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Singular Behavior of the Dirichlet Problem in Hölder Spaces of the Solutions to the Dirichlet Problem in a Cone

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Summary. - *In the present study we consider the solution of the Dirichlet problem in conical domain. For general elliptic problems in non Hilbertian Sobolev spaces built on $L^p, 1 < p < \infty$, the theory of sums of operators developed by Dore-Venni[8] provides an optimal result. Hölder spaces, as opposed to L^p spaces, are not UMD. Using the results of Da Prato-Grisvard[6] and Labbas[14] we cope with the singular behaviour of the solution in the framework of Hölder and little Hölder spaces.*

1. Introduction

The following problem

$$\begin{cases} -\Delta u = f & \text{in } Q \\ u = 0 & \text{on } \partial Q, \end{cases} \tag{1}$$

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where Q is an open set of $\mathbb{R}^3$ has been studied by several authors in the Sobolev spaces built on $L^p(Q)$ for $1 < p < \infty$. See for instance, Agmon-Douglis-Nirenberg [1][2] for regular open sets and Grisvard[10], Dauge[7] and Kontradiev[12] for open sets with conical points. The variational solution can be written as a sum

$$u = u_r + u_s, \tag{2}$$

where u_r has the optimal regularity $W^{2,p}(Q)$ and u_s is written explicitly near the singular points for a simple geometry.

For Q being a cone, the technique used in the hilbertian case ($p = 2$) is based on the Fourier's partial transform and Plancherel's theorem. For $p \neq 2$, the decomposition (2) was obtained by Clément-Grisvard[4] relying on two approaches for the sum of linear operators taken from Da Prato-Grisvard[6] and Dore-Venni[8]. The first one provides a strong solution of (1), (not necessarily coinciding with a variational solution u), and the second makes use of the UMD character of $L^p(Q)$ and yields the optimal regularity of u_r.

In the present study, problem (1) is considered in the infinite cone

$$Q = \{\rho\sigma \ / \ \rho > 0, \sigma \in G\}, \tag{3}$$

where G is a regular open set of the sphere S^2. For $k \in \mathbb{N}$, we denote by $UC^k\left(\overline{Q}\right)$ the space of the functions with uniformly continuous and bounded derivatives up to the order k in $\overline{Q}$ and by $C^\alpha\left(\overline{Q}\right)$, for $0 < \alpha < 1$, the space of the bounded and uniformly α -Hölder continuous functions u defined on $\overline{Q}$ and endowed with the norm

$$\|u\|_{C^\alpha(\overline{Q})} = \operatorname*{Max}_{x\in\overline{Q}} |u(x)| + \operatorname*{Max}_{\rho\sigma\neq\rho'\sigma'} \frac{|u(\rho\sigma) - u(\rho'\sigma')|}{\|\rho\sigma - \rho'\sigma'\|_2^\alpha} \tag{4}$$

$$= \operatorname*{Max}_{x\in\overline{Q}} |u(x)| + [u]_{\alpha,\overline{Q}}\,. \tag{5}$$

$\|\|_2$ denotes the euclidian norm. $C^{k+\alpha}\left(\overline{Q}\right)$ is the subspace of $UC^k\left(\overline{Q}\right)$ of functions whose k-th order derivatives belong to $C^\alpha\left(\overline{Q}\right)$. Similarly we define the spaces $UC^k\left(\overline{\Omega}, X\right)$, $C^\alpha\left(\overline{\Omega}, X\right)$ and $C^{k+\alpha}\left(\overline{\Omega}, X\right)$ where X is a Banach space and Ω is any open set in $\mathbb{R}^n$. These spaces are naturally normed. We shall consider also the following subspaces little Hölder continuous functions:

$$h^{\alpha}\left(\overline{\Omega},X\right)=\left\{u\in UC\left(\overline{\Omega},X\right)/\lim_{\delta\to 0}\sup_{\|x-y\|\leq\delta}\frac{\|u(x)-u(y)\|_X}{\|x-y\|^{\alpha}}=0\right\},$$

$$h^{\alpha}\left(\overline{Q}\right)=\left\{u\in UC\left(\overline{Q}\right)/\lim_{\delta\to 0}\sup_{\|x-y\|\leq\delta}\frac{|u(x)-u(y)|}{\|x-y\|_2^{\alpha}}=0\right\},$$

which are endowed respectively with the norms of $C^{\alpha}\left(\overline{\Omega},X\right)$ and $C^{\alpha}\left(\overline{Q}\right)$. The subspace $h^{\alpha}\left(\overline{\Omega},X\right)$ can be characterized as the closure of $UC^1\left(\overline{\Omega},X\right)$ in $C^{\alpha}\left(\overline{\Omega},X\right)$ or as the closure of $C^{\theta}\left(\overline{\Omega},X\right)$ in $C^{\alpha}\left(\overline{\Omega},X\right)$ for $\theta>\alpha$, see Sinestrari[19], Lunardi[15].

We then show the validity of decomposition (2) if $f\in h_0^{\alpha}\left(\overline{Q}\right)$; here, $h_0^{\alpha}\left(\overline{Q}\right)$ (resp. $C_0\left(\overline{G}\right)$) denotes the space of functions of $h^{\alpha}\left(\overline{Q}\right)$ (respectively of $C\left(\overline{G}\right)$) vanishing on ∂Q (resp. on ∂G).

We prove that

$$u_r\in C^{2+\alpha}\left(\overline{Q}\right),$$

and we describe precisely the behavior of the singular part u_s near the vertex O.

Our study in the Hölder spaces is motivated by the fact that this framework allows us the use of theorems on multipliers and the Banach algebra structure and leads to the resolution of many non linear problems via linearization and precise control of the solution near the singular points, in L^{∞}-norm.

The techniques we use are essentially based on the theory of the sums of linear operators in Banach spaces developed in Da Prato-Grisvard[6] as well as on the results for an abstract two points boundary problems of elliptic type studied in Labbas[14].

In paragraph 2 we present the main result of the theory of the sums by Da Prato-Grisvard[6] in the commutative case. In paragraph 3, we write equation (1) in the cylinder $\Sigma=\mathbb{R}\times G$ by using the spherical coordinates. In paragraphs 4 and 5, we apply the sum's strategy to the transformed equation respectively in the Banach spaces $E=L^{\infty}\left(\mathbb{R},h_0^{\alpha}\left(\overline{G}\right)\right)$ and $E=h^{\alpha}\left(\mathbb{R},C_0\left(\overline{G}\right)\right)$. In paragraph 6, some regularity results in Labbas[14] are recalled and applied to the transformed problem. Finally in section 7 we go back to our problem in the cone and give the final theorem which specifies decomposition 2.

