symmetric tensors [11, Conjecture 3.2]. She proved this stronger statement when the ambient space has dimension at most 3 [11, Figure 2]. In general, however, the answer to the (weaker) question is no, as already pointed out by Koiran in [7]. In this short paper, based on the first author's Master's thesis, we give an explicit symmetric tensor in $\left(\mathbb{C}^{12}\right)^{\otimes 3}$ that satisfies Robeva's
equations but is not approximable by complex odeco tensors. We do not know whether 12 is the minimal dimension for which this happens, but we show that if we impose a natural, additional open condition on the tensor, then Robeva's equations characterise the Zariski closure of the odeco tensors precisely up to dimension 13.

A key idea in [1] is to associate an algebra $A$ to a symmetric three-tensor $T$ and to realise that Robeva's equations express the associativity of that algebra. Furthermore, the symmetry of the tensor implies that $A$ is commutative and that the bilinear form is an invariant form on $A$; see Section 2.2 for definitions. If, in addition, $A$ contains a unit element - this turns out to be an open condition on $T$ - then $A$ is a Gorenstein algebra. Consequently, we can use the results of [2] on (ir)reducibility of the Gorenstein locus in the Hilbert scheme of points in affine space to study the variety defined by Robeva's equations.

In the opposite direction, we use this relation between algebras and tensors to give an elementary proof that the Gorenstein locus in the Hilbert scheme of $n$ points in $\mathbb{A}^{n}$ has a dimension that grows as $\Theta\left(n^{3}\right)$. This seems surprising at first, since the component containing the schemes consisting of $n$ distinct reduced points has dimension only $n^{2}$; on the other hand, it is well-known that the dimension of the Hilbert scheme itself does grow as a cubic function of $n$.

This relation between algebras and tensors is, of course, not new: a bilinear multiplication on $V$ can be thought of as an element of $V^{*} \otimes V^{*} \otimes V$, and in the presence of a bilinear form on $V$, the copies of $V^{*}$ may be identified with $V$. Further properties of the algebra, such as associativity, cut out subvarieties of the corresponding tensor space. A classical reference for varieties of algebras is [4], where the term algebraic geography is coined. Another, more closely related paper is [10], whose Remark 4.5 is closely related to Lemma 9.3, and together with [10, Theorem 9.2] gives the cubic lower bound on the dimension of the Hilbert scheme mentioned above. Finally, we note that the Zariski closure of the odeco tensors consists of tensors of minimal border rank; equations for these are studied in the recent paper [6]. In particular, [6, Proposition 1.4], which states that, in the 1 -generic locus, the $A$-Strassen equations are sufficient to characterise tensors of minimal border rank is closely related to our Theorem 2.13.

### 1.1. Organisation of this paper

In Section 2, we introduce the fundamental notions of this paper, including Robeva's equations, of which we show that they are the only quadrics that vanish on odeco tensors. We also state our main results (Theorems 2.10 and 2.13). In Section 3 we extend the well-known decomposition of finite-dimensional unital algebras into products of local algebras to the non-unital case. In Section 4 we recall that the Zariski closure of the odeco tensors is a component of the variety cut out by Robeva's equations; this was already established in [11,

Lemma 3.7]. In Section 5 we show that there are many weakly odeco tensors; combined with later results, this gives a lower bound on the dimension of the Gorenstein locus in the Hilbert scheme of $n$ points in $\mathbb{A}^{n}$. In Section 6 we show how to unitalise algebras along with an invariant bilinear form to turn them into a local Gorenstein algebra, and vice versa. In Section 7 we use this construction to motivate the search for nilpotent counterexamples to Robeva's conjecture. Then, in Section 8 we find such a counterexample for $n=12$. In Section 9 we make the connection with the Gorenstein locus in the Hilbert scheme of $n$ points and prove our second main result - that a version of Robeva's conjecture holds in the (open) unital locus precisely up to $n=13$. Finally, in Section 9.5 we show that the dimension $d(n)$ of that Gorenstein locus is lower-bounded bounded by a cubic polynomial in $n$. Since it is also trivially upper-bounded by such a polynomial, we have that $d(n)=\Theta\left(n^{3}\right)$.

### 1.2. Acknowledgement

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## 2. Set-up

### 2.1. Weakly and strongly odeco tensors

Let $V_{\mathbb{R}}$ be a finite-dimensional real vector space equipped with a positivedefinite inner product (.|.).

Definition 2.1. A symmetric tensor $T \in S^{3} V_{\mathbb{R}} \subseteq V_{\mathbb{R}} \otimes V_{\mathbb{R}} \otimes V_{\mathbb{R}}$ is called orthogonally decomposable (odeco, for short) if, for some integer $k \geq 0, T$ can be written as

$$
T=\sum_{i=1}^{k} v_{i}^{\otimes 3}
$$

where $v_{1}, \ldots, v_{k} \in V_{\mathbb{R}}$ are nonzero, pairwise orthogonal vectors. We write $Y\left(V_{\mathbb{R}}\right) \subseteq S^{3} V_{\mathbb{R}}$ for the set of odeco tensors.

Positive-definitiveness of the form implies that $\left(v_{i} \mid v_{i}\right)>0$ for each $i$, so that $v_{1}, \ldots, v_{k}$ are linearly independent. Hence $k \leq n:=\operatorname{dim}\left(V_{\mathbb{R}}\right)$, and $Y\left(V_{\mathbb{R}}\right)$ is a semi-algebraic set of dimension at most $\binom{n}{2}+n$ : the dimension of the orthogonal group plus $n$ degrees of freedom for scaling. It turns out that $Y\left(V_{\mathbb{R}}\right)$ is in fact the set of real points of an algebraic variety defined by quadratic equations that we will discuss below. Furthermore, the $k$ and the $v_{i}$ in the decomposition of an odeco tensor $T$ are unique [1, Proposition 7], which implies that the dimension of $Y\left(V_{\mathbb{R}}\right)$ is precisely $\binom{n}{2}+n$.

We now set $V:=\mathbb{C} \otimes V_{\mathbb{R}}$ and extend (.|.) to a complex symmetric bilinear form (not a Hermitian form; that setting is studied in [1] under the name udeco). We note that the extended bilinear form is nondegenerate, as the inner product (.|.) is positive-definite on the real vector space $V_{\mathbb{R}}$.

Definition 2.2. A symmetric tensor $T \in S^{3} V \subseteq V \otimes V \otimes V$ (where the tensor product is over $\mathbb{C}$ ) is weakly odeco if $T$ can be written as

$$
T=\sum_{i=1}^{k} v_{i}^{\otimes 3}
$$

where the $v_{i}$ are nonzero pairwise orthogonal vectors. It is called strongly odeco if $T$ admits such a decomposition where, in addition, $\left(v_{i} \mid v_{i}\right) \neq 0$. We write $Y(V) \subseteq S^{3} V$ for the Zariski closure of the set of strongly odeco tensors.

Remark 2.3. The set called $\operatorname{SODECO}_{n}(\mathbb{C})$ in [7] consists of the strongly odeco tensors in $S^{3} \mathbb{C}^{n}$. Koiran proves that these are precisely the set of symmetric tensors whose $n \times n$ slices are diagonalisable and commute.

As pointed out above, every element of $Y\left(V_{\mathbb{R}}\right)$, regarded as an element of $S^{3} V$, is strongly odeco. Since the real orthogonal group $O\left(V_{\mathbb{R}}\right)$ is Zariski dense in the complex orthogonal group $O(V), Y(V)$ is the Zariski closure of $Y\left(V_{\mathbb{R}}\right)$. On the other hand, due to the presence of isotropic vectors and higherdimensional spaces in $V$, the set of weakly odeco tensors strictly contains the set of strongly odeco tensors; we will return to this theme shortly. First we give the easiest example that shows the need for a Zariski closure in the definition of $Y(V)$.
Example 2.4. Consider the vector space $V=\mathbb{C}^{2}$ equipped with the symmetric bilinear form for which $\left(e_{1} \mid e_{2}\right)=1$ and all other products are zero. (Of course, since all nondegenerate symmetric bilinear forms on a finite-dimensional complex vector space are equivalent, we could have changed coordinates such that the bilinear form is the standard form.) Then

$$
\begin{aligned}
S & =e_{1} \otimes e_{2} \otimes e_{2}+e_{2} \otimes e_{1} \otimes e_{2}+e_{2} \otimes e_{2} \otimes e_{1} \\
& =\frac{1}{2} \lim _{t \rightarrow 0}\left[\left(t^{2} e_{1}+t^{-1} e_{2}\right)^{\otimes 3}+\left(t^{2} e_{1}-t^{-1} e_{2}\right)^{\otimes 3}\right]
\end{aligned}
$$

shows that $S$ is a limit of strongly odeco tensors and so $S \in Y(\mathbb{C})$. To see that the tensor $S$ is not strongly odeco, it is enough to show that its tensor rank is 3 . For order three tensors, the rank of the tensor equals the number of rank one matrices needed to span a space containing the space generated by the slices of the tensor [8, Theorem 3.1.1.1]. The tensor $S$ has slices $\left(\begin{array}{ll}0 & 0 \\ 0 & 1\end{array}\right),\left(\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right)$, which are not contained in a subspace spanned by two rank 1 matrices, so $S$ has rank 2.

### 2.2. Commutative algebras from symmetric tensors

We now associate an algebra structure on $V$ to a tensor.
Definition 2.5. We identify $V$ with $V^{*}$ via the map $v \mapsto(v \mid$.). Then any element $T=\sum_{i=1}^{k} u_{i} \otimes v_{i} \otimes w_{i} \in V^{\otimes 3}$ can also be regarded as an element of $V^{*} \otimes V^{*} \otimes V$, hence defines a bilinear map

$$
\mu_{T}: V \times V \rightarrow V ;(x, y) \mapsto \mu_{T}(x, y)=\sum_{i=1}^{k}\left(u_{i} \mid x\right)\left(v_{i} \mid y\right) w_{i}
$$

We call $V$ with $\mu_{T}$ the algebra associated to $T$.
If $T$ is symmetric, then, first, $\mu_{T}$ is commutative, and second, $\mu_{T}$ satisfies

$$
\left(\mu_{T}(x, y) \mid z\right)=\left(x \mid \mu_{T}(y, z)\right)
$$

i.e., the bilinear form (.|.) is invariant for the multiplication $\mu_{T}$. When $T$ is fixed in the context, then we will often just write $x y$ instead of $\mu_{T}(x, y)$.

### 2.3. Odeco implies associative

Proposition 2.6. If $T \in S^{3} V$ is weakly odeco, then $\mu_{T}$ is associative.
This was observed in [1] in the real case, but the argument easily generalises, as follows.

Proof. Write

$$
T=\sum_{i=1}^{k} v_{i}^{\otimes 3}
$$

where the $v_{i}$ are pairwise orthogonal. Let $x, y, z \in V$. Then

$$
\begin{aligned}
\mu_{T}\left(x, \mu_{T}(y, z)\right) & =\mu_{T}\left(x, \sum_{i=1}^{k}\left(y \mid v_{i}\right)\left(z \mid v_{i}\right) v_{i}\right)=\sum_{j=1}^{k} \sum_{i=1}^{k}\left(y \mid v_{i}\right)\left(z \mid v_{i}\right)\left(x \mid v_{j}\right)\left(v_{i} \mid v_{j}\right) v_{j} \\
& =\sum_{i=1}^{k}\left(y \mid v_{i}\right)\left(z \mid v_{i}\right)\left(x \mid v_{i}\right)\left(v_{i} \mid v_{i}\right) v_{i}
\end{aligned}
$$

where the last equality uses that $\left(v_{i} \mid v_{j}\right)=0$ whenever $i \neq j$. A similar computation for $\mu_{T}\left(\mu_{T}(x, y), z\right)$ yields the exact same result.

### 2.4. Robeva's equations

For any fixed $x, y, z$ (e.g. chosen from a basis of $V$ ), the condition that $\mu_{T}\left(x, \mu_{T}(y, z)\right)$ equals $\mu_{T}\left(\mu_{T}(x, y), z\right)$ translates into $n$ quadratic equations for $T$. All these equations together, with varying $x, y, z$, are called Robeva's equations.

Definition 2.7. We denote by $X(V) \subseteq S^{3} V$ the affine variety defined by Robeva's equations, i.e.,

$$
X(V):=\left\{T \in S^{3} V \mid \mu_{T} \text { is associative }\right\} .
$$

To prove that Robeva's equations are all quadratic equations satisfied by strongly odeco tensors, we reinterpret them as follows. The condition that $\mu_{T}\left(x, \mu_{T}(y, z)\right)$ equals $\mu_{T}\left(\mu_{T}(x, y), z\right)$ means that for every $w \in V$, we have

$$
\left(\mu_{T}\left(x, \mu_{T}(y, z)\right) \mid w\right)=\left(\mu_{T}\left(\mu_{T}(x, y), z\right) \mid w\right)
$$

which can be rewritten as

$$
\begin{equation*}
\left(\mu_{T}(y, z) \mid \mu_{T}(x, w)\right)=\left(\mu_{T}(x, y) \mid \mu_{T}(z, w)\right) . \tag{1}
\end{equation*}
$$

In other words: Robeva's equations precisely express that the 4 -linear map

$$
\begin{aligned}
& T \bullet T: V^{4} \rightarrow \mathbb{C} \\
& (x, y, z, w) \mapsto\left(\mu_{T}(x, y) \mid \mu_{T}(z, w)\right)
\end{aligned}
$$

is invariant under arbitrary permutations of $(x, y, z, w)$. This was, in fact, Robeva's original description of these quadratic equations in [11].

Proposition 2.8. The only quadratic equations vanishing on $Y(V)$ are Robeva's equations.

Proof. If we consider the natural map

$$
\begin{aligned}
S^{2}\left(S^{3} V\right) & \rightarrow S^{2}\left(S^{2} V\right) \\
\left(v_{1} \otimes v_{2} \otimes v_{3}\right) \otimes\left(w_{1} \otimes w_{2} \otimes w_{3}\right) & \mapsto\left(v_{1} \mid w_{1}\right)\left(v_{2} \otimes v_{3}\right) \otimes\left(w_{2} \otimes w_{3}\right)
\end{aligned}
$$

then for any $T \in S^{3} V$, we can identify $T \bullet T$ with the image of $T$ under the composition

$$
\begin{equation*}
S^{3} V \xrightarrow{\nu_{2}} S^{2} S^{3} V \rightarrow S^{2} S^{2} V \tag{2}
\end{equation*}
$$

where $\nu_{2}$ is the second Veronese embedding. Then $T$ satisfies Robeva's equations if and only if $T \bullet T \in S^{4} V \subseteq S^{2} S^{2} V$.

Since quadratic equations on $Y(V)$ correspond to linear equations on $\nu_{2}(Y(V))$, we want to show that the image of (2), when restricted to $Y(V)$, linearly spans $S^{4} V$. But (2) maps an odeco tensor $\sum_{i=1}^{k} v_{i}^{\otimes 3}$ to the odeco tensor $\sum_{i=1}^{k}\left(v_{i} \mid v_{i}\right) v_{i}^{\otimes 4}$, and these tensors clearly span $S^{4} V$.

### 2.5. The main question

By Proposition 2.6, $X(V)$ contains weakly odeco tensors, hence in particular the variety $Y(V)$. On the other hand, the results of [1] imply that the set of real points of $X(V)$ equals $Y\left(V_{\mathbb{R}}\right)$. This raises the following question.

Question 2.9. For which dimensions $n=\operatorname{dim}(V)$ is $X(V)$ equal to $Y(V)$ ?
Our partial answer to this question is as follows.
Theorem 2.10. For $n=\operatorname{dim}(V) \leq 3, X(V)$ equals $Y(V)$. For $n \geq 12$, we have $Y(V) \subsetneq X(V)$.

Proof. See Section 8.
In fact, the result for $n \leq 3$ is due to Robeva [11, Figure 2]. Our contribution is a counterexample to [11, Conjecture 3.2] for $n=12$.

### 2.6. The existence of a unit

The answer in Theorem 2.10 is unsatisfactory because of the large interval of dimensions $n=\operatorname{dim}(V)$ for which we do not know whether $X(V)$ is strictly larger than $Y(V)$. However, in a certain open subset of $S^{3} V$, we do know precisely where the two stop being equal.

Lemma 2.11. The condition on $T \in X(V)$ that $\mu_{T}$ has a multiplicative unit element is equivalent to the condition that there exists a $x \in V$ such that the multiplication map $L_{x}: v \mapsto \mu_{T}(x, v)$ is invertible. This is an open condition on $T$.

In the case of ordinary tensors, the analoguous condition is called 1-genericity; see, e.g., [6].

Proof. We write $x v$ instead of $\mu_{T}(x, v)$. For the implication $\Rightarrow$ take $x$ to be the unit element. For the implication $\Leftarrow$, assume that $L_{x}$ is invertible. Then in particular there exists an $e \in V$ such that $e x=x e=x$. We then find, for any $y \in V$, that

$$
e y=e x L_{x}^{-1}(y)=x L_{x}^{-1}(y)=y
$$

so that $e$ is a unit element.
It follows that $\left(V, \mu_{T}\right)$ is not unital if and only if $\operatorname{det}\left(L_{x}\right)=0$ for all $x \in V$; this is a system of degree $n$ polynomial equations on $T \in S^{3}(V)$ defining the complement of the unital locus.

We will refer to the variety in $S^{3}(V)$ defined by the degree $n$ equations in the proof above as the non-invertibility locus. If $T \in S^{3}(V)$ is not in the non-invertibility locus, then $\mu_{T}$ needs not have a unit element; but it does if
furthermore $T$ lies in $X(V)$ - we have used associativity of $\mu_{T}$ in the proof above.

We define $X^{0}(V)$ as

$$
X^{0}(V):=\left\{T \in X(V) \mid \mu_{T} \text { is unital }\right\},
$$

and similarly for $Y^{0}(V)$. Note that $\overline{Y^{0}(V)}=Y(V)$. This leads to the following weakening of Question 2.9.
Question 2.12. For which dimensions $n=\operatorname{dim}(V)$ is $X^{0}(V)$ equal to $Y^{0}(V)$ ? In other words, for which dimensions do Robeva's quadrics characterise the set of limits of strongly odeco tensors in the complement of the non-invertibility locus?

To answer this question, we will prove the following theorem.
Theorem 2.13. The number of irreducible components of $X^{0}\left(\mathbb{C}^{n}\right)$ equals that of the Gorenstein locus in the Hilbert scheme of $n$ points in $\mathbb{A}_{\mathbb{C}}^{n}$.
Proof. See Section 9.
We can now make use of the following result on the irreducibility of Gorenstein loci of Hilbert schemes.
Theorem 2.14 ([2]). The Gorenstein locus of $n$ points in $\mathbb{A}_{\mathbb{C}}^{d}$

- is irreducible if $n \leq 13$, or if $n=14$ and $d \leq 5$.
- has 2 irreducible components if $n=14$ and $d \geq 6$.

Note that the second item implies reducibility of the Gorenstein locus of $n$ points in $\mathbb{A}_{\mathbb{C}}^{d}$ for all $n \geq 14$ if $d \geq 6$. Indeed, reducibility of the Gorenstein locus means existence of a non-smoothable point, and if one finds such a point for given pair $(n, d)$, then one can always add to this point more disjoint points (increase $d$ ) or embed it in higher dimensions (increase $n$ ).

Combining the two previous theorems together with the fact that $Y\left(\mathbb{C}^{n}\right)$ is an irreducible component of $X\left(\mathbb{C}^{n}\right)$, which is proven in Section 4, gives a complete answer to Question 2.12.
Corollary 2.15. The locus $X^{0}\left(\mathbb{C}^{n}\right)$ is irreducible and equal to $Y^{0}\left(\mathbb{C}^{n}\right)$ for $n \leq 13$, and is not irreducible and not equal to $Y^{0}\left(\mathbb{C}^{n}\right)$ for $n \geq 14$.

## 3. Decomposing (the algebra of) a tensor in $X(V)$

### 3.1. Motivation

Recall that if $B$ is a unital finite-dimensional algebra over $\mathbb{C}$, then $B$ is isomorphic to a product $B_{1} \times \cdots \times B_{k}$ of local algebras; here $k=|\operatorname{Spec}(B)|$. In this section we want to establish a similar decomposition for not necessarily unital algebras.

### 3.2. The unital/nilpotent decomposition

Proposition 3.1. Let $T \in X(V)$ and equip $V$ with the corresponding commutative, associative multiplication $\mu_{T}$. Then $V$ has a unique decomposition as a direct sum

$$
V_{1} \oplus \cdots \oplus V_{k} \oplus N
$$

where the $V_{i}$ are nonzero and $N$ is potentially zero, such that $N$ and each $V_{i}$ is an ideal in $\left(V, \mu_{T}\right),\left(V_{i},\left.\mu_{T}\right|_{V_{i} \times V_{i}}\right)$ is a local unital algebra for each $i$, and $N$ is a nilpotent algebra. Furthermore, this unique direct sum decomposition is orthogonal.

Accordingly, $T$ decomposes as $T_{1}+\cdots T_{k}+T_{N}$ with $T_{i} \in X\left(V_{i}\right)$ and $T_{N} \in$ $X(N)$; and we have $T \in Y(V)$ if and only if $T_{i} \in Y\left(V_{i}\right)$ for all $i$ and $T_{N} \in$ $Y(N)$.

Proof. Note that if $a, b \in V$ belong to different factors in any decomposition of $V$ as a direct sum of ideals, and if the first factor has unit element $e$, then

$$
(a \mid b)=(a e \mid b)=(e \mid a b)=(e \mid 0)=0
$$

this proves the orthogonality of the decomposition.
That a unital finite-dimensional commutative, associative algebra $V$ has a unique product decomposition into local algebras is well-known - it is found by taking a decomposition of 1 into minimal idempotents $e_{i}$ satisfying $e_{i} e_{j}=\delta_{i j} e_{i}$ and taking $V_{i}:=e_{i} V$.

To reduce to the unital case, we proceed as follows. If $V$ is nilpotent, then we set $k:=0$ and $N:=V$. Otherwise, there exists an element $x \in V$ that is not nilpotent. Let $L_{x}: V \rightarrow V$ be multiplication with $x$. Then there exists an $m$ such that $\operatorname{im} L_{x}^{m}=\operatorname{im} L_{x}^{m+1}=\ldots$. Set $y:=x^{m}$ and $W:=y V$. Then $\left.\left(L_{y}\right)\right|_{y V}$ is invertible, so the ideal $y V$ is unital by Lemma 2.11. On the other hand, we have $V=y V \oplus \operatorname{ker}\left(L_{y}\right)$ : indeed, the dimensions of $y V=\operatorname{im}\left(L_{y}\right), \operatorname{ker}\left(L_{y}\right)$ add up to $n$, and if $L_{y}(y v)=0$, then $L_{y}^{2} v=0$, so already $L_{y} v=0$, so $y v=0$.

