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Sequential Order of Compact Sequential Spaces

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Dedicated to the memory of our friend Fabio Rossi

Summary. - *The problem of finding compact Hausdorff sequential spaces of sequential order $\alpha \leq \omega_1$ is important and highly non-trivial. A solution has been searched in ZFC, but unsuccessfully up to now. Classically it was solved under CH, and more recently under MA up to order four. We present here a construction of a space of order three that appears simpler than previous constructions.*

1. Introduction and basic definitions

A topological space X is sequential if for any non-closed subset $A \subset X$ there is a sequence $\langle x_n \rangle$ with $x_n \in A$ and a point $x \in \bar{A} \setminus A$ such that $x_n \to x$. Equivalently, we can define the sequential closure of any subset $A \subset X$ as $\hat{A} = \{x \in X : x$ is a limit of a sequence of elements in $A\}$. For any ordinal α, if α is a limit ordinal $\hat{A}^{(\alpha)} = \cup_{\beta<\alpha}\hat{A}^{(\beta)}$; if $\alpha = \gamma + 1$, then $\hat{A}^{(\alpha)} = \widehat{\hat{A}^{(\gamma)}}$; and $\hat{A}^{(0)} = A$. The sequential order of a sequential space X, denoted $\sigma(X)$, is the least ordinal λ such that $\hat{A}^{(\lambda+1)} = \hat{A}^{(\lambda)} = \bar{A}$ for any $A \subset X$. As is well known $\lambda \leq \omega_1$ in any sequential space [6, 13].

Arhangel'skiĭ and Franklin [2] have produced for any $\alpha < \omega_1$ examples of countable zero-dimensional sequential spaces K_α such

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that the sequential order $\sigma(K_\alpha) = \alpha$. Their space S_ω is a sequential, zero-dimensional, homogeneous Hausdorff space with $\sigma(S_\omega) = \omega_1$. All these spaces are countable but are not compact.

As concerns compact examples of sequential order > 1, one is known to exist in ZFC. It is the one-point compactification of a ψ-space of Isbell-Mrówka, which has sequential order 2.

Other examples of compact sequential spaces have been constructed under additional assumption of the theory of sets. Baškirov [5] and Kannan [11] have produced examples under CH and Dow [7, 8] has constructed, under MA, examples of compact Hausdorff spaces of sequentiality order three and four. Till now, it was not possible to extend the construction of compact sequential spaces to a sequentiality order greater than four, under MA.

The problem of finding compact Hausdorff spaces of sequential order α for any $\alpha \leq \omega_1$ turned out to be a highly non-trivial problem; its importance depends also on the fact that Balogh [4] has proven that under PFA every Hausdorff compact space of countable tightness is sequential. Since PFA implies MA + $\mathfrak{c} = \omega_2$, if there is a finite upper limit to the order of sequentiality under PFA (or consequences of it) it would mean that under this assumption compact sequential spaces are only few steps away from being Fréchet-Urysohn.

We want to present here a construction of a compact Hausdorff sequential space of sequential order three that seems to be easier than previous constructions (see [7], and the results to which alludes Bašikirov [5], note (5)).

2. Construction of a compact sequential space of sequential order three, under MA

It is known that MA implies $\mathfrak{b} = \mathfrak{c}$. The following construction is done under this last hypothesis. Observe that since we have

$$\omega_1 \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{b} \leq \mathfrak{a} \leq \mathfrak{c}$$

then $\mathfrak{b} = \mathfrak{c}$ implies $\mathfrak{a} = \mathfrak{c}$, where $\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A}$ is an infinite and mad family of subsets of $\omega\}$.

So if $\mathcal{A}$ is an almost disjoint family and $|\mathcal{A}| = \kappa < \mathfrak{a}$ then $\mathcal{A}$ is not maximal. This fact will be used several times, but also $\mathfrak{b} =$

$\mathfrak{c}$ will be used essentially. We want to construct an example of a compact Hausdorff sequential space of sequentiality order 3, with some modifications to the construction given by Alan Dow in [7].