2. Sums of linear operators

Let us consider a complex Banach space E and two closed linear operators A and B of domains $D(A)$ and $D(B)$. Their sum is defined by

$$Sx = Ax + Bx, \qquad x \in D(S) = D(A) \cap D(B). \tag{6}$$

We assume that these two operators verify the following hypotheses:

$$(H.1)\begin{cases} \exists r,\ C_A, C_B > 0,\ \epsilon_A,\ \epsilon_B \in]0,\pi[\text{ such that} \\ i)\ \rho(A) \supset \sum_{\epsilon_A} = \{z\ /\ |z| \geqslant r,\ |\mathrm{Arg}(z)| < \pi - \epsilon_A\} \\ \text{and } \left\|(A - zI)^{-1}\right\|_{L(E)} \leqslant C_A/|z|\ \ \forall z \in \sum_{\epsilon_A} \\ ii)\ \rho(B) \supset \sum_{\epsilon_B} = \{z\ /\ |z|\ \geqslant r,\ |\mathrm{Arg}(z)| < \pi - \epsilon_B\} \\ \text{and } \left\|(B - zI)^{-1}\right\|_{L(E)} \leqslant C_B/|z|\ \ \forall z \in \sum_{\epsilon_B} \\ iii)\ \epsilon_A + \ \epsilon_B < \pi. \\ iv)\ D(A) + D(B) \text{ is dense in } E \end{cases}$$

$$(H.2)\begin{cases} (A - \xi I)^{-1} (B - \eta I)^{-1} - (B - \eta I)^{-1} (A - \xi I)^{-1} \\ = \left[(A - \xi I)^{-1};\ (B - \eta I)^{-1}\right] = 0\,;\ \forall \xi \in \rho(A), \forall \eta \in \rho(B) \end{cases} \tag{7}$$

and

$$(H.3) \quad \sigma(A) \cap \sigma(-B) = \emptyset, \tag{8}$$

where $\sigma(A)$ and $\sigma(-B)$ denote respectively the spectrum of A and $-B$ and $\rho(A)$, $\rho(-B)$ their resolvent sets.

According to Da Prato-Grisvard[6], under hypotheses $(H.1)$, $(H.2)$, $(H.3)$ the sum $S = A + B$ is closable and the linear operator defined by the following Dunford's integral

$$x \longmapsto -\frac{1}{2i\pi} \int_\Gamma (B + zI)^{-1} (A - zI)^{-1} x dz \tag{9}$$

coincides exactly with $\left(\overline{S}\right)^{-1}$ where $\overline{S} = \overline{A + B}$ is the closure of $A + B$; , is a simple sectorial curve enclosing the spectrums of A and $(-B)$ and lying in $\rho(A) \cap \rho(-B)$. We then have the essential following result proved in [6]:

THEOREM 2.1. *Let us assume that (H.1), (H.2) and (H.3) hold. If F is a Banach subspace continuously imbedded in E and there exists a constant K such that for some $\theta \in]0,1[$ we have*

$$\|x\|_F \leqslant K\left(\|x\|_E + \|x\|_E^{1-\theta}\|Ax\|_E^{\theta}\right) \quad \forall x \in D(A),$$

then $D\left(\overline{A+B}\right) \subset F$.

The unique solution v of the equation

$$\overline{S}v = \left(\overline{A+B}\right)v = f$$

is usually called a strong solution of the equation $Sv = f$.

3. The problem in the cylinder

We assume in all this study that $f \in h_0^{\alpha}\left(\overline{Q}\right)$. The condition $f = 0$ on ∂Q is necessary in the case of Dirichlet's problem in Hölder spaces on regular open sets (see a counterexample given in Von Wahl [21]). Equation (1) is written in spherical coordinates $\rho\sigma = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$ as

$$\begin{cases} D_\rho^2 u + \frac{2}{\rho}D_\rho u + \frac{1}{\rho^2}\Delta' u = f & \text{in } Q \\ u = 0 \quad \text{on } \partial Q, \end{cases} \tag{10}$$

where Δ' denotes the Laplace-Beltrami operator on the unit sphere S^2 défined by

$$\Delta' u = \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 u}{\partial\theta^2}. \tag{11}$$

Equation (10) may be written in the form

$$\begin{cases} (\rho D_\rho)^2 u + (\rho D_\rho)u + \Delta' u = \rho^2 f = g & \text{in } Q \\ u = 0 \text{ on } \partial Q, \end{cases}$$

and the natural change of variable $\rho = e^t$ gives

$$\begin{cases} D_t^2 u + D_t u + \Delta' u = e^{2t} f = g & \text{in } \Sigma \\ u_{|\partial\Sigma} = 0, \end{cases} \tag{12}$$

where $\Sigma = \mathbb{R} \times G$. We thus set, for $(t, \sigma) \in \mathbb{R} \times G$,

$$\begin{array}{rcl} V(t,\sigma) & = & e^{-(2+\alpha)t} u(e^t \sigma), \\ H(t,\sigma) & = & e^{-\alpha t} f(e^t \sigma), \end{array} \tag{13}$$

and define the following vector-valued functions which take their values on some Banach space X

$$\begin{array}{rcl} v & : & \mathbb{R} \to X; t \longmapsto v(t); \quad v(t)(\sigma) = V(t,\sigma), \\ h & : & \mathbb{R} \to X; t \longmapsto h(t); \quad h(t)(\sigma) = H(t,\sigma), \end{array}$$

(where X shall be specified later). Then v satisfies the abstract equation

$$\left\{ \begin{array}{l} D_t^2 v(t) + (1+2\beta) D_t v(t) + \beta(\beta+1) v(t) + \Delta'(v(t)) = h(t), \ t \in \mathbb{R} \\ v(t) \in D(\Delta') \subset X, \end{array} \right. \tag{14}$$

with

$$\beta = 2 + \alpha.$$

Equation (14) may be written as a sum of two linear operators not acting with the same variable. This allows us to predict the application of the commutative case of the sum theory.

We shall need the useful following lemmas which specify the relation between a global, partial and abstract little hölderianity in the cylinder Σ.

LEMMA 3.1. *We have*

i) $h \in h^\alpha\left(\mathbb{R}, C_0\left(\overline{G}\right)\right)$ *if and only if* $H \in UC(\mathbb{R} \times \overline{G})$ *and* $H(.,\sigma) \in h^\alpha(\mathbb{R})$ *uniformly in* $\sigma \in \overline{G}$.

ii) $h \in UC\left(\mathbb{R}, C_0\left(\overline{G}\right)\right) \cap L^\infty\left(\mathbb{R}, h_0^\alpha\left(\overline{G}\right)\right)$ *if and only if* $H \in UC(\mathbb{R} \times \overline{G})$ *and* $H(t,.) \in h_0^\alpha(\overline{G})$ *uniformly in* $t \in \mathbb{R}$.

This lemma can be proved as in lemma 6.2 of Sinestrari[19].

LEMMA 3.2. *Let* $f \in h_0^\alpha(\overline{Q})$, *then the function* $H(t,\sigma) = e^{-\alpha t} f(e^t \sigma)$ *verifies* *i) $H \in UC(\mathbb{R} \times \overline{G})$ and $H(.,\sigma) \in h^{\alpha}(\mathbb{R})$ uniformly in $\sigma \in \overline{G}$.*

ii) $H \in UC(\mathbb{R} \times \overline{G})$ and $H(t,.) \in h_0^{\alpha}(\overline{G})$ uniformly in $t \in \mathbb{R}$.

