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Products of sequentially compact spaces with no separability assumption

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Abstract. *Let X be a product of topological spaces. We prove that X is sequentially compact if and only if all subproducts by $\leq \mathfrak{s}$ factors are sequentially compact. If $\mathfrak{s} = \mathfrak{h}$, then X is sequentially compact if and only if all factors are sequentially compact and all but at most $< \mathfrak{s}$ factors are ultraconnected. We give a topological proof of the inequality* $\mathrm{cf}\, \mathfrak{s} \geq \mathfrak{h}$. *Recall that $\mathfrak{s}$ denotes the* splitting number *and $\mathfrak{h}$ the* distributivity number. *Some corresponding invariants are introduced, relative to an arbitrary topological property, more generally, relative to a subset of a partial infinitary semigroup.*

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1. Introduction

A countable product of sequentially compact spaces is still sequentially compact [7, Theorem 3.10.35]. The problem whether the above assertion generalizes to uncountable products involves the so-called combinatorial cardinal characteristics of the Continuum [1, 2, 16]. These are cardinals which are provably uncountable and less than or equal to the continuum $\mathfrak{c}$, but consistently strictly smaller than $\mathfrak{c}$. In particular, they all equal $\mathfrak{c}$ if the Continuum Hypothesis holds.

A cardinal characteristic has a standard definition which involves infinite combinatorics and frequently many equivalent formulations in different settings. For example, P. Simon [15] proved that one of these characteristics, the *distributivity number* $\mathfrak{h}$, is the smallest cardinal such that every product of $< \mathfrak{h}$ sequentially compact spaces is still sequentially compact. Thus the problem mentioned at the beginning is dependent on set theory: in some models of set theory $\mathfrak{h} = \omega_1$, in which case the classical result cannot be improved, but in other models $\mathfrak{h} = \mathfrak{c} > \omega_1$ [2], hence there are uncountable products which are sequentially compact.

As another influence of cardinal characteristics on products, Booth [3] showed that the *splitting number* $\mathfrak{s}$ is the smallest cardinal such that the product $\mathbf{2}^{\mathfrak{s}}$ is not sequentially compact. Here $\mathbf{2}$ denotes the discrete 2-element space. Since any nontrivial T_1 space contains a closed subspace isomorphic to $\mathbf{2}$, we get that if $\mathfrak{h} = \mathfrak{s}$ (an identity which is relatively consistent with the usual axioms of set theory [2]), then a product of T_1 spaces is sequentially compact if and only if all factors are sequentially compact and the set of nontrivial factors has cardinality $< \mathfrak{s}$. On the other hand, we are not aware of any former result of this kind when separation axioms are not assumed, apart from some partial results in S. Brandhorst thesis [4] under the strong assumption of the Continuum Hypothesis.

While many topologists usually deal with Hausdorff spaces—possibly, even with spaces satisfying higher regularity conditions—recently the interest on spaces satisfying lower separability conditions has newly arisen, e. g. [8, 9, 19]. In particular, see [13] for an interesting recent manifesto in support of the study of spaces satisfying lower separation axioms from a purely topological point of view.

In this note we show that a product of topological spaces is sequentially compact if and only if all subproducts by $\leq \mathfrak{s}$ factors are sequentially compact. If $\mathfrak{h} = \mathfrak{s}$, then a product is sequentially compact if and only if all factors are sequentially compact, and all but at most $< \mathfrak{s}$ factors are ultraconnected. While the proofs are elementary and do not rely on set theory, apart from the mentioned known characterizations of the cardinals $\mathfrak{h}$ and $\mathfrak{s}$, we believe that the results deserve to be explicitly presented with the details of the proofs.

Finally, a longstanding open problem has been recently solved by Dow and Shelah [6] who showed that it is consistent that $\mathfrak{s}$ is singular. Here we present a simple topological proof that the cofinality of $\mathfrak{s}$ is $\geq \mathfrak{h}$. The argument has a general flavor and suggests the idea of attaching similar invariants to an arbitrary property P of topological spaces. At the end of Section 3 we argue that the right framework for the argument is the context of partial infinitary semigroups with a specified subclass. While the ideas are simple, there is the possibility that the arguments and the general framework turn out to be a useful paradigm for many disparate situations. We exemplify the methods in the case of chain compactness.

2. Products of sequentially compact spaces

For the sake of simplicity, all topological spaces are assumed to be nonempty.

Recall that a space X is called *ultraconnected* if no pair of nonempty closed sets of X is disjoint.

