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Dolcher fixed point theorem and its connections with recent developments on compressive/expansive maps [1]

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"In memory of Professor Mario Dolcher (1920-1997)"

Abstract. *In 1948 Mario Dolcher proposed an expansive version of the Brouwer fixed point theorem for planar maps. In this article we reconsider Dolcher's result in connection with some properties, such as covering relations, which appear in the study of chaotic dynamics.*

Keywords: continuous maps, fixed points, chaotic dynamics, covering relations, snap-back repellers.
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1. Introduction

The discovery of the so-called complex or chaotic dynamics, about the coexistence of periodic and non-periodic trajectories and sensitive dependence on the initial conditions, is usually attributed to Henri Poincaré [42, 43] (see also [4, 7, 35]) who found very complicated dynamics in his studies of the three body problem. According to Robert May [37, 38], the term *chaos* was introduced in a mathematical context by Li and Yorke in their famous article "Period three implies chaos" [30]. After this paper, thanks also to the preceding work by Stephen Smale on the *horseshoe* [46, 47] (see also [48]) and without forgetting the many other contributions about the so-called *strange attractors* (by Lorenz, Ruelle and Takens, Ueda, just to quote a few names), various formal definitions of chaotic dynamics were proposed (see, for instance [6, page 183], [10, page 127], [15, page 50] and [51, page 57]). A detailed analysis of these definitions, as well as a comparison about different points of view can be found in [5, 8, 9, 11, 21, 24, 34, 45, 50]. In any case, since there are thousands of references about chaotic dynamics as well as many different and interesting points of view on this topic, our discussion is clearly not exhaustive. Although some concepts of chaos may look very different from each other, it is interesting

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to observe that the notion of chaos according to Li and Yorke is implied by several other definitions. In particular, for a broad class of spaces and mappings, it follows from Devaney's definition and it holds for maps with positive topological entropy.

In 1978 Frederick R. Marotto [31] extended Li–Yorke's approach to higher dimensions, by introducing the concept of *snap-back repeller* for a given map f. One of the key assumptions in this method is the existence of a repulsive fixed point for f. This in particular implies the existence of a neighborhood U of such a point where the expansive property

$$f(U) \supseteq U \tag{1}$$

holds. Maps which are expansive (at least on some parts of their domain) are typical in the context of chaotic dynamics. Conditions of the form (1) or their generalizations (see Section 3) are usually named *covering relations*. They are, in some sense, dual with respect to the assumption $f(U) \subseteq U$ (for U a compact set homeomorphic to a closed ball) of the Brouwer fixed point theorem. It can be interesting to recall that in 1948 Mario Dolcher already considered an expansive version of the Brouwer theorem for the search of fixed points of planar maps [17].

The aim of this paper is to reconsider Dolcher's result in the context of the covering relations. In Section 2 we recall some classical facts related to Brouwer fixed point theorem and we give a comparison with their dual aspect concerning expansive properties. We also show, by means of a counterexample, that the assumptions in Dolcher's theorem are sharp. In Section 3 we survey some results about maps which are compressive/expansive only on some components of the space. With this respect, Dolcher's approach, if applied only to some components of the map, can find its interpretation in the context of the Markov partitions as presented in [52]. The content of this section is based on results obtained in [2, 3, 41, 52]. It is also strictly related to a recent paper by Jean Mawhin [36] where various fixed points theorems for such maps are unified in a generalized setting.

This paper is partially based on two lectures delivered by the authors at the University of Trieste and on the thesis [49].

2. Fixed points and periodic points

2.1. Results related to Brouwer fixed point theorem

Let $||\cdot||$ be a fixed norm in $\mathbb{R}^N$ and let $B_r := \{x \in \mathbb{R}^N : ||x|| \leq r\}$ be the closed ball of center the origin and radius $r > 0$ in $\mathbb{R}^N$. The Brouwer fixed point theorem is one of the most classical and known results about the existence of fixed points for continuous maps in finite dimensional spaces. It can be formally expressed as follows:

THEOREM 2.1 (Brouwer). *For any continuous map* $\phi : B_r \to B_r$ *there exists* $\tilde{x} \in B_r$ *such that* $\phi(\tilde{x}) = \tilde{x}$.

Usually, the presentation of this theorem is accompanied by some related results as the following.

THEOREM 2.2 (Rothe). *For any continuous map* $\phi : B_r \to \mathbb{R}^N$ *such that* $\phi(\partial B_r) \subseteq B_r$, *there exists* $\tilde{x} \in B_r$ *such that* $\phi(\tilde{x}) = \tilde{x}$.

THEOREM 2.3 (Poincaré–Bohl). *For any continuous map* $\phi : B_r \to \mathbb{R}^N$ *such that*

$$\phi(x) \neq \mu x, \quad \forall\, x \in \partial B_r \text{ and } \mu > 1,$$

there exists $\tilde{x} \in B_r$ *such that* $\phi(\tilde{x}) = \tilde{x}$.

Theorem 2.2 and Theorem 2.3, although they are apparently more general than Theorem 2.1, can be easily proven by Brouwer's theorem. Indeed, one can apply it to the continuous map

$$\psi : B_r \to B_r\,, \quad \psi(x) := P_r(\phi(x)),$$

where

$$P_r(y) := \begin{cases} y & y \in B_r \\ r\,\dfrac{y}{||y||} & y \notin B_r \end{cases}$$

is the radial projection of $\mathbb{R}^N$ onto B_r. One can easily check that if $\tilde{x} \in B_r$ is a fixed point of ψ, then the assumptions of Theorem 2.2 or Theorem 2.3 prevent the possibility that $\phi(\tilde{x}) \notin B_r$. Indeed, if $||\phi(\tilde{x})|| > r$, then, from $\psi(\tilde{x}) = \tilde{x}$, it follows that $\tilde{x} \in \partial B_r$ (that contradicts the condition of Rothe theorem) and, moreover, $\phi(\tilde{x}) = \mu\tilde{x}$ with $\mu = ||\phi(\tilde{x})||/r$ (that contradicts the assumption of Poincaré–Bohl theorem). Hence, in any case $\psi(\tilde{x}) \in B_r$ and so $\tilde{x} = \psi(\tilde{x}) = \phi(\tilde{x})$.

An equivalent manner to express the Brouwer fixed point theorem is that of saying that a closed ball in a finite dimensional normed space has the *fixed point property* (FPP). In general, we say that a topological space X has the FPP if any continuous map $f : X \to X$ has at least a fixed point. The FPP is preserved by homeomorphisms, thus, if we define a *m-dimensional cell* as a topological space which is homeomorphic to a closed ball of $\mathbb{R}^m$ (according to [40, page 4]), we can express the Brouwer fixed point theorem as follows.

