How Many Closed Structures does the Construct PRAP Admit?

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Summary. - *We will prove that the topological construct* **PRAP**, *introduced by E. and R. Lowen in [9] as a numerification supercategory of the construct* **PRTOP** *of convergence spaces and continuous maps, admits a proper class of monoidal closed structures. We will even show that under the assumption that there does not exist a proper class of measurable cardinals, it admits a proper conglomerate (i.e. one which is not codable by a class) of mutually non-isomorphic monoidal closed structures. This severely contrasts with the situation concerning symmetric monoidal closed structures, because it is shown in [13] that* **PRAP** *only admits one symmetric tensorproduct, up to natural isomorphism.*

1. Preliminaries on PRAP

In this paper we will be concerned with the topological construct **PRAP** of *pre-approach spaces* and *contractions*, which was introduced in [9] as a quantification of the well-known construct of pre-

topological spaces and continuous maps. The objective of approach theory is exactly studying such categorically well-behaved numerifying supercategories, i.e. (topological) superconstructs which contain at the same time their classical topological counterpart fully bireflectively and bicoreflectively and some category of metric objects and non-expansive maps fully bicoreflectively only. This allows for an intrinsically quantified theory, completely encompassing the corresponding topological one thanks to the initial and final closedness for the class of all "old" objects in the bigger construct and which at the same time allows the formation of canonical quantified product structures for abitrarily large sets of metric objects, also thanks to the categorical requirements stated above. For any further motivation and information about approach theory, we refer to [9] and [10]. For completeness, we briefly recall the definition of a pre-approach space. Let X be a set. We will write $\mathbf{F}(X)$ for the set of all filters on X. If $\mathcal{B}$ is a filterbase on X, stack$\mathcal{B}$ denotes the filter on X it generates and for all $x \in X$, $\dot{x} := \text{stack}\{\{x\}\}$. For a subset A of X, we denote the function on X, taking the value 0 on A and ∞ on $X \setminus A$ by θ_A. A *pre-approach limit* on X is a map $\lambda : \mathbf{F}(X) \to [0, \infty]^X$ satisfying the following properties:

(L1) $\forall x \in X : \lambda(\dot{x})(x) = 0$,

(L2) for every set-indexed family $(\mathcal{F}_i)_{i \in I}$ in $\mathbf{F}(X)$:

$$\lambda \left(\bigcap_{i \in I} \mathcal{F}_i \right) = \sup_{i \in I} \lambda(\mathcal{F}_i)$$

and the pair (X, λ) is called a *pre-approach space*. (Note that (L2) implies that $\forall \mathcal{F}, \mathcal{G} \in \mathbf{F}(X) : \mathcal{F} \subset \mathcal{G} \Rightarrow \lambda(\mathcal{F}) \geq \lambda(\mathcal{G})$.) If (X', λ') also is a pre-approach space, a function $f : X \to X'$ is called a *contraction* if

$$\forall \mathcal{F} \in \mathbf{F}(X) : \lambda'(\text{stack} f(\mathcal{F})) \circ f \leq \lambda(\mathcal{F}).$$

It was then shown in [9] that pre-approach spaces and contractions together constitute a topological construct **PRAP**, i.e. one which is at the same time initially and finally complete, into which the topological construct **PRTOP** of pre-topological convergence spaces and continuous maps (resp. pqs**MET**$^\infty$ of extended pseudo-quasi-semi-

metric spaces and non-expansive maps) are embedded as full sub-constructs and we refer to that paper for more information and for alternative descriptions of pre-approach spaces, e.g. through approach systems. For the sake of completeness, let us just recall that for an arbitrary pre-topological convergence space (X, q), the corresponding pre-approach limit λ_q is determined by $\lambda_q(\mathcal{F}) := \theta_{\{x \in X | \mathcal{F} \in q(x)\}}$. To simplify notations, we will use single characters like X to denote the objects of the considered constructs and a notation like $\underline{X}$ and λ_X for the underlying set and the additional structure on it, if no confusion can occur.

