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Periodic solutions for quasilinear complex-valued differential systems involving singular ϕ-Laplacians

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Cordially dedicated to Fabio Zanolin, for his sixtieth birthday anniversary, and the twentieth anniversary of our first joint paper

Abstract. *Topological degree is used to obtain sufficient conditions for the existence of periodic solutions of systems of second order complex-valued ordinary differential equations involving a singular ϕ-Laplacian. Corresponding results for first order equations are also obtained.*

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1. Introduction

In [8], Manásevich, Zanolin and the author have used topological degree arguments to study the existence of periodic solutions for some complex-valued differential equations of the form

$$z'' = f(t, z, z'). \tag{1}$$

or for systems of such equations, where the nonlineary $f : [0, T] \times \mathbb{C}^2 \to \mathbb{C}$ has some special structure inspired by the equations of Liénard or Rayleigh. The existence conditions, as well as the technicalities to obtain the requested a priori bounds, are rather involved.

On the other hand, Bereanu and the author [1, 2, 3] have considered the existence of solutions of quasilinear differential equations or systems of the form

$$(\phi(u'))' = f(t, u, u'), \tag{2}$$

where $f : [0, T] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ satisfies Carathéodory conditions and $\phi : B(a) \to \mathbb{R}^n$ belongs to a suitable class of so-called *singular* homeomorphisms between the open ball $B(a) \subset \mathbb{R}^n$ of center 0 and radius $a > 0$ and $\mathbb{R}^n$. A *solution* of (2) on $[0,T]$ is a function $u \in C^1([0,T],\mathbb{R}^n)$ such that $u'(t) \in B(a)$ for all $t \in [0,T]$, $\phi \circ u'$ is absolutely continuous and equation (2) holds almost everywhere. A motivating example of singular homeomorphism comes from the relativistic acceleration, associated to the homeomorphism

$$\phi : B(1) \to \mathbb{R}^n, \ v \mapsto \frac{v}{\sqrt{1-|v|^2}}\,.$$

Despite of the apparent greater complexity of equation (2) with respect to (1), existence conditions for periodic solutions of (2) are in general weaker than those for (1).

Hence it may be of interest to study the problem of the existence of periodic solutions for quasilinear complex-valued differential systems of the form

$$(\phi(z'))' = f(t,z,z'). \tag{3}$$

where $\phi : B(a) \subset \mathbb{C}^m \to \mathbb{C}^m$ is a singular homeomorphism and $f : [0,T] \times \mathbb{C}^{2m} \to \mathbb{C}^m$ is a Carathéodory function. This is done in Section 3, where we state and prove fairly general results for nonlinearities containing the Liénard or Rayleigh types. A very special case is the existence of a solution for the problem

$$\left(\frac{z'}{\sqrt{1-|z|^2}}\right)' = \alpha z^n + h(t), \quad z(0) = z(T),\ z'(0) = z'(T) \tag{4}$$

for every integer $n \geq 1$, $\alpha \in \mathbb{C}\backslash\{0\}$, and $h \in L^1([0,T],\mathbb{C})$. Such a result is sharp because, when $\alpha = 0$, problem (4) has no solution when $T^{-1}\int_0^T h(t)\,dt \neq 0$.

On the other hand, motivated by some work of Szrednicki [10, 11], Manásevich, Zanolin and the author have proved in [7] existence conditions for periodic solutions of some first order complex-valued differential equations. In the special case of the complex Riccati equation

$$z' = z^2 + h(t), \quad z(0) = z(T),$$

interesting existence and non-existence results have been subsequently obtained by Campos and Ortega [4, 5]. Hence it may be of interest to consider first order periodic problems of the type

$$(\phi(z))' = f(t,z), \quad z(0) = z(T),$$

where $\phi : B(a) \subset \mathbb{C} \to \mathbb{C}$ is a suitable singular homeomorphism. This is done in Section 4, where a very special case of the obtained results is the existence of a solution for the problem

$$\left(\frac{z}{\sqrt{1-|z|^2}}\right)' = \alpha z^n + h(t), \quad z(0) = z(T), \tag{5}$$

for every $n \geq 1$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $h \in L^1([0,T],\mathbb{C})$ such that

$$\left| T^{-1} \int_0^T h(t)\, dt \right| < |\alpha|.$$

Again, this condition is sharp because, when $\alpha = 0$, problem (5) has no solution when $T^{-1} \int_0^T h(t)\, dt \neq 0$.

