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A Remark on Long Range Effect for a System of Semilinear Wave Equations

Hideo Kubo and Motoharu Takaki

Abstract. *In this paper we consider the Cauchy problem for a system of semilinear wave equations whose nonlinearity has long range effect on the solution. Such a result was obtained by Kubo, Kubota, and Sunagawa (Math. Ann. 335 (2006)) under the assumption that the initial data are radially symmetric. The aim of this paper is to remove the radial symmetricity of the initial data for typical cases of the nonlinearities.*

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1. Introduction

This paper is concerned with the global behavior of solutions to the Cauchy problem for a system of semilinear wave equations:

$$\begin{cases} (\partial_t^2 - \Delta)u = |\partial_t v|^2 & \text{in } [0,\infty)\times\mathbb{R}^3, \\ (\partial_t^2 - \Delta)v = |\partial_t u|^p & \text{in } [0,\infty)\times\mathbb{R}^3, \end{cases} \tag{1}$$

where $p > 2$, $\Delta = \sum_{j=1}^3 \partial_j^2$, $\partial_j = \partial/\partial x_j$, and $\partial_t = \partial/\partial t$. The initial condition is posed by

$$\begin{cases} u(0) = f_1,\ \partial_t u(0) = g_1, \\ v(0) = f_2,\ \partial_t v(0) = g_2, \end{cases} \tag{2}$$

where f_i, g_i $(i = 1, 2)$ are functions in some weighted Sobolev spaces whose norms are supposed to be small enough. In [4] the Cauchy problem for (1) with a wider range of exponents of the nonlinearity was handled. Namely, the global behavior of solutions to

$$\begin{cases} (\partial_t^2 - \Delta)u = |\partial_t v|^{p_1} & \text{in } [0,\infty)\times\mathbb{R}^3, \\ (\partial_t^2 - \Delta)v = |\partial_t u|^{p_2} & \text{in } [0,\infty)\times\mathbb{R}^3 \end{cases}$$

was studied, by assuming that the initial data is radially symmetric and the exponents p_1, p_2 satisfy $1 < p_1 \le p_2$, $p_2(p_1 - 1) > 2$.

The aim of this paper is to remove the assumption that the initial data is radially symmetric for the case where $p_1 = 2$, $p_2 > 2$. We recall that in the case of the single wave equation

$$(\partial_t^2 - \Delta)u = |\partial_t u|^p \quad \text{in } [0,\infty) \times \mathbb{R}^3, \tag{3}$$

the critical exponent is $p = 2$ (see [6], [2]). We underline the following significant difference between the problems for the single wave equation and for a system of semilinear wave equations : The global solution of the single wave equation is asymptotically free for $p > 2$, i.e., there exists a solution of $(\partial_t^2 - \Delta)u_+ = 0$ in $[0,\infty) \times \mathbb{R}^3$ such that

$$\|(u - u_+)(t)\|_E \le C\varepsilon^p (1+t)^{-(p-2)},$$

for $t \ge 0$, where $\|u(t)\|_E$ stands for the energy, namely,

$$\|u(t)\|_E = \left(\int_{\mathbb{R}^3} (|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2) dx \right)^{1/2}$$

(see [2]). On the other hand, it was shown in [4] that the global solution (u, v) of (1)-(2) cannot be asymptotically free in general and tends to the solution (u_+, v_+) of

$$\begin{cases} (\partial_t^2 - \Delta)u_+ = |\partial_t v_+|^2 & \text{in } [0,\infty) \times \mathbb{R}^3, \\ (\partial_t^2 - \Delta)v_+ = 0 & \text{in } [0,\infty) \times \mathbb{R}^3 \end{cases} \tag{4}$$

in the sense of the energy, provided the solution (u, v) is radially symmetric. The presence of the right-hand side on the first equation of (4) shows the long range effect of the nonlinearity. Therefore, it is a natural question if the same is true without assuming the radial symmetricity or not.