Now ker $L_{y}$ has strictly lower dimension than $V$, and by induction we know that ker $L_{y}$ is the direct sum of a nilpotent ideal $N$ and an ideal $I$ that is unital as an algebra. Then so is $V$ : it equals the direct sum of the ideals $N$ and $y V \oplus I$, where the latter is unital as an algebra. The decomposition of $V$ into a unital ideal $V_{0}$ and a nilpotent ideal $V_{1}$ is unique, because $V_{0}$ is the space of elements $v \in V$ for which there is an idempotent $e \in V$ with $v \in \operatorname{im} L_{e}$.

The statements about $T$ are straightforward from the fact that $\mu_{T}$ is the sum of its restrictions to the ideals $V_{i}$ and $N$. Here we note that the restrictions of (.|.) to the $V_{i}$ and to $N$ are non-degenerate, so that the notation $X\left(V_{i}\right), X(N), Y\left(V_{i}\right), Y(N)$ make sense.

## 4. $Y(V)$ is a component of $X(V)$

### 4.1. Motivation

While, as we will see, $Y(V)$ is in general not equal to $X(V)$, at least it is an irreducible component. This was already observed in [11]; we paraphrase the argument here.

### 4.2. A tangent space computation

Here, and later in the paper, we will write $e_{1}, \ldots, e_{n}$ for the standard basis of $\mathbb{C}^{n}$.

Proposition 4.1. For each $V$ equipped with a nondegenerate symmetric bilinear form, the variety $Y(V)$ of limits of strongly odeco tensors is an irreducible component of the variety $X(V)$ defined by Robeva's quadrics.

Proof. It suffices to prove that for a suitable tensor $T_{0} \in Y(V)$, the tangent space to $X(V)$ at $T_{0}$ has dimension equal to $\operatorname{dim}(Y(V))=\binom{n+1}{2}$. We take $V=\mathbb{C}^{n}$ and $T_{0}=E=\sum_{i=1}^{n} e_{i}^{\otimes 3}$.

Let us first write Robeva's equations in coordinates: writing

$$
T=\sum_{i, j, k} T_{i j k} e_{i} \otimes e_{j} \otimes e_{k}
$$

equation (1) becomes

$$
\begin{equation*}
\sum_{r} T_{j k r} T_{i \ell r}=\sum_{r} T_{i j r} T_{k \ell r} \tag{3}
\end{equation*}
$$

The equations defining the tangent space at $E$ are given by substituting $T=E+\varepsilon X$ in (3) and taking the coefficients of $\varepsilon$ :

$$
\begin{equation*}
\delta_{j k} X_{i j \ell}+\delta_{i \ell} X_{i j k}=\delta_{i j} X_{i k \ell}+\delta_{k \ell} X_{i j k} \tag{4}
\end{equation*}
$$

where $\delta_{i j}$ denotes the Kronecker delta.
By taking $i=\ell \neq j \neq k \neq i$ in (4) we find that $X_{i j k}=0$ for $i, j, k$ pairwise distinct, and by taking $i=\ell \neq j=k$ we find that $X_{i i j}=-X_{i j j}$ for all $i \neq j$. But this implies that our tangent space has dimension at most $n+\binom{n}{2}=\binom{n+1}{2}$.

REmARK 4.2. In a similar manner, one finds that all strongly odeco tensors of tensor rank $n$ are smooth points of $Y(V)$.

## 5. Many weakly odeco tensors

### 5.1. Motivation

In this section, we give our first negative answer to Question 2.9 by showing that, for $n$ sufficiently large, there are many more weakly odeco tensors than strongly odeco tensors.

### 5.2. Weakly odeco tensors from isotropic spaces

Recall that $V$ is a complex vector space of dimension $n$ equipped with a symmetric bilinear form (.|.).

Proposition 5.1. The variety $X(V)$ contains the union over all (maximal) isotropic subspaces $U \subseteq V$ of $S^{3} U$. This union is an affine variety $Z(V) \subseteq S^{3} V$ of dimension

$$
\binom{\lfloor n / 2\rfloor+2}{3}+\binom{\lceil n / 2\rceil}{ 2} .
$$

Proof. For the first statement, note that if $u_{1}, \ldots, u_{k}$ are elements of an isotropic subspace $U$ of $V$, then $\sum_{i} u_{i}^{\otimes 3}$ is weakly odeco, hence in $X(V)$ by Proposition 2.6.

There is no harm in restricting our attention to maximal isotropic subspaces, i.e., those of dimension $\lfloor n / 2\rfloor$. Hence $Z(V)$ is the projection of the incidence variety

$$
\left\{(U, T) \in \operatorname{Gr}_{\text {iso }}(\lfloor n / 2\rfloor, V) \times S^{3} V \mid T \in S^{3} U\right\}
$$

onto the second factor. Since the isotropic Grassmannian is a projective variety, $Z(V)$ is closed. Furthermore, for $U \subseteq V$ isotropic of dimension $\lfloor n / 2\rfloor$ and $T \in S^{3} U$ concise, i.e. such that the associated linear map $S^{2} U^{*} \rightarrow U$ is surjective, the fibre over $T$ is the single point $(U, T)$, hence $\operatorname{dim}(Z(V))$ equals the dimension of the isotropic Grassmannian, which is the second term above, plus the dimension of $S^{3} U$ for a fixed isotropic $U \subseteq V$ of dimension $\lfloor n / 2\rfloor$, which is the first term.

Remark 5.2. Clearly, $\operatorname{dim}(Z(V))$ grows as a cubic (quasi-)polynomial in $n$, whereas $\operatorname{dim}(Y(V))$ is a quadratic polynomial in $n$. Since $X(V) \supseteq Z(V)$, this shows that $X(V) \supsetneq Y(V)$ for all $V$ of sufficiently high dimension. In fact, $\operatorname{dim}(Z(V))>\operatorname{dim}(X(V))$ for $n \geq 16$. However, we will show with an explicit example that $X(V) \supsetneq Y(V)$ holds already for $n \geq 12$ (and possibly already for smaller $n$ ).
Remark 5.3. The variety $Z(V)$ consists precisely of the tensors $T \in S^{3} V$ whose algebra is 2-step nilpotent:

$$
T \in Z(V) \Longleftrightarrow \mu_{T}\left(x, \mu_{T}(y, z)\right)=0 \quad \forall x, y, z \in V
$$

One implication is clear: if $T \in Z(V)$, we can write $T=\sum_{i} u_{i}^{\otimes 3}$ with the $u_{i}$ isotropic and pairwise orthogonal, and the computation from the proof of Proposition 2.6 gives that $\mu_{T}\left(x, \mu_{T}(y, z)\right)=0$. For the other direction we can work in coordinates: write $V=\mathbb{C}^{n}$, then the condition

$$
\mu_{T}\left(e_{i}, \mu_{T}\left(e_{j}, e_{k}\right)\right)=0 \quad \forall i, j, k \in\{1, \ldots, n\}
$$

is equivalent to

$$
\sum_{r=1}^{n} T_{i j r} T_{k \ell r}=0 \quad \forall i, j, k, \ell \in\{1, \ldots, n\}
$$

But this means that that the space $U$ spanned by the columns of $T$ is isotropic.

## 6. Unitalisation and de-unitalisation

### 6.1. Motivation

It is well known that if an associative algebra $A$, say over $\mathbb{C}$, has no multiplicative unit element, then one can turn $A$ into a unital associative algebra by setting $A^{\prime}:=\mathbb{C} 1 \oplus A$ and extending the multiplication on $A$ to $A^{\prime}$ via $1 a^{\prime}:=a^{\prime}$ for all $a^{\prime} \in A^{\prime}$. In this section, we describe a process that also extends an invariant bilinear form.

### 6.2. Unitalising algebras with invariant forms

Let $A$ be an associative algebra over $\mathbb{C}$ equipped with a bilinear form (.|.) such that $(a b \mid c)=(a \mid b c)$ for all $a, b, c \in A$. We do not require $A$ to be commutative or (.|.) to be symmetric.

We construct a new algebra

$$
\widetilde{A}:=\mathbb{C} 1 \oplus A \oplus \mathbb{C} y
$$

with multiplication determined by

$$
\begin{aligned}
& 1 * x:=x, x * 1:=x \text { for all } x \in \widetilde{A}, \\
& a * a^{\prime}:=a a^{\prime}+\left(a \mid a^{\prime}\right) y \text { for all } a, a^{\prime} \in A, \\
& a * y:=0, y * a=0 \text { for all } a \in A, \text { and } \\
& y * y:=0
\end{aligned}
$$

We also extend the form (.|.) to $\widetilde{A}$ by requiring that

$$
(1 \mid 1)=(a \mid 1)=(1 \mid a)=(a \mid y)=(y \mid a)=(y \mid y)=0 \text { for all } a \in A
$$

and $(1 \mid y)=(y \mid 1)=1$.

Remark 6.1. Let us consider the special case where the multiplication on $A$ is identically zero. If we let $a_{1}, \ldots, a_{n}$ be an orthonormal basis of $A$, then the tensor associated to $\tilde{A}$ is equal to
$\sum_{i=1}^{n}\left(1 \otimes a_{i} \otimes a_{i}+a_{i} \otimes 1 \otimes a_{i}+a_{i} \otimes a_{i} \otimes 1\right)+1 \otimes 1 \otimes y+1 \otimes y \otimes 1+y \otimes 1 \otimes 1$.
This tensor is known as the Coppersmith-Winograd tensor [3]; it has played a central role in the literature on the complexity of matrix multiplication. We refer the reader to [9] (in particular Chapter 3.4.9) for an overview. In the notation of the latter reference, the above tensor is denoted $T_{n, C W}$.

Proposition 6.2. The algebra $\widetilde{A}$ is associative, and the form (.|.) on $\widetilde{A}$ is invariant. Furthermore, if (.|.) is nondegenerate or symmetric on $A$, then its extension to $\widetilde{A}$ has the same property; and if $A$ is commutative and (.|.) is symmetric, then $\widetilde{A}$ is commutative.

Proof. It suffices to prove the identity $a *(b * c)=(a * b) * c$ for $a, b, c$ ranging over a spanning set of $\widetilde{A}$. If at least one of $a, b, c$ is 1 , then the identity is immediate. If none of them is 1 and at least one of them is $y$, then both sides are zero. So the interesting case is the case where $a, b, c$ are all in $A$. Then we have

$$
a *(b * c)=a *(b c+(b \mid c) y)=a(b c)+(a \mid b c) y+0=a(b c)+(a \mid b c) y
$$

and

$$
(a * b) * c=(a b+(a \mid b) y) * c=(a b) c+(a b \mid c) y+0=(a b) c+(a b \mid c) y
$$

These two expressions are equal by associativity of $A$ and invariance of (.|.) on $A$.

Now we turn to the identity $(a * b \mid c)=(a \mid b * c)$ for $a, b, c$ ranging over the same spanning set. If $b=1$, then the identity is immediate. If $b \in A \oplus \mathbb{C} y$ and $a=1$, then the identity reads

$$
(b \mid c)=(1 \mid b * c)
$$

Now the right-hand side is the coefficient of $y$ in $b * c$. Write $b=b^{\prime}+\beta y$ and $c=\gamma 1+c^{\prime}+\delta y$ with $\beta, \gamma, \delta \in \mathbb{C}$ and $b^{\prime}, c^{\prime} \in A$. Then the coefficient of $y$ in $b * c$ equals $\beta \gamma+\left(b^{\prime} \mid c^{\prime}\right)$, and this also equals $(b \mid c)$. Since $a, c$ play symmetric roles, the identity also holds when $c=1$. So we are left with the case where $a, b, c \in A \oplus \mathbb{C} y$. But then, since $y$ is perpendicular to $A$, we have

$$
(a * b \mid c)=(a b \mid c)=(a \mid b c)=(a \mid b * c)
$$

as desired.
That the extension of (.|.) inherits the properties of symmetry and nondegeneracy is immediate, and so is the statement about the commutativity of $\widetilde{A}$.

Remark 6.3. The space $M:=A \oplus \mathbb{C} y$ is a maximal ideal in $\widetilde{A}$, and in particular a non-unital subalgebra of $\widetilde{A}$. This subalgebra has an ideal $\mathbb{C} y$, and the natural map $A \rightarrow M / \mathbb{C} y$ is an isomorphism of algebras. We will use this construction below to de-unitalise a local Gorenstein algebra in a canonical manner.

Lemma 6.4. Suppose that $A$ is commutative and that (.|.) is symmetric. Then $A$ is nilpotent if and only if $\widetilde{A}$ is (unital and) local.
Proof. If $A$ is nilpotent, then $M$ consists of elements that are nilpotent in $\widetilde{A}$, and hence any element not in $M$ is invertible. Conversely, if $\widetilde{A}$ is local, then $M$ is the unique maximal ideal and its elements are nilpotent. This implies that $A \cong M / \mathbb{C} y$ is nilpotent.

We now show that each local, unital algebra with an invariant bilinear form arises as $\widetilde{A}$ for some $A$ equipped with a bilinear form.

Proposition 6.5. Let $B$ be a commutative, local, unital, finite-dimensional algebra with $\operatorname{dim}(B) \geq 2$, equipped with a nondegenerate invariant symmetric bilinear form. Then $B \cong \widetilde{A}$ for some nilpotent algebra $A$ equipped with a nondegenerate invariant symmetric bilinear form.

Proof. Let $M$ be the maximal ideal of $B$, and let $d$ be maximal such that $M^{d}$ is nonzero. Then $d \geq 1$ since $\operatorname{dim}(B) \geq 2$.

We claim that $M^{d}$ is one-dimensional. Indeed, if it were at least twodimensional, then $1^{\perp} \cap M^{d}$ would contain a nonzero element $x$. This element would satisfy $(x \mid 1)=0$ and $(x \mid z)=(x z \mid 1)=(0 \mid 1)=0$ for all $z \in M$, contradicting the non-degeneracy of (.|.).

Choose a spanning vector $z \in M^{d}$. Then $(1 \mid z) \neq 0$, and hence we may replace $z$ by a (unique) scalar multiple with $(1 \mid z)=1$. Furthermore, $z^{\perp}=M$. We define $A$ as the algebra $M / \mathbb{C} z$ equipped with the induced symmetric bilinear form. We claim that $\widetilde{\sim} \cong B$ as algebras with bilinear forms. The isomorphism $\phi: \widetilde{A} \rightarrow B$ sends $1 \in \widetilde{A}$ to $1 \in B, y \in \widetilde{A}$ to $z \in B$ and $m \in A$ to the unique element $m^{\prime} \in m+\mathbb{C} z \subseteq B$ that satisfies $\left(1 \mid m^{\prime}\right)=0$ in $B$. All checks are then straightforward.

Now let $V$ be an $n$-dimensional complex vector space equipped with a nondegenerate symmetric bilinear form (.|.). Define $\widetilde{V}:=\mathbb{C} 1 \oplus V \oplus \mathbb{C} y$, equipped with the symmetric bilinear form as above.

Let $T \in X(V)$ and let $V=V_{1} \oplus \cdots \oplus V_{k} \oplus N$ be the decomposition of Proposition 3.1. Let $e_{i}$ be the unit element in $V_{i}$. Now $\widetilde{V}=\mathbb{C} 1 \oplus V \oplus \mathbb{C} y$
is also a commutative, associative algebra with invariant symmetric bilinear form $(. \mid$.$) , hence it corresponds to an element \widetilde{T} \in X(\widetilde{V})$, which in turn gives a decomposition of $\widetilde{V}$ as in Proposition 3.1. The following proposition expresses the latter decomposition into the former.

Proposition 6.6. We have an orthogonal decomposition

$$
\tilde{V}=V_{1}^{\prime} \oplus \cdots \oplus V_{k}^{\prime} \oplus N^{\prime} \oplus 0
$$

into ideals, where $V_{i}^{\prime} \subseteq \widetilde{V}$ is isomorphic to $V_{i}$ via the isomorphism

$$
\phi_{i}: V_{i} \rightarrow V_{i}^{\prime}, \phi_{i}(v):=v+\left(v \mid e_{i}\right) y
$$

and where $N^{\prime}$ is a local unital algebra spanned by $N$, $y$, and the unit element $e_{k+1}:=1-e_{1}-\cdots-e_{k}$.

Proof. First, $\phi_{i}$ is clearly injective. It is also an algebra homomorphism because

$$
\begin{aligned}
\phi_{i}(v w) & =v w+\left(v w \mid e_{i}\right) y=v w+\left(v \mid w e_{i}\right) y \\
& =v w+(v \mid w) y=\left(v+\left(v \mid e_{i}\right) y\right) *\left(w+\left(w \mid e_{i}\right) y\right)=\phi_{i}(v) * \phi_{i}(w) .
\end{aligned}
$$

Now note that if $v, w$ belong to $V_{i} \neq V_{j}$, respectively, then

$$
\phi_{i}(v) * \phi_{j}(w)=\left(v+\left(v \mid e_{i}\right) y\right) *\left(w+\left(w \mid e_{j}\right) y\right)=v w+(v \mid w) y=0
$$

This shows that $V_{i}^{\prime} * V_{j}^{\prime}=\{0\}$. Similarly, we have $N^{\prime} * V_{i}=\{0\}$ for all $i-$ e.g., for $v \in V_{i}$ we have
$e_{k+1} * \phi_{i}(v)=\left(1-e_{1}-\cdots-e_{k}\right) *\left(v+\left(v \mid e_{i}\right) y\right)=v+\left(v \mid e_{i}\right) y-\left(e_{i} v+\left(e_{i} \mid v\right) y\right)=0$.
Finally, $e_{k+1}$ is clearly a unit element in $N^{\prime}$. Indeed, we even have an isomorphism $\widetilde{N} \rightarrow N^{\prime}$ of unital algebras with invariant bilinear forms that sends 1 to $1-e_{1}-\cdots-e_{k}$ and $y$ to $y$.

Proposition 6.7. The map $T \mapsto \widetilde{T}$ is a morphism from $X(V)$ into $X(\widetilde{V})$ that maps $Y(V)$ into $Y(\widetilde{V})$.

We call this morphism the unitalisation morphism.
Proof. The first statement is immediate: the algebra structure on $\tilde{V}$ depends in a polynomial manner on the algebra structure on $V$. For the last statement, we note that if $T$ is strongly odeco of tensor rank $n$, then $\left(V, \mu_{T}\right)$ is an orthogonal direct sum of $n$ one-dimensional unital ideals. By Proposition 6.6, $\left(\widetilde{V}, \mu_{\widetilde{T}}\right)$ is then an orthogonal direct sum of $n$ one-dimensional unital ideals and one two-dimensional ideal which, as an algebra with symmetric bilinear form, is
isomorphic to $\mathbb{C}[y] /\left(y^{2}\right)$ with the blinear form determined by $(1 \mid y)=1$. The latter corresponds to the tensor

$$
S:=y \otimes y \otimes 1+y \otimes 1 \otimes y+1 \otimes y \otimes y
$$

from Example 2.4, hence it is a limit of strongly odeco tensors. Consequently, by Proposition 3.1, $\widetilde{T}$ is in $Y(\widetilde{V})$. Since the map $T \mapsto \widetilde{T}$ maps the dense subset of $Y(V)$ of strongly odeco tensors of rank $n$ into $Y(\widetilde{V})$, it maps $Y(V)$ into $Y(\widetilde{V})$.

REMARK 6.8. Unfortunately, we see no reason why, if $T \in X(V)$ satisfies $\widetilde{T} \in Y(\widetilde{V}), T$ should be in $Y(V)$. Indeed, the assumption says that $\widetilde{T}$ is a limit of sums with $n+2$ pairwise orthogonal terms, and we do not see a natural construction that shows that $T$ is a limit of sums with $n$ pairwise orthogonal terms; we do not have a counterexample, though.

## 7. Nilpotent counterexamples

### 7.1. Motivation

When one studies the Hilbert scheme of $n$ points in a fixed space for increasing $n$, and $n$ is taken minimal such that the scheme has more than one irreducible component, then all components other than the main component parameterise subschemes supported in a single point. We will establish a similar result here.

### 7.2. First counterexamples are nilpotent

Theorem 7.1. Let $n=\operatorname{dim}(V)$ be minimal such that $X(V) \neq Y(V)$. Then for all $T \in X(V) \backslash Y(V)$ the algebra $\left(V, \mu_{T}\right)$ is nilpotent.

Proof. Let $T \in X(V) \backslash Y(V)$. Decompose $V=V_{1} \oplus \cdots \oplus V_{k} \oplus N$ as in Proposition 3.1, and decompose $T=T_{1}+\cdots+T_{k}+T_{N}$ accordingly. By Proposition 3.1, either some $T_{i}$ does not lie in $Y\left(V_{i}\right)$, or $T_{N}$ does not lie in $Y(N)$. By minimality of $n$, we find that either $k=0$ and we are done, or else $k=1$ and $N=\{0\}$. In the latter case, by Proposition 6.5, the algebra $\left(V, \mu_{T}\right)$ equals $\widetilde{A}$ for some nilpotent algebra $A$ of $\operatorname{dimension} \operatorname{dim}(V)-2$ equipped with a nondegenerate symmetric bilinear form. This means that $T=\widetilde{S}$ for some tensor $S \in X(A)$. By minimality of $n, S$ lies in $Y(A)$. But then, by Proposition 6.7, $T=\widetilde{S}$ lies in $Y(V)$, a contradiction. Hence $\left(V, \mu_{T}\right)$ is nilpotent, as claimed.

## 8. Proof of Theorem 2.10

Proof of Theorem 2.10. Let $V$ be a finite-dimensional complex vector space of dimension $n \geq 12$ and let (.|.) be a nondegenerate symmetric bilinear form
on $V$. We first show that $X(V)$ is not equal to $Y(V)$ when $n=12$.
In [5] an explicit 14-dimensional local Gorenstein algebra is constructed which is not smoothable. Call this algebra $B$, and let (.|.) be a nondegenerate invariant symmetric bilinear form on $B$. Let $M$ be the maximal ideal of $B$, and let $M^{d}$ be its minimal ideal. Then $A:=M / M^{d}$ is a nilpotent algebra and since $M^{d}$ is the radical of the restriction of (.|.) to $M,(. \mid$.$) induces a nondegenerate$ bilinear form on $A$. Note that $\operatorname{dim}(A)=12$, so we may assume that $V$ (with its bilinear form) is the underlying vector space of $A$ (with its bilinear form). Let $T \in X(V)$ be the tensor corresponding to the algebra $A$. We claim that $T$ does not lie in $Y(V)$. Indeed, if it does, then $T=\lim _{i \rightarrow \infty} T_{i}$ for a convergent sequence of strongly odeco tensors $T_{i}$. Applying the unitalisation morphism, we obtain $\widetilde{T}=\lim _{i \rightarrow \infty} \widetilde{T}_{i}$. Now $\widetilde{T}$ is the structure tensor of the algebra $\widetilde{A}$, which by (the proof of) Proposition 6.5 is isomorphic to $B$.

