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Optimal portfolio strategies for defined-contribution pension plans: a stochastic control approach

Paolo Battocchio

Contents

Introduction

This work contributes to the analysis of the asset allocation problem for pension funds in a stochastic continuous-time framework. In particular, we focus on the portfolio problem of a fund manager who wants to maximize the expected utility of the fund's terminal wealth, that is to say the wealth accumulated up to the retirement of a representative shareholder. We consider the case of a defined-contribution pension plan.

There are two different ways to manage a pension fund. On the one hand, we find defined-benefit pension plans (hereafter DB), where benefits are fixed in advance by the sponsor and contributions are initially set and subsequently adjusted in order to maintain the fund in balance. On the other hand, there are defined-contribution pension plans (hereafter DC), where contributions are fixed and benefits depend on the returns on fund's portfolio. In particular, DC plans allow contributors to know, at each time, the value of their retirement accounts. Although DB and DC pension plans have some similarities, they adopt different means in order to ensure the same income streams in retirement. This partly depends on the different regulatory framework which is usually applied to DB and DC plans. However, the main differences regard the risk management associated with the retirement income. Historically, fund managers have mainly proposed DB plans, which are definitely preferred by workers. In fact, in the case of DB plans, the associated financial risks are supported by the plan sponsor rather than by the individual member of the plan. Nowadays, most of the proposed pension plans are based on DC schemes involving a considerable transfer of risks to workers. Accordingly, DC pension funds provide contributors with a service of saving management, even if they do not guarantee any minimum performance. As we have already highlighted,

only contributions are fixed in advance, while the final retirement account fundamentally depends on the administrative and financial skill of the fund managers. Therefore, an efficient financial management is essential to gain contributors' trust.

The classical dynamic optimization model, initially proposed by Merton (1969, 1971), assumes a market structure with constant interest rate. We note that the optimal asset-allocation problem for a pension fund involves quite a long period, generally from 20 to 40 years. It follows that the assumption of constant interest rates does not fit with our target. For the same reason, we support the idea that also the inflation risk needs to be considered. According to this assumption, we observe that some of the leading pension funds in U.S. have from 5 to 10 percent of their portfolios allocated just to inflation-indexed instruments (Chicago Mercantile Exchange data). Moreover, the benefits proposed by DC pension plans often require the specification of the stochastic behavior of other variables, such as salaries. Thus, the fund manager must cope not only with financial risks, but also with other risk sources outside the financial market as for example salaries. In this case, we will highlight how the introduction of a stochastic non-financial income (in our case contribution) in the optimal portfolio problem causes several computational difficulties. Summing up the above considerations, with respect to the classical Merton's portfolio choice problem, here we include in the model: (i) a stochastic process for the short rate, (ii) the inflation risk, through a stochastic process for the consumer price index, and (iii) the salary risk, through a stochastic process for the contributions.

Merton (1990), Karatzas and Shreve (1998), and Duffie (2001) provide general treatments of optimal portfolio choice in continuous-time without any stochastic non-financial income. Merton (1971) analyses the effects of introducing a deterministic wage income in the consumption-portfolio problem. In the more recent literature, similar models for DC pension fund has been presented by Blake et al. (2000), Boulier et al. (2001), and Deelstra et al. (2003). Especially, Blake et al. (2000) assume a stochastic process for salary including a non-hedgeable risk component and focus on the replacement ratio as the central measure for determining the pension flow. Boulier et al. (2001) assume a

deterministic process for salary and consider a guarantee on the benefits. Accordingly, they support the real need for a downside protection of contributors who are more directly exposed to the financial risk borne by the pension fund. Also Deelstra et al. (2003) allow for a minimum guarantee in order to minimize the randomness of the retirement account, but they describe the contribution flow through a non-negative, progressively measurable, and square-integrable process. A recent model for a DC pension scheme in discrete time is proposed by Haberman and Vigna (2001). In particular, they study both the "investment risk", that is the risk of incurring a poor investment performance during the accumulation phase of the fund, and the "annuity risk", that is the risk of purchasing an annuity at retirement in a particular recessionary economic scenario involving a low conversion rate. Charupat and Milevsky (2002) and Devolder et al. (2003) analyze the problem of choosing the best investment strategy before and after retirement for life annuities and DC pension schemes respectively. Battocchio et al. (2002) consider the link between the accumulation phase (i.e. before retirement) and the decumulation phase (i.e. after retirement, when benefits are paid under the form of annuities) of a pension fund through a suitable "feasibility condition".

The problem of optimal portfolio choice for a long-term investor in presence of wage income is also treated by El Karoui and Jeanblanc-Picqué (1998), Campbell and Viceira (2002), and Franke et al. (2001). In a complete market with constant interest rates, El Karoui and Jeanblanc-Picqué (1998) present the solution of a portfolio optimization problem for an economic agent endowed with a stochastic insurable stream of labor income. Thus, they assume that the income process does not involve a new source of uncertainty. Campbell and Viceira (2002) focus on some aspects of labor income risk in discrete-time. In particular, they look at individual's labor income as a dividend on the individual's implicit holding of human wealth. Franke et al. (2001) analyze the impact of the resolution of the labor income uncertainty on portfolio choice. They show how the investor's portfolio strategy changes when the labor income uncertainty is resolved earlier or later in life. For an investor described by CARA preferences and endowed with a stochastic non-financial income, Menoncin (2002) presents both an exact solution when the non-financial in-

come can be spanned on the financial market and an approximated solution when this is not true.

The methodological approach we use to solve the optimal asset-allocation problem of a pension fund is the stochastic optimal control. Alternative approaches (see for instance Deelstra et al., 2003; and Lioui and Poncet, 2001) are based on the so-called "martingale approach" first introduced by Cox and Huang (1989, 1991), where the resulting partial differential equation is often simpler to solve than the Hamilton-Jacobi-Bellman equation coming from the dynamic programming. Nevertheless, in the martingale approach, when a stochastic process for salaries enters the optimization procedure, a submartingale is no more obtained to apply the theory.

In the first chapter we present a review of the mathematical tools required for the formal analysis of asset allocation models in continuous-time.

Chapter 2 illustrates the use of the stochastic optimal control as optimization engine in the consumption and portfolio choice problems in continuous-time.

In Chapter 3 we develop the optimal consumption and investment problem presented by Merton (1969, 1971). This model is commonly regarded as the first successful application of stochastic control in economics. Moreover, we present an explicit solution to the control problem for general hyperbolic absolute risk aversion utility functions.

In Chapter 4 we extend the classical Merton's model by allowing interest rates to be stochastic. We illustrate how the introduction of another relevant state variable (the stochastic short rate) in the control problem, in addition to the wealth, represents a delicate matter, although the methodological approach does not change. Under suitable assumptions on the value function, we derive an exact solution to the control problem by applying the Feynman-Kač Theorem directly to the Hamilton-Jacobi-Bellman equation. Then, we analyse how the short rate dynamics affects the optimal portfolio choice. Actually, the stochastic interest rate introduces a new hedging component in addition to the only speculative component characterizing the optimal portfolio strategy in the Merton's model.

Finally, Chapter 5 extends the asset allocation models presented in the

previous chapters to the case of a DC pension fund. In order to characterize the accumulation phase, we consider the case of a shareholder who, at each period $t \in [0, T]$, contributes a constant proportion of his salary to a personal pension fund. At the time of retirement T, the accumulated pension fund will be converted into an annuity. Initially, we assume a complete financial market constituted by three assets: a riskless asset, a stock and a bond which can be bought and sold without incurring any transaction costs or restriction on short sales. Then, we take into account two stochastic processes describing the behavior of salaries and the consumer price index. As we have already remarked, the presence of a stochastic process for salaries represents the chief obstacle to a complete solution of the optimal control problem. In fact, if we assume that the salary process is driven by a risk source which does not belong to those defining the financial market, that is a non-hedgeable risk, we obtain that the market is no more complete. In this case, even if we can state the control problem, the corresponding Hamilton-Jacobi-Bellman equation and the optimal portfolio, we are not able to apply the Feynman-Kač Theorem and to find the optimal value function in a closed form. Therefore, this prevents us from studying how the coefficients of the salary process affect the optimal portfolio strategies. Here, we propose a model in which the presence of stochastic salaries is consistent with the assumption of complete market. In order to justify this proposition, we link the only non-hedgeable component of the salary process to the consumer price index, whose role in the financial market will be widely investigated. By following this way, we find a closed form solution to the control problem and then we are able to analyze in detail how the risk involved by the stochastic behavior of salary and inflation affects the optimal portfolio composition. We prove that the optimal portfolio is formed by three components: (i) a speculative component proportional to both the portfolio Sharpe ratio and the reciprocal of the Arrow-Pratt risk aversion index, as the one derived in the Merton's model, (ii) a hedging component depending on the state variable parameters as the one derived in Chapter 4, and (iii) a preference-free hedging component depending only on the diffusion terms of both the financial assets and the consumer price index. Furthermore, after working out the expected values characterizing the solution, the

optimal portfolio can be simplified to the sum of only two components: one depending on the time horizon, and the other one independent of it. In particular, the optimal portfolio real composition turns out to have an absolutely time independent component. Moreover, the risk aversion parameter determines whether the portfolio is more or less affected by the time-dependent real component. The higher the risk aversion, the more the time-dependent real component affects the optimal portfolio. Accordingly, low values of the risk aversion parameter determine a real portfolio composition that becomes approximately constant through time. Finally, we present a numerical application in order to investigate the dynamic behavior of the optimal portfolio strategy more closely.

Chapter 1

Some preliminaries of stochastic calculus

The purpose of this chapter is to define the stochastic integral with respect to a Brownian motion and to introduce some fundamental results of the corresponding differential calculus. Standard references for a rigorous introduction to stochastic calculus are Karatzas and Shreve (1988), and Revuz and Yor (1991). A general approach to the theory of stochastic integration based on semimartingales as integrators can be found in Meyer (1976), Protter (1990), and Jacod and Shiryaev (2002).

1.1   Brownian motions and stochastic integration

We fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1 *A stochastic process* $W : \Omega \times [0, \infty) \to \mathbb{R}$ *is a Brownian motion on* $(\Omega, \mathcal{F}, \mathbb{P})$ *if it has the following properties:*

1. *$\mathbb{P}(W(0) = 0) = 1$;*

2. *for any $0 \leq s \leq t$, the random variable $W(t) - W(s)$ is normally distributed with mean zero and variance $(t - s)$;*

3. *the process W has independent increments; i.e., for any time sequence $0 \leq t_0 < t_1 < ... < t_n < \infty$, the random variables $W(t_0)$, $W(t_1) - W(t_0),...,W(t_n) - W(t_{n-1})$ are independent;*

4. *the process W has continuous trajectories; i.e., for almost all $\omega \in \Omega$, the map $t \to W(t,\omega)$ is continuous.*

We can generalize the above definition in order to define a Brownian motion in $\mathbb{R}^k$.

Definition 2 $W = (W^1, ..., W^k)^T$ *is a k-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ if the processes W^i, $i = 1, ..., k$, are real-valued Brownian motions and they are independent.*

Hereafter $(^T)$ indicates transposition, so that $W(t)$, $t \in [0, T]$, is a column vector in $\mathbb{R}^k$.

Let $\mathcal{F}_t$ denote the augmented filtration generated by $\mathcal{F}_t^W$ and the null subsets of $\mathcal{F}$, where $\mathcal{F}_t^W \equiv \sigma\{W(s); 0 \leq s \leq t\}$. The σ-algebra $\mathcal{F}_t$ can be interpreted as the information generated by W on the interval $[0, t]$ and available to investors at time t. We recall that $\mathcal{F}_t$ is increasing, that is $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$. We denote $E_t(\cdot) = E(\cdot|\mathcal{F}_t)$ the conditional expectation with respect to the σ-algebra $\mathcal{F}_t$.

Definition 3 *A process $\theta : \Omega \times [0, \infty) \to \mathbb{R}$ is adapted if, for all $t \geq 0$, the random variable $\theta(\cdot, t)$ is measurable with respect to $\mathcal{F}_t$.*

For the sake of simplicity, from now on we will not indicate the dependence on the state ω, unless it is strictly necessary.

Let $\mathcal{L}$ denote the set of adapted processes. We define the two following subsets of $\mathcal{L}$:

$$\mathcal{L}^1 = \left\{ \theta \in \mathcal{L} : \int_0^t |\theta(t)|\, ds < \infty \quad a.s., \text{ for all } t \geq 0 \right\},$$

$$\mathcal{L}^2 = \left\{ \theta \in \mathcal{L} : \int_0^t \theta^2(t)\, dt < \infty \quad a.s., \text{ for all } t \geq 0 \right\},$$

where, obviously, $\mathcal{L}^1 \subset \mathcal{L}^2$.

Definition 4 *A process $M : \Omega \times [0, \infty) \to \mathbb{R}$ in $\mathcal{L}$ is*

a martingale,

a supermartingale,

a submartingale,

if $M(t)$ is integrable for all $t \geq 0$, i.e. $E|M(t)| < \infty$, and if

$$E_t(M(s)) = M(t),$$

$$E_t(M(s)) \leq M(t),$$

$$E_t(M(s)) \geq M(t)$$

respectively, for any $0 \leq t \leq s$.

We note that if the process M is a martingale, then $E(M(t)) = E(M(0))$ for any $t \geq 0$. Moreover, if $t \in [0, T]$ with $T < \infty$, the process M is completely determined by its terminal value, that is $M(t) = E_t(M(T))$.

We will sometimes need to work with processes satisfying a weaker property: the local martingales.

Definition 5 *A stopping time is a random variable $\tau : \Omega \to [0, \infty]$ such that, for each $t \geq 0$, the event $\{\omega \in \Omega : \tau(\omega) \leq t\}$ is in $\mathcal{F}_t$.*

Definition 6 *An adapted process $M : \Omega \times [0, \infty) \to \mathbb{R}$ is a local martingale if there exists an increasing sequence $(\tau_n)_n$ of stopping times, with $\tau_n \to +\infty$ as $n \to +\infty$, such that the stopped process defined by $M^{\tau(n)}(t) \equiv M(t \wedge \tau_n)$, $t \geq 0$, is a martingale for all n.*

It can be proved that a martingale is a local martingale, while the converse need not be true. On the other hand, a nonnegative local martingale is a supermartingale.

We note that the properties of a standard Brownian motion imply that it is a martingale. In particular, this follows from the fact that the increments of a standard Brownian motion are independent and of zero mean.

Now, we fix a time interval $[0, T]$ and an adapted process $h : \Omega \times [0, T] \to \mathbb{R}$. Let Z be one of the processes $W^1, ..., W^k$ defining the k-dimensional Brownian

motion W. Our object is to define the stochastic integral $\int_0^T h(t) \, dZ(t)$, known as Itô integral.

In order to guarantee the existence of the stochastic integral, we assume $h \in \mathcal{H}^2$, where

$$\mathcal{H}^2 = \left\{ \theta \in \mathcal{L}^2 : E\left[\int_0^t \theta^2(t) \, dt \right] < \infty \quad, t \geq 0 \right\}.$$

The first step is to define the stochastic integral for a particular class of $\mathcal{H}^2$ processes, that is the simple (or elementary) processes.

Definition 7 *A process* $h : \Omega \times [0, T] \to \mathbb{R}$ *is simple if there exists a partition of* $[0, T]$ *given by times* $0 = t_0 < t_1 < ... < t_N = T$ *such that* h *is constant on each subinterval, in the sense that, for all* $n > 0$,

$$h(t) = h(t_n), \qquad t \in [t_{n-1}, t_n).$$

Thus, we define the stochastic integral of the simple process h with respect to Z as follows:

$$\int_0^t h(s) \, dZ(s) \equiv \sum_{i=1}^{n} h(t_{i-1}) \left[Z(t_i) - Z(t_{i-1}) \right] + h(t_n) \left[Z(t) - Z(t_n) \right],$$

if $t \in [t_n, t_{n+1})$, for any $n < N$, and

$$\int_0^T h(s) \, dZ(s) \equiv \sum_{i=1}^{N} h(t_{i-1}) \left[Z(t_i) - Z(t_{i-1}) \right],$$

if $t = t_N = T$.

Now, let us consider a general process $h \in \mathcal{H}^2$. It can be proved, see for example Karatzas and Shreve (1988), that we can always approximate a process $h \in \mathcal{H}^2$ with a sequence (h_n) of simple adapted processes such that

$$E\left(\int_0^T [h_n(t) - h(t)]^2 \, dt \right) \overset{n \to \infty}{\longrightarrow} 0$$

For each h_n, the integral $Y_n \equiv \int_0^T h_n(t) \, dZ(t)$ is a well defined random variable. Then, it follows that there exists a unique random variable Y such that

$$E\left([Y - Y_n]^2 \right) \to 0. \tag{1.1}$$

We note that Y is a uniquely defined random variable in the sense that if there exists another random variable $\overline{Y}$ satisfying property (1.1), then it must be $Y = \overline{Y}$ almost surely.

Thus, we define the stochastic integral of a process $h \in \mathcal{H}^2$ with respect to Z as the limit in $L^2(\Omega, \mathcal{F}, \mathbb{P})^1$ of the sequence (Y_n), that is

$$Y \equiv \int_0^T h(t)\, dZ(t) = \lim_{n \to +\infty} \int_0^T h_n(t)\, dZ(t). \qquad (1.2)$$

Since the time horizon T and the process Z are arbitrary, we can extend the above definition of stochastic integral to any process $h = (h^1, ..., h^k) \in (\mathcal{H}^2)^k$, that is $h^i \in \mathcal{H}^2$ for all $i = 1, ..., k$, and any $t \in [0, T]$ as follows

$$\int_0^t h(s)\, dW(s) \equiv \sum_{i=1}^k \int_0^t h^i(s)\, dW^i(s). \qquad (1.3)$$

The following results are frequently applied in stochastic models.

Proposition 1 *Given $h \in (\mathcal{H}^2)^k$, the process $M = (M(t), 0 \le t \le T)$, defined by the stochastic integral $M(t) = \int_0^t h(s)\, dW(s)$, is a martingale with mean zero and variance $E(M_t^2) = E\left[\int_0^t h^2(s)\, ds\right]$.*

In financial applications, we will sometimes need to define $\int_0^T h(t)\, dW(t)$ for an integrand h in $(\mathcal{L}^2)^k$. This because the integrability condition imposed on the set of adapted processes $(\mathcal{H}^2)^k$ can be too strong. However, this generalization can be carried out as follows. For any given $n \in \mathbb{N}_+$, let us define $\tau(n) = \inf\left\{t \in [0, T] : \int_0^t \|h(s)\|^2\, ds = n\right\}$, where $\inf \varnothing = T$ and $\|.\|$ denotes the Euclidean norm. The process $h_{\tau(n)}$, given by $h_{\tau(n)}(t) = h(t)\mathbf{1}_{\{t \le \tau(n)\}}$, belongs to $(\mathcal{H}^2)^k$. Thus, for any $t \in [0, T]$, the stochastic integral $Y_{\tau(n)} = \int_0^t h_{\tau(n)}(s)\, dW(s)$ is well defined as in Equation (1.3). Now, we note that $\tau(n) \to T$ almost surely. In fact, we have $\int_0^T \|h(s)\|^2\, ds < \infty$ almost surely. Finally, we define the stochastic integral of a process $h \in (\mathcal{L}^2)^k$ with respect to W as the limit in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of the sequence $(Y_{\tau(n)})$, that is

$$\int_0^t h(s)\, dW(s) \equiv \lim_{n \to +\infty} \int_0^t h_{\tau(n)}(s)\, dW(s) \quad \text{in } L^2, \quad t \in [0, T]. \qquad (1.4)$$

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[1]We recall that $L^2(\Omega, \mathcal{F}, \mathbb{P}) = \left\{x : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}) \text{ such that } E(x^2) < \infty\right\}$, where $\mathcal{B}$ is the Borel σ-field in $\mathbb{R}$.

Proposition 2 *Given $h \in \mathcal{L}^2$, the process $M = (M(t), 0 \leq t \leq T)$, defined by the stochastic integral $M(t) = \int_0^t h(s)\, dW(s)$, is a local martingale.*

An important property of the stochastic integral is the linearity. Given two processes h and g in $(\mathcal{L}^2)^k$ and two scalars α and β, the process $(\alpha h + \beta g)$ is also in $(\mathcal{L}^2)^k$, and, for any $t \in [0, T]$, it results that

$$\int_0^t (\alpha h(s) + \beta g(s))\, dW(s) = \alpha \int_0^t h(s)\, dW(s) + \beta \int_0^t g(s)\, dW(s).$$

In finance, the stochastic integral is often taken to model trading gains. By 1900, Louis Bachelier proposed to adopt a standard Brownian motion as the price of a security. Thus, if S denotes the (unidimensional) price process, we have $S = W$. Now, we define a trading strategy as a process $\pi \in \mathcal{L}$, where $\theta(\omega, t)$ represents the units of the security hold at each state $\omega \in \Omega$ and time $t \in [0, T]$. Under these assumptions, we suppose that the trading strategy π is piecewise constant on $[0, T]$ so that, for some time sequence $0 = t_0 < t_1 < \ldots < t_N = T$, we have $\pi(t) = \pi(t_{n-1})$ for all $t \in [t_{n-1}, t_n)$ and for any $n = 0, 1, \ldots, N$. Then, we would have no difficulty in defining the trading gains within each subinterval. In particular, the total gain involved by π will be given by

$$\int_0^T \pi(t)\, dW(t) = \sum_{n=1}^N \pi(t_{n-1})\, [W(t_n) - W(t_{n-1})].$$

However, we wish to consider trading strategies that are not necessarily piecewise constant. In this case, the stochastic integral of a trading strategy π with respect to W, as defined in (1.4), allows us to extend the model of trading gains to all processes $\pi \in \mathcal{L}^2$. In fact, for any $t \in [0, T]$, the stochastic integral $\int_0^t \pi(s)\, dW(s)$ characterizes the total gain generated up to time t by the trading strategy π. Now, a natural extension of this model regards the dynamics of the traded security. In fact, we will need to work with processes more general than standard Brownian motions. In the next section, we will define Itô processes and some fundamental results which will allow us to use Itô processes as good models of security prices.