The set ω of natural numbers is the set of 0.th level elements. We want to add to this set two families of subsets of ω, the family $\mathcal{A} = \{a_\alpha : \alpha < \mathfrak{c}\}$ corresponding to elements of the first level and the family $\mathcal{B} = \{b_\alpha : \alpha < \mathfrak{c}\}$ corresponding to elements of the second level. All this will be completed with a point ∞ which will be the element of sequential order 3. In order to obtain this result we need that the families $\mathcal{A}$ and $\mathcal{B}$ satisfy the following requirements:

1. Two different elements of the first level meet in a compact subset of elements of the 0.th level, which means in a finite subset of ω.

2. An element of the first level and an element of the second level meet in a finite subset of ω or the element $a \in \mathcal{A}$ is almost contained in an element $b \in \mathcal{B}$.

3. Two elements of the second level must meet in a compact subset obtained by a finite union of elements of ω or of $\mathcal{A}$.

4. Any sequence of elements of ω has a convergent subsequence to an element of $\mathcal{A}$, and any sequence of elements of $\mathcal{A}$ has a subsequence converging to an element of $\mathcal{B}$.

5. Finally any sequence in $\mathcal{B}$ must converge to the point ∞.

In order to achieve this purpose we introduce two auxiliary families of sets $\mathcal{X} = \{x_\alpha : \alpha < \mathfrak{c}\} = [\omega]^\omega$ and $\mathcal{U} = \{u_\alpha : u_\alpha \in [\mathfrak{c}]^\omega, u_\alpha \subset \alpha, \omega \leq \alpha < \mathfrak{c}\}$. Furthemore, to be more formal, for all $\alpha < \mathfrak{c}$, families $\mathcal{A}_\alpha = \{a_\beta : \beta < \alpha\}$ and $\mathcal{B}_\alpha = \{b_\beta : \beta < \alpha\}$ have to exist such that the following inductive hypotheses are fulfilled:

(1) $\mathcal{A}_\alpha$ is almost disjoint;

(2) $\forall \beta, \gamma < \alpha$ it is $a_\beta \subset^\star b_\gamma$ or $a_\beta \cap b_\gamma =^\star \emptyset$;

(3) $\forall \beta < \gamma < \alpha$, $\exists F_{\beta,\gamma} \subset \gamma+1$, finite or empty, such that $b_\beta \cap b_\gamma \subset^\star \cup\{a_\xi : \xi \in F_{\beta,\gamma}\}$;

(4) $\forall \gamma, \omega \leq \gamma < \alpha, \exists \beta \leq \gamma$ such that $a_\xi \subset^\star b_\beta$ for infinitely many $\xi \in u_\gamma$.

We start with elements of 0.th level which are the set ω of natural numbers. Since we want to obtain a compact sequential space of sequentiality order 3, all infinite subsets of ω must have a limit point in the first level. Therefore every infinite subset $S \subset \omega$ must have infinite intersection with some element of level one. This is fulfilled if $\mathcal{A} = \{a_\alpha : \alpha < \mathfrak{c}\}$ is a maximal almost disjoint (mad) family: $\forall S \subset \omega$, $\exists \alpha < \mathfrak{c}$ such that $S \cap a_\alpha$ is infinite. Let D be an infinite and coinfinite subset of ω and let $\{a_n : n \in \omega\}$ be a partition of $\omega \setminus D$ into infinite subsets. Let $b_n = \cup_{k \leq n} a_k$ and $b_\omega = \cup_{n \in \omega} a_n$. There is a set x_{γ_ω} with a least index γ_ω such that x_{γ_ω} is disjoint with all sets a_n, $n \in \omega$; this is possible since $D \cap \cup_{n \in \omega} a_n = \emptyset$. Let $a_\omega = x_{\gamma_\omega}$.

