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Unknotting Numbers are not Realized in Minimal Projections for a Class of Rational Knots

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Summary. - *In previous results, Bleiler and Nakanishi produced an example of a knot where the unknotting number was not realized in a minimal projection of the knot. Bernhard generalied this example to an infinite class of examples with Conway notation* $(2j+1, 1, 2j)$ *with* $j \geq 2$*. In this paper we examine the entire class of knots given in Conway notation by* $(2j+1, 2k+1, 2j)$ *where* $j \geq 1$ *and* $k \geq 0$ *and we determine that a large class of knots of this form have the unknotting number not realized in a minimal projection. We also produce an infinite class of two component links with unknotting number gap arbitrarily large.*

1. Introduction

In the early 1980s, Bleiler [3] and Nakanishi [10] independently produced an example of a knot whose unknotting number was not realized in a minimal projection. The example was the knot (5, 1, 4) in Conway notation [4]. Bleiler's method of proof involved showing that the ten crossing projection given by the Conway notation (5, 1, 4) was the minimal projection and had unknotting number 3, whereas the alternate projection with 14 crossings given by the Con-

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way notation $(2,\ -2,\ 2,\ -2,\ 2,\ 4)$ has unknotting number 2. Nakanishi's method of proof involved showing that the minimal projection had unknotting number three, whereas an alternate projection with twelve crossings had unknotting number two.

Bernhard [2] generalized Nakanishi's approach to produce an infinite class of alternating knots for which the unknotting number was not realized in a minimal projection. Specifically, he showed that the knots $(2k+1, 1, 2k)$ for $k \geq 2$ have this property.

More recent results make it easier to determine when a given projection of a knot or link is the minimal projection, and thus make it easier to analyze when gaps between the unknotting number of a minimal projection of a knot or link and the actual unknotting number of the knot or link occur. The needed definitions and background material are presented in the next section.

In this paper, we examine the unknotting number for knots given in Conway notation by (a, b, c) where a, b, and c are positive. We are able to show that for a large class of these knots, namely the knots of the form $(2j+1,\ 2k+1,\ 2j)$, for $j \geq 1$ and $k \geq 0$, the unknotting number of the minimal projection is $2j$ if $j \leq k+1$ and is $j+k+1$ if $j \geq k+2$. In addition, if $j \geq k+2$, the knot $(2j+1,\ 2k+1,\ 2j)$ has a nonminimal projection with unknotting number $\leq j+k$. As a consequence, any knot of the form $(2j+1,\ 2k+1,\ 2j)$ with $j \geq k+2$ has unknotting gap at least 1.

We are also able to show that the unknotting number of the two component links of the form $(2j,\ 1,\ 2k)$, for $j \geq k \geq 2$ is $k+j-1$. These links have a nonminimal projection with unknotting number less than or equal to j. As a consequence, the unknotting gap of these links is at least $k-1$.

Work on the results in this paper started during summer programs at Oregon State University. James Bernhard, Cassandra McGee and Eva Wailes participated in these programs and contributed to the results. Some of the results appear in preliminary form in unpublished proceedings from these programs[5], [6].

2. Definitions and Background

We use the word link to represent a knot or link in R^3. The unknotting number of a specific projection of a link is the minimal number of simultaneous crossing changes necessary in that projection to change the link into the trivial link. The unknotting number of a link is the minimum, taken over all projections of the link, of the unknotting number of the projections of that link.

We use Conway notation [4] to represent projections of rational knots and links. The continued fraction associated with a link given by Conway notation $(a_1,\ a_2,\ \ldots\ , a_n)$ is the continued fraction:

$$a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\cdots + \cfrac{1}{a_2 + \cfrac{1}{a_1}}}}$$

A main result from [4] is the following.

Proposition 2.1. *If the continued fractions associated with two links given in Conway notation are the same, then the links are equivalent.*

Figure 1 shows the knot given by Conway notation (a, b, c) where a, b, and c are positive. This is not the standard way of showing this knot, but is equivalent to the standard picture. For more information on Conway notation, consult Adams' text [1].

A projection is alternating if the sequence of crossings alternates between overcrossings and undercrossings as the knot is traversed with respect to a specific orientation. A projection is reduced alternating if it is alternating and if no crossing is reached twice successively as the knot is traversed with respect to a specific orientation. Figure 2 below shows a non reduced alternating projection. The part of the projection inside the dotted rectangle is not shown.