1.2 Itô processes and Itô's formula

In this section we will present some fundamental results of the stochastic calculus.

We fix a standard Brownian motion W defined on $\mathbb{R}$.

Definition 8 *A process $X = (X(t), 0 \le t \le T)$ is a real-valued Itô process if there exist two processes $\mu \in \mathcal{L}^1$ and $\sigma \in \mathcal{L}^2$ such that*

$$X(t) = x_0 + \int_0^t \mu(s)\,ds + \int_0^t \sigma(s)\,dW(s), \quad t \in [0, T], \qquad (1.5)$$

where $X(0) = x_0$.

It is common to write (1.5) in the following differential form

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t).$$

We will refer to μ and σ as the drift and the diffusion coefficients of X. In particular, we will interpret $\mu(t)$ as the conditional expected rate of change of X at time t and $\sigma^2(t)$ as the rate of change of the conditional variance of X at time t.

The Itô process allows us to characterize a sufficiently general type of security price processes. Let we assume that the price of a security X is described by an Itô process. If we suppose a trading strategy θ to be a process in $\mathcal{L}$ such that $(\theta\mu) \in \mathcal{L}^1$ and $(\theta\sigma) \in \mathcal{L}^2$, for any $t \in [0, T]$, the stochastic integral

$$\int_0^t \theta(s)\,dX(s) \equiv \int_0^t \theta(s)\,\mu(s)\,ds + \int_0^t \theta(s)\,\sigma(s)\,dW(s)$$

define the gain process generated by θ.

Given two Itô processes X and Y defined as follows

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t),$$
$$dY(t) = \alpha(t)\,dt + \delta(t)\,dW(t),$$
$$X(0) = Y(0) = x_0,$$

it can be proved that, for all $t \in [0, T]$, $X(t) = Y(t)$ almost surely if and only if $\mu = \alpha$ and $\sigma = \delta$ almost everywhere. This is known as the *unique*

decomposition property of Itô processes. We recall that two adapted processes μ and α are equal almost everywhere if

$$E \left(\int_0^\infty |\mu(t) - \alpha(t)| \, dt \right) = 0.$$

The next result, known as Itô's Lemma, represents the main tool in the theory of stochastic calculus. In particular, it allows us to explicitly solve most of the asset-pricing models in a continuous-time setting.

Proposition 3 *(Itô's Lemma) Let $f : \mathbb{R} \times [0, T] \to \mathbb{R}$ be in $C^{2,1} (\mathbb{R} \times [0, T])^2$ and X an Itô process as in Definition (8)*

Then, the process $Y = (Y(t), 0 \leq t \leq T)$, defined by $Y(t) = f(X(t), t)$, is an Itô process with

$$dY(t) = f_t(X(t), t) \, dt + f_x(X(t), t) \, dX(t) + \frac{1}{2} f_{xx}(X(t), t) \sigma^2(t) \, dt, \quad (1.6)$$

where $f_x(x, t) = \frac{\partial f}{\partial x}(x, t)$, $f_{xx}(x, t) = \frac{\partial^2 f}{\partial x^2}(x, t)$ and $f_t(x, t) \equiv \frac{\partial f}{\partial t}(x, t)$.

Equation (1.6) is known as Itô's formula. After substituting $dX(t)$ in Equation (1.6), we have

$$dY(t) = \left[f_t(X(t), t) + f_x(X(t), t) \mu(t) + \frac{1}{2} f_{xx}(X(t), t) \sigma^2(t) \right] dt$$
$$+ f_x(X(t), t) \sigma(t) \, dW(t).$$

In order to illustrate the use of the Itô's formula, we apply it to a very common equation which describes the so called *geometric Brownian motion* (GBM). Let $X(t)$ be the process defined by the following stochastic differential equation:

$$dX(t) = X(t) \mu dt + X(t) \sigma dW(t), \quad (1.7)$$
$$X(0) = x_0,$$

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[2]We follow here the accepted practice of denoting by $C^{2,1} (\mathbb{R} \times [0, T])$ the family of all continuous functions $f(x, t) : \mathbb{R} \times [0, T] \to \mathbb{R}$ with continuous derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$.

where μ and σ are real constants, and $x_0 > 0$. Let us suppose for the moment that $X(t)$ is a strictly positive process satisfying Equation (1.7). Then, we can define the process $Y(t) = \ln X(t)$. The Itô's formula[3] gives us

$$dY(t) = 0 + \frac{1}{X(t)}dX(t) - \frac{1}{2}\frac{1}{X(t)^2}\left(\sigma X(t)\right)^2 dt$$
$$= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW(t).$$

Thus, we have

$$Y(t) = Y(0) + \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma dW(s),$$
$$= \ln x_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t),$$

which implies that

$$X(t) = x_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}, \quad t \in [0, T]. \tag{1.8}$$

Now, it can be verified by Itô's Lemma that $X(t)$, so as defined in Equation (1.8), is actually a solution of (1.7).

We note that the variable $Y(t)$ has normal distribution with mean equal to $\left[\ln x_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t\right]$ and variance $\sigma^2 t$. Consistently with this property, a geometric Brownian motion is often indicated as a log-normal process. In the literature, it is very common to describe the price of a security as a GBM. In fact, working with GBM simplifies many technical procedures, either in the development of the probabilistic analysis, or in the solution of the differential equations involved in the problem. If we suppose that the dynamics of a price security follows a GBM, it is natural to interpret μ as the instantaneous expected rate of return and σ as the instantaneous standard deviation of the rate of return. Due to its log-normal distribution, the GBM has been widely used in most financial applications, in particular we mention the celebrated Black-Scholes option pricing formula. However, two puzzles have emerged from empirical analysis and are still subject to research. The first is due to the leptokurtic distribution of asset returns. We recall that a distribution is leptokurtic if it has a higher peak and heavier tails than those of the normal

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[3]We can note that Itô's formula can be applied because $f(x) = \ln x$ is in C^2 on $(0, \infty)$.

distribution. It is common to think that fat tails are related to the discontinuous paths of asset returns. In order to reflect this discontinuity, a widely used extension of the standard models based on GBMs is to consider jump-diffusion processes. The second puzzle is strictly related to the option pricing theory. Actually, the assumption of log-normal processes fails to explain the so-called "volatility smile" effect. Also in this case, various modifications of the standard model have been proposed in literature, particularly they focus on jump-diffusions and stochastic volatility models.

1.3   The multidimensional case

In this section we present a generalization of Itô's Lemma for multidimensional processes.

We suppose that $W = \left(W^1, ..., W^k\right)^\mathsf{T}$ is a standard Brownian motion in $\mathbb{R}^k$ as in Definition (2)

Definition 9 *A process $X = \left(X^1, X^2, ..., X^d\right)^\mathsf{T}$ is an Itô process in $\mathbb{R}^d$ if, for any $i = 1, ..., d$, there exist $\mu^i \in \mathcal{L}^1$ and the row vector $\sigma^i \in \left(\mathcal{L}^2\right)^k$ such that*

$$X^i(t) = x_0^i + \int_0^t \mu^i(s)\, ds + \int_0^t \sigma^i(s)\, dW(s), \quad t \in [0, T],$$

where

$$\int_0^t \sigma^i(s)\, dW(s) \equiv \sum_{j=1}^k \int_0^t \sigma^{i,j}(s)\, dW^j(s),$$

and $X(0) = x_0 \in \mathbb{R}^d$.

Usually, we prefer the matrix notation, then we write X in differential form as follows

$$\underset{(d \times 1)}{dX(t)} = \underset{(d \times 1)}{\mu(t)}\, dt + \underset{(d \times k)}{\Sigma(t)} \underset{(k \times 1)}{dW(t)}, \tag{1.9}$$

where $\mu = \left(\mu^1, ...\mu^d\right)^\mathsf{T}$ and $\Sigma = [\sigma^{i,j}]$..

The next result extends Itô's Lemma to the case of multidimensional processes.

Proposition 4 *(Multidimensional Itô's Lemma) Let $f : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ a function in $C^{2,1}\left(\mathbb{R}^d \times [0, T]\right)$ and X an Itô process in $\mathbb{R}^d$ as defined in Equation (1.9).*

Then, the process $Y = (Y(t), 0 \leq t \leq T)$, defined by $Y(t) = f(X(t), t)$, is an Itô process with

$$dY(t) = \left[f_t(X(t), t) + f_x(X(t), t) \mu(t) + \frac{1}{2} tr\left\{ \Sigma(t) \Sigma^{\mathsf{T}}(t) f_{xx}(X(t), t) \right\} \right] dt$$

$$\tag{1.10}$$

$$+ f_x(X(t), t) \Sigma(t) dW(t),$$

where, $f_x(x, t)$ denotes the row vector $\left[\frac{\partial f}{\partial x^i}(x, t) \right]_{i=1,..,d}$, $f_{xx}(x, t)$ denotes the matrix $\left[\frac{\partial^2 f}{\partial x^i \partial x^j}(x, t) \right]_{i,j=1,..,d}$, and $f_t(x, t) \equiv \frac{\partial f}{\partial t}(x, t)$.

In order to ease the notation, it is common to write Equation (1.10) as follows

$$dY(t) = \mathcal{D}f(X(t), t) + f_x(X(t), t) \Sigma(t) dW(t),$$

where $\mathcal{D}$ denotes the *infinitesimal generator* of X defined by

$$\mathcal{D}f = f_t + \mu f_x + \frac{1}{2} tr\left\{ \Sigma \Sigma^{\mathsf{T}} f_{xx} \right\}.$$

The operator $\mathcal{D}$ is referred also as the Dynkin operator, or the Kolmogorov backward operator of the process X.

1.4 Girsanov's Theorem

We note that an Itô process X is a martingale if $\mu = 0$ and $\sigma \in \mathcal{H}^2$, that is to say when X becomes just a stochastic integral. In many financial applications, working with martingales is not only a computational advantage but a real necessity. In order to eliminate the drift, the Girsanov's Theorem gives us a fundamental tool to move from an arbitrary Itô process to a stochastic integral by a suitable change of probability.

Definition 10 *We say that a process $h \in \left(\mathcal{L}^2\right)^k$ satisfies the Novikov's condition if*

$$E\left(e^{\frac{1}{2} \int_0^T \|h(s)\|^2 ds} \right) < \infty.$$

The following result gives us a sufficient condition to define a martingale.

Proposition 5 *Let $W = \left(W^1, ..., W^k\right)^\top$ be a standard Brownian motion in $\mathbb{R}^k$. Given a process $h \in \left(\mathcal{L}^2\right)^k$ satisfying the Novikov's condition, the process $\xi = \left(\xi\left(t\right), 0 \le t \le T\right)$ defined by*

$$\xi\left(t\right) = \exp\left\{-\int_0^t h\left(s\right) dW\left(s\right) - \frac{1}{2}\int_0^t \|h\left(s\right)\|^2 ds\right\} \qquad (1.11)$$

is a martingale.

By applying Itô's Lemma to Equation (1.11), we obtain that

$$d\xi\left(t\right) = -\xi\left(t\right) h\left(t\right) dW\left(t\right),$$

$$\xi\left(0\right) = 1.$$

Under the assumptions of Proposition (5), since the process ξ is a martingale, we have in particular that $E\left(\xi\left(T\right)\right) = 1$. Moreover, the random variable $\xi\left(T\right)$ is strictly positive, so that we can define an equivalent probability measure $\mathbb{Q}$ on $\left(\Omega, \mathcal{F}\right)$ with the following property

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \xi\left(T\right), \qquad (1.12)$$

that is

$$E^{\mathbb{Q}}\left(Z\right) = E^{\mathbb{P}}\left(\xi\left(T\right) Z\right).$$

We recall that $\mathbb{Q}$ and $\mathbb{P}$ are two equivalent probability measures on $\left(\Omega, \mathcal{F}\right)$, if they assign null probability at the same set of events. That is to say that, for any event $A \in \mathcal{F}$, $\mathbb{P}\left(A\right) = 0$ if and only if $\mathbb{Q}\left(A\right) = 0$. In probability theory, the strictly positive random variable $\xi\left(T\right)$ is called the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$, and, for any sub-tribe $\mathcal{G} \subset \mathcal{F}$, it satisfies the following property, known as Bayes rule:

$$E^{\mathbb{Q}}\left(Z \mid \mathcal{G}\right) = \frac{E^{\mathbb{P}}\left(\xi\left(T\right) Z \mid \mathcal{G}\right)}{E^{\mathbb{P}}\left(\xi\left(T\right) \mid \mathcal{G}\right)},$$

where Z is a random variable such that $E^{\mathbb{Q}}\left(|Z|\right) < \infty$.

Theorem 1 *(Girsanov's Theorem) Let $\xi = (\xi(t), 0 \le t \le T)$ be the process defined by*

$$\xi(t) = \exp\left\{-\int_0^t h(s)\, dW(s) - \frac{1}{2}\int_0^t \|h(s)\|^2\, ds\right\},$$

where $h \in (\mathcal{L}^2)^k$ satisfies the the Novikov's condition.

Let $W^* = (W^*(t), 0 \le t \le T)$ be the process defined by

$$W^*(t) = W(t) + \int_0^t h(s)\, ds.$$

If $\mathbb{Q}$ is the probability measure defined by (1.12), then the process W^* is a Brownian motion under $\mathbb{Q}$.

In particular, if X is an Itô process as defined in Equation (1.9) and $h(t) = \mu(t)\Sigma^{-1}(t)$ is bounded, we have

$$W^*(t) = W(t) + \int_0^t \mu(s)\, \Sigma^{-1}(s)\, ds,$$

and

$$dX(t) = \Sigma(t)\, dW^*(t),$$

Thus, $X(t)$ is a stochastic integral under the new probability measure $\mathbb{Q}$ and the Itô process X becomes a local martingale under $\mathbb{Q}$.

In general, Girsanov's Theorem allows us to characterize Itô processes with same diffusion and arbitrary drifts through suitable probability adjustments.

Corollary 1 *Let $X = \left(X^1, X^2, ..., X^d\right)^\top$ be an Itô process in $\mathbb{R}^d$ defined by*

$$\underset{(d\times 1)}{dX(t)} = \underset{(d\times 1)}{\mu(t)}\, dt + \underset{(d\times k)}{\Sigma(t)}\underset{(k\times 1)}{dW(t)}, \quad t \in [0, T],$$

$$X(0) = x_0.$$

Let us suppose that $\nu \in (\mathcal{L}^1)^d$ and that there exists some process $h \in (\mathcal{L}^2)^k$ such that, for any $t \in [0, T]$,

$$\Sigma(t)\, h(t) = \mu(t) - \nu(t).^4$$

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[4]This is always true if $d = k$ and σ is a full rank matrix almost surely. In this case, the process h is uniquely defined.

If the process h satisfies the Novikov's condition, then X is an Itô process also with respect to the equivalent probability measure $\mathbb{Q}$ and it results that

$$dX(t) = \nu(t)\,dt + \Sigma(t)\,dW^*(t), \quad t \in [0, T],$$
$$X(0) = x_0,$$

where

$$W^*(t) = W(t) + \int_0^t h(s)\,ds.$$

1.5 Stochastic differential equations

We fix a standard Brownian motion W in $\mathbb{R}^k$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

A stochastic differential equation (SDE) is an expression of the form

$$dX(t) = \mu(X(t), t)\,dt + \Sigma(X(t), t)\,dW(t), \tag{1.13}$$
$$X(0) = x_0,$$

where $x_0 \in \mathbb{R}^d$, $\mu : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ and $\Sigma : \mathbb{R}^d \times [0, T] \to \mathbb{R}^{d \times k}$ are given functions. Sometimes it can be useful to specify a SDE in its integral form

$$X(t) = x_0 + \int_0^t \mu(X(s), s)\,ds + \int_0^t \Sigma(X(s), s)\,dW(s).$$

The unknown is the process X in $\mathbb{R}^d$. We note that if the diffusion matrix Σ is identically equal to zero, Equation (1.13) becomes an ordinary (nonstochastic) differential equation, which can be solved path by path. Now, we want to investigate some conditions on μ and Σ under which, for each initial condition $x_0 \in \mathbb{R}^d$, there exists a unique Itô process X solving Equation (1.13). This means that if there exists another Itô process solving Equation (1.13), it must be equal to X almost everywhere. A process such as X is usually called *diffusion*.

Definition 11 *We say that the functions $\mu : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ and $\Sigma : \mathbb{R}^d \times [0, T] \to \mathbb{R}^{d \times k}$ satisfy the global Lipschitz condition in the first argument if there exists a constant K such that, for any $x, y \in \mathbb{R}^d$ and any $t \in [0, T]$,*

$$\|\mu(x, t) - \mu(y, t)\| + \|\Sigma(x, t) - \Sigma(y, t)\| \leq K\|x - y\|.$$

In the case of Σ, we note that the usual notion of Euclidean norm can be extended to any matrix $A = [a_{i,j}]$ by letting

$$\|A\| = \left(tr\left(AA^\mathsf{T}\right)\right)^{\frac{1}{2}} = \left(\sum_{i,j} a_{i,j}^2\right)^{\frac{1}{2}}.$$

.

Definition 12 *We say that the functions* $\mu : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ *and* $\Sigma : \mathbb{R}^d \times [0, T] \to \mathbb{R}^{d \times k}$ *satisfy a linear growth condition in the first argument if there exists a constant K such that, for any $x \in \mathbb{R}^d$ and any $t \in [0, T]$,*

$$\|\mu(x, t)\|^2 + \|\Sigma(x, t)\|^2 \le K\left(1 + \|x\|^2\right).$$

Since Definitions (11) and (12) apply for all $t \in [0, T]$ simultaneously, we note that these conditions are verified uniformly in the time horizon $[0, T]$.

The following theorem gives us sufficient conditions for the existence and uniqueness of a solution to Equation (1.13).

Proposition 6 *Given the SDE defined in Equation (1.13), let us suppose that the coefficients μ and Σ are measurable functions and satisfy the global Lipschitz condition and the linear growth condition in x. Then, for each $x_0 \in \mathbb{R}^d$, there exists an Itô process X in $\mathbb{R}^d$ satisfying Equation (1.13) with initial condition $X(0) = x_0$.*

Moreover, the solution is unique in the sense that, given another Itô process Y satisfying Equation (1.13), then $X = Y$ almost everywhere.

If X is the solution of (1.13), then there exists a constant C such that, for any $t \in [0, T]$,

$$E\left(\|X(t)\|^2\right) \le Ce^{Ct}\left(1 + \|x_0\|^2\right).$$

Corollary 2 *Under the assumptions of Proposition (6) the process X is Markov.*

We recall that X is a Markov process if the conditional distribution of $X(t)$, given $\mathcal{F}_s$, $s < t$, depends only on $X(s)$. Namely, for any arbitrary bounded function $f : \mathbb{R} \to \mathbb{R}$ and any $s < t \le T$, we have

$$E\left[f(X(t)) \mid \mathcal{F}_s\right] = E\left[f(X(t)) \mid X(s)\right]. \tag{1.14}$$

Under our assumptions, it can be proved that the solution X of Equation (1.13) is also a strong Markov process, roughly speaking this means that the above property (1.14) holds also if we replace time t with a stopping time τ adapted to $\mathcal{F}_t$.

The solution X defined by Proposition (6) is called a *strong* solution. This depends on the fact that the Brownian motion W is fixed in advance and the solution X built on it is consequently $\mathcal{F}_t$-adapted.

In general, it is a hard work to find a solution of a SDE. There are however some interesting nontrivial cases where it is possible to solve a SDE in explicit manner. A very common example in financial mathematics is given by the geometric Brownian motion, whose solution has been illustrated in Section 1.2. Given the result in Proposition (6), we remark that the real process

$$X(t) = x_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}, \quad t \in [0, T],$$

is the unique solution of the equation defining a GBM, that is

$$dX(t) = X(t)\,\mu dt + X(t)\,\sigma dW(t),$$
$$X(0) = x_0,$$

where we recall that μ and σ are real constants, and $x_0 > 0$.