Proof. i) Let τ, t such that $-\infty < \tau < t < +\infty$, then

$$\begin{aligned} &H(t,\sigma) - H(\tau,\sigma) \\ &\quad = \left(e^{-\alpha t} - e^{-\alpha t}\right) f(e^{\tau}\sigma) + e^{-\alpha t}\left(f(e^{t}\sigma) - f(e^{\tau}\sigma)\right) \\ &\quad = \frac{-1}{\alpha}\int_{\tau}^{t} e^{-\alpha\xi} d\xi \left(f(e^{\tau}\sigma) - f(0)\right) + e^{-\alpha t}\left(f(e^{t}\sigma) - f(e^{\tau}\sigma)\right) \\ &\quad = \Delta_1 + \Delta_2. \end{aligned}$$

and

$$|\Delta_1| \leqslant \frac{1}{\alpha}|t-\tau|\, e^{-\alpha\tau}\frac{|f(e^{\tau}\sigma) - f(0)|}{\|e^{\tau}\sigma\|_2^{\alpha}} e^{\alpha\tau} \leqslant \frac{1}{\alpha}|t-\tau|\, \|f\|_{C^{\alpha}(\overline{Q})},$$

which implies that $\Delta_1(.,\sigma) \in h^{\nu}(\mathbb{R})$ uniformly in σ for all $\nu \in]0,1[$.
For Δ_2,we have

$$\begin{aligned} |\Delta_2| &\leqslant e^{-\alpha t}\left\|e^{t}\sigma - e^{\tau}\sigma\right\|_2^{\alpha}\frac{\left|f(e^{t}\sigma) - f(e^{\tau}\sigma)\right|}{\|e^{t}\sigma - e^{\tau}\sigma\|_2^{\alpha}} \\ &\leqslant e^{-\alpha t}\left|e^{t} - e^{\tau}\right|^{\alpha}\frac{\left|f(e^{t}\sigma) - f(e^{\tau}\sigma)\right|}{\|e^{t}\sigma - e^{\tau}\sigma\|_2^{\alpha}} \\ &\leqslant e^{-\alpha t}\left(\int_{\tau}^{t} e^{\xi} d\xi\right)^{\alpha}\frac{\left|f(e^{t}\sigma) - f(e^{\tau}\sigma)\right|}{\|e^{t}\sigma - e^{\tau}\sigma\|_2^{\alpha}} \\ &\leqslant e^{-\alpha t}e^{\alpha t}(t-\tau)^{\alpha}\frac{\left|f(e^{t}\sigma) - f(e^{\tau}\sigma)\right|}{\|e^{t}\sigma - e^{\tau}\sigma\|_2^{\alpha}}, \end{aligned}$$

from which we deduce that

$$\lim_{\delta\to 0}\sup_{|t-\tau|\leqslant\delta}\frac{|\Delta_2|}{(t-\tau)^{\alpha}} = 0$$

uniformly in σ, therefore $\Delta_2(.,\sigma) \in h^{\alpha}(\mathbb{R})$.
ii)

$$H(t,\sigma) = e^{-\alpha t} f(e^{t}\sigma) = 0, \forall\sigma \in \partial G,$$

$$|H(t,\sigma)| = e^{-\alpha t}\left|f(e^{t}\sigma) - f(0)\right| \leqslant \|f\|_{C^{\alpha}(\overline{Q})}, \forall\sigma \in \overline{G},$$

and

$$\begin{aligned} \left|H(t,\sigma)-H(t,\sigma')\right| &= e^{-\alpha t}\left|f(e^t\sigma)-f(e^\tau\sigma')\right| \\ &\leqslant e^{-\alpha t}\left\|e^t\sigma-e^t\sigma'\right\|_2^\alpha \frac{\left|f(e^t\sigma)-f(e^t\sigma')\right|}{\|e^t\sigma-e^t\sigma'\|_2^\alpha} \\ &\leqslant \left\|\sigma-\sigma'\right\|_2^\alpha \frac{\left|f(e^t\sigma)-f(e^t\sigma')\right|}{\|e^t\sigma-e^t\sigma'\|_2^\alpha}, \end{aligned}$$

hence

$$\lim_{\delta\to 0}\sup_{\|\sigma-\sigma'\|\leqslant\delta}\frac{|H(t,\sigma)-H(t,\sigma')|}{\|\sigma-\sigma'\|^\alpha}=0.$$

□

LEMMA 3.3. *Let* $\phi\in L^\infty\left(\mathbb{R},h_0^\alpha\left(\overline{G}\right)\right)\cap h^\alpha\left(\mathbb{R},C_0\left(\overline{G}\right)\right)$, *then the function* F *defined by* $F(t,\sigma)=\phi(t)(\sigma)$ *belongs to* $h_0^\alpha\left(\overline{\Sigma}\right)$.

We have $F=0$ on $\partial\Sigma$. Let now (t,σ) , $(t',\sigma')\in\mathbb{R}\times\overline{G}$ ($t\neq t'$, $\sigma\neq\sigma'$) such that $\|\sigma-\sigma'\|_2\leqslant\delta/2$ and $|t-t'|\leqslant\delta/2$ for some fixed $\delta>0$, then

$$\begin{aligned} &\left|F(t,\sigma)-F(t',\sigma')\right| \\ &\leqslant \left|F(t,\sigma)-F(t,\sigma')\right|+\left|F(t,\sigma')-F(t',\sigma')\right| \\ &\leqslant \left\|\sigma-\sigma'\right\|_2^\alpha\frac{|F(t,\sigma)-F(t,\sigma')|}{\|\sigma-\sigma'\|_2^\alpha}+\left|t-t'\right|^\alpha\frac{|F(t,\sigma)-F(t',\sigma)|}{|t-t'|^\alpha} \\ &\leqslant \left(\left\|\sigma-\sigma'\right\|_2^\alpha+\left|t-t'\right|^\alpha\right)\left(\frac{|F(t,\sigma)-F(t,\sigma')|}{\|\sigma-\sigma'\|_2^\alpha}+\frac{|F(t,\sigma)-F(t',\sigma)|}{|t-t'|^\alpha}\right) \\ &\leqslant K\left\|(t,\sigma)-(t',\sigma')\right\|_2^\alpha\left(\frac{|\phi(t)(\sigma)-\phi(t)(\sigma')|}{\|\sigma-\sigma'\|_2^\alpha}+\frac{|\phi(t)(\sigma)-\phi(t')(\sigma)|}{|t-t'|^\alpha}\right) \end{aligned}$$

therefore

$$\begin{aligned}
&\frac{|F(t,\sigma)-F(t',\sigma')|}{\|(t,\sigma)-(t',\sigma')\|_2^{\alpha}} \\
&\leqslant K\left(\frac{|\phi(t)(\sigma)-\phi(t)(\sigma')|}{\|\sigma-\sigma'\|_2^{\alpha}}+\frac{|\phi(t)(\sigma)-\phi(t')(\sigma)|}{|t-t'|^{\alpha}}\right) \\
&\leqslant K\left(\sup_{\|\sigma-\sigma'\|\leq\delta/2}\frac{|\phi(t)(\sigma)-\phi(t)(\sigma')|}{\|\sigma-\sigma'\|_2^{\alpha}}+\sup_{\|t-t'\|\leq\delta/2}\frac{|\phi(t)(\sigma)-\phi(t')(\sigma)|}{|t-t'|^{\alpha}}\right)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\sup_{\|(t,\sigma)-(t',\sigma')\|_2\leq\delta}\frac{|F(t,\sigma)-F(t',\sigma')|}{\|(t,\sigma)-(t',\sigma')\|_2^{\alpha}} \\
&\leqslant K\left(\sup_{\|\sigma-\sigma'\|\leq\delta/2}\frac{|\phi(t)(\sigma)-\phi(t)(\sigma')|}{\|\sigma-\sigma'\|_2^{\alpha}}+\sup_{\|t-t'\|\leq\delta/2}\frac{|\phi(t)(\sigma)-\phi(t')(\sigma)|}{|t-t'|^{\alpha}}\right).
\end{aligned}$$

Since $\phi\in L^{\infty}\left(\mathbb{R},h_0^{\alpha}\left(\overline{G}\right)\right)\cap h^{\alpha}\left(\mathbb{R},C_0\left(\overline{G}\right)\right)$, it follows that

$$\lim_{\delta\to0}\sup_{\|(t,\sigma)-(t',\sigma')\|_2\leq\delta}\frac{|F(t,\sigma)-F(t',\sigma')|}{\|(t,\sigma)-(t',\sigma')\|_2^{\alpha}}=0,$$

from which we deduce that $F\in h^{\alpha}\left(\overline{\Sigma}\right)$.