The following result of Kauffman, Thistlethwaite, and Murasugi [8] makes it easier to check for minimal projections.

Proposition 2.2. *Any reduced alternating projection of a link is a minimal projection.*

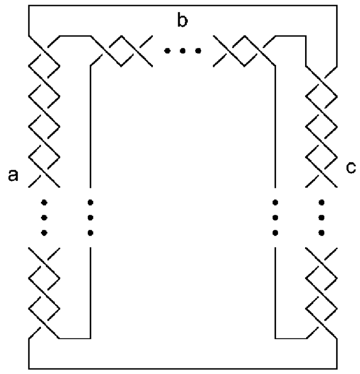

Figure 1: Knot given by Conway notation

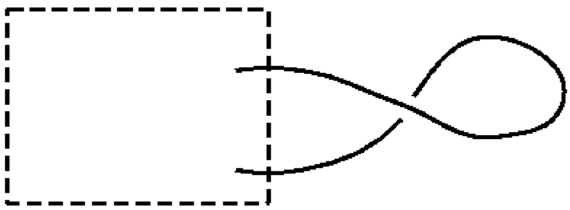

Figure 2: Unreduced alternating projection

It is easy to check that any link given in Conway notation by a sequence of length at least two of all positive, or all negative integers, is reduced alternating, and thus the corresponding projection is minimal and the link is nontrivial.

A flype is an ambient isotopy that results in changing a projection from the situation at the top of Figure 3 below to the situation at the bottom of the figure. The region of the projection inside the dotted rectangle is rotated to remove the crossing at the right and introduce the crossing at the left. Note that if two projections differ by a flype, then the unknotting number of the two projections is the same.

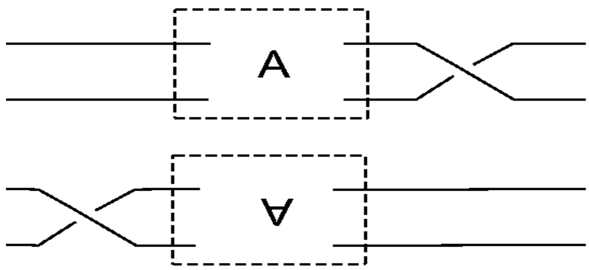

Figure 3: Flype

The following result of Menasco and Thistlethwaite [9]makes it possible to just check the unknotting number of a single reduced alternating projection of a link to find the unknotting number of minimal projections of that link.

PROPOSITION 2.3. *Any two reduced alternating projections of a link differ by a series of flypes, and so have the same unknotting number.*

Finally, the unknotting gap of a link is the difference between the unknotting number of a minimal projection of the link and the actual unknotting number of the link. The strategy for finding rational links with positive unknotting gap should now be clear. One analyzes the unknotting number of projections of links given in Conway notation by a sequence of all positive or all negative numbers. The associated projection is necessarily minimal, and the result of changes in various positions can be analyzed. If a sequence of changes results in a link that has an alternate Conway expression involving all positive or all negative integers, then that particular sequence of changes does not result in the trivial link. Analyzing all possible changes gives the unknotting number of the minimal projection of the link. Finally, one must determine if there are alternate projections that have a smaller unknotting number. If one can find such a projection, then one has a proof that the given link has a positive unknotting gap.

Much of the work involves finding a systematic procedure for analyzing all possible changes, and finding a method for producing alternate projections with possibly lower unknotting numbers.

3. Main Results

We first focus on knots given in Conway notation by (a, b, c) where a, b, and c are all positive or all negative and obtain some preliminary information about unknotting numbers of the corresponding projections. The projection of the link in this case is reduced alternating and so is minimal. The knot given by Conway notation (a, b, c) where a, b, and c are all positive is shown in Figure 1. Because of the symmetry shown in this figure, to consider all knots with notation (a, b, c), we need only consider knots with one of the patterns

(odd, even even), (odd, odd, odd), or (odd, odd, even).

Consider a knot of the form (a, b, c) where a, b, and c are all positive. We refer to the crossings as left, middle, or right crossings depending on where they occur in the diagram. (See Figure 1.) Changing k, j, and m crossings on the left, middle and right respectively results in a knot with Conway notation $(a - 2k,\ b - 2j,\ c - 2m)$. We denote such a change in crossings by $[[k, j, m]]$. The Bleiler, Nakanishi and Bernhard results all fit into the $(2j + 1,\ 1,\ 2j)$ case. We generalize this by considering knots of the form $(2j + 1,\ 2k + 1,\ 2j)$.