An important class of SDE is given by the *linear* stochastic differential equations, which have the following form

$$dX(t) = [A(t)X(t) + b(t)]\,dt + C(t)\,dW(t), \tag{1.15}$$
$$X(0) = x_0,$$

where $A : [0, T] \to \mathbb{R}^{d \times d}$, $b : [0, T] \to \mathbb{R}^d$, and $C : [0, T] \to \mathbb{R}^{d \times k}$ are continuous. Equation (1.15) can be explicitly solved. In fact it can be proved that the solution of a linear SDE is given by

$$X(t) = \Phi(t)\left[x_0 + \int_0^t \Phi^{-1}(s)\,b(s)\,ds + \int_0^t \Phi^{-1}(s)\,C(s)\,dW(s)\right],$$

where $\Phi(t)$ is a nonsingular matrix which solves the following ordinary matrix differential equation

$$\frac{d\Phi(t)}{dt} = A(t)\,\Phi(t),$$
$$\Phi(0) = I_{d \times d},$$

where $I_{d \times d}$ denotes the d-dimensional identity matrix.

An important property of this diffusion is that X is Gaussian. Given any finite sequence of times $t_1, ..., t_h$, $(X(t_1), ..., X(t_h))$ has a joint normal distribution. Moreover, for any t, we can compute the mean vector $m(t)$ and the covariance matrix $V(t)$ as solutions of the following ordinary differential equations

$$\frac{dm(t)}{dt} = A(t) m(t) + b(t), \qquad m(0) = x_0,$$

$$\frac{dV(t)}{dt} = A(t) V(t) + V(t) A(t)^{\mathsf{T}} + C(t) C(t)^{\mathsf{T}}, \qquad V(0) = 0.$$

1.6 Partial differential equations and Feynman-Kač Theorem

In this section we will investigate the close connection which exists between SDEs and certain partial differential equations. In particular, we will illustrate the so-called Cauchy problem and the probabilistic solution of a PDE.

As usual, we fix a standard Brownian motion W in $\mathbb{R}^k$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Given two functions $\mu : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ and $\Sigma : \mathbb{R}^d \times [0, T] \to \mathbb{R}^{d \times k}$ satisfying the assumption of Proposition (6), we want to find a function $f \in C^{2,1}(\mathbb{R}^d \times [0, T])$ which solves the following *Cauchy problem*:

$$\mathcal{C} : \begin{cases} \mathcal{D}f(x, t) = 0, & x \in \mathbb{R}^d, t \in [0, T], \\ f(x, T) = g(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.16}$$

where $g : \mathbb{R}^d \to \mathbb{R}$. From Section 1.3, we recall that

$$\mathcal{D}f(x, t) = f_t(x, t) + f_x(x, t) \mu(x, t) + \frac{1}{2} tr \left\{ \Sigma(x, t) \Sigma^{\mathsf{T}}(x, t) f_{xx}(x, t) \right\}.$$

The idea is to derive a probabilistic representation for the solution of the PDE (1.16). Specifically, we want to represent the Cauchy problem in terms of Brownian motions and solutions of suitable SDEs. For any fixed pair $(x, t) \in \mathbb{R}^d \times [0, T]$, we define the process X as the solution to the SDE

$$dX(s) = \mu(X(s), s) ds + \Sigma(X(s), s) dW(s), \quad s \in (t, T],$$

$$X(t) = x.$$

Let us suppose that f is a solution of the Cauchy problem $\mathcal{C}$. For any $(x,t) \in \mathbb{R}^d \times [0,T]$, we apply the Itô's formula to the process Y defined as

$$Y(s) = f(X(s),s), \quad s \in (t,T],$$
$$Y(t) = f(x,t).$$

We obtain that

$$f(X(T),T) = f(X(t),t) + \int_t^T \mathcal{D}f(X(s),s)\,ds \qquad (1.17)$$
$$+ \int_t^T f_x(X(s),s)\,\Sigma(X(s),s)\,dW(s).$$

Since f solves $\mathcal{C}$, the ordinary integral above vanishes. Moreover, if we take expectations through each side of Equation (1.17) and we suppose that the process $f_x(X,s)\,\Sigma(X,s)$ satisfies enough technical conditions as for instance in Proposition (1), then the expectation of the stochastic integral also vanishes. Thus, given the initial condition $X(t) = x$ and the boundary condition $f(\cdot,T) = g(\cdot)$, we have

$$f(x,t) = E_t[g(X(T))].$$

We have proved the following result, which is known as the *Feynman-Kač stochastic representation formula:*

Proposition 7 *Let $f \in C^{2,1}\left(\mathbb{R}^d \times [0,T]\right)$ be a solution of the Cauchy problem*

$$\mathcal{C}: \begin{cases} \mathcal{D}f(x,t) = 0, & x \in \mathbb{R}^d, t \in [0,T], \\ f(x,T) = g(x), & x \in \mathbb{R}^d, \end{cases}$$

where $g : \mathbb{R}^d \to \mathbb{R}$.

If we suppose that the process $f_x(X,s)\,\Sigma(X,s) \in \mathcal{H}^2$, then f admits the stochastic representation

$$f(x,t) = E_t[g(X(T))],$$

where, for any $s \geq t$, the process X satisfies the SDE

$$dX(s) = \mu(X(s),s)\,ds + \Sigma(X(s),s)\,dW(s),$$
$$X(t) = x.$$

We note that the Feynman-Kač stochastic representation formula, under suitable integrability conditions, gives us a "nice" solution of the Cauchy problem. However, Proposition (7) does not guarantee in general the existence of a solution of the PDE. To obtain this, we need to impose strong technical conditions. A review of these conditions is presented in Karatzas and Shreve (1988). Another important remark regards the uniqueness of the solution. In general, a parabolic PDE, like that describing the Cauchy problem, can admit infinite solutions. In our case, the integrability conditions we have assumed allow us to define only a "nice" solution of the problem, but not the unique.

We now generalize the Cauchy problem presented above and, under sufficient technical conditions, we state the uniqueness of the solution.

Let $f \in C^{2,1}\left(\mathbb{R}^d \times [0,T]\right)$ be a solution of the Cauchy problem

$$\mathcal{D}f(x,t) - r(x,t)f(x,t) + h(x,t) = 0, \quad x \in \mathbb{R}^d, t \in [0,T], \qquad (1.18)$$

$$f(x,T) = g(x), \quad x \in \mathbb{R}^d,$$

where $r : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, $h : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$.

Moreover, we assume that the following technical conditions are satisfied:

1. the functions μ, Σ, r, h, g, and f are continuous;

2. the solution f satisfies a polynomial growth condition in x; that is to say that for some positive constants K and ε,

$$|f(x,t)| \le K\left(1 + \|x\|^\varepsilon\right), \quad (x,t) \in \mathbb{R}^d \times [0,T];$$

3. the function r is nonnegative;

4. the functions g and h are either nonnegative or satisfy a polynomial growth condition in x.

The following general result extends Proposition (7).

Proposition 8 *Let $f \in C^{2,1}\left(\mathbb{R}^d \times [0,T]\right)$ be a solution of the Cauchy problem (1.18). If we suppose that the technical conditions (1-4) are satisfied and $f_x(X,s)\Sigma(X,s) \in \mathcal{H}^2$, then f admits the stochastic representation*

$$f(x,t) = E_t\left[\int_t^T \varphi(X(s),s)h(X(s),s)\,ds + \varphi(X(T),T)g(X(T))\right],$$

where, for any $s \geq t$,

$$\varphi\left(X\left(s\right),s\right) = e^{-\int_t^s r(X(u),u)du},$$

and X satisfies the SDE

$$dX\left(s\right) = \mu\left(X\left(s\right),s\right)ds + \Sigma\left(X\left(s\right),s\right)dW\left(s\right),$$
$$X\left(t\right) = x.$$

Moreover, the process X is the unique solution of the Cauchy problem (1.18) which satisfies a polynomial growth condition.

Chapter 2

Consumption and portfolio choice in continuous-time

In this chapter we illustrate the use of the dynamic programming when we face problems of optimal consumption and portfolio choice in continuous time. The dynamic programming in continuous-time is often called *optimal control* and it extends the well-known Bellman approach applied to dynamic optimization problems in discrete-time models. Moreover, when we allow for randomness, it is usually called *stochastic control*. We will see that the original optimal control problem reduces to solving a suitable PDE, known as the Hamilton-Jacobi-Bellman (HJB) equation. However, the hard work of dynamic programming consists just in solving this highly nonlinear PDE involved by the stochastic control problem. In fact, there is no general analytical method to solve it. Moreover, the study of the existence of well-behaved solutions to the HJB equation becomes harder and harder when we allow for stochastic coefficients. In Section 2.1 we will describe the theoretical structure of this approach and we highlight the basic steps necessary to specify the HJB equation associated with a general optimal control problem. Then, we will specify a market model and we apply the dynamic programming to the consumption and portfolio choice problem in continuous-time. In Chapter 3, we will present the Merton's model which is probably the first successful application of stochastic control in economics. In this case, we will solve explicitly the control problem for general hyperbolic absolute risk aversion utility functions. In Chapter 4 and 5, we will

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extend the dynamic programming approach to a more general class of market models characterized by several stochastic state processes.

2.1 The formal problem

This section develops a fairly general model for stochastic optimal control problems. Some standard references on optimal control are Åström (1970), Fleming and Rishel (1975), Krylov (1980) and Yong and Zhou (1999). A clear description of this methodology and its applications is provided by Øksendal (2000). An introduction to stochastic optimal control problems in finance can be found in Björk (1998), Dana and Jeanblanc-Picqué (1998), and Duffie (2001).

We fix a k-dimensional standard Brownian motion $W = \left(W^1, ..., W^k\right)^\mathsf{T}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. From Chapter 1, we recall that $\mathcal{F}_t$ denotes the augmented filtration generated by $\mathcal{F}_t^W$ and the null subsets of $\mathcal{F}$. We suppose that all economic activities take place on a finite horizon $[0, T]$.

The first building block of the optimal control problem is represented by a vector X in $\mathbb{R}^d$ of state variables and a vector Y in $\mathbb{R}^n$ of control variables. In the following, we will refer to $X = \left(X^1, ..., X^d\right)^\mathsf{T}$ as the *state process* of the optimal control problem, and to each real-valued stochastic process X^i, $1 \leq i \leq d$, as a *state variable* constituting X. Equivalently, we will indicate $Y = \left(Y^1, ..., Y^n\right)^\mathsf{T}$ as the *control process* and each Y^j, $1 \leq j \leq n$, as a *control variable*. For any $t \in [0, T]$ and $x_0 \in \mathbb{R}^d$, we define the state process X by the following Itô process:

$$dX\left(t\right) = \mu\left(Y\left(t\right), X\left(t\right), t\right) dt + \Sigma\left(Y\left(t\right), X\left(t\right), t\right) dW\left(t\right), \qquad (2.1)$$
$$X\left(0\right) = x_0,$$

where $\mu : \mathbb{R}^n \times \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ and $\Sigma : \mathbb{R}^n \times \mathbb{R}^d \times [0, T] \to \mathbb{R}^{d \times k}$.

The object is to "control" the state process X through a process Y in $\mathbb{R}^n$, which describes in some way the dynamics of our model. The first problem to face concerns the class of the admissible control processes. We suppose that Y is adapted to the state process X. In other words, we require that the value of the control process Y, at each time $t \in [0, T]$, depends only on past

observed values of the state process X. Here, we represent the control process through a function $\phi : \mathbb{R}^d \times [0, T] \to \mathbb{R}^n$, known as the *feedback control law*, such that $Y(t) = \phi(X(t), t)$, for all $t \in [0, T]$. Thus, at any fixed time t, we are assuming that the control process $Y(t)$ depends only on the present value of the state process $(X(t))$ and not on its full path. In order to ease the notation, for any fixed control law ϕ, we rewrite Equation (2.1) as follows

$$dX^\phi(t) = \mu\left(X^\phi(t), t\right) dt + \Sigma\left(X^\phi(t), t\right) dW(t), \qquad (2.2)$$
$$X^\phi(0) = x_0.$$

where, for the sake of simplicity, we state

$$\mu\left(X^\phi(t), t\right) \equiv \mu\left(\phi(X(t), t), X^\phi(t), t\right),$$
$$\Sigma\left(X^\phi(t), t\right) \equiv \Sigma\left(\phi(X(t), t), X^\phi(t), t\right).$$

Definition 13 *Given any initial state $x_0 \in \mathbb{R}^d$, a control law $\phi : \mathbb{R}^d \times [0, T] \to \mathbb{R}^n$ is admissible if there exists a unique Itô process X^ϕ in $\mathbb{R}^d$ solution of the SDE (2.2).*

Let we denote $\mathcal{Y}$ the set of admissible control laws.

We note that in most concrete models, we have to specify suitable technical conditions on the functions μ, Σ, and ϕ in order to guarantee the admissibility of a control law ϕ.

At this point, we introduce the objective function of the control problem. Usually, we have to deal with a *running utility function* $u : \mathbb{R}^n \times \mathbb{R}^d \times [0, T] \to \mathbb{R}$ and a *terminal utility function* $U : \mathbb{R}^d \to \mathbb{R}$. For any initial state $x_0 \in \mathbb{R}^d$, we define the *initial value function* of the optimal control problem as the function $V_0 : \mathcal{Y} \to \mathbb{R}$ given by

$$V_0(\phi) = E_0\left[\int_0^T u\left(\phi(X(t), t), X^\phi(t), t\right) dt + U\left(X^\phi(T)\right)\right], \quad \phi \in \mathcal{Y},$$

where X^ϕ is the solution to (2.2) and, for any $t \in [0, T]$, the running utility function $u(\cdot, t)$ and the terminal utility function U are assumed to be increasing and strictly concave.

Now, we can formalize our problem maximizing $V_0(\phi)$ over all $\phi \in \mathcal{Y}$. In particular, we define the optimal value V_0^* by

$$V_0^* = \sup_{\phi \in \mathcal{Y}} V_0(\phi).$$

If there exists a control law $\phi^* \in \mathcal{Y}$ such that

$$V_0^* = V_0(\phi^*),$$

then we say that ϕ^* is an *optimal control law* with respect to the initial state x_0. In general, an admissible control law may not exist. In this last case, it is common to indicate $V_0^* = -\infty$. Obviously, the object of the analysis is just to find, if it exists, the optimal control law of the problem.

As we have already highlighted, the methodology we will use here is the dynamic programming. An alternative approach is based on the martingale formulation proposed by Cox and Huang (1991).

The main idea is to extend the discrete-time Bellman equation in a continuous-time setting. By doing so, we will see that our original control problem will be embedded in the solution of a suitable partial differential equation, known as the Hamilton-Jacobi-Bellman (HJB) equation.

We fix a time $t \in [0, T]$ and a state $x \in \mathbb{R}^d$. Given the pair (x, t), we define the *value function* $V : \mathcal{Y} \times \mathbb{R}^d \times [0, T] \to \mathbb{R}$ as the function

$$V(\phi, x, t) = E_t \left[\int_t^T u\left(\phi\left(X(s), s\right), X^\phi(s), s\right) ds + U\left(X^\phi(T)\right) \right].$$

Then, we consider the problem of maximizing the value function $V(\phi, x, t)$ with respect to the control law ϕ, where the state process is given by

$$dX^\phi(s) = \mu\left(X^\phi(s), s\right) ds + \Sigma\left(X^\phi(s), s\right) dW(s), \quad s \in [t, T]$$
$$X^\phi(t) = x.$$

Let $J : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ be the *optimal value function* defined by

$$J(x, t) = \sup_{\phi \in \mathcal{Y}} V(\phi, x, t). \tag{2.3}$$

where $J(x_0, 0) = V_0^*$.

We assume that there exists an optimal control law, denoted by ϕ^*, and that $J \in C^{2,1}(\mathcal{Y} \times [0,T])$. If we start in state x at time t, the value function $V(\phi, x, t)$ can be interpreted as the expected utility involved by the control law ϕ in the time interval $[t, T]$, while the optimal value function $J(x, t)$ represents the optimal expected utility "remaining" at time t.

In order to define the HJB equation, the usual way to proceed is to compare the expected utilities computed with respect to the optimal law ϕ^* and another suitable control law, say $\widehat{\phi}$. Given a fixed but arbitrary $\phi \in \mathcal{Y}$, we define the control law $\widehat{\phi} \in \mathcal{Y}$ as follows:

$$\widehat{\phi}(z, s) = \begin{cases} \phi(z, s), & s \in [t, t + \Delta t] \\ \phi^*(z, s), & s \in (t + \Delta t, T], \end{cases}$$

where $\Delta t > 0$ represents an arbitrary time increment smaller than $(T - t)$.

The expected utility for ϕ^*, conditionally on the fact that at time t we are in state x, is simply given by $V(\phi^*, x, t) = J(x, t)$, while the expected utility for the strategy $\widehat{\phi}$ results to be

$$V\left(\widehat{\phi}, x, t\right) = E_t\left[\int_t^{t+\Delta t} u\left(\phi(X(s), s), X^\phi(s), s\right) ds + J\left(X^\phi(t + \Delta t), t + \Delta t\right)\right].$$

Actually, we note that the expected utility for the interval $[t, t + \Delta t]$, computed on ϕ, is given by

$$E_t\left[\int_t^{t+\Delta t} u\left(\phi(X(s), s), X^\phi(s), s\right) ds\right].$$

On the other hand, in the interval $(t + \Delta t, T]$ we have supposed to use the optimal law ϕ^*. Therefore, since at time $t + \Delta t$ we are in the stochastic state $X^\phi(t + \Delta t)$, the remaining expected utility is just given by

$$V\left(\phi^*, X^\phi(t + \Delta t), t + \Delta t\right) = J\left(X^\phi(t + \Delta t), t + \Delta t\right).$$

Now, if we compare the two control laws, we obtain the following inequality:

$$J(x, t) \geq E_t\left[\int_t^{t+\Delta t} u\left(\phi(X(s), s), X^\phi(s), s\right) ds + J\left(X^\phi(t + \Delta t), t + \Delta t\right)\right].$$
$$(2.4)$$

We note that we have an equality in (2.4) if and only if ϕ is also an optimal control law. In general, there is no restriction which implies that the optimal

control is unique. Since $J \in C^{2,1} (\mathcal{Y} \times [0,T])$, we can apply Itô's Lemma to $J \left(X^{\phi} (t + \Delta t), t + \Delta t \right)$. We obtain that

$$J \left(X^{\phi} (t + \Delta t), t + \Delta t \right) = J(x,t) + \int_{t}^{t+\Delta t} \mathcal{D} J \left(X^{\phi} (s), s \right) ds \qquad (2.5)$$

$$+ \int_{t}^{t+\Delta t} J_x \left(X^{\phi} (s), s \right) \Sigma \left(X^{\phi} (s), s \right) dW(s).$$

Thus, the expected value of $J \left(X^{\phi} (t + \Delta t), t + \Delta t \right)$, conditionally on the fixed pair (x,t), is given by

$$E_t \left[J(x,t) + \int_{t}^{t+\Delta t} \mathcal{D} J \left(X^{\phi} (s), s \right) ds \right].$$

We recall from Section 1.1 that the expected value of the stochastic integral in Equation (2.5) vanishes because it is a martingale.

If we apply this result to the inequality (2.4), we obtain that

$$E_t \left[\int_{t}^{t+\Delta t} \left[u \left(\phi (X(s), s), X^{\phi} (s), s \right) + \mathcal{D} J \left(X^{\phi} (s), s \right) \right] ds \right] \le 0. \qquad (2.6)$$

We now divide the expected value by Δt, and then we let Δt tend to zero. Under suitable technical conditions, we can take the limit within the expected value. This implies that

$$u(\phi, x, t) + \mathcal{D} J(x,t) \le 0.$$

Since we have fixed ϕ as an arbitrary control law in $\mathcal{Y}$, we get the equality in (2.6) if and only if ϕ is optimal. Thus, we can write

$$\sup_{\phi \in \mathcal{Y}} \left[u(\phi, x, t) + \mathcal{D} J(x,t) \right] = 0. \qquad (2.7)$$

Moreover, we have also fixed t as an arbitrary time in $[0,T]$. This allows us to establish that Equation (2.7) holds for all $t \in [0,T]$. From the definition of optimal value function (see Equation (2.3)), we note that, for any $x \in \mathbb{R}^d$, we obtain the terminal condition $J(x,T) = U(x)$.

Finally, we can define the HJB equation of our control problem as follows:

$$\begin{cases} \sup_{\phi \in \mathcal{Y}} \left[u(\phi, x, t) + \mathcal{D} J(x,t) \right] = 0, & (x,t) \in \mathbb{R}^d \times [0,T], \\ J(x,T) = U(x), & x \in \mathbb{R}^d. \end{cases}$$

In next sections we will introduce a market model and we will apply the stochastic control to the common problem of consumption and portfolio choice in continuous-time.

2.2   The financial market

In this section we consider the problem of an economic agent who wants to intertemporally allocate investment and consumption over a time horizon $[0, T]$ in order to optimize a given objective function. We consider a frictionless, arbitrage-free financial market which is continuously open over the fixed time interval $[0, T]$. We fix a k-dimensional real Brownian motion $W = \left(W^1, ..., W^k\right)^\top$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ represents the historical probability measure. We recall that $\mathcal{F}_t$ denotes the augmented filtration generated by $\mathcal{F}_t^W$ and the null subsets of $\mathcal{F}$. The process W describes the uncertainty involved by the financial market, while the natural interpretation of the σ-algebra $\mathcal{F}_t$ is to view it as the information available to investors at time $t \in [0, T]$.