2. Monoidal Closed Structures on PRAP

Monoidal closed structures in the sense of S. Eilenberg and G. M. Kelley on a category $\mathbf{A}$ are the essential basic notion one needs for doing any kind of categorically founded algebra inside the category $\mathbf{A}$. They are at the same time also the right context in which to study function spaces and to work with them, e.g. to look for Ascoli-type theorems or other settings where one needs to consider $\mathbf{A}$-enriched situations. Especially when the category $\mathbf{A}$ fails to be cartesian closed, i.e. if the categorical product functor $- \times -$ cannot be augmented to a monoidal closed structure, it is useful to look for other alternative modified tensorproducts. This is exactly the case with **PRAP**. For any categorical information we refer to [1], [3], [8]. It was already shown in [13] that the category **PRAP** admits only one such *symmetric* tensorproduct, adjoint on the left to the inner hom-functor determined by the structure of "pointwise convergence". It however follows from the approach taken by G. Greve in [5] to construct non-symmetric tensorproducts, that giving up symmetry here is by no means unnatural and indeed very useful: it helps to capture exactly that part of decent exponential behaviour present in the considered construct into the definition of the tensorproducts, in the following way. Let $\mathbf{A}$ be a topological construct. An object $B \in |\mathbf{A}|$ is called *exponential* if and only if the functor $- \times B$ has a right adjoint $-^B$, which in this particular case always can be taken to be a structured covariant hom-functor. If $\mathcal{B} \subset |\mathbf{A}|$ is a (non-empty) finitely productive class of exponential objects, it follows from theo-

rem 1.3 in G. Greve [5] that there exists a monoidal closed structure $(- \otimes_\mathcal{B} -, H_\mathcal{B}(-,-))$ on $\mathbf{A}$ with

$$\forall B \in \mathcal{B} : - \otimes_\mathcal{B} B = - \times B.$$

Such a monoidal closed structure on $\mathbf{A}$ is defined in the following way. For every $X, Y \in |\mathbf{A}|$, the object $X \otimes_\mathcal{B} Y$ is the final $\mathbf{A}$-object for the sink formed by

$$\{f \times g : A \times B \to \underline{X} \times \underline{Y} \mid A \in |\mathbf{A}|, B \in \mathcal{B}, f \in \mathbf{A}(A, X), g \in \mathbf{A}(B, Y)\}$$
$$\cup \{Y \to \underline{X} \times \underline{Y} : y \mapsto (x, y) \mid x \in X\}$$

and $- \otimes_\mathcal{B} -$ acts on morphisms by taking the cartesian product of the underlying maps. It will also be useful to work with the corresponding adjoint inner hom-functor $H_\mathcal{B}(-,-)$. If $X, Y \in |\mathbf{A}|$, $H_\mathcal{B}(X, Y)$ is the $\mathbf{A}$-object with $\mathbf{A}(X, Y)$ as underlying set carrying the initial structure for the source

$$(\mathbf{A}(f, g) : \mathbf{A}(X, Y) \to A^B : h \mapsto g \circ h \circ f)_{A \in |\mathbf{A}|, B \in \mathcal{B}, f \in \mathbf{A}(B, X), g \in \mathbf{A}(Y, A)}.$$

The action of $H_\mathcal{B}(-,-)$ on pairs of morphisms is defined to be the one of the hom-functor on the underlying functions.

The exponential objects in **PRAP** have been identified by E. Lowen-Colebunders, R. Lowen and F. Verbeeck in [11]:

THEOREM 2.1. *[11] The exponential objects in the construct* **PRAP** *are the* ∞pqs*-metric spaces.*

We will make special use of the fact that a particular pre-approach space $\mathbb{P}^\#$ which was defined in [9] was proved there to be initially dense in **PRAP**, meaning that for all $X \in |\mathbf{PRAP}|$, the source

$$(f : X \to \mathbb{P}^\#)_{f \in \mathbf{PRAP}(X, \mathbb{P}^\#)}$$

is initial in **PRAP**. This object $\mathbb{P}^\#$ is the perfect pre-approach analogue of the pre-topological space with three points and one non-trivial neighborhood, i.e. the pre-topological space **3** with $\{0, 1, 2\}$ as its underlyng set and where the neighborhood system is defined by $\mathcal{V}_\mathbf{3}(0) = \mathcal{V}_\mathbf{3}(2) := \dot{0} \cap \dot{1} \cap \dot{2}$ and $\mathcal{V}_\mathbf{3}(1) := \dot{1} \cap \dot{2}$. The only symmetric monoidal closed structure on **PRAP** is $(- \otimes -, [-, -])$ as defined

in [13]. Let us just recall how this SMC structure is defined: if $X, Y, Z \in |\mathbf{PRAP}|$, we call a map $f : \underline{X} \times \underline{Y} \to \underline{Z}$ *bi-contractive* if for all $x \in \underline{X}$ and all $y \in \underline{Y}$, both

$$f(x, \cdot) : Y \to Z : t \mapsto f(x, t)$$

and

$$f(\cdot, y) : X \to Z : s \mapsto f(s, y)$$

are contractive. For $X, Y \in |\mathbf{PRAP}|$, the object $X \otimes Y$ is defined to be the initial lift for the source

$$(f : \underline{X} \times \underline{Y} \to Z)_{Z \in \mathbf{PRAP}, f \text{ bi-contractive}}.$$