3.1. The $(2j + 1,\ 2k + 1,\ 2j)$ case.

Our knot is of the form $(2j + 1,\ 2k + 1,\ 2j)$ with k greater than or equal to 0 and j greater than or equal to 1. First note that applying the crossing change $[[j,\ 0,\ j]]$ to this knot results in a knot with Conway notation $(1,\ 2k + 1,\ 0)$ By considering the associated continued fraction, one sees that this knot is equivalent to the knot $(2k + 2,\ 0)$ which is trivial. So the unknotting number of the projection $(2j + 1,\ 2k + 1,\ 2l)$ is less than or equal to $2j$. Next note that applying the crossing change $[[j, k + 1, 0]]$ to this knot results in a knot with Conway notation $(1,\ -1,\ 2j)$ which can be seen to be trivial by examining Figure 1. So the unknotting number of the projection $(2j + 1,\ 2k + 1,\ 2l)$ is less than or equal to $j + k + 1$. The following theorem shows that one of the two crossing changes just considered is in fact minimal.

Theorem 3.1. *The unknotting number of the knot projection given by* $(2j+1,\ 2k+1,\ 2j)$, *for* $j+k \geq 2$ *is the minimum of* $2j$ *and* $j+k+1$. *The unknotting number of the projection* $(3,\ 1,\ 2)$ *is* 1.

The proof will refer to the following table of knots.

	$k=0$	$k=1$	$k=2$	$k=3$	$\cdots$
$j=1$	$(3,1,2)$	$(3,3,2)$	$(3,5,2)$	$(3,7,2)$	$\cdots$
$j=2$	$(\mathbf{5},\mathbf{1},\mathbf{4})$	$(5,3,4)$	$(5,5,4)$	$(5,7,4)$	$\cdots$
$j=3$	$(7,1,6)$	$(\mathbf{7},\mathbf{3},\mathbf{6})$	$(7,5,6)$	$(7,7,6)$	$\cdots$
$j=4$	$(9,1,8)$	$(9,3,8)$	$(\mathbf{9},\mathbf{5},\mathbf{8})$	$(9,7,8)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$

Table 1:

Proof. For the exceptional case, one can check directly that applying the crossing change $[[0,1,0]]$ to $(3,1,2)$, results in the unknot.

Consider the table of knots above, parameterized by j and k.

First row and first column:

Bernhard [2] established that the unknotting number of the projection given by $(2j+1,\ 1,\ 2j)$ is $j+1$, for $j \geq 2$. These knots, with the addition of $(3,1,2)$, are the knots in the first column of the table.

We next establish that the theorem holds for knots in the first row of the table with $k \geq 1$. By the discussion in the paragraph above the theorem, the unknotting number of a knot projection (3, $2k+1$,2) is less than or equal to 2. Consider the result of applying a single crossing change to a knot of this form. If the crossing change is of the form $[[1,0,0]]$, the resulting knot is $(1,2k+1,2)$ which is reduced alternating, and therefore not equivalent to the unknot. If the crossing change is of the form $[[0,0,1]]$ the resulting knot is $(3,2k{+}1,0)$. By examining Figure 1, one can see that this equivalent to the trefoil knot, and so is nontrivial. If the crossing change is of the form $[[0,1,0]]$, the resulting knot is $(3,2k-1,2)$. Since $k \geq 1$, this is also reduced alternating and thus nontrivial.

The inductive step

We have established that the theorem holds for knots in the first row and first column of the table. We now proceed by induction. Consider a specific knot $(2j+1,\, 2k+1,\, 2j)$ with $k > 0$, and $j > 1$. Assume inductively that for all knots in columns to the left of this knot, and for all knots in the same column as this knot, but above it, the theorem holds. There are two cases to consider.

(Case 1):

If $j \leq k+1$, $k > 0$, and $j > 1$, then the knot $(2j+1,\, 2k+1,\, 2j)$ is a knot above the diagonal in the table that starts at the knot $(5, 1, 4)$. We need to show that the unknotting number of this knot is $2j$. So we need to show that changing fewer than $2j$ crossings does not result in the trivial knot.

We proceed by considering crossing changes of the form $[[\alpha, \beta, \gamma]]$ where $\alpha + \beta + \gamma < 2j$.

Case 1a. $\alpha \geq 1$ and $\gamma \geq 1$.