We end this introduction with some notations. We denote some norm in $\mathbb{R}^n$ by $|\cdot|$, and the usual norm in $L^p := L^p(0,T;\mathbb{R}^n)$ $(1 \leq p \leq \infty)$ by $|\cdot|_p$. For $k \geq 0$, we set $C^k := C^k([0,T],\mathbb{R}^n)$ and $W^{1,1} := W^{1,1}([0,T],\mathbb{R}^n)$. The usual norm $|\cdot|_\infty$ is considered on C, and the space C^1 is endowed with the norm

$$|v|_{1,\infty} = |v|_\infty + |v'|_\infty.$$

Each $v \in C$ can be written $v(t) = v_0 + \widehat{v}(t)$, with $v_0 = v(0)$ and $\widehat{v}(0) = 0$. For $u \in W^{1,1}$ such that $u(0) = u(T)$, we have

$$\widehat{u}(t) = \int_0^t u'(s)\, ds = -\int_t^T u'(s)\, ds,$$

and $\max_{[0,T]} |\widehat{u}|$ being reached either in $[0,T/2]$ or in $[T/2,T]$, this gives

$$|\widehat{u}|_\infty \leq \frac{T}{2} |u'|_\infty \tag{6}$$

It is easily shown that the constant $T/2$ is optimal. We define the mean value $\overline{u}$ of $u \in L^1$ by

$$\overline{u} := T^{-1} \int_0^T u(t)\, dt,$$

2. A continuation theorem for periodic solutions of quasilinear systems involving singular ϕ-Laplacians

Let us consider now the periodic problem

$$(\phi(u'))' = f(t,u,u'), \quad u(0) = u(T),\ u'(0) = u'(T), \tag{7}$$

where $f : [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ is a Carathéodory function and $\phi : B(a) \to \mathbb{R}^n$ $(a < +\infty)$ satisfies the following assumption introduced in [3].

(H_Φ) *ϕ is a homeomorphism from $B(a) \subset \mathbb{R}^n$ onto $\mathbb{R}^n$ such that $\phi(0) = 0$, $\phi = \nabla\Phi$, with $\Phi : \overline{B(a)} \to \mathbb{R}$ of class C^1 on $B(a)$, continuous, strictly convex on $\overline{B(a)}$, and such that $\Phi(0) = 0$.*

The motivating example is given by the C^∞-mapping $\Phi : \overline{B(1)} \subset \mathbb{R}^n \to \mathbb{R}$ defined by

$$\Phi(u) = 1 - \sqrt{1 - |u|^2} \quad (u \in \overline{B(1)}),$$

so that

$$\phi(u) = \nabla\Phi(u) = \frac{u}{\sqrt{1 - |u|^2}} \quad (u \in B(1)).$$

Hence $(\phi(u'))'$ describes the relativistic acceleration.

Notice that the scalar problem

$$(\phi(u'))' = 1, \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has no solution, because the existence of a solution would imply, by integration over $[0, T]$ of both members of the differential equation and use of the boundary conditions, that $0 = T$. Hence we cannot expect an existence result for *any* right-hand side of the differential system in (7).

The following continuation result essentially comes from [1], and its present form is given in [9]. We denote by d_B the Brouwer degree for continuous mappings in $\mathbb{R}^n$ (see e.g. [6]).

LEMMA 1. *Assume that there exists an open bounded set $\Omega \subset C$ such that the following conditions hold :*

1. *For each $\lambda \in (0, 1]$, there is no solution of the problem*

$$(\phi(u'))' = \lambda f(t, u, u'), \quad u(0) = u(T),\ u'(0) = u'(T) \tag{8}$$

such that $u \in \partial\Omega$.

