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A study of discrete and integral transforms with 3 logarithmic separable kernels

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A study of discrete and integral transforms with logarithmic separable kernels

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Declaration

I, Thomas Futcher, declare that this thesis submitted in partial fulfilment of the requirements for the conferral of the degree Doctor of Philosophy (Mathematics), from the University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. This document has not been submitted for qualifications at any other academic institution.

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Abstract

In this thesis, we will be examining different classes of discrete and integral transforms. We start with a general class of integral transforms which include those logarithmic separable kernels. Transforms with logarithmic separable kernels include the Fourier transform, the Laplace transform and the Mellin transform. The shifting and convolution properties for this class of transforms are examined, and sufficient conditions which guarantee the existence of the convolution formula will be given. It will be shown that a subclass of these integral operators are injective and an inversion formula will be presented on some class of continuously differentiable functions. We will apply these results to second-order differential equations to obtain new analytical solutions to these equations and compare these to a numerical solution.

A class of discrete and integral transforms which are a subclass of those with logarithmic separable kernels will also be analysed. We will examine a weighted L^1 space which is related to our class of transforms. An appropriate codomain for our class of transforms as well as the continuity of our class of transforms will be determined. The shifting and convolution properties will be examined, with a focus on discrete transforms, and we will show the convolution operation is a binary relation on the appropriate weighted L^1 space. We will state conditions which makes the discrete operators in this class injective, then we will show our weighted L^1 space is a Banach algebra. It will be shown that these weighted L^1 space corresponding to integral operators do not have a unit. It will also be highlighted through an example that a weighted L^1 space corresponding to a discrete operator in out class may have a unit.

Further properties of the convolution formula will be determined. We will provide conditions on the support of the convolution of two functions as well as show the convolution of two functions is continuous given certain integrability conditions. The convolution formula is then extended to distributions, and certain differentiability conditions on the convolution of a distribution and a function will be obtained. It will be shown that the weighted L^1 spaces which correspond to integral operators can be embedded into a unital Banach algebra, namely the complex Borel measures. We show that the weighted L^1 space is a nonmaximal ideal in the complex Borel measures.

The content of this thesis is related to my work that has appeared in [36], [37] and [38].

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Chapter 1

Introduction

1.1 Discrete and integral transforms

1.1.1 General results

Discrete and integral transforms have an intrinsic relationship with difference and differential equations, respectively. Conditions in which transforms of functions exist are crucial when applying transforms to these types of equations. In terms of determining the existence of general discrete transforms, a proof was given in Schur [88] which assumes the kernel is positive and decreasing in both of its variables. Provided these assumptions hold, the transform is defined on the space of square summable sequences. It is assumed in [88] that the image of the transform is a set of sequences, and the transform is assumed to be multiplication of a sequence by an infinite matrix. Given similar assumptions, and defining a discrete transform on l^p, an upper bound for the transform was shown in Hardy et al. [48]. In the specific case when the kernel represents the Hilbert matrix, the Hilbert double series theorem, shown in Weyl [102], shows that the discrete transform corresponding to the Hilbert matrix is defined on the space of square summable functions. A simple proof which shows the Hilbert matrix is bounded can be found in Choi [21]. Furthermore, Choi showed the exact value of the norm of the Hilbert matrix is π.

It should be noted that a large class of discrete transforms exists. If one has a separable Hilbert space and a Bessel sequence in the Hilbert space, then the matrix containing all possible inner products of the elements in the Bessel sequence define a bounded linear operator on $l^2(\mathbb{N})$ (see Shapiro & Shields [92]). Another class of infinite matrices which induce a discrete transform is the class of the so-called Toeplitz matrices. These are matrices which are constant along diagonals which are parallel to the main diagonal. Suppose H^2 is the space of square integrable functions whose Fourier coefficients vanish for negative integers. Let P be the projection of $L^2[0,2\pi]$ to H^2 and let ϕ be a bounded measurable function on $L^2[0,2\pi]$. The Toeplitz operator induced by ϕ is the map which sends f to

$P(\phi \cdot f)$. The Toeplitz operators induce a class of Toeplitz matrices (see Halmos [46]).

As for existence of integral transforms, various results hold for transforms defined on arbitrary measure spaces. If a kernel is square-integrable on the product space in which the kernel is defined on, then the kernel defines a bounded integral operator between function spaces. Such an operator is known as a Hilbert-Schmidt operator (see Halmos & Sunder [47]). Given the underlying measures are finite, and that the kernel is bounded, it is guaranteed the integral operator is a Hilbert-Schmidt operator. Such operators contributed significantly to the classical theory of integral equations. We have that every Hilbert-Schmidt operator is a compact operator. So given the Hilbert-Schmidt operator is defined on a Hilbert space, the integral operator can be approximated by a sequence of operators whose members have finite rank.

The Schur Lemma can be applied to transforms whose kernels are defined on arbitrary σ-finite measure spaces. More specifically, assume the kernel of the transform is a locally integrable function on the product space. Consider the function which is equal to the integral of the absolute value of the kernel with respect to its first variable. Consider another function which is defined in a similar manner, with the difference being that the kernel is integrated with respect to its second variable. Furthermore, suppose these two functions are bounded. Now consider an integral operator whose kernel satisfies the previously stated properties and is defined on a set of bounded functions with compact support. Then the corresponding integral operator maps between two L^p spaces where the p in the domain and range are identical (see Grafakos [40]). The proof in [40] relies on the Riesz-Thorin Interpolation Theorem. Furthermore, the Schur Lemma was generalised by Grafakos & Torres [41] to a multilinear version defined on a product space where the kernel consists of $n+1$ variables. Some conditions in which a multilinear integral operator is defined on the product of weighted L^p spaces are outlined in Cwikel & Kerman [26]. A special case of the Schur Lemma is highlighted. Let the kernel of a transform be defined such that its modulus is square integrable with respect to each of its variables separately, that is to say $k(\cdot,x) \in L^2(X)$ and $k(x,\cdot) \in L^2(X)$. Suppose once the kernel has been integrated, the new function is bounded almost everywhere. Then the kernel defines a bounded integral operator on $L^2(X)$ to itself (see Conway [25]). This is highlighted as it shows that an integral operator is a map between Hilbert spaces, and the proof only requires an application of Hölder's inequality.

1.1.2 Fourier series and the Fourier transform

By focusing on specific discrete and integral transforms, more results about the conditions which guarantee a given transform exists can be determined. A classical example of an integral transform is the Fourier transform. The Hilbert spaces of square summable and square integrable functions are prevalent in the theory of general discrete and integral transforms. The square integrable functions are especially important when studying the Fourier transform. One fundamental result which holds for the Fourier transform is the so-called Plancherel Theorem [77]. Explicitly, the Plancherel Theorem states that the Fourier transform extends to an isomorphism from L^2 to itself. An example of where this result has been used is in the derivation of Heisenberg's inequality (see Folland & Sitaram [33]). In physical terms, Heisenberg's inequality states that the product of the variance of the momentum and the variance of the position is bounded below by some positive constant [50].

Additionally, the Fourier transform has also found applications to classical physics. Recall that the Schwartz functions are the class of smooth functions which vanish faster than every polynomial as the modulus of its argument approaches infinity. Assuming the solution to the heat equation is initially of the Schwartz class, the Fourier transform can be applied to the heat equation which converts a partial differential equation into an ordinary differential equation (see Folland [34]). Solving the ordinary differential equation, then applying the inverse Fourier transform gives a solution to the heat equation where the spatial variable can be taken to be an n dimensional variable and time is considered to be strictly positive. Another concept which is intrinsic to the study of integral transforms is the convolution of the two functions. This is essentially an operation such that the Fourier transform acts as an algebra homomorphism where the convolution is the underlying product.

An analogue of the Fourier transform exists when continuous functions are converted to discrete functions. Given a function f is square integrable on the interval $[0, 2\pi]$, the Fourier transform $\hat{f}$ is defined on the integers. An explicit formula for $\hat{f}$ at the nonzero integer n is the standard inner product of f with the complex exponential whose period is inversely proportional to n. The Fourier transform of f at $n = 0$ is simply the average of the function f scaled by 2π. Using the theory of the Lebesgue integral, it can be shown that the Fourier transform, which maps the periodic function f to the function $\hat{f}$ defined on the integers, is an isometry. The Fourier series of f at x is the limit of a linear combination of the functions $n \mapsto \mathrm{e}^{-2\pi \mathrm{i} nx}$ whose coefficients are $\hat{f}(n)$. Also, the Fourier series of f can be shown to converge to the underlying function f in the square integrable sense. However, stronger conditions are required to show the Fourier series converges pointwise to its underlying function. It was shown in Du Bois-Reymond [29] that a function f can be simultaneously continuous and have a Fourier series which diverges at some point. An example of such a function can be found in Bary [12].

As alluded to previously, the Fourier transform is defined on the so-called Schwartz functions. It is a well-known result that the Fourier transform is an isomorphism from the Schwartz class to itself. This allows for the Fourier transform to be extended to the Fréchet space of tempered distributions (see Schwartz [89]). It is noted that Sobolev spaces impose a level of smoothness on integrable functions. The introduction of tempered distributions, as well as the Fourier and inverse Fourier transform on these spaces, allow for a generalisation of Sobolev spaces. The inhomogeneous Sobolev space is precisely this generalisation and is defined for arbitrary real numbers. This new definition coincides with the usual definition for the Sobolev space when s is a nonnegative integer (see Grafakos [40]). A particular class of Sobolev spaces is shown to be related to the fractional Laplacian operator in Di Nezza et al. [28].

The theory of the Fourier transform can be extended to locally compact abelian (LCA) groups. Analysis on locally compact groups was first proposed by Weil [101]. One concept which is essential for extending the Fourier transform to LCA groups is the existence of the Haar measure, which is a nontrivial, translation invariant, Borel measure on locally compact groups. The proof of the existence of such a measure was first presented by Haar [43] under the assumption that the locally compact group was second countable. Several constructions of such a measure are possible. In Halmos [44], the construction of a Haar measure is possible through the use of an outer measure. In Rudin [85], an outline of a proof is given when the underlying group is an LCA group where the Reisz-Markov-Kakutani Theorem is applied. It is also shown in Rudin [85] that two Haar measures on the same group are scalar multiples of each other, and this is shown by an application of Fubini's theorem. This uniqueness result holds for locally compact topological groups, and a proof involving measures which are equal on Baire sets is presents in Halmos [44]. A proof which shows the existence of a Haar measure independent of the Axiom of Choice can be found in Cartan [19].

A character of an LCA group G is a continuous homomorphism from G to the unit circle in the complex plane. The set of characters of G forms an abelian group $\hat{G}$ which is denoted by the dual group. This allows for the Fourier transform to be extended to LCA groups, as for any $f \in L^1(G)$, the Fourier transform of f, denoted by $\hat{f}$, is defined on the space $\hat{G}$ such that $\hat{f}(\gamma)$ is equal to the integral of f multiplied by the complex conjugate of the character γ. Here, integration is over the group G and with respect to a Haar measure on G. Some results which translate to Fourier analysis on LCA groups include the Plancherel Theorem for LCA groups. Furthermore, a convolution product exists on $L^1(G)$ which makes $L^1(G)$ a Banach algebra. The Fourier transform of the convolution of two functions is the product of the Fourier transforms of the underlying functions. Results concerning Fourier analysis on local fields can be found in Taibleson [95]. Further results about the Fourier transform on LCA groups can be found in Folland [35], as well as Hewitt & Ross [52].

1.1.3 Beyond the Fourier transform

The study of the Fourier transform was the genesis of the study of more general integral transforms. An integral transform which is related to the Fourier transform is the

Mellin transform (see Marichev [68]). Sufficient conditions for when it is possible to invert the Mellin transform of a function can be found in Titchmarsh [97]. The standard representation of the inverse Mellin transform varies from the inverse Fourier transform as it involves a contour integral. There are other results which vary between the Fourier and the Mellin transform. An example of this is in the Mellin-Parseval identity, where a weighted integral of the underlying function is shown to be an integral of the transformed function. One integral is over the nonnegative real numbers whereas the other integral is a contour in the complex place (see Yakubovich & Luchko [104]).

Despite its relations to the Fourier transform, the first use of the Mellin transform was by Riemann and was used to study the zeta function (see Poularikas [81]). One such application of the Mellin transform is to solve Laplace's equation on a wedge shaped region (see Davies [27]). Further uses of the Mellin transform can be seen in its utility when solving Euler-Cauchy differential equations (see Misra & Lavoine [70]). It is possible to extend the Mellin transform to distributions. A set of test functions for these distributions can be taken to be the space of all functions defined on the positive real axis such that they are infinitely differentiable and which vanish as its argument approaches zero or infinity [81]. An application of this can be found in Zemanian [106] where the Dirichlet problem is solved on a wedge where the boundary condition is considered to be a generalised function.

The Mellin transform has been used to derive certain Leibniz-type rules for fractional calculus operators (see Luchko & Kiryakova [66]). The Mittag-Leffler function is a fundamental tool in the fractional calculus. It has been shown that the multi-index Mittag-Leffler function is equal to a scaled Mellin-Barnes-type contour integral. That is to say, the multi-index Mittag-Leffler function is equal to the inverse Mellin transform of a specified function (see Paneva-Konovska & Kiryakova [76]).

Another transform which is related to the Fourier transform is the Laplace transform. For the Laplace transform of a function to exist, it suffices to show that the function is piecewise continuous and of exponential order. Applications of the Laplace transform to ordinary differential equations with constant coefficients can be found in Carslaw & Jaeger [18]. This particular transform has found applications in physics and engineering. One such example is when determining the current and charge of a condenser at a given time. A system of linear differential equations governs this process where the coefficients are constant. Due to the relations between the current and the charge, it is possible to convert the system of equations into a single second-order differential equation. However, using the Laplace transform eliminates the need for this substitution (see Irving & Mullineux [54]). It is shown in Widder [103] that if the transform of a function is identically zero and the underying function is continuous, then the underlying function is identically zero. This can be used to invert the Laplace transform given the underlying function is known a-priori. A formula for the inverse of the Laplace transform which involves the use of contour integrals was obtained by Bromwich [16].

The Laplace transform finds applications in the fractional calculus. The computation of the Laplace transform of the Riemann-Liouville fractional derivative, the Caputo fractional derivative as well as the Grünwald-Letnikov fractional derivative can be found in Podlubny [78]. The reason is due to the fact that the Riemann-Liouville fractional integral is the so-called Laplace convolution of a function with some scaled power function. A generalisation of the Laplace transform has also found use in the fractional calculus. More specifically, the Laplace-type H-transform maps the multi-index Mittag-Leffler function to the reciprocal of a linear complex function where the linear function has leading coefficient 1 and constant term -1 (see Kiryakova [62]). Furthermore, in [62], a generalised fractional differential operator is introduced where the multi-index Mittag-Leffler function is an eigenfunction of the introduced differential operator. This is analogous to how the exponential function is an eigenfunction of the first-order derivative. Furthermore, the Laplace transform maps the exponential function to the same function the Laplace-type H-transform maps the multi-index Mittag-Leffler function.

One transform which arises through the study of the two-dimensional Fourier transform in polar coordinates is the Hankel transform. Here, the kernel involves the Bessel functions. It can be shown that the two-dimensional Fourier transform of a circular symmetric function is the Hankel transform of order zero (see Poularikas [81]). Here, circular symmetry means the function of two variables depends only on the radius of the inputs. The Hankel transform is an example of a self-invertible operator given the underlying function is analytic in some region in the complex place containing the nonnegative real line. A proof of this is given by Davies [27] and the proof is attributed to MacRobert. The proof involves the Lommel's integral which can be found in Watson [99], where the Lommel's integral involves two cylinder functions. Certain orthogonality conditions for Bessel functions hold which traditionally show that the formula for the inverse Hankel transform holds (see Ponce de Leon [80]). The purpose of [80] is to prove the existence of the inverse Hankel transform without the use of the orthogonality conditions. This is due to the fact that the inverse Hankel transform is traditionally used to derive the orthogonality relations, and this causes circular reasoning. It has been discussed previously that certain integral transforms map certain differential operators to polynomial equations. The Hankel transform is similar in this regard. More specifically, the Hankel transform maps the Bessel differential operator to a monomial in one variable of order two. This differential operator can also be obtained from the Laplacian operator in polar coordinates by applying separation of variables (see Poularikas [81]).

The previous integral transforms mentioned all involved functions defined on some subset of the Euclidean space. In Cameron & Martin [17], they defined a generalised integral transform on a set of functionals. More specifically, the set of functionals in question is the set of all complex-valued continuous functions defined on the interval

$[0,T]$, where $T > 0$, which vanish at the origin. The Fourier-Wiener transform is an example of such a transform. Here, the integral is taken with respect to the Wiener measure. The Fourier-Wiener transform has been generalised with the introduction of certain parameters by Lee [65]. In Chung & Tuan [23], the special Gaussian process, introduced in Chung et al. [22], was used to define a generalised integral transform and a generalised convolution product of functionals defined on the set of continuous, complex-valued functions on $[0,T]$ which vanish at the origin.

Several transforms have been introduced which have the purpose of generalising various other transforms. For example, an integral transform called the Sumudu transform was introduced in Watugala [100] with the purpose of solving differential equations. The Natural transform was introduced by Khan & Khan [61] and this was shown to be related the Laplace and Sumudu transforms. Following this, duality relations were shown between the Natural transform and the Fourier, Laplace, Sumudu and Mellin transforms (see Shah et al. [91]). A further generalisation of the Natural transform was introduced in Jafari [56]. The class of integral transforms introduced in [56] includes the Natural transform, the Aboodh transform (see Aboodh [1]) and the Elzaki transform (see Elzaki [31].) The transform in [56] introduced formulas which can be applied to initial value problems where the underlying differential equation was linear with variable coefficients. The transform is also applied to different Volterra integral equations, as well as fractional integral equations involving the Riemann-Liouville fractional integral. Further applications of this class of integral transforms have been explored in Meddahi et al. [69].

Generalisations of classes of integral transforms have been made which vary to that presented in [56]. An example of this is the introduction of the $\mathscr{L}_2$-transform, which has been analysed by Yürekli & Sadek [105]. The $\mathscr{L}_2$-transform of f at s is the half of the Laplace transform of the function $t \mapsto f(\sqrt{t})$ evaluated at s^2. An inversion formula for this transform involves contour integration and this expression was shown to be an inversion formula in Aghili et al. [3]. Coupled with this, it was shown in Aghili & Ansari [5] that the $\mathscr{L}_2$-transform maps a certain differential operator to a polynomial. Furthermore, a convolution product is presented in this article. Applications of these results are found in solving a system of fractional partial differential equations. Further properties of the $\mathscr{L}_2$-transform can be found in Aghili & Zeinali [4].

The $\mathscr{L}_2$-transform can be further generalised by the $\mathscr{L}_A$-transform which was introduced by Aghili & Ansari [6], which also includes the Laplace and Mellin transform as special cases. Once again, it is shown that the $\mathscr{L}_A$-transform maps a first-order differential operator to a polynomial in the transform parameter. A formula which satisfies the convolution property is introduced in this paper and another formula is shown to invert the $\mathscr{L}_A$-transform. The $\mathscr{L}_A$-transform does not include the Fourier transform. This is due to the fact that integration in the $\mathscr{L}_A$-transform is taken over the positive real numbers. However, there exists a class of transforms which are similar to those which fall under the $\mathscr{L}_A$-transform. It was shown that the $\mathscr{F}_A$-transform includes the Fourier transform and maps the same differential operator considered in [6] to a complex polynomial (see Aghili & Ansari [2]).

Another method which has been employed with the purpose of generalising a class of discrete and integral transforms can be found in Futcher & Rodrigo [37]. Here, a class of discrete and integral transforms whose kernels are logarithmic-separable are shown to have a convolution formula. For each transform in this class, there is a corresponding L^1 space such that the transform is defined and the convolution product is a binary map on the weighted L^1 space. By only considering integral transforms, a class of integral transforms was analysed in Futcher & Rodrigo [36]. It was shown that the class of transforms in [36] is injective on the intersection of the corresponding L^1 space and the set of continuous functions. The integral transforms were applied to a subclass of second-order linear differential equations with variable coefficients to derive new analytical solutions. Furthermore, these solutions appear to be the only solutions which exist for this class of differential equations.

1.1.4 Convolutions

As alluded to previously, convolution is formally a binary operation which is defined on a set of functions. Such products have been found in many areas of mathematics including, but not limited to, Fourier analysis, functional analysis, differential equations, number theory and probability theory.

With regards to Fourier analysis, the convolution of the N^{th} Dirichlet kernel and a function f is a partial sum of the Fourier series of f (see Stein & Shakarchi [94]). This becomes helpful when determining the pointwise convergence of the partial sums of a Fourier series. Given a function f is differentiable at a point x, the use of the Dirichlet kernel as well as the Riemann-Lebesgue Lemma gives information about the convergence of the Fourier series of f at x.

The use of the Fourier convolution is prevalent in functional analysis as the L^1 space on the n-dimensional Euclidean space is a Banach algebra where the convolution is the underlying product (see Rudin [86]). The convolution is present in a proof of the Fourier inversion formula as it can be shown that, for a specific function h_λ, the convolution of a bounded and continuous function f with h_λ converges pointwise to f as λ approaches 0. This limit can also be shown to be the inverse Fourier transform of f. Furthermore, the convolution formula can be used to show that the space $L^1 \cap L^2$ is dense in L^2. This gives the well-known Plancherel Theorem which states that the Fourier transform extends to a unitary map on L^2 (see Rudin [84]).

The Fourier convolution and Fourier transform have found applications to topological vector spaces, namely to Fréchet spaces. It is well known that the space L^1 has no unit.

The convolution product can be extended to distributions in several ways. It is possible to convolve distributions with smooth functions with compact support such that the resulting object is a smooth function. Otherwise, we can define the convolution such that the inputs are the same, however the output is instead a distribution. These two definitions are equivalent on an appropriate space of functions.