In this case, the resulting knot obtained after making the crossing changes is the same as one that can be obtained from the knot $(2j-1,\, 2k+1,\, 2j-2)$, the knot in the same column of the table directly above the knot under consideration, by making fewer than $2j-2$ changes. By the inductive assumption, the resulting knot is nontrivial.

Case 1b. $\alpha = 0$ and $\beta = 0$.

In this case, all of the crossing changes occur on the right. After making the crossing changes, the resulting knot is $(2j+1, 2k+1, 2j-2\gamma)$.

If $\gamma < j$, the Conway notation consists of all positive integers and so the resulting knot is reduced alternating and nontrivial.

If $\gamma = j$, we have the knot $(2j+1, 2k+1, 0)$ which is equivalent to a reduced alternating knot with $2j+1$ crossings and so is nontrivial.

If $\gamma > j$, we have a knot of the form $(a, b, -c)$ where a, b, and c are positive. By checking the associated continued fractions, one sees that this knot is equivalent to a knot with Conway notation

$$(-a, -b, -1, -(c-1))$$

which is reduced alternating since all the terms are negative. So this knot is nontrivial.

Case 1c. $\gamma = 0$ and $\beta = 0$.

In this case, all of the crossing changes occur on the left. After making the crossing changes, the resulting knot is $(2j+1-2\alpha, 2k+1, 2j)$.

If $\alpha \leq j$, the Conway notation consists of all positive integers, and so the resulting knot is nontrivial.

If $\alpha = j+1$, the resulting knot has Conway notation $(-1, 2k+1, 2j)$ which can be seen to be equivalent to the knot with Conway notation $(2k, 2j)$ which is reduced alternating and nontrivial since $k > 0$ and $j > 1$.

If $\alpha > j+1$, the resulting knot is of the form $(-a, b, c)$ with $a > 1$ and $b = 2k+1$, and $c = 2j$. This can be seen to be equivalent to the knot with Conway notation $(a-1, 1, b-1, c)$ which is reduced alternating and nontrivial.

Case 1d. $1 \leq \beta \leq 2j-1$.

Since $j \leq k+1$, this implies that $\beta \leq 2k-1$. So applying the crossing change $[[\alpha, \beta, \gamma]]$ to $(2j+1,\ 2k+1,\ 2j)$ results in the same knot as applying the crossing change $[[\alpha, \beta-1, \gamma]]$ to the knot $(2j+1,\ 2k-1,\ 2j)$ which is immediately to the left of the knot under consideration in the table.

But the knot to the immediate left in the table, by the inductive assumption, either has unknotting number $2j$ or $2j-1$. Since the sum of $\alpha, \beta-1$, and γ is less than $2j-1$, the resulting knot is nontrivial.

(Case 2):

Having completed case 1, assume the knot $(2j+1,\ 2k+1,\ 2j)$ has $j > k+1$. This is a knot on or below the diagonal in the table that starts at the knot $(5, 1, 4)$. We need to show that the unknotting number of this knot is $j+k+1$. So we need to show that making $j+k$ or fewer crossings does not result in the trivial knot.

Again we proceed by considering crossing changes of the form $[[\alpha, \beta, \gamma]]$ where $\alpha+\beta+\gamma < j+k+1$. Note that $j+k+1$ is less than $2j$.

Case 2a.$1 \leq \beta \leq 2k$.

If $1 \leq \beta \leq 2k$, applying the crossing change $[[\alpha, \beta, \gamma]]$ to $(2j+1,\ 2k+1,\ 2j)$ results in the same knot as applying the crossing change $[[\alpha, \beta-1, \gamma]]$ to the knot $(2j+1,\ 2k-1,\ 2j)$ which is immediately to the left of the knot under consideration in the table. But the knot to the immediate left in the table, by the inductive assumption, has unknotting number $j + k$. Since the sum of $\alpha, \beta - 1$, and γ is less than $j + k$ the resulting knot is nontrivial.

Case 2b.$\beta = 2k + 1$.

If $\beta = 2k + 1$, applying the crossing change $[[\alpha, \beta, \gamma]]$ to $(2j + 1,\ 2k + 1,\ 2j)$ results in the knot $(2j + 1 - 2\alpha, -2k - 1, 2j - 2\gamma)$.

