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# Computation of Poincaré-Betti Series for Monomial Rings

Mikael Johansson (*)

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Summary. - *The multigraded Poincaré-Betti series $P_R^k(\bar{x};t)$ of a monomial ring $k[\bar{x}]/\langle M\rangle$ on a finite number of monomial generators has the form $\prod_{x_i\in\bar{x}}(1+x_it)/b_{R,k}(\bar{x};t)$, where $b_{R,k}(\bar{x};t)$ is a polynomial depending only on the monomial set $M$ and the characteristic of the field $k$. I present a computer program designed to calculate the polynomial $b_{R,k}$ for a given field characteristic and a given set of monomial generators.*

## 1. Introduction

Let $Q = k[\bar{x}] = k[x_1,\dots,x_r]$ be the polynomial ring over a field $k$ with $r$ variables. The ring has a natural $\mathbb{N}^r$-grading[1] by setting $\deg(x_i) = e_i$ for the canonical basis vectors $e_i$ of $\mathbb{N}^r$. Setting $|(a_1,\dots,a_r)| = a_1+\cdots+a_r$, we can derive an $\mathbb{N}$-grading of $Q$ from this $\mathbb{N}^r$-grading. We write $\deg(m)$ for the $\mathbb{N}^r$-degree of a monomial $m$, and given $(a_1,\dots,a_r) = \alpha \in \mathbb{N}^r$, we write $x^\alpha = x_1^{a_1}x_2^{a_2}\dots x_r^{a_r}$

(*) Author's address: Mikael Johansson, Friedrich-Schiller-Univesität Jena, Facultät für Mathematik & Informatik, Matematisches Insitut, D-07737 Jena, e-mail: mik@math.su.se
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[1] Throughout this paper, $0 \in \mathbb{N}$.

The $\mathbb{N}^r$-grading and its inherent $\mathbb{N}$-grading both are inherited from $Q$ to the ring $R = Q/I$ where $I$ is a *monomial ideal*, i.e. an ideal generated by monomials in $Q$. We call $R$ a *monomial ring*.

The $\mathbb{N}^r$-grading inherits, via minimal resolutions respecting to the grading, to an $\mathbb{N}^r$-grading on $\operatorname{Tor}^R(k,k)$. Thus, we can define the multigraded Poincaré-Betti series $P_k^R(\bar{x};t)$ of an $\mathbb{N}^r$-graded ring $R$ over $k$:

$$\sum_{i\in\mathbb{N}}\sum_{\alpha\in\mathbb{N}^r}\dim_k \operatorname{Tor}_i^R(k,k)_\alpha x^\alpha t^i$$

From $P_k^R(\bar{x};t)$, the simple Poincaré-Betti series can be calculated as $P_k^R(t) = P_k^R(1,\dots,1;t)$.

Since [3], it is known that the Poincaré-Betti series of a monomial ring $R$ holds the form $P_k^R(t) = \frac{(1+t)^n}{b_{R,k}(t)}$ for some polynomial $b_{R,k}(t)$.

Moreover, Alexander Berglund proved in [4] that $\deg b_{R,k}(t) < 2n$ for monomial rings with $n$ monomial generators. It follows that there are finitely many Poincaré-Betti serie occurring for a fixed $n$ at all.

We may define a partially ordered graph, or po-graph, to be a graph with a partial order on the vertices. Two po-graphs are said to be isomorphic if there is a simultaneous isomorphism of the graph and the partial order.

Let $Q = k[\bar{x}]$ and $Q' = k[\bar{x}']$ be polynomial rings for two finite variable sets $\bar{x} = \{x_1,\dots,x_r\}$ and $\bar{x}' = \{x_1,\dots,x_{r'}\}$, and let $I$ and $I'$ be monomial ideals in $Q$ and $Q'$ respectively, with $M$ and $M'$ the sets of generators for each ideal. For some set $S$ of monomials, we denote by $L_S$ the set of all least common multiples of subsets of $S$. $L_S$ can be equipped with the structure of a po-graph, ordering monomials by divisibility and adding an edge between two elements when they have a non-trivial common factor.

