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Dimension, Inverse Limits and GF-Spaces

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Summary. - *In this paper we characterize (covering) dimension in metrizable spaces in terms of fractal structures. We will also study dimension for compact metric spaces, giving a theorem relating dimension and a certain class of inverse limits, similar to that of Freudenthal.*

1. Introduction

Recently there has been many investigations on topological structures of (strict) self-similar sets leading to the notion of symbolic self-similar set, an abstract version of the classical ones.

Looking for a generalization of symbolic self-similar sets outside compact metric spaces, we developed the concept of a GF-space (a short form, in what follows, for generalized fractal space, see [3]) and we found that it is a common framework for the study of self-similar sets, non-archimedeanly quasimetrizable spaces, inverse limits, dimension, metrization, etc. The relation between GF-spaces and self-similar sets can be found in [2].

In this paper we characterize (covering) dimension in metrizable spaces in terms of fractal structures. The study of the dimension of

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a topological space in terms of what we call *directed fractal structures* was carried out in [11] (with another notation). Thought in that paper it is proved that there is not an equivalence between the concept of dimension made with directed fractal structures and the covering dimension, we show that we can get a characterization by using fractal structures. We will also study dimension for compact metric spaces, giving a theorem similar to that of Freudenthal.

2. GF-spaces

Now, we recall from [3] some definitions and introduce some notations that will be useful in this paper.

Let $\mathbf{\Gamma} = \{\Gamma_i : i \in I\}$ be a family of coverings. Recall that

$$\mathrm{St}(x, \Gamma_i) = \bigcup_{x \in A_i, A_i \in \Gamma_i} A_i$$

we also define

$$U_{xi}^{\mathbf{\Gamma}} = \mathrm{St}(x, \Gamma_i) \setminus \bigcup_{x \notin A_i, A_i \in \Gamma_i} A_i$$

which will be denoted also by U_{xi} if there is no confusion about the family. We also denote by $\mathrm{St}(x, \mathbf{\Gamma})$ the family $\{\mathrm{St}(x, \Gamma_i) : i \in I\}$ and by $\mathcal{U}_x$ the family $\{U_{xi} : i \in I\}$.

A relation $\leq$ on a set G is called a partial order on G if it is a transitive antisymmetric reflexive relation on G. If $\leq$ is a partial order on a set G, then $(G, \leq)$ is called a partially ordered set.

$(G, \leq, \tau)$ will be called a poset (partially ordered set) or T_0-Alexandroff space if $(G, \leq)$ is a partially ordered set and τ is that in which the set $[g, \rightarrow[= \{h \in G : g \leq h\}$ forms a neighborhood base for each $g \in G$ (we say that the topology τ is induced by $\leq$). Note that then $\overline{\{g\}} =]\leftarrow, g]$ for all $g \in G$.

Let us remark that a map $f : G \to H$ between two posets G and H is continuous if and only if it is order preserving, i.e. $g_1 \leq g_2$ implies $f(g_1) \leq f(g_2)$.

Let Γ be a covering of X. Γ is said to be locally finite if for all $x \in X$ there exists a neighborhood of x which meets only a finite number of elements of Γ. Γ is said to be a tiling, if all elements of Γ are regularly closed and they have disjoint interiors (see [1]). The order of a point $x \in X$ at Γ is defined as the cardinal of the set of all elements in Γ containing x minus one, and it is denoted by $\mathrm{Ord}\,(x, \Gamma)$. If $\mathbf{\Gamma} = \{\Gamma_i : i \in I\}$, we denote

$$\mathrm{Ord}\,(x, \mathbf{\Gamma}) = \sup\{\mathrm{Ord}\,(x, \Gamma_i) : i \in I\}.$$

Definition 2.1. *Let X be a topological space, and let I be a directed set. A directed pre-fractal structure over X is a family of coverings $\mathbf{\Gamma} = \{\Gamma_i : i \in I\}$ such that $\{U_{xi} : i \in I\}$ is a open neighborhood base of x for all $x \in X$.*

We say that a directed pre-fractal structure $\mathbf{\Gamma}$ is a directed fractal structure if the following conditions hold

- *Γ_i is a closed covering for each $i \in I$.*
- *Γ_j is a refinement of Γ_i for all $j \geq i$.*
- *Given $i \in I$, $j \geq i$ and $x \in A_i$, with $A_i \in \Gamma_i$, there exists $A_j \in \Gamma_j$ with $x \in A_j \subseteq A_i$.*

If $\mathbf{\Gamma}$ is a directed (pre-) fractal structure over X, we will say that $(X, \mathbf{\Gamma})$ is a directed (pre-) GF-space. If there is no confusion about $\mathbf{\Gamma}$, we will say that X is a directed (pre-) GF-space.

Definition 2.2. *Let X be a topological space and let $\mathbb{N}$ be the set of positive integer numbers with the usual order, and suppose that $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ is a directed (pre-) fractal structure over X. We say that $\mathbf{\Gamma}$ is a (pre-) fractal structure over X, and we call $(X, \mathbf{\Gamma})$ a GF-space.*

We will denote $U_i = \{(x, y) \in X \times X : y \in U_{xi}\}$, and $U_{xi}^{-1} = U_i^{-1}(x)$.

If $\mathbf{\Gamma}$ is a directed (pre-) fractal structure over X, and for each $x \in X$, $\mathrm{St}(x, \mathbf{\Gamma})$ is a neighborhood base of x, we will say that $(X, \mathbf{\Gamma})$ is a starbase directed (pre-) GF-space.