THEOREM 2.4. *For any continuous map* $\phi : C \to C$, *where* C *is a* m*-dimensional cell, there exists* $\tilde{x} \in C$ *such that* $\phi(\tilde{x}) = \tilde{x}$.

All the above versions of the Brouwer theorem, from a geometrical point of view, describe a situation in which the image of a ball (or its boundary) is contained in the ball itself. A dual result would be naturally expected, namely the existence of fixed points when the image of a ball covers it. This is indeed true for *homeomorphisms* in finite dimensional spaces. More precisely, the following result holds.

THEOREM 2.5. *Let $C \subseteq \mathbb{R}^N$ be a m-dimensional cell and let $\phi : C \to \phi(C) \subseteq \mathbb{R}^N$ be a homeomorphism such that*

$$\phi(C) \subseteq C \quad \textit{or} \;\; \phi(C) \supseteq C. \tag{2}$$

Then there exists $\tilde{x} \in C$ such that $\phi(\tilde{x}) = \tilde{x}$.

Clearly, we are precisely in the setting of Theorem 2.4 when $\phi(C) \subseteq C$. On the other hand, when $\phi(C) \supseteq C$, we can enter again in the setting of Theorem 2.4 by observing that $C' := \phi(C)$ is a m-dimensional cell and $\phi^{-1} : C' \to C \subseteq C'$ is continuous. Hence there exists $\tilde{x} \in C'$ with $\phi^{-1}(\tilde{x}) = \tilde{x}$, so that $\tilde{x} \in C$ is also a fixed point for ϕ.

2.2. Covering relations for continuous maps

A more interesting problem arises if ϕ is only continuous and not necessarily a homeomorphism. In the one-dimensional case ($N = 1$), using the Bolzano intermediate value theorem we can provide an affirmative answer as follows.

THEOREM 2.6. *Let $I \subseteq \mathbb{R}$ be a compact interval and let $\phi : I \to \mathbb{R}$ be a continuous map such that $\phi(I) \supseteq I$. Then there exists $\tilde{x} \in I$ such that $\phi(\tilde{x}) = \tilde{x}$.*

We can find an application of this result in the classical paper of Li and Yorke “Period three implies chaos” (see [30, Lemma 2]). A second result which plays a crucial role in that paper is the following (see [30, Lemma 1]).

THEOREM 2.7. *Let $f : J \to J$ be a continuous map (where $J \subseteq \mathbb{R}$ is an interval) and let $(I_n)_n$ be a sequence of compact intervals with $I_n \subseteq J$ and $f(I_n) \supseteq I_{n+1}$ for all $n \in \mathbb{N}$. Then there is a sequence of compact intervals Q_n such that $I_0 \supseteq Q_n \supseteq Q_{n+1}$ and $f^n(Q_n) = I_n$ for all $n \geq 0$. For any $x \in Q := \cap_{n=0}^{\infty} Q_n$ we have $f^n(x) \in I_n$ for all n.*

This last result can be extended to a general setting. For example, Marotto, extending Li–Yorke’s approach to higher dimensions, used a version of Theorem 2.7 where $J = \mathbb{R}^N$ and $(I_n)_n$ is sequence of nonempty compact sets (see [31, Lemma 3.2]). The same situation has been considered by Kloeden in [23, Lemma 2], referring to Diamond [16, Lemma 1].

A more general version of Theorem 2.7 can be applied in order to prove the existence of arbitrary itineraries for a continuous map and then to obtain chaotic dynamics in the coin–tossing sense, according to [21]. More in detail, let X be a metric space and $f : X \to X$ be a continuous map. Let A_0, A_1 be two nonempty compact and disjoint sets. Following the terminology adopted in [19] we say that an *itinerary* in $\{A_0, A_1\}$ is a sequence of symbol sets $\mathcal{S} := (A_{s_0}, A_{s_1}, \dots, A_{s_n}, \dots)$ with $(s_n)_n \in \Sigma_2^+ := \{0,1\}^{\mathbb{N}}$. A point $x \in A_0 \cup A_1$ is said to *follow the itinerary* $\mathcal{S}$ if$f^n(x) \in A_{s_n}$ for all n. If, moreover, A_0 and A_1 satisfy condition

$$f(A_i) \supseteq f(A_j), \ \forall\, i, j \in \{0,1\}, \tag{3}$$

then it holds that all itineraries in $\{A_0, A_1\}$ are followed. The meaning of this result can be explained as follows: given any prescribed sequence of two symbols, for instance a sequence of *Heads* $= 1$ and *Tails* $= 0$, there is a forward orbit $(x_n)_n$ for f, i.e. $x_{n+1} = f(x_n)$, such that $x_n \in A_1$ or $x_n \in A_0$ according to the fact that $s_n =$ *Head* or $s_n =$ *Tail*. In other words, the deterministic map f is able to reproduce any outcome of a general coin flipping experiment (see [48]). We can derive many consequences from (3) which are relevant for the theory of chaotic dynamics, like the existence of a compact invariant set on which the map (or some of its iterates) is semiconjugate to the Bernoulli shift, or the positive topological entropy of f, or also the existence of ergodic invariant measures (see [5, 11, 28, 29, 44]).

In several applications, for instance when dealing with dynamical systems induced by the Poincaré map associated to a system of differential equations, it would be quite interesting to provide information also with respect to the existence of fixed points or periodic points. In this context, given a *periodic itinerary* $\mathcal{S}$, a natural question which arises is whether there exists a *periodic point* for f which follows it. In the one-dimensional setting and for A_0, A_1 compact intervals, a positive answer can be provided by applying Theorem 2.6 together with Theorem 2.7. Indeed, already in [30], periodic points of every period were found in the one-dimensional case. Extensions to higher dimensions of the covering relation, in the spirit of Li and Yorke paper, in order to provide the existence of infinitely many periodic points have been obtained by Marotto [31, 32], Kloeden [23] (see also [22]). The extension of Theorem 2.6 in the appropriate setting is made possible by assuming that the map is a homeomorphism restricted to suitable regions of its domain. In view of the above discussion, the question whether the assumption of local homeomorphism can be relaxed to the only hypothesis of continuity seems to be of some interest. As a first step in this direction, we shall focus our attention to the search of fixed points for *continuous* maps which satisfy expansivity conditions or covering relations of some kind.

2.3. Dolcher's fixed point theorem

In [17] Dolcher proposed a result of fixed points for planar maps which satisfy a covering relation. More precisely, denoting by $i(P, C)$ the topological index of a closed curve C with respect to a point $P \notin C$, the following fixed point theorem holds.

THEOREM 2.8 (Dolcher). *For $N = 2$, let $\phi : B_r \to \mathbb{R}^2$ be a continuous map such that the curve $\phi(\partial B_r)$ is external to the disc B_r and, moreover,*

$$i(P, \phi(\partial B_r)) \neq 0$$

for the points $P \in B_r$. Then, there exists $\tilde{x} \in B_r$ such that $\phi(\tilde{x}) = \tilde{x}$.