Definition 2.1. *The* splitting number $\mathfrak{s}$ *is the least cardinal such that* $\mathbf{2}^{\mathfrak{s}}$ *is not sequentially compact, where* $\mathbf{2}$ *is the two-element discrete topological space. Usually the definition of* $\mathfrak{s}$ *is given in equivalent forms, but the present one is the most suitable for our purposes. See Booth [3, Theorem 2] or van Douwen [16, Theorem 6.1] for a proof of the equivalences. See [2, 16, 18] for further information about* $\mathfrak{s}$*.*

A proof of the next lemma can be found in [12, Lemmata 4.1 and 4.2].

Lemma 2.2. *(i) A topological space* X *is both ultraconnected and sequentially compact if and only if every sequence in* X *converges.*

(ii) A product of $\geq \mathfrak{s}$ *spaces which are not ultraconnected is not sequentially compact.*

Proposition 2.3. *If a product is sequentially compact, then the set of factors with a nonconverging sequence has cardinality* $< \mathfrak{s}$*.*

Proof. Suppose by contradiction that there are $\geq \mathfrak{s}$ factors with a nonconverging sequence. Since each factor is sequentially compact, then, by Lemma 2.2(i), there are $\geq \mathfrak{s}$ factors which are not ultraconnected, and Lemma 2.2(ii) gives a contradiction. □

Theorem 2.4. *A product of topological spaces is sequentially compact if and only if all subproducts by* $\leq \mathfrak{s}$ *factors are sequentially compact.*

Proof. Necessity is trivial, since we assume that all the spaces are nonempty and sequential compactness is preserved by taking images of surjective continuous functions. For the other direction, suppose that each subproduct of $X = \prod_{j\in J} X_j$ by $\leq \mathfrak{s}$ factors is sequentially compact, and let $J' = \{j \in J \mid X_j \text{ has a nonconverging sequence}\}$. If $|J'| \geq \mathfrak{s}$, choose $J'' \subseteq J'$ with $|J''| = \mathfrak{s}$. By assumption, $\prod_{j\in J''} X_j$ is sequentially compact, and we get a contradiction from Proposition 2.3. Thus $|J'| < \mathfrak{s}$. Now X is homeomorphic to $\prod_{j\in J'} X_j \times \prod_{j\in J\setminus J'} X_j$. The first factor is sequentially compact by assumption, since we have proved that $|J'| < \mathfrak{s}$. For each $j \in J \setminus J'$, we have that every sequence on X_j converges, thus in $\prod_{j\in J\setminus J'} X_j$, too, every sequence converges; a fortiori, $\prod_{j\in J\setminus J'} X_j$ is sequentially compact. Then X is sequentially compact, being the product of two sequentially compact spaces. □

In the context of T_1 spaces, Theorem 2.4 is an immediate consequence of Definition 2.1, since any nontrivial T_1 space contains a closed subspace isomorphic to $\mathbf{2}$. Thus if a product of T_1 spaces is sequentially compact, then all but $< \mathfrak{s}$ factors are one-element spaces. Then Theorem 2.4, restricted to T_1 spaces, follows, since if all subproducts of $\leq \mathfrak{s}$ factors are sequentially compact, then all but $< \mathfrak{s}$ factors are one-element spaces and the product of the nontrivial factors is sequentially compact by hypothesis. Thus the main point of Theorem 2.4 is the case of spaces satisfying few separation axioms.

The value $\mathfrak{s}$ in Theorem 2.4 is the best possible value: by Definition 2.1, all subproducts of $\mathbf{2}^{\mathfrak{s}}$ by $< \mathfrak{s}$ factors are sequentially compact, but $\mathbf{2}^{\mathfrak{s}}$ is not.

We now show that, under a relatively weak cardinality assumption, we can replace "subproducts" with "factors" in Theorem 2.4.

DEFINITION 2.5. *The* distributivity number $\mathfrak{h}$ *is the smallest cardinal such that there are $\mathfrak{h}$ sequentially compact spaces whose product is not sequentially compact. Usually, the definition of $\mathfrak{h}$ is given in some equivalent form: see Simon [15] for the proof of the equivalence, and [2, 18], for further information. Obviously, $\mathfrak{h} \leq \mathfrak{s}$. It is known that $\mathfrak{h} < \mathfrak{s}$ is relatively consistent [2].*

THEOREM 2.6. *Assume that $\mathfrak{h} = \mathfrak{s}$. If X is a product of topological spaces, then the following conditions are equivalent.*

(i) X is sequentially compact.

(ii) All factors of X are sequentially compact, and the set of factors with a nonconverging sequence has cardinality $< \mathfrak{s}$.

(iii) All factors of X are sequentially compact, and all but at most $< \mathfrak{s}$ factors are ultraconnected.