This definition clearly is functorial in the obvious way, and remains unchanged if we restrict ourselves to only taking $Z = \mathbb{P}^{\#}$. (This way of defining a tensorproduct works in fact in any topological construct, and is due to J. Činčura [15].) The right corresponding inner hom-functor is then obtained by putting $[X, Y]$ to be the $\mathbf{PRAP}$ object one gets by endowing $\mathbf{PRAP}(X, Y)$ with the pointwise functionspace structure, i.e. the one it enherits as a subspace of the $\mathbf{PRAP}$ product $\prod_{s \in \underline{X}} Y$. It was subsequently proved in [13] that for all $(x, y) \in \underline{X} \times \underline{Y}$, the approach system $\mathcal{A}_{X \otimes Y}((x, y))$ is generated by

$$\{\underline{X} \times \underline{Y} \to [0, \infty] : (s, t) \mapsto (\varphi(s) \vee \theta_{\{y\}}(t)) \wedge (\theta_{\{x\}}(s) \vee \psi(t)) \mid$$
$$\varphi \in \mathcal{A}_X(x), \psi \in \mathcal{A}_Y(y)\}$$

For every $Y \in |\text{pqs}\mathbf{MET}^{\infty}|$, the right adjoint $-^Y$ of $- \times Y$ is completely determined by $Z^Y := (\mathbf{PRAP}(Y, Z), \lambda_c)$ with

$$\lambda_c(\Psi)(f) :=$$
$$\inf\{K \in [0, \infty] \mid \forall \varepsilon > 0, \forall y \in \underline{Y} : \lambda_Z(\text{stack}\Psi(B_Y(y, \varepsilon)))(f(y)) \leq K \vee \varepsilon\}$$
$$f \in \mathbf{PRAP}(Y, Z), \Psi \in \mathbf{F}(\mathbf{PRAP}(Y, Z))$$

for all $Z \in |\mathbf{PRAP}|$. Here $\text{stack}\Psi(B_Y(y, \varepsilon))$ denotes the filter on $\underline{Z}$ generated by

$$\{\{f(t) \mid f \in \mathcal{F}, t \in B_Y(y, \varepsilon)\} \mid \mathcal{F} \in \Psi\}.$$

We refer to [9, 11] for more details and recall that the formula above was introduced in [9] in the more general form

$$\lambda_c(\Psi)(f) = \min\{\, K \in [0, \infty] \mid \forall \mathcal{G} \in \mathbf{F}(\underline{Y}) :$$
$$\lambda_Z(\mathrm{stack}\Psi(\mathcal{G})) \circ f \leq K \vee \lambda_Y(\mathcal{G}) \,\}$$

which will be usefull in the sequell. Here $\mathrm{stack}\Psi(\mathcal{G})$ stands for the filter on $\underline{Z}$ generated by

$$\{\, \{f(t) \mid f \in \mathcal{F}, t \in G\} \mid \mathcal{F} \in \Psi, G \in \mathcal{G} \,\}.$$

We will first prove that there are "enough" mutually not naturally isomorphic tensorproducts on **PRAP**, by adapting the argument used in [12] for the pre-topological case.

THEOREM 2.2. *There exist at least as many (not naturally isomorphic) monoidal closed structures on* **PRAP** *as there are infinite cardinal numbers, whence a proper class.*