In order to state our result, we introduce notation. We set

$$\partial_0 = \partial_t,\ \partial = (\partial_t, \nabla),\ \nabla = (\partial_1, \partial_2, \partial_3),\ S = t\partial_t + x \cdot \nabla,$$
$$\Omega_{0j} = t\partial_j + x_j \partial_t \ (j = 1,2,3),\ \Omega_{ij} = x_j \partial_i - x_i \partial_j \ (1 \le i < j \le 3).$$

We denote $\Gamma = (\Gamma_1, \ldots, \Gamma_{11})$ and $\Gamma^\alpha = \Gamma_1^{\alpha_1} \ldots \Gamma_{11}^{\alpha_{11}}$, where Γ_i is one of ∂_μ $(0 \le \mu \le 3)$, $\Omega_{\mu\nu}$ $(0 \le \mu < \nu \le 3)$, and S, while $\alpha = (\alpha_1, \ldots, \alpha_{11})$ is a multi-index. For $s = 0, 1, 2, \ldots$ and $1 \le q \le \infty$, we set

$$\|u(t)\|_{s,q} = \sum_{|\alpha| \le s} \|\Gamma^\alpha u(t)\|_{L^q(\mathbb{R}^3)}.$$

Let $\mathcal{H}^s$ be the completion of $C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$ with respect to $\|\nabla f\|_{H^s} + \|g\|_{H^s}$, where $(f, g) \in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$ and $s = 0, 1, 2, \ldots$

We now define

$$Y_\delta = \{(u(t), \partial_t u(t)), (v(t), \partial_t v(t)) \in C([0,\infty); \mathcal{H}^2)\,;\, |\!|\!|(u,v)|\!|\!|_{X^2} \le \delta\},$$

where we put

$$\|\!|(u,v)\|\!|_{X^s} = \sup_{t\geq 0} \frac{\|\partial u(t)\|_{s,2}}{\log(2+t)} + \sup_{t\geq 0} \|\partial v(t)\|_{s,2}.$$

We look for a solution to the following integral equation in Y_δ:

$$\begin{cases} (u(t), \partial_t u(t)) = U(t)(f_1, g_1) + \displaystyle\int_0^t U(t-s)(0, |\partial_t v(s)|^2)ds, \\ (v(t), \partial_t v(t)) = U(t)(f_2, g_2) + \displaystyle\int_0^t U(t-s)(0, |\partial_t u(s)|^p)ds \end{cases} \tag{5}$$

for $t \in [0,\infty)$. Here $U(t)$ is the propagator associated with the wave operator $\partial_t^2 - \Delta$. In addition, we put $\langle x\rangle = \sqrt{1+|x|^2}$ for $x \in \mathbb{R}^3$. Throughout this paper, we denote by C a positive constant which may change from line by line.

Our result in this paper is the following.

Theorem 1.1. *Assume that* (f_1, g_1), $(f_2, g_2) \in \mathcal{H}^2$ *satisfy*

$$\sum_{i=1,2} (\|\langle\cdot\rangle^2 \nabla f_i\|_{H^2} + \|\langle\cdot\rangle^2 g_i\|_{H^2}) \leq \varepsilon. \tag{6}$$

If $p > 2$*, then there exist positive numbers* ε_0 *and* C_0 *such that for any* $\varepsilon \in (0, \varepsilon_0]$*, the problem* (5) *has a solution* $(u,v) \in Y_{2C_0\varepsilon}$*. Moreover, there exists a solution* (u_+, v_+) *to* (4) *satisfying*

$$\|(u-u_+)(t)\|_E \leq C\varepsilon^{p+1}(1+t)^{-(\tilde{p}-2)}, \tag{7}$$

$$\|(v-v_+)(t)\|_E \leq C\varepsilon^{p}(1+t)^{-(\tilde{p}-2)} \tag{8}$$

for $t \geq 0$*, where* $\tilde{p}$ *is an arbitrary number such that* $2 < \tilde{p} < p$*.*

If we assume, in addition to (6)*, that* $(f_i, g_i) \in C^1(\mathbb{R}^3) \times C(\mathbb{R}^3)$ *satisfy*

$$\sup_{x\in\mathbb{R}^3} [\langle x\rangle^{p-1}|f_i(x)| + \langle x\rangle^p\{|g_i(x)| + |\nabla f_i(x)|\}] \leq A \tag{9}$$

for some A *and* $i = 1, 2$*, then we have*

$$|u(t,x)| \leq \frac{C(A + \varepsilon^2(\log(1+t+|x|))^2)}{(1+t+|x|)}, \tag{10}$$

$$|v(t,x)| \leq \frac{C(A+\varepsilon^p)}{(1+t+|x|)(1+|t-|x||)^{\tilde{p}-2}} \tag{11}$$

for $(t,x) \in [0,\infty) \times \mathbb{R}^3$*.*

2. Preliminaries

In this section we prepare several lemmas.

LEMMA 2.1 ([3]). *For* $u \in C_0^\infty([0,\infty)\times\mathbb{R}^n)$ *we have*

$$|u(t,x)| \le C(1+t+|x|)^{-(n-1)/p}(1+|t-|x||)^{-1/p}\|u(t)\|_{s,p},$$

provided $s > n/p$, $s = 1, 2, \dots,$ *and* $1 \le p < \infty$.