However, by (the proof of) Proposition 6.7, each $\widetilde{T}_{i}$ is the direct product of 12 one-dimensional ideals and one copy of $\mathbb{C}[x] /\left(x^{2}\right)$. In particular, each $\widetilde{T}_{i}$ corresponds to a smoothable algebra, and $B$ is smoothable, as well. This contradicts the choice of $B$, finishing the proof in the case $n=12$. If $n>12$, one can embed the algebra $A=M / M^{d}$ from above into the vector space $V$ by $A^{\prime}:=A \oplus \mathbb{C}^{n-12}$ and then proceed as in the proof above.

## 9. Proof of Theorem 2.13

We set $V:=\mathbb{C}^{n}$, equipped with the standard symmetric bilinear form $\beta_{0}(u, v):=$ $\sum_{i} u_{i} v_{i}$. Recall that $X^{0}(V)$ is the variety of tensors corresponding to unital associative algebras on $V$ for which $\beta_{0}$ is invariant. We want to show that $X^{0}(V)$ has the same number of irreducible components as $\mathcal{H}^{\text {Gor }}$.

### 9.1. Locating the unit element

Lemma 9.1. The map $u: X^{0}(V) \rightarrow V$ that assigns to a tensor $T$ the unit element of $\left(V, \mu_{T}\right)$ is a morphism of quasi-affine varieties.

Proof. For given $T$, the unit element $u=u(T)$ is the solution to the system of linear equations $\mu_{T}\left(u, e_{i}\right)=e_{i}$ for $i=1, \ldots, n$. For each $T \in X^{0}(V)$, this system has a unique solution. This means that we can cover $X^{0}(V)$ with open affine subsets in which some subdeterminant of the coefficient matrix has nonzero determinant, and on such an open subset the map $u$ is morphism with a formula in which that determinant appears in the denominator. These morphisms glue to a global morphism $u$.

### 9.2. A map from $X^{0}(V)$ to the Hilbert scheme

We write $R:=\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$, denote by $\mathcal{H}$ the Hilbert scheme of $n$ points in $\mathbb{A}^{n}$, and by $\mathcal{H}^{\text {Gor }}$ the open subscheme of $\mathcal{H}$ parameterising Gorenstein schemes. In fact, since we care only about irreducible components, we may and will replace both of these by the corresponding reduced subvarieties, and we will only speak of $\mathbb{C}$-valued points of these varieties. Points in $\mathcal{H}$ will be regarded as ideals in $R$ of codimension $n$. To define a morphism from an affine variety $B$ over $\mathbb{C}$ to $\mathcal{H}$, it suffices to indicate a subscheme of $B \times \mathbb{A}^{n}$ (product over $\mathbb{C}$ ), flat over $B$, such that the fibre over each $b \in B$ is defined by such a codimension- $n$ ideal.

Take $B=X^{0}(V)$. A tensor $T \in X^{0}(V)$ gives rise to the ideal $I_{T}:=\operatorname{ker}\left(\phi_{T}\right)$, where $\phi_{T}: \mathbb{C}\left[x_{1}, \ldots, x_{n}\right] \rightarrow\left(V, \mu_{T}\right)$ is the homomorphism of associative algebras that maps $x_{i}$ to $e_{i}$ and 1 to the unit element $u(T)$ from Lemma 9.1. The ideals $I_{T}$ have vector space codimension $n$ in $R$ and together define a subscheme of $X^{0}(V) \times \mathbb{A}^{n}$ flat over $X^{0}(V)$. Hence we have described a morphism $\Phi: X^{0}(V) \rightarrow \mathcal{H}$. Since any algebra corresponding to a tensor in $X^{0}(V)$ has a nondegenerate invariant bilinear form, $\Phi(T) \cong\left(V, \mu_{T}\right)$ is Gorenstein for each $T \in X^{0}(V)$, so $\Phi$ is a morphism $X^{0}(V) \rightarrow \mathcal{H}^{\text {Gor }}$. We want to use $\Phi$ to compare irreducible components of $\mathcal{H}^{\text {Gor }}$ and $X^{0}(V)$. However, the map $\Phi$ is not an isomorphism, so some care is needed for this. We first describe the image of $\Phi$; the following is immediate.

Lemma 9.2. The image of $\Phi$ consists of all codimension-n ideals $I \in \mathcal{H}^{\text {Gor }}$ such that $x_{1}, \ldots, x_{n} \in R$ map to a basis of $R / I$ and moreover the bilinear form on $R / I$ for which this basis is orthonormal is invariant for the multiplication in $R / I$.

The following lemma shows that, as far as irreducible components are concerned, it is no real restriction to consider ideals $I$ modulo which $x_{1}, \ldots, x_{n}$ is a basis.

Lemma 9.3. The locus $\mathcal{H}_{0}^{\text {Gor }}$ in $\mathcal{H}^{\text {Gor }}$ of ideals $I$ in $R$ for which $x_{1}, \ldots, x_{n}$ maps to a vector space basis of $R / I$ is open and dense in $\mathcal{H}_{0}^{\text {Gor }}$. Consequently, $\mathcal{H}_{0}^{\text {Gor }}$ has the same number of irreducible components as $\mathcal{H}^{\text {Gor }}$.

This is well-known to the experts - see [10, Remark 4.5] - but we include a quick proof.

Proof. The condition on $I$ can be expressed by the non-vanishing of certain determinants; this shows that $\mathcal{H}_{0}^{\text {Gor }}$ is open. For density, suppose that some component $C$ of $\mathcal{H}^{\text {Gor }}$ does not meet $\mathcal{H}_{0}^{\text {Gor }}$, and let $I_{0}$ be a point in $C$ such that the image of $\left\langle x_{1}, \ldots, x_{n}\right\rangle_{\mathbb{C}}$ in $R / I_{0}$ has maximal dimension, say $m<n$. After a linear change of coordinates (which preserves all components of $\mathcal{H}^{\text {Gor }}$ and hence in particular $C$ ), we may assume that $x_{m+1}, \ldots, x_{n}$ are in $I_{0}$, and
there exists a monomial $r$ in $x_{1}, \ldots, x_{m}$ of degree $\neq 1$ such that $x_{1}, \ldots, x_{m}, r$ are linearly independent in $R / I_{0}$.

Now, for $a \in \mathbb{C}$, consider the nonlinear automorphism $\psi_{a}: R \rightarrow R$ that maps all $x_{i}, i \neq n$ to themselves but $x_{n}$ to $x_{n}+a \cdot r$. The map $(a, I) \mapsto \psi_{a}^{-1}(I)$ defines an action of the additive group $(\mathbb{C},+)$ on $\mathcal{H}^{\text {Gor }}$, and since the additive group is irreducible, this action preserves all components of $\mathcal{H}^{\text {Gor }}$. Since, for $a \neq 0, x_{1}, \ldots, x_{m}, x_{n}+a r$ are linearly independent modulo $I_{0}$, their pre-images $x_{1}, \ldots, x_{m}, x_{n}$ under $\psi_{a}$ are linearly independent modulo $I_{a}:=\psi_{a}^{-1}\left(I_{0}\right)$. Since $I_{a}$ is in $C$, this contradicts the maximality assumption in the choice of $I_{0}$.

The last statement is now immediate.

### 9.3. Varieties $Z_{2} \rightarrow Z_{1} \rightarrow \mathcal{H}_{0}^{\text {Gor }}$ with the same number of components

In what follows, we will identify $V$ with the space in $R$ spanned by the variables $x_{1}, \ldots, x_{n}$, via the identification $e_{i} \mapsto x_{i}$. Each point in $\mathcal{H}_{0}^{\text {Gor }}$ defines a unital, commutative, associative algebra structure on $V$. The structure constant tensor in $\left(S^{2} V^{*}\right) \otimes V$ of this algebra does not necessarily lie in $X^{0}(V)$, though, because the standard form $\beta_{0}$ may not be invariant for it.

Lemma 9.4. Let $Z_{1}$ be the subvariety

$$
\left\{(I,[\beta]) \in \mathcal{H}_{0}^{\text {Gor }} \times \mathbb{P}\left(S^{2} V^{*}\right) \mid \beta \text { is invariant for } R / I\right\} \subseteq \mathcal{H}_{0}^{\text {Gor }} \times \mathbb{P}\left(S^{2} V^{*}\right)
$$

Then the projection $Z_{1} \rightarrow \mathcal{H}_{0}^{\text {Gor }}$ is surjective and induces a bijection on irreducible components.

Proof. Indeed, every (possibly degenerate) invariant bilinear form on $R / I$ is of the form $\beta(r, s)=\ell(r s)$ for a unique linear form $\ell \in(R / I)^{*}$, namely, the form $\ell(r):=\beta(1, r)$. Moreover, since for $I \in \mathcal{H}_{0}^{\text {Gor }}$ the space $V$ is a vector space complement of $I$ in $R$, the natural map $(R / I)^{*} \rightarrow V^{*}$ is a linear bijection. We conclude that, in fact, $Z_{1}$ is isomorphic to $\mathcal{H}_{0}^{\text {Gor }} \times \mathbb{P}\left(V^{*}\right)$ via the map that sends $(I,[\beta])$ to $(I,[v \mapsto \beta(1, v)])$. So each component of $Z_{1}$ is just a component of $\mathcal{H}_{0}^{\text {Gor }}$ times the projective space $\mathbb{P}\left(V^{*}\right)$.

Lemma 9.5. Let $Z_{2}$ be the subvariety

$$
\left\{((I,[\beta]), g) \in Z_{1} \times \mathrm{GL}(V) \mid g[\beta]=\left[\beta_{0}\right]\right\} \subseteq Z_{1} \times \mathrm{GL}(V)
$$

Then the projection $Z_{2} \rightarrow Z_{1}$ has dense image and induces a bijection between irreducible components.

Proof. For the first statement, if $I \in \mathcal{H}_{0}^{\text {Gor }}$, then by definition there are nondegenerate invariant bilinear forms on $R / I$. These correspond to a dense open
subset of $\mathbb{P}\left(V^{*}\right)$ via the correspondence in the proof above. This shows that $Z_{2} \rightarrow Z_{1}$ has dense image. This image, $U$, is open in $Z_{1}$.

Next we claim that, in the analytic topology, $Z_{2} \rightarrow U$ is a fibre bundle with fibre the group $\mathbb{C}^{*} \cdot \mathrm{O}\left(\beta_{0}\right) \subseteq \mathrm{GL}(V)$; here $\mathrm{O}\left(\beta_{0}\right)$ is the orthogonal group of the form $\beta_{0}$. To see this, it consider a point $\left(I,\left[\beta_{1}\right]\right) \in U$. By definition of $U$, there exists a $g_{1} \in \mathrm{GL}(V)$ such that $g_{1}\left[\beta_{1}\right]=\left[\beta_{0}\right]$. Furthermore, there exists a holomorphic map $\gamma$ defined in an open neighbourhood $\Omega$ in $\mathbb{P} S^{2} V^{*}$ of $\left[\beta_{0}\right]$ to $\mathrm{GL}(V)$ such that $\gamma\left(\left[\beta_{0}\right]\right)=\operatorname{id}_{V}$ and $\gamma([\beta])[\beta]=\left[\beta_{0}\right]$ for all $[\beta] \in \Omega$. Essentially, $\gamma([\beta])$ is found by the Gram-Schmidt algorithm - note that in this algorithm one has to divide by square roots of complex numbers, which, since $[\beta]$ is close to $\left[\beta_{0}\right]$, are close to 1 ; this can be done holomorphically.

Now the map

$$
\begin{aligned}
& \left(U \cap\left(\mathcal{H}_{0}^{\text {Gor }} \times g_{1}^{-1} \Omega\right)\right) \times\left(\mathbb{C}^{*} \cdot \mathrm{O}\left(\beta_{0}\right)\right) \rightarrow Z_{2}, \\
& ((I,[\beta]), g) \mapsto\left((I,[\beta]), g \cdot \gamma\left(g_{1}[\beta]\right) \cdot g_{1}\right)
\end{aligned}
$$

trivialises the map $Z_{2} \rightarrow Z_{1}$ over an open neighbourhood of ( $I,\left[\beta_{1}\right]$ ); here we use that $\mathbb{C}^{*} \cdot \mathrm{O}\left(\beta_{0}\right)$ is the stabiliser of $\left[\beta_{0}\right]$ in $\mathrm{GL}(V)$.

Now since $Z_{2} \rightarrow U$ is a fibre bundle with irreducible fibre $\mathbb{C}^{*} \cdot \mathrm{O}(V)$ - this is where it is important that we work with the projective space $\mathbb{P} S^{2} V^{*}$ rather than $S^{2} V^{*}$; the orthogonal group $\mathrm{O}(V)$ itself has two components! - that map induces a bijection between irreducible components.

### 9.4. Completing the proof

Proof of Theorem 2.13. Recall that $X^{0}(V)$ parameterises the unital associative, commutative algebra structures on $V$ such that $\beta_{0}$ is invariant for the multiplication. Now consider the map

$$
\mathrm{GL}(V) \times X^{0}(V) \rightarrow \mathcal{H}_{0}^{\text {Gor }},(g, T) \mapsto g \cdot \Phi(T)
$$

By Lemma 9.2, this map is surjective. Since GL $(V)$ is irreducible, the lefthand side has as many irreducible components as $X^{0}(V)$. This shows that $X^{0}(V)$ has at least as many irreducible components as $\mathcal{H}_{0}^{\text {Gor }}$, hence as $\mathcal{H}^{\text {Gor }}$ by Lemma 9.3.

For the converse, by Lemmas 9.4 and $9.5, \mathcal{H}_{0}^{\text {Gor }}$ has as many irreducible components as $Z_{2}$. Now we claim that the morphism

$$
\begin{aligned}
Z_{2} & \rightarrow\left(S^{2} V^{*}\right) \otimes V \\
((I,[\beta]), g) & \mapsto \text { the structure constant tensor of } R /(g \cdot I)
\end{aligned}
$$

has as image the variety $X^{0}(V)$. Indeed, if $((I,[\beta]), g)$ lies in $Z_{2}$, then $\beta$ is an invariant symmetric bilinear form for the multiplication on $R / I$, and $g \beta \in \mathbb{C}^{*} \cdot \beta_{0}$ is an invariant symmetric bilinear form for the multiplication on
$R /(g \cdot I)$; this therefore corresponds to an element in $X^{0}(V)$. We conclude that the number of components of $X^{0}(V)$ is also at most that of $\mathcal{H}^{\text {Gor }}$. This concludes the proof.

### 9.5. Cubic dimension growth for the Gorenstein locus

We conclude this paper with an observation on the dimension of $\operatorname{dim}\left(\mathcal{H}^{\text {Gor }}\right)$.
Proposition 9.6. The dimension of $\mathcal{H}^{\text {Gor }}$, and hence that of the Hilbert scheme $\mathcal{H}$ of $n$ points in $\mathbb{A}^{n}$, is lower-bounded by a cubic polynomial in $n$ for $n \rightarrow \infty$.

Note that this was already known for $\mathcal{H}$ by [10, Theorem 9.2]. The algebras constructed there are of the form $A:=\mathbb{C}\left[x_{1}, \ldots, x_{d}\right] /\left(V+\mathfrak{m}^{3}\right)$ where $\mathfrak{m}$ is the maximal ideal $\left(x_{1}, \ldots, x_{d}\right)$ and where $V \subseteq \mathfrak{m}^{2} / \mathfrak{m}^{3}$ has the correct codimension $r=n-1-d$ for this quotient to have dimension $n$. Since any 1-dimensional subspace of $\mathfrak{m}^{2} /\left(V+\mathfrak{m}^{3}\right)$ is a minimal ideal in $A, A$ is not Gorenstein unless $\mathfrak{m}^{2} /\left(V+\mathfrak{m}^{3}\right)$ is one-dimensional, in which case $r=1$. However, to obtain cubic behaviour in $n$, one needs $r$ to grow linearly with $d$. So the cubic-dimensional locus in [10] is a non-Gorenstein part of the Hilbert scheme.

Proof. The unitalisation morphism sends $X\left(\mathbb{C}^{n-2}\right)$ into $X^{0}\left(\mathbb{C}^{n}\right)$ by Proposition 6.7, and it does so injectively. This means that the latter variety has dimension at least that of $Z\left(\mathbb{C}^{n-2}\right)$, which is lower-bounded by a cubic polynomial by Proposition 5.1. Furthermore, the morphism $\Phi: X^{0}\left(\mathbb{C}^{n}\right) \rightarrow \mathcal{H}^{\text {Gor }}$ is also injective.

Remark 9.7. The coefficient of $n^{3}$ in $\operatorname{dim}\left(Z\left(\mathbb{C}^{n}\right)\right)$ equals $\frac{1}{48}$, which is considerably smaller than the coefficient $\frac{2}{27}$ in [10] for the lower bound on the dimension of the Hilbert scheme of $n$ points in $\mathbb{A}^{n}$. We do not know whether the $\frac{1}{48}$ can be improved.

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# Hilbert curves of quadric fibrations over smooth surfaces 

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#### Abstract

Let $(X, L)$ be a complex polarized $n$-fold with the structure of a geometric quadric fibration over a smooth projective surface. The Hilbert curve of $(X, L)$ is a complex affine plane curve of degree $n$, containing $n-3$ evenly spaced parallel lines. This paper is devoted to a detailed study of the cubic representing the residual component. Reducibility, existence of triple points, and properties of the irreducible components are analyzed in connection with the structure of $(X, L)$.


}

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## 1. Introduction

The Hilbert curve of a polarized manifold $(X, L)$ with $\operatorname{dim}(X)=n \geq 2$ is the complex affine plane curve $\Gamma=\Gamma_{(X, L)}$, of degree $n$, defined by the Hilbertlike polynomial $\chi\left(x K_{X}+y L\right)$, where $K_{X}$ is the canonical bundle of $X$ and $x$ and $y$ are regarded as complex variables. This notion was introduced in [3] and extensively studied in $[10,11,13,6]$ for varieties which are special from the adjunction theoretic point of view. The natural expectation is that several properties of the polarized manifold that one considers are encoded by its Hilbert curve, as suggested by [3, Theorem 6.1]. In particular, if $X$ is endowed with a fibration $\varphi: X \rightarrow Y$ over a normal variety $Y$ of dimension $m<n-1$ and $K_{X}+(n-m) L=\varphi^{*} A$, for some ample $\mathbb{Q}$-line bundle $A$ on $Y$, then $\Gamma$ contains $n-m-1$ parallel lines of prescribed equations as components, and therefore it becomes important to understand the properties of the residual curve of the union of such lines in $\Gamma$.

In this paper, relying on our previous study of the Hilbert curve of threefolds which are conic fibrations over a smooth surface [6], we investigate $n$ dimensional pairs $(X, L)$ with $n \geq 4$, where $X$ is a quadric fibration in the classical sense over a smooth surface and $L$ makes it an adjunction theoretic quadric fibration at the same time. We refer to pairs $(X, L)$ of this type as
geometric quadric fibrations. In this setting, $\Gamma=\ell_{1}+\cdots+\ell_{n-3}+C$, where the $\ell_{i}$ 's are certain $n-3$ parallel lines and $C$ is the residual cubic; moreover, both $\Gamma$ and $C$ are Serre-invariant, i.e. invariant under the involution induced on the affine plane by Serre duality on $X$.

In order to make the equation of $C$ explicit in terms of the numerical invariants associated with ( $X, L$ ) (Proposition 3.1) we describe $X$ as a divisor of relative degree 2 inside the projective bundle defined by $\mathcal{E}:=\varphi_{*} L$, where $\varphi: X \rightarrow S$ is the fibration morphism. The whole Section 3 is devoted to computations involving Chern classes which lead to the equation of $C$. Various consequences of these computations are discussed in Section 4 and Section 5. A first crucial implication is that the projective closure of our cubic $C$ intersects the line at infinity transversely at a special point, say $P_{\infty}$, whose homogeneous coordinates depend on $n$ (Proposition 4.2). The cubic $C$ is irreducible in general. The above property allows us to prove that $C$ contains a special line of the affine plane whose direction is given by $P_{\infty}$ if and only if the quadric fibration has no singular fibers, and also to characterize the existence of a triple point for $C$ in terms of the structure and the numerical invariants of $X$ (Theorem 4.5). This provides a complete generalization of [6, Theorem 5.2]. Moreover, this in turn leads to investigate other significant circumstances, for instance, under what conditions: a) $\Gamma$ is nonreduced (Proposition 4.7), b) $C$ is reducible (Corollary 5.3), c) $C$ contains a general line, at least in the case when $(X, L)$ is a 4-dimensional geometric quadric fibration over $\mathbb{P}^{2}$ (Proposition 5.4).

Next we consider a special class of geometric quadric fibrations that we call "deriving from cones", in view of their construction (Section 6). They generalize the geometric conic fibrations studied in [6, Section 6]. When the base surface $S$ of such a pair $(X, L)$ is a minimal surface of Kodaira dimension zero, we prove that the residual cubic of the Hilbert curve is always irreducible unless $n \geq 4, S$ is abelian or bielliptic and the Chern classes of the vector bundle $\mathcal{E}$ satisfy a precise numerical condition depending on $n$ (Theorem 6.2). In particular, this result amends the sentence given for $n=3$ in [6, Proposition 6.3 (ii)] and at the same time provides a generalization to higher dimensions.

Clearly, $\Gamma=C$ for $n=3$, and several results established here for $C$ specialize to those proven for $\Gamma$ in [6]. As it is natural to expect, passing from threefolds to varieties of higher dimensions, new situations arise, for instance this happens when we investigate the nonreducedness of $\Gamma$ (Proposition 4.7). This fact makes case $n=4$ particularly relevant in our study. For this reason, in Section 7 we discuss several examples in the setting of fourfolds, taking also advantage of the fact that the Riemann-Roch formula, which is crucial to determine the equation of $\Gamma$, is still handleable for $n=4$. In particular, we discuss three types of geometric quadric fibrations $(X, L)$ whose underlying varieties $X$ arise in the classification of Fano fourfolds of index 2 with Picard number $\geq 2$ [14]. For all of them the residual cubic $C$ is reducible, containing a line that depends
on the polarization $L$.
In Section 8, in the framework of plane cubic curves we provide a unifying perspective for residual cubics of our $\Gamma$ 's and for Serre-invariant cubics, which constitute a dense Zariski open subset of $\mathbb{P}^{5}$. In particular we describe the varieties whose points represent the cubics satisfying the various properties discussed in the previous sections, like reducibility, existence of triple points, etc. . This offers a global view of the families in which the residual cubic of the Hilbert curve of a geometric quadric fibration $(X, L)$ can fit into. It is worth noting that while the families we describe are "continua", only points with rational coordinates on them can represent a residual cubic, because, as for $\Gamma$, its equation has rational coefficients.