If Γ_i has the property P for all $i \in I$, and $\mathbf{\Gamma}$ is a directed (pre-) fractal structure over X, we will say that $\mathbf{\Gamma}$ is a directed (pre-) fractal structure over X with the property P, and that X is a directed (pre-) GF-space with the property P. For example, if Γ_i is locally finite for all $i \in I$, and $\mathbf{\Gamma}$ is a directed fractal structure over X, we will say that $\mathbf{\Gamma}$ is a locally finite directed fractal structure over X, and that $(X, \mathbf{\Gamma})$ is a locally finite directed GF-space.

The following proposition has an easy proof and is proved in [3, Prop. 3.2] for pre-fractal structures.

Proposition 2.3. *Let X be a directed pre-GF-space. Then $U_{xi}^{-1} = \bigcap_{x \in A_i} A_i$.*

In [3], the authors introduced the following construction. Let $\mathbf{\Gamma}$ be a fractal structure, and let define $G_n = \{U_{xn}^* : x \in X\}$, and define in G_n the following order relation $U_{xn}^* \leq_n U_{yn}^*$ if $y \in U_{xn}$. It holds that G_n is a poset with this order relation and its associated topology.

Let ρ_n be the quotient map from X onto G_n which carries x in X to $U_n^*(x)$ in G_n. It holds that ρ_n is continuous.

We also consider the map $\phi_n : \mathrm{G}_n \longrightarrow \mathrm{G}_{n-1}$ defined by $\phi_n(\rho_n(x)) = \rho_{n-1}(x)$. It also holds that ϕ_n is continuous.

Let ρ be the map from X to $\varprojlim G_n$ which carries x in X to $(\rho_n(x))_n$ in $\varprojlim G_n$. Note that ρ is well defined and continuous (by definition of ϕ_n and the continuity of ρ_n and ϕ_n for all n). It holds that ρ is an embedding of X into $\varprojlim G_n$. We shall identify X with $\rho(X)$ whenever we need it.

The fractal structure, noted by $G(\mathbf{\Gamma})$, associated to $\varprojlim G_n$ is defined as $G(\Gamma_n) = \{A_n(g_n) : g_n \in G_n\}$, where for each $g_n \in G_n$, we define $A_n(g_n) = \{h \in \varprojlim G_n : h_n \leq_n g_n\}$.

If $\mathbf{\Gamma}$ is a fractal structure over a topological space X, and G_n is the associated poset for each $n \in \mathbb{N}$, then the restriction to X of the fractal structure $G(\mathbf{\Gamma})$ associated to $\varprojlim G_n$ can be described by $G(\Gamma_n) = \{(U_{xn}^{\mathbf{\Gamma}})^{-1} : x \in X\}$.

3. GF-spaces and dimension

In [11], Pears and Mack studied (among others) the following dimension functions for a (nonempty) topological space X (we will use our notation instead of theirs).

1. $\delta_1(X)$ to be the least integer n for which there exists a (locally finite) starbase directed fractal structure over X with order at most n, and $\delta_1(X) = \infty$ if such a integer there no exists.

2. $\delta_2(X)$ to be the least integer n for which there exists a (locally finite) tiling directed fractal structure over X with order at most n, and $\delta_2(X) = \infty$ if such a integer there no exists.

They proved the following results.

- X is regular and $\delta_2(X) \leq n$ if and only if there exists a locally finite starbase directed fractal structure over X with order at most n.
- $\operatorname{ind}(X) = 0$ if and only if $\delta_1(X) = 0$ if and only if $\delta_2(X) = 0$.
- For any topological space X it follows that $\operatorname{ind}(X) \leq \delta_1(X)$ and if X is also a regular space then $\operatorname{ind}(X) \leq \delta_1(X) \leq \delta_2(X)$.
- If X is a strongly metrizable space then $\operatorname{ind}(X) = \operatorname{Ind}(X) = \dim(X) = \delta_1(X) = \delta_2(X)$.

We are going to use fractal structures instead of directed fractal structures in order to get a characterization of covering dimension in metrizable spaces.

We begin with the easiest case: zero-dimensionality.

THEOREM 3.1. *Let X be a metrizable space with* $\dim(X) = 0$. *Then there exists a starbase fractal structure of order zero over X.*

Proof. Let $\{V_n : n \in \mathbb{N}\}$ be a base for a metric uniformity over X.

Since $\{V_1(x) : x \in X\}$ is an open covering of X, then there exists an open refinement Γ_1 of order zero, since $\dim(X) = 0$. If $A_1 \in \Gamma_1$, we have $A_1 = X \setminus \bigcup_{B_1 \neq A_1} B_1$, and hence A_1 is closed, and then A_1 is open and closed. We also have $U_{x1} = A_1$, where A_1 is the only element of Γ_1 which contains x. Hence U_{x1} is open and closed for all $x \in X$.

Now, $\{A_1 \cap V_2(x) : A_1 \in \Gamma_1; x \in X\}$ is an open covering of X. Then there exists an open refinement Γ_2 of order zero.

The construction of Γ_n by induction is clear. We also have:

1. Γ_{n+1} is a refinement of Γ_n by construction.
2. $U_{xn} = A_n = \operatorname{St}(x, \Gamma_n) \subseteq V_n(y)$ (for some $y \in X$). And since $U_{xn} = \operatorname{St}(x, \Gamma_n)$ is an open neighborhood of x for all $x \in X$, we have that $\mathbf{\Gamma}$ is a starbase pre-fractal structure.

3. Let $x \in A_n$; then there exists only one element $A_{n+1} \in \Gamma_{n+1}$ such that $x \in A_{n+1}$. Moreover, by construction we have that $A_{n+1} \subseteq A_n \cap V_{n+1}(y) \subseteq A_n$ for some $y \in X$. Therefore, $\mathbf{\Gamma}$ is a fractal structure over X.

4. Γ_n is of order zero by construction. □

For another relation between zero-dimensionality and GF-spaces in the realm of complete metrizable spaces, see section 2 of [4].