While, for $u \in C_0^\infty([0,\infty)\times\mathbb{R}^n)$ *we have*

$$\|u(t)\|_{L^q} \le C(1+t)^{-(n-1)(1/p-1/q)}\|u(t)\|_{s,p},$$

provided $1 \le p \le q < \infty, s \ge n(1/p-1/q)$, *and* $s = 1, 2, \dots$.

LEMMA 2.2. *Let* $F(t) \in C([0,\infty); H^m(\mathbb{R}^3))$. *If* u *solves* $(\partial_t^2 - \Delta)u = F$, *then we have for any* $m \ge 0$

$$\|\partial u(t)\|_{m,2} \le \|\partial u(0)\|_{m,2} + C\int_0^t \|F(s)\|_{m,2}ds.$$

For $g \in C(\mathbb{R}^3)$ we define its spherical mean by

$$M_t g(x) = \frac{1}{4\pi}\int_{S^2} g(x+t\omega)dS_\omega \tag{12}$$

for $t \ge 0$ and $x \in \mathbb{R}^3$. Then we have the following.

LEMMA 2.3 ([1]). *Assume that* $f \in C^1(\mathbb{R}^3)$, $g \in C(\mathbb{R}^3)$ *satisfy*

$$(1+|x|)^{\kappa-1}|f(x)| + (1+|x|)^\kappa\{|g(x)| + |\nabla f(x)|\} \le A \tag{13}$$

for any $x \in \mathbb{R}^3$, *where* $A > 0, \kappa > 2$. *If we set*

$$u(t,x) = tM_t g(x) + \partial_t(tM_t f(x)),$$

then we have for any $(t,x) \in [0,\infty)\times\mathbb{R}^3$

$$|u(t,x)| \le \frac{CA}{(1+t+|x|)(1+|t-|x||)^{\kappa-2}}.$$

LEMMA 2.4 ([5]). *If* $f \in C(\mathbb{R}^3)$, *then we have*

$$\int_{|y-x|=t} f(|y|)dS_y = \frac{2\pi t}{r}\int_{|r-t|}^{r+t} \lambda f(\lambda)d\lambda$$

for $t \ge 0$ *and* $x \in \mathbb{R}^3$ *with* $r = |x|$.

3. Linearized Problem

For $0 < \delta < 1$, we wish to define a map $L\colon Y_\delta \to Y_\delta$ by $L(w,z) = (u,v)$, where (u,v) is the solution of

$$\begin{cases} (u(t), \partial_t u(t)) = U(t)(f_1, g_1) + \int_0^t U(t-s)(0, |\partial_t z(s)|^2)ds, \\ (v(t), \partial_t v(t)) = U(t)(f_2, g_2) + \int_0^t U(t-s)(0, |\partial_t w(s)|^p)ds \end{cases} \tag{14}$$

for $t \in [0,\infty)$. Note that if (f_1, g_1), $(f_2, g_2) \in \mathcal{H}^2$ satisfy (6), then we see that

$$\|\partial u(0)\|_{2,2} + \|\partial v(0)\|_{2,2} \le C_0 \varepsilon \tag{15}$$

with a numerical constant C_0.

In the following proposition we prove a basic property of the map L.

Proposition 3.1. *Let $p > 2$ and (6) be fulfilled. Then there exists a positive number ε_0 such that L maps $Y_{2C_0\varepsilon}$ into itself for $\varepsilon \in (0, \varepsilon_0]$. Moreover, for any $(w,z), (\tilde{w}, \tilde{z}) \in Y_\delta$ with $0 < \delta < 1$ we have*

$$\|\!|L(w,z) - L(\tilde{w},\tilde{z})|\!\|_{X^1} \le C\delta \|\!|(w,z) - (\tilde{w},\tilde{z})|\!\|_{X^1} \tag{16}$$

and

$$\begin{aligned} \|\!|L(w,z) - L(\tilde{w},\tilde{z})|\!\|_{X^2} \le C\delta(&\|\!|(w,z) - (\tilde{w},\tilde{z})|\!\|_{X^2} \\ &+ \|\!|(w,z) - (\tilde{w},\tilde{z})|\!\|_{X^1}^{p-2}). \end{aligned} \tag{17}$$

Proof. We begin with proving that if $(w,z) \in Y_{2C_0\varepsilon}$, then we have $(u,v) = L(w,z) \in Y_{2C_0\varepsilon}$, provided ε is sufficiently small.