2. *There is no solution $u_0 \in \partial\Omega \cap \mathbb{R}^n$ of the system in $\mathbb{R}^n$*

$$\overline{f}(u_0) := T^{-1} \int_0^T f(t, u_0, 0)\, dt = 0, \tag{9}$$

where, in $\partial\Omega \cap \mathbb{R}^n$, $\mathbb{R}^n$ is identified with the subspace of constant functions in C.

3. $d_B[\overline{f}, \Omega \cap \mathbb{R}^n, 0] \neq 0$.

Then problem (7) has at least one solution such that $u \in \Omega$.

3. Periodic solutions of complex-valued quasilinear systems involving singular ϕ-Laplacians

In this section, let us provide $\mathbb{R}^2$ with the multiplication structure of the complex plane $\mathbb{C}$, and consider the complex-valued periodic system in $\mathbb{C}^m \simeq \mathbb{R}^{2m}$ with $m \geq 1$ an integer,

$$(\phi_k(z'))' = \alpha_k(t) z_k^{n_k} + [F_k(t,z)]' + h_k(t,z,z') \qquad (k = 1,2,\ldots,m)$$
$$z(0) = z(T),\ z'(0) = z'(T), \qquad (10)$$

where $z' = (z'_1, \ldots, z'_m)$, $z = (z_1, \ldots, z_m)$, $\phi = (\phi_1, \ldots, \phi_m) : B(a) \subset \mathbb{C}^m \to \mathbb{C}^m$ satisfies Assumption (H_ϕ), $n_k \geq 1$ is an integer, $\alpha_k \in L^1$, $F_k : [0,T] \times \mathbb{C}^m \to \mathbb{C}^m$ is of class C^1, and $h_k : [0,T] \times \mathbb{C}^{2m} \to \mathbb{C}^m$ is a Carathéodory function $(k = 1,2,\ldots,m)$. For $z = (z_1, \ldots, z_m)$, we take

$$|z| = \max\{|z_1|, \ldots, |z_m|\},$$

and for $z \in C$,

$$|z|_\infty = \max_{t \in [0,T]} |z(t)|.$$

We set

$$n = \min\{n_1, \ldots, n_m\}, \quad N = \max\{n_1, \ldots, n_m\}.$$

THEOREM 1. *Assume that, for each $k = 1,2,\ldots,m$, $\overline{\alpha}_k \neq 0$, and there exist $1 \leq \sigma_k < n$ and $\beta_k, \gamma_k \in L^1$ such that*

$$|h_k(t,z,v)| \leq \beta_k(t)|z|^{\sigma_k} + \gamma_k(t) \qquad (11)$$

for a.e. $t \in [0,T]$, all $z \in \mathbb{C}^m$ and all $v \in \mathbb{C}^m$ such that $|v| < a$. Then problem (10) has at least one solution.

Proof. Following Lemma 1, we introduce the homotopy

$$(\phi_k(z'))' = \lambda[\alpha_k(t) z^{n_k} + [F_k(t,z)]' + h_k(t,z,z')] \quad (k = 1,2,\ldots,m)$$
$$z(0) = z(T),\ z'(0) = z'(T) \quad (\lambda \in (0,1]). \qquad (12)$$

If $z(t) = z_0 + \widehat{z}(t)$ with $z_0 = z(0)$ is a possible solution of (12), then z' satisfies the inequality,

$$|z'|_\infty < a. \qquad (13)$$

and hence by (6) the inequality

$$|\widehat{z}|_\infty < \frac{aT}{2}. \qquad (14)$$

On the other hand, integrating both members of (12) over one period and using the periodicity gives

$$0=\int_0^T \alpha_k(t)[z_{0,k}+\widehat{z}_k(t)]^{n_k}\,dt+\int_0^T h_k[t,z_0+\widehat{z}(t),z'(t)]\,dt \quad (k=1,2,\dots,m),$$

and hence, letting $C_n^j=\frac{n!}{j!(n-j)!}$,

$$\begin{aligned}\overline{\alpha}_k z_{0,k}^{n_k} &= -T^{-1}\int_0^T\left[\sum_{j=0}^{n_k-1}C_{n_k}^j z_{0,k}^j\widehat{z}_k(t)^{n_k-j}\right]dt\\ &\quad -T^{-1}\int_0^T h_k(t,z_0+\widehat{z}(t),z'(t))\,dt \quad (k=1,\dots,m).\end{aligned}$$