Proof. Conditions (ii) and (iii) are equivalent by Lemma 2.2(i).

Condition (i) implies Condition (ii) by Proposition 2.3.

The proof that (ii) implies (i) is similar to the proof of Theorem 2.4. Suppose that (ii) holds, and that $X = \prod_{j \in J} X_j$. Split X as $\prod_{j \in J'} X_j \times \prod_{j \in J \setminus J'} X_j$, where $J' = \{j \in J \mid X_j \text{ has a nonconverging sequence}\}$. By (ii) and the assumption, $|J'| < \mathfrak{s} = \mathfrak{h}$, hence, by the very definition of $\mathfrak{h}$ (the one we have presented), $\prod_{j \in J'} X_j$ is sequentially compact. Moreover $\prod_{j \in J \setminus J'} X_j$ is sequentially compact, since in it every sequence converges, hence also X is sequentially compact. □

Under the stronger assumption of the Continuum Hypothesis, we have learned of the equivalence of (i) and (ii) in Corollary 2.6 from Brandhorst [4]. See also Brandhorst and Erné [5]. As mentioned in the introduction, when restricted to T_1 spaces, Theorem 2.6 follows immediately from Definitions 2.1 and 2.5 (for a T_1 space X the following are equivalent: all sequences converge; X is ultraconnected; X is trivial, that is, a one-point space). On the other hand, we are not aware of any former result of this kind when no separation axiom is assumed, apart from the mentioned partial result in [4].

Notice that the assumption $\mathfrak{h} = \mathfrak{s}$ is necessary in Theorem 2.6. Indeed, it is now almost immediate to show that Conditions (i) and (ii) in Theorem 2.6 are equivalent if and only if $\mathfrak{h} = \mathfrak{s}$.

Corollary 2.7. *The following conditions are equivalent.*

(i) $\mathfrak{h} = \mathfrak{s}$

(ii) For every product X of topological spaces, condition (i) in Theorem 2.6 holds if and only if condition (ii) there holds.

(iii) For every product X with $\mathfrak{h}$ factors, condition (ii) in Theorem 2.6 implies condition (i) there.

Proof. (i) $\Rightarrow$ (ii) is given by Theorem 2.6 itself, and (ii) $\Rightarrow$ (iii) is trivial.

To prove (iii) $\Rightarrow$ (i) we shall prove the contrapositive. Suppose that (i) fails. By the definition of $\mathfrak{h}$ there is a not sequentially compact product X by $\mathfrak{h}$ sequentially compact factors. If $\mathfrak{h} < \mathfrak{s}$, then condition (ii) in Theorem 2.6 trivially holds for such an X, while condition (i) there fails. Thus condition (iii) in the present corollary fails. □

3. A topological proof that $\operatorname{cf} \mathfrak{s} \geq \mathfrak{h}$ and a generalization

We begin this section by giving a curious and purely topological proof of the inequality $\operatorname{cf} \mathfrak{s} \geq \mathfrak{h}$. The proof does not use any of the results proved before, but relies heavily on the characterizations of the cardinals $\mathfrak{s}$ and $\mathfrak{h}$ that we have presented as Definitions 2.1 and 2.5. See Blass [1, Corollary 2.2] for another proof of $\operatorname{cf} \mathfrak{s} \geq \mathfrak{h}$. Andreas R. Blass (personal communication, June 2014) has kindly communicated us a direct simple proof which uses the combinatorial definitions of $\mathfrak{s}$ and $\mathfrak{h}$.

By the way, Dow and Shelah [6] have recently showed that it is consistent that $\mathfrak{s}$ is $\mathfrak{s}$ingular, solving a longstanding problem.

Proposition 3.1. $\operatorname{cf} \mathfrak{s} \geq \mathfrak{h}$.

Proof. Suppose by contradiction that $\operatorname{cf} \mathfrak{s} = \lambda < \mathfrak{h}$, hence we can express $\mathfrak{s}$ as $\bigcup_{\alpha \in \lambda} s_\alpha$, with $|s_\alpha| < \mathfrak{s}$, for $\alpha \in \lambda$; moreover, without loss of generality, we can take the s_α's to be pairwise disjoint. Thus $\mathbf{2}^{\mathfrak{s}}$ is (homeomorphic to) $\prod_{\alpha \in \lambda} \mathbf{2}^{s_\alpha}$. By the definition of $\mathfrak{s}$ (the one we have given) and since $|s_\alpha| < \mathfrak{s}$, for $\alpha \in \lambda$, then each $\mathbf{2}^{s_\alpha}$ is sequentially compact. By the definition of $\mathfrak{h}$, and since $\lambda < \mathfrak{h}$, we have that $\prod_{\alpha \in \lambda} \mathbf{2}^{s_\alpha}$ is sequentially compact. But then $\mathbf{2}^{\mathfrak{s}} \cong \prod_{\alpha \in \lambda} \mathbf{2}^{s_\alpha}$ would be sequentially compact, contradicting the definition of $\mathfrak{s}$. □