Note that, in virtue of assumption on f, the abstract function h defined by $h(t)(\sigma)=H(t,\sigma)=e^{-\alpha t}f(e^{t}\sigma)$ is exactly in the space $L^{\infty}\left(\mathbb{R},h_0^{\alpha}\left(\overline{G}\right)\right)\cap h^{\alpha}\left(\mathbb{R},C_0\left(\overline{G}\right)\right)$.

4. First application of the sums

We shall apply the results of section 2 to equation (14) in the Banach space $E=L^{\infty}\left(\mathbb{R},h_0^{\alpha}\left(\overline{G}\right)\right)$ normed by $\|f\|_E=\sup_{t\in R}\|f(t,.)\|_{C^{\alpha}(\overline{G})}$. Let us define the three operators A, B and C by

$$\left\{\begin{array}{l} D(A)=L^{\infty}\left(\mathbb{R},D(\Delta')\right) \\ (Av)(t)=\Delta'\left(v(t,.)\right), \end{array}\right. \tag{15}$$

with $D(\Delta') = \left\{w \in C_0^\alpha\left(\overline{G}\right) \ / \ \Delta' w \in C_0^\alpha\left(\overline{G}\right)\right\}$,

$$\left\{\begin{array}{l} D(B) = W^{2,\infty}\left(\mathbb{R}, h_0^\alpha\left(\overline{G}\right)\right) \\ Bv = D_t^2 v + (1+2\beta)\, D_t v + \left(\beta^2+\beta\right) v, \end{array}\right. \tag{16}$$

and

$$\left\{\begin{array}{l} D(C) = W^{1,\infty}\left(\mathbb{R}, h_0^\alpha\left(\overline{G}\right)\right) \\ Cv = D_t v. \end{array}\right. \tag{17}$$

Equation (14) is then equivalent to

$$Av + Bv = h$$

in E.

4.1. Spectral properties of B

Note, at first, that $\overline{D(B)} \neq E$. Moreover we have $B = P(C)$ where P is the polynomial

$$P(z) = z^2 + (1+2\beta)z + (\beta^2+\beta),$$

using the spectral mapping theorem, we have

$$\sigma(B) = \left\{-\xi^2 + (1+2\beta)\xi i + (\beta^2+\beta)\ ,\ \xi \in \mathbb{R}\right\}, \tag{18}$$

which is the parabolic curve cutting the real axis at the point $\beta(\beta+1) \in]6,12[$, oriented in the direction of the negative values of x and given by the equation

$$y^2 = -\left(1+2\beta\right)^2\left[x - \beta(\beta+1)\right]. \tag{19}$$

The two tangents at the points $(0, -\beta(\beta+1))$ and $(0, \beta(\beta+1))$ intersect on the real axis at the point $2\beta(\beta+1) \in]12,24[$ with the angle $\epsilon_B \in]0,\pi/2[$ such that $\tan\epsilon_B = \frac{(1+2\beta)^2}{2}$. So the resolvent set $\rho(B)$ contains the sector

$$S_\beta = \{z \in C \quad / \ |z| \ \geqslant \ 2\beta(\beta+1)\ ,\ |\mathrm{Arg}(z)| < \pi - \epsilon_B\}$$

On the other hand, for all given complex λ in this sector, the equation $P(z) = \lambda$ has the two complex roots

$$z_{\pm}(\lambda) = \frac{-(1+2\beta) \pm \sqrt{4\lambda+1}}{2}, \tag{20}$$

which implies that

$$(B-\lambda I)^{-1} = (C - z_{+}(\lambda) I)^{-1} (C - z_{-}(\lambda) I)^{-1}. \tag{21}$$

However we know that $\sigma(C) = i\mathbb{R}$ and for $\phi \in E$

$$\left[(C+\mu I)^{-1}\phi\right](t,\sigma) = \begin{cases} -\int_t^{\infty} e^{\mu(s-t)}\phi(s,\sigma)ds & \text{if } \operatorname{Re}\mu < 0, \\ \int_{-\infty}^{t} e^{-\mu(t-s)}\phi(s,\sigma)ds & \text{if } \operatorname{Re}\mu > 0, \end{cases}$$

from which we obtain the estimate

$$\left\|(C+\mu I)^{-1}\right\|_{L(E)} \leqslant \frac{1}{|\operatorname{Re}\mu|} \quad \forall \mu \notin i\mathbb{R}\ .$$

From (21) one finally obtains

$$\left\|(B-\lambda I)^{-1}\right\|_{L(E)} = O\left(\frac{1}{\left(\operatorname{Re}\sqrt{\lambda}\right)^2}\right) \quad \forall \lambda \in S_{\beta}.$$

Therefore the operator B verifies the statement i) of hypothesis (H.1).

4.2. Spectral properties of A

The operator A has the same properties as its realization Δ'. The domain $D(A)$ is dense in E since the closure of $D(\Delta')$ in the norm of $C_0^{\alpha}\left(\overline{G}\right)$ coincides with $h_0^{\alpha}\left(\overline{G}\right)$, see Sinestrari[19]. So the statement $iv)$ of $(H.1)$ is verified. Thanks to Campanato[3], we know that Δ' generates an analytic semigroup strongly continuous on $h_0^{\alpha}\left(\overline{G}\right)$; the same is true for A, therefore there exists $\epsilon_A \in]0,\pi/2[$ such that A verifies i) of $(H.1)$ with $r=0$. One notices that the condition $\epsilon_A + \epsilon_B < \pi$ of iii) is verified. Hypothesis $(H.1)$ is proved.

It is known that, in $L^2(G)$, $(-\Delta')$ is a non negative, self-adjoint and anti-compact operator (see Courant and Hilbert [5]), thus $\sigma(-A)$ contains only non negative isolated eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots$$

from which we deduce that hypothesis $(H.3)$ is verified if however there are no λ_j which coincides with $\beta(\beta+1) \in]6, 12[$. So we shall assume that

$$\beta(\beta+1) = (\alpha+2)\,(\alpha+3) \neq \lambda_j \quad \forall j \geqslant 1, \tag{22}$$

which is possible since λ_j are isolated, even if it means replacing α by some $\alpha' < \alpha$.

There remains to check the hypothesis of commutativity $(H.2)$. In virtue of (21) it is enough to prove that the resolvents of A and C commute. It follows easily from the formula of the resolvent of $(-C)$ and

$$\left[(A-\lambda I)^{-1}\varphi\right](t,\sigma) = \sum_{j\geqslant 1} \frac{1}{\lambda_j - \lambda}\left[\int_G \varphi(t,\xi) w_j(\xi) d\xi\right] w_j(\sigma),$$

where w_j is the eigenfunction associated to the eigenvalue λ_j and φ belonging to a dense subspace of E.

4.3. First choice of the subspace F

Due to section 2 and under the condition that $(\alpha+2)\,(\alpha+3) \neq \lambda_j \quad \forall j \geqslant 1$, the previous results implies that $(A+B)$ is closable and that the closure $\overline{A+B}$ is invertible. In order to have more regularity on the strong solution it suffices to find a subspace F such that the convexity inequality of theorem 2.1 holds. Let us consider

$$F = W^{1,\infty}\left(\mathbb{R}, h_0^\alpha\left(\overline{G}\right)\right) \subset E,$$

then by virtue of Lions-Peetre spaces of class K_θ (see the appendix) there exists a constant C such that

$$\begin{cases} \|v'\|_{L^\infty(R,h_0^\alpha(\overline{G})} \leq C\,\|v\|^{1/2}_{L^\infty(R,h_0^\alpha(\overline{G})} \cdot \|v''\|^{1/2}_{L^\infty(R,h_0^\alpha(\overline{G})} \\ \forall v \in W^{2,\infty}\left(\mathbb{R}, h_0^\alpha\left(\overline{G}\right)\right) = D(B). \end{cases}$$

Theorem 2.1 yields the following proposition.