## 2. The leading idea

Let $(X, L)$ be a quadric fibration with $\operatorname{dim}(X)=n$ over a smooth projective surface $S$, via a morphism $\varphi: X \rightarrow S$. In view of [6], we will assume that $n \geq 4$. We say that $(X, L)$ is a geometric quadric fibration, to mean that the following two facts hold. 1) The morphism $\varphi$ is equidimensional with connected fibers, and all of them are irreducible quadric hypersurfaces of $\mathbb{P}^{n-1}$ with $L$ inducing the hyperplane bundle. In particular, $\varphi$ is flat, and for the general fiber $F$ of $\varphi$ we have $\left(F, L_{F}\right)=\left(\mathbb{Q}^{n-2}, \mathcal{O}_{\mathbb{Q}}(1)\right)$, where $\mathbb{Q}^{n-2}$ stands for a smooth quadric hypersurface in $\mathbb{P}^{n-1}$. 2) $K_{X}+(n-2) L=\varphi^{*} H$ for some ample line bundle $H$ on $S$. Condition 1) means that $\varphi: X \rightarrow S$ is a fibré en quadriques in the sense of [1] and, to emphasize the role of the polarization $L$ we can say that $(X, L)$ is a classical quadric fibration, while condition 2) says that $(X, L)$ is also an adjunction theoretic quadric fibration over $S$ (in the sense of [4, p. 81]). Thanks to Grauert's theorem, conditions 1) and 2) are enough to guarantee that $\mathcal{E}:=\varphi_{*} L$ is a locally free sheaf, i.e. a vector bundle, of rank $n$ on $S$, [9, Corollary 19.2]. If we consider its projectivization $P:=\mathbb{P}(\mathcal{E})$ and we denote by $\xi$ the tautological line bundle on it, then $X$ is fiberwise embedded in the $\mathbb{P}^{n-1}$-bundle $P$ as a divisor of relative degree 2 ; more precisely, letting $\pi: P \rightarrow S$ denote the bundle projection of $P$, we have that $X \in\left|2 \xi+\pi^{*} \mathcal{B}\right|$ for some line bundle $\mathcal{B}$ on $S, \varphi=\left.\pi\right|_{X}$, and $L=\xi_{X}$.

The discriminant curve of $(X, L)$ is the possible empty curve $\mathcal{D} \subset S$ parameterizing the singular fibers of $\pi$. By $[7,(3.3)]$ we know that $\mathcal{D} \in\left|2 c_{1}(\mathcal{E})+n \mathcal{B}\right|$ (for $n=3$ see also $[6,(5)]$ ).

Let $p(x, y)=0$ be the equation of the Hilbert curve of $(X, L)$. Recall that $p(x, y)=\chi\left(x K_{X}+y L\right)$, the polynomial expressing the Euler-Poincaré characteristic of $x K_{X}+y L$, when $x$ and $y$ are regarded as complex variables.

According to [3, Theorem 6.1], we have that

$$
\begin{equation*}
p(x, y)=\prod_{i=1}^{n-3}((n-2) x-y-i) R(x, y) \tag{1}
\end{equation*}
$$

where $R(x, y)$ is a polynomial of degree 3 . From the qualitative point of view, this means that the Hilbert curve $\Gamma$ of $(X, L)$ can be written as

$$
\begin{equation*}
\Gamma=\ell_{1}+\cdots+\ell_{n-3}+C \tag{2}
\end{equation*}
$$

i.e., it consists of $n-3$ evenly spaced parallel lines with slope $(n-2)$ (the nef value of $(X, L))$ plus a cubic $C$, which we call the residual cubic.

We call Serre involution the map $s: \mathbb{A}^{2} \rightarrow \mathbb{A}^{2}$ sending $(x, y)$ to $(1-x,-y)$, induced by Serre duality. Note that $\Gamma$ is Serre-invariant, i.e., invariant under $s$. Moreover, $s$ exchanges the line $\ell_{i}$ of equation $(n-2) x-y-i=0$ with $\ell_{n-2-i}(i=1, \ldots, n-3)$, hence the set consisting of the $n-3$ lines $\ell_{1}, \ldots, \ell_{n-3}$ is globally Serre-invariant. It thus follows that the cubic $C$ itself is also Serreinvariant. We use coordinates $(u, v)$ in place of $\left(x=\frac{1}{2}+u, y=v\right)$ in order to make this invariance more evident. Since the degree of $C$ is odd, then $R\left(\frac{1}{2}+u, v\right)$ is the sum of two homogeneous polynomials in $u$ and $v$ of degrees 3 and 1 respectively [3, Lemma 7.1]. Thus we can write

$$
\begin{equation*}
R\left(\frac{1}{2}+u, v\right)=R_{3}(u, v)+R_{1}(u, v) \tag{3}
\end{equation*}
$$

where

$$
\begin{equation*}
R_{3}(u, v)=\alpha u^{3}+\beta u^{2} v+\gamma u v^{2}+\delta v^{3} \tag{4}
\end{equation*}
$$

with $(\alpha, \beta, \gamma, \delta) \neq(0,0,0,0)$, because $\operatorname{deg} C=3$, and

$$
\begin{equation*}
R_{1}(u, v)=\sigma u+\tau v \tag{5}
\end{equation*}
$$

Note that the property of having an equation of this type characterizes any Serre-invariant plane cubic, which is not necessarily the residual cubic of $\Gamma$.

Our aim is to obtain the explicit expression of $R\left(\frac{1}{2}+u, v\right)$ in our specific case, which in particular describes our cubic $C$. To do that, first recall that for any divisor $D$ on $X$,

$$
\chi(D)=\frac{1}{n!} D^{n}+\ldots,
$$

where the dots stand for lower degree terms. So, by using homogeneous coordinates $(x: y: z)$, where $z$ is the homogenizing coordinate, and letting $p_{0}(x, y, z)$
denote the homogeneous polynomial associated to $p$, we have:

$$
\begin{align*}
p_{0}(x, 1,0)= & \frac{1}{n!}\left(x K_{X}+L\right)^{n}  \tag{6}\\
= & \frac{1}{n!}\left[d_{n} x^{n}+\binom{n}{1} d_{n-1} x^{n-1}+\binom{n}{2} d_{n-2} x^{n-2}+\ldots\right. \\
& \left.\cdots+\binom{n}{n-3} d_{3} x^{3}+\binom{n}{n-2} d_{2} x^{2}+\binom{n}{n-1} d_{1} x+d\right]
\end{align*}
$$

where $d_{i}:=K_{X}^{i} \cdot L^{n-i}$ for $i=0,1, \ldots, n\left(d_{0}=d\right.$ being the degree of $\left.(X, L)\right)$. On the other hand, from (1) and (3) we see that $p_{0}(x, y, 0)=R_{3}(x, y)((n-$ 2) $x-y)^{n-3}$. Hence (4) gives

$$
\begin{equation*}
p_{0}(x, 1,0)=\left(\alpha x^{3}+\beta x^{2}+\gamma x+\delta\right)((n-2) x-1)^{n-3} \tag{7}
\end{equation*}
$$

By comparing (6) with (7), we can get the explicit expressions of $\alpha, \beta, \gamma$ and $\delta$ in terms of the natural invariants of $(X, L)$. Next, recalling that $\chi\left(\mathcal{O}_{X}\right)=\chi\left(\mathcal{O}_{S}\right)$ and Serre duality, we have

$$
\begin{equation*}
p(1,0)=\chi\left(K_{X}\right)=(-1)^{n} \chi\left(\mathcal{O}_{X}\right)=(-1)^{n} \chi\left(\mathcal{O}_{S}\right) \tag{8}
\end{equation*}
$$

On the other hand, from (1) and (3) we get

$$
\begin{equation*}
p(1,0)=\prod_{i=1}^{n-3}(n-2-i)\left(\frac{\alpha}{8}+\frac{\sigma}{2}\right)=\frac{(n-3)!}{8}(\alpha+4 \sigma) \tag{9}
\end{equation*}
$$

So, taking into account the previous discussion, we obtain the expression of $\sigma$. It remains to determine $\tau$. To do it, recall that $K_{X}+(n-2) L=\varphi^{*} H$. We have, for every $i \geq 0$,

$$
\begin{equation*}
H^{i}\left(K_{X}+(n-2) L\right)=H^{i}\left(\varphi^{*} H\right) \cong H^{i}\left(\varphi_{*}\left(\varphi^{*} H\right)\right)=H^{i}(H) \tag{10}
\end{equation*}
$$

The last equality will follow once we prove that $\left.R^{i} \varphi_{*}\left(\varphi^{*} H\right)\right)=0$ for $i>0$, see [9, p. 252, Ex. 8.1].

Because by projection formula $R^{i} \varphi_{*}\left(\varphi^{*} H\right) \cong R^{i} \varphi_{*} \mathcal{O}_{X} \otimes H$, it is enough to show that $R^{i} \varphi_{*} \mathcal{O}_{X}=0$ for $i>0$. As $X \subset P$ and $X \in\left|2 \xi+\pi^{*} \mathcal{B}\right|$, we have the following exact sequence

$$
\begin{equation*}
0 \rightarrow \mathcal{O}_{P}\left(-2 \xi-\pi^{*} \mathcal{B}\right) \rightarrow \mathcal{O}_{P} \rightarrow \mathcal{O}_{X} \rightarrow 0 \tag{11}
\end{equation*}
$$

and applying to it the direct image functor and using [9, p. 281, Ex. 11.8] we obtain the following long exact sequence

$$
\begin{array}{r}
0 \rightarrow R^{0} \pi_{*} \mathcal{O}_{P}\left(-2 \xi-\pi^{*} \mathcal{B}\right) \rightarrow R^{0} \pi_{*} \mathcal{O}_{P} \rightarrow R^{0} \varphi_{*} \mathcal{O}_{X} \rightarrow  \tag{12}\\
R^{1} \pi_{*} \mathcal{O}_{P}\left(-2 \xi-\pi^{*} \mathcal{B}\right) \rightarrow \cdots \cdots \cdots \cdots \rightarrow R^{n-2} \varphi_{*} \mathcal{O}_{X} \rightarrow \\
R^{n-1} \pi_{*} \mathcal{O}_{P}\left(-2 \xi-\pi^{*} \mathcal{B}\right) \rightarrow R^{n-1} \pi_{*} \mathcal{O}_{P} \rightarrow R^{n-1} \varphi_{*} \mathcal{O}_{X}=0,
\end{array}
$$

the last equality coming from the fact that the fibers of $\varphi$ have dimension $n-2$. By [9, p. 253, Ex. 8.4 (a)] we thus conclude that $\varphi_{*} \mathcal{O}_{X}=R^{0} \varphi_{*} \mathcal{O}_{X}=$ $R^{0} \pi_{*} \mathcal{O}_{P}=\mathcal{O}_{S}$ and $R^{i} \varphi_{*} \mathcal{O}_{X}=R^{i} \pi_{*} \mathcal{O}_{P}=0$ for $i>0$. Therefore,

$$
\begin{equation*}
p(1, n-2)=\chi\left(K_{X}+(n-2) L\right)=\chi\left(\varphi^{*} H\right)=\chi(H) \tag{13}
\end{equation*}
$$

in view of (10). Now, recalling the canonical bundle formula for $\mathbb{P}$-bundles, by adjunction we have

$$
\begin{align*}
K_{X}=\left(K_{P}+X\right)_{X} & =\left(-n \xi+\pi^{*}\left(K_{S}+\operatorname{det} \mathcal{E}\right)+2 \xi+\pi^{*} \mathcal{B}\right)_{X}  \tag{14}\\
& =\left(-(n-2) \xi+\pi^{*}\left(K_{S}+\operatorname{det} \mathcal{E}+\mathcal{B}\right)\right)_{X} \\
& =-(n-2) L+\varphi^{*}\left(K_{S}+\operatorname{det} \mathcal{E}+\mathcal{B}\right)
\end{align*}
$$

Hence, due to the injectivity of the homomorphism induced by $\varphi$ between the Picard groups of $S$ and $X$, we get

$$
\begin{equation*}
H=K_{S}+c_{1}(\mathcal{E})+\mathcal{B} \tag{15}
\end{equation*}
$$

Thus (13) allows us to express $p(1, n-2)$ in terms of $K_{S}, c_{1}(\mathcal{E})$ and $\mathcal{B}$ via the Riemann-Roch theorem. On the other hand, from (1)-(5) we get

$$
\begin{align*}
p(1, n-2)= & \prod_{i=1}^{n-3}(n-2-(n-2)-i) R(1, n-2)  \tag{16}\\
= & (-1)^{n-3}(n-3)!\left(\frac{\alpha}{8}+\frac{\beta}{4}(n-2)+\frac{\gamma}{2}(n-2)^{2}\right. \\
& \left.+\delta(n-2)^{3}+\frac{\sigma}{2}+\tau(n-2)\right)
\end{align*}
$$

So (13) and (16) give another equation, which, added to the previous ones, allows us to determine $\tau$. For the explicit computations see Section 3, which leads to Proposition 3.1.

## 3. Some computations

First of all we make explicit the coefficients of some of the powers of $x$ from (6), and precisely
$\operatorname{coeff}\left(x^{n}\right)=\frac{1}{n!} d_{n}$,
$\operatorname{coeff}\left(x^{2}\right)=\frac{1}{2(n-2)!} d_{2}$,
$\operatorname{coeff}(x)=\frac{1}{(n-1)!} d_{1}$,
$\operatorname{coeff}(1)=\frac{1}{n!} d$.
On the other hand, doing the same with (7), we get:

```
coeff \(\left(x^{n}\right)=(n-2)^{n-3} \alpha\),
\(\operatorname{coeff}\left(x^{2}\right)=(-1)^{n-5}\binom{n-3}{2}(n-2)^{2} \delta+(-1)^{n-4}(n-3)(n-2) \gamma+(-1)^{n-3} \beta\),
\(\operatorname{coeff}(x)=(-1)^{n-4}(n-3)(n-2) \delta+(-1)^{n-3} \gamma\),
\(\operatorname{coeff}(1)=(-1)^{n-3} \delta\).
```

So, by equating the corresponding expressions of the coefficients of $x^{n}, x^{2}$, $x, 1$, we obtain:

$$
\begin{gather*}
\alpha=\frac{1}{n!(n-2)^{n-3}} d_{n}  \tag{17}\\
\delta=(-1)^{n-1} \frac{1}{n!} d,  \tag{18}\\
\gamma=(-1)^{n-1} \frac{1}{(n-1)!}\left(d_{1}+\frac{(n-2)(n-3)}{n} d\right),  \tag{19}\\
\beta=(-1)^{n-1}\left(\frac{1}{2(n-2)!} d_{2}+\frac{(n-2)(n-3)}{(n-1)!} d_{1}+\frac{(n-2)^{3}(n-3)}{2 n!} d\right) . \tag{20}
\end{gather*}
$$

This shows that the values of $d, d_{1}, d_{2}$ and $d_{n}$ are enough to compute the coefficients of $R_{3}(u, v)$. As to $R_{1}(u, v)$, combining (8), (9) and (17) it follows that

$$
\begin{equation*}
\sigma=\frac{(-1)^{n} 8}{4(n-3)!} \chi\left(\mathcal{O}_{S}\right)-\frac{\alpha}{4}=(-1)^{n} \frac{2}{(n-3)!} \chi\left(\mathcal{O}_{S}\right)-\frac{1}{4 n!(n-2)^{n-3}} d_{n} \tag{21}
\end{equation*}
$$

Furthermore, using (13), (16) it follows that

$$
\tau=\frac{(-1)^{n-3} \chi(H)}{(n-2)!}-\frac{1}{(n-2)}\left(\frac{\alpha}{8}+\frac{\beta(n-2)}{4}+\frac{\gamma(n-2)^{2}}{2}+\delta(n-2)^{3}+\frac{\sigma}{2}\right)
$$

and plugging in such expression the values from (17), (20), (19), (18), (21) we get

$$
\begin{equation*}
\tau=\frac{(-1)^{n-1}}{(n-2)!}\left(\chi(H)+\chi\left(\mathcal{O}_{S}\right)-\frac{1}{8} d_{2}-\frac{n-2}{4} d_{1}-\frac{(n-2)^{2}\left(n^{2}-n+2\right)}{8 n(n-1)} d\right) \tag{22}
\end{equation*}
$$

Hence, for the time being, we obtain the following expression for the polynomial defining the residual cubic $C$ :

$$
\begin{aligned}
R\left(\frac{1}{2}+u, v\right)= & \frac{1}{n!(n-2)^{n-3}} d_{n} u^{3} \\
& +(-1)^{n-1}\left(\frac{1}{2(n-2)!} d_{2}+\frac{(n-2)(n-3)}{(n-1)!} d_{1}+\frac{(n-2)^{3}(n-3)}{2 n!} d\right) u^{2} v \\
& +(-1)^{n-1} \frac{1}{(n-1)!}\left(d_{1}+\frac{(n-2)(n-3)}{n} d\right) u v^{2}+(-1)^{n-1} \frac{1}{n!} d v^{3} \\
& +\left((-1)^{n} \frac{2}{(n-3)!} \chi\left(\mathcal{O}_{S}\right)-\frac{1}{4 n!(n-2)^{n-3}} d_{n}\right) u \\
& +\frac{(-1)^{n-1}}{(n-2)!}\left(\chi(H)+\chi\left(\mathcal{O}_{S}\right)-\frac{1}{8} d_{2}-\frac{n-2}{4} d_{1}-\frac{(n-2)^{2}\left(n^{2}-n+2\right)}{8 n(n-1)} d\right) v .
\end{aligned}
$$

To determine the value of $d_{i}$ we need several computations involving Chern classes. From now on, for simplicity we set $c_{i}=c_{i}(\mathcal{E}), i=1,2$. First of all, we recall the following facts. In the projective bundle $P:=\mathbb{P}(\mathcal{E})$, since $\operatorname{dim}(P)=n+1$, for any divisors $\mathcal{D}_{1}, \mathcal{D}_{2}$ on $S$, we have

$$
\xi^{n} \pi^{*} \mathcal{D}_{1}=c_{1} \mathcal{D}_{1} \quad \text { and } \quad \xi^{n-1} \pi^{*} \mathcal{D}_{1} \pi^{*} \mathcal{D}_{2}=\mathcal{D}_{1} \mathcal{D}_{2}
$$

Moreover, according to the Chern-Wu relation

$$
\xi^{n}-\xi^{n-1} \pi^{*} c_{1}+\xi^{n-2} \pi^{*} c_{2}=0
$$

we get

$$
\begin{equation*}
\xi^{n}=\xi^{n-1} \pi^{*} c_{1}-\xi^{n-2} \pi^{*} c_{2} \quad \text { and } \quad \xi^{n+1}=c_{1}^{2}-c_{2} . \tag{23}
\end{equation*}
$$

Then standard computations relying on the above relations lead to the following expression:

$$
\begin{equation*}
d=L^{n}=2\left(c_{1}^{2}-c_{2}\right)+c_{1} \mathcal{B} . \tag{24}
\end{equation*}
$$

Next, recalling that $X$ is contained in $P$ as an element of $\left|2 \xi+\pi^{*} \mathcal{B}\right|$ and the expression of $K_{X}$ given by (14), we get

$$
\begin{align*}
& d_{1}=K_{X} L^{n-1}=2(n-2) c_{2}-2(n-3) c_{1}^{2}+2 K_{S} c_{1}  \tag{25}\\
&-(n-5) c_{1} \mathcal{B}+K_{S} \mathcal{B}+\mathcal{B}^{2} \\
& d_{2}=K_{X}^{2} L^{n-2}= 2(n-3)^{2} c_{1}^{2}-2(n-2)^{2} c_{2}-4(n-3) K_{S} c_{1}  \tag{26}\\
&+2 K_{S}^{2}-2(n-4) K_{S} \mathcal{B}+\left(n^{2}-10 n+20\right) c_{1} \mathcal{B}-2(n-3) \mathcal{B}^{2}
\end{align*}
$$

and

$$
\begin{align*}
d_{n}= & K_{X}^{n}=(-1)^{n}(n-2)^{n-2}\left[\left(n^{2}-5 n+8\right) c_{1}^{2}-2(n-2)^{2} c_{2}\right.  \tag{27}\\
& \left.+n(n-1) K_{S}^{2}+2 n K_{S} c_{1}+n^{2} K_{S} \mathcal{B}+4 c_{1} \mathcal{B}+n \mathcal{B}^{2}\right]
\end{align*}
$$

Plugging (24), (25), (26), (27), in (21) and (22), respectively, we see that

$$
\begin{gather*}
\sigma=\frac{(-1)^{n}}{(n-3)!} 2 \chi\left(\mathcal{O}_{S}\right)-\frac{(-1)^{n}(n-2)}{4 n!}\left[\left(n^{2}-5 n+8\right) c_{1}^{2}-2(n-2)^{2} c_{2}\right.  \tag{28}\\
\left.+n(n-1) K_{S}^{2}+2 n K_{S} c_{1}+n^{2} K_{S} \mathcal{B}+4 c_{1} \mathcal{B}+n \mathcal{B}^{2}\right]
\end{gather*}
$$

and

$$
\begin{align*}
\tau= & \frac{(-1)^{n}}{4}\left(\frac{\left(3 n^{2}-9 n+8\right)}{n!} c_{1}^{2}-\frac{2(n-2)^{2}}{n!} c_{2}+\frac{\left(3 n^{2}-6 n+4\right)}{n!} c_{1} \mathcal{B}\right.  \tag{29}\\
& \left.+\frac{1}{(n-2)!}\left(\mathcal{B}^{2}+2 K_{S} \mathcal{B}\right)+\frac{1}{(n-2)!}\left(K_{S}^{2}-4 \chi\left(\mathcal{O}_{S}\right)-4 \chi(H)\right)\right) \\
= & \frac{(-1)^{n}}{4}\left[\frac{\left(n^{2}-7 n+8\right)}{n!} c_{1}^{2}-\frac{2(n-2)^{2}}{n!} c_{2}-\frac{\left(n^{2}+2 n-4\right)}{n!} c_{1} \mathcal{B}\right. \\
& \left.+\frac{1}{(n-2)!}\left(K_{S}^{2}-\mathcal{B}^{2}-8 \chi\left(\mathcal{O}_{S}\right)\right)\right]
\end{align*}
$$

after replacing $\chi(H)$ with its expression provided by the Riemann-Roch theorem.