A more general version of Theorem 2.8 consists in assuming that the curve $\phi(\partial B_r)$ has no points in the interior of the disc B_r and that $i(P, \phi(\partial B_r)) \neq 0$ for the points $P \in \text{int}B_r$ (see [17, Teorema II]). Theorem 2.8, as well as its variant, is expressed in the planar setting, however, as already observed by the author in the introduction of [17], the result can be extended to any dimension. Indeed, using the Brouwer degree on the open ball

$$\Omega_r := \text{int}B_r,$$

we can state the following.

THEOREM 2.9. *Let $\phi : B_r \to \mathbb{R}^N$ be a continuous map such that $\phi(\partial B_r) \subseteq \mathbb{R}^N \setminus \Omega_r$ and suppose that $\deg(\phi, \Omega_r, 0) \neq 0$. Then, there exists $\tilde{x} \in B_r$ such that $\phi(\tilde{x}) = \tilde{x}$.*

Proof. If there exists $\tilde{x} \in \partial B_r$ such that $\phi(\tilde{x}) = \tilde{x}$, we have the result. Therefore, we can suppose that $\phi - Id$ never vanishes on $\partial\Omega_r$. Hence, if we define the homotopy $h_\lambda(x) := \phi(x) - \lambda x$, for $x \in B_r$ and $\lambda \in [0, 1]$, we easily find that $h_\lambda(x) \neq 0, \forall\, x \in \partial B_r$. In fact $||\phi(x)|| \geq r$ for all $x \in \partial B_r$, while $||\lambda x|| < r$ for all $x \in \partial B_r$ and $0 \leq \lambda < 1$. By the homotopic invariance of the topological degree, we have $\deg(\phi - Id, \Omega_r, 0) = \deg(\phi, \Omega_r, 0) \neq 0$ and thus we conclude that there exists $\tilde{x} \in \Omega_r$ such that $\phi(\tilde{x}) - \tilde{x} = 0$. Hence, in any case, there exists a fixed point for ϕ in B_r. □

Notice that from the assumption $\phi(\partial B_r) \subseteq \mathbb{R}^N \setminus \Omega_r$ it follows that

$$\deg(\phi, \Omega_r, 0) = \deg(\phi, \Omega_r, P), \quad \forall\, P \in \Omega_r\,.$$

Indeed, for every $P \in \Omega_r$ the homotopy

$$h_\lambda(x) := \phi(x) - \lambda P, \quad \lambda \in [0, 1]$$

is admissible (i.e., the sets of zeros of h_λ in B_r is contained in Ω_r). By the same argument we can also prove that the degree condition on ϕ implies

$$\phi(B_r) \supseteq B_r \tag{4}$$

and then we can say that Dolcher's theorem provides an example of a covering relation for continuous maps (not necessarily homeomorphisms) which is accompanied by the existence of fixed points. To check (4), let us consider an arbitrary point $P \in B_r$. If $P \in \Omega_r$, it holds that $\deg(\phi, \Omega_r, P) \neq 0$ and therefore $P \in \phi(\Omega_r)$. Hence, we can suppose $P \in \partial B_r$. If $P \in \phi(\partial B_r)$, we are done. Otherwise, the whole segment $[0, P] = \{\lambda P : 0 \le \lambda \le 1\}$ is disjoint from $\phi(\partial B_r)$ and we can conclude as before by the same homotopy h_λ.

2.4. Remarks on the nonexistence of fixed points

In this section we show, by means of some examples, that the hypotheses of Dolcher's theorem are sharp.

First of all, we observe that there is a clear asymmetry in the statements of Theorem 2.1 and Theorem 2.9 due to the additional degree condition in the latter result. One can provide simple cases of nonexistence of fixed points if the degree hypothesis is not satisfied while $\phi(\partial B_r) \cap \Omega_r = \emptyset$ holds. An example is already described in [36, Remark 3] (see also [2, Example 1] for a different context) and the function involved is any translation of the form $\phi : \mathbb{R}^N \to \mathbb{R}^N$, $\phi(x) := x + \vec{v}$, with $||\vec{v}|| \ge 2r$. In this situation it is evident that the degree condition fails. Moreover ϕ is a homeomorphism with $\phi(B_r) \cap \Omega_r = \emptyset$ and also the covering relation in Theorem 2.5 fails. So, it could be interesting to provide an example in which a covering relation as (4) is satisfied with zero degree.

A possible step in this direction can be described as follows. Let $\mathcal{A} := [-1, 1]^2$ be the unit square in $\mathbb{R}^2$ and let

$$\mathcal{B} := (\,[-6, 6] \times [-2, 10]\,) \setminus (\,]-2, 2[\, \times\,]2, 6[\,)$$

be an annular region containing $\mathcal{A}$ in its interior. We define a continuous map $\phi : \mathcal{A} \to \phi(\mathcal{A}) = \mathcal{B}$ by gluing together three homeomorphisms g_1, g_2, g_3 defined as follows. The (continuous) map g_1 is defined on the rectangle $\mathcal{A}_1 := [-1, -1/3] \times [-2, 2]$ and maps its domain homeomorphically onto the pluri-rectangle $\mathcal{B}_1 := (\,[0, 6] \times [-2, 2]\,) \cup (\,[2, 6] \times [-2, 10]\,)$. It is possible to define g_1 in such a way that the part of the boundary $\partial\mathcal{A} \cap \partial\mathcal{A}_1$ is transformed onto $\partial\mathcal{B}_1 \setminus (\{2\} \times]6, 10[\,)$. In a symmetric manner, we define the (continuous) map g_3 as a homeomorphism of the rectangle $\mathcal{A}_3 := [1/3, 1] \times [-2, 2]$ onto the pluri-rectangle $\mathcal{B}_3 := (\,[-6, 0] \times [-2, 2]\,) \cup (\,[-6, -2] \times [-2, 10]\,)$. Analogously, g_3 is such that the part of the boundary $\partial\mathcal{A} \cap \partial\mathcal{A}_3$ is transformed onto $\partial\mathcal{B}_3 \setminus (\{-2\} \times]6, 10[\,)$. Finally, the (continuous) map g_2 is defined on the rectangle $\mathcal{A}_2 := [-1/3, 1/3] \times [-2, 2]$ and maps its domain homeomorphically onto the square $\mathcal{B}_2 := [-2,2] \times [6,10]$. We define g_2 in such a way that the part of the boundary $\partial\mathcal{A} \cap \partial\mathcal{A}_2$ is transformed onto $\partial\mathcal{B} \cap \partial\mathcal{B}_2$. The geometric construction for the resulting piecewise homeomorphism is sketched in Figure 1.