Luchezar Avramov shows in [2] that if two rings $R = Q/I$ and $R = Q'/I'$, with $I$ and $I'$ generated by the monomial sets $M$ and $M'$ respectively are such that $L_M \cong L_{M'}$, then $b_{R,k}(t) = b_{R',k}(t)$. From this follows that for a fixed field and a fixed number of monomial generators, only finitely many different Poincaré-Betti series can occur. Avramov further proves that the limitation to a fixed field is superfluous.

In [4], Alexander Berglund, proving the conjecture by Avramov that $\deg(b_{R,k}(t)) \leq 2n$ whenever the monomial ideal generating the

ring $R$ has $n$ generator, constructs combinatorially a minimal model for $R$ and gives a characterisation of the Poincaré-Betti series denominator polynomial in terms of the homology of associated simplicial complexes. Avramov's earlier observation that only finitely many different Poincaré-Betti series exist for a fixed number of generators for the monomial ideal $I$ follows as an immediate consequence of Berglund's construction.

I have in the course of my M.Sc. thesis work [7] continued Berglund's work by implementing his formula in a program, written in C++, capable of calculating simplicial homology over fields of arbritrary characteristic, as well as explicitly calculating $b_{R,k}(t)$ for arbritrary characteristic of the coefficient field $k$.

### 1.1. Simplicial Complexes

Since we will work a lot with simplicial homology, I will take a few moments to review definitions and terminology. A simplicial complex on a set $V$ is a set $\Delta$ of subsets of $V$ such that if $G \in \Delta$ and $F \subset G$, then $F \in \Delta$. $V$ is called the *vertex set* of $\Delta$. All simplicial complexes I shall refer to will have $V = \bigcup \Delta$ unless otherwise stated. The *i-faces* or *i-simplices* of $\Delta$ are precisely the elements in $\Delta$ of cardinality $i + 1$.

To a simplicial complex $\Delta$ we can associate an augmented chain complex $\widetilde{C}(\Delta)$ with $\widetilde{C}_i(\Delta)$ the free abelian group on the $i$-faces of $\Delta$, where we consider $\emptyset$ to be the unique $-1$-simplex. We equip $\widetilde{C}(\Delta)$ with the standard differential of degree $-1$. Thus

$$H_i(\widetilde{C}(\Delta)) = \widetilde{H}_i(\Delta)$$

As usual, for an abelian group $G$, we set $\widetilde{C}(\Delta; G) = \widetilde{C}(\Delta) \otimes_{\mathbb{Z}} G$ and $\widetilde{H}_i(\Delta; G) = H_i(\widetilde{C}(\Delta; G))$.

For a simplicial complex $\Delta$ we define the Alexander dual

$$\Delta^\vee = \{F \subseteq V \mid V \setminus F \notin \Delta\}$$

For simplicial complexes $\Delta$ and $\Delta'$ with disjoint vertex sets we define the join

$$\Delta * \Delta' = \{F \cup F' \mid F \in \Delta, F' \in \Delta'\}$$

and the dual join

$$\Delta \cdot \Delta' = (\Delta^\vee * \Delta'^\vee)^\vee$$

and note that then

$$\Delta \cdot \Delta' = \{F \in V \cup V' \mid F \cap V \in \Delta \text{ or } F \cap V' \in \Delta'\}$$

In [5, Lemma 5.5.3], it's shown that

$$\widetilde{H}_i(\Delta; k) \cong \widetilde{H}_{n-3-i}(\Delta^\vee; k)$$

for a complex $\Delta$ on $n$ vertices.

For a graded vector space $H = \bigoplus_{i \in \mathbb{N}} H_i$, we will write $H(t)$ for the *generating function* $\sum_{i \in \mathbb{N}} \dim_k H_i t^i$ of $H$. We then can find that the join of complexes as well as the Alexander dual gives rise to rather easily handled equalities on the level of generating functions for their respective homologies, again with $\Delta$ a complex on $n$ vertices, and $\Delta'$ some other complex.

$$t^n \widetilde{H}(\Delta^\vee; k)(t^{-1}) = t^3 \widetilde{H}(\Delta; k)(t) \tag{1.1}$$

$$\widetilde{H}(\Delta * \Delta'; k)(t) = t\widetilde{H}(\Delta; k)(t) \cdot \widetilde{H}(\Delta'; k)(t) \tag{1.2}$$

where the $t$ factor in the latter equation comes from the fact that a simplex with $d$ elements is considered to have dimension $d - 1$.