Proof. If α is an infinite cardinal number, we will write $\mathcal{F}_\alpha$ for the class of all finitely generated pre-topological convergence spaces (i.e. pre-topological convergence spaces such that every point has a smallest neighborhood) of which the cardinality of the underlying set is less than or equal to α. We recall from [11] that $|\mathrm{pqs}\mathbf{MET}^\infty| \cap |\mathbf{PRTOP}| = |\mathbf{FING}|$ in $|\mathbf{PRAP}|$ (where **FING** denotes the full subconstruct of **PRTOP** formed by all finitely generated objects), so all the $\mathcal{F}_\alpha$ consist of exponential objects in and therefore give rise to monoidal closed structures $(- \otimes_{\mathcal{F}_\alpha} -, H_{\mathcal{F}_\alpha}(-, -))$ on **PRAP**. Now fix $\alpha < \beta$ infinite cardinal numbers. It is now our intention to prove that the monoidal closed structures $(- \otimes_{\mathcal{F}_\alpha} -, H_{\mathcal{F}_\alpha}(-, -))$ and $(- \otimes_{\mathcal{F}_\beta} -, H_{\mathcal{F}_\beta}(-, -))$ are not naturally isomorphic. Next define X^β to be the pre-topological convergence space used in [12]: therefore we fix two sets $\emptyset \neq A \subset B$ with $\mathrm{Card}(B) = \beta$ and $\mathrm{Card}(A) \leq \alpha$ and a point $b \in B \setminus A$. Define a pre-topological neighborhoodsystem on B by

$$\mathcal{V}^\beta(x) := \begin{cases} \dot{x} & \text{if } x \in B \setminus \{b\}, \\ \mathrm{stack}\{B \setminus A\} & \text{if } x = b \end{cases}$$

and put $X^\beta := (B, \mathcal{V}^\beta)$. We are done if we show that $H_{\mathcal{F}_\alpha}(X^\beta, \mathbf{3},) \neq H_{\mathcal{F}_\beta}(X^\beta, \mathbf{3})$. Because $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$, it is clear that the former is coarser than the latter, so we have to show that

$$1_{\mathbf{PRAP}(X^\beta, \mathbf{3})} : H_{\mathcal{F}_\alpha}(X^\beta, \mathbf{3}) \to H_{\mathcal{F}_\beta}(X^\beta, \mathbf{3})$$

is not a contraction. By construction of the space $H_{\mathcal{F}_\beta}(X^\beta, \mathbf{3})$ by means of initial lifting, this means we have to look for some particular choice of Z, F, f and g for which

$$H_{\mathcal{F}_\alpha}(X^\beta, \mathbf{3}) \to Z^F : k \mapsto g \circ k \circ f$$

fails to be a contraction. By choosing $Z := \mathbf{3}, F := X^\beta, f := 1_{X^\beta}$ and $g := 1_{\mathbf{3}}$, we are done if we can show that $H_{\mathcal{F}_\alpha}(X^\beta, \mathbf{3})$ is not finer than $\mathbf{3}^{X^\beta}$.

We write $f_1 : X^\beta \to \mathbf{3}$ for the constant 1-function (which clearly is a contraction) and for each $S \subset B$ with $\mathrm{Card}(S) \leq \alpha$, we define

$$\Gamma_S := \{l \in \mathbf{PRAP}(X^\beta, \mathbf{3}) = \mathbf{PRTOP}(X^\beta, \mathbf{3}) \mid l(S) \subset \{1, 2\} \text{ and}$$
$$0 \in l(B \setminus (A \cup S))\}.$$

Clearly, $\Gamma_S \neq \emptyset$ and $\Gamma_{S \cup T} \subset \Gamma_S \cap \Gamma_T$ for all subsets S, T of B with cardinality less than or equal to α, so $\{\Gamma_S \mid S \subset B, \mathrm{Card}(S) \leq \alpha\}$ generates a filter Ψ_α on $\mathbf{PRAP}(X^\beta, \mathbf{3})$ for which

$$\mathrm{stack}\Psi_\alpha(\mathcal{V}^\beta(b)) \not\in q_{\mathbf{3}}(1),$$

where by $\mathrm{stack}\Psi_\alpha(\mathcal{V}^\beta(b))$ we mean the filter on $\{0, 1, 2\}$ generated by

$$\{\{f(t) \mid f \in \Gamma_S, t \in V\} \mid S \subset B, \mathrm{Card}(S) \leq \alpha, V \in \mathcal{V}^\beta(b)\}.$$

Remember that $q.$ denotes the pre-topological convergence structure itself. So we obtain that, because $\mathcal{V}^\beta(b) \in q_{X^\beta}(b)$,

$$\lambda_{\mathbf{3}^{X^\beta}}(\Psi_\alpha)(f_1) \geq \inf\{K \in [0, \infty] \mid$$
$$\lambda_{\mathbf{3}}(\mathrm{stack}\Psi_\alpha(\mathcal{V}^\beta(b))(f_1(b)) \leq K \vee \lambda_{X^\beta}(\mathcal{V}^\beta(b))(b)\} = \infty,$$

whence $\lambda_{\mathbf{3}^{X^\beta}}(\Psi_\alpha)(f_1) = \infty$. We are done if we can prove that

$$\lambda_{H_{\mathcal{F}_\alpha}(X^\beta, \mathbf{3})}(\Psi_\alpha)(f_1) \in \mathbb{R}^+.$$