In [6] it is proved the following theorem.

THEOREM 3.2. *Let* $\mathbf{\Gamma}$ *be a locally finite fractal structure over a regular space* X *and let* $\Gamma'_n = \operatorname{reg}(\Gamma_n) = \{A'_n = \operatorname{Cl}(A_n^\circ) : A_n \in \Gamma_n\}$. *Then* $\mathbf{\Gamma}' = \{\Gamma'_n : n \in \mathbb{N}\}$ *(called the regularization* $\operatorname{reg}(\mathbf{\Gamma})$ *of* $\mathbf{\Gamma}$*) is a locally finite fractal structure over* X. *Moreover, if* $\mathbf{\Gamma}$ *is starbase, then* $\mathbf{\Gamma}'$ *also is.*

The next result is of a technical nature and yields the construction, from a starbase fractal structure, of another one with the same properties, the same order, but a tiling.

THEOREM 3.3. *Let* $\mathbf{\Gamma}$ *be a locally finite starbase fractal structure over* X. *Then there exists a locally finite starbase tiling fractal structure* $\mathbf{\Gamma}'$ *over* X. *Moreover the order of* $\mathbf{\Gamma}'$ *is less or equal than the order of* $\mathbf{\Gamma}$ *and if* $\mathbf{\Gamma}$ *is starbase, so is* $\mathbf{\Gamma}'$.

Proof. By Theorem 3.2, we can assume that each member of Γ_n is regularly closed (that is, $Cl(A_n^\circ) = A_n$).

Now, we are going to modify the proof of Lemma 1.1 in [9] (see also Proposition 6.1.2 in [10]) to get the desired result.

Let Λ_n be an ordinal which indexes the members of Γ_n.

For each $\lambda_1 \in \Lambda_1$, let $F_1^{\lambda_1} = \operatorname{Cl}((A_1^{\lambda_1})^\circ \setminus \bigcup_{\mu_1 < \lambda_1} A_1^{\mu_1})$. Let $\Gamma'_1 = \{F_{A_1^{\lambda_1}} : A_1^{\lambda_1} \in \Gamma_1\}$. By [9, Lemma 1.1], Γ'_1 is a tiling with $F_{A_1^{\lambda_1}} \subseteq A_1^{\lambda_1}$ (so the order of Γ'_1 is less or equal than the order of Γ_1).

For all $A_2^{\lambda_2}$ let $\lambda_1(\lambda_2) \in \Lambda_1$ be the minimum element $\lambda_1 \in \Lambda_1$ (note that this element exists, since ordinals are well-ordered) such that $A_2^{\lambda_2} \subseteq A_1^{\lambda_1}$.

For each $\lambda_{n+1} \in \Lambda_{n+1}$ ($n \geq 2$) we define by recursion $\lambda_n(\lambda_{n+1})$ as $\mu_n \in \Lambda_n$ such that $A_{n+1}^{\lambda_{n+1}} \subseteq A_n^{\mu_n}$ and $(\lambda_1(\mu_n), \ldots, \lambda_{n-1}(\mu_n), \mu_n)$

is the minimum in $\Lambda_1 \times \cdots \times \Lambda_n$ (with the lexicographic order); and we define $\lambda_i(\lambda_{n+1}) = \lambda_i(\lambda_n(\lambda_{n+1}))$ for all $i \leq n-1$.

Then let

$$F_n^{\lambda_n} = \mathrm{Cl}((A_n^{\lambda_n})^\circ \setminus \bigcup_{\substack{\mu_n \in \Lambda_n \\ (\lambda_1(\mu_n),\ldots,\lambda_{n-1}(\mu_n),\mu_n)<(\lambda_1(\lambda_n),\ldots,\lambda_{n-1}(\lambda_n),\lambda_n)}} A_n^{\mu_n})$$

Next, we define $\Gamma'_n = \{F_n^{\lambda_n} : \lambda_n \in \Lambda_n\}$. By [9, Lemma 1.1], Γ'_n is a tiling such that $F_n^{\lambda_n} \subseteq A_n^{\lambda_n}$ (and hence the order of Γ'_n is less or equal than the order of Γ_n). It is also clear that Γ'_n is locally finite, since Γ_n is.

Claim: $F_n^{\lambda_n} = \bigcup_{\lambda_n(\lambda_{n+1})=\lambda_n} F_{n+1}^{\lambda_{n+1}}$ for all $\lambda_n \in \Lambda_n$

First, we have that

$$(A_{n+1}^{\lambda_{n+1}})^\circ \setminus \bigcup_{\substack{\mu_{n+1} \in \Lambda_{n+1} \\ (\lambda_1(\mu_{n+1}),\ldots,\lambda_n(\mu_{n+1}),\mu_{n+1})<(\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1}),\lambda_{n+1})}} A_{n+1}^{\mu_{n+1}}$$
$$\subseteq (A_n^{\lambda_n})^\circ \setminus \bigcup_{\substack{\mu_n \in \Lambda_n \\ (\lambda_1(\mu_n),\ldots,\lambda_{n-1}(\mu_n),\mu_n)<(\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1}))}} A_n^{\mu_n}$$

for all $\lambda_{n+1} \in \Lambda_{n+1}$ with $\lambda_n(\lambda_{n+1}) = \lambda_n$. Let see it: $(A_{n+1}^{\lambda_{n+1}})^\circ \subseteq A_n^{\lambda_n}$, since $\lambda_n(\lambda_{n+1}) = \lambda_n$; on the other hand

$$\bigcup_{\substack{\mu_n \in \Lambda_n \\ (\lambda_1(\mu_n),\ldots,\lambda_{n-1}(\mu_n),\mu_n)<(\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1}))}} A_n^{\mu_n}$$
$$\subseteq \bigcup_{\substack{\mu_{n+1} \in \Lambda_{n+1} \\ (\lambda_1(\mu_{n+1}),\ldots,\lambda_n(\mu_{n+1}),\mu_{n+1})<(\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1}),\lambda_{n+1})}} A_{n+1}^{\mu_{n+1}}$$