In regards to differential equations, the Mellin convolution product is present in the solution to the Black & Scholes [14] partial differential equation solved by Rodrigo & Mamon [83]. The existence of a solution where the payoff, the terminal condition, is an arbitrary function is made possible through the use of a convolution formula. It is not obvious that the usual solution derived by Black & Scholes for the European call and put options can be unified under a single formula. However, the Mellin convolution makes this possible. Another advantage of the formula presented in [83] is the fact that the payoff functions are not restricted to those of a call or a put option. Furthermore, the standard Black & Scholes equation is often solved where the interest-rate, the dividend yield and the volatility are constant. In [83], by using the Mellin transform and the Mellin convolution, a solution for the partial differential equation could be derived where the interest rate and volatility were assumed to be positive, continuous functions of time and the dividend yield was assumed to be a nonnegative function of time.

Such appearances of a convolution formula in number theory occur due to the so-called Dirichlet convolution which is defined on a class of sequences. This product, also known as the Dirichlet product, provides an elementary proof of the Möbius Inversion Theorem (see Ireland & Rosen [53]). This, in turn can be used to provide an expression for the value of Euler's totient function for every natural number. Another use of the Dirichlet convolution formula is the fact that it can construct multiplicative functions. That is, functions which satisfy the homomorphism property given the underlying integers which are being multiplied together are relatively prime. It can be shown if two arithmetic functions are multiplicative, then the Dirichlet convolution of these two functions is a multiplicative function (see Apostol [9]).

In regards to probability theory, the Fourier convolution extends to measures. Included in this, the Fourier transform also extends to measures, and the Fourier transform of a measure is often called the characteristic function of the probability measure. It can be shown that if two probability measures have the same characteristic functions, then the underlying measures are equal. We can gain information on a characteristic function that is not obvious when analysing the measure directly. For example, it can be shown that if a sequence of characteristic functions converges pointwise to a characteristic function, then the underlying sequence of probability measures converges weakly to the probability measure corresponding to the limiting characteristic function (see Billingsley [13]). In fact, it is shown that the converse is true. That is to say, a sequence of measures converges weakly to a measure if and only if it is the corresponding sequence of characteristic functions converges pointwise to the Fourier transform of the limiting measure. Furthermore, certain statements about independent random variables can instead be written in the language which involves convolutions of distribution functions (see Lamperti [64]).

Convolutions have been studied for their own interest. One such concept which unifies several convolution formulas is the so-called φ-convolution which was analysed by Nhan et al. [75]. More specifically, the Fourier and Mellin convolution formulas over the n-dimensional Euclidean space are special cases of the φ-convolution formula. Unlike many other convolution operations, the φ-convolution has been studied without an associated discrete or integral transform. In this paper, several inequalities of convolutions on L^p spaces were presented. Such results determined in the paper were applied to the Bernoulli-Euler beam equation, which is a fourth-order differential equation which measures the vertical deflection of an infinite beam. Further applications of convolution products and L^p norm inequalities can be found in Nhan & Duc [73].

The φ-convolution can be thought of as a generalisation of the Fourier convolution. A new convolution formula was obtained by Saitoh [87] which involves functions belonging to different weighted L^p spaces where p is held constant. Applications of Hölder's inequality and Fubini's Theorem are used to derive this inequality. The inequality has been applied in various ways. In Nhan & Duc [74], this inequality as well as further inequalities are presented and then applied to the heat and wave equations. Further convolution inequalities have been derived which do not consider the Fourier convolution as the underlying product. The Mellin convolution product is considered in Nhan & Duc [72]. Here, suppose that one is given two weight functions ρ_1 and ρ_2, and another weight function ρ which is related to the Mellin convolution of the ρ_j functions where $j \in \{1,2\}$. Assume $F_j \in L^p(\rho_j)$, then the Mellin convolution is in $L^p(\rho)$ and the $L^p(\rho)$ norm of the convolution of the F_j functions is bounded by the multiplication of the $L^p(\rho_j)$ norm of the F_j functions. Castro & Saitoh [20] introduced several convolutions which included different kinds of translations, as well as complex conjugation of the underlying functions. The proofs of these new convolution inequalities was based off the theory of reproducing kernels (see Aronszajn [10]). Further inequalities were derived in Jain & Jain [57]. A combination of these convolution formulas, as well as the φ-convolution is given in Duc & Nhan [30], and a proof involving the sum of these convolution formulas and an upper bound for the norm of this sum is presented.

1.2 Discrete transforms: an alternative approach

It was hinted at previously that discrete transforms are considered mappings from sequences to sequences. More specifically, a discrete transform maps functions defined on countable spaces to functions defined on countable spaces. For our purposes, we will consider discrete transforms to be operators which map functions defined on countable spaces to functions defined on some uncountable subset of the complex plane. More specifically, the types of discrete transforms we will analyse will be analogous to either a Z-transform or a generalised Dirichlet series.

1.2.1 Z-transform

The Z-transform is a discrete transform which takes functions defined on the integers to a function defined on some subset of the complex plane. The subset of the plane the transformed function is defined on is often considered to be an open ball, the complement of a closed ball or an annulus. For example, when considering the Z-transform of a causal function, that is a function that is zero when its argument is less than zero, then the set in which the transformed function is defined on can be taken to be the complement of some ball in the complex plane (see Poularikas [81]).

In terms of applications, the Z-transform has been used to determine the solution to linear difference equations with periodically time-varying coefficients (see Jury [58]). This includes the class of linear difference equations with constant coefficients. Despite being a linear operator, the Z-transform has found applications in nonlinear difference equations. Jury & Pai [60] applied the Z-transform to a certain class of nonlinear difference equations. More specifically, the nonlinear difference equations could be written as a linear and a nonlinear component which was autonomous. The nonlinear component contains terms of the dependent variable of degree two and higher. Furthermore, the difference equation which the Z-transform was applied to in [60] was assumed to have an asymptotically stable equilibrium point and the nonlinear component was assumed to be analytic in its variables around the equilibrium point. In regards to difference equations, the Z-transform has found further applications when determining the solution to linear difference-differential equations and the so-called double Z-transformation has been applied to partial difference equations (see Jury [59]).

A relationship between the Laplace transform and the Z-transform can be found in Grove [42]. Similarly to how the Laplace transform finds applications in probability theory in the moment generating function, the Z-transform appears in probability theory through the concept of a generating function. Examples of generating functions can be found in Knuth [63]. Through the use of a substitution, the Z-transform can be shown to be almost identical to the discrete-time Fourier transform.

The Z-transform has also found applications in the analysis of electrical signals. It was shown in Lynn [67] that a linear time-invariant system can be represented by a linear, autonomous difference equation which consists of two dependent variables. The inverse transform is helpful when given a transfer function, which is the ratio of the Z-transform of the output signal to the Z-transform of the input signal. The transfer function is exactly the Z-transform of the impulse response. In [67], an example of a transfer function of order two is given and the inverse Z-transform is applied to this function to obtain an equation for the signal in the original time domain.

1.2.2 Dirichlet series

The Dirichlet series is a way to transform sequences to complex functions that has found many uses in number theory. As mentioned previously, its use provides an elementary proof of the Möbius Inversion Theorem. Existence conditions for the Dirichlet series are well known. For example, it can be shown if the Dirichlet series exists for a complex number s, then the Dirichlet series exists for every complex number in the plane such that its real part is greater than or equal to $\mathrm{Re}(s)$ (see Hardy [49]). In [49], further properties are shown about the region in which a given Dirichlet series is defined on. Examples include the constant σ such that the series converges when the real part of its argument is greater than σ and diverges when the real part is less than σ. This is under the assumption that the Dirichlet series does not converge everywhere. A proof showing that two Dirichlet series are equal in a half plane implies the two underlying sequences are equal can be found in Serre [90].

A special case of the Dirichlet series are the L-functions, which are the Dirichlet series of a character modulo k. Given the character is the Dirichlet character modulo 1, the character function is the constant one function, then the L-function reduces to the well-known Riemann zeta function. It is possible to write the L-functions as a linear combination of Hurwitz zeta functions (see Apostol [9]). The L-functions have found applications in number theory. Suppose one is given any two nonnegative integers n and m such that their greatest common divisor is 1. Through the use of the L-functions, it can be shown that an infinite amount of primes occur in the arithmetic progression with initial term n and common difference m (see Ireland & Ross [53]). Applying techniques from complex analysis, it is possible to apply an analytic continuation to the L-functions such that they are defined on the negative integers. Simple techniques which give an analytic continuation of the Riemann zeta function can be found in Goss [39]. The L-function evaluated at the negative integers are directly related to the generalised Bernoulli numbers (see Iwasawa [55]). More specifically, given a natural number n and a Dirichlet character, the Dirichlet L-function associated with the Dirichlet character evaluated at $(1-n)$ is the ratio of the nth generalized Bernoulli number associated with the Dirichlet character to $-n$.

1.3 Structure of the thesis

The purpose of the work in this thesis is to examine a class of discrete and integral transforms with logarithmic separable kernels. Some integral transforms which have logarithmic separable kernels include the Fourier transform, the Laplace transform and the Mellin transform. We show that various concepts associated with the Fourier transform which have been thoroughly studied have analogous versions for any integral transform with a logarithmic separable kernel. Topics such as the convolution formula also extend to discrete transforms with logarithmic separable kernels, which we analyse in this thesis. The work presented here provides an alternative framework for deriving general results about a class discrete and integral transforms.

In Chapter 2, we introduce the prerequisites which are required to understand the work in this thesis. We start with introducing the Axiom of Choice in set theory and derive some of its implications which turn out to be equivalent to the Axiom of Choice. Further concepts introduced are the Fourier transform which is done with the purpose of introducing the reader to integral transforms as well as convolution products. Furthermore, we present the required topology and measure theory background which includes Fréchet spaces, distributions, complex measures and the total variation of a measure. We finish the section by introducing a new class of discrete and integral transforms as well as a type of weighted L^p space which will become particularly important in Chapters 4 and 5.

In Chapter 3, we analyse a more general class of integral transform than that defined in Chapter 2. The so-called shifting and convolution properties are introduced and a link is shown between the two through formal calculations. The shifting property is established for the purposes of simplifying computations when transforming the convolution of two functions. Conditions are presented which show when the convolution of two functions exists almost everywhere. Moreover, conditions which guarantee the convolution of two functions is continuous are given. Sufficient conditions in which the integral operators are injective are given and a proof is derived. Furthermore, by restricting the class of integral transforms, we give a proof of an inversion formula for our class of integral transforms. An application is given to second-order differential equations and our analytical solutions are compared to numerical results derived using software.

In Chapter 4, the class of discrete and integral transforms that was formulated in Chapter 2 is analysed. We start by proving certain properties of the range of the discrete and integral operators. Such properties include the transformed functions being bounded and continuous given the underlying function is an element of the weighted L^1 space given in Chapter 2. As the shifting property also extends to discrete transforms, we substantiate that the shifting property is related to the convolution property and sufficient conditions are given such that the shifting property is satisfied. Following this, the convolution operation is shown to be a binary operation on the weighted L^1 space. Examples of convolution formulas for specific transforms are given. It is then established that the discrete operators defined are injective on the weighted L^1 space introduced in Chapter 2. This is then used to give an elementary proof that the convolution operation satisfies the properties of a commutative ring given the corresponding transform is discrete. Another proof showing that the convolution operation is commutative, associative and distributes over the addition is given when the underlying transform is an integral transform. The results are all unified when it is shown that the weighted L^1 space introduced in Chapter 2 is a commutative Banach algebra and the discrete and integral transforms are continuous homomorphisms between Banach algebras.

In Chapter 5, further properties of the convolution are determined, specifically in the case where the underlying transform is an integral transform. This includes deriving properties of the support of the convolution of two functions, as well as showing the convolution is a binary operation on the set of continuous functions with compact support. The convolution operation is then extended to distributions. We define our general convolution formula between a distribution and a smooth function with compact support to be a function whose domain is some set depending on the underlying function with compact support. Basic properties of this new function are shown including the function being defined on an open set and the function being smooth. Certain properties of our class of integral transforms are highlighed and these are used to define our integral transforms on distributions. The set of complex Borel measures is analysed in this chapter. The convolution is then defined between two measures. It is shown that the set of Borel measures is a commutative Banach algebra where the convolution is the underlying product. It will be shown that the inclusion map from our weighted L^1 space to the set of complex Borel measures is a homomorphism, and the image of this inclusion map is an ideal in the set of complex Borel measures. We show how the set of Borel measures provides a way of embedding the weighted L^1 space into an algebra which contains a unit, namely the Dirac measure at a specific point.

In Chapter 6 we offer some concluding remarks, as well as state various further areas where our results can be applied. This includes the work in Chapter 3, where the class of transforms can be applied to certain integral equations. An alternative definition of an integral transform of a distribution is given, and a justification for this new definition is presented. An example where the Fourier convolution is applied to probability theory are highlighted here, and a potential area where our analysis of complex Borel measures could be applied is discussed.

Chapter 2

Preliminaries and fundamental information

It is assumed the reader is familiar with the basic principles of analysis, such as various ε-δ arguments, uniform continuity, pointwise convergence, uniform convergence, Cauchy sequences and Riemann integration. We make the further assumption that the reader is familiar with abstract measure spaces, as well as integration with respect to measures, particularly the Lebesgue and the counting measures. Certain concepts in measure theory that are assumed knowledge include the Monotone Convergence Theorem and the Dominated Convergence Theorem, as such they will not be explicitly stated here. However, we will introduce various ideas in measure theory which the author believes are less well-known. It is expected the reader is familiar with concepts in complex analysis. These concepts include general facts about holomorphic functions, contour integraion, Cauchy's Integral Formula, and Morera's Theorem. While the reader has certainly been introduced to set theory, for the sake of completeness, we will recall some set theoretic properties.

2.1 Set theory

It is assumed that the reader has seen elementary set theory. The concepts introduced here will become relevant when discussing nets in Section 2.3.1 and when showing nonmeasurable sets exists in Section 2.5.1. First, some terminology is introduced to make sense of expressions of the form $\{X_\alpha\}_{\alpha\in A}$.

Definition 1. Let $\mathfrak{X}$ be a nonempty collection of sets. An *indexed family of sets* is the image of a surjective function $g : A \to \mathfrak{X}$, known as the indexing function. Here, A is called the index set. Given $g(\alpha) = X_\alpha$, we will denote the range of g by $\{X_\alpha\}_{\alpha\in A}$.

The above definition lets us write a nonempty collection of sets $\mathfrak{X}$ as $\{X_\alpha\}_{\alpha\in A}$ given we can define a surjective function from some set A to $\mathfrak{X}$. As this is always possible if we let $A = \mathfrak{X}$ and let g be the identity function, we will from now on write a nonempty collection of sets as $\{X_\alpha\}_{\alpha\in A}$. For the sake of clarity we inform the reader that the work in this thesis is based off the assumption that the Axiom of Choice is true.

Assumption (the Axiom of Choice)**.** *The Cartesian product of a nonempty collection of nonempty sets is nonempty.*

In mathematical notation, this means that if A is nonempty and X_α is nonempty for every $\alpha \in A$, then

$$\prod_{\alpha\in A} X_\alpha \tag{2.1}$$

is nonempty. Recall that when A infinite, the Cartesian product of a nonempty collection of sets is defined to be the set of all functions of the form

$$x : A \to \bigcup_{\alpha\in A} X_\alpha$$

such that $x(\alpha) \in X_\alpha$ for every $\alpha \in A$.

There are several formulations of the Axiom of Choice. The reason for starting with this assumption is that this axiom is easier to comprehend as opposed to other formulations. We will use an equivalent formulation of this axiom in this thesis.

Proposition 2. *Suppose the Axiom of Choice is true. Then for any nonempty set X, there exists a function $f : \mathscr{P}(X)\setminus\{\emptyset\} \to X$ such that $f(Y) \in Y$ for every $Y \in \mathscr{P}(X)\setminus\{\emptyset\}$, where $\mathscr{P}(X)$ denotes the power set of X.*

Proof. Let $g : \mathscr{P}(X)\setminus\{\emptyset\} \to \mathscr{P}(X)\setminus\{\emptyset\}$ be the identity function. This turns $\{Y\}_{Y\in\mathscr{P}(X)\setminus\{\emptyset\}}$ into an indexed family of sets. By the Axiom of Choice, the Cartesian product of these sets is nonempty. That is, there exists a function

$$f : \mathscr{P}(X)\setminus\{\emptyset\} \to \bigcup_{S\in\mathscr{P}(X)\setminus\{\emptyset\}} S$$

such that $f(Y) \in Y$ for every $Y \in \mathscr{P}(X)\setminus\{\emptyset\}$. As the union of all nonempty subsets of X is just X, this completes the proof. $\square$

Definition 3. A function defined as in Proposition 2 is known as a *choice function* on X.

We note that if $\mathfrak{Y}$ is a subcollection of sets in $\mathscr{P}(X)\setminus\{\emptyset\}$, then $f|_{\mathfrak{Y}}$ is a function defined on $\mathfrak{Y}$ such that $f(Y) \in Y$ for every $Y \in \mathfrak{Y}$. While it will not be proven here, it should be highlighted that Proposition 2 is equivalent to the Axiom of Choice. It can be shown that Proposition 2 implies that Zorn's Lemma is true, which in turn implies the Well Ordering Principle is true (see Halmos [45]). That is, if Zorn's Lemma is true, then every set X can be equipped with a total order such that every subset of X has a smallest element. It is straightforward to show the Well Ordering Principle implies the

Axiom of Choice. Indeed, if $\{X_\alpha\}_{\alpha\in A}$ is an indexed family of sets, where A and each X_α is nonempty, observe that $\cup_{\alpha\in A}X_\alpha$ can be well ordered. Let x be a function defined on A such that $x(\alpha)$ is the smallest element of X_α. As this is a well-defined function, we have that the Cartestian product of $\{X_\alpha\}_{\alpha\in A}$ is nonempty. From here, we can see that the Axiom of Choice is equivalent to Proposition 2.

We highlight for the benefit of the reader that the Axiom of Choice becomes relevant when defining our weighted L^1 space in Section 2.6. The relevance will occur because we will only consider Borel functions in our weighted L^1 space. This is due to the fact that the Axiom of Choice is used to show the existence of non-measurable sets. In fact, the Axiom of Choice is required to show the existence of sets which are not Lebesgue measurable (see Solovay [93]). Moreover, Solovay showed that the Axiom of Countable Choice, together with the other axioms of Zermelo-Fraenkel set theory do not posses the necessary assumptions to create a non Lebesgue measurable set. Furthermore, if sets which are not Lebesgue measurable exist, then it is possible to show that the composition of Lebesgue measurable functions is not necessarily Lebesgue measurable. Since our new convolution formula will consist of the product of several functions, one of these underlying functions consists of the composition of two functions, we restrict the functions which will be transformed and convolved to Borel functions.

The Axiom of Choice also becomes relevant when we discuss the topological prerequisites. More specifically, when nets are introduced, a choice function will be used to define a net which converges. This serves the purpose of giving the reader a concrete example of a net which does not reduce to a sequence.

2.2 The Fourier transform

Definition 4. The *Fourier transform* is an operator taking f to the function $\mathscr{F}\{f\}$, where $\mathscr{F}\{f\}$ is defined by
$$\mathscr{F}\{f\}(\omega) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi\mathrm{i}\langle\omega,x\rangle} f(x)\,\mathrm{d}\lambda(x).$$
Here, we impose that $\omega\in\mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^n$.

Let λ denote the Lebesgue measure on $\mathbb{R}^n$. It is straightforward to show that f being a function in $L^1(\lambda)$ is a sufficient condition for $\mathscr{F}\{f\}$ to exist almost everywhere on $\mathbb{R}^n$.

Definition 5. We say a function f *vanishes at infinity* if for every $\varepsilon>0$, the set
$$\{x : |f(x)| \geq \varepsilon\} \tag{2.2}$$
is compact.

Example 6. Let $f:[1,\infty)\to\mathbb{R}$ be defined by $f(x)=1/x$. Fix $\varepsilon>0$. If $\varepsilon>1$, then the set in (2.2) is empty. Suppose $0 < \varepsilon \leq 1$, we have

$$\left\{x \in [1,\infty) : \frac{1}{x} \geq \varepsilon\right\} = \left[1, \frac{1}{\varepsilon}\right].$$

So f is a function which vanishes at infinity.

It should be noted for $f \in L^1(\lambda)$, we have $\mathscr{F}\{f\} \in C_0(\mathbb{R}^n;\mathbb{C})$, where $C_0(\mathbb{R}^n;\mathbb{C})$ denotes the space of all complex-valued, continuous functions defined on $\mathbb{R}^n$ which vanish at infinity.

Definition 7. The *inverse Fourier transform* is an operator $\mathscr{F}^{-1}$ defined by the following formula

$$\mathscr{F}^{-1}\{f\}(x) = \int_{\mathbb{R}^n} \mathrm{e}^{2\pi\mathrm{i}\langle\omega,x\rangle} f(\omega)\,\mathrm{d}\lambda(\omega).$$

It is important to note that the space $L^1(\lambda)$ is not closed under multiplication. If we let f be the function defined by $f(x) = x^{-1/2}\chi_{(0,1)}(x)$, then $f \in L^1(\lambda)$. The function χ_E is the characteristic function on E and is defined pointwise by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

However, f^2 is defined by $f(x)^2 = x^{-1}\chi_{(0,1)}(x)$ which is not in $L^1(\lambda)$. We require a new operation, which varies from addition, which is a binary relation on $L^1(\lambda)$.