Consequently, using (11), (13) and (14),

$$|\overline{\alpha}_k||z_{0,k}|^{n_k}\le\sum_{j=0}^{n_k-1}C_{n_k}^j(aT/2)^{n_k-j}|z_{0,k}|^j+\overline{\beta}_k 2^{\sigma_k}[|z_0|^{\sigma_k}+(aT/2)^{\sigma}]+\overline{\gamma}_k \quad (k=1,\dots,m). \tag{15}$$

Let $k_0\in\{1,\dots,m\}$ be such that $|z_{0,k_0}|=|z_0|$. Then, either $|z_0|<1$ or, using (15) with $k=k_0$, $|z_0|\ge 1$ and

$$\alpha|z_0|^n\le\sum_{j=0}^{N-1}C_N^j\eta(a,T)^{N-j}|z_0|^j+2^\sigma\beta[|z_0|^\sigma+\eta(a,T)^\sigma]+\gamma,$$

where

$$\alpha=\min\{|\overline{\alpha}_1|,\dots,|\overline{\alpha}_m|\},\ \beta=\max\{\beta_1,\dots,\beta_m\},\ \gamma=\max\{\gamma_1,\dots,\gamma_m\},$$
$$\sigma=\max\{\sigma_1,\dots,\sigma_m\},\ \eta(a,T)=\max\{1,aT/2\}.$$

Hence there exists $\rho>0$ depending only upon a, T, α, β and γ such that

$$|z_0|<\rho$$

which, together with (14) gives

$$|z|_\infty<\max\{1,\rho\}+\frac{aT}{2}:=R. \tag{16}$$

Thus Assumption (1) of Lemma 1 holds with $\Omega=B(R)\subset C$. System (9) can be written

$$\overline{f}_k(z_0):=\overline{\alpha}_k z_{0,k}^{n_k}+T^{-1}\int_0^T h_k(t,z_0,0)\,dt=0 \quad (k=1,\dots,m),$$

and any of its possible solution is such that either $|z_0| < 1$ or $|z_0| \geq 1$ and

$$\alpha |z_0|^n \leq \beta |z_0|^\sigma + \gamma. \qquad (17)$$

Consequently, $|z_0| < \max\{1, \rho\} < R$ and Assumption (2) of Lemma 1 is satisfied. Finally, introducing the homotopy $\mathcal{F} : \mathbb{C} \times [0,1] \to \mathbb{C}$ defined by

$$\mathcal{F}_k(z_0, \mu) = \overline{\alpha}_k z_{0,k}^{n_k} + \frac{\mu}{T} \int_0^T h_k(t, z_0, 0)\, dt \quad (k = 1, \dots, m;\ \mu \in [0,1])$$

we see that any possible solution z_0 of $\mathcal{F}(z_0, \mu) = 0$ again is such that (17) holds, so that $|z_0| < R$ and, by the homotopy invariance of Brouwer degree, with

$$p(z) = (z_1^{n_1}, z_2^{n_2}, \dots, z_m^{n_m})$$

and A is the diagonal matrix

$$A = \mathrm{diag}(\overline{\alpha}_1, \dots, \overline{\alpha}_m),$$

we obtain

$$\begin{aligned} d_B[\overline{f}, B(R), 0] &= d_B[\mathcal{F}(\cdot,1), B(R), 0] = d_B[\mathcal{F}(\cdot,0), B(R), 0] \\ &= d_B[Ap, B(R), 0] = d_B[p, B(R), 0] = n_1\, n_2 \dots n_m, \end{aligned}$$