In order to see this, if $x \in A_n^{\mu_n}$ with $\mu_n \in \Lambda_n$ and

$$(\lambda_1(\mu_n),\ldots,\lambda_{n-1}(\mu_n),\mu_n) < (\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1})),$$

then, since $\mathbf{\Gamma}$ is a fractal structure, there exists $A_{n+1}^{\alpha_{n+1}} \in \Gamma_{n+1}$ such that $x \in A_{n+1}^{\alpha_{n+1}} \subseteq A_n^{\mu_n}$, then it is clear that $(\lambda_1(\alpha_{n+1}),\ldots,\lambda_n(\alpha_{n+1})) \leq (\lambda_1(\mu_n),\ldots,\lambda_{n-1}(\mu_n),\mu_n) < (\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1}))$, and hence

$$x \in \bigcup_{\substack{\mu_{n+1} \in \Lambda_{n+1} \\ (\lambda_1(\mu_{n+1}),\ldots,\lambda_n(\mu_{n+1}),\mu_{n+1})<(\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1}),\lambda_{n+1})}} A_n^{\mu_{n+1}}.$$

Taking closures, $F_{n+1}^{\lambda_{n+1}} \subseteq F_n^{\lambda_n(\lambda_{n+1})}$ for all $\lambda_{n+1} \in \Lambda_{n+1}$ with $\lambda_n(\lambda_{n+1}) = \lambda_n$.

In order to get the reverse inclusion, let

$$x \in (A_n^{\lambda_n})^\circ \setminus \bigcup_{\substack{\mu_n \in \Lambda_n \\ (\lambda_1(\mu_n),\ldots,\lambda_{n-1}(\mu_n),\mu_n)<(\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1}))}} A_n^{\mu_n}$$

and let $\lambda_{n+1} \in \Lambda_{n+1}$ be such that $(\lambda_1(\lambda_{n+1}), \ldots, \lambda_n(\lambda_{n+1}), \lambda_{n+1})$ is minimum in $\Lambda_1 \times \cdots \times \Lambda_{n+1}$ with $x \in A_{n+1}^{\lambda_{n+1}}$.

Then it is clear (by the definition of λ_{n+1}) that

$$x \in A_{n+1}^{\lambda_{n+1}} \setminus \bigcup_{\substack{\mu_{n+1} \in \Lambda_{n+1} \\ (\lambda_1(\mu_{n+1}),\ldots,\lambda_n(\mu_{n+1}),\mu_{n+1})<(\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1}),\lambda_{n+1})}} A_{n+1}^{\mu_{n+1}}$$

Then if W is an open neighborhood of x,

$$W \setminus \bigcup_{\substack{\mu_{n+1} \in \Lambda_{n+1} \\ (\lambda_1(\mu_{n+1}),\ldots,\lambda_n(\mu_{n+1}),\mu_{n+1})<(\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1}),\lambda_{n+1})}} (A_{n+1}^{\mu_{n+1}})$$

$$\cap (A_{n+1}^{\lambda_{n+1}})^\circ \neq \emptyset$$

(note that $x \in \mathrm{Cl}((A_{n+1}^{\lambda_{n+1}})^\circ)$, since $A_{n+1}^{\lambda_{n+1}}$ is regularly closed and

$$W \setminus \bigcup_{\substack{\mu_{n+1} \in \Lambda_{n+1} \\ (\lambda_1(\mu_{n+1}),\ldots,\lambda_n(\mu_{n+1}),\mu_{n+1})<(\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1}),\lambda_{n+1})}} A_{n+1}^{\mu_{n+1}}$$

is an open neighborhood of x, since Γ_n is locally finite and hence

$$\bigcup_{\substack{\mu_{n+1} \in \Lambda_{n+1} \\ (\lambda_1(\mu_{n+1}),\ldots,\lambda_n(\mu_{n+1}),\mu_{n+1})<(\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1}),\lambda_{n+1})}} A_{n+1}^{\mu_{n+1}}$$

is closed). Therefore x is in

$$\mathrm{Cl}((A_{n+1}^{\lambda_{n+1}})^\circ \setminus \bigcup_{\substack{\mu_{n+1} \in \Lambda_{n+1} \\ (\lambda_1(\mu_{n+1}),\ldots,\lambda_n(\mu_{n+1}),\mu_{n+1})<(\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1}),\lambda_{n+1})}}$$

$$A_{n+1}^{\mu_{n+1}}) = F_{n+1}^{\lambda_{n+1}}$$

On the other hand, since

$$x \in (A_n^{\lambda_n})^\circ \setminus \bigcup_{\substack{\mu_n \in \Lambda_n \\ (\lambda_1(\mu_n),\ldots,\lambda_{n-1}(\mu_n),\mu_n)<(\lambda_1(\lambda_n),\ldots,\lambda_{n-1}(\lambda_n),\lambda_n)}} A_n^{\mu_n}$$

it follows that

$$(\lambda_1(\lambda_n),\ldots,\lambda_{n-1}(\lambda_n),\lambda_n) \leq (\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1})).$$