Definition 8. Suppose that $f, g \in L^1(\lambda)$. The *Fourier convolution* of f and g is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-u)g(u)\,\mathrm{d}\lambda(u). \tag{2.3}$$

It is a well-known result that $L^1(\lambda)$ is a commutative Banach algebra (see Rudin [84]). We remind the reader that a Banach algebra is a Banach space that is also an algebra whose elements satisfy the following

$$||fg|| \leq ||f|| \cdot ||g||.$$

Part of showing $L^1(\lambda)$ is a Banach algebra is showing the convolution operation is a binary relation on $L^1(\lambda)$. This is evident from the fact that $u \mapsto f(x-u)g(u)$ is Lebesgue measurable due to the Lebesgue measure being a translation invariant measure. A proof involving Fubini's Theorem shows that the formula in (2.3) exists almost everywhere on $\mathbb{R}^n$ (see Rudin [84]). It is worth pointing out that this is the most we can say for a general f and g in $L^1(\lambda)$. We can not guarantee that the formula in (2.3) is defined almost everywhere on $\mathbb{R}^n$. To see why this distinction is important, we will present an example from Apostol [8].

Example 9. Let $n = 1$ and consider the following functions

$$f(x) = \frac{1}{\sqrt{x}}\chi_{(0,1)}(x), \qquad g(x) = \frac{1}{\sqrt{1-x}}\chi_{(0,1)}(x).$$

An elementary computation shows that

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}\lambda(x) = 2, \qquad \int_{\mathbb{R}} g(x)\,\mathrm{d}\lambda(x) = 2.$$

Now, if we convolve f and g and evaluate this function at $x = 1$, then this yields

$$\begin{aligned}(f*g)(1) &= \int_{\mathbb{R}} \frac{1}{\sqrt{1-u}}\chi_{(0,1)}(1-u)\frac{1}{\sqrt{1-u}}\chi_{(0,1)}(u)\,\mathrm{d}\lambda(u)\\ &= \int_0^1 \frac{1}{1-u}\,\mathrm{d}\lambda(u). \end{aligned} \tag{2.4}$$

Note that the integral in (2.4) is not finite. Hence $f*g$ does not define a complex-valued function on the whole of $\mathbb{R}$.

We also note that the Fourier transform is an isomorphism between the algebra $L^1(\lambda)$ and some subalgebra of $C_0(\mathbb{R}^n;\mathbb{C})$. More specifically, the Fourier convolution is defined such that for every $f,g \in L^1(\lambda)$, we have

$$\mathscr{F}\{f*g\} = \mathscr{F}\{f\}\mathscr{F}\{g\}.$$

It is important to recognise that the Fourier transform is also an algebra isomorphism from $L^1(\mathbb{R}^n,\mathscr{B},\lambda)$ to some subalgebra of $C_0(\mathbb{R}^n;\mathbb{C})$, where $\mathscr{B}$ denotes the Borel σ-algebra on $\mathbb{R}^n$. This will be important when examining a broader class of discrete and integral transforms and weighted L^1 spaces in which these transforms are defined on. More specificaly, in Chapter 4, a convolution formula will be presented which makes sense given the underlying functions are Borel measurable. However, the formula may not be defined if the functions being convolved are Lebesgue measurable.

The Fourier transform has other elegant properties, for example, it is a unitary map on $L^2(\lambda)$. That is, $\mathscr{F} : L^2(\lambda) \to L^2(\lambda)$ is a bijection which preserves the inner product. One other important property is that the Fourier transform is an isomorphism from the collection of Schwartz functions to itself. The Schwartz functions are specifically the class of infinitely differentiable functions which vanish faster than any polynomial. We introduce our class of discrete and integral transforms in Section 2.6, and then extend the integral transforms to distributions in Section 5.2. It will then be highlighted the elegance of a transform being an automorphism on some space. That is, the transform being an isomorphism where the domain and range are the same space.

2.3 Topological prerequisites

We assume the reader is already familiar with concepts such as topological spaces, continuous maps between topological spaces, sequences and compactness. We now introduce the topological prerequisites which become vital when studying distributions in Chapter 5.

2.3.1 Nets

Definition 10. A *directed set* is a set A with a binary relation $\preceq$ which has the following properties:

(i) $x \preceq x$ for every $x \in A$;

(ii) For every $x, y, z \in A$, if $x \preceq y, y \preceq z$ then $x \preceq z$;

(iii) For every $x, y \in A$, there exists $z \in A$ such that $x \preceq z$ and $y \preceq z$.

It is important to recognise the need for assumption (iii). If the assumption were relaxed to say that for every $x \in A$ there is a $y \in A$ such that $x \preceq y$, then $y = x$ suffices. Also, it will be convenient to use the notation $y \succeq x$ which is equivalent to $x \preceq y$.

We now introduce the concept of a net, which generalises the concept of a sequence.

Definition 11. A *net* in a topological space X is a mapping $x : A \to X$ from a directed set into X, where we often denote the term $x(i)$ by x_i.

Much like in the case for sequences, we will sometimes denote a net by its range. We say a net $x = \{x_a\}$ *converges* to x_0 if for every open set U containing x_0, there exists $b \in A$ such that $a \succeq b$ implies $x_a \in U$. We now highlight how convergent nets are a nontrivial generalisation of convergent sequences.

Example 12. Let A be the collection of all open subsets of $\mathbb{R}$ that contain 0. We will order A by reverse inclusion. That is to say for $a, b \in A$, $a \succeq b$ if and only if $a \subseteq b$. By the typical rules of set theory, A is a directed set. Now, by the Axiom of Choice, there exists a choice function on $\mathbb{R}$, and we will denote its restriction to A by x. We will show this net converges to 0.

Let U be an open set containing 0. By definition, we have $U \in A$. Let $V \in A$ such that $V \succeq U$, or equivalently, $V \subseteq U$. The function x is the restriction of a choice function to A, so $x(V) = x_V \in V \subseteq U$. This shows for every open set U containing 0, there exists $b = U \in A$ such that $V \succeq U$ implies $x_V \in U$. Hence the net x converges to 0.

This concept is an appropriate generalisation of sequences in a topological space. This is apparent as if $f : X \to Y$ is a map between topological spaces and $\{x_a\}$ is any net in X which converges to x_0, then $\{f(x_a)\}$ converges to $f(x_0)$ if and only if f is continuous at x_0. We show that it is not necessarily true if $x_n \to x$ implies $f(x_n) \to f(x)$ for every sequence $\{x_n\}$, then f is continuous.

Let $f : X \to Y$ be a map where $U \subset X$ is open if and only if $X \backslash U$ is countable. We assume X is an uncountable set. Let the topology on Y be the power set on Y. Now, let $Y = X$ and $f(x) = x$. Let $\{x_n\}$ be a sequence in X which converges to x. Since $\{x_n\}$ converges in x, it is necessary that $\{x_n\}$ is eventually constant. If we assume it is not eventually constant, then $X \backslash \mathrm{Im}(x_n) \cup \{x\}$ is an open set which contains x and no x_n whenever $x_n \neq x$. We note that this set is open due to the fact that $\mathrm{Im}(x_n)$ is a countable set. Therefore, we must have that $\{x_n\}$ is eventually constant. Because of this, we deduce that

$$f(x_n) = x_n \to x = f(x).$$

That is, for every sequence $\{x_n\}$ which converges to x, the sequence $\{f(x_n)\}$ converges to $f(x)$. Consider the preimage of the set $\{x\}$ under f. That is, the set

$$f^{-1}(\{x\}) = \{x\}$$

which is not open in X. Therefore f is not continuous despite being sequentially continuous.

We will now show if for every net $\{x_a\}$ which converges to x, the net $\{f(x_a)\}$ converges to $f(x)$, then f is continuous at x. We state for the sake of clarity that for a set $S \subseteq X$, S^o is the union of all open sets U such that $U \subseteq S$ and $\overline{S}$ is the intersection of all closet set C such that $S \subseteq C$.

We will establish the contrapositive, that is, if $f : X \to Y$ is not continuous at x then there exists a net $\{x_i\}$ which converges to x such that $f(x_i) \not\to f(x)$. Since f is not continuous at x, there exists a neighbourhood U of $f(x)$ such that $f^{-1}(U)$ is not a neighbourhood of x. So $x \notin (f^{-1}(U))^o$ which implies $x \in \overline{f^{-1}(Y \backslash U)}$. Observe it is not possible that $x \in f^{-1}(Y \backslash U)$, as this would imply that $f(x) \notin U$. So x is an accumulation point of $f^{-1}(Y \backslash U)$. Let A be the directed set of all neighbourhoods of x ordered by reverse inclusion. Consider any net defined on A which sends $V \in A$ to a point in $(V \backslash \{x\}) \cap f^{-1}(Y \backslash U)$. It is evident that $\{x_V\}$ is a net in $f^{-1}(Y \backslash U)$ which converges to x. However, $f(x_V) \notin U$, so $f(x_V) \not\to f(x)$.

2.3.2 Topological vector spaces

In the previous section, we introduced open sets. For the sake of revision, we introduce the concepts of a ‘neighbourhood base’ and a ‘first-countable topological space’.

Definition 13. Let X be a topological space. A *neighbourhood base* at a point $x \in X$ is a collection $\mathscr{N}(x)$ of open sets such that

(i) $V \in \mathscr{N}(x)$ implies $x \in V$;

(ii) for every open set U containing x, there exists $V \in \mathscr{N}(x)$ such that $V \subseteq U$.

Example 14. Let $\mathscr{P}(X)$ be the collection of all subsets of some nonempty set X, turning X into a topological space. If $\mathscr{N}(x)$ is the collection of all sets containing x, then $\mathscr{N}(x)$ is a neighbourhood base at x. Another neighbourhood base at x is the set $\{\{x\}\}$.

Definition 15. A topological space X is *first-countable* if for every point $x \in X$, there exists a countable neighbourhood base at x.

Example 16. Consider the standard topology on $\mathbb{R}$ and fix $x \in \mathbb{R}$. Define

$$\mathscr{N}(x) = \left\{ B\left(x; \frac{1}{n}\right) : n \in \mathbb{N} \right\},$$

where $B(x;\varepsilon) = \{y \in \mathbb{R} : |y-x| < \varepsilon\}$. By the definition of $\mathscr{N}(x)$, every $V \in \mathscr{N}(x)$ contains x. Now, for every open set U containing x, there exists some $\varepsilon > 0$ such that $B(x;\varepsilon) \subseteq U$. By choosing $N \in \mathbb{N}$ such that $N > 1/\varepsilon$, we guarantee that the inclusion $B(x;1/N) \subseteq B(x;\varepsilon) \subseteq U$ holds. Hence $\mathscr{N}(x)$ is a neighbourhood base for x. Therefore $\mathbb{R}$ is a first-countable space.

Definition 17. A *topological vector space* is a vector space over a field such that the maps of addition and scalar multiplication are continuous.

Definition 18. Let V be a vector space over a field $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. A *seminorm* is a map $p : V \to [0,\infty)$ which satisfies the following properties:

(i) $p(x+y) \leq p(x) + p(y)$ for every $x, y \in V$;

(ii) $p(cx) = |c|p(x)$ for every $x \in V, c \in \mathbb{F}$.

Suppose that $\mathscr{P}$ is a family of seminorms. Then we define the topology on V to be that where the sets

$$U_{x,p,\varepsilon} = \{y \in V : p(y-x) < \varepsilon\} \tag{2.5}$$

are a subbasis for the topology given $p \in \mathscr{P}$ and $\varepsilon > 0$. Any vector space with this topology is a topological vector space.

Example 19. Let $S : V \times V \to V$ be defined by $S(x,y) = x+y$. Fix $(x_0, y_0) \in V \times V$ and let W be a neighbourhood of $S(x_0,y_0) = x_0 + y_0$. There exists some set $U_{x_0+y_0,p,\varepsilon}$ such that $U_{x_0+y_0,p,\varepsilon} \subseteq W$. Observe that $U_{x_0,p,\varepsilon/2} \times U_{y_0,p,\varepsilon/2}$ is an open set in $V \times V$. If $(x,y) \in U_{x_0,p,\varepsilon/2} \times U_{y_0,p,\varepsilon/2}$, we have

$$p((x+y) - (x_0+y_0)) \leq p(x-x_0) + p(y-y_0) < \varepsilon.$$

That is to say

$$S(U_{x_0,p,\varepsilon/2} \times U_{y_0,p,\varepsilon/2}) \subseteq U_{x_0+y_0,p,\varepsilon} \subset W.$$

To summarise, for every neighbourhood W of $S(x_0,y_0)$, there exists a neighbourhood U of (x_0,y_0) such that $S(U) \subseteq W$. So S is continuous at (x_0,y_0). As $(x_0,y_0) \in V \times V$ was arbitrary, we deduce that S is continuous on $V \times V$.

Now, suppose that V is a vector space over $\mathbb{F}$ and let $T : V \to V$ be defined by $T(v) = cv$, where $c \in \mathbb{F}$. Consider the case where $c \neq 0$. Fix $x_0 \in V$ and let $W \subseteq V$ be an open set such that $cx_0 \in W$. Since W is an open set, there exists some open set $U_{cx_0,p,\varepsilon}$ such that $U_{cx_0,p,\varepsilon} \subseteq W$. Observe that $x_0 \in U_{x_0,p,\varepsilon/|c|}$. Let $x \in U_{x_0,p,\varepsilon/|c|}$, we deduce that

$$p(T(x) - T(x_0)) = p(cx - cx_0) \leq |c|p(x - x_0) < \varepsilon.$$

So $T(U_{x_0,p,\varepsilon/|c|}) \subseteq U_{cx_0,p,\varepsilon} \subseteq W$. If $c = 0$, then T is a constant map. Hence T is continuous and V is a topological vector space.

The concept of 'completeness' extends to topological vector spaces.

Definition 20. A net $\{x_a\}$ in a topological vector space V is called a *Cauchy net* if the net $\bar{x} : A \times A \to V$ defined by $\bar{x}(a,b) = x_a - x_b$ converges to 0, where $(a_1,b_1) \preceq (a_2,b_2)$ if and only if $a_1 \preceq a_2$ and $b_1 \preceq b_2$.

We now prove a result which reduces the problem of determining if all Cauchy nets converge in a space to checking if all Cauchy sequences converge in a space.

Theorem 21. *Let V be a first-countable topological vector space in which all Cauchy sequences in V converge in V. Then every Cauchy net in V converges in V.*

Proof. Let $\{x_a - x_b\}$ be a Cauchy net in V. Let $\{N_{0,j} : j \in \mathbb{N}\}$ be a countable neighbourhood base for the point 0, where $N_{0,j+1} \subseteq N_{0,j}$ for every $j \in \mathbb{N}$. Then for every $N_{0,n}$, there exists $(a',b') \in A \times A$ such that $(\hat{a},\hat{b}) \preceq (a,b)$ implies $x_a - x_b \in N_{0,n}$. Choose $k(n) \in A$ such that $a' \preceq k(n)$ and $b' \preceq k(n)$, which is possible as A is a directed set. Since x is a Cauchy net, for every $a,b \succeq k(n)$ the following holds:

$$x_a - x_b \in N_{0,n}. \tag{2.6}$$

By the way the neighbourhood base for 0 has been constructed, the inclusion $N_{0,m} \subseteq N_{0,n}$ holds for every $n < m$. Moreover, the previous discussion showed that we can define a sequence $k : \mathbb{N} \to A$ such that $k(n) \preceq k(m)$ for $n < m$ and if $a,b \succeq k(n)$ then (2.6) holds. From here we see that if $n,m \geq M$, then

$$x_{k(n)} - x_{k(m)} \in N_{0,M}. \tag{2.7}$$

So the sequence that we have constructed, $\{x_{k(n)}\}$, is a Cauchy sequence in V. Therefore $\{x_{k(n)}\}$ converges to some value $x \in V$. Now, let U be an open set containing x. Since the maps $(x_0,y_0) \mapsto x_0 + y_0$ and $x_0 \mapsto (x_0,y_0)$ are continuous, it follows that the translation map $\tau_x : V \to V$, where $\tau_x(y) = x + y$, is continuous. Since $\tau_x^{-1} = \tau_{-x}$, the map τ_x is a homeomorphism. This implies that $\tau_{-x}(U) = U - x$ is an open set which contains 0.

We know that the addition map is continuous. A consequence of this is that the set $\{(a,b) \in V \times V : a+b \in U - x\}$ is open. So there are open sets in V containing 0, say W_1, W_2 such that

$$W_1 \times W_2 \in \{(a,b) \in V \times V : a+b \in U - x\}.$$

This implies that $W_1 + W_2 \subseteq U - x$. As W_1 and W_2 are open sets which contain 0, there exists $M \in \mathbb{N}$ such that $N_{0,M} \subseteq W_1 \cap W_2$. Since $N_{0,M} + x$ is an open set containing x and $\{x_{k(n)}\}$ converges to x, there exists $M' \in \mathbb{N}$ such that $n \geq M'$ implies $x_{k(n)} \in N_{0,M} + x$. This in turn gives us $x_{k(n)} - x \in N_{0,M}$ for $n \geq k(M')$. Choose $R = \max\{M, M'\}$. For $a \succeq k(R)$,it is evident from (2.6) and (2.7) that

$$x_a = (x_a - x_{k(R)}) + (x_{k(R)} - x) + x \in N_{0,M} + N_{0,M} + x.$$

Since the inclusion $N_{0,M} + N_{0,M} + x \subseteq W_1 + W_2 + x \subset U$ holds, it follows $a \succeq k(R)$ then $x_a \in U$. Recall that U was an arbitrary set containing x, so the net $\{x_a\}$ converges to x. This completes the proof. □

The previous result was essential due to the fact that a topological vector space is complete when all Cauchy nets converge. In Chapter 5, a first-countable topological vector space will be utilised and certain continuous functions will be defined which are related to these spaces. When considering if a sequence is continuous on a first-countable topological vector space, it is sufficient to determine if the sequential characterisation of continuity is satisfied, as opposed to using any arbitrary net.

Suppose that $\mathscr{P} = \{p_m : m \in \mathbb{N}\}$, that is to say, $\mathscr{P}$ is a countable family of seminorms. The finite intersection of the sets

$$\{x \in V : p_m(x-y) < 1/n\}$$

form a countable neighbourhood base for $y \in V$. If we denote

$$U_{x,m,\varepsilon} = \{y \in V : p_m(x-y) < \varepsilon\} \tag{2.8}$$

then it is sufficient to show that every set of the form

$$\bigcap_{j=1}^{k} U_{x_j,m_j,\varepsilon_j}$$

which contains x also contains a finite intersection of sets of the form $U_{x,m,1/n}$. Choose n_j

such that $1/n_j < \varepsilon_j - p_{m_j}(x - x_j)$. Then for $y \in U_{x,m_j,1/n_j}$, there holds

$$p_{m_j}(y - x_j) \leq p_{m_j}(y - x) + p_{m_j}(x - x_j) < \varepsilon_j.$$

From here, we see that

$$\bigcap_{j=1}^{k} U_{x,m_j,1/n_j} \subseteq \bigcap_{j=1}^{k} U_{x_j,m_j,\varepsilon_j}. \tag{2.9}$$

Hence the finite intersection of sets of the form $U_{x,m_j,1/n_j}$ form a neighbourhood base for the topology of V at x. Therefore, with this topology, V is a first-countable topoogical vector space.

2.3.3 The weak* topology

Let V be a vector space over a field $\mathbb{F}$, where $\mathbb{F}$ is either the real or complex numbers. We define the *dual* of V by $V^* = \mathrm{Hom}(V, \mathbb{F})$, where $\mathrm{Hom}(V, \mathbb{F})$ is the space of all linear maps from V to $\mathbb{F}$. We highlight if V is a Banach space over $\mathbb{F}$, the dual of V is instead defined by $V^* = \mathrm{B}(V, \mathbb{F})$. That is, the space of all bounded linear maps from V to $\mathbb{F}$.

Suppose now that V is a vector space. Consider the space $\hat{V}$ of all maps $\hat{x} : V^* \to \mathbb{F}$, defined by $\hat{x}(f) = f(x)$ for every $f \in V^*$. We note that $\hat{V} \subseteq (V^*)^*$. It is now possible to define the topology of pointwise convergence. The *weak* topology* on V^* is the weakest topology such that all maps in $\hat{V}$ are continuous. We now show that this is the topology of pointwise convergence.

Fix a net $\{f_a\}$ in V^*. Assume $\{f_a\}$ converges to f. Since each $\hat{x}$ is continuous, it is evident that $\hat{x}(f_a) \to \hat{x}(f)$ for every $\hat{x} \in \hat{V}$. Therefore $f_a(x) \to f(x)$ for every $x \in V$. Now, suppose $f_a(x) \to f(x)$ for every $x \in V$. Let U be an open set containing f. By the definition of the topology on V^*, there exists $x_1, \ldots, x_n \in V$ and open sets $U_1, \ldots, U_n \subseteq \mathbb{F}$ such that

$$\bigcap_{i=1}^{n} \hat{x}_i^{-1}(U_i) \subseteq U.$$

Each U_i contains $f(x_i)$. Let a_i be defined such that $a \succeq a_i$ implies $f_a(x_i) \to f(x_i)$ for every i. Choose α such that $\alpha \succeq a_i$ for every i. We have that $a \succeq \alpha$ implies $f_a \in U$. Therefore $f_a \to f$.

Therefore, $f_a \to f$ if and only if $f_a(x) \to f(x)$ for every $x \in V$.

2.3.4 Fréchet spaces

For convenience we define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Fréchet spaces are introduced for the purpose of defining distributions. The Fréchet spaces we examine here will become relevant in Chapter 5. Particularly throughout Section 5.2.

Definition 22. A *Fréchet space* is a complete Hausdorff topological vector space whose topology is generated by countably many seminorms.

By the previous discussion, every Fréchet space is first-countable, hence Theorem 21 applies. This simplifies calculations when considering continuous functions defined on Fréchet spaces. We give an example of a Fréchet space which will be of importance to us.