and Assumption (3) of Lemma 1 holds. □

The special case of Theorem 1 with $m = 1$ states as follows. Consider the complex-valued periodic equation

$$(\phi(z'))' = \alpha(t) z^n + [F(t,z)]' + h(t, z, z'), \quad z(0) = z(T),\ z'(0) = z'(T), \quad (18)$$

where $\phi : B(a) \subset \mathbb{C} \to \mathbb{C}$ satisfies Assumption (H_ϕ), $n \geq 1$ is an integer, $\alpha \in L^1$, $F : [0,T] \times \mathbb{C} \to \mathbb{C}$ is of class C^1 and $h : [0,T] \times \mathbb{C}^2 \to \mathbb{C}$ is a Carathéodory function.

Corollary 1. *Assume that $\overline{\alpha} \neq 0$, and that there exist $1 \leq \sigma < n$ and $\beta, \gamma \in L^1$ such that*

$$|h(t, z, v)| \leq \beta(t) |z|^\sigma + \gamma(t)$$

for a.e. $t \in [0,T]$, all $z \in \mathbb{C}$ and all $v \in \mathbb{C}$ such that $|v| < a$. Then problem (18) has at least one solution.

Remark 1. Such a result does not hold in classical case. The problem

$$z'' = -z + \sin t, \quad z(0) = z(2\pi),\ z'(0) = z'(2\pi),$$

has no solution, as shown by multiplying each member by $\sin t$ and integrating the result over $[0, 2\pi]$.

Remark 2. Such a result does not hold in the real case. The problem

$$(\phi(u'))' = u^2 + 1, \quad u(0) = u(T),\ u'(0) = u'(T)$$

has no solution, as shown by integrating each member of the differential equation over $[0, 2\pi]$ and using the boundary conditions.

Remark 3. The periodic problem (18) is of course equivalent to a periodic problem for a system of two *real-valued* differential equation. Getting the requested a priori bounds for the solutions from the real form is less apparent, showing the help of the complex structure in their obtention.

It follows from Corollary 1 that, for any integer $n \geq 1$, any C^1 function $F : \mathbb{C} \to \mathbb{C}$ and any $h \in L^1$ the periodic problem for the Liénard-type equation

$$(\phi(z'))' = \alpha(t)z^n + [F(z)]' + h(t), \quad z(0) = z(T),\ z'(0) = z'(T),$$

has a solution when $\overline{\alpha} \neq 0$. This is in particular the case for the complex-valued relativistic van der Pol equation

$$\left(\frac{z'}{1 - |z'|^2}\right)' + (\beta + \gamma z^2)z' + \alpha z = h(t), \quad z(0) = z(T),\ z'(0) = z'(T) \tag{19}$$

when $\alpha \neq 0$, $\beta, \gamma \in \mathbb{R}$ and $h \in L^1$. When $\alpha = 0$, problem (19) has no solution when $\overline{h} \neq 0$.

Another consequence of Corollary 1 is that the problem

$$(\phi(z'))' = \alpha_n(t)z^n + \sum_{k=0}^{n-1} \alpha_k(t, z')z^k, \quad z(0) = z(T),\ z'(0) = z'(T),$$

where $n \geq 1$, $\alpha_n \in L^1$ and the $\alpha_k : [0, T] \times \mathbb{C} \to \mathbb{C}$ are Carathéodory functions $(k = 1, \ldots, n-1)$, has at least one solution if $\overline{\alpha}_n \neq 0$.

In particular, for any integer $n \geq 1$ and any $h \in L^1$, the periodic problem

$$(\phi(z'))' = \alpha(t)z^n + h(t), \quad z(0) = z(T),\ z'(0) = z'(T)$$

has a solution for any $\alpha \in L^1$ such that $\overline{\alpha} \neq 0$, and the periodic problem for the complex-valued relativistic Rayleigh equation

$$\left(\frac{z'}{1 - |z'|^2}\right)' + \beta z' + \gamma z'^3 + \alpha z = h(t), \quad z(0) = z(T),\ z'(0 = z'(T),$$

has a solution when $\alpha \neq 0$, $\beta, \gamma \in \mathbb{R}$ and $h \in L^1$.