The financial market is characterized by $(d+1)$ securities. The first security is a riskless asset whose price is given by

$$dS^0\left(t\right) = r\left(t\right) S^0\left(t\right) dt, \tag{2.8}$$
$$S^0\left(0\right) = 1,$$

where $r \in \mathcal{L}^1$ denotes the short rate. We note that the riskless asset can be interpreted as a bank account paying the instantaneous interest rate $r(t)$ without any default risk.

The prices of the d risky assets, for any $t \in [0, T]$, are defined by an Itô process $S = \left(S^1, ..., S^d\right)^\top$ in $\mathbb{R}^d$:

$$dS^i\left(t\right) = S^i\left(t\right) \left\{\mu^i\left(t\right) dt + \sum_{j=1}^k \sigma^{i,j}\left(t\right) dW^j\left(t\right)\right\}, \quad i = 1, ..., d,$$
$$S^i\left(0\right) > 0,$$

where $\mu^i \in \mathcal{L}^1$ and $\sigma^{i,j} \in \mathcal{L}^2$ are deterministic functions.

In the present model we assume that μ, σ and r are all deterministic functions, and that $r\left(t\right)$ is positive for all $t \in [0, T]$. In the next chapters, we will generalize this setting by introducing in the model a stochastic interest rate.

In order to simplify the notation, we summarize the risky components of

the financial market as follows

$$dS\left(t\right) = \underset{d\times 1}{I_S\left(t\right)} \left\{ \underset{d\times 1}{\mu\left(t\right)} + \underset{d\times d}{\Sigma\left(t\right)} \underset{d\times k}{dW\left(t\right)} \underset{k\times 1}{} \right\}, \quad t \in [0,T] \tag{2.9}$$

where $\Sigma\left(t\right) = [\sigma^{i,j}\left(t\right)]$ and $I_S\left(t\right) \in \mathbb{R}^{d\times d}$ is the diagonal matrix with respect to the elements of $S\left(t\right)$, namely,

$$I_S\left(t\right) = \begin{bmatrix} S^1\left(t\right) & 0 & \ldots & 0 \\ 0 & S^2\left(t\right) & 0 & \ldots \\ \ldots & 0 & \ldots & 0 \\ 0 & \ldots & 0 & S^d\left(t\right) \end{bmatrix}.$$

We recall that the standard Brownian motions $W^1, W^2, ..., W^k$ are independent. We can impose this constraint without loss of generality. In fact, we can always shift from uncorrelated to suitable correlated Wiener processes (and vice versa) via the Cholesky decomposition of the correlation matrix. A description of the Cholesky decomposition of the correlation matrix is provided in Appendix A.

At time $t = 0$ the agent is endowed with an initial wealth $x_0 > 0$. For any $i = 0, 1, ..., d$, we define $\pi^i(t)$ as the number of shares of asset S^i hold in portfolio at time t. Let $X\left(t\right)$ denote the wealth of the investor at time $t \in [0, T]$, then

$$X\left(t\right) = \sum_{i=0}^{d} \pi^i\left(t\right) S^i\left(t\right),$$

where the process π is assumed to be adapted with $\pi^i \in \mathcal{L}^2$ for any $0 \leq i \leq d$.

We allow the agent to consume over the given time horizon. His consumption at time t is denoted by $c\left(t\right)$. We assume that c is an adapted nonnegative process with $\int_0^T c\left(t\right) dt < \infty$ almost surely. Moreover, we restrict the agent's investment-consumption strategies to be *budget-feasible*: given an initial wealth $x_0 \in \mathbb{R}_+$, we say that an investment-consumption pair (π, c) is budget-feasible if the wealth process $X\left(t\right)$ satisfies, for any $t \in [0, T]$, the self-financing condition given by

$$X\left(t\right) = x_0 + \sum_{i=0}^{d} \left(\int_0^t \pi^i\left(s\right) dS^i\left(s\right) \right) - \int_0^t c\left(s\right) ds, \tag{2.10}$$

which is equivalent to the differential form

$$dX(t) = \sum_{i=0}^{d} \pi^i(t) \, dS^i(t) - c(t) \, dt. \tag{2.11}$$

This means that variations of wealth due to changes in portfolio composition at time t must compensate what the investor consumes in t.

Now, we rewrite the number of shares of the riskless asset hold in portfolio at time t as follows

$$\pi^0(t) = \frac{X(t) - \sum_{i=1}^{d} \pi^i(t) \, S^i(t)}{S^0(t)}.$$

Substituting for $\pi^0(t)$ into Equation (2.11), we obtain that

$$dX(t) = \left[X(t) - \sum_{i=1}^{d} \pi^i(t) \, S^i(t) \right] \frac{dS^0(t)}{S^0(t)} + \sum_{i=1}^{d} \pi^i(t) \, dS^i(t) - c(t) \, dt.$$

In order to simplify the wealth dynamics and to ease the following computations, we define $\theta^i(t)$ as the amount of wealth invested in S^i at time t, then

$$\theta^i(t) = \pi^i(t) \, S^i(t), \quad i = 1, ..., d, \ t \in [0, T].$$

Thus, an investor's trading strategy is fully defined by a process $\theta = (\theta^1, ..., \theta^d)^{\mathsf{T}}$. For any $i = 1, ..., d$, we assume $\theta^i \in \mathcal{L}^2$ so that the stochastic integral $\int \theta dW$ exists (see Section 1.1). Now, we can write the wealth dynamics as follows

$$dX(t) = \left[X(t) - \theta(t)^{\mathsf{T}} \mathbf{1} \right] \frac{dS^0(t)}{S^0(t)} + \theta(t)^{\mathsf{T}} I_S^{-1}(t) \, dS(t) - c(t) \, dt,$$

where $\mathbf{1} \in \mathbb{R}^d$ is a vector containing only ones. Finally, substituting for $S^0(t)$ and $S(t)$ from Equations (2.8) and (2.9), the wealth process becomes

$$dX(t) = \left[X(t) r(t) + \theta(t)^{\mathsf{T}} (\mu(t) - r(t) \mathbf{1}) - c(t) \right] dt \tag{2.12}$$
$$+ \theta(t)^{\mathsf{T}} \Sigma(t) \, dW(t),$$

where we recall that $X(0) = x_0$. We note that the wealth process X is an Itô process. By applying the Ito's formula defined in Proposition (4) to Equation

(2.12), we find the solution

$$X(t) S^0(t) = x_0 + \int_0^t S^0(s) \left[\theta^\mathsf{T}(s) (\mu(s) - r(s) \mathbf{1}) - c(s) \right] ds$$
$$+ \int_0^t S^0(s) \theta^\mathsf{T}(s) \Sigma(s) \, dW(s)$$

where $S^0(t) = \exp\left(\int_0^t r(s) \, ds \right)$.

2.3 The optimal control problem

Since μ, Σ and r are all deterministic functions, we can leave the asset prices out of the definition of the state process for the optimal control problem. Thus, the unique state variable directly involved in the dynamic programming procedure is the investor's wealth. On the other hand, the control process is fully characterized by all the investment-consumption pairs $(\theta, c) \in \mathcal{Y}$. We recall that $\mathcal{Y}$ represents the set of admissible control processes (see Section 2.1). In this case, a pair (θ, c) is admissible if the self-financing condition (2.10) is verified and, given an initial wealth x_0, if there exists a unique Itô process X satisfying Equation (2.12).

For any $t \in [0, T]$, we assume a running utility function $u : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ and a terminal utility function $U : \mathbb{R}_+ \to \mathbb{R}$. With respect to the general setting in Section 2.1, here we have

$$u(\theta(X(t), t), c(X(t), t), X(t), t) = u(c(X(t), t), t).$$

In order to simplify the notation, from now on we will indicate

$$c(X(t), t) \equiv c(t),$$
$$\theta(X(t), t) \equiv \theta(t).$$

Thus, for any initial state $x_0 \in \mathbb{R}_+$, we define the *initial value function* of the optimal control problem as the function $V_0 : \mathcal{Y} \to \mathbb{R}$ given by

$$V_0(\theta, c) = E_0 \left[\int_0^T u(c(t), t) \, dt + U(X(T)) \right], \quad (\theta, c) \in \mathcal{Y},$$

where, for any $t \in [0, T]$, the running utility $u(\cdot, t)$ and the terminal utility U are increasing and strictly concave functions.

In this case, we have supposed that the investor's preferences are represented by a utility function separable with respect to consumption and terminal wealth. We note that a utility function allows us to characterize the risk aversion of investors. In particular, we are not interested in the value itself assumed by an utility function, but in the preferences ordering which is involved by such a value. Under our assumptions, when we force utility functions to be concave with respect to consumption and wealth, we implicitly suppose that all investors are risk averse economic agents. Another important remark concerns the evolution of investor's preferences over the time horizon. In most economic applications, it is very common to assume that utility functions are separable with respect to time. This implies that investor's preferences are independent with respect to time. In the next chapters, we will adopt this assumption.

The goal of the investor is to make the optimal consumption and portfolio choice in order to maximize his utility in the finite horizon $[0, T]$. The optimal control problem can be formally stated as follows

$$
\begin{cases}
\sup\limits_{(\theta, c) \in \mathcal{Y}} V_0(\theta) \\
dX = \left[Xr + \theta^{\mathsf{T}} (\mu - r\mathbf{1}) - c \right] dt + \theta^{\mathsf{T}} \Sigma dW, \\
X(0) = x_0.
\end{cases} \tag{2.13}
$$

For the sake of simplicity, from now on we will not indicate the functional time dependence, unless it is strictly necessary.

2.4 The Hamilton-Jacobi-Bellman equation

In this section, we apply the Bellman approach described in Section 2.1 to the optimal control problem (2.13).

The optimal value function $J \in C^{2,1}(\mathbb{R}_+ \times [0, T])$ relative to the control problem (2.13) is defined by

$$
J(x, t) = \sup\limits_{(\theta, c) \in \mathcal{Y}} V(\theta, c, x, t),
$$

where

$$V(\theta, c, x, t) = E_t \left[\int_t^T u(c, s) \, ds + U(X(T)) \right].$$

For any $t \in [0, T]$, we note that $J(\cdot, t)$ can be read as the investor's indirect utility function for the wealth at time t. Then, we can define the HJB equation corresponding to the control problem (2.13) as follows:

$$\sup_{(\theta, c) \in \mathcal{Y}} [u(c, t) + \mathcal{D}J(x, t)] = 0, \quad (x, t) \in \mathbb{R}^d \times [0, T], \quad (2.14)$$

with the boundary condition

$$J(x, T) = U(x), \quad x \in \mathbb{R}_+.$$

Let H be the Hamiltonian corresponding to control problem (2.13) and defined by

$$\mathsf{H}(\theta, c, t) = u(c, t) + J_x \left[Xr + \theta^\mathsf{T} (\mu - r\mathbf{1}) - c \right] + \frac{1}{2} J_{xx} \theta^\mathsf{T} \Sigma \Sigma^\mathsf{T} \theta. \quad (2.15)$$

Thus, the HJB equation (2.14) can be rewritten as follows

$$J_t(x, t) + \sup_{(\theta, c) \in \mathcal{Y}} \mathsf{H}(\theta, c, t) = 0.$$

It is quite common to work directly with the Hamiltonian. Actually, since J is the optimal value function, we note that maximizing $[u(c, t) + \mathcal{D}J(x, t)]$ is equivalent to maximizing the Hamiltonian.

Therefore, we want to maximize the Hamiltonian H with respect to the control variables c and θ. We recall that $u(\cdot, t)$ and U are both increasing and strictly concave functions. Computing the first order condition with respect to consumption, we obtain that

$$\left. \frac{\partial \mathsf{H}}{\partial c} \right|_{c=c^*} = \left. \frac{\partial u}{\partial c} \right|_{c=c^*} - J_x = 0 \quad (2.16)$$

$$\implies c^*(t) = I_u [J_x(x, t), t], \quad t \in [0, T],$$

where c^* denotes the optimal consumption choice and $I_u(\cdot, t)$ is the inverse of $\frac{\partial}{\partial c} u(\cdot, t)$, meaning that $I_u \left[\frac{\partial}{\partial c} u(x, t), t \right] = x$ for all x and t. When $u = 0$, we

let $I_u = 0$. On the other hand, the first order condition with respect to trading strategies gives us

$$\frac{\partial \mathsf{H}}{\partial \theta}\bigg|_{\theta = \theta^*} = J_x \left(\mu - r\mathbf{1} \right) + J_{xx} \Sigma \Sigma^{\mathsf{T}} \theta^* = 0 \tag{2.17}$$

$$\implies \theta^* = -\frac{J_x}{J_{xx}} \left(\Sigma \Sigma^{\mathsf{T}} \right)^{-1} \left(\mu - r\mathbf{1} \right),$$

where θ^* denotes the optimal portfolio strategy. We note that the optimal portfolio is given by a time dependent vector $\left[\left(\Sigma \Sigma^{\mathsf{T}} \right)^{-1} \left(\mu - r\mathbf{1} \right) \right]$, proportional to the Sharpe ratio, and the reciprocal of the Arrow-Pratt absolute risk aversion index with respect to the value function J. In order to guarantee the invertibility of the matrix $\left(\Sigma \Sigma^{\mathsf{T}} \right)$ it is sufficient to impose that Σ is a full rank matrix.

For any $(x,t) \in \mathbb{R}_+ \times [0,T]$, the PDE associated with our optimal control problem, the so-called Hamilton-Jacobi-Bellman equation, is defined by

$$\begin{cases} J_t + u\left(c^*, t \right) + J_x \left[Xr + \left(\theta^* \right)^{\mathsf{T}} \left(\mu - r\mathbf{1} \right) - c^* \right] \\ + \frac{1}{2} J_{xx} \left(\theta^* \right)^{\mathsf{T}} \Sigma \Sigma^{\mathsf{T}} \left(\theta^* \right) = 0, \\ \\ J\left(x, T \right) = U\left(x \right), \quad x \in \mathbb{R}^d. \end{cases}$$

Finally, after substituting c^* and θ^* into the Hamiltonian (2.15), the HJB equation can be written as

$$\begin{cases} J_t + u\left(c^*, t \right) + J_x \left[Xr - c^* \right] - \frac{1}{2} \frac{J_x^2}{J_{xx}} \xi^{\mathsf{T}} \xi = 0, \\ \\ J\left(x, T \right) = U\left(x \right), \quad x \in \mathbb{R}_+, \end{cases} \tag{2.18}$$

where ξ denotes the market price of risk which solves the following equation:[1]

$$\Sigma \xi = \mu - r\mathbf{1}. \tag{2.19}$$

---

[1] As the market is arbitrage-free, there exists always a solution ξ to Equation (2.19). Moreover, if the financial market is complete and $d = k$, the diffusion matrix Σ is invertible and the market price of risk can be written as

$$\xi = \Sigma^{-1} \left(\mu - r\mathbf{1} \right).$$

We recall that the solution of this PDE is just the value function corresponding to our optimal control problem. The hard work of stochastic dynamic programming consists just in solving the nonlinear PDE involved by the associated optimal control problem. There is no general analytical method to solve a parabolic PDE as (2.18). In this case, one usually tries to guess a priori a suitable parametrized form for the solution to the PDE, and then he uses the PDE itself in order to specify the parameters. However, this analytical procedure is hardly ever successful. Thus, we must often try for a numerical approximation. In the next chapter, we will develop a particular case in which it is quite easy to solve the HJB equation: the Merton's model with log-normal asset-prices.

Chapter 3

The Merton's model with HARA utility

In this chapter we present the optimal consumption and portfolio choice problem presented by Merton (1969, 1971). This model is probably the first successful application of stochastic control in economics. In particular, we aim to solve explicitly the control problem for general hyperbolic absolute risk aversion utility (HARA) functions. On the other hand, we suppose that asset-prices are log-normal. Although this condition on asset-prices represents a strong restriction in terms of interpretation of the model, it is assumed very often in financial literature. Actually, working with geometric Brownian motions allow us to solve quite easily the HJB equation associated with the optimal consumption and portfolio choice problem.

3.1 The optimal control problem

The asset-prices are assumed to be log-normal, that is to say that the process S is defined by a geometric Brownian motion in $\mathbb{R}^d$ (see Section 1.2). Then, we have

$$dS(t) = I_S(t) \{\mu dt + \Sigma dW(t)\}, \quad t \in [0, T],$$
$$S(0) = S_0.$$

where μ and Σ are both constituted by constant elements. The short rate r is also constant, then the riskless asset is given by

$$\frac{dS^0(t)}{S^0(t)} = rdt, \quad t \in [0, T],$$

with $S^0(0) = 1$.

As we have remarked in Section 2.3, it is common to take utility functions which are separable with respect to time. This means that investor's preferences do not change over the time horizon. Then, we assume that the running utility function u can be written as

$$u(c, t) = e^{-\rho t} v(c),$$

where ρ is a real nonnegative constant. We note that ρ can be interpreted as an intertemporal preference rate which allows us to compare different consumption budgets over time. Now, we suppose that the function $v : \mathbb{R}_+ \to \mathbb{R}$ belongs to the family of the hyperbolic absolute risk aversion functions, that is

$$v(c) = h_1 (\alpha + \gamma c)^{1-\frac{\beta}{\gamma}},$$

where h_1, α, β, and $\gamma \neq 0$ are all constant and $(\alpha + \gamma c) > 0$. The family of HARA functions allows us to generalize the present model in several ways. Actually, taking suitable adjustment of the parameters, we can define utility functions with absolute or relative risk aversion which is increasing, decreasing, or constant. In particular, if $\alpha = 0$ we obtain a constant relative risk aversion (CRRA) function, while if we take the limit for $\gamma \to 0$ and $\alpha = 1$ we find a constant absolute risk aversion (CARA) function.

In the present model we have assumed that the utility function u is increasing and strictly concave with respect to consumption. Then, $v(c)$ must satisfy the following conditions:

$$\begin{cases} h_1 (\alpha + \gamma c)^{-\frac{\beta}{\gamma}} (\gamma - \beta) > 0, \\ h_1 (\alpha + \gamma c)^{-\frac{\beta+\gamma}{\gamma}} (\beta - \gamma) \beta < 0, \end{cases}$$

This implies that

$$h_1 (\gamma - \beta) > 0,$$

$$\beta > 0.$$

Thus, the associated Arrow-Pratt absolute risk aversion index is given by

$$-\left(\frac{\partial^2 v}{\partial c^2}\right)\left(\frac{\partial v}{\partial c}\right)^{-1} = \frac{\beta}{\alpha + \gamma c} > 0.$$

We assume that the terminal utility U is also an hyperbolic function given by

$$U\left(x\right) = e^{-\rho T} h_2 \left(\alpha + \gamma x\right)^{1-\frac{\beta}{\gamma}}, \quad x \in \mathbb{R}_+,$$

where h_2 is a constant. The parameters α, β, γ, and ρ are the same defined above. Since consumption is measured in monetary unit, it is consistent to adopt the same functional form for the utility function u, whose argument is consumption, and the utility function U, whose argument is the terminal wealth. Moreover, the parameters h_1 and h_2 allow us to scale each utility function as we want. We note that the parameters h_1 and h_2 measure the relative weight, in terms of utility, of consumption and terminal wealth respectively in the optimization model. Obviously, if $h_1 = 0$, the control problem reduces to maximize the terminal wealth. On the other hand, if $h_2 = 0$, the terminal utility disappears. In the following, we will see that only the ratio $\frac{h_1}{h_2}$ will be really relevant in the optimal control problem. We recall that T is not a time variable, but a fixed terminal date.