Similarly, plugging (24), (25), (26), (27), in (17), (20), (19), (18), respectively, we see that

$$
\begin{gather*}
\alpha=\frac{(-1)^{n}(n-2)}{n!}\left[\left(n^{2}-5 n+8\right) c_{1}^{2}-2(n-2)^{2} c_{2}\right.  \tag{30}\\
\left.+n(n-1) K_{S}^{2}+2 n K_{S} c_{1}+n^{2} K_{S} \mathcal{B}+4 c_{1} \mathcal{B}+n \mathcal{B}^{2}\right] \\
\delta=(-1)^{n-3} \frac{1}{n!}\left(2 c_{1}^{2}-2 c_{2}+c_{1} \mathcal{B}\right)  \tag{31}\\
\gamma=\frac{(-1)^{n-1}}{n!}\left[4(3-n) c_{1}^{2}+6(n-2) c_{2}+6 c_{1} \mathcal{B}\right.  \tag{32}\\
\left.\quad+n\left(2 K_{S} c_{1}+K_{S} \mathcal{B}+\mathcal{B}^{2}\right)\right] \\
\beta=(-1)^{n-1}\left[\frac{1}{n!}\left(3 n^{2}-17 n+24\right) c_{1}^{2}-\frac{6}{n!}(n-2)^{2} c_{2}+\frac{1}{(n-2)!} K_{S}^{2}\right.  \tag{33}\\
\left.-\frac{(n-3)}{(n-1)!}\left(2 K_{S} c_{1}+\mathcal{B}^{2}\right)+\frac{2}{(n-1)!} K_{S} \mathcal{B}-\frac{\left(n^{2}+2 n-12\right)}{n!} c_{1} \mathcal{B}\right]
\end{gather*}
$$

The above discussion proves the following result.
Proposition 3.1. Let $(X, L)$ be a geometric quadric fibration over a smooth surface $S$, as in Section 2. Then the residual cubic of its Hilbert curve is defined by (3), where the homogeneous part of degree 3 is

$$
\begin{align*}
R_{3}(u, v)= & \frac{(-1)^{n-1}}{n!}\left\{-(n-2)\left[\left(n^{2}-5 n+8\right) c_{1}^{2}-2(n-2)^{2} c_{2}\right.\right.  \tag{34}\\
& \left.+n^{2} K_{S} \mathcal{B}+n(n-1) K_{S}^{2}+2 n K_{S} c_{1}+4 c_{1} \mathcal{B}+n \mathcal{B}^{2}\right] u^{3} \\
& +\left[\left(3 n^{2}-17 n+24\right) c_{1}^{2}-6(n-2)^{2} c_{2}+n(n-1) K_{S}^{2}\right. \\
& \left.-n(n-3)\left(2 K_{S} c_{1}+\mathcal{B}^{2}\right)+2 n K_{S} \mathcal{B}-\left(n^{2}+2 n-12\right) c_{1} \mathcal{B}\right] u^{2} v \\
& +\left[4(3-n) c_{1}^{2}+6(n-2) c_{2}+6 c_{1} \mathcal{B}+n\left(2 K_{S} c_{1}+K_{S} \mathcal{B}+\mathcal{B}^{2}\right)\right] u v^{2} \\
& \left.+\left(2 c_{1}^{2}-2 c_{2}+c_{1} \mathcal{B}\right) v^{3}\right\}
\end{align*}
$$

while the homogenous part of degree 1 is

$$
\begin{align*}
R_{1}(u, v)= & \frac{(-1)^{n}}{4 n!}\left\{\left(8 n(n-1)(n-2) \chi\left(\mathcal{O}_{S}\right)-(n-2)\left[\left(n^{2}-5 n+8\right) c_{1}^{2}\right.\right.\right.  \tag{35}\\
& -2(n-2)^{2} c_{2}+n(n-1) K_{S}^{2}+2 n K_{S} c_{1}+n^{2} K_{S} \mathcal{B}+4 c_{1} \mathcal{B} \\
& \left.\left.+n \mathcal{B}^{2}\right]\right) u+\left[\left(n^{2}-7 n+8\right) c_{1}^{2}-2(n-2)^{2} c_{2}-\left(n^{2}+2 n-4\right) c_{1} \mathcal{B}\right. \\
& \left.+4 n(n-1)\left(K_{S}^{2}-\mathcal{B}^{2}-8 \chi\left(\mathcal{O}_{S}\right)\right] v\right\} .
\end{align*}
$$

We like to point out that if we plug $n=3$ in (34) and (35) then their sum gives the equation of $\Gamma$ in [6, Proposition 4.1].

## 4. First properties of the Hilbert curve

Let $\ell_{\infty}$ be the line at infinity of the $(u, v)$ plane. We denote by $\ell_{0}$ the line of equation $(n-2) u-v=0$, whose point at infinity is $P_{\infty}:=(1: n-2: 0)$.

Lemma 4.1. Let $C$ be any Serre-invariant plane cubic and let (3) be its equation, with $R_{3}$ and $R_{1}$ given by (4) and (5) respectively.
a) The projective closure $\bar{C}$ of $C$ contains the point $P_{\infty}$ if and only if

$$
\begin{equation*}
\alpha+(n-2) \beta+(n-2)^{2} \gamma+(n-2)^{3} \delta=0 . \tag{36}
\end{equation*}
$$

b) $C$ contains the line $\ell_{0}$ if and only if, in addition to (36), we have

$$
\begin{equation*}
\sigma+(n-2) \tau=0 \tag{37}
\end{equation*}
$$

Proof. If we put $v=(n-2) u$ in (3) then (36) and (37) express the vanishing of the homogeneous parts of degree 3 and 1 of the polynomial in (3), respectively. This proves a) and b).

The computations done for Proposition 3.1 have the following consequence.
Proposition 4.2. Let $(X, L)$ be a geometric quadric fibration over $S$ as in Section 2 and let $C$ be the residual cubic of its Hilbert curve. Then condition (36) is always satisfied for $C$. Moreover, $\bar{C}$ intersects $\ell_{\infty}$ at $P_{\infty}$, transversely.
Proof. If we plug the values (30), (33), (32), (31) in (36) we get an expression involving $c_{1}^{2}, c_{2}, K_{S}^{2}, K_{S} c_{1}, K_{S} \mathcal{B}, \mathcal{B}^{2}, c_{1} \mathcal{B}$, with appropriate coefficients. At a close look such coefficients are all zeroes, hence our former claim follows. To prove the latter, suppose, by contradiction, that $\bar{C}$ intersects $\ell_{\infty}$ at $P_{\infty}$ with multiplicity $>1$. Then, dividing $R_{3}(u, v)$ by $u^{3}$ and letting $t:=v / u$, the value $n-2$ has to be a common root of the polynomial

$$
\delta t^{3}+\gamma t^{2}+\beta t+\alpha
$$

and of its derivative. Hence

$$
\begin{equation*}
3(n-2)^{2} \delta+2(n-2) \gamma+\beta=0 \tag{38}
\end{equation*}
$$

However, taking into account (31), (32) and (33), the term on the left hand side of (38) becomes

$$
3(n-2)^{2} \delta+2(n-2) \gamma+\beta=\frac{(-1)^{n}}{(n-2)!}\left(K_{S}+c_{1}+\mathcal{B}\right)^{2}
$$

and since $H=K_{S}+c_{1}+\mathcal{B}$ is the ample divisor in (15), this cannot be zero, a contradiction.

As to the residual intersections of $\bar{C}$ with $\ell_{\infty}$ we have the following result.
Proposition 4.3. Let $C$ be a Serre-invariant plane cubic as in Lemma 4.1 such that $P_{\infty} \in \bar{C}$, and let $Q_{\infty}$ be a point at infinity distinct from $P_{\infty}$. The cubic $\bar{C}$ intersects $\ell_{\infty}$ at $Q_{\infty}$ with multiplicity 2 if and only if the following condition

$$
\begin{equation*}
4 \alpha \delta+(n-2)(\gamma+(n-2) \delta)^{2}=0 \tag{39}
\end{equation*}
$$

is satisfied, in addition to (36). Moreover, in this case, if $\bar{C}$ is irreducible, then $\bar{C}$ is singular at $Q_{\infty}$.

Proof. Let $Q_{\infty}=(-a: b: 0)$. Dividing $R_{3}(u, v)$ by $u^{3}$ and letting $t=v / u$, as before, we see that

$$
\begin{equation*}
\bar{C} \cap \ell_{\infty}=P_{\infty}+2 Q_{\infty} \tag{40}
\end{equation*}
$$

if and only if

$$
\begin{equation*}
\delta t^{3}+\gamma t^{2}+\beta t+\alpha=(t-n+2)(a t+b)^{2} \tag{41}
\end{equation*}
$$

identically with respect to $t$. This is equivalent to

$$
\begin{equation*}
\delta=a^{2}, \quad \gamma=2 a b-(n-2) a^{2}, \quad \beta=b^{2}-2 a b(n-2), \quad \alpha=-(n-2) b^{2} \tag{42}
\end{equation*}
$$

and eliminating $a, b$ from these equations gives (36) and (39). Now suppose that $\bar{C}$ is irreducible and smooth at $Q_{\infty}$. The Serre involution $(u, v) \mapsto$ $(-u,-v)$ induces an involution $\iota: \bar{C} \rightarrow \bar{C}$ such that $\iota\left(Q_{\infty}\right)=Q_{\infty}$. Then, as in [3, Lemma 3.3], we see that either $\iota$ is the identity map, or its differential, acting on the tangent space $T_{Q_{\infty}}(\bar{C})$ to $\bar{C}$ at $Q_{\infty}$, is the multiplication by -1 . But the projective closure of $T_{Q_{\infty}}(\bar{C})$ is $\ell_{\infty}$ itself because the intersection index of $\bar{C}$ and $\ell_{\infty}$ at $Q_{\infty}$ is 2 . We thus get a contradiction since the Serre involution induces the identity on $\ell_{\infty}$ but not on $\bar{C}$. It thus follows that $Q_{\infty}$ is a singular point of $\bar{C}$.

Actually, more can be said about the singular point $Q_{\infty}$.
Proposition 4.4. Let $C$ be a Serre-invariant plane cubic such that $P_{\infty} \in \bar{C}$. If $C$ is irreducible and $Q_{\infty}$ is a double point of $\bar{C}$, then $Q_{\infty}$ is necessarily a node. Moreover the Serre involution exchanges the principal tangents to $\bar{C}$ at $Q_{\infty}$.

Proof. Let $Q_{\infty}=(-a: b: 0)$ be a double point and suppose that $a \neq b$. Up to the change of homogenous coordinates $u=U-a W, v=V+b W, w=U+V$, with $a \neq b$ we can assume that $Q_{\infty}$ is the origin. In the new affine coordinates $U, V$ (if we set $W=1$ ) the equation of $\bar{C}$, after using (42), is:

$$
\begin{gathered}
(b U+a V)^{2}[b+(n-2) a]-(U+V)^{2}(a \sigma-b \tau)+\left[\sigma-b^{2}(n-2)\right] U^{3}+\left[b^{2}\right. \\
-2 a b(n-2)+2 \sigma+\tau] U^{2} V+\left[2 a b-a^{2}(n-2)+\sigma+2 \tau\right] U V^{2}+\left(a^{2}+\tau\right) V^{3}=0 .
\end{gathered}
$$

The coefficient of the first term is not zero because $Q_{\infty} \neq P_{\infty}$. We thus see that $Q_{\infty}$ is a cusp, the line of equation $b U+a V=0$ being the unique principal tangent to $\bar{C}$ at $Q_{\infty}$, if and only if $\sigma a-\tau b=0$. Next, note that the point $O$, the origin of the affine coordinates $(u, v)$, is a smooth point of $C$, due to the assumptions. Thus the condition $\sigma a-\tau b=0$ is equivalent to saying that the line tangent to $\bar{C}$ at $O$ (whose equation is $\sigma u+\tau v=0$ ) contains the point $Q_{\infty}$. But then, the intersection index of this line and $\bar{C}$ would be greater than 3 (2 intersections at $O$, due to the tangency plus 2 intersections at least at the singular point $Q_{\infty}$ ), a contradiction. If $a=b$, the same argument as above works by using the following change of homogeneous coordinates: $u=U+V-W, v=U+W, w=U+V$.

To see that the Serre involution exchanges the principal tangents at the node $Q_{\infty}$, let $\bar{s}$ denote the extension of the Serre involution to $\mathbb{P}^{2}$. If $\ell$ is a principal tangent at $Q_{\infty}$, then $\bar{s}(\ell)$ is also a principal tangent. But if $\bar{s}(\ell)=\ell$, then necessarily $\ell$ must contain $O$. This comes from the fact that the only lines fixed by $\bar{s}$ are those in the pencil through $O$ plus $\ell_{\infty}$. The latter, however, cannot be a principal tangent to $\bar{C}$ at $Q_{\infty}$, since the multiplicity of intersection is just 2. But then the intersection index of $\ell$ and $\bar{C}$ would be greater than 3 (1
intersection at $O$ and 3 at $Q_{\infty}$, since it is a principal tangent), a contradiction again.

The following result is a generalization of [6, Theorem 5.2].
Theorem 4.5. Let $(X, L)$ be a geometric quadric fibration of dimension $n$ over $S$ and let $C$ be the residual cubic of the Hilbert curve $\Gamma$ of $(X, L)$. Then
(i) $\ell_{0}$ is contained in $C$ if and only if $X$ is a bundle.
(ii) $C$ has a triple point if and only if $X$ is a bundle and

$$
K_{X}^{n}+(-1)^{n-1} 8 n(n-1)(n-2)^{n-2} \chi\left(\mathcal{O}_{X}\right)=0
$$

(iii) If $\ell_{0}$ is contained in $C$, then it is an irreducible component of multiplicity 1 of $C$.
Proof. A tedious check shows that

$$
\begin{equation*}
\sigma+(n-2) \tau=\frac{(-1)^{n+1}}{4 n!} n(n-2)\left(2 c_{1}+n \mathcal{B}\right)\left(K_{S}+c_{1}+\mathcal{B}\right) \tag{43}
\end{equation*}
$$

Recalling that the discriminant curve $\mathcal{D} \in\left|2 c_{1}+n \mathcal{B}\right|$ and that $H=K_{S}+c_{1}+\mathcal{B}$ is the ample divisor in (15) this shows that

$$
\begin{equation*}
\sigma+(n-2) \tau=\frac{(-1)^{n+1}}{4 n!} n(n-2) \mathcal{D} H \tag{44}
\end{equation*}
$$

Therefore $\sigma+(n-2) \tau=0$ if and only if $\mathcal{D}=0$, i.e. $X$ has no singular fibers. Then (i) follows from Lemma 4.1, b) taking into account Proposition 4.2. To prove (ii) note that if $C$ has a triple point, then the origin must be a triple point, and this happens if and only if $\sigma=\tau=0$. This is equivalent to $\sigma=\sigma+(n-2) \tau=0$ and we know from (i) that the latter of these two conditions is equivalent to $X$ being a bundle, hence to the fact that $\mathcal{B}=-\frac{2}{n} c_{1}$. Replace $\mathcal{B}$ with this value in the expression of $\sigma$ provided by (21). Recalling that $\chi\left(\mathcal{O}_{X}\right)=\chi\left(\mathcal{O}_{S}\right)$ since $X$ is a bundle, we thus get (ii). Finally, (iii) follows from the latter assertion in Proposition 4.2.

The next question we want to address is about the nonreducedness of $\Gamma$, where $\Gamma$ is the Hilbert curve of a geometric quadric fibration $(X, L)$ as in Section 2. First of all consider the residual cubic $C$. As a consequence of Theorem 4.5 we have the following result.
Corollary 4.6. Let $(X, L)$ be a geometric quadric fibration of dimension $n$ over $S$ and let $C$ be the residual cubic of the Hilbert curve $\Gamma$ of $(X, L)$. Then $C$ is nonreduced if and only if $C=\ell_{0}+2 \ell^{\prime}$, with $\ell^{\prime}$ a line through the origin transverse to $\ell_{0}$. This happens if and only if (40) holds, where $Q_{\infty} \neq P_{\infty}$, i.e. if and only if, letting $Q_{\infty}=(-a: b: 0)$, the coefficients of $R_{3}(u, v)$ satisfy conditions (36) and (39).

Proof. Suppose that $C$ is nonreduced. Clearly for no line $\ell$ it can happen that $C=3 \ell$, in view of Proposition 4.2. Therefore $C=\ell+2 \ell^{\prime}$ where $\ell$ and $\ell^{\prime}$ are two distinct lines, which cannot be parallel, by Proposition 4.2. Thus $C$ has a single triple point at $\ell \cap \ell^{\prime}$, which necessarily has to be the origin, and then Theorem 4.5 and Proposition 4.2 again imply that $X$ is a bundle and $\ell=\ell_{0}$. Moreover, (40) holds, where $Q_{\infty}$ is the point at infinity of $\ell^{\prime}$. Then the last assertion follows from Proposition 4.3. The converse is obvious.

For an example of the situation described in Corollary 4.6, see [6, Example 5.3, case (ii) on p. 556 and Remark 5.4].

Next look at $\Gamma$. In view of Corollary 4.6 we can suppose that $C$ is reduced. Assume that $\Gamma=\ell_{1}+\cdots+\ell_{n-3}+C$ is nonreduced; then $C=\ell+\gamma$ is necessarily reducible into a line $\ell$ and a conic $\gamma$ which could possibly be reducible. Recall that $\ell_{1}, \ldots, \ell_{n-3}$ have the same point at infinity, which is $P_{\infty}$. Due to Proposition 4.2 there are two possibilities: either
i) $P_{\infty}$ is the point at infinity of $\ell$ but it does not belong to $\bar{\gamma}$, or
ii) $P_{\infty}$ is a point at infinity of $\gamma$ but not of $\ell$.

In case i ), even if $\gamma$ is reducible no line constituting $\gamma$ can overlap one of the $\ell_{i}$ 's, having a point at infinity distinct from $P_{\infty}$. On the other hand, $\ell$ has to contain the origin $O$ regardless of the rank of $\gamma$, in view of the symmetry of $C$, hence $\ell=<O, P_{\infty}>=\ell_{0}$. Therefore $\ell_{0}$ must coincide with one of the $\ell_{i}$ 's $(i=1, \ldots, n-3)$. Since $\ell_{i}$ is described, in coordinates $u, v$ by $(n-2) u-v-$ $\left(i+1-\frac{n}{2}\right)=0$, we have that $\ell_{0}=\ell_{i}$ if and only if $n=2 m \geq 4$ and $i=m-1$.

In case ii), since $C$ is reduced, the nonreducedness of $\Gamma$ implies that $\gamma$ is reducible in two lines, one of which, say $\ell^{\prime}$, has $P_{\infty}$ as point at infinity. Since $C$ also contains the line $\ell$, we conclude that $C$ has a triple point, which is the origin $O$, due to the symmetry, hence $\ell^{\prime}=<O, P_{\infty}>=\ell_{0}$. Then up to exchanging $\ell$ with $\ell^{\prime}$ we fall in case i) again and we get the same conclusion.

As we have seen, if $\Gamma$ is nonreduced, then $\ell_{0} \subset C$ regardless of the fact that $C$ is reduced or not; hence $X$ is a bundle; moreover, if $C$ is reduced, $\ell_{0}$ is the unique irreducible multiple component of $\Gamma$.

We want to stress the following fact. Suppose that $C$ has no triple point (or, equivalently, that $\gamma$ has not a double point at the origin). This is equivalent to requiring that the polynomial $R_{1}(u, v)$ is not identically zero. In this case, it represents $\ell_{0}$, hence it divides $R_{3}(u, v)$, since $C=\ell_{0}+\gamma$. So $R_{3}(u, v)=$ $Q(u, v) R_{1}(u, v)$, where $Q$ is a homogeneous polynomial in $u, v$ of degree 2 and then, recalling (3), $C$ has equation

$$
\begin{equation*}
R\left(\frac{1}{2}+u, v\right)=(Q(u, v)+1) R_{1}(u, v)=0 \tag{45}
\end{equation*}
$$

Therefore the conic $\gamma$ is described by $Q(u, v)+1=0$; this clearly shows that its rank is $\geq 2$. In particular, if equality holds, our assumptions imply that $\gamma$ consist of two parallel lines, symmetric with respect to the origin. This situation does really occur, as [6, Example 7.1, equation (41) at p. 563] shows.

In conclusion, we have
Proposition 4.7. Let $(X, L)$ be a geometric quadric fibration of dimension $n$ over a smooth surface $S$ as in Section 2. Its Hilbert curve $\Gamma$ is nonreduced if and only if either

1. $C=\gamma+\ell_{0}$, where $\gamma$ is a conic of rank $\geq 2$ and $n=2 m \geq 4$, or
2. $C$ is non reduced.

In both cases $X$ is a bundle. In the former case $\ell_{0}$ is the only multiple component of $\Gamma$ and its multiplicity is 2 ; in the latter, $\gamma=2 \ell^{\prime}$, where $\ell^{\prime} \neq \ell_{0}$ is a line; $\ell^{\prime}$ is the only component of multiplicity 2 of $\Gamma$, unless $n=2 m \geq 4$, in which case $\ell_{0}$ is a further component of multiplicity 2.

Case 1. in Proposition 4.7 is clearly a novelty with respect to what is known for $n=3$.

## 5. More on the residual cubic $C$

In this Section we analyze further the reducibility of the residual cubic $C$. More generally, we first look at reducible Serre-invariant plane cubics.

Proposition 5.1. Let $C \subset \mathbb{A}^{2}$ be a Serre-invariant plane cubic such that $\bar{C}$ meets $\ell_{\infty}$ transversely at $P_{\infty}$, and let $O$ be the origin of coordinates $(u, v)$. If $C$ is reducible then $C=\ell+\gamma$, where $\ell$ is a line passing through $O$ and $\gamma$ is a conic, possibly reducible. Moreover, either
a) $\gamma$ is of hyperbolic type, with center at $O$ (in particular it has two distinct points at infinity), or
b) $\gamma$ consist of two parallel lines.

Proof. Clearly, if $C$ is reducible, then it contains a line, say $\ell$. There are two possibilities: either
i) the line $\ell$ contains the origin $O$, or
ii) the line $\ell$ does not contain $O$.