Let $I \subsetneq \mathbb{R}$ be an open interval. We say a set is σ-compact if it is the countable union of compact sets. Since I is homeomorphic to the σ-compact set $(-\pi/2, \pi/2)$, it follows that that I is σ-compact. If we redefine I such that $I = [a,b)$ or $I = (a,b]$, then I is once again σ-compact. We retain the assumption that $I \subsetneq \mathbb{R}$. Recall that X° and $\overline{X}$ denote the interior and the closure of the set X respectively. As I is a locally compact Hausdorff space, there exists a sequence of compact sets $\{F_m\}$ which are subsets of I such that $F_m \subseteq F_{m+1}^\circ$ for every $m \in \mathbb{N}$ and

$$I = \bigcup_{m=1}^{\infty} F_m^\circ. \tag{2.10}$$

Consider the space $C_c^\infty(I;\mathbb{C})$ of all functions $f : I \to \mathbb{C}$ which have compact support and are infinitely differentiable on I. Note that for every $f \in C_c^\infty(I;\mathbb{C})$, f and its derivatives of all orders are bounded. If $F \subseteq I$, we denote the space $C_c^\infty(I,F;\mathbb{C})$ to be all functions $f \in C_c^\infty(I;\mathbb{C})$ where $\operatorname{supp}(f) \subseteq F$. Now, let F be a compact set. Consider the function $||\cdot||_{(n,m)} : C_c^\infty(I,F;\mathbb{C}) \to [0,\infty)$ defined by

$$||f||_{(n,m)} = \sup_{x \in F \cap F_m} |f^{(n)}(x)|. \tag{2.11}$$

For every $(n,m) \in \mathbb{N}_0 \times \mathbb{N}$, the function $||\cdot||_{(n,m)}$ is a norm. So it is also a seminorm. We let $\mathscr{P} = \{||\cdot||_{(n,m)} : n \in \mathbb{N}_0, m \in \mathbb{N}\}$ and define the topology on $C_c^\infty(I,F;\mathbb{C})$ to be the one generated by finite intersections of sets of the form $U_{f,(n,m),\varepsilon}$, where

$$U_{f,(n,m),\varepsilon} = \{g \in C_c^\infty(I,F;\mathbb{C}) : ||f-g||_{(n,m)} < \varepsilon\}.$$

We guarantee the topology is not only generated by countably many seminorms, but it is also Hausdorff. We see this by letting $f, g \in C_c^\infty(I,F;\mathbb{C})$, where $f \neq g$. Then there exists $m \in \mathbb{N}$ such that

$$\operatorname{supp}(f) \cup \operatorname{supp}(g) \subseteq F_m \cap F.$$

By letting

$$\varepsilon = \frac{||f-g||_{(0,m)}}{2} > 0,$$

it is apparent that the open sets $U_{f,(0,m),\varepsilon}$ and $U_{g,(0,m),\varepsilon}$ are disjoint. We now show that for every compact $F \subseteq I$, the collection $C_c^\infty(I,F;\mathbb{C})$ is a Fréchet space.

Theorem 23. *Let $F \subseteq I$ be a compact set. Then the space $C_c^\infty(I,F;\mathbb{C})$ is complete.*

Proof. If $\inf(F) = \operatorname{supp}(F)$, then $C_c^\infty(I,F;\mathbb{C})$ consists of the zero function which is complete. So we may assume $\inf(F) \neq \operatorname{supp}(F)$. Let $\{f_j\}$ be a sequence in $C_c^\infty(I,F;\mathbb{C})$ which is Cauchy. So for every $\varepsilon > 0$ and $(n,m) \in \mathbb{N}_0 \times \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $i, j \geq N$ implies

$$f_i - f_j \in U_{0,(n,m),\varepsilon}.$$

This is equivalent to saying

$$\sup_{x \in F \cap F_m} ||f_i^{(n)}(x) - f_j^{(n)}(x)|| = ||f_i - f_j||_{(n,m)} < \varepsilon. \tag{2.12}$$

This shows that the sequence $\{f_j^{(n)}\}$ is uniformly Cauchy on F_m, hence converges uniformly to some continuous function $g_{(n,m)}$ on F_m. Let $g_n : F \to \mathbb{R}$ be the function defined by $g_n(x) = g_{(n,m)}(x)$ when $x \in F \cap F_m$. The sequence $\{f_j^{(n)}\}$ converges to g_n uniformly on F. For every $n \in \mathbb{N}_0$, since $\operatorname{supp}(f_j^{(n)}) \subseteq F$ for every j, we see that $\operatorname{supp}(g_n) \subseteq F$. We will show $\{f_j\}$ converges to some $g \in C_c^\infty(I,F;\mathbb{C})$, where $g(x) = g_0(x)$ for $x \in F$ and $g(x) = 0$ whenever $x \in I \backslash F$. We define $s_i = \inf(F)$ and $s_s = \sup(F)$. Fix $\varepsilon' > 0$. Choose $N \in \mathbb{N}$ such that $j \geq N$ implies for every $x \in F$,

$$|f_j^{(n)}(x) - g_n(x)| < \frac{\varepsilon'}{3}, \qquad |f_j^{(n-1)}(x) - g_{n-1}(x)| < \frac{\varepsilon'}{2(s_s - s_i)}.$$

From here the following holds for every $x \in F$,

$$\begin{aligned} &\left| g_{n-1}(x) - g_{n-1}(s_i) - \int_{s_i}^x g_n(t)\,\mathrm{d}t \right| \\ &\qquad \leq |g_{n-1}(x) - f_j^{(n-1)}(x)| + |f_j^{(n-1)}(s_i) - g_{n-1}(s_i)| + \left| \int_{s_i}^x f_j^{(n)}(t) - g_n(t)\,\mathrm{d}t \right| < \varepsilon'. \end{aligned}$$

Since g_{n-1} is the integral of a continuous function, g_{n-1} is continuously differentiable on F. A simple induction proof from here shows that $g \in C_c^\infty(I,F;\mathbb{C})$. Therefore, the sequence $\{f_j\}$ converges to g in $C_c^\infty(I,F;\mathbb{C})$. □

2.4 Distributions

We start by defining what it means for a sequence to converge in $C_c^\infty(I,U;\mathbb{R})$ where $U \subseteq I$ is an open set. A sequence $\{f_j\} \subseteq C_c^\infty(I,U;\mathbb{R})$ converges in $C_c^\infty(I,U;\mathbb{R})$ if there exists a compact set $F \subseteq I$ such that for every $j \in \mathbb{N}$, the supports of f_j are subsets of F and $f_j \to f$ in $C_c^\infty(I,F;\mathbb{R})$. Using this, it is possible to define continuity of a linear functional on $C_c^\infty(I,U;\mathbb{C})$. Let $L : C_c^\infty(I,U;\mathbb{C}) \to \mathbb{C}$ be a linear map. We call the map L *continuous* if for every compact $F \subseteq U$, the map $L|_{C_c^\infty(I,F;\mathbb{R})}$ is continuous. An (I,U)-*distribution* is a continuous linear functional on $C_c^\infty(I,U;\mathbb{C})$. We denote the space of (I,U)-distributions by $\mathscr{D}'(I,U;\mathbb{C})$. We simply refer to these functionals as distributions when the underlying sets are clear. There will be times when L may denote either a function or a distribution. When L denotes a function, we will denote the function evaluated at the real number x as $L(x)$. When L denotes a distribution, we will denote the distribution L evaluated at g by $L[g]$.

2.4.1 Linear maps on $L^1_{\text{loc}}(I,U)$

Recall the space $L^1_{\text{loc}}(I,U)$ of locally integrable functions denotes the space of all Lesbesgue measurable functions defined on I which are integrable on every bounded subset F of U. We note that every $f \in L^1_{\text{loc}}(I,U)$ defines a distribution. More specifically, the following holds for every $g \in C_c^\infty(I,U;\mathbb{C})$

$$\int_I |f(t)g(t)|\,\mathrm{d}t \leq ||g||_\infty \int_{\text{supp}(g)} |f(t)|\,\mathrm{d}t < \infty.$$

Therefore, if f is a locally integrable, then we define $f \in \mathscr{D}'(I,U;\mathbb{C})$ by

$$f[g] = \int_I f(t)g(t)\,\mathrm{d}t.$$

Now, assume $U,V \subseteq I$ are open and X is a vector space such that $X \subseteq L^1_{\text{loc}}(I,U)$. Consider two linear maps $L : X \to L^1_{\text{loc}}(I,V)$ and $\tilde{L} : C_c^\infty(I,V;\mathbb{C}) \to C_c^\infty(I,U;\mathbb{C})$, where

$$\int_I (Lf)(t)g(t)\,\mathrm{d}t = \int_I f(t)(\tilde{L}g)(t)\,\mathrm{d}t.$$

Then we may extend the map to distributions. That is, $L : D'(I,U;\mathbb{C}) \to D'(I,V;\mathbb{C})$, where

$$(Lf)[g] = f[\tilde{L}(g)].$$

2.4.2 Fourier convolution of distributions and bump functions

A similar theory of distributions is known for the real line. The Fourier convolution of a distribution with a smooth function with compact support is defined using a similar method to that when extending a linear map $L : L^1_{\text{loc}}(\mathbb{R},U) \to L^1_{\text{loc}}(\mathbb{R},V)$ to distributions. We define a *bump function* to be the set of all smooth functions on $\mathbb{R}$ which have compact support. In this subsection, when we say distribution we mean a continuous linear functional on $C_c^\infty(\mathbb{R};\mathbb{C})$, where continuity is defined in a similar way to that which was introduced previously. A classic example of such a function is given by

$$f(x) = \begin{cases} \mathrm{e}^{-\frac{1}{1-x^2}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

We wish to define the Fourier convolution of a distribution and a function as a function defined on some open set. Before we define this, we introduce the set

$$V_g = \{x \in \mathbb{R} : x - y \in U \text{ for every } y \in \operatorname{supp}(g)\}. \tag{2.13}$$

We will eventually define the convolution involving distributions on V_g

Theorem 24. *Let $g \in C_c^\infty(\mathbb{R}, U; \mathbb{C})$, where $U \subseteq \mathbb{R}$ is an open set. Then V_g is an open set.*

Proof. Fix $x \in V_g$. As $x - y \in U$ for every $y \in \operatorname{supp}(g)$, we may choose for every $y \in \operatorname{supp}(g)$ a number $\varepsilon_y > 0$ such that $B(x-y; \varepsilon_y) \subseteq U$. Throughout this proof, we will frequently use the following facts

$$B(x+y; \varepsilon) = B(x; \varepsilon) + y, \qquad B(-x; \varepsilon) = -B(x; \varepsilon).$$

Observe that since $B(x-y; \varepsilon_y) \subseteq U$ for every $y \in \operatorname{supp}(g)$, it is evident that $B(y; \varepsilon_y) \subseteq -U + x$ for every $y \in \operatorname{supp}(g)$. A trivial consequence of this is

$$B\left(y; \frac{\varepsilon_y}{2}\right) \subseteq -U + x.$$

Since $\{B(y; \varepsilon_y/2)\}$ is an open cover of $\operatorname{supp}(g)$, there exists $\{y_1, \dots, y_n\}$ such that

$$\operatorname{supp}(g) \subseteq \bigcup_{i=1}^{n} B\left(y_i; \frac{\varepsilon_{y_i}}{2}\right) \subseteq -U + x.$$

Now, let $\varepsilon = \min\{\varepsilon_{y_i}/2 : i = 1, \dots, n\}$. Fix $z \in \operatorname{supp}(g)$ and let $r \in B(z; \varepsilon)$. As $z \in \operatorname{supp}(g)$, there exists i such that $z \in B(y_i; \varepsilon_{y_i}/2)$. We have

$$|y_i - r| \leq |y_i - z| + |z - r| < \frac{\varepsilon_{y_i}}{2} + \varepsilon < \varepsilon_{y_i}.$$

Recall that $r \in B(z; \varepsilon)$ was arbitrary. A consequence of this is that $B(z; \varepsilon) \subseteq B(y_i; \varepsilon_{y_i})$. Since $z \in \operatorname{supp}(g)$ was arbitrary, it can be seen that for every $z \in \operatorname{supp}(g)$:

$$B(z; \varepsilon) \subseteq \bigcup_{i=1}^{n} B(y_i; \varepsilon_{y_i}) \subseteq -U + x.$$

This implies for every $z \in \operatorname{supp}(g)$,

$$B(x; \varepsilon) - z \subset U. \tag{2.14}$$

Thus $B(x; \varepsilon) \subseteq V_g$. This completes the proof. $\square$

Due to Theorem 24 it is possible to define the convolution of a distribution on the set of bump functions and a bump function in a natural way. Observe first if $f \in L^1_{\text{loc}}(\mathbb{R}, U)$

and $g \in C_c^\infty(\mathbb{R}, U; \mathbb{C})$, we have that the Fourier convolution exists everywhere on $\mathbb{R}$. More specifically, if $x \in V_g$, then

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{d}y. \tag{2.15}$$

By making a substitution, the convolution formula is equivalent to

$$\int_{\mathbb{R}} f(y)g(x+y)\,\mathrm{d}y = \int_{\mathbb{R}} f(y)(g \circ \tau_x)(y)\,\mathrm{d}y,$$

where τ_x is the translation map $\tau_x(y) = x + y$. From here, we see that if $f \in D'(\mathbb{R}, U; \mathbb{C})$ and $g \in C_c^\infty(\mathbb{R}, U; \mathbb{C})$, we may define the convolution of f and g as follows

$$(f * g)(x) = f[g \circ \tau_x],$$

where $f * g$ is defined on V_g. A similar process will be used to define convolutions of distributions and smooth functions with compact support.

As mentioned previously, the Fourier transform maps the space of Schwartz functions to itself. As such we may define the Fourier transform on *tempered distributions*, which are continuous linear functionals on $\mathscr{S}$. If f is a tempered distribution, we define $\mathscr{F}\{f\}$ by

$$\mathscr{F}\{f\}[g] = f[\mathscr{F}\{g\}].$$

For the sake of clarity, we highlight the importance that a tempered distribution is defined on a space P, where $\mathscr{F}(P) \subseteq P$. A classical example of a tempered distribution is the Dirac distribution δ_a, where $\delta_a[f] = f(a)$. Observe that if f is a distribution with compact support, then we may identity $\mathscr{F}\{f\}$ with a C^∞ function g, where $g(\omega) = f[x \mapsto \mathrm{e}^{-2\pi\mathrm{i}\omega x}]$. This gives us

$$\mathscr{F}\{\delta_a\}(\omega) = \delta_a[x \mapsto \mathrm{e}^{-2\pi\mathrm{i}\omega x}] = \mathrm{e}^{-2\pi\mathrm{i}\omega a}.$$

We note that this is the typical expression one derives for the Fourier transform of the Dirac delta function when the following property is used

$$\int_{\mathbb{R}^n} f(t)\delta_a(t)\,\mathrm{d}t = f(a).$$

2.5 Measure theory prerequisites

2.5.1 Nonmeasurable sets

In this subsection, the discussion regarding the Axiom of Choice becomes relevant. It is a well-known fact that if $\mathscr{M}$ denotes the Lebesgue measurable sets on $\mathbb{R}$, then $\mathscr{B}$, which denotes the Borel σ-algebra on $\mathbb{R}$, is a proper subset of $\mathscr{M}$. We will now prove that $\mathscr{M}$ is a proper subset of the power set of $\mathbb{R}$.

Theorem 25. *Let λ be the Lebesgue measure on $\mathbb{R}$ and let $E \in \mathscr{M}$, where $\lambda(E) > 0$. Then there exists $F \subseteq E$ such that $F \notin \mathscr{M}$.*

Proof. We will establish the contrapositive, that is, if every subset of a Lebesgue measurable set E is measurable, then $\lambda(E) = 0$. The proof will be similar to that given in Rudin [84]. Consider the quotient group $\mathbb{R}/\mathbb{Q}$ and let f be a choice function on $\mathbb{R}$. Let $S = f(\mathbb{R}/\mathbb{Q})$, so $f(x+\mathbb{Q}) \in x+\mathbb{Q}$ for every $x \in \mathbb{R}$. From elementary group theory, we have that the cosets of $\mathbb{Q}$ in $\mathbb{R}$ form a partition of $\mathbb{R}$, so S contains one and only one element of each coset of $\mathbb{Q}$ in $\mathbb{R}$. We now prove two properties of S:

(P1) If $p, q \in \mathbb{Q}$ are distinct, then $(p+S) \cap (q+S) = \emptyset$;

(P2) For every $x \in \mathbb{R}$, $x \in p+S$ for some $p \in \mathbb{Q}$.

Let $x \in (p+S) \cap (q+S)$, where $p \neq q$. There exists $y, z \in S$ with $y \neq z$ such that $a = p+y = q+z$. This implies that $p-q = z-y \in \mathbb{Q}$. Recall that S contains one and only one element from each coset of $\mathbb{Q}$ in $\mathbb{R}$, and since $y-z \in \mathbb{Q}$, it holds that y and z belong to the same coset, therefore $y = z$. Hence $(p+S) \cap (q+S) = \emptyset$ and (P1) is proven.

Now, fix $x \in \mathbb{R}$. There exists a coset of $\mathbb{Q}$ in $\mathbb{R}$ which x lies in, say, $x \in y+\mathbb{Q}$. Let $z = f(y+\mathbb{Q}) \in y+\mathbb{Q}$. As x and z lie in the same coset, we have $x-z \in \mathbb{Q}$. We now see that if $p = x-z$, then $x = (x-z)+z \in p+S$. Hence (P2) is true.

Let $r \in \mathbb{Q}$ and examine the subset $F_r = E \cap (r+S)$ of E. Let $K \subseteq F_r$ be compact and define the set

$$G = \bigcup_{p \in \mathbb{Q} \cap [0,1]} (p+K).$$

As G is bounded, G is a set of finite Lebesgue measure. Observe that $K \subseteq F_r \subseteq r+S$. This implies that $p+K \subseteq (p+r)+S$. By (P1),

$$(p+K) \cap (q+K) \subseteq (p+r+S) \cap (q+r+S) = \emptyset$$

for $p \neq q$. Hence G is given by

$$\lambda(G) = \sum_{r \in \mathbb{Q} \cap [0,1]} \lambda(K+r) = \sum_{r \in \mathbb{Q} \cap [0,1]} \lambda(K),$$

as λ is translation invariant. Using the fact that G has finite measure, the only possible value for $\lambda(K)$ is 0. As K was an arbitrary compact subset of F_r, this gives us $\lambda(F_r) = 0$. Observe by (P2) that

$$E = E \cap \left(\bigcup_{r \in \mathbb{Q}} r+S \right) = \bigcup_{r \in \mathbb{Q}} F_r.$$

Therefore $\lambda(E) \leq \sum_{r \in \mathbb{Q}} \lambda(F_r) = 0$. This completes the proof. $\square$

2.5.2 Complex measures

Definition 26. A *complex measure* on a measurable space (X,Ω) is a map $\mu : \Omega \to \mathbb{C}$ satisfying the following properties:

- $\mu(\emptyset) = 0$;
- For every $\{E_n\}_{n=1}^{\infty} \subseteq \Omega$, where $E_i \cap E_j = \emptyset$ for $i \neq j$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Recall if μ is a signed measure, then by the Jordan Decomposition Theorem,

$$\mu = \mu^{+} - \mu^{-},$$

where μ^{+} and μ^{-} are mutually singular, positive measures. If $f \in L^1(\mu) = L^1(\mu^{+}) \cap L^1(\mu^{-})$, then we define the integral of f with respect to μ by

$$\int_X f \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu^{+} - \int_X f \,\mathrm{d}\mu^{-}.$$

Now, suppose μ is a complex measure, that is, $\mu = \mu_r + \mathrm{i}\mu_i$, where μ_r and μ_i are real-valued measures. If $f \in L^1(\mu) = L^1(\mu_r) \cap L^1(\mu_i)$, we define the integral of f with respect to μ by

$$\int_X f \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu_r + \mathrm{i} \int_X f \,\mathrm{d}\mu_i.$$

We now show an important result which will be crucial throughout our work.

Lemma 27. *Let μ, ν be measures on X, where $\nu \ll \mu$ and μ is a positive measure. Let $f = \mathrm{d}\nu/\mathrm{d}\mu$. For every $g \in L^1(\nu)$ and $fg \in L^1(\mu)$, the following equality holds*

$$\int_X g \,\mathrm{d}\nu = \int_X gf \,\mathrm{d}\mu. \tag{2.16}$$

Proof. Observe for every $E \in \Omega$, the measure ν can be represented as an integral as follows

$$\int_E 1 \,\mathrm{d}\nu = \nu(E) = \int_E f \,\mathrm{d}\mu.$$

A consequence of this is that the equation in (2.16) holds whenever g is a simple function. Now, suppose $g \in L^1(\nu)$ and let s_n be a sequence of functions which converge pointwise to g almost everywhere on X and $|s_n| \leq |g|$. By the Dominated Convergence Theorem, the integral of g with respect to ν can be written as a limit. Namely

$$\int_X g \; d\nu = \int_X g \,\mathrm{d}\nu_r^{+} - \int_X g \,\mathrm{d}\nu_r^{-} + \mathrm{i} \int_X g \,\mathrm{d}\nu_i^{+} - \mathrm{i} \int_X g \,\mathrm{d}\nu_i^{-}$$

$$\begin{aligned}
&= \lim_{n\to\infty} \left(\int_X s_n \,\mathrm{d}\nu_r^+ - \int_X s_n \,\mathrm{d}\nu_r^- + \mathrm{i} \int_X s_n \,\mathrm{d}\nu_i^+ - \mathrm{i} \int_X s_n \,\mathrm{d}\nu_i^- \right) \\
&= \lim_{n\to\infty} \int_X s_n \,\mathrm{d}\nu \\
&= \lim_{n\to\infty} \int_X s_n f \,\mathrm{d}\mu.
\end{aligned}$$

As $|s_n f| \leq |gf|$ almost everywhere on X, another application of the Dominated Convergence Theorem gives us

$$\int_X g \,\mathrm{d}\nu = \lim_{n\to\infty} \int_X s_n \,\mathrm{d}\nu = \lim_{n\to\infty} \int_X s_n f \,\mathrm{d}\mu = \int_X gf \,\mathrm{d}\mu.$$

This completes the proof. □

2.5.3 Total variation of measures

We introduce another concept which will help us define a space in which a class of integral transforms is defined on. If μ is a complex measure, the *total variation* of $\mu : \Omega \to [0,\infty)$ is the measure $|\mu|$, where

$$|\mu|(E) = \sup \left\{ \sum_{n=1}^{\infty} |\mu(E_n)| : \{E_n\}_{n=1}^{\infty} \text{ are pairwise disjoint and } \bigcup_{n=1}^{\infty} E_n = E \right\}.$$

It is easy to see that this expression guarantees that the total variation of a measure exists for every complex measure. However, it can be difficult to compute the specific value of the total variation of a measure from this formula. We introduce other functions on Ω and show these are equivalent. Define the maps $\tilde{\mu}$, $\overline{\mu}$ on X in the following way:

- $\tilde{\mu}(E) = \sup \left\{ \left| \int_E f \,\mathrm{d}\mu \right| : |f| \leq 1 \right\}$;
- If $d\mu = f \,\mathrm{d}\nu$ where ν is a positive measure, then $\mathrm{d}\overline{\mu} = |f| \,\mathrm{d}\nu$.