To do so, we fix $F^\alpha \in \mathcal{F}_\alpha$, $Z \in |\mathbf{PRAP}|$ and morphisms $f : F^\alpha \to X^\beta$, $g : \mathbf{3} \to Z$. Let us write $C : \mathbf{PRAP} \to \mathbf{PRTOP}$ for the concrete coreflector (which is the forgetful functor introduced in [9].) Because $\mathbf{3}$ is a pre-topological pre-approach object, saying that $g : \mathbf{3} \to Z$ is a contraction, means exactly that $g : \mathbf{3} \to CZ$ is continuous. For every $p \in F^\alpha, \mathcal{H} \in q_{F^\alpha}(p)$, it follows from the continuity of $g : \mathbf{3} \to CZ$, the construction of Ψ_α and the fact that $\mathrm{Card}(f(F^\alpha)) \leq \mathrm{Card}(F^\alpha) \leq \alpha$, that $\mathrm{stack}(\mathbf{PRAP}(f,g)(\Psi_\alpha))(\mathcal{H})$ converges to $g(1)$ in CZ, whence $\lambda_Z(\mathrm{stack}(\mathbf{PRAP}(f,g)(\Psi_\alpha))(\mathcal{H}))((g \circ f_1 \circ f)(p)) = 0$ On the other hand, since λ_{F^α} only takes values in $\{0, \infty\}$ because F^α is a pre-topological object, it is clear that

$$
\begin{aligned}
& \lambda_{Z^{F^\alpha}}(\mathrm{stack}\mathbf{PRAP}(f,g)(\Psi_\alpha))(g \circ f_1 \circ f) \\
= \ & \min\{ K \in [0,\infty] \mid \forall \mathcal{H} \in \mathbf{F}(\underline{F^\alpha}) : \\
& \lambda_Z(\mathrm{stack}(\mathbf{PRAP}(f,g)(\Psi_\alpha))(\mathcal{H})) \circ (g \circ f_1 \circ f) \leq K \vee \lambda_{F^\alpha}(\mathcal{H}) \} \\
= \ & \min\{ K \in [0,\infty] \mid \forall p \in \underline{F^\alpha}, \forall \mathcal{H} \in q_{F^\alpha}(p) : \\
& \lambda_Z(\mathrm{stack}(\mathbf{PRAP}(f,g)(\Psi_\alpha))(\mathcal{H}))((g \circ f_1 \circ f)(p) \leq K\} \\
= \ & 0.
\end{aligned}
$$

We thus have obtained that

$$
\begin{aligned}
& \lambda_{H_{\mathcal{F}_\alpha}(X^\beta,\mathbf{3})}(\Psi_\alpha)(f_1) \\
= \ & \sup_{F^\alpha \in \mathcal{F}_\alpha} \sup_{Z \in |\mathbf{PRAP}|} \sup_{f \in \mathbf{PRAP}(F^\alpha,X^\beta)} \sup_{g \in \mathbf{PRAP}(\mathbf{3},Z)} \\
& \lambda_{Z^{F^\alpha}}(\mathrm{stack}\mathbf{PRAP}(f,g)(\Psi_\alpha))(g \circ f_1 \circ f) \\
= \ & 0.
\end{aligned}
$$

and this concludes the proof. $\qquad\square$

Next we will try to improve on this result, under the (not very strong) set-theoretical hypothesis

(H) "There does not exist a proper class of measurable cardinals."

To do this we will use another technique for building proper conglomerates of tensorproducts, given a strongly rigid class of objects. Let $\mathbf{A}$ be a topological construct. We will call a class $\mathcal{C} \subset |\mathbf{A}|$ *strongly rigid* if it contains no singleton sets and if for every $X, Y \in \mathcal{C}$, the set $\mathbf{A}(X,Y)$ consists only of identity maps or constant maps. Finally

a result of V. Trnková from [14] will ensure the existence of such a "large enough" strongly rigid class of metric spaces under (H).

