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On J. Deák's Construction for Quasi-uniform Extensions

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Summary. - *Let* $(X,\mathcal{U})$ *be a quasi–uniform space,* $Y \supset X$, $\mathcal{T}$ *a topology on* Y. *An extension compatible with* $(\mathcal{U},\mathcal{T})$ *is a quasi–uniformity* $\mathcal{W}$ *on* Y *such that the restriction* $\mathcal{W} \mid X$ *of* $\mathcal{W}$ *to* X *coincides with* $\mathcal{U}$ *and the topology* $\mathcal{W}^{tp}$ *induced by* $\mathcal{W}$ *equals* $\mathcal{T}$.

The paper [1] contains a construction of such extensions. The purpose of the present paper is to give some applications of the result in [1]. Without explicit mention of the contrary, we shall use the terminology and notation of [2].

1. Preliminaries

Let $\mathcal{V}$ be a quasi–uniformity on Y and suppose $\mathcal{U} \subset \mathcal{V} \mid X$. According to [1], we define, for $V \in \mathcal{V}$, $U \in \mathcal{U}$,

$$V + U = V \cup V \circ U \circ V.$$

Evidently $V + U$ is an entourage on Y, and

$$\{V + U : V \in \mathcal{V}, U \in \mathcal{U}\}$$

is a base for a quasi–uniformity $\mathcal{V} + \mathcal{U}$ on Y ([1], Lemma 1.5). By [1], Lemma 1.4, we have $(\mathcal{V} + \mathcal{U}) \mid X = \mathcal{U}$ and, by [1], Lemma 1.2, $(\mathcal{V} + \mathcal{U})^{tp} = \mathcal{V}^{tp}$ provided the traces on X of the $\mathcal{V}^{tp}$–neighbourhood filters of the points of Y are $\mathcal{U}$–round (if a point does not belong to the $\mathcal{V}^{tp}$–closure of X then this trace coincides with the zero filter

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exp X); a filter $\mathfrak{s}$ on X is said to be $\mathcal{U}$–round iff $S \in \mathfrak{s}$ implies the existence of $S' \in \mathfrak{s}$ and $U \in \mathcal{U}$ such that $U(S') \subset S$. Therefore, in this case, $\mathcal{V}+\mathcal{U}$ is an extension compatible with $(\mathcal{U}, \mathcal{V}^{tp})$. Moreover, if $\mathcal{W}$ is a $(\mathcal{U}, \mathcal{T})$–compatible extension coarser than $\mathcal{V}$, then $\mathcal{W} \subset \mathcal{V}+\mathcal{U}$ ([1], Lemma 1.6).

It is not difficult to prove with the help of these results:

Theorem 1.1. *([1], Theorem 1.7). Let $(X,\mathcal{U})$ be a quasi–uniform space, $Y \supset X$, $\mathcal{T}$ a topology on Y. There exists a $(\mathcal{U}, \mathcal{T})$–compatible extension iff $\mathcal{U}^{tp} = \mathcal{T} \mid X$, the trace on X of the $\mathcal{T}$–neighbourhood filters of the points of Y are $\mathcal{U}$–round, and $\mathcal{U} \subset \mathcal{V} \mid X$ for the fine quasi–uniformity $\mathcal{V}$ of $\mathcal{T}$.*

We shall establish similar existence theorems for special classes of quasi–uniformities.

We first consider transitive quasi–uniformities.

Lemma 1.2. *Let U and V be transitive entourages on X and $Y \supset X$, respectively, and suppose $V \mid X = V \cap (X \times X) \subset U$. Then $V + U$ is a transitive entourage on Y.*

Proof. Assume (a,b), $(b,c) \in V + U = V \cup (V \circ U \circ V)$. If (a,b), $(b,c) \in V$ then clearly $(a,c) \in V \subset V + U$. If $(a,b) \in V$, $(b,c) \in V \circ U \circ V$ then there are $x, y \in X$ such that $(b,x) \in V$, $(x,y) \in U$, $(y,c) \in V$. Hence $(a,x) \in V$ and $(a,c) \in V \circ U \circ V \subset V + U$.

The case $(a,b) \in V \circ U \circ V$, $(b,c) \in V$ is similar.