The optimal control problem can be summarized as follows:

$$\begin{cases} \sup_{(\theta,c)\in\mathcal{Y}} E_0 \left[\int_t^T e^{-\rho s} h_1 \left(\alpha + \gamma c\left(s\right)\right)^{1-\frac{\beta}{\gamma}} ds + e^{-\rho T} h_2 \left(\alpha + \gamma X\left(T\right)\right)^{1-\frac{\beta}{\gamma}}\right] \\ dX = \left[Xr + \theta^{\mathsf{T}}\left(\mu - r\mathbf{1}\right) - c\right] dt + \theta^{\mathsf{T}}\Sigma dW, \\ X\left(0\right) = x_0. \end{cases}$$

3.2 The Hamilton-Jacobi-Bellman equation

We will now use the results of the previous chapter in order to derive the Hamilton-Jacobi-Bellman equation associated with the Merton's model with HARA utilities and log-normal asset-prices. For any $t \in [0, T]$, we recall that the Hamiltonian H is given by

$$\mathsf{H} = u\left(c, t\right) + J_x \left[Xr + \theta^{\mathsf{T}}\left(\mu - r\mathbf{1}\right) - c\right] + \frac{1}{2}J_{xx}\theta^{\mathsf{T}}\Sigma\Sigma^{\mathsf{T}}\theta, \qquad (3.1)$$

where $J(\cdot, t)$ is the investor's indirect utility function which satisfies the boundary condition

$$J(x, T) = U(x), \quad x \in \mathbb{R}_+.$$

In this case, the first order condition with respect to consumption (Equation (2.16)) implies that

$$\left. \frac{\partial H}{\partial c} \right|_{c=c^*} = e^{-\rho t} h_1 (\alpha + \gamma c^*)^{-\frac{\beta}{\gamma}} (\gamma - \beta) - J_x = 0 \tag{3.2}$$

$$\implies c^* = \frac{1}{\gamma} \left(\frac{J_x}{e^{-\rho t} h_1 (\gamma - \beta)} \right)^{-\frac{\gamma}{\beta}} - \frac{\alpha}{\gamma}, \quad t \in [0, T].$$

On the other hand, the first order condition with respect to trading strategies, Equation (2.17), is not directly affected by the functional form of the utility functions. Then, the optimal portfolio choice is given by

$$\theta^* = -\frac{J_x}{J_{xx}} \left(\Sigma \Sigma^\mathsf{T} \right)^{-1} (\mu - r\mathbf{1}). \tag{3.3}$$

After substituting the optimal value c^* and θ^* into the HJB Equation (3.1), we find, for any $(x, t) \in \mathbb{R}_+ \times [0, T]$, the following PDE:

$$\begin{cases} J_t + e^{-\rho t} h_1 (\alpha + \gamma c^*)^{1-\frac{\beta}{\gamma}} + J_x (Xr - c^*) \\ -\frac{1}{2} \frac{J_x^2}{J_{xx}} \xi^\mathsf{T} \xi = 0, \\ \\ J(x, T) = e^{-\rho T} h_2 (\alpha + \gamma x)^{1-\frac{\beta}{\gamma}}, \quad x \in \mathbb{R}_+. \end{cases} \tag{3.4}$$

3.3   The optimal consumption and portfolio choice

We aim to solve explicitly the parabolic PDE (3.4). Given the functional form of the corresponding boundary condition, we try for a solution, here called *guess function*, defined as follows

$$J(x, t) = e^{-\rho t} h(t) [a(t) + g(t) x]^{1-\frac{\beta}{\gamma}}, \quad t \in [0, T], \tag{3.5}$$

where $h(t)$, $a(t)$, and $g(t)$ are the functions we need to explicit. After replacing the guess function (3.5) into the HJB Equation (3.4), we obtain that

$$0 = -J\rho + J\frac{h_t}{h} + \frac{1}{2}J\left(\frac{\gamma}{\beta} - 1\right)\xi^{\mathsf{T}}\xi$$
$$+ J\left(1 - \frac{\beta}{\gamma}\right)[a + gX]^{-1}[a_t + g_tX + g(Xr - c^*)]$$
$$+ e^{-\rho t}h_1(\alpha + \gamma c^*)^{1-\frac{\beta}{\gamma}}.$$

Now, we substitute c^* from Equation (3.2) and divide for the value function J. Then, the HJB equation can be rewritten as

$$0 = -\rho + \frac{1}{h}\left[h_t + \beta\gamma^{\frac{\gamma}{\beta}-2}h_1^{\frac{\gamma}{\beta}}(hg)^{1-\frac{\gamma}{\beta}}\right] + \frac{1}{2}\left(\frac{\gamma}{\beta} - 1\right)\xi^{\mathsf{T}}\xi$$
$$+ \left(1 - \frac{\beta}{\gamma}\right)(a + gX)^{-1}\left[(g_t + gr)X + a_t + \frac{\alpha}{\gamma}g\right].$$

In this case, the solution of the HJB (3.4) is equivalent[1] to the solution of the system of first order differential equations defined by

$$\begin{cases} h_t + h\left[\frac{1}{2}\left(\frac{\gamma}{\beta} - 1\right)\xi^{\mathsf{T}}\xi - \rho\right] + \beta\gamma^{\frac{\gamma}{\beta}-2}h_1^{\frac{\gamma}{\beta}}(hg)^{1-\frac{\gamma}{\beta}} = 0 \\ g_t + gr = 0 \\ a_t + \frac{\alpha}{\gamma}g = 0. \end{cases} \quad (3.6)$$

In order to solve exactly this system, we need a consistent set of boundary conditions. However, we note that, for any $x \in \mathbb{R}_+$, it must be $J(x,T) = U(x)$, that is

$$e^{-\rho T}h_2(\alpha + \gamma x)^{1-\frac{\beta}{\gamma}} = e^{-\rho T}h(T)[a(T) + g(T)x]^{1-\frac{\beta}{\gamma}}.$$

This implies that

$$\begin{cases} h(T) = h_2, \\ g(T) = \gamma, \\ a(T) = \alpha. \end{cases}$$

Thus, we are able to solve the three differential equations in System (3.6). In particular, we note that the equation for $g(t)$ and $a(t)$ are two ODE whose solution is simply given by

$$g(t) = \gamma e^{r(T-t)},$$
$$a(t) = \frac{\alpha}{r}\left(e^{r(T-t)} + r - 1\right).$$

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[1]Using a well-known argument of "separation of variables".

On the other hand, the equation for $h(t)$ is a Bernoulli equation whose solution is represented by

$$h(t) = h_2 \, e^{\left[\frac{\beta}{\gamma}(r-A)-r\right](T-t)} \left\{ 1 - \left(\frac{h_1}{h_2}\right)^{\frac{\gamma}{\beta}} \frac{1-e^{A(T-t)}}{A} \right\}^{\frac{\beta}{\gamma}},$$

where we let

$$A = \frac{\beta}{\gamma \left[\frac{1}{2}\left(\frac{\gamma}{\beta}-1\right)\xi^{\mathsf{T}}\xi - \rho + r\right] - r\beta}.$$

Now, we can compute explicitly the optimal values c^* and θ^*. After tedious computation, the optimal consumption can be written as

$$c^*(t) = \frac{1}{\gamma} \frac{\left(\frac{\alpha}{r} + \gamma X(t)\right)e^{A(T-t)} - \frac{\alpha}{r}(1-r)e^{(A-r)(T-t)}}{\left(\frac{h_2}{h_1}\right)^{\frac{\gamma}{\beta}} - \frac{1-e^{A(T-t)}}{A}} - \frac{\alpha}{\gamma}, \quad t \in [0,T] \quad (3.7)$$

Finally, we determine the optimal investment strategy θ^*. Given the value function in Equation (3.5), we note that

$$-\frac{J_x}{J_{xx}} = \frac{\gamma}{\beta}\left(\frac{a(t)}{g(t)} + X\right)$$
$$= \frac{1}{\beta r}\left[\alpha - \alpha(1-r)e^{-r(T-t)} + \gamma r X\right].$$

Then, we have

$$\theta^*(t) = \frac{1}{\beta r}\left[\alpha - \alpha(1-r)e^{-r(T-t)} + \gamma r X(t)\right]\left(\Sigma\Sigma^{\mathsf{T}}\right)^{-1}(\mu - r\mathbf{1}). \quad (3.8)$$

We can note that the optimal values (3.7) and (3.8) are both linear in wealth. Merton (1971) proves that the HARA family is the only class of concave utility functions which implies this property for the solutions of the control problem.

Chapter 4

Portfolio choice with stochastic interest rate

In the previous chapters, since μ, Σ and r were assumed as deterministic functions, we left the asset prices out of the definition of the state process, then the unique state variable directly involved in the dynamic programming procedure was the investor's wealth. Thus, the state process underlying the control problem was simply a scalar. Now, we generalize the control problem in the sense that we allow interest rates to be stochastic. By doing so, we will introduce in the state process a new variable, the short rate, in addition to the wealth. Although the basic analysis is the same as for the case of a deterministic interest rate, the introduction of another relevant state variable causes many computational difficulties. In particular, finding a well-behaved solution to the HJB equation associated with the control problem becomes a delicate matter. However, we will see that under suitable separability conditions it will be possible to solve the HJB equation directly through the Feynman-Kač Theorem (see Section 1.6).

4.1   The financial market

In this section we will not include the consumption process in the stochastic control problem. Thus, we focus only on the optimal portfolio choice problem.

The financial market is frictionless, arbitrage-free and continuously open

over the fixed time interval $[0, T]$. Moreover, we assume that the market is complete. We fix a d-dimensional Brownian motion $W = \left(W^1, ..., W^d\right)^\mathsf{T}$ on $(\Omega, \mathcal{F}, \mathbb{P})$.

For any $t \in [0, T]$, we assume that the short rate process $r(t)$ is defined by

$$dr(t) = \alpha(r, t)\, dt + \sigma(r, t)\, dW^r(t),\qquad (4.1)$$
$$r(0) = r_0,$$

where α and σ are two functions such that a unique solution of the above SDE exists.

Without loss of generality, we agree that the first component of the d-dimensional Brownian motion W is just given by W^r (i.e $W^1 \equiv W^r$), then the standard Brownian motion W^r in $\mathbb{R}$ determines the uncertainty involved in the financial market by the interest rate.

The financial market is still characterized by $(d+1)$ securities. The first is a riskless asset, whose price is given by

$$dS^0(t) = r(t)\, S^0(t)\, dt,\qquad (4.2)$$

where r is defined in Equation (4.1) and $S^0(0) = 1$.

The prices of the d risky assets, for any $t \in [0, T]$, are defined by an Itô process $S = \left(S^1, ..., S^d\right)^\mathsf{T}$ in $\mathbb{R}^d$:

$$dS(t)_{\underset{d \times 1}{}} = I_S(t)_{\underset{d \times d}{}} \left\{ \mu(r, t)_{\underset{d \times 1}{}} + \Sigma(r, t)_{\underset{d \times d}{}} dW(t)_{\underset{d \times 1}{}} \right\}, \quad t \in [0, T],\qquad (4.3)$$
$$S(0) = S_0,$$

where $\mu \in \left(\mathcal{L}^1\right)^d$ and $\Sigma \in \left(\mathcal{L}^2\right)^{d \times d}$ are deterministic functions. In this case, we have as much risky assets as risky sources. We note that the term ΣdW can be written more explicitly as follows

$$\Sigma dW = \begin{bmatrix} \sigma_{[r,S]} & \Sigma_{[S]} \end{bmatrix} \begin{bmatrix} dW^r \\ d\overline{W} \end{bmatrix},$$

where $\overline{W} = \left(W^2, ..., W^d\right)^\mathsf{T}$. In this case, the elements of the vector $\sigma_{[r,S]} \in \mathbb{R}^d$ describe how the uncertainty involved by interest rate affects each risky asset,

while the elements of the matrix $\Sigma_{[S]} \in \mathbb{R}^{d \times (d-1)}$ relate the d risky assets to the other risk sources inside the financial market. For any $t \in [0,T]$, we suppose that the diffusion matrix $\Sigma(r,t)$ has full rank equal to d. Thus, consistently with the assumption of complete market, the diffusion matrix $\Sigma(t,r)$ is invertible.

Let now define the wealth process X. At time $t = 0$ the representative investor is endowed with an initial wealth $x_0 > 0$. From Section 2.2, we recall that $\theta^i(t)$, for any $i = 0, 1, ..., d$, denotes the amount of wealth invested in the asset S^i at time $t \in [0,T]$. In this case, the dynamic budget constraint is given by

$$dX(t) = \sum_{i=0}^{d} \theta^i(t) \frac{dS^i(t)}{S^i(t)}, \quad t \in [0,T]. \tag{4.4}$$

This means that the investor neither adds any cash to finance changes in portfolio composition nor withdraws any cash to consume until $t = T$.

Now, for any $t \in [0,T]$, we write the amount of wealth invested in the riskless asset as the difference between the accumulated wealth and the total amount invested in the risky assets, namely,

$$\theta^0(t) = X(t) - \sum_{i=1}^{d} \theta^i(t). \tag{4.5}$$

Thus, we define an investor's trading strategy by the process $\theta = \left(\theta^1, ..., \theta^d \right)^\mathsf{T}$ in $(\mathcal{L}^2)^d$.

For any time $t \in [0,T]$, substituting $\theta^0(t)$, $S^0(t)$ and $S(t)$ from Equations (4.5), (4.2) and (4.3) into the budget constraint (4.4), the dynamics of the wealth process X is given by

$$dX(t) = \left[X(t)r(t) + \theta(t)^\mathsf{T}(\mu(t) - r(t)\mathbf{1}) \right] dt \tag{4.6}$$
$$+ \theta(t)^\mathsf{T} \Sigma(t) dW(t),$$
$$X(0) = x_0.$$

Moreover, we can note that X is an Itô process. By applying the Ito's formula derived in Proposition (4) to Equation (4.6), we find the explicit equation

for the wealth

$$X(t) S^0(t) = x_0 + \int_0^t S^0(s) \left[\theta^{\mathsf{T}}(s) (\mu(s) - r(s) \mathbf{1})\right] ds$$
$$+ \int_0^t S^0(s) \theta^{\mathsf{T}}(s) \Sigma(s) dW(s).$$

4.2 The optimal control problem

Now, we move to define the optimal control problem. In the present model, the state process is defined by the pair $(r, X) \in \mathcal{S}$, where $\mathcal{S}$ denotes the set of the state processes involved in the dynamic programming. As we have already remarked, when we assume the interest rate to be stochastic instead of deterministic, the wealth process by itself is not sufficient to fully describe the state dynamics characterizing our financial market. We need to specify explicitly the short rate in the state process entering in the control problem. In this case, although the basic analysis is the same as for the case of a deterministic interest rate, we will see that the introduction of another relevant state variable (in addition to the wealth) causes many computational difficulties. In particular, finding a well-behaved solution to the HJB equation associated with the control problem becomes harder and harder. On the other hand, the control process is fully characterized by all the trading strategies $\theta \in \mathcal{Y}$, where $\mathcal{Y}$ denotes the set of admissible controls. In this case, a trading strategy θ is admissible if it verifies the dynamic budget constraint (4.4) and, given an initial wealth x_0, if there exists a unique Itô process X satisfying Equation (4.6).

Since there is no consumption, we set the running utility function u identically equal to zero. For any $t \in [0, T]$, the investor's preferences are represented only by the terminal utility function $U : \mathbb{R}_+ \to \mathbb{R}$, where U is supposed to be an increasing and concave function. We assume that U is a CARA function defined by

$$U(x) = \eta e^{\gamma x},$$

where η and γ are strictly negative parameters. We note that the associated

absolute risk aversion index is just given by the parameter γ, namely

$$-\left(\frac{\partial^2 U}{\partial x^2}\right)\left(\frac{\partial U}{\partial x}\right)^{-1} = -\gamma.$$

Then, for any initial state $(r_0, x_0) \in \mathcal{S}$, the initial value function $V_0 : \mathcal{Y} \to \mathbb{R}$ is given by

$$V_0(\theta) = E_0\left[\eta e^{\gamma X(T)}\right], \quad \theta \in \mathcal{Y}.$$

In order to ease the notation, for any $t \in [0, T]$, the interest rate dynamics can be rewritten as

$$dr(t) = \alpha(r, t)\, dt + \delta(r, t)^{\mathsf{T}}\, dW(t),$$

where $\delta(r, t)$ is a vector in $\mathbb{R}^d$ given by $\delta(r, t) = (\sigma(r, t), 0, ..., 0)^{\mathsf{T}}$.

Thus, we can define the optimal control problem as follows

$$\begin{cases} \sup_{\theta \in \mathcal{Y}} V_0(\theta) \\ d\begin{bmatrix} r \\ X \end{bmatrix} = m\, dt + M\, dW, \\ X(0) = x_0, \ r(0) = r_0, \end{cases} \qquad (4.7)$$

where

$$\underset{(2\times 1)}{m} = \begin{bmatrix} \alpha \\ Xr + \theta^{\mathsf{T}}(\mu - r\mathbf{1}) \end{bmatrix},$$

$$\underset{(2\times d)}{M} = \begin{bmatrix} \delta^{\mathsf{T}} \\ \theta^{\mathsf{T}}\Sigma \end{bmatrix}.$$

The scalar variables r and X represent the two state variables, while the elements of θ represent the d control variables.

4.3 The Hamilton-Jacobi-Bellman equation

Now, we want to apply the dynamic programming to the control problem defined in (4.7). The corresponding optimal value function $J \in C^{2,1}(\mathcal{S} \times [0, T])$ is defined by

$$J(r, x, t) = \sup_{\theta \in \mathcal{Y}} V(\theta, r, x, t),$$

where

$$V(\theta, r, x, t) = E_t \left[\eta e^{\gamma X(T)} \right].$$

and by the boundary condition

$$J(r, x, T) = \eta e^{\gamma x}, \quad x \in \mathbb{R}_+.$$

Given the value function J, we define the Hamiltonian H associated with (4.7) by

$$\mathsf{H} = \alpha J_r + \left(Xr + \theta^{\mathsf{T}} (\mu - r\mathbf{1}) \right) J_x + \frac{1}{2} tr \left(MM^{\mathsf{T}} \begin{bmatrix} J_{rr} & J_{rx} \\ J_{xr} & J_{xx} \end{bmatrix} \right).$$

Working out the Hamiltonian, we obtain that

$$\mathsf{H} = \alpha J_r + \left(Xr + \theta^{\mathsf{T}} (\mu - r\mathbf{1}) \right) J_x + \frac{1}{2} \delta^{\mathsf{T}} \delta J_{rr} + \theta^{\mathsf{T}} \Sigma \delta J_{xr} + \frac{1}{2} \theta^{\mathsf{T}} \Sigma \Sigma^{\mathsf{T}} \theta J_{xx}.$$

Thus, the first order conditions give us

$$\left. \frac{\partial \mathsf{H}}{\partial \theta} \right|_{\theta = \theta^*} = (\mu - r\mathbf{1}) J_x + \Sigma \delta J_{xr} + (\theta^*)^{\mathsf{T}} \Sigma \Sigma^{\mathsf{T}} J_{xx} = 0. \qquad (4.8)$$

Solving the linear System (4.8), we obtain that

$$\theta^* = \underbrace{-\frac{J_x}{J_{xx}} \left(\Sigma \Sigma^{\mathsf{T}} \right)^{-1} (\mu - r\mathbf{1})}_{\theta_{(1)}^*} \underbrace{-\frac{J_{xr}}{J_{xx}} \left(\Sigma^{\mathsf{T}} \right)^{-1} \delta}_{\theta_{(2)}^*}, \qquad (4.9)$$

where the value function J must satisfy the HJB equation

$$\begin{cases} J_t + \mathsf{H}|_{\theta = \theta^*} = 0 \\ J(r, x, T) = \eta e^{\gamma x}, \quad x \in \mathbb{R}_+. \end{cases} \qquad (4.10)$$

Thus, we can formally state the following result.

Proposition 9 *Given the financial market defined by (4.1), (4.2), (4.3), and the wealth process (4.6), the portfolio strategy solving the optimal control problem (4.7) is formed by two components:*

1. *a speculative component $\theta_{(1)}^*$ proportional to both the portfolio Sharpe ratio and the reciprocal of the Arrow-Pratt absolute risk aversion index;*

2. an hedging component $\theta^*_{(2)}$ depending on the state variable parameters.

Comparing θ^* in Equation (4.9) with the optimal portfolio strategy derived in the Merton's model (see Section 3.2), we note that the first component $\theta^*_{(1)}$ is just the same speculative component derived in Equation (3.3). In addition, the assumption of stochastic interest rate has introduced a new hedging component $\theta^*_{(2)}$ depending on the state variable parameters. Actually, if we let $\sigma = 0$, then $\theta^*_{(2)}$ disappears and we find again the Merton's model with constant interest rate.

After substituting θ^* from (4.9) into the HJB Equation (4.10), the PDE we need to solve is given by

$$0 = J_t + J_r \alpha + J_x X r \qquad (4.11)$$
$$+ \frac{1}{2} \left(J_{rr} - \frac{(J_{xr})^2}{J_{xx}} \right) \delta^\mathsf{T} \delta$$
$$- \frac{J_x J_{xr}}{J_{xx}} \delta^\mathsf{T} \xi - \frac{1}{2} \frac{(J_x)^2}{J_{xx}} \xi^\mathsf{T} \xi,$$

with the boundary condition

$$J(r, x, T) = \eta e^{\gamma x}, \quad x \in \mathbb{R}_+.$$

We recall that ξ denotes the market price of risk defined by

$$\xi = \Sigma^{-1} \left(\mu - r\mathbf{1} \right).$$

As we have already remarked, there is no general analytical method to solve a PDE such as (4.11). The usual strategy is to try for a parametrized guess function and then to use the PDE itself in order to specify the parameters. In this case, the presence of a stochastic interest rate causes further difficulties in specifying a consistent form for the guess function. We will see that the usual separability conditions on the utility function fails. However, we will specify a guess function which will allow us to simplify the HJB equation and therefore to solve it explicitly through the Feynman-Kač Theorem.

4.4 An exact solution

A standard approach to solve a PDE such as (4.11) is to try for a separability condition. In the financial literature, since Merton (1969, 1971), the condition

of separability in wealth by product represents a common assumption in the attempt to solve explicitly optimal portfolio problems. Here, the presence of a stochastic interest rate prevents us from applying successfully such a separability assumption. In other words, this means that the PDE (4.11) does not simplify as much as we need to solve it explicitly. Before presenting an exact solution of our control problem, we highlight why in this case the usual separability condition does not allow us to simplify and to solve explicitly the PDE (4.11).