THEOREM 2.3. *([14], Corollary c), p. 283) Under the assumption (H), there exists a proper class of metric spaces which is strongly rigid in* **MET**.

Note that because **MET** is embedded in **AP** as a full subcategory, a strongly rigid class in **MET** is strongly rigid in **AP** as well. We write $\mathbf{PRAP}_0$ for the full subcategory of **PRAP** defined by the object class

$$\{ X \in |\mathbf{PRAP}| \mid \mathbf{PRAP}(X, \mathbb{P}^{\#}) \text{ separates points} \}.$$

Note that obviously $|\mathbf{MET}| \cup \{\mathbb{P}^{\#}\} \subset |\mathbf{PRAP}_0|$ and that $|\mathbf{PRAP}_0|$ is closed with respect to taking finer structures.

LEMMA 2.4. *For every* $X \in |\mathbf{MET}|$, *the following assertions are equivalent:*
   (1) $[X, \mathbb{P}^{\#}] = (\mathbb{P}^{\#})^X$,
   (2) $\forall Y \in |\mathbf{PRAP}_0| : Y \otimes X = Y \times X$,
   (3) $\mathrm{Card}(\underline{X}) = 1$,
   (4) $\forall Y \in |\mathbf{PRAP}| : Y \otimes X = Y \times X$

Proof. $\boxed{(1) \Rightarrow (2)}$ From the initial density of $\mathbb{P}^{\#}$ in **PRAP**, and the fact that both $[X, -]$ and $-^X$ preserve initial mono-sources in **PRAP** because they are both right-adjoint (to resp. $-\otimes X$ and $-\times X$) and because in the topological construct **PRAP** the initial embeddings coincide with the regular monomorphisms, it follows that for each $Y \in |\mathbf{PRAP}_0|$, both of the sources

$$(f \circ - : [X, Y] \to [X, \mathbb{P}^{\#}])_{f \in \mathbf{PRAP}(Y, \mathbb{P}^{\#})}$$

and

$$(f \circ - : Y^X \to (\mathbb{P}^{\#})^X)_{f \in \mathbf{PRAP}(Y, \mathbb{P}^{\#})}$$

are initial in **PRAP**. Therefore, (1) implies that $[X, -] = -^X$ on $\mathbf{PRAP}_0$. It is now easy to see that restricting the argument $-$ to $\mathbf{PRAP}_0$ we obtain well defined functors with values in $\mathbf{PRAP}_0$

together with adjoint situations (for these restricted functors) $(- \otimes X, [X, -])$ and $(- \times X, -^X)$, so we obtain (2).

$\boxed{(2) \Rightarrow (3)}$ Suppose that X contains at least two different points x_1, x_2. Then $K := d_X(x_1, x_2) \in \mathbb{R}_0^+$.

It now follows from [13] that

$$\varphi := ((d_X(x_1, \cdot) \circ \mathrm{pr}_1) \vee (\theta_{\{x_1\}} \circ \mathrm{pr}_2)) \wedge ((\theta_{\{x_1\}} \circ \mathrm{pr}_1) \vee (d_X(x_1, \cdot) \circ \mathrm{pr}_2))$$
$$\in \mathcal{A}_{X \otimes X}((x_1, x_1)),$$

where pr_1 and $\mathrm{p}r_2$ stand for the projection from $\underline{X} \times \underline{X}$ to $\underline{X}$ onto the first and the second co-ordinate. Because $\varphi((x_2, x_2)) = \infty$, $K < \infty$ and the fact that

$$\mathcal{A}_{X \times X}((x_1, x_1)) = \langle \{(d_X(x_1, \cdot) \circ \mathrm{pr}_1) \vee (d_X(x_1, \cdot) \circ \mathrm{pr}_2)\} \rangle$$

it is impossible for $K + 1 < L < \infty$, to find $\psi \in \mathcal{A}_{X \times X}((x_1, x_1))$ such that $\varphi \wedge L \leq 1 + \psi$, yielding that

$$\varphi \notin \mathcal{A}_{X \times X}((x_1, x_1))$$

contradicting (2).

$\boxed{(3) \Rightarrow (4)}$ Obvious because singleton spaces are identities with respect to $\times$ and $\otimes$.