Observe the set function $\overline{\mu}$ is well defined. Suppose $f_1 \,\mathrm{d}\nu_1 = f_2 \,\mathrm{d}\nu_2$ and let $\nu = \nu_1 + \nu_2$. By Lemma 27, the following equation holds

$$f_1 \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu} = f_2 \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu}. \tag{2.17}$$

Using the fact that the Radon-Nikodym derivative of ν_j with respect to ν is nonnegative, we deduce that

$$|f_1| \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu} = \left| f_1 \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu} \right| = \left| f_2 \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu} \right| = |f_2| \frac{\mathrm{d}\nu_2}{\mathrm{d}\nu}.$$

This gives us $|f_1|\,\mathrm{d}\nu_1 = |f_2|\,\mathrm{d}\nu_2$. To guarantee the existence of $\overline{\mu}$, observe that $\mu \ll |\mu|$. Also, an important property of this function is

$$|\mu(E)| = \left| \int_E f\,\mathrm{d}|\mu| \right| \leq \int_E |f|\,\mathrm{d}|\mu| = \overline{\mu}(E) \tag{2.18}$$

for every $E \in \Omega$. This implies $\mu \ll \overline{\mu}$. A consequence of the Radon-Nikodym theorem is

$$f\,\mathrm{d}\nu = \mathrm{d}\mu = \frac{\mathrm{d}\mu}{\mathrm{d}\overline{\mu}}\,\mathrm{d}\overline{\mu} = \frac{\mathrm{d}\mu}{\mathrm{d}\overline{\mu}}|f|\,\mathrm{d}\nu.$$

From here, we see that $|\mathrm{d}\mu/\mathrm{d}\overline{\mu}| = 1$ almost everywhere with respect to $\overline{\mu}$. A similar calculation shows that $|\mathrm{d}\mu/\mathrm{d}|\mu|| = 1$ almost everywhere with respect to $|\mu|$.

Theorem 28. *The functions $|\mu|$, $\tilde{\mu}$ and $\overline{\mu}$ are equivalent.*

Proof. We will show $|\mu| \leq \tilde{\mu} \leq \overline{\mu} \leq \tilde{\mu} \leq |\mu|$. Let $\{E_n\}_{n=1}^{\infty}$ be a partition of E. We define the function f almost everywhere on X by

$$f(x) = \sum_{n=1}^{\infty} \frac{|\mu(E_n)|}{\mu(E_n)} \chi_{E_n}(x).$$

For the sake of clarity we highlight some abuse of notation. If one of the E_n in the partition of E is a set of measure 0, then we define $f(x) = 1$ for every $x \in E_n$. Fix $x \in E$, observe that $x \in E_n$ for some $n \in \mathbb{N}$. Furthermore, x belongs to only one set in the collection $\{E_n\}_{n=1}^{\infty}$. If $x \in E_m$ where $\mu(E_m) = 0$, then $f(x)$ was defined to be 1, so $|f(x)| = 1$. Suppose $x \in E_m$ given $\mu(E_m) \neq 0$. A straightforward calculation gives us

$$\begin{aligned} |f(x)| &= \left| \sum_{n=1}^{\infty} \frac{|\mu(E_n)|}{\mu(E_n)} \chi_{E_n}(x) \right| \\ &= \left| \frac{|\mu(E_m)|}{\mu(E_m)} \chi_{E_m}(x) \right| \\ &= 1. \end{aligned}$$

Therefore $|f| \leq 1$, where 1 denotes the constant function taking every value to the number 1. A straightforward computation gives us

$$\left| \int_E f\,\mathrm{d}\mu \right| = \left| \sum_{n=1}^{\infty} \int_E \frac{|\mu(E_n)|}{\mu(E_n)} \chi_{E_n}\,\mathrm{d}\mu \right| = \sum_{n=1}^{\infty} |\mu(E_n)|.$$

From here, we see that $|\mu|(E) \leq \tilde{\mu}(E)$ for every $E \in \Omega$. Observe that for every $|f| \leq 1$ almost everywhere with respect to μ, the following inequality holds

$$\left| \int_E f\,\mathrm{d}\mu \right| = \left| \int_E f \frac{\mathrm{d}\mu}{\mathrm{d}\overline{\mu}}\,\mathrm{d}\overline{\mu} \right| \leq \int_E \left| \frac{\mathrm{d}\mu}{\mathrm{d}\overline{\mu}} \right| \mathrm{d}\overline{\mu} = \overline{\mu}(E).$$

This implies $\tilde{\mu}(E) \le \overline{\mu}(E)$. Now, let $f = \overline{\mathrm{d}\mu/\mathrm{d}\overline{\mu}}$, that is, the complex conjugate of g where $\mathrm{d}\mu = g\,\mathrm{d}\overline{\mu}$. This gives us

$$\left|\int_E f\,\mathrm{d}\mu\right| = \left|\int_E \frac{\mathrm{d}\mu}{\mathrm{d}\overline{\mu}}\overline{\frac{\mathrm{d}\mu}{\mathrm{d}\overline{\mu}}}\,\mathrm{d}\overline{\mu}\right| = \left|\int_E \left|\frac{\mathrm{d}\mu}{\mathrm{d}\overline{\mu}}\right|^2 \mathrm{d}\overline{\mu}\right| = \overline{\mu}(E).$$

Therefore $\overline{\mu}(E) \le \tilde{\mu}(E)$. Finally, the following holds for every $|f| \le 1$ on E

$$\left|\int_E f\,\mathrm{d}\mu\right| = \left|\int_E f\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\,\mathrm{d}|\mu|\right| \le \int_E \left|f\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}\right| \mathrm{d}|\mu| \le |\mu|(E).$$

Hence $\tilde{\mu} \le |\mu|$ on Ω. □

Because of this theorem, we will denote $\overline{\mu} = |\mu|$ and will mainly focus on this representation for the measure $|\mu|$. We will now show a property which helps identify functions which are integrable with respect to complex measures.

Proposition 29. *If μ is a complex measure, then $L^1(\mu) = L^1(|\mu|)$.*

Proof. Fix $f \in L^1(\mu)$. It is apparent from the definition of the space $L^1(\mu)$ that

$$\int_X |f|\,\mathrm{d}\mu_r^+ + \int_X |f|\,\mathrm{d}\mu_r^- + \int_X |f|\,\mathrm{d}\mu_i^+ + \int_X |f|\,\mathrm{d}\mu_i^- < \infty. \tag{2.19}$$

Observe the fact that $|\mu| \le |\mu_r| + |\mu_i|$. A consequence of this and the above inequality is that $f \in L^1(|\mu|)$. So $L^1(\mu) \subseteq L^1(|\mu|)$. Now, since $\mu \ll |\mu|$, it follows that $\mu_r \ll |\mu|$ and $\mu_i \ll |\mu|$. This in turn implies $\mu_r^+, \mu_r^-, \mu_i^+, \mu_i^- \ll |\mu|$, which shows that $L^1(|\mu|) \subseteq L^1(\mu)$. This completes the proof. □

2.5.4 The complex Borel measures

The space of complex Borel measures on I where $I = [a,b)$ or $I = (a,b]$ is denoted by $M(I)$. We let $C_c(I;\mathbb{C})$ be the space of all complex-valued, continuous functions on I which have compact support. The purpose of this section is to introduce the necessary tools to show that $M(I)$ is a Banach space under an appropriate norm. We start by introducing a theorem about linear maps on normed vector spaces which can be found in Reed & Simon [82]. As a proof does not follow the statement of this theorem in [82], we will present a proof here.

Theorem 30. *Let $T : V \to W$ be a bounded linear map between two real or complex, normed vector spaces, where W is complete. Then T can be extended to a bounded linear map which is defined on the completion of V and maps to W. Moreover, this extension is unique.*

Proof. Let $\overline{V}$ be the completion of V. We will assume V and W are complex vector spaces. Let $x \in \overline{V} \backslash V$, and let $\{x_n\} \subseteq V$ be a sequence such that $x_n \to x$. As T is continuous, we have that $\{T(x_n)\}$ is a Cauchy sequence in W, so $T(x_n) \to y$ for some $y \in W$. Define $T(x) = y$. We will start by showing that T is well defined. That is, if $\{x_n\}$ and $\{x'_n\}$ are two sequences which converge to x, then $\{T(x_n)\}$ and $\{T(x'_n)\}$ converge to the same value. Observe that

$$||T(x_n) - T(x'_n)||_W \le ||T|| \cdot ||x_n - x'_n||_{\overline{V}} \le ||T||(||x_n - x||_{\overline{V}} + ||x - x'_n||_{\overline{V}}).$$

From here, it is straightforward to show that $\{T(x_n)\}$ and $\{T(x'_n)\}$ converge to the same value. Hence this extension is well defined.

We will now show that T is linear. Let $x, y \in \overline{V}$ and let $\alpha, \beta \in \mathbb{F}$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in V such that $x_n \to x$ and $y_n \to y$. The definition of T gives us

$$T(\alpha x_n + \beta y_n) \to T(\alpha x + \beta y) \qquad \text{and} \qquad \alpha T(x_n) + \beta T(y_n) \to \alpha T(x) + \beta T(y).$$

Since $T(\alpha x_n + \beta y_n) = \alpha T(x_n) + \beta T(y_n)$ for every n and by uniqueness of limits we have that T is linear on $\overline{V}$.

We will now show that T is bounded. Fix $\varepsilon > 0$. For $x \in \overline{V}$ such that $||x||_{\overline{V}} \le 1$, it follows that

$$||T(x)||_W = ||T(x) - T(x_n) + T(x_n)||_W \le ||T(x) - T(x_n)||_W + ||T(x_n)||_W,$$

where $\{x_n\}$ is a sequence in V which converges to x. By choosing n to be significantly large, we obtain the following upper bound on the norm of $T(x)$

$$||T(x)||_W < \varepsilon + ||T||(1 + \varepsilon).$$

As ε was an arbitrary positive number, the inequality $||T(x)||_W \le ||T||$ holds for all $||x||_{\overline{V}} \le 1$. So T satisfies the properties of the theorem.

Finally, we will show uniqueness. Let T_1 and T_2 be two extensions of T satisfying the statements of the theorem. For any $\{x_n\}$ in V which converges to $x \in \overline{V}$, the continuity of the maps T_1 and T_2 show that

$$T_1(x) = \lim_{n \to \infty} T_1(x_n) = \lim_{n \to \infty} T_2(x_n) = T_2(x).$$

Hence the map T is unique. □

We introduce the concept of 'Radon measures' for the sake of the next theorem.

Definition 31. A *Radon measure* on I is a Borel measure μ that is finite on all compact sets, which satisfies the following

$$\mu(E) = \inf\{\mu(U) : U \text{ is open and } E \subset U\}$$

for all Borel sets $E \subseteq I$, and

$$\mu(U) = \sup\{\mu(K) : K \text{ is compact and } K \subset U\}$$

for all open sets $U \subseteq I$.

Theorem 32. *(The Riesz Representation Theorem) Let Λ be a positive linear functional on $C_c(I;\mathbb{C})$. That is, $\Lambda(f) \geq 0$ for every $f \geq 0$. Then there is a unique Radon measure μ on I such that for all $f \in C_c(I;\mathbb{C})$, the following formula holds*

$$\Lambda(f) = \int_I f \,\mathrm{d}\mu.$$

The proof for this theorem can be found in Folland [32]. Due to its technical nature, the proof is not presented here.

Several other facts about linear functionals are relevant for our analysis. More specifically, if Λ is a real linear functional on $C_0(I;\mathbb{R})$, then there exist positive linear functionals $\Lambda^{\pm}$ on $C_0(I;\mathbb{R})$ such that $\Lambda = \Lambda^+ - \Lambda^-$. Now, if $\Lambda \in C_0(I;\mathbb{C})$, then $\Lambda(f + \mathrm{i}g) = \Lambda_{\mathbb{R}} f + \mathrm{i}\Lambda_{\mathbb{R}} g$, where $\Lambda_{\mathbb{R}}$ is the restriction of Λ to $C_0(I;\mathbb{R})$. Since $\Lambda_{\mathbb{R}} : C_0(I;\mathbb{R}) \to \mathbb{C}$ is linear over $\mathbb{R}$, then

$$\Lambda_r = \frac{\Lambda_{\mathbb{R}} + \overline{\Lambda_{\mathbb{R}}}}{2} \qquad \text{and} \qquad \Lambda_i = \frac{\Lambda_{\mathbb{R}} - \overline{\Lambda_{\mathbb{R}}}}{2\mathrm{i}}$$

are real linear functionals, and $\Lambda_{\mathbb{R}} = \Lambda_r + \mathrm{i}\Lambda_i$. As such, the linear functional Λ is a linear combination of positive linear functionals defined on $C_0(I;\mathbb{R})$. Note that every continuous linear functional L on $C_0(I;\mathbb{R})$ can be restricted to a continuous linear functional on $C_c(I;\mathbb{R})$. Recall by the Riesz Representation Theorem that

$$L(f) = \int_I f \,\mathrm{d}\mu,$$

for some unique Radon measure on I and for all $f \in C_c(I;\mathbb{R})$. By Theorem 30, the functional L may be extended continuously to all of $C_0(I;\mathbb{R})$. This gives us

$$L(f) = \int_I f \,\mathrm{d}\mu$$

for all $f \in C_0(I;\mathbb{R})$. It is true that every complex Borel measure on I is a complex Radon measure on I. So it holds that for every complex linear functional Λ on $C_0(I;\mathbb{C})$, there exists μ which is a complex Borel measure such that for every $f \in C_0(I;\mathbb{C})$, we have

$$\Lambda(f) = \int_I f \,\mathrm{d}\mu. \tag{2.20}$$

From now on, we will denote the complex linear functional Λ in (2.20) by Λ_μ. We define the norm on $M(I)$ as follows

$$||\mu||_{M(I)} = |\mu|(I).$$

Recall if V is a Banach space over $\mathbb{F}$, where $\mathbb{F}$ is either the real or complex numbers, then the dual of V is defined by $V^* = \mathrm{B}(V,\mathbb{F})$. The following theorem has been proven in Folland [32], where I is replaced with an arbitrary locally compact Hausdorff space.

Theorem 33. *The map $\mu \mapsto \Lambda_\mu$ is an isometric isomorphism from $M(I)$ to $C_0(I;\mathbb{C})^*$.*

Using the above theorem, we can finally show that $M(I)$ is a Banach space.

Theorem 34. *The space of all complex Borel measures $M(I)$ is complete.*

Proof. As $C_0(I;\mathbb{C})^* = \mathrm{B}((C_0(I;\mathbb{C}),\mathbb{C})$ and $\mathbb{C}$ are complete, the space $C_0(I;\mathbb{C})$ is complete. Let $\{\mu_n\}$ be a Cauchy sequence in $M(I)$. Since the map $\mu \to \Lambda_\mu$ is an isometry, the sequence $\{\Lambda_{\mu_n}\}$ is a Cauchy sequence in $C_0(I;\mathbb{C})^*$. As such, $\Lambda_{\mu_n} \to L$ for some $L \in C_0(I;\mathbb{C})^*$. Using the fact that $\mu \mapsto \Lambda_\mu$ is an isomorphism, there exists $\nu \in M(I)$ such that $L = \Lambda_\nu$. This yields

$$||\mu_n - \nu||_{M(I)} = ||\Lambda_{\mu_n} - \Lambda_\nu||.$$

Taking n to be sufficiently large shows that $\mu_n \to \nu$. Hence $M(I)$ is complete. □

2.5.5 Fourier convolution of measures

It is possible to generalise the Fourier convolution of two functions by extending it to the concept to measures. We denote by $M(\mathbb{R}^n)$ the collection of complex Borel measures on $\mathbb{R}^n$. For any μ, $\nu \in M(\mathbb{R}^n)$, we define the Fourier convolution of μ and ν by

$$(\mu * \nu)(E) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \chi_E(x+y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y).$$

On the set of complex Borel measures on $\mathbb{R}^n$, the norm is given by

$$||\mu|| = |\mu|(\mathbb{R}^n)$$

which turns $M(\mathbb{R}^n)$ into a Banach space. By considering the underlying product on $M(\mathbb{R}^n)$ to be the convolution, it is evident that $M(\mathbb{R}^n)$ is a commutative Banach algebra. We also have that the embedding which takes $f \in L^1(\lambda)$ to the measure defined by $f\,\mathrm{d}\lambda$ is a Banach algebra homomorphism, where λ is the Lebesgue measure. It should be noted that the set of all measures of the form $f\,\mathrm{d}\lambda$, where $f \in L^1(\lambda)$ is an ideal in $M(\mathbb{R}^n)$.

Included in this, we can extend the Fourier transform to complex Borel measures. For $\mu \in M(\mathbb{R}^n)$, define $\mathscr{F}\{\mu\} : \mathbb{R}^n \to \mathbb{C}$ as the map which has the pointwise formula

$$\mathscr{F}\{\mu\}(\omega) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi\mathrm{i}\langle\omega,x\rangle} \,\mathrm{d}\mu(x).$$

It is not too difficult to show that $\mathscr{F}\{\mu * \nu\} = \mathscr{F}\{\mu\}\mathscr{F}\{\nu\}$.

2.6 A new class of discrete and integral transforms

Now that the preliminaries have been introduced, we apply some of the knowledge to create a new class of discrete and integral transforms. In this section, we will also define a space in which the discrete and integral operators are defined on.

We begin by properly defining an integral transform. Let I be an open interval which is a subset of $\mathbb{R}$. An *integral transform* is a linear operator $\mathscr{K}$ which takes a function f to a complex function $\mathscr{K}\{f\}$, where $\mathscr{K}\{f\}$ is defined pointwise by

$$\mathscr{K}\{f\}(s) = \int_I K(x,s)f(x)\,\mathrm{d}x,$$

where $s \in J \subset \mathbb{C}$. We refer to K as the *kernel* of the transform. Currently, the kernel of the transform is rather general. We impose some restrictions which allow for the transforms to have interesting properties. We say a kernel K is *logarithmic separable* if $K(x,s) = K_1(x)^{K_2(s)}$ or $K(x,s) = K_2(s)^{K_1(x)}$. Given $K(x,s) = K_1(x)^{K_2(s)}$, we impose that $\mathrm{Im}(K_1) \subseteq (0,\infty)$. Note that formally, the logarithm of K is the product of two functions whose underlying variables are independent of each other. Furthermore, we will assume $K_2(s) = s - w$, where w is some fixed complex number. That is to say, we will assume the kernel is of two types, namely

(i) $K(x,s) = K_1(x)^{s-w}$;

(ii) $K(x,s) = (s-w)^{K_1(x)}$.

Further assumptions will be imposed on K_1 such that the formulas are well defined and satisfy well-behaved properties.

Let $\mathrm{card}(I) = \aleph_0$ and suppose I is a set which is bounded below but not bounded above. A *discrete transform* is an operator $\mathscr{K}$ which inputs a function f and outputs some function $\mathscr{K}\{f\} : J \to \mathbb{C}$ where

$$\mathscr{K}\{f\}(s) = \sum_{x\in I} K(x,s)f(x). \tag{2.21}$$

For our purposes, we will assume K is bounded on $I \times \overline{J}$ if $\mathscr{K}$ is either a discrete or integral transform and f is defined such that the sum in (2.21) converges absolutely.

Currently, we do not have a space in which, given a function belongs to this space, the transform of said function exists. We will first introduce some more concepts which will help us define such a space. Let λ be the Lebesgue measure if I is an interval and let λ be the counting measure if $\operatorname{card}(I) = \aleph_0$. We introduce the measure $\tilde{\mu}_s$ defined by

$$\tilde{\mu}_s(E) = \int_E |K(x,s)| \,\mathrm{d}\lambda(x),$$

where $s \in \overline{J}$ and E is a Borel set on I. We will consider kernels where

$$\sup_{s \in \overline{J}} \tilde{\mu}_s(I) < \infty.$$

Now, we define the measure $\tilde{\mu}_s$ by

$$\mu_s(E) = \int_E \frac{K(x,s)}{|K(x,s)|} \,\mathrm{d}\tilde{\mu}_s(x), \tag{2.22}$$

given that $K(x,s) \neq 0$. If the kernel is of type (ii) and $s = w$, we define μ_s as the trivial measure.