First we show that for any $(w,z) \in Y_\delta$ we have

$$\sup_{t \ge 0} \frac{\|\partial u(t)\|_{2,2}}{\log(2+t)} \le \|\partial u(0)\|_{2,2} + C\delta^2. \tag{18}$$

By using Lemma 2.1 we get

$$\begin{aligned} \||\partial_t z(s)|^2\|_{L^2} &\le \|\partial_t z(s)\|_{L^\infty} \|\partial_t z(s)\|_{L^2} \le C(1+s)^{-1} \|\partial_t z(s)\|_{2,2}^2 \\ &\le C\delta^2 (1+s)^{-1}, \\ \|\Gamma(|\partial_t z(s)|^2)\|_{L^2} &\le C \|\partial_t z(s)\|_{L^\infty} \|\Gamma \partial_t z(s)\|_{L^2} \\ &\le C(1+s)^{-1} \|\partial_t z(s)\|_{2,2}^2 \\ &\le C\delta^2 (1+s)^{-1}, \end{aligned}$$

and

$$\begin{aligned}\|\Gamma^2(|\partial_t z(s)|^2)\|_{L^2} &\le C(\|\partial_t z(s)\|_{L^\infty}\|\Gamma^2\partial_t z(s)\|_{L^2}+\|\Gamma\partial_t z(s)\|_{L^4}^2)\\ &\le C(1+s)^{-1}\|\partial_t z(s)\|_{2,2}^2\\ &\le C\delta^2(1+s)^{-1}.\end{aligned}$$

Therefore it follows from Lemma 2.2 that

$$\|\partial u(t)\|_{2,2} \le \|\partial u(0)\|_{2,2} + C\delta^2\int_0^t (1+s)^{-1}ds, \tag{19}$$

which implies (18).

Next we show that for any $(w,z)\in Y_\delta$ with $0<\delta<1$, we have

$$\sup_{t\ge 0}\|\partial v(t)\|_{2,2} \le \|\partial v(0)\|_{2,2} + C\delta^2. \tag{20}$$

By Lemma 2.1 we get

$$\begin{aligned}\||\partial_t w(s)|^p\|_{L^2} &\le \|\partial_t w(s)\|_{L^\infty}^{p-1}\|\partial_t w(s)\|_{L^2}\\ &\le C(1+s)^{-(p-1)}\|\partial_t w(s)\|_{2,2}^p\\ &\le C\delta^p(1+s)^{-(p-1)}(\log(2+s))^p,\\ \|\Gamma(|\partial_t w(s)|^p)\|_{L^2} &\le C\|\partial_t w(s)\|_{L^\infty}^{p-1}\|\Gamma\partial_t w(s)\|_{L^2}\\ &\le C(1+s)^{-(p-1)}\|\partial_t w(s)\|_{2,2}^p\\ &\le C\delta^p(1+s)^{-(p-1)}(\log(2+s))^p,\end{aligned}$$

and

$$\begin{aligned}\|\Gamma^2(|\partial_t w(s)|^p)\|_{L^2} &\le C(\|\partial_t w(s)\|_{L^\infty}^{p-1}\|\Gamma^2\partial_t w(s)\|_{L^2}\\ &\qquad + \|\partial_t w(s)\|_{L^\infty}^{p-2}\|\Gamma\partial_t z(s)\|_{L^4}^2)\\ &\le C\delta^p(1+s)^{-(p-1)}(\log(2+s))^p.\end{aligned}$$

Since $p>2$ and $0<\delta<1$ we have

$$\begin{aligned}\|\partial v(t)\|_{2,2} &\le \|\partial v(0)\|_{2,2} + C\delta^p\int_0^t (1+s)^{-(p-1)}(\log(2+s))^p ds\\ &\le \|\partial v(0)\|_{2,2} + C\delta^2.\end{aligned}$$

Thus we obtain (20), and hence

$$|||(u,v)|||_{X^2} \le C_0\varepsilon + C\delta^2,$$

by (15). If we choose $\delta = 2C_0\varepsilon$ and taking ε_0 so small that $4CC_0\varepsilon_0 \le 1$ and $2C_0\varepsilon_0 < 1$, we find the desired conclusion.