Let μ_1 be the minimum of Λ_1 with $x \in A_1^{\mu_1}$, μ_2 be the minimum of Λ_2 with $x \in A_2^{\mu_2} \subseteq A_1^{\mu_1}$ (it is clear that then we have that $\lambda_1(\mu_2) = \mu_1$), and for $2 \leq i \leq n$, let μ_{i+1} be the minimum of Λ_{i+1} with $x \in A_{i+1}^{\mu_{i+1}} \subseteq A_i^{\mu_i}$ (it is easy to see that it holds that $\lambda_i(\mu_{i+1}) = \mu_i$). Then, by definition of λ_{n+1}, it holds that $(\mu_1,\ldots,\mu_{n+1}) \geq (\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1}),\lambda_{n+1})$, and by definition of μ_i it is clear that $(\mu_1,\ldots,\mu_{n+1}) \leq (\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1}),\lambda_{n+1})$. Therefore $(\mu_1,\ldots,\mu_{n+1}) = (\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1}),\lambda_{n+1})$. Since $x \in A_i^{\lambda_i(\lambda_n)}$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n-1$, then it is clear (by the previous reasoning) that

$$(\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1})) \leq (\lambda_1(\lambda_n),\ldots,\lambda_{n-1}(\lambda_n),\lambda_n),$$

which join with the inclusion obtained before, yields that

$$(\lambda_1(\lambda_{n+1}),\ldots,\lambda_n(\lambda_{n+1})) = (\lambda_1(\lambda_n),\ldots,\lambda_{n-1}(\lambda_n),\lambda_n),$$

and hence $\lambda_n(\lambda_{n+1}) = \lambda_n$.

Therefore

$$(A_n^{\lambda_n})^\circ \setminus \bigcup_{\substack{\mu_n \in \Lambda_n \\ (\lambda_1(\mu_n),\ldots,\lambda_{n-1}(\mu_n),\mu_n)<(\lambda_1(\lambda_n),\ldots,\lambda_{n-1}(\lambda_n),\lambda_n)}} A_n^{\mu_n}$$

$$\subseteq \bigcup_{\lambda_n(\lambda_{n+1})=\lambda_n} F_{n+1}^{\lambda_{n+1}}$$

for every $\lambda_n \in \Lambda_n$, and since Γ'_{n+1} is locally finite (and hence

$$\bigcup_{\lambda_n=\lambda_n(\lambda_{n+1})} F_{n+1}^{\lambda_{n+1}}$$

is closed for all $\lambda_n \in \Lambda_n$), then taking closures it follows that

$$F_n^{\lambda_n} \subseteq \bigcup_{\lambda_n = \lambda_n(\lambda_{n+1})} F_{n+1}^{\lambda_{n+1}}$$

which proves the claim.

The claim proves then that $\mathbf{\Gamma}'$ is a locally finite tiling fractal structure over X, with order less or equal than the order of $\mathbf{\Gamma}$.

Finally it is clear that $\mathbf{\Gamma}'$ is starbase, since $\mathbf{\Gamma}'$ is a refinement of $\mathbf{\Gamma}$, if $\mathbf{\Gamma}$ is starbase. $\square$

The next lemma can be found in [5].

Lemma 3.4. *Let $\mathbf{\Gamma}$ be a starbase fractal structure over X, K be a compact subset of X and F be a closed subset of X disjoint from K. Then there exists $n \in \mathbb{N}$ such that* $\mathrm{St}(K, \Gamma_n) \cap F = \emptyset$.

The definition of perfect map involves two conditions: compact inverse images and closed map. The checking of the second can be skipped in some cases.

Proposition 3.5. *Let (Y, Δ) be a starbase GF-space, $(X, \mathbf{\Gamma})$ be a GF-space, and let $f : Y \to X$ be a continuous map such that $f^{-1}(x)$ is compact for all $x \in X$ and such that $f(B_n) \in \Gamma_n$ for all $B_n \in \Delta_n$. Then f is a perfect map.*

Proof. Let see that f is a closed map.

Let F be a closed set in Y, let $x \in \overline{f(F)}$ and let $n \in \mathbb{N}$. Then there exists $z \in U_{xn} \cap f(F)$. Hence there exists $t \in F$ such that $z = f(t)$. Let $B_n \in \Delta_n$ be such that $t \in B_n$, then there exists $A_n \in \Gamma_n$ such that $z = f(t) \in f(B_n) = A_n$. Hence $x \in U_{zn}^{-1} = \bigcap_{z \in C_n} C_n \subseteq A_n = f(B_n)$, and then $f^{-1}(x) \cap B_n \neq \emptyset$, and hence $t \in B_n \subseteq \mathrm{St}(f^{-1}(x), \Delta_n)$ and we conclude that $F \cap \mathrm{St}(f^{-1}(x), \Delta_n) \neq \emptyset$.

Summing up, we have that $F \cap \mathrm{St}(f^{-1}(x), \Delta_n) \neq \emptyset$ for all $n \in \mathbb{N}$. Suppose that $F \cap (f^{-1}(x)) = \emptyset$. Since $f^{-1}(x)$ is compact and F is closed, then by Lemma 3.4 there exists $k \in \mathbb{N}$ such that $F \cap \mathrm{St}(f^{-1}(x), \Delta_k) = \emptyset$, but this is a contradiction with what we have proved.

Therefore $F \cap (f^{-1}(x)) \neq \emptyset$ and then $x \in f(F)$, obtaining that $f(F)$ is closed, and f is a closed map. $\square$

Definition 3.6. *Let G be a poset. We define the dimension of G and we denote it by* $\dim(G)$*, as the supremum of the lengths of all the chains in G, minus one (where a chain is a totally ordered subset of G).*

We define the width-dimension of G, and we denote it by $\operatorname{wdim}(G)$*, as the natural number l, such that there exist no $g_1, \ldots, g_{l+2}$ maximal in G, such that g_i is not related by $\leq$ to g_j for any $i \neq j$, and there exists $g \in G$ with $g \leq g_i$ for all $i = 1, \ldots, l+2$.*

We can relate the order of the fractal structure with the concepts of dimension for posets just stated.