## 2. Berglund's work

In [4], Berglund treats the theoretical aspects of computation of the Poincaré-Betti series denominator for monomial rings. I will not repeat all of his proof here, but rather reference his work to establish the vocabulary and touch the results I will need for my own work. $Q = k[\bar{x}]$ is $\mathbb{N}^r$-graded by assigning to a monomial $x_1^{a_1} \dots x_r^{a_r}$ the element $\alpha = (a_1, \dots, a_r)$. We write $x^\alpha$ for $x_1^{a_1} \dots x_r^{a_r}$. The monomial $x^\alpha$ is said to be squarefree if $\alpha \in \{0, 1\}^r$.

By a construction by Weyman and Fröberg [8, 6], it is enough to treat squarefree monomial sets, since an easy procedure can be used to go from a monomial ring to a squarefree monomial ring with the same homological properties. So we can assume that $R = Q/I$

is a squarefree monomial ring with $I$ generated by the monomial antichain[2] $M$ of cardinality $n$.

For a set $S$ of monomials, set $m_S = \operatorname{lcm}(m : m \in S)$. In particular $m_\emptyset = 1$. Thus $L_M = \{m_S \mid S \subset M\}$. For a specific monomial $m$ and a monomial set $M$ set $M_m = \{m' \in M \mid m'|m\}$.

Now, for $I = \langle M \rangle$ a monomial ideal in $Q$ generated minimally by the antichain $M$, Berglund introduces the complex

$$\Delta_M = \{S \subseteq M \mid m_S \neq m_M \text{ or } S \text{ disconnected}\}$$

where connectivity is for $S$ as a subgraph of $L_M$.

Using multigraded ring-deviations

$$\epsilon_{i,\alpha} = \dim_k \widetilde{H}_{i-3}(\Delta_{M_{x^\alpha}}; k)$$

Berglund gives the squarefree part of the multigraded Poincaré-Betti polynomial

$$b_{R,k}(\bar{x}, t) \equiv \prod_{x\alpha \in L_M} (1 - x^\alpha p_\alpha(t)) \pmod{\langle x_1^2, \ldots, x_r^2 \rangle} \tag{2.1}$$

with $p_\alpha(t) = t^3 \widetilde{H}(\Delta_{M_{x^\alpha}}; k)(t)$. Backelin demonstrated already in [3] that the denominator polynomial will be squarefree whenever the monomials generating the ideal all are.

Berglund then goes on to find several more theoretically pliable forms of this particular formula; expanding the product and taking the irrelevance of non-squarefree terms into account, he arrives at the form

$$b_{R,k}(\bar{x}; t) = 1 + \sum_S m_S (-t)^{c(S)+2} \widetilde{H}(\Delta_S; k)(t) \tag{2.2}$$

where the sum is taken over all non-empty *saturated* subsets of $M$ and $c(S)$ counts the number of graph components of $S$ as a subgraph to $L_M$. We define the *saturation* of a subset $S \subset M$ as the set of all monomials in $M$ that divide the least common multiple of some connected component of $S$ as subgraph to $L_M$. A set is saturated if it is equal to its saturation.

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[2]Recall that in a partially ordered set $P$, an antichain $A$ is a subset such that all elements of $A$ are mutually incomparable. The monomials are partially ordered by divisibility.

## 3. The resulting application – `poincare`

My own achievement is that I have constructed a computer program to calculate simplical complex homology and mainly to calculate the denominator polynomials of Poincaré-Betti series using Berglunds methods. I will devote this section to a discussion of the program, which can be fetched in its latest version under the MIT software license from `http://www.math.su.se/~mik/poincare/`.