Next we prove (16) and (17). In the following, we denote by $(\tilde{u}, \tilde{v}) = L(\tilde{w}, \tilde{z})$. Analogously to (18), we can prove

$$\sup_{t\ge 0} \frac{\|\partial(u-\tilde{u})(t)\|_{s,2}}{\log(2+t)} \le C\delta \|\!|(w,z)-(\tilde{w},\tilde{z})\|\!|_{X^s} \tag{21}$$

for $s = 1, 2$. Therefore it suffices to show

$$\sup_{t\ge 0} \|\partial(v-\tilde{v})(t)\|_{1,2} \le C\delta \|\!|(w,z)-(\tilde{w},\tilde{z})\|\!|_{X^1} \tag{22}$$

and

$$\begin{aligned}\sup_{t\ge 0} \|\partial(v-\tilde{v})(t)\|_{2,2} \le C\delta(&\|\!|(w,z)-(\tilde{w},\tilde{z})\|\!|_{X^2} \\ &+ \|\!|(w,z)-(\tilde{w},\tilde{z})\|\!|_{X^1}^{p-2}).\end{aligned} \tag{23}$$

For this, we need to evaluate $\||\partial_t w(s)|^p - |\partial_t \tilde{w}(s)|^p\|_{s,2}$ $(s = 0, 1, 2)$.

Since $p > 1$, we have from Lemma 2.1

$$\begin{aligned}&\||\partial_t w(s)|^p - |\partial_t \tilde{w}(s)|^p\|_{L^2} \\ &\le p(\|\partial_t w(s)\|_{L^\infty} + \|\partial_t \tilde{w}(s)\|_{L^\infty})^{p-1}\|\partial_t(w-\tilde{w})(s)\|_{L^2} \\ &\le C(1+s)^{-(p-1)}(\|\partial_t w(s)\|_{2,2} + \|\partial_t \tilde{w}(s)\|_{2,2})^{p-1}\|\partial_t(w-\tilde{w})(s)\|_{1,2} \\ &\le C\delta^{p-1}\|\!|(w,z)-(\tilde{w},\tilde{z})\|\!|_{X^1}(1+s)^{-(p-1)}(\log(2+s))^p.\end{aligned}$$

Similarly we get

$$\begin{aligned}&\|\Gamma(|\partial_t w(s)|^p - |\partial_t \tilde{w}(s)|^p)\|_{L^2} \\ &\le C\delta^{p-1}\|\!|(w,z)-(\tilde{w},\tilde{z})\|\!|_{X^1}(1+s)^{-(p-1)}(\log(2+s))^p.\end{aligned}$$

Thus we get (22), since $0 < \delta < 1$ and $p > 2$.

When $p \ge 3$, we have

$$\begin{aligned}&\|\Gamma^2(|\partial_t w(s)|^p - |\partial_t \tilde{w}(s)|^p)\|_{L^2} \\ &\le C\delta^{p-1}\|\!|(w,z)-(\tilde{w},\tilde{z})\|\!|_{X^2}(1+s)^{-(p-1)}(\log(2+s))^p.\end{aligned}$$

While, when $2 < p < 3$, observing that

$$\begin{aligned}&\|(|\partial_t w(s)|^{p-2} - |\partial_t \tilde{w}(s)|^{p-2})(\Gamma\partial_t \tilde{w}(s))^2\|_{L^2} \\ &\le \||\partial_t(w-\tilde{w})(s)|^{p-2}(\Gamma\partial_t \tilde{w}(s))^2\|_{L^2} \\ &\le \|\partial_t(w-\tilde{w})(s)\|_{L^6}^{p-2}\|\Gamma\partial_t \tilde{w}(s)\|_{L^q}^2 \\ &\le C\delta^2\|\!|(w,z)-(\tilde{w},\tilde{z})\|\!|_{X^1}^{p-2}(1+s)^{-(2(p-2)/3)-((p+1)/3)}(\log(2+s))^p\end{aligned}$$

with $q = 12/(5-p)$, we find

$$\begin{aligned}&\|\Gamma^2(|\partial_t w(s)|^p - |\partial_t \tilde{w}(s)|^p)\|_{L^2}\\ &\le C(\delta^2 |\!|\!|(w,z)-(\tilde{w},\tilde{z})|\!|\!|_{X^1}^{p-2} + \delta^{p-1}|\!|\!|(w,z)-(\tilde{w},\tilde{z})|\!|\!|_{X^2})\\ &\qquad \times (1+s)^{-(p-1)}(\log(2+s))^p.\end{aligned}$$

Since $0<\delta<1$ and $p>2$, we obtain (23), and hence (16) and (17) follow. This completes the proof. □

4. Proof of Theorem 1.1

Let C_0 and ε_0 be the numbers from (15) and Proposition 3.1. In the following we always assume $\varepsilon \in (0, \varepsilon_0]$.