Observe that the total variation of μ_s is simply the measure $\tilde{\mu}_s$ by Theorem 28. From now on we will denote $\tilde{\mu}_s$ by the measure $|\mu_s|$. An alternative respresentation of a discrete or integral transform of a function is

$$\mathscr{A}\{f\}(s) = \int_I f(x) \,\mathrm{d}\mu_s(x) \tag{2.23}$$

given $x \mapsto f(x)K(x,s) \in L^1(I, \mathscr{B}, \lambda)$. This is sufficient due to the fact that for every $s \in \overline{J}$,

$$\int_I |f| \,\mathrm{d}|\mu_s| = \int_I |f(x)||K(x,s)| \,\mathrm{d}\lambda(x) < \infty.$$

It can be seen that similar calculations to that which were present in Lemma 27 justify the above statement. Using the fact that $L^1(\mu_s) = L^1(|\mu_s|)$, we have that $\mathscr{A}\{f\}$ exists on $\overline{J}$. By an abuse of notation, we will consider the function $\mathscr{A}\{f\}$ to have domain $\overline{J}$ or J whenever it is convenient for us.

Now, we introduce the following weighted L^1 space

$$L^1(I, \Omega, |\mu_s|) = \left\{ f : I \to \mathbb{R} : \int_I |f(t)| \,\mathrm{d}|\mu_s|(t) < \infty \right\}, \tag{2.24}$$

where Ω is simply a σ-algebra on I such that μ_s defines a measure on I. We observe that if $f \in L^1(I, \Omega, |\mu_s|)$, then the transform of f exists. Also, observe if

$$f \in \bigcap_{s \in \overline{J}} L^1(I, \Omega, |\mu_s|),$$

then $\mathscr{A}\{f\}$ exists on $\overline{J}$. It is now important for us to consider some more conditions on the kernels and the regions J.

2.6.1 Kernels of type (i)

When considering integral transforms, we will only consider kernels which are of type (i). This is to avoid the use of a complex logarithm. There are three main assumptions we will place on integral transforms. These are:

- The function K_1 is strictly monotonic on I;
- The image of I under K_1 satisfies $K_1(I) \subseteq (0,1)$ or $K_1(I) \subseteq (1,\infty)$;
- If $K_1(I) \subseteq (0,1)$ then $K_1(b-) = 0$ and if $K_1(I) \subseteq (1,\infty)$ then $K_1(b-)$[a] $= \infty$.

Due to the fact that K_1 is continuous, it is evident that $K_1(I)$ is connected. All of the assumptions we have imposed guarantee that the product of any two elements in $K_1(I)$ is another element in $K_1(I)$. This will become significant when we consider the convolution of two functions which correspond to the so-called $\mathscr{A}$-transform. Furthermore, the assumption on the range of K_1 guarantees that $x \mapsto \log(K_1(x))$ is monotonic and does not change sign.

Now, consider discrete transforms whose kernel is of type (i). Given

$$I = \{r_i : i \in \mathbb{N} \text{ and } r_i < r_{i+1} \text{ for every } i \in \mathbb{N}\},$$

we assume K_1 has the same properties mentioned previously, with the exception that $K_1(I) \subseteq (0,1]$ or $K_1(I) \subseteq [1,\infty)$. Also, given $b = \infty$, the value $K_1(b-)$ is computed using a discrete limit.

Currently, we have not imposed any structure on the set J. For type (i) kernels, we define

$$J = \{s \in \mathbb{C} : c < \mathrm{Re}(s) < d\}, \tag{2.25}$$

for some values $c, d \in \overline{\mathbb{R}}$ such that J is a proper subset of the complex plane. Observe for both discrete and integral transforms of type (i), we have that $x \mapsto \log(K_1(x))$ does not change sign on I. We also have $|K_1(x)^{s-w}| = \mathrm{e}^{\mathrm{Re}(s-w)\log(K_1(x))}$. If $\log(K_1(\cdot))$ is nonnegative on I, then we see that $s \mapsto \mathrm{e}^{\mathrm{Re}(s-w)\log(K_1(x))}$ increases as the real part of s increases. Since $|K|$ is bounded on $I \times \overline{J}$, we deduce that, for a fixed $x \in I$, that $K(x,\cdot)$ achieves a maximum on the line $\mathrm{Re}(s) = d$, where $d < \mathrm{Re}(w)$. We note that since $\log(K_1(\cdot))$ does not change sign on I, it is evident that the modulus of the kernel of our transform achieves a maximum on any point on the line $\mathrm{Re}(s) = d$. By a similar calculation, given that $K_1(I) \subseteq (0,1]$, it follows that $|K(x,\cdot)|$ achieves a maximum on the line $\mathrm{Re}(s) = c$

[a] We define $K_1(b) = K_1(b-)$.

for every $x \in I$ where $c > \mathrm{Re}(w)$. What is important to note is that $|K(x,\cdot)|$ achieves its maximum on ∂J for every $x \in I$.

2.6.2 Kernels of type (ii)

When a transform has a kernel of type (ii), we will only be working with the case when $\mathscr{A}$ is a discrete transform. We will assume once again that K_1 is strictly monotonic. One variation between the assumptions for these types of kernels is the behaviour at ∞. We will assume $K_1(\infty) = -\infty$ or $K_1(\infty) = \infty$. Much like for type (i) kernels, we will assume K_1 does not change sign on I. Furthermore, we will assume K_1 is an integer valued function. We have two cases to consider.

Suppose $K_1(\infty) = \infty$. Consider the set $J = B_r(w) = \{s \in \mathbb{C} : |z - w| < r\}$, where $r < 1$. We have for every $x \in I$,

$$|(s-w)^{K_1(x)}| = |s-w|^{K_1(x)} < r^{K_1(x)}.$$

As $x \in I$ was arbitrary, the function $|K(x,\cdot)|$ achieves its maximum on ∂J for every $x \in I$. Now, given $K_1(-\infty) = \infty$ and $J = \mathbb{C} \setminus \overline{B_r(w)}$, we have for every $x \in I$,

$$|(s-w)^{K_1(x)}| = |s-w|^{K_1(x)} < r^{K_1(x)}.$$

Once again, the modulus of $K(x,\cdot)$ achieves a maximum on ∂J for each value x.

2.6.3 Properties of $L^1(I, \mathscr{B}, |\mu|)$

We can now find an alternative formulation for the space $L^1(I, \mathscr{B}, |\mu|)$, where $\mathscr{B}$ is the Borel σ-algebra on I.

Proposition 35. *Let K be a kernel of either type (i) or type (ii), then we have*

$$L^1(I, \mathscr{B}, |\mu|) = \left\{ f : (I, \mathscr{B}, |\mu|) \to \mathbb{C} : \sup_{s \in \overline{J}} \int_I |f(x)| \,\mathrm{d}|\mu_s|(x) < \infty \right\}.$$

Proof. Fix $f \in L^1(I, \mathscr{B}, |\mu|)$. Let $\tilde{s}$ be a value in ∂J such that $|K(x,s)| \le |K(x,\tilde{s})|$ for every $(x,s) \in I \times \overline{J}$. We have

$$\int_I |f(x)||K(x,s)| \,\mathrm{d}\lambda(x) \le \int_I |f(x)||K(x,\tilde{s})| \,\mathrm{d}\lambda(x).$$

An application of the Monotone Convergence Theorem gives us

$$\int_I |f(x)| \,\mathrm{d}|\mu_s|(x) \le \int_I |f(x)| \,\mathrm{d}|\mu_{\tilde{s}}|(x).$$

This, in turn, implies

$$\sup_{s\in\overline{J}} \int_I |f(x)|\,\mathrm{d}|\mu_s|(x) \leq \int_I |f(x)|\,\mathrm{d}|\mu_{\tilde{s}}|(x) < \infty,$$

which shows that

$$L^1(I,\mathscr{B},|\mu|) \subseteq \left\{ f:(I,\mathscr{B},|\mu|)\to\mathbb{C} : \sup_{s\in\overline{J}} \int_I |f(x)|\,\mathrm{d}|\mu_s|(x) < \infty \right\}.$$

Now, suppose f is a Borel function such that the supremum of the integral of $|f(\cdot)||K(\cdot,s)|$ taken over $s\in\overline{J}$ is finite. Clearly, for every $s\in\overline{J}$, we have

$$\int_I |f(x)|\,\mathrm{d}|\mu_s|(x) \leq \sup_{s\in\overline{J}} \int_I |f(x)|\,\mathrm{d}|\mu_s|(x) < \infty.$$

1 Hence $f\in L^1(I,\mathscr{B},|\mu|)$ and this completes the proof. □

It is easy to see that if $\tilde{s}\in\partial J$ such that $|K(x,\tilde{s})|$ is maximised for every $x\in I$, then $L^1(I,\mathscr{B},|\mu|) = L^1(I,\mathscr{B},|\mu_{\tilde{s}}|)$. From here, we see that our L^1 space is a special case of a general L^1 space, where the norm on $L^1(I,\mathscr{B},|\mu|)$ is defined by

$$||f||_\mu = \max_{s\in\partial J} \int_I |f(x)|\,\mathrm{d}|\mu_s|(x).$$

Chapter 3

A general class of integral transforms and an expression for their convolution formulas

3.1 Assumptions on the kernel

We make the following assumptions regarding the kernel K of the integral transform $\mathscr{A}$:

(A1) For every $u \in I$, there exists an open interval $J_u \subseteq I$ and a C^1 function $\psi : J_u \times I \to \mathbb{R}$, strictly monotonic in the first argument, such that

$$K(\psi(x,u),s) = \frac{K(x,s)}{K(u,s)}, \qquad (x,u) \in J_u \times I. \tag{3.1}$$

(A2) For every $u \in I$, we have $\psi(J_u,u) = I$.

Note that we assume throughout that $K(\cdot,s) \in C(I)$ and is never zero.

Example 36. Consider the Laplace transform, where $K(x,s) = \mathrm{e}^{-sx}$ and $I = (0,\infty)$. Choose $J_u = (u,\infty) \subseteq I$ and $\psi(x,u) = x - u$ for every $(x,u) \in J_u \times I$. Then $D_1\psi(x,u) = 1 > 0$ where D_j denotes the partial derivative with respect to the variable in the jth position. It is easy to see that (3.1) is satisfied. Similarly, $\psi(J_u,u) = (0,\infty) = I$. Hence (A2) is satisfied.

Example 37. For the Mellin transform, where $K(x,s) = x^{s-1}$ and $I = (0,\infty)$, let $J_u = (0,\infty)$ and $\psi(x,u) = x/u$. We have $D_1\psi(x,u) = 1/u > 0$ so ψ is strictly monotonic in the first argument and (3.1) holds. Also (A2) is satisfied since $\psi(J_u,u) = (0,\infty) = I$.

Example 38. For

$$K(x,s) = \frac{1}{(1+x)^s}, \qquad I = (0,\infty),$$

taking

$$J_u = (u,\infty), \qquad \psi(x,u) = \frac{x-u}{1+u}$$

shows that (A1) and (A2) are verified since $D_1\psi(x,u) = 1/(1+u) > 0$. Note that this kernel is similar to the Laplace transform kernel as, if we extend I to include 0, then $K(0,s) = 1$, $K(x,s) \to 0$ as $x \to \infty$ and $K(\cdot,s)$ is strictly decreasing given $s > 0$, but the decay at infinity is algebraic.

Example 39. Next we look at an integral transform when I is a bounded interval. Let

$$K(x,s) = \left(\frac{1-x}{1+x}\right)^s, \qquad I = (0,1).$$

Choose

$$J_u = (u,1), \qquad \psi(x,u) = \frac{x-u}{1-xu}, \qquad D_1\psi(x,u) = \frac{1-u^2}{(1-xu)^2} > 0.$$

From here it is easy to show that (A1) and (A2) are satisfied.

Example 40. For

$$K(x,s) = \mathrm{e}^{-s(x^2-1)}, \qquad I = (1,\infty),$$

the choice

$$J_u = (u,\infty), \qquad \psi(x,u) = \sqrt{1+x^2-u^2}$$

verifies (A1) and (A2).

We have yet to determine conditions for when the $\mathscr{A}$-transform exists. We denote by $L^1(I,K(\cdot,s))$ the weighted $L^1(I)$ space consisting of all functions $f : I \to \mathbb{C}$ such that

$$\int_I |K(x,s)||f(x)|\,\mathrm{d}x < \infty.$$

It is easy to see that $f \in L^1(I,K(\cdot,s))$ is a sufficient condition for the existence of $\mathscr{A}\{f\}(s)$. Henceforth we will assume that $f \in L^1(I,K(\cdot,s))$.

3.2 Shifting property

Assuming (A1) and (A2), we now show that the following shifting property holds

$$\mathscr{A}\{f(\psi(\cdot,u))D_1\psi(\cdot,u)\chi_{J_u}(\cdot)\}(s) = K(u,s)\mathscr{A}\{f\}(s), \quad u \in I. \tag{3.2}$$

Let $z = \psi(x,u)$, so that $\mathrm{d}z = D_1\psi(x,u)\,\mathrm{d}x$. Then

$$\begin{aligned}&\mathscr{A}\{f(\psi(\cdot,u))D_1\psi(\cdot,u)\chi_{J_u}(\cdot)\}(s)\\&\quad = \int_{J_u} \frac{K(x,s)}{K(u,s)} K(u,s) f(\psi(x,u)) D_1\psi(x,u)\,\mathrm{d}x\end{aligned}$$

$$= K(u,s)\int_{J_u} K(\psi(x,u),s)f(\psi(x,u))D_1\psi(x,u)\,\mathrm{d}x$$
$$= K(u,s)\int_I K(z,s)f(z)\chi_{\psi(J_u,u)}(z)\,\mathrm{d}z.$$

But $\psi(J_u,u) = I$, which yields

$$\begin{aligned}\mathscr{A}\{f(\psi(\cdot,u))D_1\psi(\cdot,u)\chi_{J_u}(\cdot)\}(s) &= K(u,s)\int_I K_s(z)f(z)\,\mathrm{d}z\\ &= K(u,s)\mathscr{A}\{f\}(s).\end{aligned}$$

Remark 41. There is a similar shifting property which holds for the Fourier cosine transform but it is different from (3.2) since the function to be transformed appears more than once on the right-hand side. Nevertheless, (3.2) encapsulates the shifting properties for the Fourier, Laplace and Mellin transforms.

Example 42. From Example 36 we recover the well-known shifting property

$$\int_0^\infty \mathrm{e}^{-sx}f(x-u)\chi_{(u,\infty)}(x)\,\mathrm{d}x = \mathrm{e}^{-su}\int_0^\infty \mathrm{e}^{-sx}f(x)\,\mathrm{d}x$$

for the Laplace transform.

Example 43. Continuing with Example 39, the shifting property reads

$$\int_0^1 \Big(\frac{1-x}{1+x}\Big)^s f\Big(\frac{x-u}{1-xu}\Big)\frac{1-u^2}{(1-xu)^2}\chi_{(u,1)}(x)\,\mathrm{d}x = \Big(\frac{1-u}{1+u}\Big)^s\int_0^1\Big(\frac{1-x}{1+x}\Big)^s f(x)\,\mathrm{d}x.$$

3.3 Convolution definition and property

We again assume that (A1) and (A2) hold. Define the convolution of two functions $f : I \to \mathbb{C}$ and $g : I \to \mathbb{C}$ by

$$(f*g)(x) = \int_I f(\psi(x,u))g(u)D_1\psi(x,u)\chi_{J_u}(x)\,\mathrm{d}u, \tag{3.3}$$

provided it exists.

Example 44. For Example 37, the expression in (3.3) simplifies to

$$(f*g)(x) = \int_0^\infty \frac{1}{u}f\Big(\frac{x}{u}\Big)g(u)\,\mathrm{d}u,$$

which is the Mellin convolution formula.

Example 45. For the kernel of Example 38, the convolution definition (3.3) becomes

$$(f*g)(x) = \int_0^\infty \frac{1}{1+u}f\Big(\frac{x-u}{1+u}\Big)g(u)\chi_{(u,\infty)}(x)\,\mathrm{d}u$$

$$= \int_0^x \frac{1}{1+u} f\left(\frac{x-u}{1+u}\right) g(u)\, \mathrm{d}u.$$

The substitution $z = (x-u)/(1+u)$ leads to

$$\begin{aligned}(f*g)(x) &= \int_0^x \frac{1}{1+z} g\left(\frac{x-z}{1+z}\right) f(z)\, \mathrm{d}z \\ &= \int_0^\infty \frac{1}{1+z} g\left(\frac{x-z}{1+z}\right) f(z) \chi_{(z,\infty)}(x)\, \mathrm{d}z = (g*f)(x),\end{aligned}$$

i.e. the convolution operator is commutative.

Example 46. For the kernel of Example 39, the equation in (3.3) is

$$\begin{aligned}(f*g)(x) &= \int_0^1 \frac{1-u^2}{(1-xu)^2} f\left(\frac{x-u}{1-xu}\right) g(u) \chi_{(u,1)}(x)\, \mathrm{d}u \\ &= \int_0^x \frac{1-u^2}{(1-xu)^2} f\left(\frac{x-u}{1-xu}\right) g(u)\, \mathrm{d}u.\end{aligned}$$

Letting $z = (x-u)/(1-xu)$, we see that

$$\begin{aligned}(f*g)(x) &= \int_0^x \frac{1-z^2}{(1-xz)^2} g\left(\frac{x-z}{1-xz}\right) f(z)\, \mathrm{d}z \\ &= \int_0^1 \frac{1-z^2}{(1-xz)^2} g\left(\frac{x-z}{1-xz}\right) f(z) \chi_{(z,1)}(x)\, \mathrm{d}z = (g*f)(x)\end{aligned}$$

and again the convolution operator is commutative.

Remark 47. We will show later that if f and g are both uniformly continuous on every bounded subinterval of I, then the convolution operator is in fact commutative with some extra assumptions.

We claim that if an integral transform has a shifting property, then it has a convolution property. The idea for this proof was inspired by Davies [27], where the formula for the convolution was shown to satisfy the convolution property by using the operational notation as opposed to standard calculus techniques. It should be noted that while the proof here is based off the one presented in [27], to the authors' knowledge there is no proof showing this holds for a general transform (such as one where (A1) and (A2) are satisfied) in the literature.

It was shown in Section 3.2 that the shifting property (3.2) is true for kernels that satisfy (A1) and (A2). We wish to show that the shifting property (3.2) implies the convolution property

$$\mathscr{A}\{f*g\}(s) = \mathscr{A}\{f\}(s)\mathscr{A}\{g\}(s). \tag{3.4}$$

Indeed, the convolution definition (3.3) gives

$$\begin{aligned}\mathscr{A}\{f*g\}(s) &= \int_I K(x,s)\int_I f(\psi(x,u))g(u)D_1\psi(x,u)\chi_{J_u}(x)\,\mathrm{d}u\,\mathrm{d}x\\ &= \int_I\int_I K(x,s)f(\psi(x,u))g(u)D_1\psi(x,u)\chi_{J_u}(x)\,\mathrm{d}u\,\mathrm{d}x.\end{aligned}$$

Assuming that the interchange of the order of integration is valid, we obtain with the help of (3.2) that

$$\begin{aligned}\mathscr{A}\{f*g\}(s) &= \int_I g(u)\int_I K(x,s)f(\psi(x,u))D_1\psi(x,u)\chi_{J_u}(x)\,\mathrm{d}x\,\mathrm{d}u\\ &= \int_I g(u)\mathscr{A}\{f(\psi(\cdot,u))D_1\psi(\cdot,u)\chi_{J_u}(\cdot)\}(s)\,\mathrm{d}u\\ &= \int_I g(u)K(u,s)\mathscr{A}\{f\}(s)\,\mathrm{d}u\\ &= \mathscr{A}\{f\}(s)\mathscr{A}\{g\}(s),\end{aligned}$$

which proves the claim. The above argument shows that we only need to compute a one-dimensional integral to derive a convolution formula given that the shifting property holds. This helps in determining a formula which satisfies the convolution property for a given class of integral transforms. More specifically, we take the expression which satisfies the shifting property, multiply it by $g(u)$ and then integrate the new expression over I with respect to u.

3.4 Existence of the convolution integral

Up until now our calculations were purely formal, e.g. we assumed that the convolution integral (3.3) exists. We will introduce a lemma which will help determine the existence of the convolution formula.

Lemma 48. *Suppose that $J_u = (\xi_1(u), \xi_2(u))$, where $\xi_1 < \xi_2$ on I and satisfy exactly one of the three following properties:*

(E1) *Both ξ_1 and ξ_2 are continuous functions from I to I.*

(E2) *One of the functions is continuous from I to I and the other function is a constant function from I to its closure $\bar{I} \subseteq \overline{\mathbb{R}}$.*

(E3) *Both functions are constant from I to $\bar{I}$.*

Then the function $(x,u) \mapsto \chi_{J_u}(x)$ is $\mathscr{M}\otimes\mathscr{M}$-measurable.