PROPOSITION 4.1. *For any $h \in L^{\infty}\left(\mathbb{R}, h_0^{\alpha}\left(\overline{G}\right)\right)$ problem (14) admits a unique strong solution v. Moreover $v \in W^{1,\infty}\left(\mathbb{R}, h_0^{\alpha}\left(\overline{G}\right)\right)$.*

4.4. Second choice of the subspace F

Now, let us choose the space

$$F = L^{\infty}\left(\mathbb{R}, C^{1+\alpha}\left(\overline{G}\right) \cap h_0^{\alpha}\left(\overline{G}\right)\right) \subset E,$$

It is known that there exists a constant C such that

$$\|w\|_{C^{2+\alpha}\left(\overline{G}\right)} \leq C \left\|\Delta' w\right\|_{C_0^{\alpha}\left(\overline{G}\right)} \quad \forall w \in D(\Delta'),$$

(see Campanato[3] for example). Interpolation yields

$$\begin{aligned} &\|v(t,.)\|_{C^{1+\alpha}\left(\overline{G}\right)} \\ &\qquad \leq C \|v(t,.)\|_{C_0^{\alpha}\left(\overline{G}\right)}^{1/2} \left\|\Delta'\left(v(t,.)\right)\right\|_{C_0^{\alpha}\left(\overline{G}\right)}^{1/2} \quad \forall v(t,.) \in D(\Delta'), \end{aligned}$$

and hence

$$\|v\|_F \leq C \|v\|_E^{1/2} \|Av\|_E^{1/2} \quad \forall v \in D(A).$$

Theorem 2.1 leads to Proposition 4.2

PROPOSITION 4.2. *For any $h \in L^{\infty}\left(\mathbb{R}, h_0^{\alpha}\left(\overline{G}\right)\right)$ problem (14) admits a unique strong solution v which belongs to $L^{\infty}\left(\mathbb{R}, C^{1+\alpha}\left(\overline{G}\right) \cap h_0^{\alpha}\left(\overline{G}\right)\right)$.*

From propositions 4.1 and 4.2 it follows that

$$v \in W^{1,\infty}\left(\mathbb{R}, h_0^{\alpha}\left(\overline{G}\right)\right) \cap L^{\infty}\left(\mathbb{R}, C^{1+\alpha}\left(\overline{G}\right) \cap h_0^{\alpha}\left(\overline{G}\right)\right). \tag{23}$$

5. Second application of the sums

Let us now consider the Banach space $E = h^{\alpha}\left(\mathbb{R}, C_0\left(\overline{G}\right)\right)$ and set

$$\begin{cases} D(A) = h^{\alpha}\left(\mathbb{R}, D(\Delta')\right) \\ (Av)(t) = \Delta'\left(v(t,.)\right), \end{cases} \tag{24}$$

where

$$D(\Delta') = \left\{ w \in C_0\left(\overline{G}\right) \cap W^{2,q}(G)\ ,\ q > 3\ ,\ \Delta' w \in C_0\left(\overline{G}\right) \right\}; \quad (25)$$

and

$$\left\{ \begin{array}{l} D(B) = C^{2+\alpha}\left(\mathbb{R}, C_0\left(\overline{G}\right)\right) \\ Bv = D_t^2 v + (1+2\beta)\, D_t v + \left(\beta^2 + \beta\right) v. \end{array} \right. \quad (26)$$

Here we have $\overline{D(B)} = E$. The same previous spectral properties are true; for the operator A, we use Stewart[19]. The convexity inequality of theorem 2.1 is respectively true for $F = C^{1+\alpha}\left(\mathbb{R}, C_0\left(\overline{G}\right)\right)$ and $F = h^{\alpha}\left(\mathbb{R}, W_0^{1,q}\left(\overline{G}\right)\right)$, $\forall q > 3$, (see Lunardi[16]). A consequence is the following proposition 5.1

Proposition 5.1. *For any $h \in h^{\alpha}\left(\mathbb{R}, C_0\left(\overline{G}\right)\right)$ problem (14) admits a unique strong solution v verifying*

$$v \in C^{1+\alpha}\left(\mathbb{R}, C_0\left(\overline{G}\right)\right) \cap h^{\alpha}\left(\mathbb{R}, W_0^{1,q}(G)\right), \forall q > 3. \quad (27)$$

Summing up we have proved

Theorem 5.2. *For any $h \in L^{\infty}\left(\mathbb{R}, h_0^{\alpha}\left(\overline{G}\right)\right) \cap h^{\alpha}\left(\mathbb{R}, C_0\left(\overline{G}\right)\right)$ there exists a unique strong solution v of problem (14) such that (23) and (27) hold.*

So v is a unique solution of equation

$$\overline{S} v = \left(\overline{A + B}\right) v = h \quad (28)$$

verifying (23) and (27). In the case $E = h^{\alpha}\left(\mathbb{R}, C_0\left(\overline{G}\right)\right)$, it means from equation (28) that there exists a sequence

$$v_n \in D(A) \cap D(B) = C^{2+\alpha}\left(\mathbb{R}, C_0\left(\overline{G}\right)\right) \cap h^{\alpha}\left(\mathbb{R}, D\left(\Delta'\right)\right),$$

($D\left(\Delta'\right)$ is defined in (25)), such that

$$\left\{ \begin{array}{l} v_n \xrightarrow{E} v \\ D_t^2 v_n + (1+2\beta)\, D_t v_n + \beta\,(\beta+1)\, v_n + \Delta' v_n \xrightarrow{E} h \\ v_n = 0 \quad \text{on } \partial\Sigma. \end{array} \right. \quad (29)$$

Similarly, in $E = L^{\infty}\left(\mathbb{R}, h_0^{\alpha}\left(\overline{G}\right)\right)$, there exists

$$\varphi_n \in D(A) \cap D(B) = W^{2,\infty}\left(\mathbb{R}, h_0^{\alpha}\left(\overline{G}\right)\right) \cap L^{\infty}\left(\mathbb{R}, D\left(\Delta'\right)\right),$$

($D\left(\Delta'\right)$ is defined in (15)), such that

$$\begin{cases} \varphi_n \xrightarrow{E} v \\ D_t^2\varphi_n + (1+2\beta)\, D_t\varphi_n + \beta\,(\beta+1)\,\varphi_n + \Delta'\varphi_n \xrightarrow{E} h \\ \varphi_n = 0 \quad \text{on } \partial\Sigma. \end{cases} \tag{30}$$

which implies that v is a distribution solution of (14).

6. The strong solution

6.1. Recall

For $\rho_0 > 0$, set

$$Q_{\rho_0} = Q \cap \{\rho\sigma \;/\; \rho \leqslant \rho_0\},$$

then problem (1) and so problem (14) admits a unique variational solution u in Q_{ρ_0} which does not necessarily coincide with the strong solution v on this bounded open set. In order to analyse u near the vertex of the cone we need the optimal regularity of v. Therefore, Labbas' results[14] will be essential, and we briefly recall them.