Let us assume that the value function J is given by the product of two terms: an increasing and concave function of the wealth X, and an exponential function depending on time and on the interest rate r. Thus, the value function J can be written as follows

$$J(r, x, t) = U(x)e^{h(r,t)}.$$

Now, we suppose that the function $U : \mathbb{R}_+ \to \mathbb{R}$ belongs to the family of the CARA functions, that is

$$U(x) = \eta e^{\gamma x},$$

where η and γ are both strictly negative constants. Thus, the HJB Equation (4.11) can be written as

$$0 = h_t + h_r \left(\alpha - \delta^\mathsf{T}\xi\right) + \frac{1}{2}h_{rr}\delta^\mathsf{T}\delta - \frac{1}{2}\xi^\mathsf{T}\xi + \gamma X r, \qquad (4.12)$$

with the boundary condition

$$h(r, T) = 0.$$

We note that the term depending directly on the wealth X makes difficult to find a solution of (4.12). On the other hand, if we assumed $\gamma = 0$, the PDE (4.12) would satisfy the assumption of the Feynman-Kač Theorem (see Section 1.6), but the value function would become independent of the wealth.

Therefore, we will try for a different guess function not satisfying the usual separability condition. Initially, we suppose that the the value function J has the following form

$$J(r, x, t) = \eta e^{g(r,t)x + h(r,t)}.$$

After substituting for J into (4.11), the HJB equation can be written as

$$0 = \left[g_t + g_r \left(\alpha - \delta^\mathsf{T}\xi \right) + \frac{1}{2} \left(g_{rr} - \frac{2g_r^2}{g} \right) \delta^\mathsf{T}\delta + gr \right] X \qquad (4.13)$$

$$+ h_t + h_r \left(\alpha - \delta^\mathsf{T}\xi - \frac{g_r}{g}\delta^\mathsf{T}\delta \right) + \frac{1}{2} h_{rr}\delta^\mathsf{T}\delta$$

$$- \frac{g_r}{g}\delta^\mathsf{T}\xi - \frac{1}{2} \left(\frac{g_r}{g} \right)^2 \delta^\mathsf{T}\delta - \frac{1}{2}\xi^\mathsf{T}\xi,$$

with the boundary conditions

$$\begin{cases} g\left(r, T \right) = \gamma, \\ h\left(r, T \right) = 0. \end{cases}$$

As we have already remarked in Section 3.2, the solution of the HJB Equation (4.13) is equivalent to the solution of the system of PDEs defined by

$$\begin{cases} g_t + g_r \left(\alpha - \delta^\mathsf{T}\xi \right) + \frac{1}{2} \left(g_{rr} - \frac{2g_r^2}{g} \right) \delta^\mathsf{T}\delta + gr = 0, \\ h_t + h_r \left(\alpha - \delta^\mathsf{T}\xi - \frac{g_r}{g}\delta^\mathsf{T}\delta \right) + \frac{1}{2} h_{rr}\delta^\mathsf{T}\delta + b = 0, \end{cases} \qquad (4.14)$$

where

$$b \equiv - \frac{g_r}{g}\delta^\mathsf{T}\xi - \frac{1}{2} \left(\frac{g_r}{g} \right)^2 \delta^\mathsf{T}\delta - \frac{1}{2}\xi^\mathsf{T}\xi.$$

In order to solve the first PDE in System (4.14), we apply the procedure exposed in Zariphopoulou (2001). The object is to find a function g such that the term given by $\left(g_{rr} - \frac{2g_r^2}{g} \right)$ simplifies in a term depending only on g_{rr}. Thus, we let that

$$g\left(r, t \right) = q\left(r, t \right)^\beta . \qquad (4.15)$$

After substituting for (4.15) into (4.14) and dividing by $\left(\beta q^{\beta-1} \right)$, we obtain that

$$q_t + q_r \left(\alpha - \delta^\mathsf{T}\xi \right) + \frac{1}{2} \left[q_{rr} - \frac{q_r^2}{q} \left(\beta + 1 \right) \right] \delta^\mathsf{T}\delta + qr = 0. \qquad (4.16)$$

If we let $\beta = -1$, Equation (4.16) can be rewritten as

$$q_t + q_r \left(\alpha - \delta^\mathsf{T}\xi \right) + \frac{1}{2} q_{rr}\delta^\mathsf{T}\delta + qr = 0, \qquad (4.17)$$

with the boundary condition

$$q\left(r, T \right) = \frac{1}{\gamma}.$$

Now, we apply the Feynman-Kač Theorem stated in Section 1.6 to the PDE (4.17). So, the solution to (4.17) can be represented as follows

$$q\left(r, t\right) = \frac{1}{\gamma} E_t \left[e^{-\int_t^T \widetilde{r}(s) ds} \right],$$

where the stochastic process $\widetilde{r}$ is defined by

$$d\widetilde{r}\left(s\right) = \left(\alpha - \delta^\mathsf{T} \xi\right) ds + \delta^\mathsf{T} dW\left(s\right),$$
$$\widetilde{r}\left(t\right) = r\left(t\right).$$

We note that the process $\widetilde{r}$ has the same diffusion term as the stochastic interest rate r, while the drift is different. From Section 1.4, we recall that we can always modify the drift term of an Itô process by a suitable change of probability as stated in Girsanov's Theorem. In particular, it can be verified that the probability measure $\mathbb{Q}$ which links $\widetilde{r}$ to r is just the *equivalent martingale measure* given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left\{ -\int_0^T \xi^\mathsf{T} dW\left(t\right) - \frac{1}{2} \int_0^T \|\xi\|^2 dt \right\}.$$

Accordingly, the standard Brownian motion $W_\mathbb{Q}$ is defined by

$$dW_\mathbb{Q} = \xi dt + dW.$$

We recall that $\mathbb{Q}$ is an equivalent martingale measure if it is a probability measure equivalent to $\mathbb{P}$ with the property that the discounted price process $\left(\frac{S}{S^0}\right)$ is a martingale. We note that $q\left(r, t\right)$ is just the price of a zero-coupon bond under $\mathbb{Q}$ up to a constant $\left(\frac{1}{\gamma}\right)$, namely

$$q\left(r, t\right) = \frac{1}{\gamma} E_t^Q \left[e^{-\int_t^T r(s) ds} \right].$$

Finally, given the solution q to the first PDE in System (4.14), we can substitute for it in the second PDE and then solve it directly by the Feynman-Kač Theorem. Thus, its solution can be represented by

$$h\left(r, t\right) = E_t \left[\int_t^T b\left(\widehat{r}, s\right) ds \right],$$

where the stochastic process $\widehat{r}$ is defined by

$$d\widehat{r}(t) = \left(\alpha - \delta^\mathsf{T}\xi + \frac{q_r}{q}\delta^\mathsf{T}\delta\right)dt + \delta^\mathsf{T}dW(t),$$

$$\widehat{r}(t) = r(t).$$

In the next chapter we will present a complete application of this methodology when the interest rate is specified by an Ornstein -Uhlenbeck process. Under this assumption, we will be able to compute explicitly the expected values involved in the value function solving the optimal control problem.

Chapter 5

Portfolio choice for defined-contribution pension plans

The introduction of a stochastic non-financial income in the optimal portfolio problem causes several computational difficulties, although the underlying methodological approach is the same as that presented in Chapter 4. When we consider a risk source outside the financial market which directly affects the wealth level (for example salaries), the HJB equation associated with the control problem becomes harder to solve. On the other hand, when we focus on the optimal portfolio strategies for a DC pension fund, we cannot overlook the leading role of the salary process.

Since Merton (1971), where a deterministic wage income enters the consumption and portfolio choice problem, the introduction of a non-financial income in the optimal portfolio problem has represented a key issue. In the recent literature, Haberman and Vigna (2001) suggest a model for DC pension funds in discrete-time with a fixed contribution rate. Some models for DC pension fund in continuous-time are provided by Boulier et al. (2001) and Deelstra et al. (2001), who both describe salaries through a deterministic process, and by Blake et al. (2000), who consider a stochastic process for salaries

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[1]This chapter reproduces the main results presented in Battocchio and Menoncin (2002), and then extended in Battocchio and Menoncin (2004).

including a "non-hedgeable" component, in the sense that the salary risk is outside the financial market and cannot be hedge. Under these assumptions, Blake et al. (2000) do not find a complete solution of the model. Here, we will also introduce in the salary process a non-hedgeable component, but we will model it in a suitable way through the consumer price index in order to solve explicitly the control problem. We will note how the assumption of stochastic salaries instead of deterministic ones allows us to really consider the effects associated with the labor income, and in particular to its resolution over time.

The problem of optimal portfolio choice for a long-term investor endowed with a wage income goes beyond the features of the pension fund management. El Karoui and Jeanblanc-Picqué (1998), under the assumptions of complete market and constant interest rate, solve a portfolio optimization problem for an economic agent endowed with a "stochastic insurable stream" of labor income. In this way, they introduce an income process which adds no other source of uncertainty to those existing in the financial market. Campbell and Viceira (2002) consider a discrete-time model where the labor income is viewed as a dividend on the individual's implicit holding of human wealth. Franke et al. (2001) analyze how the portfolio choice changes when the labor income uncertainty is resolved early or late in the time horizon. In their model the labor income is considered as a cumulated value added to the terminal wealth.

In this chapter, we will describe and solve a stochastic control problem for a DC pension fund whose accumulation phase is characterized by a stochastic process for contributions. We will link the only non-hedgeable component of the salary process to a consumer price index, whose role will be widely discussed. In this setting, we will completely solve the HJB equation associated with the control problem and therefore we will be able to analyze how the optimal portfolio is affected by the uncertainty involved by the labor market.

This chapter is organized as follows. Sections 5.1-3 describe the general structure of the model: the stochastic processes describing the behavior of asset prices, salaries and the consumer price index. In Section 5.4 we derive the wealth equation of the fund both in nominal and real terms. In Section 5.5 we formally state the optimal control problem, while in Section 5.6 we define the HJB equation and we introduce the main results. An explicit solution to

the HJB equation is derived in Section 5.7. In Section 5.8 we discuss some important properties of the optimal portfolio. Finally, a numerical application is presented in Section 5.9.

5.1 The financial market

In this section we introduce the market structure under which the optimal asset allocation problem is defined.

We consider an arbitrage-free, complete and frictionless financial market which is continuously open over the fixed time interval $[0, T]$. We will assume that $T > 0$ coincides with the retirement time of a representative shareholder. The uncertainty involved by the financial market is described by two standard and independent Brownian motions $W^r(t)$ and $W^S(t)$, with $t \in [0, T]$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We assume a stochastic interest rate $r(t)$ defined by an Ornstein-Uhlenbeck process (see Vasiček, 1977). Then, under the historical probability measure $\mathbb{P}$, the process $r(t)$ is given by

$$dr(t) = \alpha (\beta - r(t)) dt + \sigma dW^r(t), \tag{5.1}$$
$$r(0) = r_0,$$

where α, β, and σ are strictly positive constants. Thus, the interest rate presents a mean-reverting effect where the parameter β is the "mean" level attracting the interest rate while the parameter α measures the strength of this attraction.

Given the differential equation of the interest rate we can derive both its closed form expression and the value of a zero coupon bond with fixed maturity. A proof of the following proposition is illustrated in Vasiček (1977).

Proposition 10 *Suppose that the interest rate $r(t)$ satisfies the SDE (5.1), then:*

1. *the explicit solution of (5.1) is*

$$r(t) = (r_0 - \beta) e^{-\alpha t} + \beta + \sigma \int_0^t e^{-\alpha(t-u)} dW^r(u);$$

2. the price of a zero coupon bond with maturity $\tau > t$ is given by

$$B(t, \tau, r) = e^{b(t,\tau) - a(t,\tau)r(t)},$$

where

$$a(t, \tau) = \frac{1 - e^{-\alpha(\tau - t)}}{\alpha},$$

$$b(t, \tau) = -R(\infty)(\tau - t) + a(t, \tau)\left[R(\infty) - \frac{\sigma^2}{2\alpha^2}\right] + \frac{\sigma^2}{4\alpha^3}\left(1 - e^{-2\alpha(\tau - t)}\right),$$

$R(\infty) = \beta + \frac{\sigma\xi}{\alpha} - \frac{\sigma^2}{2\alpha^2}$ *represents the long-term yield, and ξ denotes the constant market price of risk.*

The financial market is characterized by a riskless asset $S^0(t)$ and by two risky assets. Given the initial condition $S^0(0) = 1$, the price process $S^0(t)$ is given by

$$\frac{dS^0(t)}{S^0(t)} = r(t)dt, \tag{5.2}$$

where the dynamics of the short rate $r(t)$, under the real probability measure $\mathbb{P}$, is defined in Equation (5.1).

The first risky asset we assume is a stock whose price $S(t)$ satisfies the following SDE:

$$\frac{dS(t)}{S(t)} = \mu_S(t, r)dt + \nu\sigma dW^r(t) + \sigma_S dW^S(t), \tag{5.3}$$

$$S(0) = S_0,$$

where $\nu \neq 0$ represents a volatility scale factor measuring how the interest rate volatility affects the stock volatility, and $\sigma_S \neq 0$ is the stock own volatility. Thus, the whole stock instantaneous volatility is given by $\sqrt{\nu^2\sigma^2 + \sigma_S^2}$. Moreover, we assume that the instantaneous mean has the form $\mu_S(t, r) = r(t) + m_S$, where $m_S > 0$ can be interpreted as a risk premium. The parameter m_S is assumed strictly positive so that the stock return is higher than the return on the riskless asset. For the sake of simplicity, we introduce in our model only one stock, which can be interpreted as a stock market index. Nevertheless, if we allow for a complete market with a finite number of stocks, no further difficulties are added to the model. In this case, the symbol S stays for "stock"

and denotes a scalar process instead of a vector process as in the previous chapters.

Now, we assume that there exists a market for zero coupon bonds for every maturity $\tau \in [0, T]$. According to Proposition (10), the return of a zero coupon bond with maturity $\tau \in [0, T]$ is given by

$$\frac{dB(t, \tau, r)}{B(t, \tau, r)} = (r(t) + \mathrm{a}(t, \tau)\,\sigma\xi)\,dt - \mathrm{a}(t, \tau)\,\sigma dW^r(t), \tag{5.4}$$

$$B(\tau, \tau, r) = 1,$$

where $\mathrm{a}(t, \tau)$ was defined in Proposition (10).

Nevertheless, as pointed out in Boulier et al. (2001), assuming the existence of infinite zero coupon bonds is quite unrealistic. However, since the interest rate defined by (5.1) is a one-factor model, at any time t, we only need one zero coupon bond price to obtain the other ones. If all bonds are regarded as derivatives of the underlying interest rate r, then they are all characterized by the same market price of risk (see for example Björk, 1998). Therefore, when the market has specified the price process of a "basic" bond, say with maturity τ_K, the market has also indirectly specified its market price of risk, but, as we have just noted, this is the same for all bonds. Then, the basic τ_K-bond and the stochastic interest rate r fully determine the price of all bonds.

As in Boulier et al. (2001), the second risky asset we introduce is a bond "rolling over" zero coupon bonds with constant time to maturity τ_K. The price $B_K(t, r)$ of such a bond is given by

$$\frac{dB_K(t, r)}{B_K(t, r)} = (r(t) + \mathrm{a}_K\sigma\xi)\,dt - \mathrm{a}_K\sigma dW^r(t), \tag{5.5}$$

where

$$\mathrm{a}_K = \frac{1 - e^{-\alpha\tau_K}}{\alpha}.$$

Boulier et al. (2001) claim that the asset allocation problem can be solved, without any loss of generality, just taking into account this "rolling bond". Actually, the instantaneous return on $B(t, \tau, r)$ and $B_K(t, r)$ are linked (through the riskless asset $S^0(t)$) by the following linear equation:

$$\frac{dB(t, \tau, r)}{B(t, \tau, r)} = \left(1 - \frac{\mathrm{a}(t, \tau)}{\mathrm{a}_K}\right)\frac{dS^0(t)}{S^0(t)} + \frac{\mathrm{a}(t, \tau)}{\mathrm{a}_K}\frac{dB_K(t, r)}{B_K(t, r)}.$$

Accordingly, we can state that in the model it is indifferent to use B or B_K. Here, we choose to work B_K because it allows us both to simplify the following computations and to enhance the interpretation of the model.

The diffusion matrix Σ associated with our financial market is given by:

$$\Sigma = \begin{bmatrix} \nu\sigma & \sigma_S \\ -a_K\sigma & 0 \end{bmatrix},$$

and, since σ_S and σ are different from zero by hypothesis, and $a_K \neq 0$ by construction, it follows that

$$\det \Sigma = \sigma_S a_K \sigma \neq 0.$$

Since we have as many risky assets as risk sources and the diffusion matrix Σ is invertible, the financial market defined by (5.2), (5.3) and (5.5) is complete.

5.2 The inflation risk

Usually, portfolio choice problems do not consider the inflation risk. However, when we study the optimal portfolio choice for a pension fund, the time period we consider is too long for neglecting the role of the inflation risk. Actually, although fund managers have to invest the nominal fund, they are interested in maximizing the growth rate of the real fund. In this work, we will solve the portfolio choice problem for a DC pension fund taking into account both the inflation risk and the salary risk. This last one is the risk associated with the contribution flow during the accumulation phase of the fund.

We will model the inflation risk by introducing a consumer price index (CPI) process, which can be interpreted as the price of the only consumption good in the economy. Consistently with our target, we will have that the CPI affects only the wealth growth rate, but not the amount of wealth that can be invested. Let us assume that the CPI process, denoted by p, is defined by the following SDE:

$$\frac{dp(t)}{p(t)} = \mu_\pi(t,r)dt + \rho_r\sigma dW^r(t) + \rho_S\sigma_S dW^S(t) + \sigma_\pi dW^\pi(t), \qquad (5.6)$$

$$p(0) = 1,$$

where $\mu_\pi(t, r) = r(t) + m_\pi$. Since the price level can be always normalized, we can fix $p(0) = 1$ without any loss of generality. The parameters ρ_r and ρ_S are two volatility scale factors measuring how the volatility of interest rate and stock affect the CPI. The real standard Brownian motion $W^\pi(t)$ defines the inflation risk characterizing the CPI process and it is assumed to be independent of $W^r(t)$ and $W^S(t)$. The reader is referred to Cox et al. (1985) for two particular functional forms which can be used for modeling inflation. A general equilibrium diffusion model with inflation uncertainty is presented by Moriconi (1994, 1995).

In the next section, after introducing the contribution process, we will widely discuss the fundamental role of the process W^π in our model.

We have supposed that the CPI is affected by both the risk sources of interest rate and stock. This means that we are assuming the stock index and the interest rate level as "good" explaining factors.

After applying the Itô's Lemma (see Section 1.2) to the logarithm of $p(t)$, we can find the explicit solution of Equation (5.6):

$$p(t) = \exp\left\{ \int_0^t r(\tau)\, d\tau + \left(m_\pi - \frac{1}{2}\rho_r^2\sigma^2 - \frac{1}{2}\rho_S^2\sigma_S^2 - \frac{1}{2}\sigma_\pi^2 \right) t \right.$$
$$\left. + \rho_r\sigma W^r(t) + \rho_S\sigma_S W^S(t) + \sigma_\pi W^\pi(t) \right\}.$$

In the next section we will state formally the role of the CPI process. We will see that it represents the key process which allow us to solve the portfolio choice problem in presence of stochastic non-financial income.

5.3 The defined-contribution process

In this section, we will introduce the defined-contribution process and we will link its dynamics to the CPI process. Then, we will reconsider the financial setting in order to define a market which is both complete and consistent with our target.

Let L denote the labor income process defined by the following SDE

$$\frac{dL(t)}{L(t)} = \mu_L(t, r)\, dt + \kappa_r\sigma dW^r(t) + \kappa_S\sigma_S dW^S(t) + \sigma_L dW^\pi(t), \qquad (5.7)$$

$$L(0) = L_0,$$

where κ_r and κ_S are two volatility scale factors measuring how the risk sources of interest rate and stock affect the salaries, while $\sigma_L \neq 0$ is the salary own volatility. Moreover, we assume that the instantaneous mean of salaries is such that $\mu_L(t,r) = r(t) + m_L$, where m_L is a real constant.

We note that the salary risk and the inflation risk are characterized by the same source of randomness (W^π). This assumption interprets consistently the following fact: a shock in the inflation rate usually generates a shock also to the labor market, and vice versa.