The form deemed most promising for implementation as I started was the form given in (2.1) – mainly since the formulation in terms of saturated subsets had at that time not yet matured. Thus, I have implemented specific C++ classes for calculating in the ring $Q/\langle x_1^2, x_2^2, \ldots, x_r^2\rangle$ and let the final product forming the polynomial take place in that particular ring. The only part forming any kind of complexity for the straightforward implementation is that of forming the complex $\Delta_{M_{x^\alpha}}$ and calculating its homology over the specified characteristic.

The construction of $\Delta_{M_{x^\alpha}}$ is done with a modified kind of breadth -first search: monomials are stored in a queue along with an index keeping track of which of the monomials covering the particular monomial that have already been tried. Thus, for each monomial in the queue, all later covering monomials are tried one after the other, and upon compliance with the two conditions – that the least common multiple of all generating monomials dividing the candidate is equal to $x^\alpha$ and that those generating monomials are connected as a graph – the monomial is added to the queue carrying a testing index one higher than the index that produced it. This algorithm does yield a speed increase compared to the earlier algorithm that simply tested all monomials for both conditions; but still is not optimal by far.

Once the simplicial complex as such has been constructed, the calculation of its homology commences. This is calculated degree by degree, constructing a matrix with entries in $\{0, \pm 1\}$ and fetching its rank from the matrix routines in the Pari library [1].

This is wrapped in a text-mode user interface, using the GNU Readline library to facilitate command history and command editing. The user interface reads in space-separated lists of monomials as

input to the `add simplex` and `add monomial` commands. A monomial, to the program, is a `*`-separated list of strings of characters, where each separated string is taken to be the name of a variable. The variable names must avoid `+-*/^,.` and whitespace, but can use any other characters. Any string occuring in such a position will be interpreted as a variable and added to an internal dynamic variable pool.

The user interface wraps, among other things, around the Weyman-Fröberg method for conversion to a squarefree monomial ring. The conversion is done transparently, using several internal variables that are easily converted back to the original variables before printing the answer. The output rendered by the program is written in such a way that other computer algebra systems should have an easy time handling it.

As an example on how the program works, I give in example 3.1 a session, calculating first the simplicial homology of the projective plane over $\mathbb{Q}$ as well as over $\mathbb{Z}_2$ and then calculating the Poincaré-Betti denominator polynomial of $\mathbb{Q}[x,y,z]/\langle x^2, xy, yz\rangle$.

EXAMPLE 3.1.

```
Welcome to the Poincaré calculator.
You can use this program to calculate simplicial
homology over prime fields and to calculate the
denominator polynomial of the Poincaré-Betti series
of monomial rings.
(c) 2004 Mikael Johansson
This program is released under the MIT License
> add simplex a*b*e a*b*f a*c*d a*c*f a*d*e
> add simplex b*c*d b*c*e b*d*f c*e*f d*e*f
> homology
Calculating homology ranks...
***** Hilbert series of simplicial homology *****
0
> char 2
New characteristic:  2
> homology
Calculating homology ranks...
***** Hilbert series of simplicial homology *****
ZZ + ZZ^2
> add monomial x^2 x*y y*z
```

```
> char 0
New characteristic:  0
> denom
1 - x^2*ZZ^2 - x*y*ZZ^2 - y*z*ZZ^2 - x^2*y*ZZ^3 - x*y*z*ZZ^3
> set multigrade false
> denom
1 - 3*ZZ^2 - 2*ZZ^3
> quit
Thanks for visiting.
   Calculation with poincare
```

Among the things we may observe in example 3.1 is the basic set of commands – `add simplex` and `add monomial` to build simplicial complexes or monomial ideals, `homology` and `denominator` (or an abbreviation `denom` thereof) to calculate simplicial homology and the Poincaré-Betti denominator respectively, as well as the command `char`, which changes the field characteristic over which all homology calculations take place and the command sequence `set multigrade false`, which sets a flag that causes the program to change the way it prints the polynomials output by the `denominator` command, so that instead of the polynomial $b_{R,k}(\bar{x}, t)$ the program prints the polynomial $b_{R,k}(1, \ldots, 1, t)$. The program ends upon receiving `quit`.