Let us consider the non homogenous abstract second order differential equation

$$\begin{cases} y''(t) + Ly(t) = l(t) \in X \\ y(0) = y_0 \\ y(1) = y_1, \end{cases} \tag{31}$$

where $y_0, y_1 \in X$ and L is a closed linear operator of domain $D(L)$ not necessarily dense in a complex Banach space X and verifying the following unique hypothesis of ellipticity in the Krein's sense[13]:

$$\exists C > 0 \;\; \forall r \geqslant 0 \quad \exists (L - rI)^{-1} \;/\; \left\|(L - rI)^{-1}\right\|_{L(X)} \leq \frac{C}{1+r}. \tag{32}$$

For $\theta \in]0,1[$, let us consider the real interpolation Banach space characterized in Grisvard[9] by

$$D_L(\theta,+\infty) = \left\{ x \in X \quad / \quad \sup_{r>0} r^\theta \left\| L(L-rI)^{-1} x \right\|_X < \infty \right\},$$

and its closed subspace $D_L(\theta)$ (See Sinestrari[19] and Lunardi[15]) defined by

$$D_L(\theta) = \left\{ x \in X \quad / \quad \lim_{r\to\infty} r^\theta \left\| L(L-rI)^{-1} x \right\|_X = 0 \right\}.$$

Let θ be fixed in $]0,1/2[$. Then from Labbas[14] one has:

THEOREM 6.1. *For y_0, $y_1 \in D(L)$, $l \in C^{2\theta}([0,1],X)$ there exists a unique solution y of problem (31) such that*

i) $y \in C^2([0,1],X) \cap C([0,1],D(L))$ if and only if $l(0)$ $-Ly_0$ and $l(1)$ $-Ly_1$ belong to $\overline{D(L)}$.

ii) y'' , Ly belong to $C^{2\theta}([0,1],X)$ if and only if $l(0)$ $-Ly_0$ and $l(1) - Ly_1$ belong to $D_L(\theta,+\infty)$.

iii) $y'' \in L^\infty(0,1;D_L(\theta,+\infty))$ if and only if $l(0)$ $-Ly_0$ and $l(1)$ $-Ly_1$ belong to $D_L(\theta,+\infty)$.

THEOREM 6.2. *For $l \in C([0,1],X) \cap L^\infty(0,1;D_L(\theta,+\infty))$ and y_0, $y_1 \in D(L)$, there exists a unique solution y of problem (31) such that*

i) $y \in W^{2,\infty}(0,1;X) \cap L^\infty(0,1;D(L))$ if and only if $l(0) - Ly_0$ and $l(1) - Ly_1$ belong to $\overline{D(L)}$.

ii) y'' and Ly belong to $L^\infty(0,1;D_A(\theta,+\infty))$ if and only if $l(y_0)$ $-Ly_1$ and $l(1) - Ly_1$ belong to $D_L(\theta,+\infty)$.

iii) $Ly \in C^{2\theta}([0,1];X)$ if and only if $l(0)-Ly_0$ and $l(1)-Ly_1$ belong to $D_L(\theta,+\infty)$.

The same results are true if we replace $C^{2\theta}([0,1],X)$ by $h^{2\theta}([0,1],X)$ and $D_L(\theta,+\infty)$ by $D_L(\theta)$. By using the same techniques as in [14] for the same equation on the semi-infinite interval $[0,\ +\infty[$

$$\begin{cases} y''(t)+Ly(t)=l(t)\in X \\ y(0)=y_0 \\ y \text{ bounded on } [0,+\infty[. \end{cases} \tag{33}$$

one has the following theorem

THEOREM 6.3. *For* $y_0\in D(L)$, $l\in C^{2\theta}([0,+\infty[,X)$, *there exists a unique solution* y *of problem (33) such that*

i) $y\in C^2([0,+\infty[;X)\cap C([0,+\infty[;D(L))$ *if and only if* $l(0)-Ly_0$ *belongs to* $\overline{D(L)}$.

ii) y'' *and* Ly *belong to* $C^{2\theta}([0,+\infty[;X)$ *if and only if* $l(0)-Ly_0$ *belongs to* $D_L(\theta,+\infty)$.

iii) $y''\in L^\infty([0,+\infty[;D_L(\theta,+\infty))$ *if and only if* $l(0)-Ly_0$ *belongs to* $D_L(\theta,+\infty)$.

We have an analogous theorem if we replace $C^{2\theta}$ by $h^{2\theta}$ and $D_L(\theta,+\infty)$ by $D_L(\theta)$. The problem on $]-\infty,\ 0])$ is similar.

6.2. Back to the strong solution v

We recall that the solution v of (14) verifies

$$v''(t)+\Delta'(v(t))=h(t)-(1+2\beta)v'(t)-(\beta^2+\beta)v(t)=k(t), \tag{34}$$

and thus, from (23) and (27), we have

$$k\in C^\alpha\left(\mathbb{R},C_0\left(\overline{G}\right)\right)\cap L^\infty\left(\mathbb{R},h_0^\alpha\left(\overline{G}\right)\right). \tag{35}$$

We are going to study the equation (34) on the two half-axis $[t_0,+\infty[$, $]-\infty,t_0]$ for some fixed $t_0>0$. Let Ψ be a scalar function in $C^\infty(\mathbb{R})$ verifying

$$\begin{cases} \Psi\equiv 1 & \text{if } t\geqslant t_0 \\ \Psi\equiv 0 & \text{if } t\leqslant 0, \end{cases}$$

then the function $w = \Psi.v$ verifies the equation

$$\begin{cases} w''(t) + \Delta'(w(t)) = \Psi(t)k(t) + \Psi'(t)v'(t) + \Psi''(t)v(t) \\ \qquad\qquad\qquad\quad = l(t) \qquad \text{on } (0,\infty) \\ w(0) = 0 \\ w \qquad \text{bounded on } [0,\infty[, \end{cases} \tag{36}$$

where we have, in virtue of (35),

$$l \in C^{\alpha}\left([0,\infty[, C_0\left(\overline{G}\right)\right) \cap L^{\infty}\left(0,\infty; h_0^{\alpha}\left(\overline{G}\right)\right).$$

We obtain a similar equation on $]-\infty, 0]$.

In $X = C_0\left(\overline{G}\right)$ we define L by

$$\begin{cases} D(L) = \left\{w \in C_0\left(\overline{G}\right) \cap W^{2,q}(G) \;\; , q > 3 \; , \Delta' w \in C_0\left(\overline{G}\right)\right\} \\ Lw = \Delta' w. \end{cases} \tag{37}$$

Then theorem 6.3, as well as the first regularity of l, that is $l \in C^{\alpha}\left([0,\infty[, C_0\left(\overline{G}\right)\right)$ lead to the following optimal regularity result for the strong solution v.

PROPOSITION 6.4. *The strong solution v verifies*

i) v'' and $\Delta' v$ belong to $C^{\alpha}\left([0,\infty[, C_0\left(\overline{G}\right)\right)$,

ii) v'' belongs to $L^{\infty}\left(0,\infty; D_{\Delta'}(\alpha/2,+\infty)\right)$.

In fact it is enough to verify hypothesis (32) and the compatibility condition

$$l(0) \in D_{\Delta'}(\alpha/2, +\infty). \tag{38}$$

In the case of real-valued functions, (32) is a simple application of maximum principle whereas in the complex field it comes from Miranda[17] and Stewart[20]. The interpolation space $D_{\Delta'}(\alpha/2,+\infty)$ coincides with $C_0^{\alpha}\left(\overline{G}\right)$ (see Lunardi [15]). Since $v \in W^{1,\infty}\left(\mathbb{R}, h_0^{\alpha}\left(\overline{G}\right)\right)$ we have

$$v(t) \;,\; v'(t) \in h_0^{\alpha}\left(\overline{G}\right) \text{ a-e in } t \in \mathbb{R},$$

and hence $l(0) \in h_0^{\alpha}\left(\overline{G}\right) = D_{\Delta'}(\alpha/2) \subset D_{\Delta'}(\alpha/2,+\infty)$.