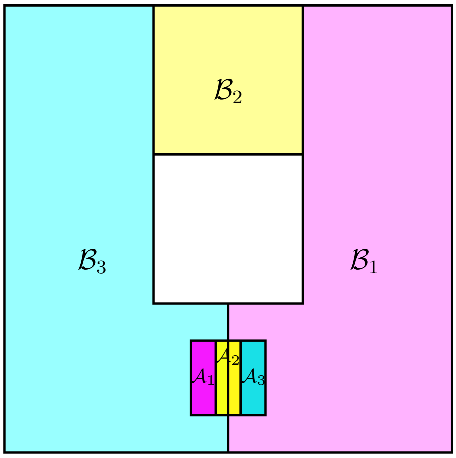

Figure 1: Action of the map ϕ defined on the square $\mathcal{A}$ onto the annular region $\mathcal{B}$.

If the g_i's are glued together correctly, a continuous function ϕ such that $\phi(\mathcal{A}) = \mathcal{B} \supseteq \mathcal{A}$ can be exhibited. Moreover $\phi(x) \neq x$, for each $x \in \mathcal{A}$, since $g_i(\mathcal{A}_i) \cap \mathcal{A}_i = \emptyset$ for $i \in \{1,2,3\}$. We also have $\deg(\phi, \mathrm{int}\mathcal{A}, 0) = 0$. Indeed, while a point P moves on the boundary of $\mathcal{A}$ in the counterclockwise sense, its image $\phi(P)$ makes a loop along $\partial B \cup \Gamma$, where $\Gamma := \{0\} \times [-2,2]$. In fact, the point $\phi(P)$ moves on the vertical segment twice in opposite directions.

The example above, even if it goes well with regard to the properties of covering, is not suitable in the context of Dolcher's theorem, because of the nonempty intersection $\phi(\partial\mathcal{A}) \cap \mathcal{A} = \Gamma$. Therefore, we provide a more elaborate construction of a continuous map which expands a smaller disk to a larger concentric one, it sends the boundary onto the boundary, but it has no fixed points. In this way, the following result holds.

Proposition 2.10. *For $N \geq 2$ and given $0 < r < R$, there exists a continuous map $\phi : B_r \to B_R$ satisfying*

$$\phi(B_r) = B_R\,, \quad \phi(\partial B_r) = \partial B_R \tag{5}$$

and without fixed points.

Proof. We start by proving the claim for $N = 2$. The trick of the proof is to find two rectangles

$$\mathcal{R} := [-a_0, a_0] \times [-b_0, b_0], \quad \mathcal{R}' := [-a_1, a_1] \times [-b_1, b_1], \tag{6}$$

with

$$0 < b_0 < a_0 \quad \text{and} \quad a_1 = ka_0 \,, b_1 = kb_0 \,, \quad \text{for some } k > 1 \tag{7}$$

and define a continuous map $f : \mathcal{R} \to \mathbb{R}^2$ (actually a piecewise homeomorphism) such that

- $f(\mathcal{R}) = \mathcal{R}'$,
- $f(p) \neq p, \ \forall\, p \in \mathcal{R}$,
- $f(\partial\mathcal{R}) \subseteq \mathbb{R}^2 \setminus \mathcal{R}$,
- $\{t\vec{v} : t > 0\} \cap f(\partial\mathcal{R}) \neq \emptyset, \ \forall\, \vec{v} \in S^1$.

The rectangle $\mathcal{R}$ is the closed unit disc for the norm

$$|||(x, y)||| = \max\{|x|/a_0, |y|/b_0\}$$

and, consequently, $\mathcal{R}' = \{z \in \mathbb{R}^2 : |||z||| \leq k\}$. Once we have introduced the map f, we can define $\phi_0 : B_1 \to B_k$ as

$$\phi_0(z) := h^{-1}(f(h(z))), \quad \forall\, z \in B_1 \,,$$

where $h : \mathbb{R}^2 \to \mathbb{R}^2$ is the homeomorphism such that

$$h(z) := \begin{cases} \frac{||z||}{|||z|||}\, z & \text{for } z \neq 0, \\ 0 & \text{for } z = 0. \end{cases}$$

The continuous function ϕ_0 maps B_1 onto B_k , with $\phi_0(\partial B_1) \subseteq \mathbb{R}^2 \setminus B_1$ and, moreover, any half-ray from the origin meets $\phi_0(\partial B_1)$. Now, we take $\delta \in]1, k[$ such that $\phi_0(\partial B_1) \subseteq \mathbb{R}^2 \setminus B_\delta$ and define

$$\phi_1 := P_\delta \circ \phi_0 \,,$$

where P_δ is the projection of $\mathbb{R}^2$ onto B_δ . In this manner, we have a fixed point free continuous map ϕ_1 of B_1 onto B_δ such that $\phi_1(\partial B_1) = \partial B_\delta$. The definition of the map $\phi : B_r \to B_R$ immediately follows from that of $\phi_1 : B_1 \to B_\delta$, by a suitable rescaling.

With this in mind, our problem reduces to the construction contained in the following example.

Example 2.11. We define f with respect to the rectangles $\mathcal{R}$ and $\mathcal{R}'$ defined as in (6)-(7) with

$$a_0 := 6, \quad b_0 := 2, \quad k := 2.$$

We evenly divide the rectangle $\mathcal{R}$ into six closed sub-rectangles $\mathcal{R}_1, \ldots, \mathcal{R}_6$, as described in Figure 2.

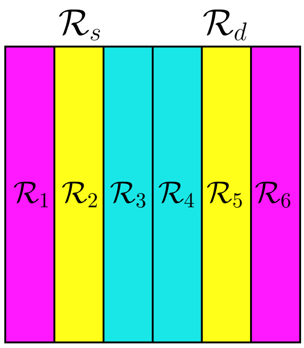

Figure 2: Subdivision of the rectangle $\mathcal{R}$. We have denoted by $\mathcal{R}_s := \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 = [-6, 0] \times [-2, 2]$ the left half of $\mathcal{R}$ and, analogously, by $\mathcal{R}_d$ its right hand half.

On each rectangle $\mathcal{R}_i$ we define a continuous map g_i, which is indeed a homeomorphism of $\mathcal{R}_i$ onto $g_i(\mathcal{R}_i) =: \mathcal{R}'_i$.

First of all, we introduce the map $g_s : \mathcal{R}_s \to \mathcal{R}'_1 := [-2, 12] \times [-4, 4]$ which is obtained by the gluing of the continuous maps g_1, g_2, g_3. In a symmetric manner, one defines a continuous map $g_d : \mathcal{R}_d \to \mathcal{R}'_6 := [-12, 2] \times [-4, 4]$ and, finally f is the result of the pasting lemma applied to g_s and g_d. At each step, we carefully avoid the presence of fixed points.