$\boxed{(4) \Rightarrow (1)}$ Again obvious from the fact that

$$(- \otimes X, [X, -]) \text{ and } (- \times X, -^X)$$

are pairs of adjoint functors. $\qquad \square$

LEMMA 2.5. *Let $\mathcal{C} \subset |\mathbf{MET}|$ be a strongly rigid class and write $\mathcal{C}_{\mathrm{fin}}^\times$ for the class of all finite products of $\mathcal{C}$-objects. Then any contraction between two $\mathcal{C}_{\mathrm{fin}}^\times$-objects is either constant or a projection on co-ordinates.*

Proof. First note that because $\mathbf{MET}$ is finitely productive in $\mathbf{PRAP}$, $\mathcal{C}_{\mathrm{fin}}^\times$ still consists of exponential objects in $\mathbf{PRAP}$. The proof here is exactly the same one, mutatis mutandis, as the proof of lemma 2.3 in [5], because for the case $X = Y = Z$ which is treated in [6], only Hausdorffness is required. $\qquad \square$

LEMMA 2.6. *If $\mathcal{C} \subset |\mathbf{MET}|$ is a proper strongly rigid class of non-singleton spaces, then there are at least as many mutually not naturally isomorphic closed structures on $\mathbf{PRAP}$ as there are non-empty subclasses of $\mathcal{C}$, whence a conglomerate which is not codable by a class.*

Proof. Take $\emptyset \neq \mathcal{D} \subsetneq \mathcal{E} \subset \mathcal{C}$. Applying the technique developed in [5], we now obtain two monoidal closed structures

$$(- \otimes_{\mathcal{D}_{\mathrm{fin}}^{\times}} -, H_{\mathcal{D}_{\mathrm{fin}}^{\times}}(-,-)) \text{ and } (- \otimes_{\mathcal{E}_{\mathrm{fin}}^{\times}} -, H_{\mathcal{E}_{\mathrm{fin}}^{\times}}(-,-))$$

on $\mathbf{PRAP}$. Now fix

$$E \in \mathcal{E} \setminus \mathcal{D}.$$

We are done if we show that

$$H_{\mathcal{D}_{\mathrm{fin}}^{\times}}(E, \mathbb{P}^{\#}) \neq H_{\mathcal{E}_{\mathrm{fin}}^{\times}}(E, \mathbb{P}^{\#}).$$

By definition

$$(\mathbf{PRAP}(f,g) : H_{\mathcal{D}_{\mathrm{fin}}^{\times}}(E, \mathbb{P}^{\#}) \to A^{B} :$$

$$k \mapsto g \circ k \circ f)_{A \in |\mathbf{PRAP}|, B \in \mathcal{D}_{\mathrm{fin}}^{\times}, f \in \mathbf{PRAP}(B,E), g \in \mathbf{PRAP}(\mathbb{P}^{\#}, A)}$$

is initial in $\mathbf{PRAP}$. On the other hand, 2.5 and the fact that $E \notin \mathcal{D}$ imply that $\mathbf{PRAP}(B, E)$ consists of only constant maps, so this boils down to initiality of the source

$$(H_{\mathcal{D}_{\mathrm{fin}}^{\times}}(E, \mathbb{P}^{\#}) \to A : k \mapsto g(k(e)))_{A \in |\mathbf{PRAP}|, e \in E, g \in \mathbf{PRAP}(\mathbb{P}^{\#}, A)},$$

is initial in $\mathbf{PRAP}$. Because initiality of composite sources is inherited by the first factor, we obtain that

$$(H_{\mathcal{D}_{\mathrm{fin}}^{\times}}(E, \mathbb{P}^{\#}) \to \mathbb{P}^{\#} : k \mapsto k(e))_{e \in E},$$

is initial in $\mathbf{PRAP}$ and hence

$$H_{\mathcal{D}_{\mathrm{fin}}^{\times}}(E, \mathbb{P}^{\#}) = [E, \mathbb{P}^{\#}]$$

(because $[E, \mathbb{P}^{\#}]$ is, as shown in [13], precisely the $\mathbf{PRAP}$ object we get by equipping $\mathbf{PRAP}(E, \mathbb{P}^{\#})$ with the pointwise or product

structure). On the other hand, the definition of $H_{\mathcal{E}_{\text{fin}}^{\times}}$ implies that $H_{\mathcal{E}_{\text{fin}}^{\times}}(E, \mathbb{P}^{\#})$ is finer than $(\mathbb{P}^{\#})^E$. If now the equality

$$H_{\mathcal{D}_{\text{fin}}^{\times}}(E, \mathbb{P}^{\#}) = H_{\mathcal{E}_{\text{fin}}^{\times}}(E, \mathbb{P}^{\#})$$