Lemma 3.7. *Let $\mathbf{\Gamma}$ be a fractal structure over X, and let $G_k = G(\Gamma_k)$ be the associated posets. If the order of $\mathbf{\Gamma}$ is less or equal than n then* $\dim(G_k) \leq n$*, for all $k \in \mathbb{N}$. If $\mathbf{\Gamma}$ is also irreducible, then* $\operatorname{wdim}(G_k) \leq n$*, for all $k \in \mathbb{N}$.*

Proof. Suppose that the order of $\mathbf{\Gamma}$ is less or equal than n. Let $k \in \mathbb{N}$, and let $\rho_k(x_1) < \ldots < \rho_k(x_l)$ be a proper chain in G_k. Let $x_l \in A_k^{x_l}$, then since $x_{l-1} \in U_{x_l}^{-1} = \bigcap_{x_l \in A_k} A_k$ and $\rho_k(x_{l-1}) \neq \rho_k(x_l)$, there exists $A_k^{x_{l-1}} \in \Gamma_k$, such that $x_{l-1} \in A_k^{x_{l-1}}$ and $x_l \notin A_k^{x_{l-1}}$. Since $\rho_k(x_{l-1}) \neq \rho_k(x_{l-2})$, then there exists $A_k^{x_{l-2}}$ such that $x_{l-2} \in A_k^{x_{l-2}}$ and $x_{l-1} \notin A_k^{x_{l-2}}$ (and then $x_l \notin A_k^{x_{l-2}}$, since $x_{l-2} \in U_{x_{l-1}}^{-1} = \bigcap_{x_{l-1} \in A_k} A_k$). Recursively, we can construct $A_k^{x_i}$, with $i \in \{1, \ldots, l\}$ and $x_i \in A_k^{x_i}$, but $x_i \notin A_k^{x_{i-1}}$ with $i \in \{2, \ldots, l\}$. Therefore $x_1 \in A_k^{x_i}$ for all $i = 1, \ldots, l$ and $A_k^{x_i} \neq A_k^{x_j}$ for $i \neq j$. Then, since the order of Γ_k is less or equal than n, we have that $l \leq n+1$, and so $\dim(G_k) \leq n$.

Let see that $\operatorname{wdim}(G_k) \leq n$. Let $\rho_k(x) \in G_k$, and let $\{\rho_k(x_i) : \rho_k(x_i)$ is maximal in G_k; $i = 1, \ldots, l\}$ be an antichain with $\rho_k(x) \leq \rho_k(x_i)$ for all $i = 1, \ldots, l$. If $x_i \in A_k \cap B_k$, then $\rho_k(x_i) < \rho_k(z)$ for all $z \in A_k \setminus \bigcup_{C_k \neq A_k} C_k$ (note that there exists such z, since Γ_k is irreducible), and then $\rho_k(x_i)$ is not a maximal element in G_k. Therefore, for each $i \in \{1, \ldots, l\}$, there exists $A_k^i \in \Gamma_k$, such that $x_i \in A_k^i \setminus \bigcup_{B_k \neq A_k^i} B_k$, and hence, since all the $\rho_k(x_i)$ are unrelated by $\leq_k$, we have that $A_k^i \neq A_k^j$ for $i \neq j$. On the other hand, since $x \in U_{x_i k}^{-1} = \bigcap_{x_i \in B_k} B_k$ for all $i = 1, \ldots, l$, we have that $x \in A_k^i$ for all $i = 1, \ldots, l$, and since Γ_k is of order less or equal than n, then $l \leq n+1$. Therefore $\operatorname{wdim}(G_k) \leq n$ for all $k \in \mathbb{N}$. $\square$

Now we have the characterization of dimension in terms of fractal structures. Note that the equivalence between statements 1 and 5 in the next theorem is given in Theorem 4.3.17 of [7] and Theorem 7.1.8 of [10].

THEOREM 3.8. *The following statements are equivalent:*

1. *X is metrizable and* $\dim(X) \leq n$*.*
2. *X is metrizable and* $\mathrm{Ind}\,(X) \leq n$*.*
3. *There exists a metrizable space Y, such that* $\dim(Y) = 0$*, and a closed map f from Y onto X such that* $\mathrm{Ord}\ (f) \leq n$ *(where* $\mathrm{Ord}\ (f) = max\{Card(f^{-1}(x)) : x \in X\} - 1$ *).*
4. *There exists a tiling starbase fractal structure over X of order less or equal than n.*
5. *There exists a starbase fractal structure over X of order less or equal than n.*

Proof. The equivalence among 1, 2, and 3 is known, see for instance [8, Theorem 12.6], and it is trivial that 4) implies 5).

For 5) implies 4), use Theorem 3.3.

Let see 3) implies 4).

Suppose there exists a metrizable space Y, such that $\dim(Y) = 0$, and a closed map f from Y onto X such that $\mathrm{Ord}\ (f) \leq n$.

By Theorem 3.1, there exists Γ_n, a starbase fractal structure of order zero over Y. Then it is straightforward to check that $f(\Gamma_n) = \{f(A_n) : A_n \in \Gamma_n\}$ is a tiling starbase fractal structure of order less or equal than n (see [9, Theorem 2.5]).

Let see 4) implies 3).

Let $\mathbf{\Gamma}$ be a tiling starbase fractal structure over X of order less or equal than n. Consider Γ_n with the discrete topology. For each $A_{n+1} \in \Gamma_{n+1}$, there exists only one (since Γ_n is a tiling, see [9, Lemma 1.3]) $A_n \in \Gamma_n$ such that $A_{n+1} \subseteq A_n$ and define $A_n(A_{n+1})$ as that A_n. Let $p_{n+1} : \Gamma_{n+1} \to \Gamma_n$ defined as $p_{n+1}(A_{n+1}) = A_n(A_{n+1})$. Let $Y = \{(A_n) \in \varprojlim(\Gamma_n, p_n) : \bigcap_{n\in\mathbb{N}} A_n$ is non empty$\}$.