Finally if (a,b), $(b,c) \in V \circ U \circ V$ then there are $x,y,z,u \in X$ such that

$$(a,x) \in V,\ (x,y) \in U,\ (y,b) \in V,\ (b,z) \in V,\ (z,u) \in U,\ (u,c) \in V.$$

Therefore $(y,z) \in V \mid X \subset U$ by hypothesis, hence $(x,u) \in U$, and $(a,c) \in V \circ U \circ V \subset V + U$. □

Theorem 1.3. *Let $(X,\mathcal{U})$ be a transitive quasi–uniform space, $Y \supset X$, $\mathcal{T}$ a topology on Y. There exists a transitive $(\mathcal{U}, \mathcal{T})$–compatible extension iff $\mathcal{U}^{tp} = \mathcal{T}$, the traces on X of the $\mathcal{T}$–neighbourhood filters of the points of Y are $\mathcal{U}$–round, and $\mathcal{U} \subset \mathcal{V} \mid X$ for the fine transitive quasi–uniformity $\mathcal{V}$ of $\mathcal{T}$.*

Proof. The necessity is evident, and the hypotheses imply that the $(\mathcal{U},\mathcal{T})$–compatible extension $\mathcal{V}+\mathcal{U}$ is transitive: the entourages $V+U$ such that $V \in \mathcal{V}$, $U \in \mathcal{U}$ and $V \mid X \subset U$ clearly constitute a base for $\mathcal{V}+\mathcal{U}$, so that 1.2 applies. □

COROLLARY 1.4. *Under the above hypotheses, $\mathcal{V}+\mathcal{U}$ is the finest transitive $(\mathcal{U},\mathcal{T})$–compatible extension.*

We next turn to the case of totally bounded spaces.

Let $\mathfrak{r}$ be a cover of the set X, and define

$$U(\mathfrak{r}) = \{(x,y) \in X \times X : y \in \cap\{C \in \mathfrak{r} : x \in C\}\}.$$

Then $U(\mathfrak{r})$ is an entourage on X, and clearly $U(\mathfrak{r}) \mid X_0 = U(\mathfrak{r} \mid X_0)$ for $X_0 \subset X$ and $\mathfrak{r} \mid X_0 = \{C \cap X_0 : C \in \mathfrak{r}\}$. It is easily seen (and well–known) that, it $\mathcal{T}$ is a topology on X, the entourages $U(\mathfrak{r})$, where $\mathfrak{r}$ runs over the finite open covers of X satisfying $X \in \mathfrak{r}$, constitute a base for the Pervin quasi–uniformity of $\mathcal{T}$, i.e. (see [2], 2.2) the finest totally bounded quasi–uniformity inducing $\mathcal{T}$.

LEMMA 1.5. *Let $\mathcal{T}$ be a topology on $Y \supset X$, $\mathcal{T}_0 = \mathcal{T} \mid X$, $\mathcal{P}$ and $\mathcal{P}_0$ the Pervin quasi–uniformities of $\mathcal{T}$ and $\mathcal{T}_0$, respectively. Then $\mathcal{P}_0 = \mathcal{P} \mid X$.*

Proof. If $\mathfrak{r}$ runs over the finite $\mathcal{T}$–open covers of Y then $\mathfrak{r} \mid X$ runs over the finite $\mathcal{T}_0$–open covers of X. □

THEOREM 1.6. *Let $(X,\mathcal{U})$ be a totally bounded quasi–uniform space, $\mathcal{T}$ a topology on $Y \supset X$. There exists a totally bounded $(\mathcal{U},\mathcal{T})$–compatible extension iff $\mathcal{U}^{tp} = \mathcal{T} \mid X$ and the traces on X of the $\mathcal{T}$–neighbourhood filters of the points in Y are $\mathcal{U}$–round.*

Proof. The necessity is contained in Theorem 1.1. Sufficiency: if $\mathcal{P}$ is the Pervin quasi–uniformity of $\mathcal{T}$, then $\mathcal{P} \mid X$ is the Pervin quasi–uniformity on $\mathcal{U}^{tp}$ by 2.4. Hence $\mathcal{U} \subset \mathcal{P} \mid X$ and $\mathcal{P}+\mathcal{U}$ is a $(\mathcal{U},\mathcal{T})$–compatible extension. By [1](5a), we have $\mathcal{P}+\mathcal{U} \subset \mathcal{P}$, so that $\mathcal{P}+\mathcal{U}$ is totally bounded. □

COROLLARY 1.7. *Under the above hypotheses, $\mathcal{P}+\mathcal{U}$ is the finest totally bounded $(\mathcal{U},\mathcal{T})$–compatible extension.*

References

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