Conditions (1) to (4) hold (where applicable) for $\alpha \leq \omega$.

Since an almost disjoint (a.d.) family $\{a_\alpha : \alpha < \kappa (< \mathfrak{c} = \mathfrak{a})\}$ is not maximal there is some x_{γ_κ} which is a.d. with all elements $\{a_\alpha : \alpha < \kappa\}$.

From this fact it follows that there is some $x_{\gamma_{\omega+1}}$ which is a.d. with all a_γ, with $\gamma \leq \omega$. Let us define $a_{\omega+1} = x_{\gamma_{\omega+1}}$.

Now $\{a_\alpha : \alpha \leq \omega + 1\}$ and $\{b_\beta : \beta \leq \omega\}$ are given. Taken $u_{\omega+1} \subset (\omega + 1)$, for all $\xi \in u_{\omega+1} \setminus \{\omega\}$, we have $a_\xi \subset^\star b_\omega$. According to Dow, let us put $b_{\omega+1} = \emptyset$. Anyway we can define $a_{\omega+2} = x_{\gamma_{\omega+2}}$, where $x_{\gamma_{\omega+2}}$ is the set with minimum index which is a.d. to all a_α, with $\alpha \leq \omega + 1$. For many indexes to follow we have $b_\beta = \emptyset$, precisely for all $\beta = \omega + n$, $n \in \omega$, $n \geq 1$.

Suppose now that all sets $\mathcal{A}_\alpha$ and $\mathcal{B}_\alpha$ satisfying conditions (1) to (4) are known. As remarked before there is x_{γ_α} which is a.d. with the family $\mathcal{A}_\alpha$. Let $a_\alpha = x_{\gamma_\alpha}$. Consider $u_\alpha \subset \alpha$. If there is $\beta < \alpha$ such that $a_\xi \subset^\star b_\beta$ for infinitely many $\xi \in u_\alpha$, then $b_\alpha = \emptyset$. Otherwise, for all $\beta < \alpha$, there is a finite subset $F_{\alpha,\beta} \subset u_\alpha$ such that $a_\xi \cap b_\beta$ is finite for all $\xi \in u_\alpha \setminus F_{\alpha,\beta}$. The set u_α is countable; fix one enumeration for it: let $u_\alpha = \{\xi_n : n \in \omega\}$. For all $\gamma \leq \alpha$, if $\gamma \neq \xi_n$, $a_\gamma \cap a_{\xi_n}$ is finite, possibly empty. Define a function $f_\gamma : \omega \to \omega$ as follows: if $a_\gamma \cap a_{\xi_n} \neq \emptyset$ and $\gamma \neq \xi_n$, $f_\gamma(n) = \max(a_\gamma \cap a_{\xi_n})$; otherwise if $a_\gamma \cap a_{\xi_n} = \emptyset$ or $\gamma = \xi_n$, $f_\gamma(n) = 0$. Furthermore, for $\delta < \alpha$, define a function $g_\delta : \omega \to \omega$ as follows: $g_\delta(n) = \max(b_\delta \cap a_{\xi_n})$, if $\xi_n \notin F_{\alpha,\delta}$ and $b_\delta \cap a_{\xi_n} \neq \emptyset$; otherwise $g_\delta(n) = 0$.

Cardinality of $\{f_\gamma : \gamma \leq \alpha\} \cup \{g_\delta : \delta < \alpha\}$ is not greater than $|\alpha| < \mathfrak{c}$. As $\mathfrak{b} = \mathfrak{c}$, there is $h : \omega \to \omega$ majorizing all f_γ and g_δ. Remark that for all $\delta < \alpha$ there is m_δ such that $(a_{\xi_n} \setminus h(n)) \cap b_\delta = \emptyset$ if $\xi_n \in u_\alpha \setminus F_{\alpha,\delta}$ and $n > m_\delta$. Furthermore for all $\gamma \leq \alpha$, there is n_γ such that $(a_{\xi_n} \setminus h(n)) \cap a_\gamma = \emptyset$, if $\gamma \neq \xi_n$ and $n > n_\gamma$. (Note that $(a_{\xi_n} \setminus h(n)) \cap b_\delta =^\star \emptyset$ if $n \leq m_\delta$ and $(a_{\xi_n} \setminus h(n)) \cap a_\gamma =^\star \emptyset$ if $n \leq n_\gamma$).