First of all, we define a sequence $\{(u_n, v_n)\}_{n=0}^\infty \subset Y_{2C_0\varepsilon}$ by

$$\begin{cases}(u_{n+1}, v_{n+1}) = L(u_n, v_n) \quad \text{for } n = 0,1,2\dots,\\ (u_0(t), \partial_t u_0(t)) = U(t)(f_1, g_1), \quad (v_0(t), \partial_t v_0(t)) = U(t)(f_2, g_2).\end{cases}$$

From (15) we have $(u_0, v_0) \in Y_{2C_0\varepsilon}$, so that $\{(u_n, v_n)\}_{n=0}^\infty \subset Y_{2C_0\varepsilon}$ for all n. It follows from (16) and (17) with $(w,z) = (u_n, v_n)$, $(\tilde{w},\tilde{z}) = (u_{n-1}, v_{n-1})$ that

$$\begin{aligned}&|\!|\!|(u_{n+1}, v_{n+1}) - (u_n, v_n)|\!|\!|_{X^2} + |\!|\!|(u_{n+1}, v_{n+1}) - (u_n, v_n)|\!|\!|_{X^1}^{p-2}\\ &\le 2CC_0\varepsilon(|\!|\!|(u_n, v_n) - (u_{n-1}, v_{n-1})|\!|\!|_{X^2} + |\!|\!|(u_n, v_n) - (u_{n-1}, v_{n-1})|\!|\!|_{X^1}^{p-2}).\end{aligned}$$

We may assume $2CC_0\varepsilon < 1/2$, by taking ε_0 to be smaller if necessary. Thus we get

$$\begin{aligned}&|\!|\!|(u_{n+1}, v_{n+1}) - (u_n, v_n)|\!|\!|_{X^2} + |\!|\!|(u_{n+1}, v_{n+1}) - (u_n, v_n)|\!|\!|_{X^1}^{p-2}\\ &\le 2^{-n}(|\!|\!|(u_1, v_1) - (u_0, v_0)|\!|\!|_{X^2} + |\!|\!|(u_1, v_1) - (u_0, v_0)|\!|\!|_{X^1}^{p-2}),\end{aligned}$$

which implies that $\{(u_n, v_n)\}_{n=0}^\infty$ is a Cauchy sequence in $Y_{2C_0\varepsilon}$. Hence there exists $(u,v) \in Y_{2C_0\varepsilon}$ such that $\{(u_n, v_n)\}_{n=0}^\infty$ converges to it in $Y_{2C_0\varepsilon}$, so that it is the solution of (5).

Next we deduce the asymptotic behavior of the solution (u,v) to (1)-(2) obtained in the above. Let $\delta = 2C_0\varepsilon$ in the following.

Notice that $|\partial_t u(t)|^p \in L^1([0,\infty); L^2(\mathbb{R}^3))$, since we have $\||\partial_t u(t)|^p\|_{L^2} \le C\delta^p(1+s)^{-(p-1)}(\log(2+s))^p$. Therefore, we can define

$$v_+(t) = v(t) + \int_t^\infty \mathcal{F}^*\Big(\frac{\sin(t-s)|\xi|}{|\xi|}\mathcal{F}(|\partial_t u(s)|^p)\Big)ds, \tag{24}$$

and we see that $v - v_+$ solves

$$(\partial_t^2 - \Delta)(v - v_+) = |\partial_t u|^p$$

in the sense of distribution. Hence v_+ satisfies $(\partial_t^2 - \Delta)v_+ = 0$. Moreover, we have

$$\begin{aligned}\|(v-v_+)(t)\|_E &\le \int_t^\infty \||\partial_t u(s)|^p\|_{L^2}ds \\ &\le C\delta^p \int_t^\infty (1+s)^{-(p-1)}(\log(2+s))^p ds,\end{aligned}$$

which leads to (8).

In order to proceed further we need the following lemma.

LEMMA 4.1. *Let v_+ be the function defined by* (24)*. Then we have*

$$\|\partial_t v_+(t)\|_{1,2} \le C\delta, \tag{25}$$

$$\|\partial_t (v-v_+)(t)\|_{1,2} \le C\delta^p(1+t)^{-(\tilde{p}-2)} \tag{26}$$

for $t \ge 0$.