Now from the second regularity of l, that is $l \in C\left([0,\infty[, C_0\left(\overline{G}\right)\right) \cap L^{\infty}\left(\mathbb{R}, h_0^{\alpha}\left(\overline{G}\right)\right)$ and the equivalent of theorem 6.3 we deduce the following proposition in a same way as above.

PROPOSITION 6.5. *The strong solution v verifies*

i) v'' *and* $\Delta' v$ *belong to* $L^\infty\left(0,\infty; h_0^\alpha\left(\overline{G}\right)\right)$,

ii) $\Delta' v$ *belongs to* $h^\alpha\left([0,\infty[, C_0\left(\overline{G}\right)\right)$.

After the analogous study on $]-\infty,0]$ we summarize all the regularities

$$\left\{\begin{array}{l} i)\ v\in W^{1,\infty}\left(\mathbb{R},h_0^\alpha\left(\overline{G}\right)\right)\cap L^\infty\left(\mathbb{R},C^{1+\alpha}\left(\overline{G}\right)\cap h_0^\alpha\left(\overline{G}\right)\right),\\ ii)\ v\in C^{1+\alpha}\left(\mathbb{R},C_0\left(\overline{G}\right)\right)\cap h^\alpha\left(\mathbb{R},W_0^{1,q}\left(\overline{G}\right)\right),\forall q>3,\\ iii)\ v\in C^{2+\alpha}\left(\mathbb{R},C_0\left(\overline{G}\right)\right)\cap C\left(\mathbb{R},D(\Delta')\right)\cap W^{2,\infty}\left(\mathbb{R},h_0^\alpha\left(\overline{G}\right)\right),\\ iv)\ \Delta' v\in L^\infty\left(\mathbb{R},h_0^\alpha\left(\overline{G}\right)\right)\cap h^\alpha\left(\mathbb{R},C_0\left(\overline{G}\right)\right). \end{array}\right. \tag{39}$$

The statements $iii)$ and $iv)$ and Najmi's results[18] imply that

$$V(t,\sigma)=v(t)(\sigma)\in C^{2+\alpha}\left(\overline{\Sigma}\right).$$

Summing up we have proved

THEOREM 6.6. *Let* $h\in L^\infty\left(\mathbb{R},h_0^\alpha\left(\overline{G}\right)\right)\cap h^\alpha\left(\mathbb{R},C_0\left(\overline{G}\right)\right)$ *with* $\alpha\in\ ]0,1[$ *such that*

$$(2+\alpha)(3+\alpha)\neq\lambda_j \quad \forall j\geqslant 1,$$

where the λ_j , $j=1,2,...$ *are the eigenvalues of the operator* $(-\Delta')$ *on* G *under Dirichlet's condition. Then the problem*

$$\left\{\begin{array}{l} D_t^2 v+(5+2\alpha)D_t v+(\alpha+2)(\alpha+3)v+\Delta' v=h \text{ in } \Sigma\\ v_{|\partial\Sigma}=0, \end{array}\right. \tag{40}$$

has a unique solution v *such that* $V(t,\sigma)=v(t)(\sigma)\in C^{2+\alpha}\left(\overline{\Sigma}\right)$ $\cap C_0\left(\overline{\Sigma}\right)$.

REMARK 6.7. *Let us set* $u_0=e^{(\alpha+2)t}v$ *, so by only using the regularity properties in (23) and (27) on* v, *we deduce that* u_0 *is solution of equation (1) in the sense of distributions. The cut off function* Ψ *allows to study* u_0 *far from the vertex* O.

7. Back to the problem in the cone

Equation (36) has been obtained by the following change of variables and functions

$$\rho = e^t \ ; \ V(t,\sigma) = e^{-(\alpha+2)t}u(e^t\sigma) \ ; \ H(t,\sigma) = e^{-\alpha t}f(e^t\sigma)$$

where, in virtue of assumption on f, we have $H \in L^\infty\left(\mathbb{R}, h_0^\alpha\left(\overline{G}\right)\right) \cap h^\alpha\left(\mathbb{R}, C_0\left(\overline{G}\right)\right)$. The previous theorem implies the existence of $u_0 = e^{(\alpha+2)t}v$, solution of (1) and verifying the following converse properties:

$$\frac{1}{\rho^2}u_0 \in C^\alpha\left(\overline{Q \cap B_R}\right), \quad \frac{1}{\rho}D_i u_0 \in C^\alpha\left(\overline{Q \cap B_R}\right)$$

and

$$D_{ij}u_0 \in C^\alpha\left(\overline{Q \cap B_R}\right),$$

where $B_R = B(O,R)$. This implies that

$$u_0 \in C^{2+\alpha}\left(\overline{Q \cap B_R}\right).$$

If now u is a variational solution (whenever it exists) of the problem

$$\begin{cases} \Delta u = f \in C_0^\alpha\left(\overline{Q}\right) \\ u \in H_0^1(Q), \end{cases}$$

then, in $B_R \cap Q$, the function defined by

$$Z = u - u_0$$

is harmonic and belongs to $H^1(B_R \cap Q)$. Consequently it can be expanded, near the neighborhood of the origin, over the system of the eigenfunctions w_j of $(-\Delta')$ in L^2 . So there exists two sequences $(a_j)_{j\geqslant 1}$ and $(b_j)_{j\geqslant 1}$ such that

$$Z = \sum_{j\geqslant 1} a_j \rho^{-\frac{1}{2}+\beta_j} w_j(\sigma) + \sum_{j\geqslant 1} b_j \rho^{-\frac{1}{2}-\beta_j} w_j(\sigma)$$

with

$$\beta_j^2 = \lambda_j + 1/4.$$

Since $Z \in H^1_{loc}(Q)$, all the coefficients b_j are necessarily zero. On the other hand we knows that

$$\rho^\nu w_j \in C^{2+\alpha} \Longleftrightarrow \operatorname{Re}\nu \geqslant 2+\alpha,$$

from which it follows that the variational solution u may be written as

$$\begin{aligned} u &= u_0 + Z \\ &= \Big(u_0 + \big(Z - \sum_{j\in I} a_j \rho^{-\frac{1}{2}+\sqrt{\lambda_j+\frac{1}{4}}} w_j(\sigma)\big)\Big) + \Big(\sum_{j\in I} a_j \rho^{-\frac{1}{2}+\sqrt{\lambda_j+\frac{1}{4}}} w_j(\sigma)\Big) \\ &= u_r + u_s, \end{aligned}$$

where

$$\begin{aligned} u_r &= u_0 + \big(Z - \sum_{j\in I} a_j \rho^{-\frac{1}{2}+\sqrt{\lambda_j+\frac{1}{4}}} w_j(\sigma)\big) \in C^{2+\alpha}\left(\overline{Q\cap B_R}\right), \\ u_s &= \sum_{j\in I} a_j \rho^{-\frac{1}{2}+\sqrt{\lambda_j+\frac{1}{4}}} w_j(\sigma) \end{aligned}$$

and

$$I = \{ j \geqslant 1 \ / \ \lambda_j < (\alpha+2)(\alpha+3)\}.$$

The final conclusion is summarized by

THEOREM 7.1. *Let u be the variational solution of the problem $-\Delta u = f$ in the cone $Q = \{\rho\sigma \ / \ \rho > 0, \sigma \in G\}$ where G is an open regular set of the unit sphere S^2 and $f \in h_0^\alpha\left(\overline{Q}\right)$. Let $(\lambda_j)_{j\geqslant 1}$ be the sequence of eigenvalues of $(-\Delta')$ on G under Dirichlet's condition and w_j their corresponding eigenfunctions. Assume that $(\alpha+2)(\alpha+3) \neq \lambda_j$ for all $j \geqslant 1$. Then there exists a sequence (a_j) such that*

$$\left[u - \sum_{\lambda_j < (\alpha+2)(\alpha+3)} a_j \rho^{-\frac{1}{2}+\sqrt{\lambda_j+\frac{1}{4}}} w_j(\sigma) \right] \in C^{2+\alpha}\left(\overline{Q\cap B_R}\right)$$

for every $R > 0$.