4. The case of first order equations

Let us consider the periodic problem for first order quasilinear systems of the form

$$(\phi(u))' = f(t,u), \quad u(0) = u(T) \tag{20}$$

where $\phi : B(a) \subset \mathbb{R}^n \to \mathbb{R}^n$ satisfies Assumption (H_ϕ) and $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function. By *solution* of (20) we mean a continuous function $u : [0,T] \to B(a)$ such that $\phi \circ u \in W^{1,1}$ and equation (20) holds almost everywhere. We keep the notations of the previous sections, and define the mapping $N_f : C \to W^{1,1}$ by

$$N_f(u)(t) := \int_0^t f(s,u(s))\,ds \quad (t \in [0,T]).$$

The following result is the analog of Lemma 1 for problem (20).

Lemma 2. *Assume that the following conditions hold.*

(i) There is no solution $u_0 \in \partial B(a) \subset \mathbb{R}^n$ of equation

$$\overline{f}(u_0) := T^{-1} \int_0^T f(t,u_0)\,dt = 0.$$

(ii) $d_B[\overline{f}, B(a) \cap \mathbb{R}^n, 0] \neq 0$.

Then problem (20) has at least one solution in $B(a)$.

Proof. Let us consider the family of problems

$$(\phi(u))' = \lambda f(t,u), \quad u(0) = u(T) \quad (\lambda \in [0,1]). \tag{21}$$

We first show that, for $\lambda \in (0,1]$, problem (21) is equivalent to the fixed point problem in C

$$u(t) = \phi^{-1} \circ [\phi(u(0)) - N_f(u)(T) + \lambda N_f(u)(t)] \quad (t \in [0,T]). \tag{22}$$

Indeed, if u is a solution of (21), then by integrating the differential equation from 0 to t, and from 0 to T and using boundary conditions, we get

$$\phi(u(t)) - \phi(u(0)) - \lambda N_f(u)(t) = 0, \quad N_f(u)(T) = 0,$$

hence, both equations taking values in supplementary subspaces,

$$\phi(u(t)) = \phi(u(0)) - N_f(u)(T) + \lambda N_f(u)(t),$$

which is equivalent to (22). Conversely, if u satisfies (22), then $u \in B(a)$ (as $\phi^{-1} : \mathbb{R}^n \to B(a)$), and

$$\phi(u(t)) = \phi(u(0)) - N_f(u)(T) + \lambda N_f(u)(t) \quad (t \in [0,T]). \tag{23}$$

Differentiating, we get the differential equation in (21), taking $t = 0$ we obtain

$$N_f(u)(T) = 0, \tag{24}$$

and taking $t = T$ and using (24) we get

$$\phi(u(T)) = \phi(u(0)),$$

which is equivalent to the boundary condition in (21).

For $\lambda = 0$, equation (22) reduces to

$$u(t) = \phi^{-1} \circ [\phi(u(0)) - N_f(u)(T)] \quad (t \in [0,T])$$

which means that any solution $u = u(0)$ is constant with $u(0) \in B(a) \subset \mathbb{R}^n$ and $u(0)$ solution of (24). Conversely, the solutions of (24) in $B(a)$ are the solutions of (22) with $\lambda = 0$.

Now, the operator $\mathcal{M} : C \times [0,1] \to B(a) \subset C$ defined by

$$\mathcal{M}(u)(t) := \phi^{-1} \circ [\phi(u(0)) - N_f(u)(T) + \lambda N_f(u)(t)] \quad (t \in [0,T])$$

is easily seen to be completely continuous on C, using Arzela-Ascoli's theorem. Hence, if Assumption (i) holds, we have

$$u \neq \mathcal{M}(u,\lambda) \quad \forall\, (u,\lambda) \in \partial B(a) \times [0,1],$$