Since we were making at most $j + k$ crossing changes altogether, since $j > k+1$, and since $2k+1$ of the crossing changes are made on the middle section, at most $(j + k) - (2k + 1) = j - (k + 1)$ changes can be made on the left and right portions of the knot. So the resulting knot is of the form $(a, -2k-1, c)$ with a and c greater than 1. But this knot is equivalent to the knot with Conway notation $(a-1, 1, 2k-1, 1, c-1)$ which is reduced alternating and nontrivial.

Case 2c. $\beta = 0, \alpha \geq 1$ and $\gamma \geq 1$.

Applying the crossing change $[[\alpha, 0, \gamma]]$ to $(2j + 1,\ 2k + 1,\ 2j)$ results in the same knot as applying the crossing change $[[\alpha-1, 0, \gamma-1]]$ to the knot $(2j - 1,\ 2k + 1,\ 2j - 2)$ which is immediately above the knot under consideration in the table. The unknotting number of the knot $(2j-1,\ 2k+1,\ 2j-2)$ is $2j-2$ if $j = k+2$, and is $j+k$ if $j > k + 2$. But $\alpha - 1 + \gamma - 1 < j + k - 1$. So in either case, the resulting knot is not the trivial knot.

Case 2d: $\beta = 0, \alpha = 0$.

Applying the crossing change $[[0, 0, \gamma]]$ to $(2j + 1,\ 2k + 1,\ 2j)$ results in the knot with Conway notation $(2j + 1,\ 2k + 1,\ 2j - 2\gamma)$. If γ is less than j, this knot is reduced alternating and nontrivial. If $\gamma = j$, the knot is $(2j + 1,\ 2k + 1,\ 0)$ which is equivalent to a reduced alternating knot with $2j + 1$ crossings and so is nontrivial. If $\gamma > j$, we have a knot of the form $(a, b, -c)$ where a, b, and c are positive. By checking the associated continued fractions, one sees that this knot is equivalent to a knot with Conway notation continued fraction associated with this knot is

$$(-a, -b, -1, -(c - 1))$$

which is reduced alternating since all the terms are negative. So this knot is nontrivial.

Case 2e: $\beta = 0, \gamma = 0$.

Applying the crossing change $[[\alpha, 0, 0]]$ to $(2j+1,\ 2k+1,\ 2j)$ results in the knot with Conway notation $(2j+1-2\alpha,\ 2k+1,\ 2j)$.

If $\alpha \leq j$, the Conway notation consists of all positive integers, and so the resulting knot is nontrivial.

If $\alpha = j+1$, the resulting knot has Conway notation $(-1, 2k+1, 2j)$ which can be seen to be equivalent to the knot with Conway notation $(2k, 2j)$ which is reduced alternating and nontrivial since $k > 0$ and $j > 1$.

If $\alpha > j+1$, the resulting knot is of the form $(-a, b, c)$ with $a > 1$ and $b = 2k+1$, and $c = 2j$. This can be seen to be equivalent to the knot with Conway notation $(a-1, 1, b-1, c)$ which is reduced alternating and nontrivial.

This completes the second case of the inductive step and the proof of the theorem. □

Next, we show that for all knots on or below the diagonal referred to in the previous proof, there is a different projection of the knot with a smaller unknotting number than the unknotting number that occurs in the minimal projection.

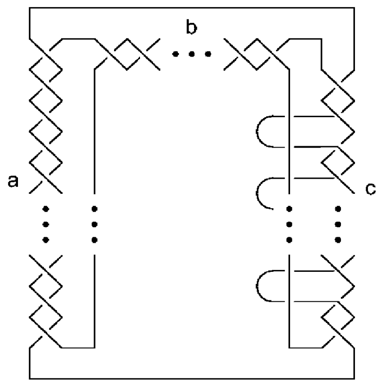

Figure 4: K'

THEOREM 3.2. *For knots $(2j+1,\ 2k+1,\ 2j)$ with $j > k+1$, there is a projection that has unknotting number $j+k$, whereas the minimal projection has unknotting number $j+k+1$.*

Proof. Let K represent the knot projection given by $(2j+1,\ 2k+1,\ 2j)$. Create a new projection K' of this knot with $2(j-1)$ more crossings by dragging $j-1$ strands along the right hand side under the adjacent vertical strand as indicated in Figure 4. By changing k of the middle crossings, one obtains a knot projection K'' that can be obtained from the knot with Conway notation $(2j+1,\ 1,\ 2j)$ by dragging the corresponding $j-1$ strands under the adjacent vertical strand. This knot projection K'' is shown in Figure 5. Bernhard [2] shows that this knot projection K''has unknotting number $\leq j$ and so the knot projection that we described, K' has unknotting number $\leq k+j$. □