Remark 7.2. *Our study can be extended to conical open sets* Ω *of* $\mathbb{R}^n$, $n \geqslant 3$. *The condition on* α *becomes* $(\alpha+2)(\alpha+n) \neq \lambda_j$, $\forall j \geqslant 1$. *Notice that* $(\alpha+2)$ *is the Sobolev exponent corresponding to Hölder spaces* $C^{2+\alpha}$. *This condition allows us to think that the sum considered in equation (14) is not closable for* $\alpha \in \left]0,1\right[$ *such that* $\lambda_j = (\alpha+2)(\alpha+3)$ *for some* j.

Remark 7.3. *Let* f *be a function in* $C_0^{\beta}\left(\overline{Q}\right)$ *with compact support with* $\beta \in \left]0,1\right[$. *Suppose that a function* u *in* $H_0^1(Q)$ *is a variational solution of problem (1). Choose* $\alpha = \beta - \epsilon$ *with* $\epsilon > 0$ *arbitrary small in such a way that (22) holds. Then* $f \in h_0^{\beta}\left(\overline{Q}\right)$ *and decomposition of the solution given in theorem 7.1 apply for* u.

Appendix.

In this paragraph we recall the definition of the spaces of classes K'_θ and the proof of the convexity inequality given in section 5 in the case of the Banach space $C^\alpha\left(\mathbb{R}, C_0\left(\overline{G}\right)\right)$.

Let E_0 and E_1 be two Banach spaces imbedded in a separate topological space T. According to Lions-Peetre the Banach space X belongs to class $K'_\theta(E_0, E_1)$ if and only if

$$\begin{cases} i)\ E_0 \cap E_1 \subset X \subset E_0 + E_1 \\ ii)\ \exists C > 0\ /\ \|x\|_X \leqslant C\, \|x\|_{E_0}^{1-\theta}\, \|x\|_{E_1}^{\theta} \quad \forall x \in E_0 \cap E_1. \end{cases}$$

The following proposition describes a frequent situation where we obtain examples of X verifying $i)$ and $ii)$.

Proposition 7.4. *Let* Λ *be a closed linear operator of domain* $D(\Lambda) \subset E$ *,where* E *is a Banach space. Assume that* $\rho(\Lambda) \supset \mathbb{R}_+$ *and there exists* $C_\Lambda > 0$ *such that*

$$\left\|(\Lambda - \lambda I)^{-1}\right\|_{L(E)} \leqslant \frac{C_\Lambda}{\lambda} \quad \forall \lambda > 0,$$

then $D(\Lambda) \in K'_{1/2}\left(D\left(\Lambda^2\right), E\right)$.

Indeed for $x \in D\left(\Lambda^2\right)$, $x \neq 0$, one has for every $\lambda > 0$

$$x = (\Lambda - \lambda I)^{-1} \Lambda x - \lambda (\Lambda - \lambda I)^{-1} x,$$

and thus

$$\begin{aligned} \Lambda x &= (\Lambda - \lambda I)^{-1} \Lambda^2 x - \lambda \Lambda (\Lambda - \lambda I)^{-1} x \\ \|\Lambda x\| &\leqslant \frac{C_\Lambda}{\lambda} \left\|\Lambda^2 x\right\| + (C_\Lambda + 1) \lambda \|x\| . \end{aligned}$$

Now for

$$\lambda = \lambda_0 = \sqrt{\frac{C_\Lambda}{C_\Lambda + 1} \frac{\|\Lambda^2 x\|}{\|x\|}},$$

we get

$$\|\Lambda x\| \leqslant 2\sqrt{C_\Lambda (C_\Lambda + 1)} \left\|\Lambda^2 x\right\|^{1/2} \|x\|^{1/2} ,$$

the proposition is then proved. Notice that $D(\Lambda)$ and $D(\Lambda^2)$ are equiped with their respective graph norm.

Let us go back to section 5. Put $E = C^\alpha\left(\mathbb{R}, C_0\left(\overline{G}\right)\right)$ and define Λ by

$$\begin{cases} D(\Lambda) = \{u \in E \ / \ u' \in E\} = C^{1+\alpha}\left(\mathbb{R}, C_0\left(\overline{G}\right)\right) \\ \Lambda u = u', \end{cases}$$

then

$$\begin{cases} D(\Lambda^2) = C^{2+\alpha}\left(\mathbb{R}, C_0\left(\overline{G}\right)\right) \\ \Lambda^2 u = u'', \end{cases}$$

and it is easy to see that Λ is a closed linear operator such that for any $\lambda > 0$

$$\left[(\Lambda - \lambda I)^{-1} f\right](x) = -\int_x^\infty e^{-\lambda(s-x)} f(s) ds = -\int_0^\infty e^{-\lambda\xi} f(x+\xi) d\xi,$$

from which it follows that:

$$\left\|(\Lambda - \lambda I)^{-1} f\right\|_{C(R, C_0(\overline{G}))} \leqslant \frac{1}{\lambda} \|f\|_{C(R, C_0(\overline{G}))}$$

and

$$\begin{aligned} &\left|\left[(\Lambda - \lambda I)^{-1} f\right](x) - \left[(\Lambda - \lambda I)^{-1} f\right](y)\right| \\ &\qquad \leqslant \int_0^\infty e^{-\lambda\xi} \left|f(x+\xi) - f(y+\xi)\right| d\xi \leqslant \frac{1}{\lambda} |x-y|^\alpha [f]_\alpha . \end{aligned}$$

Thus

$$\left\| (\Lambda - \lambda I)^{-1} f \right\|_{C^\alpha(R, C_0(\overline{G}))} \leqslant \frac{1}{\lambda} \|f\|_{C^\alpha(R, C_0(\overline{G}))}.$$

Using the above proposition, there exists $C > 0$ such that

$$\begin{aligned} &\|u'\|_{C^\alpha(R, C_0(\overline{G}))} \\ &\leqslant C \|u\|^{1/2}_{C^\alpha(R, C_0(\overline{G}))} \|u''\|^{1/2}_{C^\alpha(R, C_0(\overline{G}))} \quad \forall u \in C^{2+\alpha}\left(\mathbb{R}, C_0\left(\overline{G}\right)\right) \end{aligned}$$

and then

$$\begin{aligned} &\|u\|_{C^{1+\alpha}\left(R, C_0(\overline{G})\right)} \\ &\leqslant \sup(1, C) \left(\|u\|_{C^\alpha(R, C_0(\overline{G}))} + \|u\|^{1/2}_{C^\alpha(R, C_0(\overline{G}))} \|u''\|^{1/2}_{C^\alpha(R, C_0(\overline{G}))} \right). \end{aligned}$$

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