Now, let $f : Y \to X$ be defined by $f(A_n) = \bigcap_{n\in\mathbb{N}} A_n$. Clearly f is onto, since $\mathbf{\Gamma}$ is a fractal structure and since $\mathbf{\Gamma}$ is starbase, it is well defined and continuous: To see that, if $x, y \in \bigcap_{n\in\mathbb{N}} A_n$, since $\mathbf{\Gamma}$ is starbase, there exists k such that $\mathrm{St}(y, \Gamma_k) \cap \mathrm{St}(x, \Gamma_k) = \emptyset$, which contradicts that $x, y \in \bigcap_{n\in\mathbb{N}} A_n$. Therefore $\bigcap_{n\in\mathbb{N}} A_n$ is a point and f is well defined. On the other hand, let $y = (A_n) \in Y$ and $k \in \mathbb{N}$. Then $f(A'_k) = A_k \subseteq \mathrm{St}(f(y), \Gamma_k)$, where $A'_k = \{(B_n) \in Y : B_k = A_k\}$ is an open neighborhood of y. Therefore f is continuous.

Clearly $\dim(Y) = 0$, since it is a subset of a countable product of discrete spaces and $\mathrm{Ord}\ (f) \leq n$ since the order of each Γ_k is less or equal than n for all k. To see this, let $x \in X$ and suppose that there exist different $y_1, \ldots, y_{n+2}$ such that $f(y_i) = x, i = 1, \ldots, n+2$. Let $y_i = (A^i_k)_k$ and let $k \in \mathbb{N}$ be such that $A^i_k \neq A^j_k$ for all $i \neq j$. Then $x \in A^i_k$ for all $i = 1, \ldots, n+2$ and hence $\mathrm{Ord}\,(x, \Gamma_k) \geq n+1$, what contradicts that $\mathrm{Ord}\,(\Gamma_m) \leq n$ for all $m \in \mathbb{N}$.

To see that f is closed, we note that $\Delta_k = \{A'_k : A_k \in \Gamma_k\}$ where A'_k is defined as above, is a starbase fractal structure over Y which verifies the hypotheses of Proposition 3.5. So 3) is proved. ☐

Remark 3.9. *If we redefine the dimension functions used in [11] as follows,*

1. *$\delta'_1(X)$ to be the least integer n for which there exists a (locally finite) starbase fractal structure over X with order at most n, and $\delta'_1(X) = \infty$ if there exists no such integer.*

2. *$\delta'_2(X)$ to be the least integer n for which there exists a (locally finite) tiling fractal structure over X with order at most n, and $\delta'_2(X) = \infty$ if there exists no such integer.*

then the previous theorem can be stated as: if X is a metrizable space, it holds that $\dim(X) = \mathrm{Ind}\,(X) = \delta'_1(X) = \delta'_2(X)$.

Compare with the results of Mack and Pears at the beginning of this section. Also note that we can get neither $\dim(X) = \delta_1(X)$ *nor* $\dim(X) = \delta_2(X)$*, since it follows that* $\delta_1(X) = 0$ *(respectively* $\delta_2(X) = 0$*) if and only if* $\mathrm{ind}\,(X) = 0$*, and it is know that* $\mathrm{ind}\,(X) = 0$ *does not imply dim$(X) = 0$, see [11, Example 3.5] (Roy's space).*

Note that from this result we can easily see that the dimension of Sierpinski's triangle is 1, and analogously for other fractal spaces, whose Hausdorff dimension is difficult to calculate.

The next result provides a representation theorem for metrizable spaces with $\dim(X) \leq n$.

PROPOSITION 3.10. *Let X be a metrizable space with* $\dim(X) \leq n$. *Then X can be embedded into the inverse limit of a sequence of locally finite posets of dimension and width-dimension less or equal than n.*

Proof. X admits a tiling (and so irreducible) starbase fractal structure of order less or equal than n by Theorem 3.8, and then it follows the thesis from the relationship between fractal structures and subsets of an inverse limit of a sequence of posets, since the posets associated to that fractal structure are locally finite and they have dimension and width-dimension less or equal than n by Lemma 3.7. □

Compactness allows good properties for $G(\mathbf{\Gamma})$.

THEOREM 3.11. *Let $\mathbf{\Gamma}$ be a fractal structure over a compact Hausdorff space X. Then $G(\mathbf{\Gamma})$ is starbase.*

Proof. Suppose that $G(\mathbf{\Gamma})$ is not starbase, then there exist $x \in X$ and $l \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ there exists $x_n \in \mathrm{St}(x, G(\Gamma_n)) \setminus U_{xl}$. Since X is compact, then there exists $y \in X$ such that y is an adherent point of (x_n). By construction of the sequence it is clear that $y \neq x$, and then, since X is Hausdorff, there exists $m \in \mathbb{N}$ such that $U_{xm} \cap U_{ym} = \emptyset$. Let $k \geq m$ be such that $x_k \in U_{ym}$. Since $x_k \in \mathrm{St}(x, G(\Gamma_k))$, then there exists $z \in X$ such that $x, x_k \in U_{zk}^{-1}$. Then $z \in U_{xk} \cap U_{x_k k} \subseteq U_{xm} \cap U_{ym}$ (note that $x_k \in U_{ym}$, and then $U_{x_k k} \subseteq U_{x_k m} \subseteq U_{ym}$), and this contradicts that $U_{xm} \cap U_{ym} = \emptyset$. Therefore $G(\mathbf{\Gamma})$ is starbase. □

The next result is a sort of converse of Proposition 3.10.