As we mentioned in the introduction, the arguments in the proofs of Proposition 3.1 have a general form and work for every property P of topological spaces. We could work as well with some property (= a subclass) of objects in a category in which some infinite products or coproducts are defined. However, the right ambient in which the results can be stated in their full generality appears to be the context of partial infinitary semigroups. We shall sketch here a basic result. For more details and further invariants, see Section 7 in the unpublished manuscript [11] from which the present note has been extracted.

DEFINITION 3.2. *A* partial infinitary semigroup *is a* Σ*-algebra satisfying properties (U) and (P), in the terminology from [10].*

For short, in a partial infinitary semigroup we have a partially defined infinitary operation $\sum_{i \in I} a_i$*, for every index set* I*. Property (U) asserts that if* $|I| = 1$*, then* $\sum_{i \in I} a_i$ *is defined and its outcome is the only element* a_i *of the sequence.*

Property (P) asserts that if $\sum_{i \in I} a_i$ *is defined, then, for every partition* $(J_k)_{k \in K}$ *of* I*, all the sums in the following equation are defined, and equality actually holds:* $\sum_{i \in I} a_i = \sum_{k \in K} \sum_{i \in J_k} a_i$*.*

With the customary foundational caution, classes of topological spaces modulo homeomorphism and with the Tychonoff product form a partial infinitary semigroups.

DEFINITION 3.3. *If* S *is a partial infinitary semigroup and* $P \subseteq S$*, let* $\mathfrak{H}(P)$ *be the class of all cardinals* $\kappa \geq 2$ *such that the following holds. There are some* I *of cardinality* κ *and some sum* $\sum_{i \in I} a_i$ *which is defined, whose outcome is not in* P*, while* $\sum_{i \in J} a_i \in P$*, for every* $J \subseteq I$ *with* $|J| < \kappa$*,* $J \neq \emptyset$*.*

Notice that property (P) implies that if $\sum_{i \in I} a_i$ *is defined, then* $\sum_{i \in J} a_i$ *is defined, for every nonempty* $J \subseteq I$*.*

Let $\mathfrak{H}^*(P)$ *be the class of all cardinals* $\kappa \geq 2$ *such that there are some* I *of cardinality* κ *and some sum* $\sum_{i \in I} a_i$ *which is defined, whose outcome is not in* P*, while* $a_i \in P$*, for every* $i \in I$*. In most examples, if* $\kappa \in \mathfrak{H}^*(P)$*, then* $\lambda \in \mathfrak{H}^*(P)$*, for every* $\lambda \geq \kappa$*. In this case* $\mathfrak{H}^*(P)$*, if nonempty, is determined by* $\mathfrak{h}(P) = \inf \mathfrak{H}^*(P)$*. However, we shall not need to assume this further property of* $\mathfrak{H}^*(P)$ *in what follows.*

PROPOSITION 3.4. *Suppose that* S *is a partial infinitary semigroup and* $P \subseteq S$*. Then*

(i) $\mathfrak{H}(P) \subseteq \mathfrak{H}^*(P)$*.*

(ii) If $\kappa \in \mathfrak{H}(P)$*, then* $1 + \mathrm{cf}\, \kappa \in \mathfrak{H}^*(P)$*.*

(iii) If $\mathfrak{H}^*(P)$ *is not empty, then* $\inf \mathfrak{H}^*(P) \in \mathfrak{H}(P)$*, thus* $\mathfrak{H}(P) \neq \emptyset$*,* $\inf \mathfrak{H}^*(P) = \inf \mathfrak{H}(P)$*, and* $\inf \mathfrak{H}^*(P)$ *is a regular cardinal.*

Proof. (i) follows from the definitions and Property (U).

(ii) If κ is an infinite regular cardinal, then $\kappa = \mathrm{cf}\,\kappa = 1 + \mathrm{cf}\,\kappa$, hence (ii) follows from (i).

If κ is finite, say, $\kappa = n \geq 2$ and $\sum_{i<n} a_i$ witnesses $\kappa \in \mathfrak{H}(P)$, then $a_{n-1} + \sum_{i<n-1} a_i$ witnesses $1 + \mathrm{cf}\,n = 1 + 1 = 2 \in \mathfrak{H}^*(P)$.