The map $g_1 : [-6, -4] \times [-2, 2] =: \mathcal{R}_1 \to g_1(\mathcal{R}_1) = \mathcal{R}'_1$ is defined as

$$g_1(x, y) := (-7x - 30, 2y).$$

The sides $\ell_1, \ell_2, \ell_3, \ell_4$ of $\mathcal{R}_1$ are mapped by g_1 onto the sides $\ell'_1, \ell'_2, \ell'_3, \ell'_4$ of $\mathcal{R}'_1$ as in Figure 3. By construction, $g_1(p) \neq p$, for each $p \in \mathcal{R}_1$.

The map g_2 transforms the rectangle $\mathcal{R}_2 := [-4, -2] \times [-2, 2]$ onto a Jordan domain $\mathcal{R}'_2$ which is bounded by a simple closed curve that we describe as follows.

We denote by $\ell_5, \ell_6, \ell_7, \ell_8$ the sides of $\mathcal{R}_2$ (traversed counterclockwise) and by $\ell'_i := g_2(\ell_i)$ their images (which are traversed in a counterclockwise manner, too). With this notation, first of all, we impose that

$$g_2(z) = g_1(z), \quad \forall\, z \in \ell_3 = \ell_5 = \mathcal{R}_1 \cap \mathcal{R}_2 .$$

In order to properly define g_2, it is important to notice that a point which moves in the up down direction along the segment ℓ_5 is transformed by g_1 to a point which moves in the same direction along the segment ℓ'_5.

Next, we take as ℓ'_7 the segment $\{11\} \times [5/2, 3]$ and, finally, we take as ℓ'_6 and ℓ'_8 two simple arcs contained in $\mathcal{R}'_1 \setminus \mathcal{R}$. A possible choice of ℓ'_6 is given by an arc of cubic with $(-2, -4)$ and $(11, 5/2)$ as endpoints and passing through $(0, 7/2)$

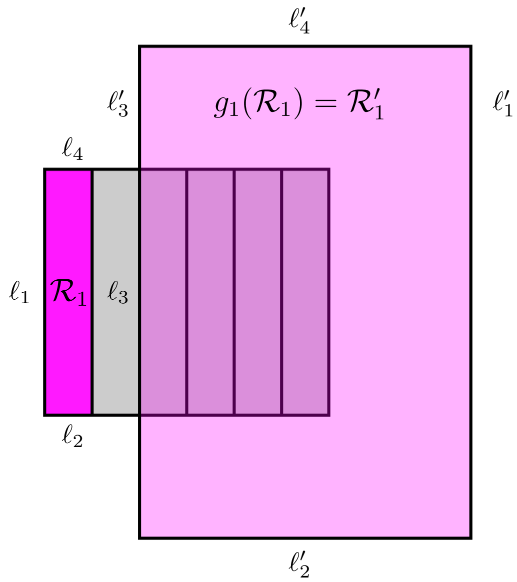

Figure 3: Transformation of the sub-rectangle $\mathcal{R}_1$ onto $\mathcal{R}_1'$ by g_1. The part of the boundary of $\mathcal{R}_1$ given by $\ell_4\ell_1\ell_2$ (traversed counterclockwise) is transformed into $\ell_4'\ell_1'\ell_2'$ (in the clockwise sense).

and $(13/2, -2)$. For ℓ_8' we have taken an arc of hyperbola with endpoints $(11, 3)$ and $(-2, 4)$ and passing through $(6, 7/2)$. The specific definition of g_2 is given by

$$g_2(x, y) := \left(\tfrac{13}{2}x + 24, \tfrac{2-y}{4} h_1\left(\tfrac{13}{2}x + 24\right) + \tfrac{2+y}{4} h_2\left(\tfrac{13}{2}x + 24\right)\right),$$

where

$$h_1(x) := -\frac{7}{2} + \frac{1685}{9724}x - \frac{265}{9724}x^2 + \frac{27}{4862}x^3,$$
$$h_2(x) := \frac{\sqrt{31678 - 859x - 29x^2}}{4\sqrt{130}}.$$

In Figure 4 we show $\mathcal{R}_2$ and its image $\mathcal{R}_2'$. The map g_2 is a homeomorphism. Moreover, $\mathcal{R}_2 \cap \mathcal{R}_2' = \ell_7$ and $g_2(\ell_7) = \ell_7' \cap \ell_7 = \emptyset$. Hence, also g_2 is fixed point free.

The map $g_3 : [-2, 0] \times [-2, 2] =: \mathcal{R}_3 \to g_3(\mathcal{R}_3) = \mathcal{R}_3' := [0, 11] \times [5/2, 3]$ is defined as

$$g_3(x, y) := \left(-\tfrac{11}{2}x, \tfrac{1}{8}y + \tfrac{11}{4}\right).$$

The sides ℓ_9, ℓ_{10}, ℓ_{11}, ℓ_{12} of $\mathcal{R}_3$ are mapped by g_3 onto the sides ℓ_9', ℓ_{10}', ℓ_{11}', ℓ_{12}'

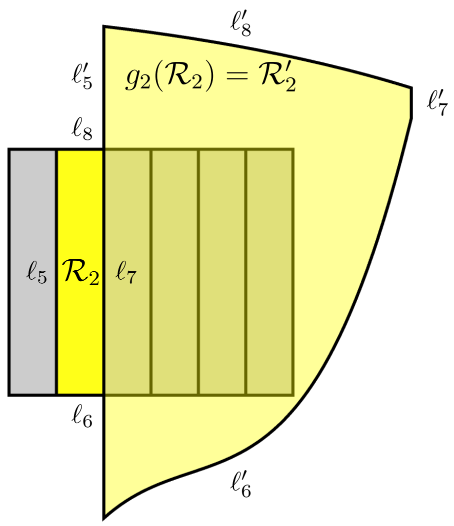

Figure 4: Transformation of the sub-rectangle $\mathcal{R}_2$ onto $\mathcal{R}_2'$ by g_2.

of $\mathcal{R}_3'$ as in Figure 5. By construction,

$$g_2(z) = g_3(z), \quad \forall\, z \in \ell_7 = \ell_9 = \mathcal{R}_2 \cap \mathcal{R}_3$$

and, moreover, $g_3(p) \neq p$, for each $p \in \mathcal{R}_3$.

Gluing together g_1, g_2, g_3 we obtain the map

$$g_s(x,y) := \begin{cases} g_1(x,y) & \text{for } -6 \leq x < -4, \\ g_2(x,y) & \text{for } -4 \leq x < -2, \\ g_3(x,y) & \text{for } -2 \leq x \leq 0 \end{cases}$$

which maps the rectangle $\mathcal{R}_s = [-6,0] \times [-2,2]$ onto $\mathcal{R}_1'$.