We claim that in case ii) $\ell$ is an irreducible component of a conic residual with respect to another line, which is also contained in $C$, and passes through $O$. So, up to renaming, case ii) reduces to i), which gives the first assertion in the
statement. To prove the claim, note that the map $\iota: C \rightarrow C$ induced by the Serre involution maps $\ell$ to another line $\iota(\ell)$, which also does not contain $O$. Thus $C$ consists of three lines, two of which do not contain $O$, hence $O$ belongs to the third line, say $\lambda$. Note that $\ell$ and its conjugate $\iota(\ell)$ are parallel, due to the symmetry properties of $C$. Thus their projective closures cannot contain the point $P_{\infty}=(1: n-2: 0)$, in view of the assumption on $\bar{C}$. It thus follows from Lemma 4.1 a) that $P_{\infty}$ is the point at infinity of $\lambda$ and therefore $\lambda=\ell_{0}$, since it contains both the origin and $P_{\infty}$. In conclusion, in case ii) we have that $C=\ell_{0}+\ell^{\prime}+\ell^{\prime \prime}$, where $\ell^{\prime}$ and $\ell^{\prime \prime}$ are two parallel lines, and letting $\gamma=\ell^{\prime}+\ell^{\prime \prime}$ this gives $b$ ) in the statement. Next come to case i). Clearly $\gamma:=C-\ell$ is symmetric with respect to $O$. Hence $\gamma$ is as in $a$ ) (regardless the fact that it is irreducible or not) if $O$ is its unique center. Otherwise it is as in $b$ ), since it cannot be a parabola, because it is Serre-invariant itself.

Now let $C$ be any Serre-invariant plane cubic. If $C$ has a triple point then necessarily it has a triple point at the origin, hence assuming that $C$ has not a triple point is equivalent to requiring that $(\sigma, \tau) \neq(0,0)$. So, let $C$ be a Serre-invariant reducible plane cubic again. Suppose that $C$ has not a triple point. Then $R_{1}(u, v)=0$ represents a line $\ell$ through $O$. Moreover, since $C$ is reducible, $R_{1}(u, v)$ divides $R_{3}(u, v)$, hence $R_{3}(u, v)=Q(u, v) R_{1}(u, v), Q$ being a homogeneous (nontrivial) polynomial of degree two in $u$ and $v$. Thus, $C$ is described by (45). This shows that $\ell$ is a component of $C$ and the conic residual of $\ell$ in $C$ has rank $\geq 2$, in accordance with the assumption that $C$ has not a triple point. Now, by applying the same argument as in [6, p. 551] we see that the existence of a polynomial $Q$ as above is equivalent to the condition

$$
\begin{equation*}
\sigma^{2}(\sigma \delta-\tau \gamma)+\tau^{2}(\sigma \beta-\tau \alpha)=0 \tag{46}
\end{equation*}
$$

Note that (46) is trivially satisfied also when $C$ has a triple point. On the other hand (45) obviously implies reducibility. Therefore, we have

Proposition 5.2. Let $C$ be a Serre-invariant plane cubic and let (3) be its equation, with $R_{3}$ and $R_{1}$ given by (4) and (5) respectively. Then $C$ is reducible if and only if (46) holds.

In particular, we get the following consequence.
Corollary 5.3. Let $(X, L)$ be a geometric quadric fibration over a smooth surface, as in Section 2, and let $C$ be the residual cubic of its Hilbert curve with respect to the lines $\ell_{1}, \ldots, \ell_{n-3}$. Then $C$ is reducible if and only if (46) holds.

For $n=4$, assuming that $S=\mathbb{P}^{2}$, we can characterize the fact that $C$ contains a given line $\ell$ through the origin even more explicitly. In view of Theorem 4.5(i), we can suppose that $\ell \neq \ell_{0}$.

Proposition 5.4. Let $(X, L)$ be a 4-dimensional geometric quadric fibration over $\mathbb{P}^{2}$ and let $\Gamma$ be its Hilbert curve. Then $\Gamma$ contains the line $\ell: p u-q v=0$ $((p, q) \neq(0,0))$, with $p \neq 2 q$, if and only if $\left[p\left(c_{1}+b-1\right)-4 q\right]\left[p\left(c_{1}+b+1\right)-8 q\right]=$ 0 , where $c_{1}$ and $b$ are such that $c_{1}(\mathcal{E})=\mathcal{O}_{\mathbb{P}^{2}}\left(c_{1}\right)$ and $\mathcal{B}=\mathcal{O}_{\mathbb{P}^{2}}(b)$.
Proof. Because $\operatorname{dim}(X)=4$ then $\Gamma=\ell_{1}+C$, where $\ell_{1}: 2 u-v=0$ and $C$ is the residual cubic. Thus if the line $\ell: p u-q v=0$, with $p \neq 2 q$ is contained in $\Gamma$ it follows that it is a component of the residual cubic $C$. Because the base of the geometric quadric fibration $(X, L)$ is $\mathbb{P}^{2}, c_{1}(\mathcal{E})=\mathcal{O}_{\mathbb{P}^{2}}\left(c_{1}\right)$ and $c_{2}(\mathcal{E})=c_{2}$, for some $c_{1}, c_{2} \in \mathbb{Z}$. Thus the coefficients of the terms in $R_{3}(u, v)$ become, up to the factor $\frac{1}{24}$ :

$$
\begin{gathered}
\alpha=8 c_{1}^{2}-16 c_{2}-48 c_{1}-96 b+8 c_{1} b+8 b^{2}+216, \\
\beta=-\left(4 c_{1}^{2}-24 c_{2}+24 c_{1}-12 c_{1} b-4 b^{2}-24 b+108\right), \\
\gamma=4 c_{1}^{2}-12 c_{2}+24 c_{1}-6 c_{1} b+12 b-4 b^{2} \\
\delta=-2 c_{1}^{2}+2 c_{2}-c_{1} b
\end{gathered}
$$

Likewise the coefficients of $u$ and $v$ in $R_{1}(u, v)$ are, up to the factor $\frac{1}{24}$, respectively

$$
\begin{gathered}
\sigma=-\left(2 c_{1}^{2}-4 c_{2}-12 c_{1}-24 b+2 c_{1} b+2 b^{2}+6\right) \\
\tau=-\left(c_{1}^{2}+2 c_{2}+5 c_{1} b+3 b^{2}-3\right)
\end{gathered}
$$

In view of Proposition 5.3, the line $\ell: p u-q v=0$ is a component of $C$ if and only if (46) holds with $\sigma=p k$ and $\tau=-q k$ for some non zero $k \in \mathbb{Z}$ (since $\sigma$ and $\tau$, expressed by the above equalities, are integers). Recalling that $p \neq 2 q$, the last two conditions, combined with the above expressions of $\sigma$ and $\tau$, give

$$
\begin{aligned}
c_{2}=-\frac{1}{2} & \frac{1}{p-2 q}\left(2 q c_{1}^{2}+2 q b^{2}-12 q c_{1}-24 q b+2 q c_{1} b+p c_{1}^{2}+5 p c_{1} b+3 p b^{2}\right. \\
& -3 p+6 q)
\end{aligned}
$$

The relation (46), after replacing $\sigma=p k, \tau=-q k$ and $\alpha, \beta, \gamma, \delta$ with the above expressions, becomes

$$
\begin{align*}
& k^{3}(2 q-p)\left(4 q^{2} c_{1}^{2}-8 q^{2} c_{2}+108 q^{2}-24 q^{2} c_{1}-48 q^{2} b+4 q^{2} b c_{1}\right. \\
&+4 q^{2} b^{2}+8 q p c_{2}-24 q p c_{1}+8 q p c_{1} b+4 q p b^{2} \\
&\left.-12 q p b-2 p^{2} c_{2}+2 p^{2} c_{1}^{2}+p^{2} c_{1} b\right)=0 \tag{47}
\end{align*}
$$

Because $k \neq 0$, after dividing out (47) with $k^{3}$ and replacing the value of $c_{2}$, we see that (47) can be rewritten as

$$
3(2 q-p)\left(-4 q-p+c_{1} p+b p\right)\left(-8 q+p+c_{1} p+b p\right)=0
$$

and this proves the assertion, since, as we said, $p-2 q \neq 0$.

As a further comment, we note the following. If $p\left(c_{1}+b-1\right)-4 q=0$ then $b=-c_{1}+\frac{4 q}{p}+1$ and in this case $C=\ell+\gamma_{1}$, where the equation of $\gamma_{1}$ is

$$
\begin{aligned}
\left(16 p+4 p c_{1}-32 q\right) u^{2}+\left(-4 p c_{1}-10 p+20 q\right) u v+\left(p c_{1}\right. & +p-2 q) v^{2} \\
& +2 p-p c_{1}+8 q=0
\end{aligned}
$$

On the other hand, if $p\left(c_{1}+b+1\right)-8 q=0$ then $b=-c_{1}+\frac{8 q}{p}-1$ and in this case $C=\ell+\gamma_{2}$, where the equation of $\gamma_{2}$ is

$$
\begin{aligned}
\left(20 p+4 p c_{1}-64 q\right) u^{2}+\left(-4 p c_{1}-8 p+40 q\right) u v+( & \left.p c_{1}-p-4 q\right) v^{2} \\
& -2 p-p c_{1}+16 q=0
\end{aligned}
$$

For instance, as to $\gamma_{2}$, the determinant of its matrix is $36(p-2 q)^{2}\left(p\left(c_{1}+2\right)-\right.$ $16 q)$. Hence $\gamma_{2}$ is reducible if and only if $p=16 q /\left(c_{1}+2\right)$.

## 6. A special class of geometric quadric fibrations

Here we introduce a special class of quadric fibrations $(X, L)$ which generalize conic fibrations considered in [6, Section 6]. In line with [6], we call them quadric fibrations deriving from cones since they are defined by generic quadric sections of a cone with vertex a point over a scroll on a surface. As we will see, they can never be quadric bundles, however the equation of the corresponding Hilbert curve simplifies considerably with respect to that of a general quadric fibration. The construction goes as follows.

### 6.1. Construction.

Let $S$ be a smooth surface and let $\mathcal{V}$ be a very ample vector bundle of rank $n-1 \geq 2$ on $S$. Set $T:=\mathbb{P}(\mathcal{V})$ and denote by $h$ the tautological line bundle. Then $h$ embeds $T$ as an $n$-dimensional scroll over $S$ in some projective space, say $\mathbb{P}^{m}$. Now set $\mathcal{E}:=\mathcal{V} \oplus \mathcal{O}_{S}$, and $R:=\mathbb{P}(\mathcal{E})$. Then $R$ is a $\mathbb{P}^{n-1}$-bundle over $S$, with projection $\pi: R \rightarrow S$. Let $\xi$ be the tautological line bundle of $\mathcal{E}$ on $R$ and denote by $\phi: R \rightarrow \mathbb{P}^{N}$ the map defined by $\xi$. Clearly $\phi$ is a morphism, since $\mathcal{E}$ is spanned. We have

$$
\begin{equation*}
c_{i}(\mathcal{E})=c_{i}(\mathcal{V}) \quad \text { for } i=1,2, \tag{48}
\end{equation*}
$$

by construction. Furthermore, from the additivity of $H^{0}$, we get $h^{0}(\xi)=$ $h^{0}(\mathcal{E})=h^{0}(\mathcal{V})+1=h^{0}(h)+1$, hence $N=m+1$. Note that $\xi$ restricts trivially to the section, say $\sigma$, of $R$ corresponding to the obvious surjection $\mathcal{E} \rightarrow \mathcal{O}_{S}$. Hence $\phi$ contracts $\sigma$ to a point, say $v$, of the image $Y:=\phi(R)$. Due to the properties of $\phi, Y \subset \mathbb{P}^{m+1}$ is the cone over $T$ with vertex $v, \phi: R \rightarrow Y$ being
the desingularization morphism; in fact, any fiber $F_{s}=\pi^{-1}(s)$ of $\pi: R \rightarrow S$ is a $\mathbb{P}^{n-1}$, and $\phi$ maps it isomorphically to the linear subspace of $\mathbb{P}^{N}$ spanned by $v$ and the $\mathbb{P}^{n-2}$ which is the fiber of the scroll $T$ over $s$. Now consider a general quadric hypersurface $Q \subset \mathbb{P}^{N}$ (i.e., not containing $v$ ) and let $X \subset R$ be its inverse image via $\phi$. Then $X \in|2 \xi|$, because $\xi=\phi^{*} \mathcal{O}_{\mathbb{P}^{m+1}}(1)$. Note that $X$ is smooth and $L:=\xi_{X}$ is ample since $X \subset R \backslash \sigma$. Moreover, $X$ intersects every fibre $F_{s}$ of $\pi$ along a quadric. Therefore, by restricting $\pi$ to $X$ we get a fibration $\varphi:=\left.\pi\right|_{X}: X \rightarrow S$ in quadrics over $S$.

Because $L=\xi_{X}$ we have that $\mathcal{E}=\pi_{*} \xi=\varphi_{*} L$, thus $R$ is exactly the $\mathbb{P}^{n-1}$-bundle $P$ introduced in Section 2.

By the canonical bundle formula, recalling (48), we know that $K_{R}=-n \xi+$ $\pi^{*}\left(K_{S}+\operatorname{det} \mathcal{V}\right)$, thus, since $X \in|2 \xi|$, we get by adjunction

$$
\begin{aligned}
K_{X}=\left(K_{R}+X\right)_{X} & =\left(-(n-2) \xi+\pi^{*}\left(K_{S}+\operatorname{det} \mathcal{V}\right)\right)_{X} \\
& =-(n-2) L+\varphi^{*}\left(K_{S}+\operatorname{det} \mathcal{V}\right)
\end{aligned}
$$

The fact that $\mathcal{B}$ is trivial implies that $H:=K_{S}+\operatorname{det} \mathcal{E}=K_{S}+\operatorname{det} \mathcal{V}$, hence $H$ is ample unless $(S, \mathcal{V})$ is in a precise list of exceptions [8, Main Theorem]. Therefore,

Proposition 6.1. Let $(X, L)$ be a quadric fibration over $S$ deriving from cones. Then $(X, L)$ is a geometric quadric fibration if and only if $(S, \mathcal{V})$ is not one of the following pairs:
(i) $\left.(S, \mathcal{V})=\left(\mathbb{P}^{2}, \mathcal{O}_{\mathbb{P}^{2}}(1)^{\oplus r}\right)\right)$, with $r=2,3$,
(ii) $\left.(S, \mathcal{V})=\left(\mathbb{P}^{2}, \mathcal{O}_{\mathbb{P}^{2}}(2) \oplus \mathcal{O}_{\mathbb{P}^{2}}(1)\right)\right)$,
(iii) $(S, \mathcal{V})=\left(\mathbb{P}^{2}, T_{\mathbb{P}^{2}}\right)$,
(iv) $S$ is a $\mathbb{P}^{1}$-bundle over a smooth curve and $\mathcal{V}$ restricts as $\mathcal{O}_{\mathbb{P}^{1}}(1)^{\oplus 2}$ to every fiber.

Note that the exception $\left.(S, \mathcal{V})=\left(\mathbb{Q}^{2}, \mathcal{O}_{\mathbb{Q}^{2}}(1)^{\oplus 2}\right)\right)($ case 6 in $[8$, Main Theorem]) is included in case (iv).

According to what we said before, for quadric fibrations $(X, L)$ as in the above construction, the line bundle $\mathcal{B}$ is trivial. This has strong implications. As a first thing, we observe the following fact.

Remark 6.1. If $(X, L)$ is quadric fibration deriving from cones then $X$ cannot be a bundle. Actually, were $X$ a bundle, the fact that $\mathcal{D} \in\left|2 c_{1}(\mathcal{E})+n \mathcal{B}\right|$ would imply $c_{1}(\mathcal{E})=0$, hence $c_{1}(\mathcal{V})=0$ by (48), but this would prevent $\mathcal{V}$ from being ample.

By Proposition 3.1 the equation of the residual cubic of its Hilbert curve is such that the homogeneous part of degree 3 is

$$
\begin{align*}
R_{3}(u, v)= & \frac{(-1)^{n}}{n!}\{(n-2)
\end{aligned} \begin{aligned}
& \left.\left(n^{2}-5 n+8\right) c_{1}^{2}-2(n-2)^{2} c_{2}+n(n-1) K_{S}^{2}+2 n K_{S} c_{1}\right] u^{3} \\
+\left[\left(3 n^{2}-17 n\right.\right. & \left.+24) c_{1}^{2}-6(n-2)^{2} c_{2}+n(n-1) K_{S}^{2}-2 n(n-3) K_{S} c_{1}\right] u^{2} v  \tag{49}\\
& \left.+\left[4(3-n) c_{1}^{2}+6(n-2) c_{2}+2 n K_{S} c_{1}\right] u v^{2}+\left(2 c_{1}^{2}-2 c_{2}\right) v^{3}\right\}
\end{align*}
$$

while the homogenous part of degree 1 is

$$
\begin{align*}
R_{1}(u, v)= & \frac{(-1)^{n}}{4 n!}\left\{\left(8 n(n-1)(n-2) \chi\left(\mathcal{O}_{S}\right)-(n-2)\left[\left(n^{2}-5 n+8\right) c_{1}^{2}\right.\right.\right. \\
& \left.\left.-2(n-2)^{2} c_{2}+n(n-1) K_{S}^{2}+2 n K_{S} c_{1}\right]\right) u  \tag{50}\\
& \left.+\left[\left(n^{2}-7 n+8\right) c_{1}^{2}-2(n-2)^{2} c_{2}+4 n(n-1)\left(K_{S}^{2}-8 \chi\left(\mathcal{O}_{S}\right)\right)\right] v\right\} .
\end{align*}
$$

Moreover, if the base surface $S$ is a minimal surface of Kodaira dimension zero, then the fact that $K_{S}$ is numerically trivial produces a further simplification, which leads to the following result.

Theorem 6.2. Let $(X, L)$ be a quadric fibration deriving from cones over a minimal surface $S$ with $\kappa(S)=0$ and $\operatorname{dim}(X)=n \geq 3$.
(1) If $n=3$ then $C=\Gamma$ is always irreducible.
(2) If $n \geq 4$ then $C$ is irreducible in the following cases:
(i) if $S$ is a K3 surface or an Enriques surface, and
(ii) if $S$ is an abelian or a bielliptic surface and $d \neq \frac{2 n(n-3)}{n^{2}-5 n+8} c_{2}$.

Proof. Due to Proposition 5.2, and Proposition 4.2 we know that (46) can be rewritten as

$$
[\sigma+(n-2) \tau] U=0
$$

where

$$
U:=n^{2} \delta \tau^{2}-n \delta \sigma \tau-4 n \delta \tau^{2}+n \gamma \tau^{2}+\beta \tau^{2}+\delta \sigma^{2}+2 \delta \sigma \tau+4 \delta \tau^{2}-\gamma \sigma \tau-2 \gamma \tau^{2}
$$

Now, taking into account that $\mathcal{B}$ is trivial, we can compute the coefficients $\sigma, \tau, \beta, \gamma, \delta$ from (50) and (49). Moreover, adding the information that $K_{S}$ is numerically trivial and recalling (15), we get

$$
\sigma+(n-2) \tau=\frac{(-1)^{n} n(n-2)}{2 n!} c_{1}^{2}=\frac{(-1)^{n} n(n-2)}{2 n!} H^{2} \neq 0
$$

Therefore $C$ is reducible if and only if $U=0$. Plugging in the expression of $U$ the values of the coefficients $\sigma, \tau, \beta, \gamma, \delta$ we see that, up to the scalar factor
$-\frac{(-1)^{3 n} n(n-1)}{16 n!^{3}} c_{1}^{2}$

Thus for every $n \geq 3, U$ is different from zero if $S$ is a $K 3$ surface or an Enriques surface, being $U$ the sum of two quantities in which the first one is greater than or equal to zero and the second one is strictly greater than zero because $c_{1}^{2}=H^{2}>0$. If $S$ is an abelian or a bielliptic surface we see that if $n=3$ then $U=\left[\left(n^{2}-5 n+8\right) c_{1}^{2}-\left(2 n^{2}-8 n+8\right) c_{2}\right]^{2}=2 c_{1}^{2}-2 c_{2}$ and recalling that $d=L^{3}=2\left(c_{1}^{2}-c_{2}\right)$ our claim follows. This proves (1) and (i) of (2). On the other hand, if $n \geq 4$ then $U=\left[\left(n^{2}-5 n+8\right) c_{1}^{2}-\left(2 n^{2}-8 n+8\right) c_{2}\right]^{2}=0$ if and only if $c_{1}^{2}=\frac{2 n^{2}-8 n+8}{n^{2}-5 n+8} c_{2}$, in which case $d=2\left(c_{1}^{2}-c_{2}\right)=\left[2+\frac{4(n-4)}{n^{2}-5 n+8}\right] c_{2}$. This proves (ii) of (2).