The remaining case is similar to Proposition 3.1. Suppose that κ is singular, thus $\kappa = \bigcup_{k \in K} J_k$, for some sets K and pairwise disjoint J_k such that $|K|, |J_k| < \kappa$, for $k \in K$. Let $c = \sum_{\gamma \in \kappa} a_\gamma$ witness $\kappa \in \mathfrak{H}(P)$. For $k \in K$, let $b_k = \sum_{\gamma \in J_k} a_\gamma$. Since $|J_k| < \kappa$, for $k \in K$, then, by the definition of $\mathfrak{H}(P)$, each b_k is in P. By Property (P), $c = \sum_{k \in K} b_k$ and this sum witnesses $\mathrm{cf}\,\kappa \in \mathfrak{H}^*(P)$.

(iii) Let $\kappa = \inf \mathfrak{H}^*(P)$ and let $\sum_{\gamma \in \kappa} a_\gamma$ witness $\kappa \in \mathfrak{H}^*(P)$. By assumption, $\kappa \geq 2$ and each a_γ is in P. If there is $J \subseteq \kappa$ such that $2 \leq |J| < \kappa$ and $\sum_{j \in J} a_j \notin P$, then $\sum_{j \in J} a_j$ witnesses $|J| \in \mathfrak{H}^*(P)$, contradicting the minimality of κ. Thus, by (U), for every $J \subseteq \kappa$ with $1 \leq |J| < \kappa$, we have $\sum_{j \in J} a_j \in P$. This means that $\sum_{\gamma \in \kappa} a_\gamma$ witnesses $\kappa \in \mathfrak{H}(P)$. The rest follows from (i) and (ii). □

If S is the class of topological spaces modulo homeomorphism with Tychonoff products and P is the class of sequentially compact spaces, then $\mathfrak{h} = \inf \mathfrak{H}^*(P)$, by Definition 2.5. Moreover, $\mathfrak{s} \in \mathfrak{H}(P)$, by Definition 2.1. Thus Proposition 3.4(ii) generalizes Proposition 3.1. Moreover, the last assertion in Proposition 3.4(iii) generalizes the known fact that $\mathfrak{h}$ is a regular cardinal.

By Theorem 2.4, $\mathfrak{s} = \sup \mathfrak{H}(P)$, hence $\mathfrak{H}(P) \subseteq [\mathfrak{h}, \mathfrak{s}]$, where $[\mathfrak{h}, \mathfrak{s}]$ is the set of those cardinals λ such that $\mathfrak{h} \leq \lambda \leq \mathfrak{s}$. It is an open problem whether the inclusion $\mathfrak{H}(P) \subseteq [\mathfrak{h}, \mathfrak{s}]$ may be strict (of course, this is a nontrivial problem only when $\mathfrak{h} < \mathfrak{s}$).

As an application of Proposition 3.4, one can consider chain compactness. If $\lambda \leq \mu$ are infinite cardinals, a topological space X is $[\lambda, \mu]$-*chain compact* [17] if, for every cardinal ν such that $\lambda \leq \nu \leq \mu$, every ν-indexed sequence of elements of X has a converging cofinal subsequence. Thus $[\omega, \omega]$-chain compactness is the same as sequential compactness.

A product of countably many $[\lambda, \mu]$-chain compact spaces is still $[\lambda, \mu]$-chain compact [17]. Thus if $P_{[\lambda,\mu]\text{-}c}$ is the property of being $[\lambda, \mu]$-chain compact, then $\mathfrak{h}(P_{[\lambda,\mu]\text{-}c}) = \inf \mathfrak{H}^*(P_{[\lambda,\mu]\text{-}c}) > \omega$. By Proposition 3.4, $\mathfrak{h}(P_{[\lambda,\mu]\text{-}c})$ is a regular cardinal, and if $\kappa \in \mathfrak{H}(P_{[\lambda,\mu]\text{-}c})$, then $\mathrm{cf}\,\kappa \geq \mathfrak{h}(P_{[\lambda,\mu]\text{-}c})$. To the best of our knowledge, it is an open problem to explicitly characterize the cardinal $\mathfrak{h}(P_{[\lambda,\mu]\text{-}c})$ and the class $\mathfrak{H}(P_{[\lambda,\mu]\text{-}c})$. Some results about products of $[\omega, \mu]$-chain compact spaces can be found in [14]. If follows from [11, Theorem 3.1] that $\sup \mathfrak{H}(P_{[\lambda,\mu]\text{-}c}) \leq 2^\mu$.

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