Using the symmetry $S_x : (x,y) \mapsto (-x,y)$, we can define the continuous map $g_d : \mathcal{R}_d \to \mathcal{R}_6'$ as

$$g_d(z) := S_x(g_s(S_x(z))), \quad \forall\, z \in \mathcal{R}_d = [0,6] \times [-2,2]$$

and then

$$f(z) := \begin{cases} g_s(z) & \text{for } z \in \mathcal{R}_s, \\ g_d(z) & \text{for } z \in \mathcal{R}_d. \end{cases}$$

See Figure 6 for a visualization of deformation of $\mathcal{R}$ through the final map.

At this point it is easy to check that f maps continuously $\mathcal{R}$ onto $\mathcal{R}'$, without fixed points and with $f(\partial\mathcal{R})$ a loop external to $\mathcal{R}$ which intersects any ray starting from the origin. $\lhd$

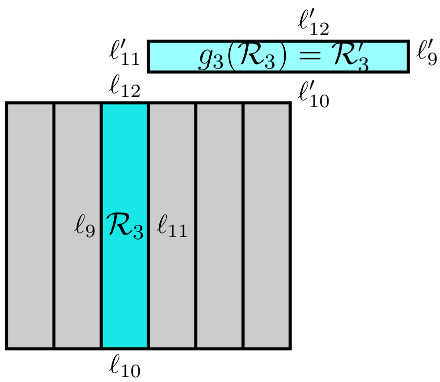

Figure 5: Transformation of the sub-rectangle $\mathcal{R}_3$ onto $\mathcal{R}_3'$ by g_3. The boundary of $\mathcal{R}_3$ given by $\ell_9\ell_{10}\ell_{11}\ell_{12}$ (traversed counterclockwise) is transformed into $\ell_9'\ell_{10}'\ell_{11}'\ell_{12}'$ (in the clockwise sense).

This example concludes the proof for $N = 2$ and now we consider an arbitrary dimension $N \geq 3$.

Applying Proposition 2.10 for $N = 2$ and the $||\cdot||_\infty$-norm, there exists a continuous and surjective planar map

$$\psi : [-r, r]^2 \to [-R, R]^2 \quad \text{with} \;\; \psi(x_1, x_2) = (\psi_1(x_1, x_2), \psi_2(x_1, x_2)),$$

without fixed points and such that $\psi(\partial[-r, r]^2) = \partial[-R, R]^2$. Next, we define $\Psi : [-r, r]^N \to [-R, R]^N$ by

$$\Psi(x_1, x_2, x_3, \ldots, x_N) := \left(\psi_1(x_1, x_2), \psi_2(x_1, x_2), \tfrac{R}{r}x_3, \ldots, \tfrac{R}{r}x_N\right).$$

Clearly, also Ψ is continuous, surjective, without fixed points and maps the boundary onto the boundary. Finally, we set

$$\phi(z) := h^{-1}(\Psi(h(z))), \quad \forall\, z \in B_r\,,$$

where $h : \mathbb{R}^N \to \mathbb{R}^N$ is the homeomorphism defined by

$$h(z) := \begin{cases} \frac{||z||}{||z||_\infty}\, z & \text{for } z \neq 0, \\ 0 & \text{for } z = 0. \end{cases}$$

This concludes our proof. □

Remark 2.12. *With the same approach, one can obtain a version of Proposition 2.10 in the case $r = R$, for a continuous map ϕ satisfying* (5) *and such*

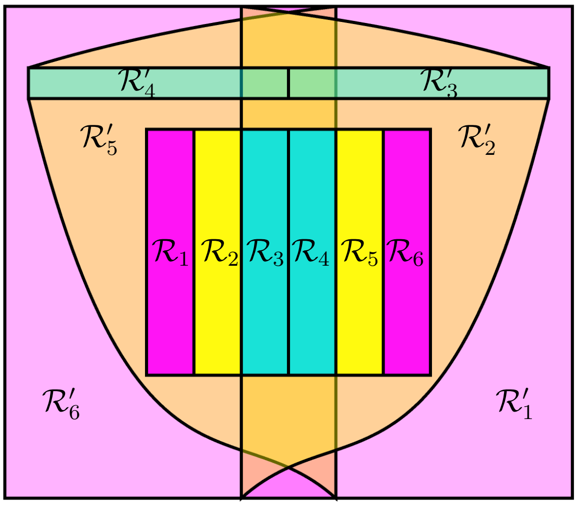

Figure 6: Action of the map f on $\mathcal{R}$.

that $\phi(z) \neq z$ for each $z \in \Omega_r$. With this respect, it may be interesting to recall a result by Brown and Greene [12, Theorem 1] where it is proved (for $N = 2$ and the Euclidean norm) that given a continuous map $f : \partial B_r \to \partial B_r$ with at least one fixed point, there exists a continuous map $\phi : B_r \to B_r$ which extends f and without fixed points on Ω_r . One can also see that if f is surjective, then ϕ is surjective, too.

In the two-dimensional case (which was the original setting in [17]) our construction can be used to provide a counterexample to Theorem 2.5 for maps which are only continuous when the second instance in (2) holds.

PROPOSITION 2.13. *Let $C \subseteq \mathbb{R}^2$ be a 2-dimensional cell. There exists a continuous map $\psi : C \to \psi(C) =: D$, where $D \subseteq \mathbb{R}^2$ is a 2-dimensional cell such that $C \subseteq \text{int}D$ and $\psi(\partial C) = \partial D$, with $\psi(z) \neq z$, $\forall\, z \in C$.*

Proof. Let $C \subseteq \mathbb{R}^2$ be a 2-dimensional cell and let $h : C \to h(C) = B_1$ be a homeomorphism, which exists by definition and is such that $h(\partial C) = \partial B_1$. Using the Schoenflies theorem (see [39]) we extend h to a homeomorphism $\tilde{h}$ of the whole plane. Let $\phi : B_1 \to B_2$ be a continuous map as in Proposition 2.10 for $r = 1$ and $R = 2$. Taking $\psi := \tilde{h}^{-1} \circ \phi \circ \tilde{h}$ and $D := \tilde{h}^{-1}(B_2)$, we achieve the thesis. □

3. Compressive/expansive maps

The search of fixed points for continuous maps which have, at the same time, a compressive and an expansive property is a field of research which has gradually attracted increasing interest in recent years. Besides that there be an intrinsic interest for this type of problems (see [18, 26, 27, 33]), motivations come from the study of Markov partitions and their generalizations (see [52]), as well as from the researches about topological horseshoes [13, 19, 20]. Another natural motivation comes from the search of periodic solutions for nonautonomous differential systems which are dissipative only with respect to some components of the phase space (see [2] and the references therein, as well as [1, 3, 25]).