After applying the Itô's Lemma (see Section 1.2) to the logarithm of $L(t)$, we can find the explicit solution of Equation (5.7):

$$
L(t) = L_0 \exp \left\{ \int_0^t r(\tau) \, d\tau + \left(m_L - \frac{1}{2}\kappa_r^2\sigma^2 - \frac{1}{2}\kappa_S^2\sigma_S^2 - \frac{1}{2}\sigma_L^2 \right) t \right.
$$
$$
\left. + \kappa_r \sigma W^r(t) + \kappa_S \sigma_S W^S(t) + \sigma_L W^\pi(t) \right\}.
$$

Now, we assume that each employee puts a constant proportion ψ of his salary into the personal pension fund. Then, the defined-contribution level is characterized as follows

$$
C(t) = \psi L(t).
$$

Thus, we have that the contribution growth equals the wage growth, namely

$$
dC(t) = \psi dL(t).
$$

As we have already remarked, the introduction in the optimal portfolio problem of a stochastic non-financial income causes several problems in the application of the dynamic programming. In fact, if we assume that the contribution process is driven by a risk source which does not belong to those defining the financial market, that is a non-hedgeable risk, we obtain that the market is no more complete. In this case, even if we can state both the HJB equation and the corresponding optimal portfolio, we are not able to apply the Feynman-Kač Theorem and to find the optimal value function in a close form. Therefore, this prevents us from studying how the coefficients of the salary process affects the optimal portfolio strategies.

What we propose here is a model in which the presence of a stochastic contribution is consistent with the assumption of complete market. We note

that the only non-hedgeable component of the salary process (5.7) is just the diffusion term (W^π) linked to the CPI process. Now, if the inflation risk is considered as a risk outside the financial market, that is to say that p is not a tradable asset, the risk source represented by W^π does not belong to the set of financial market risk sources. In this case, the parameters σ_π and σ_L can be interpreted as non-hedgeable volatilities, the new diffusion matrix we obtain is not of full rank and the market becomes incomplete.

On the opposite, here we consider the CPI process p as the price of a tradable asset. Nowadays, this assumption is well supported, for example, by the presence at the Chicago Mercantile Exchange of CPI futures[2]. Since 1997, the launch of Inflation-Indexed Treasury Notes (TIPS) satisfied the strong demand for a U.S. dollar-denominated asset able to hedge against a rise in inflation. Moreover, an over-the-counter (OTC) inflation-indexed derivative market is already established both in U.S. and in Europe and it is characterized by a strong growth. At the end of third-quarter 2003, over \$160 billion in TIPS were outstanding. In U.S., some of the leading pension fund have from 5 to 10 percent of their portfolios allocated to inflation-indexed instruments. The European OTC inflation-indexed derivative market actually moves approximately \$8 billion in notional value. Therefore, these new opportunities not only justify our technical assumptions, but also value the interpretation of the present model.

In the next section, we will verify that assuming p tradable implies that the financial market is still complete and, given the invertibility of the corresponding diffusion matrix, we will be able to carry on the control problem until the explicit solution of the HJB equation.

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[2]The CPI futures contract traded at the Chicago Mercantile Exchange (www.cme.com) represents the inflation on a notional value of \$1,000,000 for a period of three months, implied by the U.S. Consumer Price Index.

5.4   The nominal and the real wealth processes

Let us summarize the whole economy as follows:

$$
\begin{cases}
dr\left(t\right) = \alpha\left(\beta - r(t)\right)dt + \sigma dW^r\left(t\right), \\[2ex]
\frac{dS^0(t)}{S^0(t)} = r(t)dt, \\[2ex]
\frac{dS(t)}{S(t)} = \mu_S(t,r)dt + \nu\sigma dW^r(t) + \sigma_S dW^S(t), \\[2ex]
\frac{dB_K(t,r)}{B_K(t,r)} = \left(r(t) + a_K\sigma\xi\right)dt - a_K\sigma dW^r(t), \\[2ex]
\frac{dL(t)}{L(t)} = \mu_L\left(t,r\right)dt + \kappa_r\sigma dW^r(t) + \kappa_S\sigma_S dW^S(t) + \sigma_L dW^\pi(t), \\[2ex]
\frac{dp(t)}{p(t)} = \mu_\pi(t,r)dt + \rho_r\sigma dW^r(t) + \rho_S\sigma_S dW^S(t) + \sigma_\pi dW^\pi\left(t\right).
\end{cases}
\tag{5.8}
$$

Let $\theta_S\left(t\right)$, $\theta_B\left(t\right)$, and $\theta_0(t)$ denote the amount of money invested in the two risky assets (i.e. the stock and the bond) and in the riskless asset respectively. Thus, the accumulated fund's nominal wealth X_N at any time $t \in [0,T]$ must verify the following condition:

$$
X_N\left(t\right) = \theta_0\left(t\right) + \theta_S\left(t\right) + \theta_B\left(t\right). \tag{5.9}
$$

From Section 2.2, we recall that θ_i, for any $i \in \{0,S,B\}$, is given by the number of asset i hold in the portfolio (π_i) times the corresponding asset value. Since the quantity π_i is a stochastic variable, in order to differentiate this product we must apply the Itô's formula. It follows that

$$
\begin{aligned}
dX_N = {} & d\pi_0 S^0 + \pi_0 dS^0 + d\pi_0 dS^0 \\
& + d\pi_S S + \pi_S dS + d\pi_S dS \\
& + d\pi_B B + \pi_B dB + d\pi_B dB.
\end{aligned}
$$

At each time $t \in [0,T]$, the only deposit in the fund is constituted by contributions and there is no withdraw. Then, we have the self-financing condition

$$
d\pi_0\left(S^0 + dS^0\right) + d\pi_S\left(S + dS\right) + d\pi_B\left(B + dB\right) = \psi dL,
$$

and the dynamic budget constraint

$$dX_N = \pi_0 dS^0 + \pi_S dS + \pi_B dB + \psi dL.$$

Finally, in terms of amount invested in each asset, we obtain that

$$dX_N = \theta_0 \frac{dS^0}{S^0} + \theta_S \frac{dS}{S} + \theta_B \frac{dB}{B} + \psi dL.$$

After substituting the values of the differentials, dX_N can be written as

$$
\begin{aligned}
dX_N = {}& \left(\theta_0 r + \theta_S \mu_S + \psi L \mu_L + \theta_B \left(r + \mathrm{a}_K \sigma \xi \right) \right) dt \\
& + \left(\theta_S \nu \sigma - \theta_B \mathrm{a}_K \sigma + \psi L \kappa_r \sigma \right) dW^r \\
& + \left(\theta_S \sigma_S + \psi L \kappa_S \sigma_S \right) dW^S + \psi L \sigma_L dW^\pi.
\end{aligned}
$$

Now, the real wealth can be defined as the ratio between the nominal fund's wealth and the CPI. Thus, we have:

$$X = \frac{X_N}{p},$$

and, after applying the Itô's Lemma (see Section 1.2) and substituting the value of X_N given in Equation (5.9), we can write:[3]

$$dX = \left(\theta^{\mathsf{T}} M + u \right) dt + \left(\theta^{\mathsf{T}} \Gamma^{\mathsf{T}} + \Lambda^{\mathsf{T}} \right) dW, \tag{5.10}$$

---

[3]We note that the Jacobian of the real wealth is given by

$$\nabla_{X_N,p} X = \begin{bmatrix} \frac{1}{p} \\ -\frac{X_N}{p^2} \end{bmatrix},$$

while its Hessian matrix is

$$\nabla^2_{X_N,p} F = \begin{bmatrix} 0 & -\frac{1}{p^2} \\ -\frac{1}{p^2} & 2\frac{X_N}{p^3} \end{bmatrix}.$$

where,

$$\theta \equiv \begin{bmatrix} \theta_0 & \theta_S & \theta_B \end{bmatrix}^\mathsf{T},$$

$$M \equiv \frac{1}{p} \begin{bmatrix} -m_\pi + \rho_r^2 \sigma^2 + \rho_S^2 \sigma_S^2 + \sigma_\pi^2 \\ m_S - m_\pi + \rho_r^2 \sigma^2 (1 - \nu) + \rho_S \sigma_S^2 (\rho_S - 1) + \sigma_\pi^2 \\ a_K \sigma \xi - m_\pi + \rho_r \sigma^2 (\rho_r + a_K) + \rho_S^2 \sigma_S^2 + \sigma_\pi^2 \end{bmatrix},$$

$$u \equiv \frac{\psi}{p} L \left(m_L - m_\pi + \rho_r \sigma^2 (\rho_r - \kappa_r) + \rho_S \sigma_S^2 (\rho_S - \kappa_S) + \sigma_\pi (\sigma_\pi - \sigma_L) \right),$$

$$\Gamma^\mathsf{T} \equiv \frac{1}{p} \begin{bmatrix} -\rho_r \sigma & -\rho_S \sigma_S & -\sigma_\pi \\ \sigma (\nu - \rho_r) & \sigma_S (1 - \rho_S) & -\sigma_\pi \\ -\sigma (a_K + \rho_r) & -\rho_S \sigma_S & -\sigma_\pi \end{bmatrix},$$

$$\Lambda \equiv \frac{\psi}{p} L \begin{bmatrix} \sigma (\kappa_r - \rho_r) & \sigma_S (\kappa_S - \rho_S) & \sigma_L - \sigma_\pi \end{bmatrix}^\mathsf{T},$$

$$W \equiv \begin{bmatrix} W^r & W^S & W^\pi \end{bmatrix}^\mathsf{T}.$$

We underline that the new diffusion matrix for the financial market is given by Γ which must be invertible if we want this market to be complete. In this case, we have:

$$\det(\Gamma) = -\frac{1}{p^3} a_K \sigma \sigma_\pi \sigma_S,$$

which is different from zero because σ, σ_S, and σ_π are different from zero by hypothesis, while $a_K \neq 0$ by construction (see Equation (5.5)). Thus, since p can be considered as the price of a tradable asset, the financial market is complete even after the introduction of W^π.

5.5   The optimal control problem

The target of the fund manager is to choose a portfolio strategy in order to maximize the expected value of a terminal utility function. The argument of this utility function is the real fund's wealth. We assume an exponential utility function of the form

$$U(X) = \eta e^{\gamma X},$$

where, in order to have an increasing and concave utility function, η and γ are strictly negative parameters. In this case, we think that the fund manager

takes in the preferences of a representative shareholder. Thus, the utility parameters describe an homogeneous group of shareholders with the same risk profile.

Now, we move to define the optimal control problem. Under the market structure (5.8), we note that the state variables involved in the dynamic programming are the real wealth X, the interest rate r, the labor income L and the consumer price index p. Here, the state process is defined by a vector $(r, L, p, X) \in \mathcal{S}$, where $\mathcal{S}$ denotes the set of the admissible state processes. On the other hand, the control process is fully characterized by all the trading strategies $\theta \in \mathcal{Y}$, where $\mathcal{Y}$ denotes the set of admissible controls. In this case, a trading strategy θ is admissible if it is self-financing and, given an initial wealth x_0, if there exists a unique Itô process X satisfying Equation (5.10).

Let we define $z \equiv \begin{bmatrix} r & L & p \end{bmatrix}^\mathsf{T}$ the vector containing all the state variable but the wealth. Then, for any initial state $(z_0, x_0) \in \mathcal{S}$, the initial value function $V_0 : \mathcal{Y} \to \mathbb{R}$ is given by

$$V_0(\theta) = E_0 \left[\eta e^{\gamma X(T)} \right], \quad \theta \in \mathcal{Y}.$$

Thus, we can define the optimal control problem as follows

$$\begin{cases} \sup_{\theta \in \mathcal{Y}} V_0(\theta) \\ \\ d \begin{bmatrix} z \\ X \end{bmatrix} = \begin{bmatrix} \mu_z \\ \theta^\mathsf{T} M + u \end{bmatrix} dt + \begin{bmatrix} \Omega^\mathsf{T} \\ \theta^\mathsf{T} \Gamma^\mathsf{T} + \Lambda^\mathsf{T} \end{bmatrix} dW, \\ \\ z(0) = z_0, \quad X(0) = x_0, \quad \forall 0 \le t \le T, \end{cases} \quad (5.11)$$

where,

$$\underset{3 \times 1}{\mu_z} \equiv \begin{bmatrix} \alpha(\beta - r) & L\mu_L & p\mu_\pi \end{bmatrix}^\mathsf{T},$$

$$\underset{3 \times 3}{\Omega^\mathsf{T}} \equiv \begin{bmatrix} \sigma & 0 & 0 \\ L\kappa_r \sigma & L\kappa_S \sigma_S & L\sigma_L \\ p\rho_r \sigma & p\rho_S \sigma_S & p\sigma_\pi \end{bmatrix}.$$

We will see that in the present model the exponential (CARA) utility function leads to a value function that is separable by product in the real wealth

(X) and all the other state variables (here z). As we have already remarked in Chapter 4, since Merton (1971), separability conditions have been widely used in order to solve the PDEs associated with control problems. When we describe by stochastic processes both wage income and inflation, it can be verified (see Menoncin (2002)) that the family of the CARA utility functions is the only one consistent with a condition of separability by product on the value function.

5.6 The Hamilton-Jacobi-Bellman equation

The optimal value function J associated with the control problem (5.11) is defined by

$$J(z, x, t) = \sup_{\theta \in \mathcal{Y}} V(\theta, z, x, t),$$

where

$$V(\theta, z, x, t) = E_t\left[\eta e^{\gamma X(T)}\right],$$

and the boundary condition

$$J(z, x, T) = \eta e^{\gamma x}, \quad x \in \mathbb{R}_+.$$

Thus, the Hamiltonian H associated with (5.11) is given by

$$\mathsf{H} = \mu_z^\mathsf{T} J_z + J_x\left(\theta^\mathsf{T} M + u\right) + \frac{1}{2}tr\left(\Omega^\mathsf{T}\Omega J_{zz}\right) + \left(\theta^\mathsf{T}\Gamma^\mathsf{T} + \Lambda^\mathsf{T}\right)\Omega J_{zx}$$
$$+ \frac{1}{2}J_{xx}\left(\theta^\mathsf{T}\Gamma^\mathsf{T}\Gamma\theta + 2\theta^\mathsf{T}\Gamma^\mathsf{T}\Lambda + \Lambda^\mathsf{T}\Lambda\right).$$

The system of the first order conditions on H is:[4]

$$\left.\frac{\partial\mathsf{H}}{\partial\theta}\right|_{\theta=\theta^*} = J_x M + J_{zx}\Gamma^\mathsf{T}\Omega + J_{xx}\left(\Gamma^\mathsf{T}\Gamma\theta + \Gamma^\mathsf{T}\Lambda\right) = 0, \qquad (5.12)$$

---

[4]The second order conditions hold if the Hessian matrix of H:

$$\frac{\partial\mathsf{H}}{\partial\theta^\mathsf{T}\partial\theta} = J_{xx}\Gamma^\mathsf{T}\Gamma,$$

is negative definite. Since the quadratic form $\Gamma^\mathsf{T}\Gamma$ is positive definite, the second order conditions are satisfied if and only if $J_{xx} < 0$, that is if the value function is concave in X. The reader is referred to Stokey and Lucas (1989) for the assumptions that must hold on the objective function for having a strictly concave value function.

where θ^* denotes as usual the optimal portfolio. Solving Equation (5.12), we obtain that

$$\theta^* = -\frac{J_x}{J_{xx}} \left(\Gamma^{\mathsf{T}} \Gamma \right)^{-1} M - \frac{J_{zx}}{J_{xx}} \left(\Gamma^{\mathsf{T}} \Gamma \right)^{-1} \Gamma^{\mathsf{T}} \Omega - \left(\Gamma^{\mathsf{T}} \Gamma \right)^{-1} \Gamma^{\mathsf{T}} \Lambda. \qquad (5.13)$$

Since the matrix Γ is invertible, θ^* simplifies as follows.

$$\theta^* = \underbrace{-\frac{J_x}{J_{xx}} \left(\Gamma^{\mathsf{T}} \Gamma \right)^{-1} M}_{\theta^*_{(1)}} \underbrace{-\frac{J_{zx}}{J_{xx}} \Gamma^{-1} \Omega}_{\theta^*_{(2)}} \underbrace{-\Gamma^{-1} \Lambda}_{\theta^*_{(3)}}. \qquad (5.14)$$

We note here why it is so important to obtain a diffusion matrix (in this case Γ) which is invertible. Without introducing the CPI process p, the salary risk remains non-hedgeable and the resulting diffusion matrix cannot be inverted. In this case, the term $\left(\Gamma^{\mathsf{T}} \Gamma \right)^{-1} \Gamma^{\mathsf{T}}$ in Equation (5.13) does not simplify and therefore it enters directly in the HJB equation, preventing us to apply the Feynman-Kač Theorem.

Thus, we can state the following result.

Proposition 11 *Under market structure (5.8), the portfolio composition solving the control problem (5.11) is formed by three components:*

1. *a speculative component $\theta^*_{(1)}$ proportional to both the portfolio Sharpe ratio and the reciprocal of the Arrow-Pratt risk aversion index;*

2. *a hedging component $\theta^*_{(2)}$ depending on the state variable parameters;*

3. *a preference-free hedging component $\theta^*_{(3)}$ depending only on the diffusion terms of assets and other variables.*

The preference free portfolio component $\theta^*_{(3)}$ has an important property: it minimizes the instantaneous variance of the wealth differential. In fact, from Equation (5.10) we can see that the variance of the growth in the investor's

wealth is given by[5]

$$Var_t\left(dX\right) = \left(\theta^\mathsf{T}\Gamma^\mathsf{T}\Gamma\theta + 2\theta^\mathsf{T}\Gamma^\mathsf{T}\Lambda + \Lambda^\mathsf{T}\Lambda\right)dt,$$

from which we can immediately formulate the following result.[6]

Proposition 12 *Given the optimal portfolio solving the control problem (5.11), the preference-free component $\theta^*_{(3)}$ minimizes the instantaneous variance of the wealth differential.*

For the first portfolio component $\theta^*_{(1)}$, we just outline that it increases when the real returns on assets (M) increase and decreases when the risk aversion $(-J_{FF}/J_F)$ or the asset variance $(\Gamma^\mathsf{T}\Gamma)$ increase. From this point of view, we can argue that this component of the optimal portfolio has just a speculative role.

The second part $\theta^*_{(2)}$ is the only optimal portfolio component explicitly depending on the diffusion terms of the state variables (Ω). Thus, while $\theta^*_{(3)}$ covers the investor from the risk associated with W^π, $\theta^*_{(2)}$ covers the investor also from the other risks associated with the financial market. In the next sections, after computing the functional form of the value function, we will investigate the precise role of this component.

5.7 An exact solution

For studying the exact role of the portfolio components $\theta^*_{(1)}$ and $\theta^*_{(2)}$ (see Equation (5.14)), we need to compute the value function $J\left(z, x, t\right)$. It can be demonstrated (see Menoncin, 2002) that, given the exponential utility function, the

---

[5]Let Y denote an Itô process defined by

$$dY = \mu_Y dt + \sigma_Y dW.$$

It is quite common to write $Var_t\left(dY\right) = \sigma_Y^2 dt$, although this is an abuse of notation because dY is not a random variable. Therefore, the above characterization is not rigorously justified, but it is used purely for its intuitive contents.

[6]We underline that the second derivative of $Var\left(dX\right)$ with respect to θ is $2\Gamma'\Gamma$, which is always positive definite because the quadratic form $\Gamma'\Gamma$ is positive definite.

value function is separable by product in wealth and in the other state variables as follows:

$$J(z, x, t) = \eta e^{\gamma X + h(z,t)},$$

where the function $h(z, t)$ must satisfy the PDE

$$\begin{cases} h_t + \left(\mu_z^{\mathsf{T}} - M^{\mathsf{T}}\Gamma^{-1}\Omega\right) h_z + b(z, t) + \frac{1}{2}tr\left(\Omega^{\mathsf{T}}\Omega h_{zz}\right) = 0, \\ \\ h(z, T) = 0. \end{cases} \tag{5.15}$$

and where,

$$b(z, t) \equiv \gamma u - \gamma M^{\mathsf{T}}\Gamma^{-1}\Lambda - \frac{1}{2}M^{\mathsf{T}}\left(\Gamma^{\mathsf{T}}\Gamma\right)^{-1}M.$$

Since we can apply the Feynman-Kač theorem to Equation (5.15), we can state the following result.

Proposition 13 *Under market structure (5.8), the portfolio composition maximizing the expected exponential utility of fund's terminal real wealth is as follows:*

$$\theta^* = -\Gamma^{-1}\Lambda - \frac{1}{\gamma}\left(\Gamma^{\mathsf{T}}\Gamma\right)^{-1}M - \frac{1}{\gamma}\Gamma^{-1}\Omega\int_t^T \frac{\partial}{\partial z_t}E_t\left[b\left(\widetilde{z}_s, s\right)\right]ds,$$

where

$$d\widetilde{z}_s = \left(\mu_{\widetilde{z}} - \Omega^{\mathsf{T}}\left(\Gamma^{-1}\right)^{\mathsf{T}}M\right)ds + \Omega\left(\widetilde{z}_s, s\right)^{\mathsf{T}}dW,$$

$$\widetilde{z}_t = z_t,$$

$$b(z, t) \equiv \gamma u - \gamma\Lambda^{\mathsf{T}}\left(\Gamma^{-1}\right)^{\mathsf{T}}M - \frac{1}{2}M^{\mathsf{T}}\left(\Gamma^{\mathsf{T}}\Gamma\right)^{-1}M.$$

We underline that, in our market structure, the quadratic term $M^{\mathsf{T}}\left(\Gamma^{\mathsf{T}}\Gamma\right)^{-1}M$ does not depend on the state variables. Thus, its derivative with respect to z_t is zero and so the second optimal portfolio component in Proposition (13) can be written as

$$\theta_{(2)}^* = -\Gamma^{-1}\Omega\int_t^T \frac{\partial}{\partial z_t}E_t^{\widetilde{z}}\left[u - \Lambda^{\mathsf{T}}\left(\Gamma^{-1}\right)^{\mathsf{T}}M\right]ds. \tag{5.16}$$

In the next section we compute the expected value characterizing the second optimal portfolio component. Once we will have completely solved the control problem we will be able to determine how the time horizon T affects the optimal portfolio composition.