Define

$$b_\alpha = \bigcup_{\xi_n \in u_\alpha} (a_{\xi_n} \setminus h(n)) . \tag{1}$$

We have in this way defined the families $\mathcal{A}_{\alpha+1} = \{a_\beta : \beta < \alpha + 1\}$ and $\mathcal{B}_{\alpha+1} = \{b_\beta : \beta < \alpha + 1\}$.

We can check that properties (1) to (4) hold.

Property (1) is true because of the construction of the family $\mathcal{A}_{\alpha+1}$.

Property (2). We must prove that for all $\beta, \gamma < \alpha+1$, $a_\beta \subset^\star b_\gamma$ or $a_\beta \cap b_\gamma =^\star \emptyset$. If $\beta, \gamma < \alpha$ property (2) holds because of the inductive hypothesis. If $\beta = \gamma = \alpha$, we have

$$a_\alpha \cap b_\alpha = a_\alpha \cap \bigcup_{\xi_n \in u_\alpha} (a_{\xi_n} \setminus h(n)) = \bigcup_{\xi_n \in u_\alpha} [(a_{\xi_n} \setminus h(n)) \cap a_\alpha] . \tag{2}$$

The sets in this union are empty if $n > n_\alpha$ otherwise they are finite. Then $a_\alpha \cap b_\alpha =^\star \emptyset$.

If $\gamma < \beta = \alpha$,

$$a_\alpha \cap b_\gamma = \bigcup_{\xi_n \in u_\gamma} [(a_{\xi_n} \setminus h(n)) \cap a_\alpha] . \tag{3}$$

Then, as before, the sets in this union are empty if $n > n_\gamma$, and are finite for $n \leq n_\gamma$.

If $\beta < \gamma = \alpha$.

$$a_\beta \cap b_\alpha = \bigcup_{\xi_n \in u_\alpha} [(a_{\xi_n} \setminus h(n)) \cap a_\beta] , \tag{4}$$

then $a_\beta \subset^\star b_\alpha$ if $\beta = \xi_{\bar{n}} \in u_\alpha$. Otherwise we can repeat the previous proof: there is an α such that $[(a_{\xi_n} \setminus h(n)) \cap a_\beta] = \emptyset$ if $n > n_\alpha$ and finite if $n \leq n_\alpha$. Finally we have $a_\beta \cap b_\gamma =^\star \emptyset$ or $a_\beta \subset^\star b_\gamma$ for all $\beta, \gamma < \alpha + 1$.

Property (3). We must show that if $\beta < \gamma < \alpha + 1$ there is a finite set (possibly empty) $F_{\gamma,\beta}$ of $\gamma + 1$ such that $b_\beta \cap b_\gamma =^\star \cup\{a_\xi : \xi \in F_{\gamma,\beta}\}$; for all $\beta < \gamma < \alpha$ property (3) holds by the inductive hypothesis. If $\beta < \gamma = \alpha < \alpha+1$, then there is a finite set $F_{\alpha,\beta} \subset \alpha + 1$ such that $b_\beta \cap b_\alpha =^\star \cup\{a_\xi : \xi \in F_{\alpha,\beta}\}$; in fact

$$b_\beta \cap b_\alpha = b_\beta \cap \bigcup_{\xi_n \in u_\alpha} (a_{\xi_n} \setminus h(n)) = \bigcup_{\xi_n \in u_\alpha} [b_\beta \cap (a_{\xi_n} \setminus h(n))] \quad .$$

For all $\xi_n \in u_\alpha \setminus F_{\alpha,\beta}$ if $n > m_\beta$ it is true that $(a_{\xi_n} \setminus h(n)) \cap b_\beta = \emptyset$, while for $n \le m_\beta$, $(a_{\xi_n} \setminus h(n)) \cap b_\beta$ is finite and we can conclude that $b_\beta \cap b_\alpha =^\star \cup\{a_\xi : \xi \in F_{\alpha,\beta}\}$.