The study of this type of maps leads to consider, as in [52], some generalized rectangles which are homeomorphic to the product of two closed balls whose dimensions correspond to the dimensions of the compressive and expansive directions, respectively. More formally, we consider two nonnegative integers $u = u(N)$ and $s = s(N)$ with $u + s = N$ and decompose the vector space $\mathbb{R}^N$ as $\mathbb{R}^u \times \mathbb{R}^s$, with the canonical projections

$$p_u : \mathbb{R}^N \to \mathbb{R}^u,\ p_u(x, y) = x,\ \ p_s : \mathbb{R}^N \to \mathbb{R}^s,\ p_s(x, y) = y$$

with $x \in \mathbb{R}^u$ and $y \in \mathbb{R}^s$. We denote by $||\cdot||_u$ and $||\cdot||_s$ two norms in $\mathbb{R}^u$ and $\mathbb{R}^s$, respectively, which are inherited by a given norm $||\cdot||$ in $\mathbb{R}^N$. In other words, we set $||x||_u := ||(x, 0)||$ and $||y||_s := ||(0, y)||$. In the sequel, we'll avoid to indicate the subscripts u, s when there is no possibility of misunderstanding. Then we introduce a compact set

$$\mathcal{R}[a, b] := B^u_a \times B^s_b = \{(x, y) \in \mathbb{R}^u \times \mathbb{R}^s\ :\ ||x||_u \le a,\ ||y||_s \le b\},$$

where $a, b > 0$ are fixed real numbers. Let $\phi : \mathcal{R}[a, b] \to \mathbb{R}^N$ be a continuous map. We want to think of the number u as a dimension for which the map is expansive and s as a dimension for which the map is compressive, in analogy to the case of the unstable and stable manifolds for the saddle points. For this reason, it is convenient to split ϕ as

$$\phi(x, y) = (\phi_u(x, y), \phi_s(x, y)), \quad \text{with}\ \ \phi_u : \mathcal{R}[a, b] \to \mathbb{R}^u,\ \phi_s : \mathcal{R}[a, b] \to \mathbb{R}^s,$$

defined as $\phi_u := p_u \circ \phi$ and $\phi_s := p_s \circ \phi$. At this point, a natural approach is to consider the conditions of the theorem of Brouwer (or the Rothe's one) in the compressive direction $s(N)$ and those of the theorem of Dolcher in the expansive direction $u(N)$. This leads to the following result previously considered in [36, Corollary 1], [41, Lemma 1.1], where we denote by $\Omega^u_a = \{x \in \mathbb{R}^u\ :\ ||x|| < a\}$ and $\Omega^s_b = \{y \in \mathbb{R}^s\ :\ ||y|| < b\}$ the interiors of the balls B^u_a and B^s_b, respectively.

Theorem 3.1. *Let $u \ge 1$ and let $\phi = (\phi_u, \phi_s) : \mathcal{R}[a, b] \to \mathbb{R}^N$ be a continuous map such that*

$$\phi_u(\partial B^u_a \times B^s_b) \subseteq \mathbb{R}^u \setminus \Omega^u_a\,, \quad \phi_s(B^u_a \times \partial B^s_b) \subseteq B^s_b\,. \tag{8}$$

Suppose moreover that $\deg(\phi_u(\cdot,0),\Omega^u_a,0)\neq 0$. *Then there exists* $\tilde z\in\mathcal{R}[a,b]$ *such that* $\phi(\tilde z)=\tilde z$.

Proof. We combine the proof of Theorem 2.9 with a classical degree argument for the Brouwer fixed point theorem. If there is already a fixed point on the boundary of $\mathcal{R}[a,b]$, we are done. Hence, we suppose that $z\neq\phi(z)$ for all $z\in\partial\mathcal{R}[a,b]$, so that the degree $\deg(Id-\phi,\Omega^u_a\times\Omega^s_b,0)$ is well defined. Next, we consider the homotopy

$$(z,\lambda)\mapsto h_\lambda(z):=(\lambda x-\phi_u(x,\lambda y),y-\lambda\phi_s(x,y)),$$

for $z=(x,y)$ and $\lambda\in[0,1]$. In order to prove that the homotopy is admissible on $\mathcal{R}[a,b]$ (that is $h_\lambda(z)\neq 0$ for each $z\in\partial\mathcal{R}[a,b]$ and $\lambda\in[0,1]$) it is sufficient to check that there are no solutions of the system

$$\begin{cases}\phi_u(x,\lambda y)=\lambda x\\ y=\lambda\phi_s(x,y)\end{cases}\qquad\forall\,\lambda\in[0,1[\ \text{ and }\ z=(x,y)\in\partial\mathcal{R}[a,b].$$

If $(x,y)\in(\partial\Omega^u_a)\times B^s_b$, then also $(x,\lambda y)\in(\partial\Omega^u_a)\times B^s_b$ and hence $||\phi_u(x,\lambda y)||\geq a>||\lambda x||$, so that the first equation in the system has no solutions. If $(x,y)\in B^u_a\times(\partial\Omega^s_b)$, then $||\lambda\phi_s(x,y)||<b=||y||$, so that the second equation in the system has no solutions. By the homotopic invariance of the topological degree we obtain

$$\begin{aligned}\deg(Id-\phi,\Omega^u_a\times\Omega^s_b,0) &= \deg((-\phi_u(\cdot,0),Id|_{\mathbb{R}^s}),\Omega^u_a\times\Omega^s_b,0)\\ &= (-1)^u\deg(\phi_u(\cdot,0),\Omega^u_a,0)\neq 0\end{aligned}$$

and thus we conclude that there exists a $\tilde z\in\Omega^u_a\times\Omega^s_b=\operatorname{int}(\mathcal{R}[a,b])$ such that $\tilde z-\phi(\tilde z)=0$. Hence, in any case, there exists a fixed point for ϕ in $\mathcal{R}[a,b]$. □

We have considered the case $u=u(N)\geq 1$, since for $u=0$ the result reduces to Rothe fixed point theorem. On the other hand, Theorem 3.1 reduces to Theorem 2.9 for $u=N$. We refer to [52] and [41] for variants of Theorem 3.1 and we also recommend [36] for recent extensions as well as connections with Poincaré–Miranda theorem.

It may be interesting to investigate whether the assumptions of Theorem 2.9 imply a covering relation analogous to (4) for the expansive component. In fact, we prove the following.

PROPOSITION 3.2. *In the setting of Theorem 2.9 the inclusion*

$$\phi_u(B^u_a\times\{y\})\supseteq B^u_a\,,\quad\forall\,y\in B^s_b \tag{9}$$

holds.