Proof. We state for the sake of clarity that

$$\mathscr{M}\otimes\mathscr{M} = \sigma(A\times B : A, B \in \mathscr{M}).$$

That is, $\mathscr{M} \otimes \mathscr{M}$ is the smallest σ-algebra which contains all Lebesgue measurable rectangles which are subsets of I. We will prove the case when ξ_1 and ξ_2 are continuous from I to I. We show that $J_u \times I$ is an $\mathscr{M} \otimes \mathscr{M}$-measurable set, where $J_u = (\xi_1(u), \xi_2(u))$. Fix $(x_0, u_0) \in (\xi_1(u_0), \xi_2(u_0)) \times I$. As $(\xi_1(u_0), \xi_2(u_0))$ is an open set, we have the following inequality for some $\varepsilon > 0$

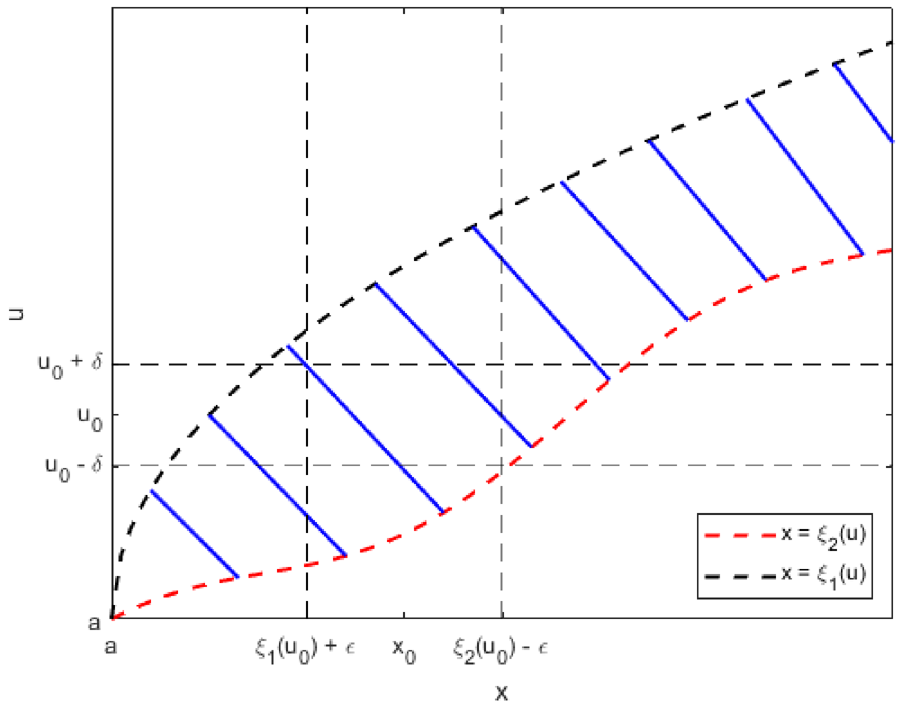

Figure 3.1: Visualisation of the set $J_u \times I = (\xi_1(u), \xi_2(u)) \times I$, where a is finite.

$$\xi_1(u_0) + \varepsilon < x_0 < \xi_2(u_0) - \varepsilon.$$

By our hypothesis, the function ξ_1 is continuous, so there exists $\delta_1 > 0$ such that for every $u \in I$ and $|u - u_0| < \delta_1$ implies $|\xi_1(u) - \xi_1(u_0)| < \varepsilon$. Similarly, there exists $\delta_2 > 0$ such that for every $u \in I$ and $|u - u_0| < \delta_2$ implies $|\xi_2(u) - \xi_2(u_0)| < \varepsilon$. Take $\delta = \min\{\delta_1, \delta_2\}$. If $|u - u_0| < \delta$, we have

$$|\xi_1(u) - \xi_1(u_0)| < \varepsilon, \quad |\xi_2(u) - \xi_2(u_0)| < \varepsilon,$$

which in turn implies

$$\xi_1(u_0) - \varepsilon < \xi_1(u) < \xi_1(u_0) + \varepsilon, \quad \xi_2(u_0) - \varepsilon < \xi_2(u) < \xi_2(u_0) + \varepsilon.$$

This can be rewritten as for every $u \in (u_0 - \delta, u_0 + \delta)$, the following inclusion holds

$$(\xi_1(u_0) + \varepsilon, \xi_2(u_0) - \varepsilon) \subseteq (\xi_1(u), \xi_2(u)).$$

Consider the set

$$(\xi_1(u_0)+\varepsilon, \xi_2(u_0)-\varepsilon) \times (u_0-\delta, u_0+\delta) \subseteq (\xi_1(u), \xi_2(u)) \times I.$$

Now let

$$r = \frac{1}{2}\min\{\delta, |x_0-(\xi_1(u_0)+\varepsilon)|, |x_0-(\xi_2(u_0)-\varepsilon)|\}.$$

We state for the sake of clarity that the graph in Figure 3.2 is similar to that in Figure 3.1.

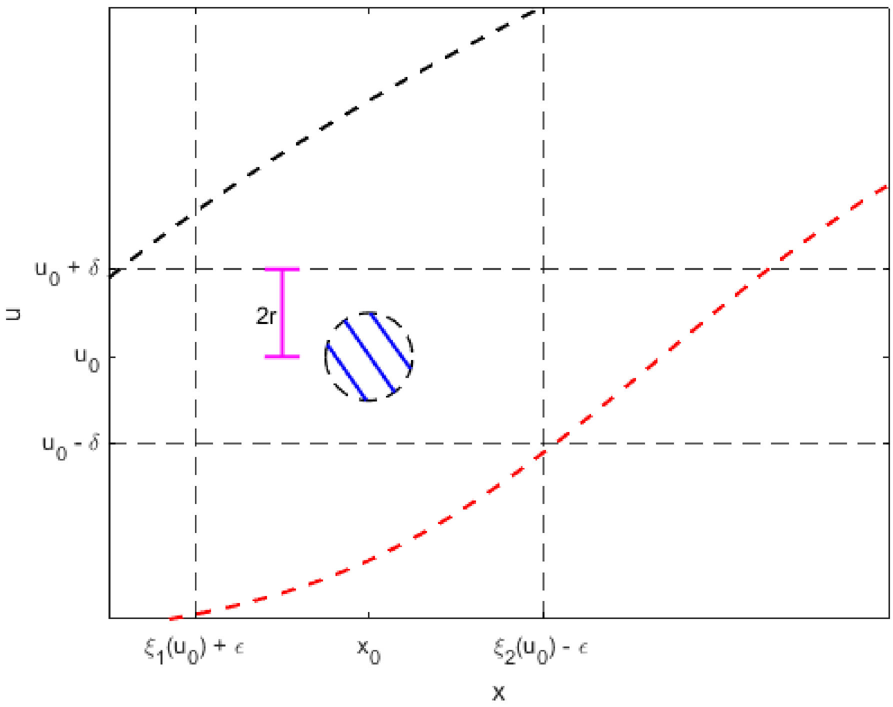

Figure 3.2: Illustration of the ball of radius r around the point (x_0, u_0).

However, in Figure 3.2, the focus is placed on the region

$$(\xi_1(u_0)+\varepsilon, \xi_2(u_0)-\varepsilon) \times (u_0-\delta, u_0+\delta).$$

The magenta line in Figure 3.2 is a line of radius $2r$, and is included to highlight why the value of r was chosen. That is, r is chosen to be half of the smallest distance between the two vertical lines and the two horizontal lines seen in Figure 3.2. We then have

$$B((x_0,u_0);r) \subseteq (\xi_1(u_0)+\varepsilon, \xi_2(u_0)-\varepsilon) \times (u_0-\delta, u_0+\delta) \subseteq (\xi_1(u), \xi_2(u)) \times I = J_u \times I.$$

As the point (x_0, u_0) was arbitrary, we have that $J_u \times I$ is an open set.

Now recall $\mathscr{B}$ is the Borel σ-algebra on the interval I. It is clear from this definition that the σ-algebra $\mathscr{B} \otimes \mathscr{B}$ contains all the rectangles in $\mathbb{R}^2$ which are subsets of $I \times I$. Since any open set in $\mathbb{R}^2$ can be expressed as a countable union of rectangles (see Tao

[96]), the same is true for open subset of $I \times I$. Therefore every open subset of $I \times I$ is in $\mathscr{B} \otimes \mathscr{B}$.

Now, let $h : I \times I \to \mathbb{R}$ be the function defined by

$$h(x,u) = \chi_{J_u}(x).$$

Let U be an open subset of $\mathbb{R}$, we have

$$\begin{aligned} h^{-1}(U) &= \{(x,u) \in I \times I : h(x,u) \in U\} \\ &= \{(x,u) \in J_u \times I : h(x,u) \in U\} \\ &\qquad \cup \{(x,u) \in (I \times I) \backslash (J_u \times I) : h(x,u) \in U\}. \end{aligned}$$

If $(x,u) \in J_u \times I$, then $h(x,u) = 1$, else $h(x,u) = 0$. This gives us

$$h^{-1}(U) = \{(x,u) \in J_u \times I : 1 \in U\} \cup \{(x,u) \in (I \times I) \backslash (J_u \times I) : 0 \in U\}.$$

Observe that

$$\{(x,u) \in J_u \times I : 1 \in U\} = \begin{cases} \emptyset & \text{if } 1 \notin U, \\ J_u \times I & \text{if } 1 \in U, \end{cases}$$

and

$$\{(x,u) \in (I \times I) \backslash (J_u \times I) : 0 \in U\} = \begin{cases} \emptyset & \text{if } 0 \notin U, \\ (I \times I) \backslash (J_u \times I) & \text{if } 0 \in U. \end{cases}$$

It is clear that $\emptyset \in \mathscr{B} \otimes \mathscr{B}$. Since $(I \times I) \backslash (J_u \times I)$ is the complement of the set $J_u \times I$, we deduce that $(I \times I) \backslash (J_u \times I) \in \mathscr{B} \otimes \mathscr{B}$. As $h^{-1}(U)$ is the union of two sets in $\mathscr{B} \otimes \mathscr{B}$ for every open set $U \subseteq \mathbb{R}$, we conclude that h is $\mathscr{M} \otimes \mathscr{M}$-measurable. □

A similar proof can be used which shows that Lemma 48 is true when the functions ξ_1 and ξ_2 satisfy the assumptions in (E2) or (E3). The next theorem gives sufficient conditions for the convolution integral to exist.

Theorem 49 (Existence of the convolution integral)**.** *Suppose that f belongs to $L^1(I, K(\cdot, s)) \cap C(I)$ and $g \in L^1(I, K(\cdot, s))$. Furthermore, suppose that $J_u = (\xi_1(u), \xi_2(u))$, where $\xi_1 < \xi_2$ on I and satisfy exactly one of the three properties stated in Lemma 48. Then $f * g$ defined in* (3.3) *exists almost everywhere on I and $f * g \in L^1(I, K(\cdot, s))$.*

Proof. Consider the measure space $(I \times I, \mathscr{M} \otimes \mathscr{M}, \lambda \times \lambda)$, where λ denotes the Lebesgue measure restricted to I. First we show that the function defined by

$$\begin{aligned} (x,u) \mapsto K(x,s) f(\psi(x,u)) g(u) D_1 \psi(x,u) \chi_{J_u}(x) \\ = K(\psi(x,u), s) f(\psi(x,u)) K(u,s) g(u) D_1 \psi(x,u) \chi_{J_u}(x) \end{aligned} \tag{3.5}$$

is $\mathscr{M}\otimes\mathscr{M}$-measurable. For definiteness, assume that $D_1\psi(x,u)>0$; the case $D_1\psi(x,u)<0$ can be proved similarly. For each $(x,u)\in I\times I$, define

$$h_1(x,u)=K(\psi(x,u),s)f(\psi(x,u))\chi_{J_u}(x), \quad h_2(x,u)=K(u,s)g(u)\chi_{J_u}(x)$$
$$h_3(x,u)=D_1\psi(x,u)\chi_{J_u}(x).$$

Note that $K(\cdot,s)\cdot f$ and $K(\cdot,s)\cdot g$ are $\mathscr{M}$-measurable since $f,g\in L^1(I,K(\cdot,s))$. The function h_1 is either a constant function or the composition of the continuous function $K(\cdot,s)\cdot f$ with the measurable function ψ. Fix $U\subseteq\mathbb{R}$, where U is open. We have

$$\begin{aligned}h_1^{-1}(U)&=\{(x,u)\in J_u\times I : h_1(x,u)\in U\}\\ &\qquad\cup\{(x,u)\in(I\times I)\backslash(J_u\times I) : h_1(x,u)\in U\}\\ &=\{(x,u)\in J_u\times I : K(\psi(x,u),s)f(\psi(x,u))\in U\}\\ &\qquad\cup\{(x,u)\in(I\times I)\backslash(J_u\times I) : 0\in U\}.\end{aligned}$$

Recall the proof in Lemma 48 showed that $J_u\times I$ is an open set. So $\mathscr{B}_{J_u\times I}\subseteq\mathscr{B}_{I\times I}=\mathscr{B}\otimes\mathscr{B}$. This implies

$$\{(x,u)\in J_u\times I : K(\psi(x,u),s)f(\psi(x,u))\in U\}\in\mathscr{B}\otimes\mathscr{B}.$$

Similarly, by Lemma 48 we have that $\{(x,u)\in(I\times I)\backslash(J_u\times I) : 0\in U\}\in\mathscr{B}\otimes\mathscr{B}$. We deduce that h_1 is $\mathscr{M}\otimes\mathscr{M}$-measurable. A similar argument shows that h_2 and h_3 are $\mathscr{M}\otimes\mathscr{M}$-measurable. Thus the function defined in (3.5) is $\mathscr{M}\otimes\mathscr{M}$-measurable.

Next we consider the integral

$$\begin{aligned}&\int_I\int_I|K(x,s)f(\psi(x,u))g(u)D_1\psi(x,u)\chi_{J_u}(x)|\,\mathrm{d}x\,\mathrm{d}u\\ &\qquad=\int_I\int_{J_u}|K(x,s)||f(\psi(x,u))||g(u)|D_1\psi(x,u)\,\mathrm{d}x\,\mathrm{d}u.\end{aligned}$$

Using (A1), we have

$$\begin{aligned}&\int_I\int_{J_u}|K(x,s)||f(\psi(x,u))||g(u)|D_1\psi(x,u)\,\mathrm{d}x\,\mathrm{d}u\\ &\qquad=\int_I\int_{J_u}|K(\psi(x,u),s)||K(u,s)||f(\psi(x,u))||g(u)|D_1\psi(x,u)\,\mathrm{d}x\,\mathrm{d}u.\end{aligned}$$

Letting $z=\psi(x,u)$ and $\mathrm{d}z=D_1\psi(x,u)\,\mathrm{d}x$, we see that

$$\begin{aligned}&\int_I\int_{J_u}|K(\psi(x,u),s)||f(\psi(x,u))||K(u,s)||g(u)|D_1\psi(x,u)\,\mathrm{d}x\,\mathrm{d}u\\ &\qquad=\int_I\int_{\psi(J_u,u)}|K(z,s)||f(z)||K(u,s)||g(u)|\,\mathrm{d}z\,\mathrm{d}u\end{aligned}$$

$$= \int_I \int_I |K(z,s)||f(z)||K(u,s)||g(u)| \,\mathrm{d}z \,\mathrm{d}u,$$

which follows from (A2). Therefore

$$\begin{aligned}
&\int_I \int_I |K(x,s) f(\psi(x,u)) g(u) D_1 \psi(x,u) \chi_{J_u}(x)| \,\mathrm{d}x \,\mathrm{d}u \\
&\qquad = \left[\int_I |K(u,s)||g(u)| \,\mathrm{d}u \right] \left[\int_I |K(z,s)||f(z)| \,\mathrm{d}z \right],
\end{aligned}$$

which is finite since $f, g \in L^1(I, K(\cdot, s))$. By an application of one form of Fubini's Theorem [51], we deduce that the function

$$u \mapsto K(x,s) f(\psi(x,u)) g(u) D_1 \psi(x,u) \chi_{J_u}(x)$$

is in $L^1(I)$ for almost all $t \in I$, i.e. $f * g$ exists almost everywhere on I. Moreover, the function

$$\begin{aligned}
x \mapsto &\int_I K(x,s) f(\psi(x,u)) g(u) D_1 \psi(x,u) \chi_{J_u}(x) \,\mathrm{d}u \\
&= K(x,s) \int_I f(\psi(x,u)) g(u) D_1 \psi(x,u) \chi_{J_u}(x) \,\mathrm{d}u
\end{aligned}$$

is in $L^1(I)$. Thus we conclude that $f * g \in L^1(I, K(\cdot, s))$. □

We note that due to the set $J_u \times I$ being some arbitrary open set in $I \times I$, it appears difficult to determine the existence of the formula which satisfies the convolution property on the whole interval I. By making some further assumptions, we can show the formula is continuous.

Theorem 50. *Suppose the convolution formula* (3.3) *exists on the interval I. Impose the following conditions:*

(C1) *The functions f and g are uniformly continuous on every bounded subinterval of I.*

(C2) *The functions ψ and $D_1\psi$ are uniformly continuous on every bounded subset of $J_u \times I$.*

(C3) *Let $J_u \in \{(a, \xi_2(u)), (\xi_1(u), b)\}$, where ξ_1 and ξ_2 are strictly monotonic increasing, continuous functions from I to I such that $\xi_1(I) = I$ and $\xi_2(I) = I$.*

(C4) *If $J_u = (a, \xi_2(u))$, assume b is finite, and if $J_u = (\xi_1(u), b)$, assume a is finite.*

Then $f * g$ as defined in (3.3) *is continuous on I.*

Proof. Fix $\varepsilon > 0$. We will consider the case when $J_u = (\xi_1(u), b)$. Based on the assumptions for ξ_1, its inverse ξ_1^{-1} exists and since $\xi_1(I) = I$, we have that $\chi_{(\xi_1(u),b)}(x) =$

$\chi_{(a,\xi_1^{-1}(x))}(u)$. We will show the convolution formula is continuous at c for an arbitrary $c \in I$. Now, there exists $\delta_1 > 0$ such that $[c-\delta_1, c+\delta_1] \subset I$. Using (C1), g is uniformly continuous on both $(a, \xi_1^{-1}(c)]$ and $[c-\delta_1, c+\delta_1]$, so that g is bounded on $(a, \xi_1^{-1}(c)] \cup [c-\delta_1, c+\delta_1]$, say $|g| \leq M_1$ for some constant $M_1 > 0$. By the assumption of uniform continuity of $f \circ \psi$ and $D_1\psi$ in (C1) and (C2), for every fixed $x \in [c-\delta_1, c+\delta_1]$ we have that the functions $f(\psi(x,\cdot))$ and $D_1\psi(x,\cdot)$ are uniformly continuous on $[\xi_1^{-1}(c-2\delta_1), \xi_1^{-1}(x))$ (by choosing δ_1 small enough such that $c-2\delta_1 \in I$, we guarantee the interval $[\xi_1^{-1}(c-2\delta_1), \xi_1^{-1}(x))$ is well defined and nonempty). From here we can see that, for every fixed $x \in [c-\delta_1, c+\delta_1]$ the functions $f(\psi(x,\cdot))$ and $D_1\psi(x,\cdot)$ are bounded on $[\xi_1^{-1}(c-2\delta_1), \xi_1^{-1}(x))$. So the functions $f \circ \psi$ and $D_1\psi$ are bounded on the set

$$[c-\delta_1, c+\delta_1] \times [\xi_1^{-1}(c-2\delta_1), \xi_1^{-1}(x)).$$

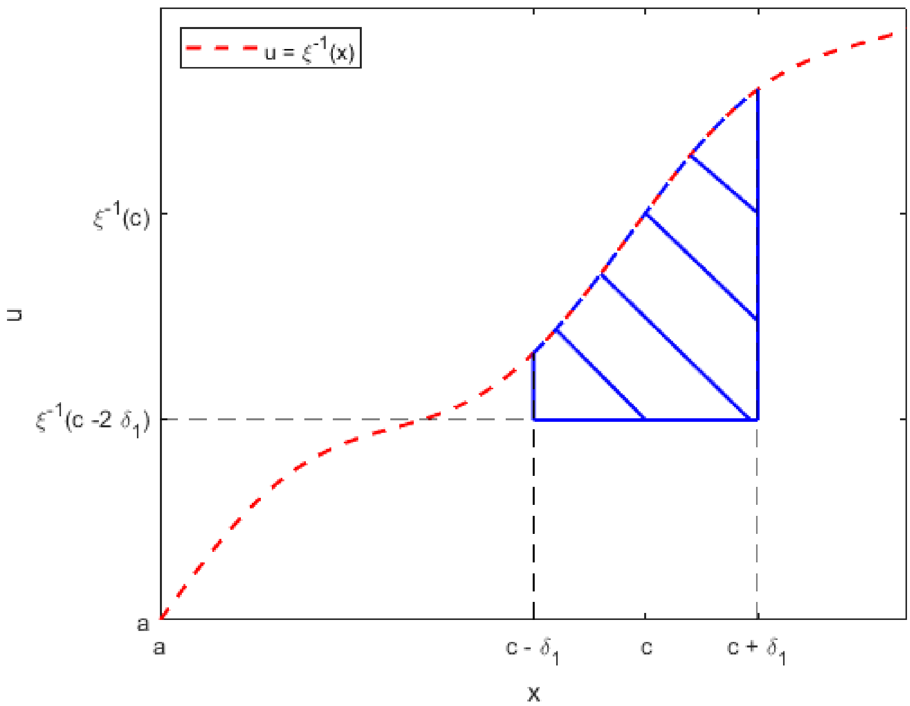

Figure 3.3: Visualisation of the set $[c-\delta_1, c+\delta_1] \times [\xi_1^{-1}(c-2\delta_1), \xi_1^{-1}(x))$.