Once we find the above lemma, we can deduce (7). In fact, (25) and (26) together with Lemma 2.1 yield

$$\begin{aligned}&\||\partial_t v(t)|^2 - |\partial_t v_+(t)|^2\|_{L^2} \\ &\le (\|\partial_t v(t)\|_{L^4} + \|\partial_t v_+(t)\|_{L^4})\|\partial_t(v-v_+)(t)\|_{L^4} \\ &\le C(1+s)^{-1}(\|\partial_t v(t)\|_{1,2} + \|\partial_t v_+(t)\|_{1,2})\|\partial_t(v-v_+)(t)\|_{1,2} \\ &\le C\delta^{p+1}(1+s)^{-(\tilde{p}-1)}.\end{aligned} \tag{27}$$

This estimate allows us to define

$$u_+(t) = u(t) + \int_t^\infty \mathcal{F}^*\Big(\frac{\sin(t-s)|\xi|}{|\xi|}\mathcal{F}(|\partial_t v(s)|^2 - |\partial_t v_+(s)|^2)\Big)ds.$$

Then we see that $u - u_+$ solves

$$(\partial_t^2 - \Delta)(u-u_+) = |\partial_t v|^2 - |\partial_t v_+|^2$$

in the sense of distribution, and hence u_+ satisfies

$$(\partial_t^2 - \Delta)u_+ = |\partial_t v_+|^2.$$

Moreover, by (27) we have

$$\begin{aligned}\|(u-u_+)(t)\|_E &\le C\int_t^\infty \||\partial_t v(s)|^2 - |\partial_t v_+(s)|^2\|_{L^2}ds \\ &\le C\delta^{p+1}\int_t^\infty (1+s)^{-(\tilde{p}-1)}ds,\end{aligned}$$

which leads to (7).

Proof of Lemma 4.1. Notice that (26) yields

$$\|\partial_t v_+(t)\|_{1,2} \le \|\partial_t v(t)\|_{1,2} + \|\partial_t (v_+ - v)(t)\|_{1,2} \le C\delta,$$

since $0 < \delta < 1$ and $p > 2$. Therefore, it is enough to show (26). In view of (8), we see that (26) follows from

$$\|\Gamma \partial_t (v - v_+)(t)\|_{L^2} \le C\delta^p (1+t)^{-(\tilde{p}-2)}. \tag{28}$$

We are going to show (28). For simplicity, we put $\tilde{v} = v - v_+$ and $F(t,x) = |\partial_t u(t,x)|^p$. It follows from (24) that

$$\begin{aligned}
\partial_t^2 \tilde{v}(t) &= \int_t^\infty \mathcal{F}^*\big(|\xi|(\sin(t-s)|\xi|)\mathcal{F}(F(s))\big)ds + F(t,x),\\
\partial_j \partial_t \tilde{v}(t) &= -\int_t^\infty \mathcal{F}^*\big((\cos(t-s)|\xi|)\mathcal{F}(\partial_j F(s))\big)ds \quad (j=1,2,3),\\
\Omega_{jl} \partial_t \tilde{v}(t) &= -\int_t^\infty \mathcal{F}^*\big((\cos(t-s)|\xi|)\mathcal{F}(\Omega_{jl} F(s))\big)ds \quad (1 \le j < l \le 3),\\
\Omega_{0j} \partial_t \tilde{v}(t) &= -\int_t^\infty \mathcal{F}^*\big((\cos(t-s)|\xi|)\mathcal{F}(\Omega_{0j} F(s))\big)ds\\
&\quad + i\int_t^\infty \mathcal{F}^*\Big((\sin(t-s)|\xi|)\frac{\xi_j}{|\xi|}\mathcal{F}(F(s))\Big)ds \quad (j=1,2,3),\\
S\partial_t \tilde{v}(t) &= -\int_t^\infty \mathcal{F}^*\big((\cos(t-s)|\xi|)\mathcal{F}(SF(s))\big)ds\\
&\quad -\int_t^\infty \mathcal{F}^*\big((\cos(t-s)|\xi|)\mathcal{F}(F(s))\big)ds.
\end{aligned}$$

Thus we get

$$\|\Gamma \partial_t \tilde{v}(t)\|_{L^2} \le \int_t^\infty \sum_{|\alpha|\le 1} \|\Gamma^\alpha F(s)\|_{L^2}\, ds + \|F(t)\|_{L^2},$$

which implies (28). This completes the proof of Lemma 4.1. □

We turn back to the proof of Theorem 1.1. It remains to prove (10) and (11) under the assumption (9). Since $(f_i, g_i) \in C^1(\mathbb{R}^3) \times C(\mathbb{R}^3)$ $(i = 1,2)$, we see that

$$\begin{aligned}
u(t,x) &= u_0(t,x) + \frac{1}{4\pi}\int_0^t \frac{1}{t-s}\int_{|y-x|=t-s} |\partial_t v(s,y)|^2 dS_y ds,\\
v(t,x) &= v_0(t,x) + \frac{1}{4\pi}\int_0^t \frac{1}{t-s}\int_{|y-x|=t-s} |\partial_t u(s,y)|^p dS_y ds
\end{aligned}$$

with

$$u_0(t,x) = tM_t g_1(x) + \partial_t(tM_t f_1(x)),$$
$$v_0(t,x) = tM_t g_2(x) + \partial_t(tM_t f_2(x)).$$