Proof. Let $y \in B^s_b$ be fixed and let $P \in B^u_a$. We claim that there exists $x \in B^u_a$ such that the equation $\phi_u(x,y) = P$ has a solution. If there exists $\tilde{x} \in \partial B^u_a$ with $\phi_u(\tilde{x},y) = P$, we are done. Otherwise, $\phi_u(x,y) \neq P$, $\forall\, x \in \partial B^u_a$ and thus the degree $\deg(\phi_u(\cdot,y),\Omega^u_a,0)$ is defined. For $\lambda \in [0,1[$ and $x \in \partial B^u_a$ we have (by the second condition in (8)) $||\phi_u(x,\lambda y)|| \geq a > \lambda a \geq ||\lambda P||$. Hence the homotopy $h_\lambda(x) := \phi_u(x,\lambda y) - \lambda P$, for $\lambda \in [0,1]$ and $x \in B^u_a$, is admissible. Therefore,

$$\begin{aligned}&\deg(\phi_u(\cdot,y) - P,\Omega^u_a,0) = \deg(h_1,\Omega^u_a,0)\\ &= \deg(h_0,\Omega^u_a,0) = \deg(\phi_u(\cdot,0),\Omega^u_a,0) \neq 0\end{aligned}$$

and thus the equation $\phi_u(x,y) = P$ has a solution with $x \in \Omega^u_a$. In any case, P is the image through $\phi_u(\cdot,y)$ of some point in B^u_a. □

The examples in Section 2.4 can be easily adapted to the context of Theorem 2.9. In particular, one can provide examples of continuous maps satisfying (8) and (9), but without fixed points in $\mathcal{R}[a,b]$.

An application of Theorem 3.1 to ordinary differential equations can be described as follows.

Let $F : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a continuous vector field such that, for some $T > 0$,

$$F(t+T,w) = F(t,w), \quad \forall\, t \in \mathbb{R},\ \forall\, w \in \mathbb{R}^N.$$

Let $\mathcal{D} \subseteq \mathbb{R}^N$ be a nonempty set such that for each $w \in \mathcal{D}$ there is a unique solution of the Cauchy problem

$$\begin{cases}\zeta' = F(t,\zeta)\\ \zeta(0) = w\end{cases}$$

which is defined on $[0,T]$. We denote such a solution by $\zeta(\cdot,w)$. The fundamental theory of ODEs ensures that the Poincaré map

$$\Psi : \mathcal{D} \to \mathbb{R}^N, \quad \Psi(w) := \zeta(T,w)$$

is continuous, actually a homeomorphism of $\mathcal{D}$ onto $\Psi(\mathcal{D})$. The existence of a T-periodic solution $\zeta(t)$ for system

$$\zeta' = F(t,\zeta), \tag{10}$$

with $\zeta(0) \in \mathcal{D}$ is equivalent to the existence of a fixed point (in $\mathcal{D}$) for the map Ψ. In this context, every time we are in the presence of a flow induced by (10) which possesses an expansive property on the first j $(1 \leq j \leq N)$ components and it has a compressive property on the remaining $k := N - j$ components, we can enter into the scheme of Theorem 3.1 with $u = u(N) = j$ and $s = s(N) = k$.

In particular, the condition on the degree may be replaced by a more simple one to control, by adequately exploiting the properties of Poincaré map. In this manner, we obtain the following result where, for notational convenience, we represent a solution $\zeta(t)$ of (10) as

$$\zeta(t) = (E(t), C(t)),$$

with

$$E(t) := (\zeta_1(t), \dots, \zeta_j(t)) \in \mathbb{R}^j, \ C(t) := (\zeta_{j+1}(t), \dots, \zeta_N(t)) \in \mathbb{R}^k.$$

Similarly, we write $\zeta(t, w)$ as $(E(t, w), C(t, w))$, with reference to an initial value problem with $\zeta(0) = w$.

Corollary 3.3. *Suppose there exists two positive real numbers a, b such that $\mathcal{R}[a, b] \subseteq \mathcal{D}$ and, moreover, for each $w = (x, y) \in \mathcal{R}[a, b]$, we have:*

(i_1) $\quad ||x|| = a, ||y|| \leq b \implies ||E(T, w)|| \geq a,$

(i_2) $\quad ||x|| \leq a, ||y|| = b \implies ||C(T, w)|| \leq b,$

(i_3) $\quad ||x|| = a, y = 0 \implies ||E(t, w)|| > 0, \quad \forall\, t \in]0, T[\,.$

Then system (10) *has a T-periodic solution $\zeta(t)$ with $\zeta(0) \in \mathcal{R}[a, b]$.*

Proof. As previously observed, we consider the case $u = j$, $s = N - j$ and we apply Theorem 3.1 to the map $\phi := \Psi$, by the obvious decomposition $\phi_u(w) := E(T, w)$, $\phi_s(w) := C(T, w)$, for $w = (x, y)$. With this notation, it is immediate to check that the two assumptions in (8) follow from (i_2) and (i_1), respectively. Condition (i_3) implies $E(\lambda T, (x, 0)) \neq 0$ for every $x \in \partial\Omega^u_a$, for all $\lambda \in]0, 1[\,$. Moreover, $E(\lambda T, (x, 0)) \neq 0$ for $\lambda = 1$ (by (i_1)). For $x \in \partial\Omega^u_a$, also $E(\lambda T, (x, 0)) \neq 0$ for $\lambda = 0$. In fact, $\zeta(0, (x, 0)) = (x, 0)$ and hence $E(0, (x, 0)) = x$. We have thus verified that the homotopy $(x, \lambda) \mapsto h_\lambda(x) := E(\lambda T, (x, 0))$, is admissible on B^u_a . Hence

$$\deg(\phi_u(\cdot, 0), \Omega^u_a, 0) = \deg(h_1, \Omega^u_a, 0) = \deg(h_0, \Omega^u_a, 0) = \deg(Id|_{\mathbb{R}^j}, \Omega^u_a, 0) = 1$$

and so the degree condition in Theorem 3.1 is satisfied. We conclude that there exists a fixed point for Ψ in $\mathcal{R}[a, b]$ that is a T-periodic solution $\zeta(t)$ of (10) with $\zeta(0) \in \mathcal{R}[a, b]$. $\square$

Remark 3.4. *Corollary 3.3 is contained (in a slightly different form) in [2], which, in turn, is based on the continuation theorems developed in [14]. The assumptions $(i_1), (i_2), (i_3)$ are meaningful independently from the fact that they refer to the components of a Poincaré map. From this point of view, an abstract version of this result concerning the search of fixed points for multivalued maps depending on a parameter $t \in [0, T]$ has been obtained in [3, Theorem 3] and [1, Lemma 3.3, Theorem 3.4]. Applications to differential inclusions have been proposed in [1, 3], as well.*

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