and the homotopy invariance and reduction property of Leray-Schauder degree d_{LS}, together with Brouwer degree results for homeomorphisms (see e.g. [6]), imply, with $P : C \to C \cap \mathbb{R}^n, u \mapsto u(0)$, that

$$\begin{aligned}
d_{LS}[I - \mathcal{M}(\cdot,1), B(a), 0] &= d_{LS}[I - \mathcal{M}(\cdot,0), B(a), 0] \\
&= d_{LS}[I - \phi^{-1} \circ \{\phi \circ P - N_f(\cdot)(T)\}, B(a), 0] \\
&= d_B[(I - \phi^{-1} \circ \{\phi - N_f(\cdot)(T)\})|_{\mathbb{R}^n}, B(a) \cap \mathbb{R}^n, 0] \\
&= \pm d_B[\phi \circ \{I - \phi^{-1} \circ [\phi - N_f(\cdot)(T)], B(a), 0] \\
&= \pm d_B[N_f(\cdot)(T), B(a), 0] = \pm d_B[\overline{f}, B(a), 0] \neq 0,
\end{aligned}$$

using Assumption (ii). The result follows from the existence property of Leray-Schauder's degree. □

Let us apply Lemma 2 to the periodic problem for the complex-valued differential equation

$$(\phi(z))' = \alpha(t) z^n + h(t,z), \quad z(0) = z(T) \tag{25}$$

where $\phi : B(a) \subset \mathbb{C} \to \mathbb{C}$ satisfies condition (H_ϕ), $\alpha \in L^1$, $n \geq 1$ is an integer, and $h : [0,T] \times \mathbb{C} \to \mathbb{C}$ is a Carathéodory function.

THEOREM 2. *Assume that* $\overline{\alpha} \neq 0$ *and that there exists* $0 \leq \sigma < n$ *and* $\beta \geq 0$, $\gamma \geq 0$ *such that*

(a) $\left|T^{-1} \int_0^T h(t,z)\, dt\right| \leq \beta |z|^\sigma + \gamma$ *for all* $z \in B(a) \subset \mathbb{C}$.

(b) *the unique positive root* u_0 *of equation*

$$|\overline{\alpha}| u^n = \beta u^\sigma + \gamma$$

is such that $u_0 < a$.

Then problem (25) has at least one solution z.

Proof. With the notations of Lemma 2, we have

$$\overline{f}(z_0) = \overline{\alpha} z_0^n + T^{-1} \int_0^T h(t, z_0)\, dt,$$

so that any possible zero z_0 of $\overline{f}$ is such that

$$|\overline{\alpha}||z_0|^n \leq \beta |z_0|^\sigma + \gamma, \tag{26}$$

and hence, by Assumption (b), $|z_0| < a$. Now, let us consider the homotopy

$$\mathcal{F} : \mathbb{C} \times [0,1] \to \mathbb{C},\ (z_0, \mu) \mapsto \overline{\alpha} z_0^n + \mu T^{-1} \int_0^T h(t, z_0)\, dt \quad (\mu \in [0,1]).$$

If $\mathcal{F}(z_0, \mu) = 0$, then z_0 satisfies inequality (26) and hence $|z_0| < a$. By the homotopy invariance of Brouwer degree, we get , with $p(z) := z^n$,

$$\begin{aligned} d_B[\overline{f}, B(a), 0] &= d_B[\mathcal{F}(\cdot,1), B(a), 0] = d_B[\mathcal{F}(\cdot,0), B(a), 0] \\ &= d_B[\overline{\alpha} p, B(a), 0] = d_B[p, B(a), 0] = n. \end{aligned}$$

The result follows from Lemma 2. □

COROLLARY 2. *Let* $\phi : B(a) \to \mathbb{C}$ *satisfy condition* (H_ϕ), $n \geq 1$ *be an integer and* $\alpha \in L^1$. *Then the periodic problem*

$$(\phi(z))' = \alpha(t) z^n + h(t), \quad z(0) = z(T) \tag{27}$$

has at least one solution when $\overline{\alpha} \neq 0$ *and* $|\overline{h}| < |\overline{\alpha}| a^n$.