We have to point out that in [6, Proposition 6.3, (ii)] the statement is not correct, in fact no condition on $L^{3}$ is needed in order to have the irreducibility of $\Gamma$. As to case $n \geq 4$ with $S$ abelian or bielliptic we observe that $C$ is certainly irreducible if $(n-1) c_{1}^{2}>2 n c_{2}$ (that is if $\mathcal{E}$ is not Bogomolov stable, being $\operatorname{rk}(\mathcal{E})=n$ ), because this prevents the term $U$ from being zero.
EXAMPLE 6.2. If in the construction 6.1 , as $T=\mathbb{P}(\mathcal{V})$ we take the 5 -dimensional scroll in $\mathbb{P}^{11}$, over $\mathbb{P}^{2}$, of degree 10 and sectional genus 3 , that is $T=\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^{2}}(1)^{\oplus 4}\right)$ $=\mathbb{P}^{2} \times \mathbb{P}^{3}$, then $(X, L)$ is a geometric quadric fibration, with $c_{1}(\mathcal{E})=c_{1}(\mathcal{V})=$ $\mathcal{O}_{\mathbb{P}^{2}}(4)$ and $c_{2}(\mathcal{E})=c_{2}(\mathcal{V})=6$ and plugging such values in (49) and (50) we get

$$
p_{(X, L)}\left(\frac{1}{2}+u, v\right)=-\frac{1}{12}(2 u-v)\left(12 u^{2}-10 u v+2 v^{2}+3\right) .
$$

Note that the linear factor $2 u-v$ is not the one defining the line $\ell_{0}$ whose equation is $(n-2) u-v=0$, that is $3 u-v=0$.
Example 6.3. If in the construction 6.1 , as $T=\mathbb{P}(\mathcal{V})$ we take the 4-dimensional scroll in $\mathbb{P}^{10}$, over $\mathbb{P}^{2}$, of degree 10 and sectional genus 3 , that is $T=\mathbb{P}\left(T_{\mathbb{P}^{2}} \oplus\right.$ $\left.\mathcal{O}_{\mathbb{P}^{2}}(1)\right)$, then $(X, L)$ is a geometric quadric fibration, with $c_{1}(\mathcal{E})=c_{1}(\mathcal{V})=$ $\mathcal{O}_{\mathbb{P}^{2}}(4)$ and $c_{2}(\mathcal{E})=c_{2}(\mathcal{V})=6$ and plugging such values in (49) and (50) we get

$$
p_{(X, L)}\left(\frac{1}{2}+u, v\right)=\frac{7}{3} u^{3}-\frac{31}{6} u^{2} v+\frac{11}{3} u v^{2}-\frac{5}{6} v^{3}+\frac{17}{12} u-\frac{25}{24} v .
$$

We like to stress the following fact. Let $T_{1}$ and $T_{2}$ be the scrolls in Example 6.2 and Example 6.3 respectively. Adding $\mathcal{O}_{\mathbb{P}^{2}}(1)$ to the three terms of the Euler sequence on $\mathbb{P}^{2}$, we get the following exact sequence

$$
0 \rightarrow \mathcal{O}_{\mathbb{P}^{2}} \oplus \mathcal{O}_{\mathbb{P}^{2}}(1) \rightarrow \mathcal{O}_{\mathbb{P}^{2}}(1)^{\oplus 4} \rightarrow T_{\mathbb{P}^{2}} \oplus \mathcal{O}_{\mathbb{P}^{2}}(1) \rightarrow 0
$$

where $T_{\mathbb{P}^{2}}$ is the tangent bundle to $\mathbb{P}^{2}$. Then by using, for instance, [12, Lemma 0.7], we see that $T_{2} \subset \mathbb{P}^{10}$ is the general hyperplane section of $T_{1}$, Segre embedded in $\mathbb{P}^{11}$.

## 7. Examples

Let $(X, L)$ be a geometric quadric fibration over a smooth surface as in Section 2. The key point in passing from the case of threefolds studied in [6] to higher dimensions is clearly $n=4$, as already (1), and what we proved in the previous sections, show. For this reason the examples we discuss in this Section are concerned with $n=4$. First of all, note that if $(X, L)$ is general, then the residual cubic $C$ of its Hilbert curve is irreducible according to Proposition 5.3. Here is an example.

Example 7.1. Let $\mathbb{P}(\mathcal{E})$ be the $\mathbb{P}^{3}$-bundle over the smooth quadric $\mathbb{Q}^{2}$, defined by the rank four vector bundle $\mathcal{E}=\mathcal{O}_{\mathbb{Q}^{2}}(1,2)^{\oplus 4}$. Let $\pi: \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{Q}^{2}$ be the projection morphism, let $X$ be a general element in $\left|2 \xi+\pi^{*} \mathcal{O}_{\mathbb{Q}^{2}}(1,2)\right|$, where $\xi$ denotes the tautological line bundle of $\mathcal{E}$ on $\mathbb{P}(\mathcal{E})$, and call $\varphi: X \rightarrow \mathbb{Q}^{2}$ the restriction of $\pi$ to $X$. On $X$ we consider the polarization given by $L=(\xi)_{X}$ Note that $K_{X}=\left(-4 \xi+\pi^{*} \mathcal{O}_{\mathbb{Q}^{2}}(4,8)+\pi^{*} \mathcal{O}_{\mathbb{Q}^{2}}(-2,-2)+2 \xi+\pi^{*} \mathcal{O}_{\mathbb{Q}^{2}}(1,2)\right)_{X}=$ $\left(-2 \xi+\pi^{*} \mathcal{O}_{\mathbb{Q}^{2}}(3,8)\right)_{X}$. The polarized pair $(X, L)$ is a geometric quadric fibration over $\mathbb{Q}^{2}$. Using (28) through (33) and the fact that $c_{1}=\mathcal{O}_{\mathbb{Q}^{2}}(4,8)$, $c_{2}=24$ and $\mathcal{B}=\mathcal{O}_{\mathbb{Q}^{2}}(1,2)$, we see that the residual cubic $C$ of the Hilbert curve $\Gamma$ of $(X, L)$ has equation

$$
\begin{equation*}
4 u^{3}-12 u^{2} v-3 u v^{2}+4 v^{3}-3 u+\frac{17}{2} v=0 \tag{51}
\end{equation*}
$$

In this case $C$ is irreducible, the term on the left hand side of (46) taking the value 266 .

The remainder of this Section is devoted to examples for which $C$ is reducible.

Example 7.2 . Let $Y:=\mathbb{P}^{2} \times \mathbb{P}^{3}$ and let $\pi$ and $\rho$ be the projections onto the first and the second factors respectively. Set $A:=\pi^{*} \mathcal{O}_{\mathbb{P}^{2}}(1), B:=\rho^{*} \mathcal{O}_{\mathbb{P}^{3}}(1)$, and write $\mathcal{O}(r, s)$ for $r A+s B$. Let $X \subset Y$ is a smooth element in the linear system $|\mathcal{O}(1,2)|$, hence

$$
\begin{equation*}
X \sim A+2 B \tag{52}
\end{equation*}
$$

By adjunction, $K_{X}=[\mathcal{O}(-3,-4)+X]_{X}=\mathcal{O}(-2,-2)_{X}=-2 \mathcal{O}(1,1)_{X}$, so that $X$ is a Fano 4 -fold of index 2 ; moreover, taking into account that $A^{3}=B^{4}=0$ and $A^{2} B^{3}=1$, we have

$$
\left(\mathcal{O}(1,1)_{X}\right)^{4}=\mathcal{O}(1,1)^{4} X=(A+B)^{4}(A+2 B)=(4+12) A^{2} B^{3}=16
$$

(i. e. $\left(X, \mathcal{O}(1,1)_{X}\right)$ has degree 16$)$.

Up to now $\varphi:=\pi_{\mid X}: X \rightarrow \mathbb{P}^{2}$ is only a classical quadric fibration. To make it a geometric quadric fibration consider on $X$ the ample line bundle $L:=\mathcal{O}(a, 1)_{X}$ for some positive integer $a$. Clearly $L$ induces the hyperplane bundle on every quadric surface, fiber of $\varphi$. Moreover,

$$
K_{X}+2 L=\mathcal{O}(2(a-1), 0)_{X}=\varphi^{*} \mathcal{O}_{\mathbb{P}^{2}}(2(a-1))
$$

Therefore, for $(X, L)$ to be a geometric quadric fibration we need $a \geq 2$.
Note that $Y=\mathbb{P}(\mathcal{V})$, where $\mathcal{V}=\mathcal{O}_{\mathbb{P}^{2}}(1)^{\oplus 4}$, the corresponding tautological line bundle being $\zeta:=\mathcal{O}(1,1)$. This is clear once we compare the two expression of the canonical bundle of $Y$, viewed both as a product and as $\mathbb{P}(\mathcal{V})$ respectively. Then $\mathcal{V}=\pi_{*} \zeta$ and, recalling that $\varphi=\pi_{\mid X}$, we also have $\mathcal{V}=\varphi_{*}\left(\zeta_{X}\right)$. Next let us determine the vector bundle $\mathcal{E}:=\varphi_{*} L$. Since $L=\left(\zeta+(a-1) \pi^{*} \mathcal{O}_{\mathbb{P}^{2}}(1)\right)_{X}$, we get

$$
\mathcal{E}=\varphi_{*}\left[\left(\zeta+\pi^{*} \mathcal{O}_{\mathbb{P}^{2}}(a-1)\right)_{X}\right]=\mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^{2}}(a-1)=\mathcal{O}_{\mathbb{P}^{2}}(a)^{\oplus 4}
$$

In particular, this gives

$$
\begin{equation*}
c_{1}=\mathcal{O}_{\mathbb{P}^{2}}(4 a) \tag{53}
\end{equation*}
$$

Since $\mathcal{E}$ is $\mathcal{V}$ twisted by a line bundle, we see that $P:=\mathbb{P}(\mathcal{E}) \cong Y$ itself; note however that the tautological line bundle corresponding to $\mathcal{E}$ is $\xi=\zeta+(a-1) A$ (in accordance with the fact that $L=\xi_{X}$ ). Now recall that, in our setting, $X \in\left|2 \xi+\pi^{*} \mathcal{B}\right|$. So, letting $\mathcal{B}=\mathcal{O}_{\mathbb{P}^{2}}(b)$, we get

$$
\begin{aligned}
2 \xi+\pi^{*} \mathcal{B} & =(2 \zeta+2(a-1) A+b A)=[2(A+B)+(2 a+b-2) A] \\
& =[(2 a+b) A+2 B]
\end{aligned}
$$

and from a comparison with (52) we deduce that $b=-2 a+1$, i.e. $\mathcal{B}=$ $\mathcal{O}_{\mathbb{P}^{2}}(-2 a+1)$. Finally, look at $\mathcal{D}$, the discriminant curve of our quadric fibration $(X, L)$. From a general result already mentioned in Section 2 combined with (53), since $n=4$ we get

$$
\mathcal{D} \in\left|2 c_{1}+n \mathcal{B}\right|=\left|\mathcal{O}_{\mathbb{P}^{2}}(4)\right|
$$

Therefore certainly $X$ is not a $\mathbb{Q}^{2}$-bundle over $\mathbb{P}^{2}, \mathcal{D}$ being non-trivial. It remains to determine the canonical equation of the Hilbert curve $\Gamma$ of $(X, L)$. By using the Riemann-Roch-Hirzebruch formula in the following form (see [2, (8)])

$$
\begin{equation*}
\chi(D)=\frac{1}{24} E^{4}+\frac{1}{48}\left(2 c_{2}(X)-K_{X}^{2}\right) E^{2}+\frac{1}{384}\left(K_{X}^{2}-4 c_{2}(X)\right) K_{X}^{2}+\chi\left(\mathcal{O}_{X}\right) \tag{54}
\end{equation*}
$$

where $D=\frac{1}{2} K_{X}+E$ and $E=u K_{X}+v L$, after standard Chern class computations we get the canonical equation of $\Gamma$, which is

$$
p\left(\frac{1}{2}+u, v\right)=\frac{1}{6}(v-2 u)(a v-2 u)\left(16 u^{2}-2(3 a+5) u v+(3 a+1) v^{2}+2\right)=0
$$

We note that $\Gamma$ is reducible, but, in addition to the line $2 u-v=0$, which is a prescribed component of $\Gamma$ according to [3, Theorem 6.1], there is another linear component for any $a \geq 2$, namely the line $a v-2 u=0$.

As to the conic component, say $\gamma$, note that it is irreducible since $a \geq 2$. On the other hand, if $a=1$, then $(X, L)$ is not a geometric quadric fibration, as already observed; moreover the equation of $\Gamma$ becomes $(2 u-v)^{2}\left(4(2 u-v)^{2}+\right.$ $2)=0$. In particular we see that the projective closure $\bar{\Gamma}$ of $\Gamma$ has a singular point of multiplicity 4 at $(2: 1: 0)$. Note that this is in accordance with [3, Lemma 3.2], since for $a=1$, we have $K_{X}+2 L=0$, hence ( $X, L$ ) fits into the degenerate case.

Example 7.3. Consider $\mathbb{P}^{2} \times \mathbb{Q}^{3}$ and let $p_{1}$ and $p_{2}$ be the projections onto the first and the second factor respectively. Set $H_{1}:=p_{1}^{*}\left(\mathcal{O}_{\mathbb{P}^{2}}(1)\right)$ and $H_{2}:=$ $p_{2}^{*}\left(\mathcal{O}_{\mathbb{Q}^{3}}(1)\right)$, and write $\mathcal{O}(r, s)$ for $r H_{1}+s H_{2}$. Let $X \subset \mathbb{P}^{2} \times \mathbb{Q}^{3}$ be a smooth element in the linear system $|\mathcal{O}(1,1)|$. By adjunction, $K_{X}=[\mathcal{O}(-3,-3)+$ $X]_{X}=-2 \mathcal{O}(1,1)_{X}$, so that $X$ is a Fano 4 -fold of index 2. Taking into account that $H_{1}^{3}=H_{2}^{4}=0$ and $H_{1}^{2} H_{2}^{3}=2$, we see that $\left(X, \mathcal{O}(1,1)_{X}\right)$ has degree 20. Let $\varphi: X \rightarrow \mathbb{P}^{2}$ be the restriction of $p_{1}$ to $X$, and take on $X$ the polarization given by $L:=\mathcal{O}(a, 1)_{X}$ for some positive integer $a$. Clearly $L$ induces the hyperplane bundle on every quadric surface, fiber of $\varphi$. Moreover,

$$
K_{X}+2 L=\mathcal{O}(2(a-1), 0)_{X}=\varphi^{*} \mathcal{O}_{\mathbb{P}^{2}}(2(a-1))
$$

Therefore, $(X, L)$ will be a geometric quadric fibration as soon as $a \geq 2$.
In order to compute the canonical equation of $\Gamma$, we tensor the structure sequence

$$
0 \rightarrow \mathcal{O}(-1,-1) \rightarrow \mathcal{O}(0,0) \rightarrow \mathcal{O}_{X} \rightarrow 0
$$

with $\mathcal{O}(a y-2 x, y-2 x)$ and we get

$$
\begin{equation*}
0 \rightarrow \mathcal{O}(a y-2 x-1, y-2 x-1) \rightarrow \mathcal{O}(a y-2 x, y-2 x) \rightarrow x K_{X}+y L \rightarrow 0 \tag{55}
\end{equation*}
$$

Using the fact that $\chi\left(\mathbb{P}^{2} \times \mathbb{Q}^{3}, \mathcal{O}(r, s)\right)=\chi\left(\mathbb{P}^{2}, \mathcal{O}_{\mathbb{P}^{2}}(r)\right) \cdot \chi\left(\mathbb{Q}^{3}, \mathcal{O}_{\mathbb{Q}^{3}}(s)\right)$ and standard computations, after replacing $x=u+\frac{1}{2}, y=v$ we see that the canonical equation of $\Gamma$ is

$$
p\left(\frac{1}{2}+u, v\right)=\frac{1}{6}(v-2 u)(a v-2 u)\left(20 u^{2}-2(3 a+7) u v+(3 a+2) v^{2}+1\right)=0 .
$$

Thus the residual cubic $C$ has equation

$$
\frac{1}{6}(a v-2 u)\left(20 u^{2}-2(3 a+7) u v+(3 a+2) v^{2}+1\right)=0
$$

Even in this case the conic $\gamma$ is irreducible, since $a \geq 2$. Because $X$ sits in $P:=\mathbb{P}(\mathcal{E})$ as a divisor, $X \in\left|2 \xi+\pi^{*} \mathcal{B}\right|$ where $\xi$ is the tautological line bundle,
we compute the values of $c_{i}$. Let $\mathcal{B}=\mathcal{O}_{\mathbb{P}^{2}}(b)$ for some integer $b$. From (15) we have

$$
2 a-2=c_{1}-3+b
$$

and thus $c_{1}=2 a+1-b$. Easy computations show that

$$
\begin{gather*}
d=12 a^{2}+8 a  \tag{56}\\
d_{1}=-24 a-4-12 a^{2} . \tag{57}
\end{gather*}
$$

On the other hand from (24) and (25), since $K_{S}=\mathcal{O}_{\mathbb{P}^{2}}(-3), \mathcal{B}=\mathcal{O}_{\mathbb{P}^{2}}(b)$, $n=4$, and $c_{1}=2 a+1-b$ we get that

$$
\begin{equation*}
d=8 a^{2}+8 a-6 a b+2-3 b+b^{2}-2 c_{2} \tag{58}
\end{equation*}
$$

and

$$
\begin{equation*}
d_{1}=-8 a^{2}-20 a+10 a b-8+8 b-2 b^{2}+4 c_{2} \tag{59}
\end{equation*}
$$

Using (56),(57),(58),(59) we get that $b=-2 a, c_{1}=4 a+1, c_{2}=6 a^{2}+3 a+1$. As to the discriminant curve $\mathcal{D}$ of our quadric fibration $(X, L)$, we have

$$
\mathcal{D} \in\left|2 c_{1}+4 \mathcal{B}\right|=\left|\mathcal{O}_{\mathbb{P}^{2}}(2)\right|
$$

EXAMPLE 7.4. Let $\pi: X \rightarrow \mathbb{P}^{2} \times \mathbb{P}^{2}$ be a double cover of $\mathbb{P}^{2} \times \mathbb{P}^{2}$, branched along a smooth divisor of type $(2,2)$ and let $\mathscr{R} \subset X$ be the ramification divisor. Then $\mathscr{R}$ is a smooth hypersurface and $\pi(\mathscr{R}) \in|2 H|$, with $H=\mathcal{O}_{\mathbb{P}^{2} \times \mathbb{P}^{2}}(1,1)$. We have a short exact sequence

$$
\begin{equation*}
0 \rightarrow \pi^{*} T_{\mathbb{P}^{2} \times \mathbb{P}^{2}}^{*} \rightarrow T_{X}^{*} \rightarrow N_{\mathscr{R} / X}^{*} \rightarrow 0 \tag{60}
\end{equation*}
$$

where $N_{\mathscr{R} / X}^{*}$ is the conormal bundle of $\mathscr{R} \subset X$. It comes from a local computation combined with the fact that $N_{\mathscr{R} / X}^{*}=\mathcal{J} / \mathcal{J}^{2}$, where $\mathcal{J}$ is the ideal sheaf of $\mathscr{R}$ in $X$. We will use (60) and the short exact sequence

$$
\begin{equation*}
0 \rightarrow-2 \mathscr{R} \rightarrow-\mathscr{R} \rightarrow N_{\mathscr{R} / X}^{*} \rightarrow 0 \tag{61}
\end{equation*}
$$

to determine $c_{i}(X)$. In fact arguing as in [5, Lemma 2.6] (which holds in any dimension) we see that

$$
\begin{gather*}
c_{1}(X)=\pi^{*}\left(c_{1}\left(\mathbb{P}^{2} \times \mathbb{P}^{2}\right)-H\right)  \tag{62}\\
c_{2}(X)=\pi^{*}\left(c_{2}\left(\mathbb{P}^{2} \times \mathbb{P}^{2}\right)-c_{1}\left(\mathbb{P}^{2} \times \mathbb{P}^{2}\right) H+2 H^{2}\right) \tag{63}
\end{gather*}
$$

Let $p_{i}: \mathbb{P}^{2} \times \mathbb{P}^{2} \rightarrow \mathbb{P}^{2}$ be the projection onto the $i$-th factor. Let $p_{1}^{*}\left(\mathcal{O}_{\mathbb{P}^{2}}(1)\right)=H_{1}$ and $p_{2}^{*}\left(\mathcal{O}_{\mathbb{P}^{2}}(1)\right)=H_{2}$, where $H_{1}$ and $H_{2}$ satisfy $H_{1}^{3}=H_{2}^{3}=0$ and $H_{1}^{2} H_{2}^{2}=1$.

In order to compute $c_{i}\left(\mathbb{P}^{2} \times \mathbb{P}^{2}\right)$ we use the following exact sequence, deriving from the Euler sequence,

$$
\begin{array}{r}
0 \rightarrow p_{1}^{*}\left(\mathcal{O}_{\mathbb{P}^{2}}\right) \oplus p_{2}^{*}\left(\mathcal{O}_{\mathbb{P}^{2}}\right) \rightarrow p_{1}^{*}\left(\mathcal{O}_{\mathbb{P}^{2}}(1)\right)^{\oplus 3} \oplus p_{2}^{*}\left(\mathcal{O}_{\mathbb{P}^{2}}(1)\right)^{\oplus 3} \rightarrow \\
p_{1}^{*}\left(T_{\mathbb{P}^{2}}\right) \oplus p_{2}^{*}\left(T_{\mathbb{P}^{2}}\right)=T_{\mathbb{P}^{2} \times \mathbb{P}^{2}} \rightarrow 0
\end{array}
$$

and we see that

$$
c_{1}\left(\mathbb{P}^{2} \times \mathbb{P}^{2}\right)=3\left(H_{1}+H_{2}\right), c_{2}\left(\mathbb{P}^{2} \times \mathbb{P}^{2}\right)=3\left(H_{1}^{2}+H_{2}^{2}+3 H_{1} H_{2}\right) .
$$

Thus

$$
\begin{gathered}
c_{1}(X)=\pi^{*}\left(c_{1}\left(\mathbb{P}^{2} \times \mathbb{P}^{2}\right)-H\right)=\pi^{*}\left(2 H_{1}+2 H_{2}\right) \\
c_{2}(X)=\pi^{*}\left(c_{2}\left(\mathbb{P}^{2} \times \mathbb{P}^{2}\right)-c_{1}\left(\mathbb{P}^{2} \times \mathbb{P}^{2}\right) H+2 H^{2}\right)=\pi^{*}\left(2 H_{1}^{2}+2 H_{2}^{2}+7 H_{1} H_{2}\right)
\end{gathered}
$$

Let $\varphi=p_{1} \circ \pi: X \rightarrow \mathbb{P}^{2}$, which is a classical quadric fibration. To make it a geometric quadric fibration we consider on $X$ the ample line bundle $L:=$ $\pi^{*}\left(\mathcal{O}_{\mathbb{P}^{2} \times \mathbb{P}^{2}}(a, 1)\right)$ for some positive integer $a$. Because

$$
K_{X}+2 L=\pi^{*} \mathcal{O}_{\mathbb{P}^{2} \times \mathbb{P}^{2}}(2(a-1), 0)=\varphi^{*} \mathcal{O}_{\mathbb{P}^{2}}(2(a-1))
$$

it follows that ( $X, L$ ) will be a geometric quadric fibration if $a \geq 2$.
By (54), after standard computations, we get the canonical equation of $\Gamma$, which is

$$
p\left(\frac{1}{2}+u, v\right)=\frac{1}{2}(v-2 u)(a v-2 u)\left(4 u^{2}-2(a+1) u v+a v^{2}+1\right)=0 .
$$

Thus the residual cubic $C$ has equation

$$
\frac{1}{2}(2 u-a v)\left(4 u^{2}-2(a+1) u v+a v^{2}+1\right)=0
$$

Even in this case the conic $\gamma$ is irreducioble, provided that $a \geq 2$. For such $(X, L)$ we see that

$$
\begin{equation*}
d=L^{4}=\left(\pi^{*}\left(\mathcal{O}_{\mathbb{P}^{2} \times \mathbb{P}^{2}}(a, 1)\right)\right)^{4}=2 \cdot 6\left(a H_{1}\right)^{2} H_{2}^{2}=12 a^{2} \tag{64}
\end{equation*}
$$

Similarly

$$
\begin{equation*}
d_{1}=2\left(-2 H_{1}-2 H_{2}\right)\left(a H_{1}+H_{2}\right)^{3}=-12 a^{2}-12 a \tag{65}
\end{equation*}
$$

We now compute the values of $c_{i}$. Let $\mathcal{B}=\mathcal{O}_{\mathbb{P}^{2}}(b)$ for some integer $b$. From (15) we have

$$
2 a-2=c_{1}-3+b
$$

and thus $c_{1}=2 a+1-b$.