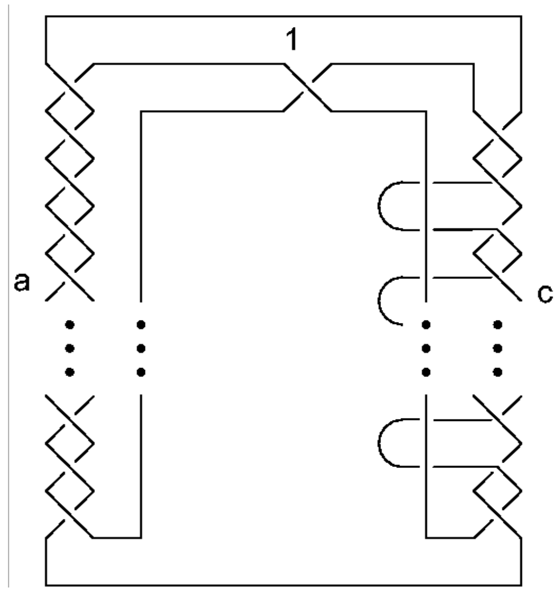

Figure 5: K''

3.2. Links with arbitrary unknotting gap.

Consider links of the form $(2j, 1, 2k)$ with $j \geq k \geq 2$. See the following table.

	$k=2$	$k=3$	$k=4$	$k=5$	$\cdots$
$j=2$	$(4,1,4)$				$\cdots$
$j=3$	$(6,1,4)$	$(6,1,6)$			$\cdots$
$j=4$	$(8,1,4)$	$(8,1,6)$	$(8,1,8)$		$\cdots$
$j=5$	$(10,1,4)$	$(10,1,6)$	$(10,1,6)$	$(10,1,10)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$

Table 2:

Using techniques similar to that used in the proof of theorems one and two, one can obtain the following result.

THEOREM 3.3. *For a links of the form $(2j,1,2k)$ with $j \geq k \geq 2$, the unknotting number of the minimal projection is $k+j-1$. These links have alternate projections with unknotting number $\leq j$. Thus, for the link $(2j,1,2k)$, the unknotting gap is at least $k-1$.*

4. Questions

A number of questions and problems remain concerning knots given by Conway notation.

1. Analyze all knots and links given in Conway notation (a,b,c) where a, b, and c are all positive or all negative to determine which knots have an unknotting gap.
2. Develop a criterion for determining when a knot or link given in Conway notation $(a_1,a_2,\cdots,a_n)$ with each a_i positive or each a_i negative has an unknotting gap and develop a method for determining what the unknotting gap is.
3. Find classes of prime knots with arbitrarily large unknotting gap.

Regarding the third question, it is easy to form non prime knots with large unknotting gaps by taking connected sums of knots with unknotting gap one. The author and Peterson [7] have in preparation a paper giving prime knots of the form (a, b, c, d, e) in Conway notation with arbitrarily large unknotting gap.

References

[1] C. Adams, *The knot book, an elementary introduction to the mathematical theory of knots*, W. H. Freeman and Co., New York, 1994.

[2] J. Bernhard, *Unknotting numbers and minimal knot diagrams*, Journal of Knot Theory and Its Ramifications **3** (1994), 1–5.

[3] S. Bleiler, *A note on the unknotting number*, Math. Proc. Camb. Phil. Soc. **96** (1984), 469–471.

[4] J. H. Conway, *On enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra (Oxford), 1967, pp. 329–358.

[5] D. J. Garity, *Proceedings of the 1992 Research Experiences for Undergraduates Program at Oregon State University*, editor.

[6] D. J. Garity, *Proceedings of the 1993 Research Experiences for Undergraduates Program at Oregon State University*, editor.

[7] D. J. Garity and P. Peterson, *Prime knots with large unknotting gaps*, in preparation.

[8] L. Kauffman, *New invariants in the theory of knots*, Amer. Math. Monthly **95** (1988), no. 3, 195–242.

[9] W. Menasco and M. Thistlethwaite, *The tait flyping conjecture*, Bull. Amer. Math. Soc. **25** (1991), no. 2, 403–412.

[10] Y. Nakanishi, *Unknotting numbers and knot diagrams with the minimum crossings*, Math. Sem. Notes, Kobe Univ **11** (1983), 257–258.

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