In addition to these, there are the commands `clear`: clearing the stored simplicial complex and monomial ideal, but not changing the characteristics used and `var`: which changes the implicit homology variable, which in my review in this paper has been called $t$, and which by default in `poincare` is called `ZZ`. Should you wish to use `ZZ` as a ring variable in your calculations, a change of homology variable will be necessary. For this, the command `var` is provided, with which you can change the string that `poincare` uses for the homology variable.

## 4. Questions and future directions

There are several things that I want to improve upon on the system herein presented, and also several questions that can be posed.

There are numerous complexity issues associated to the program in its current form. Mainly, these issues are related to the size of

the resulting po-graphs for larger sets of monomials. As an example, the initial ideal of a Gröbner basis with revlex ordering of the homogenized cyclic 6-root ideal, i.e. the ideal generated by

$$
\begin{aligned}
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6,\\
&x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_1x_6,\\
&x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_6 + x_1x_5x_6 + x_1x_2x_6,\\
&x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + x_1x_4x_5x_6 + x_1x_2x_5x_6 +\\
&\quad + x_1x_2x_3x_6,\\
&x_1x_2x_3x_4x_5 + x_2x_3x_4x_5x_6 + x_1x_3x_4x_5x_6 +\\
&\quad + x_1x_2x_4x_5x_6 + x_1x_2x_3x_5x_6 + x_1x_2x_3x_4x_6,\\
&x_1x_2x_3x_4x_5x_6 - y^6
\end{aligned}
$$

invariably becomes larger than Pari's working memory. This initial ideal has 100 monomial generators and produces a po-graph with 11443 elements. The calculations normally halt after between 200 and 400 lattice point calculations.

The problem I have observed with for instance this example is that the calculation of homology of large simplicial complexes is memorywise unfeasible. The most visible problem is when the homology calculations turn out to be too hard, since this results in a crash in the Pari library; whereas too hard construction of the simplicial complexes merely result in slow running of the program.

QUESTION 4.1. *Given the rather special structure of the matrices that are used to calculate field homology of a simplicial complex, can anything be said about the sizes of elements of a matrix as some sort of reduction algorithm is used to deduce its rank? Can some variant of row-reduction be found such that the matrices that produce "large" entries (for instance, larger than a standard 32-bit word) have some easily recognizable feature?*

If such a method could be found, then it would be possible to single out the specific homology matrices that would actually need treatment with some sort of bignum library, and produce fast 32-bit (sparse) matrix arithmetic to deal with all other instances. Such a separate treatment would also obliterate the need to rely on Pari's

internal memory allocation structures and would give the user more control over acceptable memory consumption for the calculations at hand.

An obvious further point of attack is the generation algorithm for the Berglund complexes. This would be vastly improved if a search algorithm would be constructed that minimizes the number of multiple checks done on each candidate monomial; since the graph connectivity checks are not, in the context, very fast.

Finally, an interesting direction to take would be to look at the APIs for larger computer algebra systems and try to adapt the code here written to work as a pluggable module to those systems; for instance providing an interface to calculate Poincaré-Betti series of monomial rings directly from Singular or Magma or Macaulay 2.

## References

[1] *Pari/gp, version* `2.2.7`$\alpha$, Tech. report, The PARI Group, Bordeaux, 2004, available from `http://pari.math.u-bordeaux.fr/`.

[2] L. L. Avramov, *Homotopy Lie algebras and Poincaré series of algebras with monomial relations*, Homology Homotopy Appl. **4** (2002), no. 2, 17–27.

[3] J. Backelin, *Les anneaux locaux à relations monomiales ont des séries de Poincaré-Betti rationelles*, C. R. Acad. Science Paris **295** (1982), 607–610.

[4] A. Berglund, *Poincaré series of monomial rings*, arXiv:math.AC 0412282, 2004.

[5] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, 1993, revised edition.

[6] R. Fröberg, *A study of graded extremal rings and of monomial rings*, Math. Scand. **51** (1982), 22–34.

[7] M. Johansson, *Computation of denominator polynomials for Poincaré series on monomial rings*, Master's thesis, Department of Mathematics, Stockholm University, SE-10691 Stockholm, Sweden, 2004.

[8] J. Weyman, *Ideały generowane przez jednomiany*, Master's thesis, Nicholas Copernicus University, Torun, Poland, 1977.

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