would hold, $[E, \mathbb{P}^{\#}]$ would be finer than $(\mathbb{P}^{\#})^E$. On the other hand, the former is coarser than the latter, because $\forall \Psi \in \mathbf{F}(\mathbf{PRAP}(E, \mathbb{P}^{\#}))$ and $\forall f \in \mathbf{PRAP}(E, \mathbb{P}^{\#})$

$$
\begin{aligned}
&\lambda_c(\Psi)(f)\\
&= \quad \min\{\, K \in [0, \infty] \mid \forall \mathcal{F} \in \mathbf{F}(\underline{E}) :\\
&\qquad \lambda_{\mathbb{P}^{\#}}(\operatorname{stack}\Psi(\mathcal{F})) \circ f \leq K \vee \lambda_E(\mathcal{F})\,\}\\
&\geq \quad \inf\{K \in [0, \infty] \mid \forall x \in \underline{E} :\\
&\qquad \lambda_{\mathbb{P}^{\#}}(\operatorname{stack}\Psi(\dot{x}))(f(x)) \leq K \vee \lambda_E(\dot{x})(x) = K\,\}\\
&= \quad \sup_{x \in \underline{E}} \lambda_{\mathbb{P}^{\#}}(\operatorname{stack}\Psi(\dot{x}))(f(x))\\
&= \quad \lambda_{[E, \mathbb{P}^{\#}]}(\Psi)(f).
\end{aligned}
$$

So we would obtain that $(\mathbb{P}^{\#})^E = [E, \mathbb{P}^{\#}]$ which would be a contradiction because $\underline{E}$ is infinite. $\square$

Pasting the previously described results together and using the result quoted from [14] we obtain:

THEOREM 2.7. *Under the assumption* (H), *there exists a conglomerate, which is not codable by a class, of mutually not naturally isomorphic monoidal closed structures on the construct* **PRAP**.

3. Acknowledgements

References

[1] J. ADÁMEK, H. HERRLICH, AND G. STRECKER, *Abstract and Concrete Categories*, John Wiley, New-York, 1990.

[2] P. BOOTH AND J. TILLOTSON, *Monoidal closed, Cartesian Closed and convenient categories of topological spaces*, Pacific. J. Math. **88** (1980), no. 1, 35 – 53.

[3] F. BORCEUX, *Handbook of Categorical Algebra*, Encyclopedia of Mathematics **50, 51 and 52** (1994), Parts I, II and III.

[4] R. BROWN, *Function spaces and product topologies*, J. Math. Oxford **15** (1964), 238–250.

[5] G. GREVE, *General construction of monoidal closed structures in topological, uniform and nearness spaces*, Category Theory, vol. L.N.M. 962, Springer, Berlin, 1982, pp. 100 – 114.

[6] H. HERRLICH, *Topologische Reflexionen und Coreflexionen*, L.N.M. 78, Springer, Berlin, 1968.

[7] H. HERRLICH AND M. HUŠEK, *Some open categorical problems in* **TOP**, Appl. Categ. Struct. **1** (1993), 1–19.

[8] S. MAC LANE, *Categories for the Working Mathematician*, Graduate Texts in Mathematics, Springer Verlag, New-York, 1971.

[9] E. LOWEN AND R. LOWEN, *Topological quasitopos hulls of categories containing topological and metric objects*, Cahiers Top. et Géom. Diff. Catég. **XXX** (1989), no. 3, 213 – 228.

[10] R. LOWEN, *Approach Spaces: the Missing Link in the Topology-Uniformity-Metric Triad*, Oxford Mathematical Monographs, 1997.

[11] E. LOWEN-COLEBUNDERS, R. LOWENAND, AND F. VERBEECK, *Exponential objects in the construct* **PRAP**, Appl. Categ. Struct. **1** (1993), 345 – 360.

[12] M. SIOEN, *Non-symmetric exponential laws in* **PRTOP**, To appear in Appl. Categ. Struct.

[13] M. SIOEN, *Symmetric monoidal closed structures in* **PRAP**, To appear in *Cahiers Top. et Géom. Diff. Catég.*.

[14] V. TRNKOVÁ, *Non-constant continuous maps of metric or compact Hausdorff spaces*, Comm. Math. Univ. Carolinae **13** (1972), no. 2, 283 – 295.

[15] J. ČINČURA, *On a tensorproduct in initially structured categories*, Math. Slovaca **29** (1979), no. 4, 245–255.