Property (4): For all $\omega \le \gamma < \alpha + 1$ there is $\beta \le \gamma$ such that $a_\xi \subset^\star b_\beta$ for infinitely many $\xi \in u_\gamma$: for all $\gamma < \alpha$ property holds by inductive hypothesis. If $\gamma = \alpha$ there is $\alpha \le \alpha$ such that, because of the definition of b_α, $a_\xi \subset^\star b_\alpha$ for infinitely many (in fact all) $\xi \in u_\alpha$.

We identify every element a_α with a point p_α and every $b_\beta \ne \emptyset$ with a point q_β. Let $\mathcal{P} = \{p_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{Q} = \{q_\alpha : \alpha < \mathfrak{c}, b_\alpha \ne \emptyset\}$. The set $X = \omega \cup \mathcal{P} \cup \mathcal{Q}$, with the topology to be described, is a Hausdorff sequential and locally compact space, and its Alexandroff's compactification $X^\star = X \cup \{\infty\}$ is a sequential compact Hausdorff space with sequentiality order 3.

The topology of X is the following.

Points of ω are isolated.

Fundamental neighbourhoods of p_α are $\{p_\alpha\} \cup (a_\alpha \setminus F)$ where F is a finite subset of ω.

Define, for every $b \in \mathcal{B}$, $[b] = \{p_\alpha : a_\alpha \subset^\star b\}$. Fundamental neighbourhoods of q_β are $\{q_\beta\} \cup (b_\beta \setminus (\bigcup_{\xi \in G}\{p_\xi\} \cup a_\xi \cup F) \cup [b_\beta \setminus \bigcup\{a_\xi : \xi \in G\}]$, where G is a finite subset of u_β and F is a finite subset of ω.

Fundamental neighbourhoods of p_α and q_α are open compact sets. The space is Hausdorff as can be easily proved. For example, for $x = p_\alpha$ and $y = q_\beta$, $U = \{p_\alpha\} \cup a_\alpha$ and $V = \{q_\beta\} \cup (b_\beta \setminus (\{p_\alpha\} \cup a_\alpha)) \cup [b_\beta \setminus a_\alpha]$ are disjoint neighbourhoods of x and y respectively. If $x = q_\beta$ and $y = q_\gamma$ (with $\beta < \gamma$) then $U = \{q_\beta\} \cup (b_\beta \setminus b_\gamma) \cup [b_\beta \setminus b_\gamma]$ and $V = \{q_\gamma\} \cup b_\gamma \cup [b_\gamma]$ are the neighbourhoods we need.

In $X^\star$ fundamental neighbourhoods of the point ∞ are of type $\{\infty\} \cup (X \setminus K)$, K compact in X.

Let $Z \subset X^\star$ be a non-closed set. We will prove that there is a point $z \in \bar{Z} \setminus Z$ and a sequence $\langle z_k \rangle$ in Z converging to z. The point z cannot be in ω; if $Z \cap (\omega \cup \mathcal{P})$ is not closed in the subspace $\omega \cup \mathcal{P}$, then there is a point $p_\alpha \in \bar{Z} \setminus Z$. Since points of $\mathcal{P}$ have countable character there is a sequence of points in Z converging to p_α. If $Z \cap (\omega \cup \mathcal{P})$ is closed and $Z \cap (\omega \cup \mathcal{P} \cup \mathcal{Q})$ is not closed, let δ be the smallest index such that $q_\delta \in \bar{Z} \setminus Z$. Now, the set $[b_\delta \cap Z]$ is certainly infinite and any countable subset of it is a sequence converging to q_δ. Let finally $z = \infty$. Every its neighbourhood of type $W = \{\infty\} \cup (X \setminus K)$ meets Z in infinitely many points q_γ of the second level, then a sequence extracted from it converges to z.