PROPOSITION 3.12. *Let X be a compact Hausdorff space which is a subset of an inverse limit of a sequence of locally finite posets of width-dimension less or equal than d. Let $\Gamma_n = \{\overline{\{g_n\}} : g_n$ is maximal in $G_n\}$. Then $\mathbf{\Gamma}$ is a starbase pre-fractal structure of order less or equal than d.*

Proof. For each maximal element $g_k \in G_k$, we define $A_k(g_k) = \{h \in X : h_k \leq g_k\}$. Let $\Gamma_k = \{A_k(g_k) : g_k$ is a maximal element in $G_k\}$.

Let see that $\mathbf{\Gamma}$ (restricted to X) is a pre-fractal structure over X of order less or equal than d.

It is clear that $A_k(g_k)$ is closed and that Γ_k is a covering for all $k \in \mathbb{N}$ (note that since G_k is locally finite, each element in G_k has a maximal element greater or equal than it).

Let g_{k+1} be a maximal element of G_{k+1}. Let g_k be a maximal element of G_k greater or equal than $\phi_{k+1}(g_{k+1})$ (note that this element exists, since G_k is locally finite). Then it is clear that $A_{k+1}(g_{k+1}) \subseteq A_k(g_k)$, and this proves that Γ_{k+1} is a refinement of Γ_k.

Let $\mathbf{\Delta}$ be the fractal structure over X associated to the inverse limit of the posets associated to $\mathbf{\Gamma}$. Since X is compact, then $\mathbf{\Delta}$ is starbase (by Theorem 3.11), and hence $\mathbf{\Gamma}$ is starbase, too, since it can be easily checked that $\mathrm{St}(x, \Delta_n) = \mathrm{St}(x, \Gamma_n)$ for all $x \in X$ and all $n \in \mathbb{N}$ (since G_n is locally finite).

Therefore, $\mathbf{\Gamma}$ is a starbase pre-fractal structure over X. Let see that its order is less than or equal to n.

Suppose that $x \in A_k(g_k^i)$, with $i \in \{1, \ldots, l\}$ and g_k^i being maximal in G_k, then $\rho_k(x) \leq g_k^i$ for all $i = 1, \ldots, l$, and since g_k^i is maximal and $\mathrm{wdim}\,(G_k) \leq n$ then $l \leq n+1$, and thus the order of Γ_k is less or equal than n. □

The last brick for our theorem is a slight modification of a result from [11], mentioned previously.

LEMMA 3.13. *Let $\mathbf{\Gamma}$ be a starbase pre-fractal structure of order less or equal than n over X. Then* $\mathrm{ind}\,(X) \leq n$.

Proof. The proof follows from a slight modification of the proof of Theorem 3.3 of [11].

It is enough to establish that for every $n \in \mathbb{N}$, the following statement (a_n) is true: if X is a space such that there exists a starbase pre-fractal structure of order less or equal than n then $\mathrm{ind}\,(X) \leq n$. (a_0) is true by Theorem 3.8 (note that a pre-fractal structure of order zero is a fractal structure). Suppose that (a_{n-1}) is true and that there exists a starbase pre-fractal structure $\mathbf{\Gamma}$ over X of order less or equal than n.

Let $x \in X$ and $k \in \mathbb{N}$. Let $\Delta_m = \{A_m \cap \mathrm{Fr}(U_{xk}) : A_m \in \Gamma_m; A_m \cap U_{xk} = \emptyset\}$. It is easy to prove that $\mathbf{\Delta} = \{\Delta_m : m \in \mathbb{N}\}$

is a starbase pre-fractal structure over $\mathrm{Fr}(U_{xk})$. Let us show that it has order less or equal than $n-1$. Let $y \in \mathrm{Fr}(U_{xk})$, then U_{ym} is an open neighborhood of y, and hence there exists $z \in U_{ym} \cap U_{xk}$. Since $z \in U_{ym} \subseteq \mathrm{St}(y, \Gamma_m)$ then it follows that there exists $A_m \in \Gamma_m$ such that $y, z \in A_m$, and hence $z \in A_m \cap U_{xk} \neq \emptyset$, and since the order of Γ_m is less or equal than n, then it follows that the order of Δ_m is less or equal than $n-1$, and by (a_{n-1}) it follows that $\mathrm{ind}\,(\mathrm{Fr}(U_{xk})) \leq n-1$ for all $x \in X$ and $k \in \mathbb{N}$, and hence $\mathrm{ind}\,(X) \leq n$, and so (a_n) is true. □

Finally, in compact metric spaces we can get the following result, that reminds Freudenthal's Theorem.

COROLLARY 3.14. *Let X be a compact metrizable space. Then the following statements are equivalent:*

1. *$ind(X) = Ind(X) = \dim(X) \leq n$*
2. *There exists a finite tiling starbase fractal structure over X of order less or equal than n.*
3. *X can be embedded into the inverse limit of a sequence of finite posets of width-dimension less or equal than n.*
4. *There exists a finite starbase pre-fractal structure over X of order less or equal than n.*

Proof. 1) implies 2). It is as 3) implies 4) in Theorem 3.8. Note that if X is compact, then Y must be compact (since f is perfect), and then we can take a finite starbase fractal structure over Y of order zero (note that with a slight modification of the proof of Theorem 3.1, we can get a finite starbase fractal structure over a compact metrizable strongly zero-dimensional space).

2) implies 3). If the fractal structure is finite, then so are the associated posets, and then we can apply Lemma 3.7 and the result follows from the relation between fractal structures and subsets of inverse limits of a sequence of posets.

3) implies 4) by Proposition 3.12.

4) implies 1) by the previous lemma. □

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