On the other hand from (24) and (25), since $K_{S}=\mathcal{O}_{\mathbb{P}^{2}}(-3), \mathcal{B}=\mathcal{O}_{\mathbb{P}^{2}}(b)$, $n=4$, and $c_{1}=2 a+1-b$ we get that

$$
\begin{equation*}
d=8 a^{2}+8 a-6 a b+2-3 b+b^{2}-2 c_{2} \tag{66}
\end{equation*}
$$

and

$$
\begin{equation*}
d_{1}=-8 a^{2}-20 a+10 a b-8+8 b-2 b^{2}+4 c_{2} \tag{67}
\end{equation*}
$$

Using (64), (66), (65), (67) we obtain $b=-2 a+2, c_{1}=4 a-1, c_{2}=6 a^{2}-3 a$ and thus for the discriminant curve $\mathcal{D}$, of our quadric fibration $(X, L)$, we have

$$
\mathcal{D} \in\left|2 c_{1}+4 \mathcal{B}\right|=\left|\mathcal{O}_{\mathbb{P}^{2}}(6)\right| .
$$

We like to point out that in Examples $7.2-7.4 X$ is a Fano 4 -fold as in [14, Table 0.3 , No. $5,8,4$, respectively]. For $a \geq 2(X, L)$ is a 4-dimensional geometric quadric fibration over $\mathbb{P}^{2}$ and fits into the situation described by Proposition 5.4, satisfying the condition $p\left(c_{1}+b-1\right)-4 q=0$ in all three cases. For instance, in Example 7.2, $p=2, b=-\frac{c_{1}}{2}+1, q=\frac{c_{1}}{4}$. On the other hand, if $a=1$ then $K_{X}=-2 L$, hence $\Gamma$ itself is reducible into 4 parallel lines, in accordance with [13, Lemma 3.1].
Example 7.5. Let $\mathbb{P}(\mathcal{E})$ be the $\mathbb{P}^{3}$-bundle over the Segre-Hirzebruch surface $\mathbb{F}_{e}$ of invariant $e(\geq 0)$, defined by the rank four vector bundle $\mathcal{E}=\left[C_{0}+(e+1) f\right]^{\oplus 4}$, where $C_{0}$ is a section of self-intersection $C_{0}^{2}=-e$ and $f$ a fiber of the bundle projection $\mathbb{F}_{e} \rightarrow \mathbb{P}^{1}$. Let $\pi: \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{F}_{e}$ be the projection morphism, let $X$ be a general element in $\left|2 \xi+\pi^{*} \mathcal{B}\right|$, where $\xi$ denotes the tautological line bundle of $\mathcal{E}$ on $\mathbb{P}(\mathcal{E}), \mathcal{B}=a C_{0}+b f$ and call $\varphi: X \rightarrow \mathbb{F}_{e}$ the restriction of $\pi$ to $X$. Note that $K_{X}=\left(-4 \xi+\pi^{*}\left(4 C_{0}+4(e+1) f-2 C_{0}-(e+2) f\right)+2 \xi+\pi^{*} \mathcal{B}\right)_{X}=$ $\left(-2 \xi+\pi^{*}\left((2+a) C_{0}+(3 e+2+b) f\right)\right)_{X}$. The polarized pair $(X, L)$ where $L=(\xi)_{X}$ is a geometric quadric fibration over $\mathbb{F}_{e}$. Using (28) through (33) and the fact that $c_{1}=4 C_{0}+4(e+1) f, c_{2}=6(e+2)$, we see that the residual cubic $C$ of the Hilbert curve $\Gamma$ of $(X, L)$ has equation

$$
\begin{equation*}
\frac{2}{3}(2 u-3 v)\left(v^{2}-u v-2 u^{2}+2\right)=0 \tag{68}
\end{equation*}
$$

if the base surface is $\mathbb{F}_{0}$ and $\mathcal{B}=2 C_{0}$, and

$$
\begin{equation*}
\frac{1}{6}(2 u-3 v)\left(5 v^{2}-8 u v-4 u^{2}+7\right)=0 \tag{69}
\end{equation*}
$$

if the base surface is $\mathbb{F}_{1}$ and $\mathcal{B}=\mathcal{O}_{\mathbb{F}_{1}}$.

## 8. A unifying perspective

Here we discuss a natural framework in which the residual cubics of Hilbert curves of geometric quadric fibrations over a smooth surface fit into, offering
a unifying perspective to many results proved in the previous Sections. To start with let $\mathscr{V}$ be the family of Serre-invariant cubics, see for instance [3, Section 7]. As observed in Section 2, a cubic $C$ in the complex affine plane of coordinates $u$ and $v$ belongs to $\mathscr{V}$ if and only if it is described by an equation of type (3), with $R_{3}(u, v)$ and $R_{1}(u, v)$ as in (4) and (5), respectively, for some complex numbers $\alpha, \ldots, \tau$, with $(\alpha, \beta, \gamma, \delta) \neq(0,0,0,0)$. So we can look at $C$ as the point $(\alpha: \beta: \gamma: \delta: \sigma: \tau)$ of $\mathbb{P}^{5}$ lying outside the line, say $\Lambda$, defined by $\alpha=\beta=\gamma=\delta=0$. Thus we have a natural identification

$$
\begin{equation*}
\mathscr{V}=\mathbb{P}^{5} \backslash \Lambda . \tag{70}
\end{equation*}
$$

According to Proposition 5.2, we can identify the reducible $C \in \mathscr{V}$ with the points of the quartic hypersurface $V \subset \mathbb{P}^{5}$ of equation (46), lying outside $\Lambda$. Let us rewrite the equation of $V$ in the form

$$
\begin{equation*}
f(\alpha, \beta, \gamma, \delta, \sigma, \tau)=\sigma^{2}(\sigma \delta-\tau \gamma)+\tau^{2}(\sigma \beta-\tau \alpha)=0 \tag{71}
\end{equation*}
$$

Clearly, $V$ contains $\Lambda$ and also the 3-plane $\sigma=\tau=0$. In fact we have
Proposition 8.1. The singular locus $\operatorname{Sing}(V)$ is exactly the 3 -plane of equations $\sigma=\tau=0$.

Proof. The assertion follows immediately from the Jacobian criterion. Actually, from (71) we get

$$
\operatorname{grad}(f)=\left(-\tau^{3}, \tau^{2} \sigma,-\sigma^{2} \tau, \sigma^{3}, 3 \delta \sigma^{2}-2 \gamma \sigma \tau+\beta \tau^{2},-\gamma \sigma^{2}+2 \beta \sigma \tau-3 \alpha \tau^{2}\right)
$$

This shows that condition $\operatorname{grad}(f)=0$ is equivalent to the vanishing of the first and the fourth components only.

As a consequence of Proposition 8.1 we have that also the singular locus of $V \backslash \Lambda$ is the 3-plane $\sigma=\tau=0$, since the latter and the line $\Lambda$ are skew.

Now we consider another relevant locus in $\mathbb{P}^{5}$. We denote by $\mathscr{S}$ the family of cubics $C \in \mathscr{V}$ such that the projective closure $\bar{C}$ contains the point at infinity $P_{\infty}=(1: n-2: 0)$ and intersects the line at infinity $\ell_{\infty}$ at that point transversely. The first condition says that the point corresponding to $C$ lies on the hyperplane $\mathscr{H}$ of equation (36), while the latter means that it does not belong to the hyperplane $h$ of equation (38), according to the proof of Proposition 4.2. So $\mathscr{S}$ seems the most appropriate locus of $\mathscr{V}$ to include the residual cubics of the Hilbert curves of geometric quadric fibrations as in Section 2.

Because the coefficients in (1) are rationals, all residual cubics of the Hilbert curves of our quadric fibrations correspond to rational points of the locus $\mathscr{S}$.

By definition $\mathscr{S}$ is the complement of $h$ in $\mathscr{V} \cap \mathscr{H}$, hence it is a quasiprojective variety of dimension 4 . In fact, noting that both $\mathscr{H}$ and $h$ contain $\Lambda$, we have that

$$
\begin{equation*}
\mathscr{S}:=\mathscr{V} \cap(\mathscr{H} \backslash h)=\left(\mathbb{P}^{5} \backslash \Lambda\right) \cap(\mathscr{H} \backslash h)=\mathscr{H} \backslash h . \tag{72}
\end{equation*}
$$

In particular, $\overline{\mathscr{S}}=\mathscr{H}$. Next, consider

$$
\begin{equation*}
\mathscr{T}:=\{C \in \mathscr{S} \mid C \text { has a triple point }\} . \tag{73}
\end{equation*}
$$

According to the discussion in Section 4, any $C \in \mathscr{T}$ has a triple point at the origin, hence it is reducible into three lines through the origin. Moreover, since $\bar{C}$ contains $P_{\infty}$, any such $C$ consists of the line $\ell_{0}$ and two other lines distinct from $\ell_{0}$ belonging to the same pencil. Thus $\mathscr{T}=S^{(2)}\left(\mathbb{P}^{1} \backslash\{o\}\right)$, the second symmetric product of $\mathbb{P}^{1} \backslash\{o\}$ with itself, o representing the line $\ell_{0}$, which is removed from the pencil. Recall that for a $C \in \mathscr{V}$, having a triple point at the origin is equivalent to satisfying the equations $\sigma=\tau=0$ with the further condition that $C \in \mathscr{H}$. Removing the intersection with the hyperplane $h$, this shows that $\mathscr{T}$ is a $\mathbb{P}^{2}$ minus a line, which agrees with the previous description, since $S^{(2)}\left(\mathbb{P}^{1}\right)=\mathbb{P}^{2}$. Proposition 8.1 has the following consequence.
Corollary 8.2. We have $\operatorname{Sing}(V) \cap \mathscr{S}=\mathscr{T}$.
Remark 8.1. Since all reducible cubics of $\mathscr{S}$ lie in $V \cap \mathscr{H}$, one could be tempted to think that the singular locus $\operatorname{Sing}(V \cap \mathscr{H})$ is more related to our analysis than $\operatorname{Sing}(V)$. However, this is not the case, as we will see in a moment. Of course $\operatorname{Sing}(V \cap \mathscr{H})$ is larger than $\operatorname{Sing}(V) \cap \mathscr{H}$ and using the Jacobian criterion one can see that it consists of two components. Precisely, $\operatorname{Sing}(V \cap \mathscr{H})=\mathscr{T} \cup \mathscr{Z}$, where $\mathscr{Z}$ is defined by the following three equations: $\sigma=-(n-2) \tau, \beta=$ $-(n-2)(2 \gamma+3(n-2) \delta)$ and $\alpha=(n-2)^{2}(\gamma+2(n-2) \delta)$. However

$$
\begin{equation*}
(\mathscr{Z} \backslash \mathscr{T}) \cap \mathscr{S}=\emptyset \tag{74}
\end{equation*}
$$

which says that the component $\mathscr{Z}$ is irrelevant for $\mathscr{S}$. To see this, suppose that $C \in \mathscr{Z}$. Then its equation is

$$
\begin{aligned}
(n-2)^{2}(\gamma+2(n-2) \delta) u^{3}-(n-2)( & 2 \gamma+3(n-2) \delta) u^{2} v \\
& +\gamma u v^{2}+\delta v^{3}-(n-2) \tau u+\tau v=0
\end{aligned}
$$

where $\tau \neq 0$ if, in addition, $C \notin \mathscr{T}$. Note that the polynomial at the left hand side is divisible by $(n-2) u-v$, hence the above equation can be rewritten as

$$
\begin{equation*}
((n-2) u-v)\left[\gamma u((n-2) u-v)+\delta\left(2(n-2)^{2} u^{2}-(n-2) u v-v^{2}\right)-\tau\right]=0 \tag{75}
\end{equation*}
$$

This shows that $C=\ell_{0}+G$, where $G$ is the conic described by the factor in brackets. Looking for the points at infinity of $\bar{G}$ we immediately see that they
are $Q_{\infty}^{\prime}=(1: n-2: 0)$ and $Q_{\infty}^{\prime \prime}=(-\delta: \gamma+2(n-2) \delta: 0)$, up to renaming. But $Q_{\infty}^{\prime}=P_{\infty}$, the point at infinity of $\ell_{0}$, hence $\bar{C}$ intersects $\ell_{\infty}$ at $P_{\infty}$ with multiplicity $\geq 2$. Therefore $C \notin \mathscr{S}$.

We can also revisit Proposition 4.3 in the current setting. Let $\mathscr{Q} \subset \mathbb{P}^{5}$ be the quadric hypersurface of equation (39). According to Proposition 4.3, the section of $\mathscr{Q}$ with the hyperplane $\mathscr{H}$ of equation (36), outside of $\Lambda$, that is

$$
\mathscr{Q}^{\prime}:=\mathscr{Q} \cap \mathscr{H} \backslash \Lambda,
$$

describes the Serre-invariant cubics $C$ such that $\bar{C} \cap \ell_{\infty}=P_{\infty}+2 Q_{\infty}$, where $Q_{\infty}$ is a point at infinity, distinct from $P_{\infty}$. An immediate check shows that $\operatorname{Sing}(\mathscr{Q})$ is the plane of equations $\delta=\gamma=\alpha=0$, and combining them with (36), we see that $\operatorname{Sing}(\mathscr{Q}) \cap \mathscr{H}=\Lambda$. But $\Lambda$ is not included in $\mathscr{S}$, hence

$$
\operatorname{Sing}(\mathscr{Q}) \cap \mathscr{S}=\emptyset
$$

On the other hand, replacing $\alpha$ in (39) with its expression provided by (36) we get an equation in $\beta, \gamma, \delta$, mute in $\sigma$ and $\tau$, representing a quadric hypersurface of $\mathbb{P}^{4}$, say $\mathscr{Q}^{\prime \prime}$, which is the image of $\mathscr{Q}^{\prime}$ in the hyperplane $\Pi$ of equation $\alpha=0$ via the projection $\rho: \mathbb{P}^{5} \backslash\{A\} \rightarrow \Pi$ from the point $A=(1: 0: 0: \cdots: 0) \in \mathbb{P}^{5}$. A straightforward computation shows that the singular locus of $\mathscr{Q}^{\prime \prime}$ in the hyperplane $\Pi$ is described by the equations $\beta=\gamma=\delta=0$, i.e. $\operatorname{Sing}\left(\mathscr{Q}^{\prime \prime}\right)=\Lambda$. So, coming back to $\mathscr{Q}^{\prime}$ through the projection $\rho$ to describe $\operatorname{Sing}\left(\mathscr{Q}^{\prime}\right)$, we see that

$$
\operatorname{Sing}\left(\mathscr{Q}^{\prime}\right) \cap \mathscr{S}=\emptyset
$$

To complete the picture it remains to understand which loci of $\mathscr{Q}^{\prime}$ represent the different kinds of cubics $C$ fitting in Proposition 4.3. They are:

1) Serre-invariant cubics (the general being irreducible) passing through $P_{\infty}$ and with a node on $\ell_{\infty}$, not in $P_{\infty}$;
2) reducible Serre-invariant cubics passing through $P_{\infty}$ and having a double intersection with $\ell_{\infty}$, not in $P_{\infty}$.

Now, come back to reducible Serre-invariant cubics. According to Proposition 5.1, in case 2) we have $C=\ell+\gamma$; moreover, if $\gamma$ is irreducible, then $\ell$ contains $O$ and $\gamma$ has center in $O$, due to the Serre invariance, hence it cannot be a parabola. As a consequence, $\bar{\gamma} \cap \ell_{\infty}$ consists of two distinct points. Since we are dealing with cubics $C$ such that $\bar{C} \cap \ell_{\infty}=P_{\infty}+2 Q_{\infty}$, we have that $\ell \neq \ell_{0}$ and $\gamma$ is a hyperbola whose asymptotes are $\ell_{0}$ and the line $<O, Q_{\infty}>$. In this case $\bar{C}$ intersects $\ell_{\infty}$ with multiplicity one in $P_{\infty}$ and two in $Q_{\infty}$. Note that for these cubics $C$, the admissible conics $\gamma$ constitute a pencil; however,
since $Q_{\infty}$ can vary on $\ell_{\infty}$, with the only restriction of being different from $P_{\infty}$, the family of such conics depends on two parameters. In addition, the line $\ell$ moves in a pencil, since it has to contain $O$. Therefore the family of cubics $C$ of this type is 3 -dimensional. Let us call this case $2 a$ ). Clearly, case 2) also includes the following two possibilities.

2b) $\ell=\ell_{0}$ and $\gamma$ consists of two lines through $O$. This is the situation in which $C$ has a triple point, i.e., $C \in \mathscr{T}$. We already know that this is a 2-dimensional family.

2c) $\ell=\ell_{0}$ again, but $C$ has no triple points. In this case, according to Proposition 5.1(b), $\gamma$ consists of two parallel lines $\ell^{\prime}$ and $\ell^{\prime \prime}$ with direction corresponding to $Q_{\infty}$ (and obviously symmetric with respect to $O$ ). This family too has dimension 2 , as one can see from the following computation.

Concerning the dimensions of the various families, we note the following. We know that case 1) is effective, as [6, Example 6.3] shows. Suppose that the node is $Q_{\infty}=(-b: a: 0)$ and for simplicity, to treat equations in affine coordinates, call $m=-a / b$ the slope; then $Q_{\infty}=(1: m: 0)$. The equation of $\bar{C}$, in homogeneous coordinates $u, v, w$ is

$$
f_{0}:=f_{0}(u, v, w)=\alpha u^{3}+\beta u^{2} v+\gamma u v^{2}+\delta v^{3}+\sigma u w^{2}+\tau v w^{2}=0 .
$$

Imposing the vanishing of the three partial derivatives of $f_{0}$ evaluated at $Q_{\infty}$, we get the following system:

$$
\left\{\begin{array}{l}
f_{0, u}=3 \alpha+2 \beta m+\gamma m^{2}=0  \tag{76}\\
f_{0, v}=\beta+2 \gamma m+3 \delta m^{2}=0
\end{array}\right.
$$

Note that we get only two nontrivial equations, since the derivative of $f_{0}$ with respect to $w$ evaluated at any point of $\ell_{\infty}$ is zero, because only the last two terms of $f_{0}$ contain $w$, and in fact only $w^{2}$. Equivalently, the homogeneous equation of the tangent line to $\bar{C}$ at any of its points at infinity does not contain $w$, hence it passes through $O$ (cf. [3, Theorem 3.4]). Now, the occurrence of a singular point of $\bar{C}$ at $Q_{\infty}$ is equivalent to the fact that $m$ is a common root of both equations in (76), This happens if and only if the resultant $\operatorname{Res}\left(f_{0, u}, f_{0, v}\right)$ of the two polynomials in $m$ at the left hand side of the two equations vanishes. On the other hand,

$$
\operatorname{Res}\left(f_{0, u}, f_{0, v}\right)=\left|\begin{array}{cccc}
\gamma & 2 \beta & 3 \alpha & 0 \\
0 & \gamma & 2 \beta & 3 \alpha \\
3 \delta & 2 \gamma & \beta & 0 \\
0 & 3 \delta & 2 \gamma & \beta
\end{array}\right|
$$

which, up to the multiplicative factor 3 , is given by

$$
F:=4 \alpha \gamma^{3}+4 \beta^{3} \delta-\beta^{2} \gamma^{2}+27 \alpha^{2} \delta^{2}-18 \alpha \beta \gamma \delta
$$

An easy check shows that this quartic polynomial is irreducible. It defines a quartic hypersurface in $\mathbb{P}^{5}$ and then the locus of the cubics $C$ as in 1) corresponds to the section of the quadric $\mathscr{Q}^{\prime}$ with the quartic of equation $F=0$. In particular this says that the family corresponding to case 1) depends on three parameters. This is in accordance with the following fact: Serre-invariant cubics passing through $P_{\infty}$ depend on 4 parameters (general point of the hyperplane $\mathscr{H}$ ) and imposing a singularity at a point that can vary on $\ell_{\infty}$ requires only one condition; hence the dimension is $4-1=3$. We already said about the dimensions of the families corresponding to cases $2 a$ ) and $2 b$ ), hence we come to case $2 c$ ). Since $\ell_{0}$ is fixed, having equation $(n-2) u-v=0, C$ is determined by the slope of $\ell^{\prime}$ (the same as that of $\ell^{\prime \prime}$ ) and e. g. the distance between $\ell^{\prime}$ and $\ell^{\prime \prime}$; hence the family depends on two parameters. In fact, letting $Q_{\infty}=(-a: b: 0), \ell^{\prime}$ has equation $b u+a v-c=0$ for some $c \in \mathbb{C}$, and then the equation of $C$ has the form:

$$
[(n-2) u-v]\left[(b u+a v)^{2}-c^{2}\right]=0
$$

Therefore $C$ depends on the two parameters $m:=-b / a$ and $c / a$. In conclusion, letting $F_{i)}$ denote the family of the cubics $C$ as in case $i$, where $i=1,2 a$, etc., we have

$$
\operatorname{dim}\left[F_{1)}\right]=\operatorname{dim}\left[F_{2 a)}\right]=3, \text { while } \operatorname{dim}\left[F_{2 b}\right]=\operatorname{dim}\left[F_{2 c)}\right]=2
$$

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[^0]:    ${ }^{1}$ In fact, the polarization depends only on the first Chern class $c_{1}(L)$

[^1]:    ${ }^{2} \mathcal{O}_{C}(1)$ denotes the restriction of the line bundle $\mathcal{O}_{\mathbb{P}^{2}}(1)$ to $C$.
    ${ }^{3}$ In order to facilitate further reading, we keep the notation as in [15].

[^2]:    ${ }^{1}$ we will call such a morphism a fibration throughout the rest of these exercises.

[^3]:    ${ }^{2}$ you may assume both $X$ and $B$ are projective, although the results presented here work in a more general setting.

[^4]:    ${ }^{1}$ After this paper was finished wonderful rays have been exhibited in the preprint "Irrational nef rays at the boundary of the Mori cone for very general blowups of the plane" (arXiv:2201.08634), by J. Roé and the two authors of the present paper.