Clearly every infinite set of points of ω contains a subsequence converging to some point p_α. Every infinite set of points q_β (of the second level) has a subsequence converging to the point ∞. Given a countably infinite set of points p_{α_i} (of level one), there is a set u_α such that $\alpha_i \in u_\alpha \subset \alpha$ for all $i \in \omega$; if $\alpha > \omega$, then $\alpha \geq \sup\{\alpha_i\}$ and, by condition (4), there is some $\beta \leq \alpha$ such that $a_{\alpha_i} \subset^\star b_\beta$ for infinitely many $\alpha_i \in u_\alpha$, i.e. a subsequence of $\langle p_{\alpha_i} \rangle$ converges to q_β. If $\alpha = \omega$, because of the construction, $\langle p_{\alpha_i} \rangle$ converges to q_ω, corresponding to b_ω.

Then no sequence of level one points converges to ∞. The space that we have constructed is a sequential compact Hausdorff space of sequentiality order three.

We can also remark that the space is a scattered space of scattered heigth 4, and that the scattering levels 0, 1, 2, 3 correspond to the sequential levels 0, 1, 2, 3 respectively. In fact, in $X \setminus \omega$ points of $\mathcal{P}$ are isolated [8].

3. The sequential space of order three as a Stone space

Consider the Boolean subalgebra S of $\mathcal{P}(\omega)$ generated by $\omega \cup \mathcal{A} \cup \mathcal{B}$. Let $\mathcal{S}$ be the Stone space of S (see e.g. [12]). For all $a \in \mathcal{A}$, $\overline{a}$ in $\mathcal{S}$ is $\{p_a\} \cup a$ and for all $b \in \mathcal{B}$, $\overline{b}$ in $\mathcal{S}$ is $\{q_b\} \cup b \cup \{p_a : a \subset^\star b, a \in \mathcal{A}\}$. Because of the construction we see that $\omega \setminus (a_\alpha \cup a_\beta)$ is infinite, in fact if $\gamma \neq \alpha, \beta$ then $a_\gamma \cap (\omega \setminus (a_\alpha \cup a_\beta))$ is infinite. Analogously $\omega \setminus (b_\alpha \cup b_\beta)$ and $\omega \setminus (a_\alpha \cup b_\beta)$ are infinite. In fact, if $\gamma > \max\{\alpha, \beta\}$, according to the construction of b in (1), $a_\gamma \cap (\omega \setminus (b_\alpha \cup b_\beta))$ is infinite.

The same is true because of formulas (2), (3), (4) for $\delta > \alpha, \beta, \gamma$: $a_\delta \cap (\omega \setminus (a_\alpha \cup b_\alpha))$ is infinite, $a_\delta \cap (\omega \setminus (a_\alpha \cup b_\gamma))$ is infinite and $a_\delta \cap (\omega \setminus (a_\beta \cup b_\alpha))$ is infinite. Then there is a unique ultrafilter generated by the complements of elements of $\mathcal{A}, \mathcal{B}$ and finite subsets of ω. This last element is the point of sequential order three in the previous construction.

4. Final remarks

We remark that it is an open problem to find compact sequential spaces of order five or higher without CH. In fact requirements 1. to 5. have obviously to be satisfied, but nobody was able to assure them without additional assumptions of the theory of sets, even for sequential order three. To construct examples of higher sequential order an iteration of these requirements should be satisfied up to the requested order.

Finally, to construct examples of higher sequential order, it would be useful to find inductive hypotheses simpler then those presented in [8], which appear difficult to extend beyond order four.

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