By (9) and Lemma 2.3 we get

$$|u_0(t,x)| + |v_0(t,x)| \le \frac{CA}{(1+t+r)(1+|t-r|)^{p-2}},$$

where we have set $r = |x|$. While, observing that Lemma 2.1 leads to

$$|\partial_t v(s,y)|^2 \le \frac{\|\partial_t v(s)\|_{2,2}^2}{(1+s+|y|)^2(1+|s-|y||)},$$

we have from Lemma 2.4

$$\begin{aligned}
&\left|\frac{1}{4\pi}\int_0^t \frac{1}{t-s}\int_{|y-x|=t-s} |\partial_t v(s,y)|^2 dS_y ds\right| \\
&\le \frac{C}{r}\int_0^t \int_{|r-t+s|}^{r+t-s} \frac{\lambda\|\partial_t v(s)\|_{2,2}^2}{(1+s+\lambda)^2(1+|s-\lambda|)} d\lambda ds \\
&\le \frac{C\delta^2}{r}\int_0^t \int_{|r-t+s|}^{r+t-s} \frac{1}{(1+s+\lambda)(1+|s-\lambda|)} d\lambda ds.
\end{aligned}$$

Changing the variables by $\alpha = s+\lambda$, $\beta = s-\lambda$, the last quantity is bounded by

$$\frac{C\delta^2}{r}\int_{|r-t|}^{r+t}\int_{-t-r}^{t-r} \frac{d\beta d\alpha}{(1+\alpha)(1+|\beta|)} \le \frac{C\delta^2}{r}\log(2+t+r)\int_{|r-t|}^{r+t}\frac{d\alpha}{(1+\alpha)}.$$

When $t \le 2r$ and $r \ge 1$, we have $r \ge C(1+r+t)$, so that

$$\frac{1}{r}\int_{|r-t|}^{r+t}\frac{1}{1+\alpha}d\alpha \le \frac{C\log(2+r+t)}{1+t+r}.$$

Therefore we get

$$\left|\frac{1}{4\pi}\int_0^t \frac{1}{t-s}\int_{|y-x|=t-s} |\partial_t v(s,y)|^2 dS_y ds\right| \le \frac{C\delta^2(\log(2+r+t))^2}{1+t+r}.$$

While, when $t > 2r$ or $0 < r < 1$, we have

$$\frac{1}{r}\int_{|r-t|}^{r+t}\frac{1}{1+\alpha}d\alpha \le \frac{r+t-|r-t|}{r(1+|r-t|)} \le \frac{1}{1+|r-t|}.$$

Since $1+r+t \le C(1+|r-t|)$, we find (10).

Similarly, we get

$$|\partial_t u(s,y)|^p \le \frac{\|\partial_t u(s)\|_{2,2}^p (\log(2+s))^p}{(1+s+|y|)^p(1+|s-|y||)^{p/2}},$$

so that

$$\begin{aligned}
&\left|\frac{1}{4\pi}\int_0^t \frac{1}{t-s}\int_{|y-x|=t-s} |\partial_t u(s,y)|^p dS_y ds\right| \\
&\le C\delta^p \int_0^t \frac{1}{t-s}\int_{|y-x|=t-s} \frac{(\log(2+s))^p}{(1+s+|y|)^p(1+|s-|y||)^{p/2}} dS_y ds \\
&\le \frac{C\delta^p}{r}\int_0^t \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{(1+s+\lambda)^{\tilde p}(1+|s-\lambda|)^{p/2}} d\lambda ds,
\end{aligned}$$

where $\tilde p$ is an arbitrary number such that $2 < \tilde p < p$. Changing the variables as before, we obtain (11) in a similar fashion. This completes the proof of Theorem 1.1. □

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Authors' addresses:

Hideo Kubo
Division of Mathematics
Graduate School of Information Sciences, Tohoku University
Sendai 980-8579, Japan
E-mail: kubo@math.is.tohoku.ac.jp

Motoharu Takaki
Nishinomiya 662-0811, Japan

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