5.8   The optimal portfolio

First, we explicitly compute the second optimal portfolio component $\theta^*_{(2)}$. In particular, the argument of the expected value in (5.16) is given by

$$u - \Lambda^\mathsf{T} \left(\Gamma^{-1}\right)^\mathsf{T} M = \psi \frac{L}{p} q,$$

where q depends only on constant parameters and not on the state variables r, L, and p. Actually, its value is

$$q \equiv -\kappa_S m_S - \frac{\sigma_L}{\sigma_\pi} \left(m_\pi - \rho_S m_S\right) + m_L + \left(\kappa_r - \kappa_S \nu\right) \sigma\xi - \frac{\sigma_L \sigma\xi}{\sigma_\pi} \left(\rho_r - \rho_S \nu\right).$$

Accordingly, the derivative of the expected value in Equation (5.16) can be written as follows:

$$\begin{bmatrix} \frac{\partial}{\partial r(t)} E_t \left[u - \Lambda^\mathsf{T} \left(\Gamma^{-1}\right)^\mathsf{T} M\right] \\ \frac{\partial}{\partial L(t)} E_t \left[u - \Lambda^\mathsf{T} \left(\Gamma^{-1}\right)^\mathsf{T} M\right] \\ \frac{\partial}{\partial p(t)} E_t \left[u - \Lambda^\mathsf{T} \left(\Gamma^{-1}\right)^\mathsf{T} M\right] \end{bmatrix} = \psi q \begin{bmatrix} 0 \\ \frac{\partial}{\partial L(t)} E_t \left[\frac{\tilde{L}}{\tilde{p}}\right] \\ \frac{\partial}{\partial p(t)} E_t \left[\frac{\tilde{L}}{\tilde{p}}\right] \end{bmatrix}.$$

The only term we have to compute is the expected value of the ratio between the modified processes of salaries and prices, that is to say the modified real contribution. Therefore, we carry out the necessary computations for the modified processes of L and p. In particular, we have to compute the matrix product

$$\Omega^\mathsf{T} \left(\Gamma^{-1}\right)^\mathsf{T} M.$$

According to what we have already presented in the previous sections, we can write:

$$\Omega^\mathsf{T} \left(\Gamma^{-1}\right)^\mathsf{T} M = \begin{bmatrix} w_1 \\ L w_2 \\ p w_3 \end{bmatrix},$$

where w_1, w_2, and w_3 are constant parameters given by

$$w_1 \equiv -\sigma\xi - \rho_r \sigma^2,$$

$$w_2 \equiv \kappa_S m_S - \frac{1}{\sigma_\pi} \rho_S \sigma_L m_S - \kappa_r \sigma\xi + \nu \kappa_S \sigma\xi + \frac{1}{\sigma_\pi} \rho_r \sigma_L \sigma\xi$$
$$- \frac{1}{\sigma_\pi} \rho_S \nu \sigma_L \sigma\xi - \rho_r \sigma^2 \kappa_r - \rho_S \sigma_S^2 \kappa_S + \frac{1}{\sigma_\pi} \sigma_L m_\pi - \sigma_L \sigma_\pi,$$

$$w_3 = m_\pi - \rho_r^2 \sigma^2 - \rho_S^2 \sigma_S^2 - \sigma_\pi^2.$$

Thus, the modified differential of the state variables $\tilde{z}_s$ can be written as

$$
\begin{bmatrix} d\tilde{r} \\ \frac{d\tilde{L}}{\tilde{L}} \\ \frac{d\tilde{p}}{\tilde{p}} \end{bmatrix} = \begin{bmatrix} \alpha(\beta - \tilde{r}) - w_1 \\ \tilde{r} + m_L - w_2 \\ \tilde{r} + m_\pi - w_3 \end{bmatrix} ds + \begin{bmatrix} \sigma & 0 & 0 \\ \kappa_r\sigma & \kappa_S\sigma_S & \sigma_L \\ \rho_r\sigma & \rho_S\sigma_S & \sigma_\pi \end{bmatrix} \begin{bmatrix} dW^r \\ dW^L \\ dW^\pi \end{bmatrix}.
$$

All these processes have a closed form solution. In particular, for $s \geq t$, the solution of the interest rate process is

$$
\tilde{r}(s) = \tilde{r}(t)e^{\alpha(t-s)} + \frac{\alpha\beta - w_1}{\alpha}\left(1 - e^{\alpha(t-s)}\right) + \sigma e^{-\alpha s}\int_t^s e^{\alpha\tau}dW^r(\tau),
$$

while the solutions of the other two processes are

$$
\tilde{L}(s) = \tilde{L}(t)\exp\left\{\int_t^s \tilde{r}(\tau)\,d\tau + \left(m_L - w_2 - \frac{1}{2}\kappa_r^2\sigma^2 - \frac{1}{2}\kappa_S^2\sigma_S^2 - \frac{1}{2}\sigma_L^2\right)(s-t)\right.
$$
$$
\left. + \kappa_r\sigma\left(W^r(s) - W^r(t)\right) + \kappa_S\sigma_S\left(W^L(s) - W^L(t)\right) + \sigma_L\left(W^\pi(s) - W^\pi(t)\right)\right\},
$$

$$
\tilde{p}(s) = \tilde{p}(t)\exp\left\{\int_t^s \tilde{r}(\tau)\,d\tau + \left(m_\pi - w_3 - \frac{1}{2}\rho_r^2\sigma^2 - \frac{1}{2}\rho_S^2\sigma_S^2 - \frac{1}{2}\sigma_\pi^2\right)(s-t)\right.
$$
$$
\left. + \rho_r\sigma\left(W^r(s) - W^r(t)\right) + \rho_S\sigma_S\left(W^L(s) - W^L(t)\right) + \sigma_\pi\left(W^\pi(s) - W^\pi(t)\right)\right\}.
$$

From these equations we can immediately derive the value of the modified real contribution, that is to say the ratio between $\tilde{L}$ and $\tilde{p}$, whose expected value is

$$
E_t\left[\frac{\tilde{L}(s)}{\tilde{p}(s)}\right] = \frac{\tilde{L}(t)}{\tilde{p}(t)}e^{q(s-t)},
$$

where the boundary condition $(\tilde{z}_t = z_t)$ in Proposition (13) assures that

$$
\tilde{L}(t) = L(t),
$$
$$
\tilde{p}(t) = p(t).
$$

Accordingly, we have

$$
E_t\left[\frac{\tilde{L}(s)}{\tilde{p}(s)}\right] = \frac{L(t)}{p(t)}e^{q(s-t)}.
$$

Thus, the integral defining $\theta^*_{(2)}$ is given by

$$\int_t^T \frac{\partial}{\partial z} E_t \left[u - \Lambda^{\mathsf{T}} \left(\Gamma^{-1} \right)^{\mathsf{T}} M \right] ds = \psi q \begin{bmatrix} 0 \\ \int_t^T \frac{1}{p(t)} e^{q(s-t)} ds \\ \int_t^T -\frac{L(t)}{p^2(t)} e^{q(s-t)} ds \end{bmatrix}$$

$$= \left(e^{q(T-t)} - 1 \right) \frac{\psi}{p(t)} \begin{bmatrix} 0 \\ 1 \\ -\frac{L(t)}{p(t)} \end{bmatrix}.$$

Finally, we can write the second optimal portfolio component as

$$\theta^*_{(2)}(t) = \left(1 - e^{q(T-t)} \right) \psi \frac{1}{p(t)} \Gamma^{-1} \Omega \begin{bmatrix} 0 \\ 1 \\ -\frac{L(t)}{p(t)} \end{bmatrix},$$

from which it is evident that its absolute weight on the total optimal portfolio decreases when the time t becomes closer to the horizon T. In fact, when $t = T$ we have $\theta^*_{(2)}(T) = 0$. Finally, after noting that the following identity holds:

$$\psi \frac{1}{p(t)} \Omega \begin{bmatrix} 0 \\ 1 \\ -\frac{L(t)}{p(t)} \end{bmatrix} = \Lambda,$$

the second optimal portfolio component can be written as

$$\theta^*_{(2)} = \left(1 - e^{q(T-t)} \right) \Gamma^{-1} \Lambda.$$

Now, we are able to simplify Proposition (13), and we obtain the following result.

Proposition 14 *Under market structure (5.8), the portfolio composition solving the control problem (5.11) is given by*

$$\theta^* = -\frac{1}{\gamma} \left(\Gamma^{\mathsf{T}} \Gamma \right)^{-1} M - e^{q(T-t)} \Gamma^{-1} \Lambda,$$

where

$$q \equiv -\kappa_S m_S - \frac{\sigma_L}{\sigma_\pi} \left(m_\pi - \rho_S m_S \right) + m_L + \left(\kappa_r - \kappa_S \nu \right) \sigma \xi - \frac{\sigma_L \sigma \xi}{\sigma_\pi} \left(\rho_r - \rho_S \nu \right).$$

This result shows that the optimal portfolio is actually formed by two components: one depending on the time horizon T and the other one independent of T. Furthermore, we underline that it is possible to write the matrix terms of θ^* as

$$-\frac{1}{\gamma}\left(\Gamma^\mathsf{T}\Gamma\right)^{-1}M \equiv \frac{1}{\gamma}p\left(t\right)\phi_1,$$

$$-\Gamma^{-1}\Lambda \equiv \psi L\left(t\right)\phi_2,$$

where $\phi_1, \phi_2 \in \mathbb{R}^{3\times1}$ are two vectors of parameters which do not depend on time. Thus, the optimal portfolio can be written in real term as follows:

$$\frac{\theta^*}{p\left(t\right)} = \frac{1}{\gamma}\phi_1 + \psi\frac{L\left(t\right)}{p\left(t\right)}e^{q(T-t)}\phi_2,$$

from which we see that the first component of the optimal portfolio real composition is time independent. The risk aversion parameter γ determines if the portfolio is more or less affected by the time-dependent real component. The higher γ (in absolute value), the more the time-dependent real component affects the optimal portfolio. Accordingly, low values of γ determines a real portfolio composition that tends to be constant through time. In the next section, where we carry out a simulation of the model, we highlight the necessity of assigning a numerical value for γ which is consistent with the initial value given to the salary process. In particular, a too low absolute value of γ (with respect to L_0) leads to an optimal strategy which is practically constant through time.

Now it can be interesting to investigate which is the total amount of wealth invested in the financial assets, that is

$$\mathbf{1}^\mathsf{T}\theta^* = \frac{1}{\gamma}p\left(t\right)\mathbf{1}^\mathsf{T}\phi_1 + \psi L\left(t\right)e^{q(T-t)}\mathbf{1}^\mathsf{T}\phi_2, \tag{5.17}$$

where $\mathbf{1} \in \mathbb{R}^{3\times1}$ is a vector containing only ones. After computing the products $\mathbf{1}^\mathsf{T}\phi_1$ and $\mathbf{1}^\mathsf{T}\phi_2$ we have

$$\mathbf{1}^\mathsf{T}\theta^* = \frac{1}{\gamma}p\left(t\right)\left(1 - \frac{1}{\sigma_\pi^2}\sigma\xi\left(\rho_r - \rho_S\nu\right) - \frac{1}{\sigma_\pi^2}\left(m_\pi - \rho_S m_S\right)\right)$$
$$+ \psi L\left(t\right)e^{q(T-t)}\left(\frac{\sigma_L}{\sigma_\pi} - 1\right).$$

We see that the sign of the time-dependent component is determined by the ratio between the volatility terms σ_L and σ_π. In particular, since ψ is always positive and $L(t)$ is positive too,[7] if $\sigma_L > \sigma_\pi$ then the time-dependent component is positive, on the contrary it is negative.

In the numerical simulation which follows we have assumed $\sigma_\pi > \sigma_L$ because it seems more reasonable that the inflation own volatility is higher than the salaries own volatility. This means that when the time t approaches the horizon T the amount of wealth invested in the financial asset tends to increase. This is consistent with the hypothesis that the fund's wealth mainly increases thanks to the contributions. Thus, at the beginning, the amount of wealth invested in financial assets is low with respect to the contributions while at the end of the accumulation period, the financial wealth dominates the contributions.

5.9 A numerical application

In this section we set a numerical application in order to verify the dynamic behavior of the optimal portfolio strategy derived in the previous section. Table 5.1 reports the set of parameters characterizing the financial market, the DC process and the CPI process. In particular, the set of parameters representing the financial market is consistent with the numerical analysis presented by Boulier et al. (2001).

We consider a contribution period before retirement equal to 40 years. The absolute risk aversion of the investor is given by $\gamma = -20$. The value we have assigned to γ is consistent with the initial value of the salary process (L_0). In fact, from Equation (5.17), we note that there must be a suitable trade-off between the initial value of the salary process and the scale of values characterizing the risk aversion index. This allows us to avoid the case of an optimal portfolio rule practically constant trough time.

The optimal proportion invested in the riskless asset increases from an initial value close to 3%, to about 56%. On the other hand, the optimal proportion invested in the two risky assets progressively decrease with respect

---

[7]We recall that $L(t)$ is log-normally distributed and it cannot take negative values.

Interest rate		Defined-contribution process	
Mean reversion, α	0.2	Risk premium, m_L	0.01
Mean rate, β	0.05	Volatility scale factor, κ_r	0.7
Volatility, σ	0.02	Volatility scale factor, κ_S	0.9
Initial rate, r_0	0.03	Volatility, σ_L	0.01
		Initial salary, L_0	100
Fix-maturity bond		Contribution rate, ψ	0.12
Maturity, τ_K	20		
Market price of risk, ξ	0.15	**Inflation process**	
		Risk premium, m_π	-0.01
Stock		Volatility scale factor, ρ_r	0.9
Risk premium, m_S	0.06	Volatility scale factor, ρ_S	0.8
Volatility scale factor, ν	3	Volatility, σ_π	0.015
Stock own volatility, σ_S	0.19		

Table 5.1: Values of model parameters

to time. In particular, the stock proportion declines from an initial value of about 73%, to about 47%, while the proportion invested in the long-term bond declines from an initial value close to 24%, to about −3%. The investment trends of the three assets are consistent with the portfolio managers experience and the conventional wisdom. During the beginning of the contribution period, the fund manager realizes a more aggressive investment policy in order to boost the fund. Consistently, as the time approaches the retirement in T, Figure 5.1 shows a shift of wealth from the investment in risky assets to the money account. However, the fund manager maintains a diversified portfolio until retirement.

Figure 5.1 highlights how the evolution of the optimal portfolio strategy is actually affected by the realization of the stochastic variables characterizing the economy. The dotted lines in the figure represent the mean and the extremes of an estimated range including, at each time t, at least 90% of the values of the optimal portfolio components. Consistently, the uncertainty related to the fund managers' decisions augments as the retirement approaches, or better as
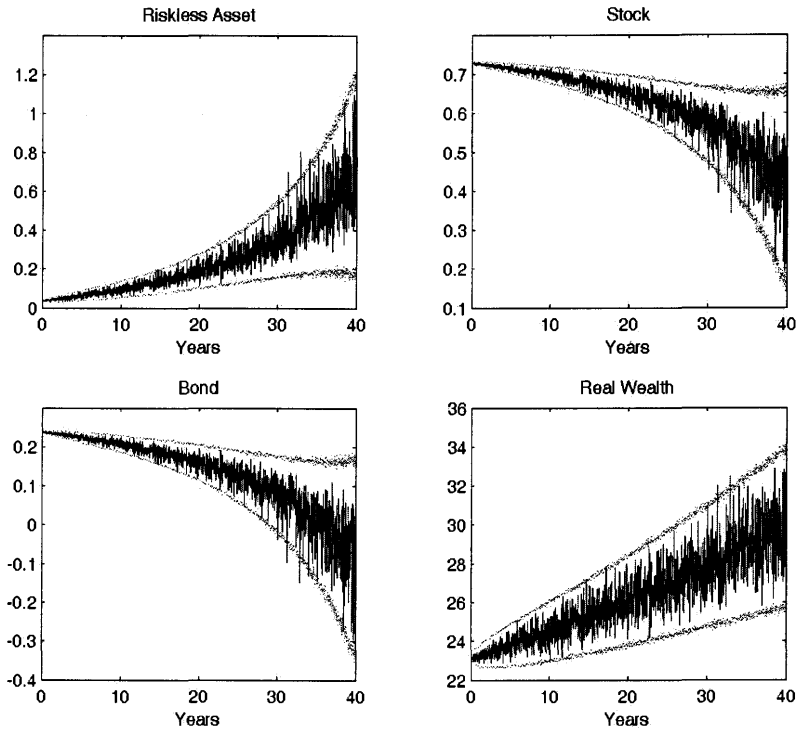
Figure 5.1: Percentage composition of optimal portfolio and real wealth

we deviate from the present.

We recall that the bond, at its expiration date, gives the right to receive a fixed amount of money (generally its nominal value). This means that the amount of wealth invested in the bond at the beginning of the accumulation phase must be relatively high because it may guarantee a flow of money. On the contrary, when the time horizon T approaches, then this need for a certain flow becomes weaker and, very close to T, the amount of money invested in the bond can become even negative. Moreover, we can see that the riskless asset plays a residual role in the optimal portfolio composition. At the beginning of the accumulation phase, the need of an aggressive strategy for reaching a higher wealth level leads to a high percentage of stock in the optimal portfolio, while the need of a guarantee for a financial flow leads to a relative high investment in the bond. Consequently, the investment in the riskless asset is very low.

While the riskiness of the strategy decreases and so the need for a guarantee at time T, both the investments in stock and bond decrease and so, the percentage of wealth invested in the riskless asset increases. Moreover, we note an increasing slope, in absolute terms, for all assets. This evidence suggests the necessity for a more frequent adjustment of the investment strategies in the last years of the accumulation phase.

In Boulier et al. (2001), the mean composition of the pension fund is characterized by deterministic trends. On the opposite, given the length of the accumulation phase and the central role of the contribution flow, we strongly support the need for a stochastic framework. In contrast with Boulier et al. (2001), who find a hefty short position in cash, our model implies persistent long position in the riskless asset. Nevertheless, we underline that both Boulier et al. (2001) and Deelstra et al. (2003) include a guarantee and use power utility. In their numerical results, they obtain first a negative cash investment since the guarantee is an option which involves a short position.

Appendix A

The Cholesky Decomposition of the Correlation Matrix

Let $\begin{bmatrix} W_x(t) & W_y(t) \end{bmatrix}^{\mathsf{T}}$ denote a vector of two independent standard Brownian motions. Thus, we have

$$cov\,[dW_x, dW_y] = E\,[dW_x, dW_y] = 0,$$

and variance-covariance matrix $\Sigma = tI_{(2)}$, where $I_{(2)}$ denotes the identity matrix of dimension two. We can transform $\begin{bmatrix} W_x & W_y \end{bmatrix}^{\mathsf{T}}$ into a vector of two correlated Brownian motions $\begin{bmatrix} \widetilde{W}_x & \widetilde{W}_y \end{bmatrix}^{\mathsf{T}}$ with the same mean (i.e. zero mean) but with variance-covariance matrix

$$\widetilde{\Sigma} = \begin{bmatrix} \sigma_x^2 & \varphi\sigma_x\sigma_y \\ \varphi\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix},$$

by applying to the original vector of uncorrelated processes the Cholesky decomposition as follows

$$\begin{bmatrix} \widetilde{W}_x \\ \widetilde{W}_y \end{bmatrix} = C'_{\widetilde{\Sigma}} \begin{bmatrix} W_x \\ W_y \end{bmatrix},$$

where $C_{\widetilde{\Sigma}}$ is just the Cholesky decomposition of the matrix $\widetilde{\Sigma}$. The matrix $C_{\widetilde{\Sigma}}$ is an upper-triangular matrix such that $\widetilde{\Sigma} = C_{\widetilde{\Sigma}}^{\mathsf{T}} C_{\widetilde{\Sigma}}$. Finally, we have

$$\begin{bmatrix} \widetilde{W}_x \\ \widetilde{W}_y \end{bmatrix} = \begin{bmatrix} \sigma_x & \varphi\sigma_y \\ 0 & \sigma_y\sqrt{1-\varphi^2} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} W_x \\ W_y \end{bmatrix} = \begin{bmatrix} \sigma_x W_x \\ \sigma_y\varphi W_x + \sigma_y\sqrt{1-\varphi^2}W_y \end{bmatrix}.$$

In conclusion, the following general result holds: given a set of Brownian motions, correlated or uncorrelated, we can always change it into a vector of Brownian motions with diffusion term equal to the transpose of the Cholesky matrix calculated with respect to the variance-covariance matrix of the initial processes.

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