In particular, the problem

$$\left(\frac{z}{\sqrt{1 - |z|^2}}\right)' = \alpha z^n + h(t), \quad z(0) = z(T) \tag{28}$$

has at least one solution when $\alpha \in \mathbb{C} \setminus \{0\}$ and $|\overline{h}| < |\alpha|$. This result is sharp because if (28) has a solution z, then letting

$$y = \frac{z}{\sqrt{1-|z|^2}} \quad \text{so that} \quad z = \frac{y}{\sqrt{1+|y|^2}}$$

we have

$$y' = \alpha \left(\frac{y}{\sqrt{1+|y|^2}} \right)^n + h(t), \quad y(0) = y(T).$$

Hence, taking the mean value of the differential equation and using the boundary conditions,

$$0 = \alpha T^{-1} \int_0^T \left(\frac{y(t)}{\sqrt{1+|y(t)|^2}} \right)^n dt + \overline{h},$$

which gives

$$|\overline{h}| \leq |\alpha| T^{-1} \int_0^T \left(\frac{|y(t)|}{\sqrt{1+|y(t)|^2}} \right)^n dt < |\alpha|.$$

Remark 4. A result like Corollary 2 does not hold in the classical case

$$z' = \alpha(t) z^n + h(t), \quad z(0) = z(T),$$

as shown by

$$z' = iz + e^{it}, \quad z(0) = z(2\pi)$$

which has no solution, because if it were the case, we would have

$$(e^{-it}z)' = e^{-it}z' - ie^{-it}z = 1, \quad z(0) = z(2\pi)$$

leading to a contradiction by integration over $[0, 2\pi]$.

Remark 5. By analogy with the results of Section 3, the reader will easily state and proof the extension of Theorem 2 to complex-valued systems of the form

$$(\phi_k(z))' = \alpha_k(t) z_k^{n_k} + h_k(t, z), \quad z(0) = z(T) \quad (k = 1, \dots, m).$$

References

[1] C. Bereanu and J. Mawhin, *Existence and multiplicity results for some nonlinear problems with singular ϕ-laplacian*, J. Differential Equations **243** (2007), 536–557.

[2] C. Bereanu and J. Mawhin, *Boundary value problems for some nonlinear systems with singular ϕ-laplacian*, J. Fixed Point Theory Appl. **4** (2008), 57–75.

[3] C. Bereanu and J. Mawhin, *Periodic solutions of nonlinear perturbations of ϕ-laplacian with possibly bounded ϕ*, Nonlinear Anal. **68** (2008), 1668–1681.

[4] J. Campos, *Möbius transformation and periodic solutions of complex riccati equations*, Bull. London Math. Soc. **9** (1997), 205–213.

[5] J. Campos and R. Ortega, *Nonexistence of periodic solutions of a complex riccati equation*, Differential Integral Equations **9** (1996), 247–250.

[6] K. Deimling, *Nonlinear functional analysis*, Springer, Berlin, 1985.

[7] R. Manásevich, Mawhin J., and F. Zanolin, *Periodic solutions of complex-valued differential equations and systems with periodic coefficients*, J. Differential Equations **126** (1996), 355–373.

[8] R. Manásevich, J. Mawhin, and Zanolin F., *Periodic solutions of some complex-valued liénard and rayleigh equations*, Nonlinear Anal. **36** (1999), 997–1014.

[9] J. Mawhin, *Resonance problems for some non-autonomous ordinary differential equations*, Non-autonomous differential equations, Cetraro 2011 (Berlin), CIME Lecture Notes in Math., vol. 2065, Springer, 2012, pp. 103–184.

[10] R. Srzednicki, *On periodic solutions of planar polynomial differential equations with periodic coefficients*, J. Differential Equations **114** (1994), 77–100.

[11] R. Srzednicki, *Periodic and bounded solutions in blocks for time-periodic non-autonomous ordinary differential equations*, Nonlinear Anal. **22** (1994), 707–737.

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