For the sake of clarity, say $|f \circ \psi| \leq M_2$ and $|D_1\psi| \leq M_3$ for some constants M_2, $M_3 > 0$. By the assumptions of uniform continuity, the function $(f \circ \psi) \cdot D_1\psi$ is uniformly continuous on every bounded subset of $J_u \times I$. So there exists $\delta_2 > 0$ such that if S is a bounded subset of $J_u \times I$, then if $(x_1, u_1), (x_2, u_2) \in S$ such that $|(x_1, u_1) - (x_2, u_2)| < \delta_2$ we have

$$|f(\psi(x_1,u_1))D_1\psi(x_1,u_1) - f(\psi(x_2,u_2))D_1\psi(x_2,u_2)| < \frac{\varepsilon}{4M_1[\xi_1^{-1}(c) - a]}.$$

Now, ξ_1 is a strictly monotonic continuous function defined on an interval I as assumed in (C3). It can be shown that ξ_1^{-1} is continuous on I. So if $|x-c| < \delta_3$ then

$$|\xi_1^{-1}(x) - \xi_1^{-1}(c)| < \frac{\varepsilon}{4M_1M_2M_3}.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. We see that if $|x-c| < \delta$, then

$$\begin{aligned}
|(f*g)(x) - (f*g)(c)| &= \bigg| \int_I f(\psi(x,u))g(u)D_1\psi(x,u)\chi_{(a,\xi_1^{-1}(x))}(u)\,\mathrm{d}u \\
&\quad - \int_I f(\psi(c,u))g(u)D_1\psi(c,u)\chi_{(a,\xi_1^{-1}(c))}(u)\,\mathrm{d}u \bigg| \\
&= \bigg| \int_a^{\xi_1^{-1}(x)} f(\psi(x,u))g(u)D_1\psi(x,u)\,\mathrm{d}u \\
&\quad - \int_a^{\xi_1^{-1}(c)} f(\psi(c,u))g(u)D_1\psi(c,u)\,\mathrm{d}u \bigg|.
\end{aligned}$$

Suppose $c \le x$. Using the Triangle Inequality, we have

$$\begin{aligned}
&|(f*g)(x) - (f*g)(c)| \\
&\le \int_{\xi_1^{-1}(c)}^{\xi_1^{-1}(x)} |f(\psi(x,u))g(u)D_1\psi(x,u)|\,\mathrm{d}u \\
&\quad + \int_a^{\xi_1^{-1}(c)} |g(u)||f(\psi(x,u))D_1\psi(x,u) - f(\psi(c,u))D_1\psi(c,u)|\,\mathrm{d}u \\
&\quad \le \int_{\xi_1^{-1}(c)}^{\xi_1^{-1}(x)} M_1M_2M_3\,\mathrm{d}u + \int_a^{\xi_1^{-1}(c)} \frac{\varepsilon}{4[\xi_1^{-1}(c)-a]}\,\mathrm{d}u.
\end{aligned}$$

The above expression makes sense as a is finite by (C4). From here we can see that $|(f*g)(x) - (f*g)(c)| \le \varepsilon/2 < \varepsilon$. Similar calculations can be made for the case when $x < c$. By modifying the argument, analogous techniques can also be employed when $J_u = (a, \xi_2(u))$. This proves that $f*g$ is continuous on I. □

3.5 Injective nature of $\mathscr{A}$

Here we will provide a proof of the injective nature of the operator $\mathscr{A}$ under certain conditions, namely when the kernel K is logarithmic separable. Due to the fact that the convolution exists almost everywhere on I, there is no guarantee that this is the only formula which satisfies the convolution property. In fact, suppose that there is a set $E \subseteq I$ of measure zero and the convolution exists and is not zero on E. If we were to define the convolution to be zero on E, then we would have two formulas which are clearly different but both satisfy the convolution property. This is not just true in terms of the convolution formula, however, as if we have two functions $f = g$ almost everywhere on I, then we will have $\mathscr{A}\{f\}(s) = \mathscr{A}\{g\}(s)$. This gives us an indication on what extra condition we need in order to construct a class of functions where the transform of those such functions exists and is unique. Recall from Chapter 2 that K is logarithmic separable if it can be expressed as

$$K(x,s) = K_1(x)^{K_2(s)}$$

for some functions $K_1 = K_1(x)$ and $K_2 = K_2(s)$. Here we impose that $\operatorname{Im}(K_1) \subseteq (0,\infty)$.

Example 51. We have $K_1(x) = \mathrm{e}^{-x}$ and $K_2(s) = \mathrm{i}s$ for the Fourier transform, $K_1(x) = \mathrm{e}^{-x}$ and $K_2(s) = s$ for the Laplace transform, and $K_1(x) = x$ and $K_2(s) = s-1$ for the Mellin transform.

Example 52. The kernel of Example 38 is logarithmic separable with

$$K_1(x) = 1+x, \quad K_2(s) = -s.$$

Example 53. The kernel of Example 39 is logarithmic separable with

$$K_1(x) = \frac{1-x}{1+x}, \quad K_2(s) = s.$$

Remark 54. For a logarithmic separable kernel, (A1) is replaced by

(A1)' For every $u \in I$, there exist an open interval $J_u \subseteq I$ and a C^1 function $\psi : J_u \times I \to \mathbb{R}$, strictly monotonic in the first argument, such that

$$K_1(\psi(x,u)) = \frac{K_1(x)}{K_1(u)}, \quad (x,u) \in J_u \times I. \tag{3.6}$$

Note that K_2 is arbitrary, and (A2) remains unchanged.

We first prove a lemma which will aid us in showing when $\mathscr{A}$ is injective.

Lemma 55. *Let K_s be a logarithmic separable kernel, i.e.*

$$\log(K(x,s)) = K_1(x)K_2(s), \quad K_2(s) = c_1 s + c_0,$$

where $c_0, c_1 \in \mathbb{C}$ and $c_1 \neq 0$, such that $K_1 \in C^1(I)$ is strictly monotonic and both[a] *$K_1(a)$ and $K_1(b)$ are finite. Suppose that*

$$\hat{f}(s) = \mathscr{A}\{f\}(s) = \int_I K(x,s) f(x)\,\mathrm{d}x$$

---

[a]Here and throughout this chapter, for a function f defined on (a,b) we let

$$f(a) = \lim_{x\to a^+} f(x) \text{ and } f(b) = \lim_{x\to b^-} f(x).$$

exists for all $\mathrm{Re}(s) > s_0$ *for some* $s_0 \in \mathbb{R}$*. Assume also that* f *is continuous on* I*, and that there exists* $P > 0$ *such that*

$$\hat{f}(s_0 + nP) = \mathscr{A}\{f\}(s_0 + nP) = 0, \quad n = 0, 1, 2, \ldots. \tag{3.7}$$

Then $f(x) = 0$ *for all* $x \in I$*.*

Proof. We introduce the auxiliary function

$$g(x) = \int_a^x K(u, s_0) f(u)\,\mathrm{d}u.$$

It is easy to see that $g(a) = 0$. Similarly, by (3.7) we have $g(b) = \mathscr{A}\{f\}(s_0) = 0$ if we take $n = 0$. As such we conclude that g vanishes at the endpoints of I. We now consider the transform of f, namely

$$\begin{aligned}
\mathscr{A}\{f\}(s_0 + nP) &= \int_I K(x, s_0 + nP) f(x)\,\mathrm{d}x \\
&= \int_I K_1(x)^{c_1(s_0+nP)+c_0} f(x)\,\mathrm{d}x \\
&= \int_I K(x, s_0) \mathrm{e}^{c_1 nP \log(K_1(x))} f(x)\,\mathrm{d}x.
\end{aligned}$$

Here we used the fact that K_2 is an affine function of s. We assume that $K_1'(x) > 0$ since the case $K_1'(x) < 0$ can be similarly handled. Applying integration by parts and recalling that g vanishes at the endpoints of I gives us

$$\begin{aligned}
\mathscr{A}\{f\}(s_0 + nP) &= \Big[K_1(x)^{c_1 nP} g(x)\Big]_I - c_1 nP \int_I g(x) K_1(x)^{c_1 nP - 1} K_1'(x)\,\mathrm{d}x \\
&= -c_1 nP \int_I g(x) K_1(x)^{c_1 nP - 1} K_1'(x)\,\mathrm{d}x \\
&= -n \int_I g(x) K_1(x)^{c_1(n-1)P} K_1(x)^{c_1 P - 1} c_1 P K_1'(x)\,\mathrm{d}x.
\end{aligned}$$

We introduce a map which takes the interval I to $(0, 1)$. This is achieved by making the substitutions

$$\begin{gathered}
u = \frac{K_1(x)^{c_1 P} - K_1(b)^{c_1 P}}{K_1(a)^{c_1 P} - K_1(b)^{c_1 P}}, \qquad \mathrm{d}u = \frac{K_1(x)^{c_1 P - 1} c_1 P K_1'(x)}{K_1(a)^{c_1 P} - K_1(b)^{c_1 P}}\,\mathrm{d}x, \\
K_1(x)^{c_1 P} = [K_1(a)^{c_1 P} - K_1(b)^{c_1 P}]u + K_1(b)^{c_1 P}, \\
x = h(u) = K_1^{-1}\Big(\big\{[K_1(a)^{c_1 P} - K_1(b)^{c_1 P}]u + K_1(b)^{c_1 P}\big\}^{1/c_1 P}\Big),
\end{gathered}$$

to obtain the equivalent expression

$$0 = \mathscr{A}\{f\}(s_0 + nP)$$

$$= n[K_1(a)^{c_1P} - K_1(b)^{c_1P}] \int_0^1 g(h(u)) \left\{[K_1(a)^{c_1P} - K_1(b)^{c_1P}]u + K_1(b)^{c_1P}\right\}^{n-1} \mathrm{d}u.$$

By using the binomial expansion, the integrand can then be written as a linear combination of the moments of the auxiliary function, i.e.

$$\begin{aligned} 0 &= n[K_1(a)^{c_1P} - K_1(b)^{c_1P}] \\ &\quad \times \int_0^1 g(h(u)) \sum_{j=0}^{n-1} \binom{n-1}{j} [K_1(b)^{c_1P}]^{n-1-j}[K_1(a)^{c_1P} - K_1(b)^{c_1P}]^j u^j \,\mathrm{d}u \\ &= n\sum_{j=0}^{n-1} \binom{n-1}{j} [K_1(b)^{c_1P}]^{n-1-j}[K_1(a)^{c_1P} - K_1(b)^{c_1P}]^{j+1} \int_0^1 u^j g(h(u))\,\mathrm{d}u. \end{aligned}$$

We want to show that the integral term causes the equation to be zero. For this we apply the Principle of Complete Mathematical Induction to show that the moments of $g \circ h$ are always trivial. Now, by invoking Theorem 9 in Chapter 13 of [103], we deduce that $g(h(u)) = 0$ for all $u \in (0,1)$. Thus $g(x) = 0$ for all $x \in I$, i.e.

$$\int_a^x K(u,s_0)f(u)\, du = 0.$$

An application of the Fundamental Theorem of Calculus implies that $K(x,s_0)f(x) = 0$ for all $x \in I$. Because of this we have that $f(x) = 0$ for all $x \in I$ since $K(x,s_0)$ never vanishes by assumption. □

The above lemma has some interesting consequences, one of which is that for a logarithmic separable kernel where K_2 is a linear function of s, $\mathscr{A}$ cannot map f to a trigonometric function given f is continuous. It is apparent that f needs to be continuous in order for this proof to work. By simply assuming that $f \in L^1(I, K(\cdot,s))$, then the proof only shows that $f = 0$ almost everywhere on I, which will become important when showing that $\mathscr{A}$ is injective. While it has not yet been proven directly that the operator $\mathscr{A}$ is injective, the difficult part has been completed. An application of the previous result provides a simple proof below that $\mathscr{A}$ is injective.

Theorem 56. *Let K be a logarithmic separable kernel, i.e.*

$$K(x,s) = K_1(x)^{K_2(s)}, \quad K_2(s) = c_1 s + c_0,$$

where $c_0, c_1 \in \mathbb{C}$ and $c_1 \neq 0$, such that $K_1 \in C^1(I)$ is strictly monotonic and both $K_1(a)$ and $K_1(b)$ are finite. If $f : I \to \mathbb{C}$ and $g : I \to \mathbb{C}$ are continuous and $\mathscr{A}\{f\}(s) = \mathscr{A}\{g\}(s)$ for all $\mathrm{Re}(s) > s_0$ *for some $s_0 \in \mathbb{R}$, then $f(x) = g(x)$ for all $x \in I$.*

Proof. Define h by $h = f - g$ on I. A simple computation shows that $\mathscr{A}\{h\}(s) = 0$ for all $s > s_0$. Lemma 55 gives $h(x) = 0$, or $f(x) = g(x)$, for all $x \in I$. □

Corollary 57. *Let K be a logarithmic separable kernel, i.e.*

$$K(x,s) = K_1(x)^{K_2(s)}, \quad K_2(s) = c_1 s + c_0,$$

where $c_0, c_1 \in \mathbb{C}$ and $c_1 \neq 0$, such that $K_1 \in C^1(I)$ is strictly monotonic and both $K_1(a)$ and $K_1(b)$ are finite. If $f : I \to \mathbb{C}$ and $g : I \to \mathbb{C}$ are in the set $L^1(I, K(\cdot, s))$ and are uniformly continuous on every bounded subinterval of I, then $f * g = g * f$ given they exist on the whole set I.

Proof. Observe the following calculation

$$\mathscr{A}\{f * g\}(s) = \mathscr{A}\{f\}(s)\mathscr{A}\{g\}(s) = \mathscr{A}\{g\}(s)\mathscr{A}\{f\}(s) = \mathscr{A}\{g * f\}(s).$$

Then $f * g = g * f$ follows from the previous theorem. □

3.6 Inversion formulas

Now that it has been shown that $\mathscr{A}$ is injective when $K(x,s) = K_1(x)^{K_2(s)}$ (where $K_1 \in C^1(I)$ is strictly monotonic and K_2 is a linear function of s), it makes sense to determine a formula for the inverse transform. It may have been possible to do so previously, however, the proof of uniqueness guarantees that if one formula can be determined, then this is sufficient as all other forms are equivalent. An example of where multiple formulas of an inverse exist is that of the Laplace transform. As seen in Widder [103], the inverse formula can be represented as the limit of an integral over the positive real line, whereas the classical formula determined by Bromwich involves an integral over a contour in the complex plane. For the sake of consistency, we will present the inverse $\mathscr{A}$-transform as a contour integral.

Theorem 58. *Fix $x \in I$, and assume the following:*

(I1) *Suppose that $f \in L^1(I, K(\cdot, (\mu - c_2)/c_1)) \cap C^1(I)$, where $\mu \in \mathbb{R}$, $c_1, c_2 \in \mathbb{C}$, $c_1 \neq 0$ and $K(x,s) = K_1(x)^{c_1 s + c_2}$.*

(I2) *The function K_1 is monotonic increasing in t and $K_1 \in C^1(I)$.*

(I3) *We have $[f(b)K_1(b)]/K_1'(b) \cdot [K_1(b)/K_1(x)]^{\mu} = 0$.*

(I4) *The quantity $[f(a)K_1(a)]/K_1'(a) \cdot [K_1(a)/K_1(x)]^{\mu}$ is finite.*

(I5) *The function $t \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{f(t)K_1(t)}{K_1'(t)}\left[\frac{K_1(t)}{K_1(x)}\right]^{\mu}\right\}$ is in $L^1(I)$.*

Then the following formula holds

$$f(x) = \frac{c_1 K_1'(x)}{2\pi \mathrm{i} K_1(x)} \lim_{T\to\infty} \int_{\frac{\mu - c_2 - \mathrm{i}T}{c_1}}^{\frac{\mu - c_2 + \mathrm{i}T}{c_1}} K_1(x)^{-(c_1 s + c_2)} \mathscr{A}\{f\}(s)\,\mathrm{d}s, \tag{3.8}$$

where $T \in \mathbb{R}$.

Proof. Define an auxillary function F by

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\mathrm{Si}(x), \quad \mathrm{Si}(x) = \int_0^x \frac{\sin(y)}{y}\,\mathrm{d}y. \tag{3.9}$$

Let $\{T_n\}$ be a real-valued, positive sequence which diverges to ∞. Obverse that the expression

$$\frac{c_1 K_1'(x)}{2\pi \mathrm{i} K_1(x)} \int_{\frac{\mu - c_2 - \mathrm{i}T_n}{c_1}}^{\frac{\mu - c_2 + \mathrm{i}T_n}{c_1}} K_1(x)^{-(c_1 s + c_2)} \mathscr{A}\{f\}(s)\,\mathrm{d}s$$

can be written as the following iterated integral

$$\frac{c_1 K_1'(x)}{2\pi \mathrm{i} K_1(x)} \int_{\frac{\mu - c_2 - \mathrm{i}T_n}{c_1}}^{\frac{\mu - c_2 + \mathrm{i}T_n}{c_1}} \int_I K_1(x)^{-(c_1 s + c_2)} K_1(t)^{c_1 s + c_2} f(t)\,\mathrm{d}t\,\mathrm{d}s.$$

Using the definition of an integral over the complex plane, the expression can be written as follows

$$\frac{K_1'(x)}{2\pi K_1(x)} \int_{-T_n}^{T_n} \int_I K_1(x)^{-(\mu + \mathrm{i}\omega)} K_1(t)^{\mu + \mathrm{i}\omega} f(t)\,\mathrm{d}t\,\mathrm{d}\omega. \tag{3.10}$$

We have that the following functions

$$(t, \omega) \mapsto K_1(x)^{-(\mu + \mathrm{i}\omega)}, \qquad (t, \omega) \mapsto K_1(t)^{\mu + \mathrm{i}\omega}, \qquad (t, \omega) \mapsto f(t),$$

are all continuous on $I \times [-T_n, T_n]$ and therefore are measurable on the same domain. By the assumption (I1), we can show that $\mathscr{A}\{f\}$ is continuous on its domain. Furthermore, a similar proof to that in Proposition 93 shows that, due to the assumption in (I1), $\mathscr{A}\{f\}$ is holomorphic for $\mathrm{Re}(c_1 s + c_2) \geq \mu$. As such the contour integral in (3.8) is well defined. Because of this, we have that the integral of the function

$$\omega \mapsto \left| K_1(x)^{-(\mu + \mathrm{i}\omega)} \mathscr{A}\{f\}\left(\frac{\mu - c_2 + \mathrm{i}\omega}{c_1}\right) \right|$$

over $[-T_n, T_n]$ is finite. So, by Fubini's Theorem, we may interchange the order of integration in the expression (3.10), which gives us

$$\begin{aligned}
&\frac{K_1'(x)}{2\pi K_1(x)} \int_I f(t) \int_{-T_n}^{T_n} \left[\frac{K_1(t)}{K_1(x)}\right]^{\mu + \mathrm{i}\omega} \mathrm{d}\omega\,\mathrm{d}t \\
&\quad = \frac{K_1'(x)}{2\pi K_1(x)} \int_I f(t) \int_{-T_n}^{T_n} \mathrm{e}^{(\mu + \mathrm{i}\omega)[\log(K_1(t)) - \log(K_1(x))]}\,\mathrm{d}\omega\,\mathrm{d}t \\
&\quad = \frac{K_1'(x)}{K_1(x)} \int_I f(t) \mathrm{e}^{\mu[\log(K_1(t)) - \log(K_1(x))]} \frac{\sin(T_n[\log(K_1(t)) - \log(K_1(x))])}{\pi[\log(K_1(t)) - \log(K_1(x))]}\,\mathrm{d}t
\end{aligned}$$

$$= \frac{K_1'(x)}{K_1(x)} \int_I \frac{f(t)K_1(t)}{K_1'(t)} \left[\frac{K_1(t)}{K_1(x)}\right]^{\mu} \frac{\mathrm{d}}{\mathrm{d}t} [F(T_n[\log(K_1(t)) - \log(K_1(x))])] \,\mathrm{d}t,$$

where F is given in (3.9). We note that the Si function is defined on $\mathbb{R}$, with finite limits as the argument approaches $\pm\infty$. Using this, as well as (I3) and (I4), we may apply integration by parts and the expression for the above integral is equal to

$$\begin{aligned} &-\frac{K_1'(x)}{K_1(x)} \frac{f(a)K_1(a)}{K_1'(a)} \left[\frac{K_1(a)}{K_1(x)}\right]^{\mu} F(T_n[\log(K_1(a)) - \log(K_1(x))]) \\ &\quad -\frac{K_1'(x)}{K_1(x)} \int_I F(T_n[\log(K_1(t)) - \log(K_1(x))]) \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \frac{f(t)K_1(t)}{K_1'(t)} \left[\frac{K_1(t)}{K_1(x)}\right]^{\mu} \right\} \mathrm{d}t. \end{aligned} \tag{3.11}$$

It can be shown that there exists some constant $M > 0$ such that $|F(x)| \le M$ for every $x \in \mathbb{R}$. This combined with the fact that the integrand in (3.11) is measurable for every $n \in \mathbb{N}$ allows us to apply Lebesgue's Dominated Convergence Theorem. We also introduce an auxillary function given by

$$H(x) = \lim_{T\to\infty} F(Tx) = \begin{cases} 0 & \text{if } x < 0, \\ 1/2 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

As we used an arbitrary sequence $\{T_n\}$ which diverges to ∞, we may replace the limit with a continuous limit. This gives us

$$\begin{aligned} &\frac{c_1 K_1'(x)}{2\pi \mathrm{i} K_1(x)} \lim_{T\to\infty} \int_{\frac{\mu - c_2 - \mathrm{i}T}{c_1}}^{\frac{\mu - c_2 + \mathrm{i}T}{c_1}} K_1(x)^{-(c_1 s + c_2)} \mathscr{A}\{f\}(s) \,\mathrm{d}s \\ &\quad = -\frac{K_1'(x)}{K_1(x)} \int_I H(\log(K_1(t)) - \log(K_1(x))) \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \frac{f(t)K_1(t)}{K_1'(t)} \left[\frac{K_1(t)}{K_1(x)}\right]^{\mu} \right\} \mathrm{d}t \\ &\quad = -\frac{K_1'(x)}{K_1(x)} \int_{(x,b)} \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \frac{f(t)K_1(t)}{K_1'(t)} \left[\frac{K_1(t)}{K_1(x)}\right]^{\mu} \right\} \mathrm{d}t. \end{aligned}$$

Finally, an application of the Fundamental Theorem of Calculus gives us

$$\frac{c_1 K_1'(x)}{2\pi \mathrm{i} K_1(x)} \lim_{T\to\infty} \int_{\frac{\mu - c_2 - \mathrm{i}T}{c_1}}^{\frac{\mu - c_2 + \mathrm{i}T}{c_1}} K_1(x)^{-(c_1 s + c_2)} \mathscr{A}\{f\}(s) \,\mathrm{d}s = f(x).$$

□

The proof here only considered the case when K_1 is a monotonic increasing function. For the sake of clarity, we present an expression for the inverse transform which accounts for the case when K_1 is monotonic decreasing. Furthermore, we impose the following conditions:

- We have $[f(b)K_1(b)]/K_1'(b)\cdot[K_1(b)/K_1(x)]^\mu$ is finite.
- The quantity $[f(a)K_1(a)]/K_1'(a)\cdot[K_1(a)/K_1(x)]^\mu = 0$,

given K_1 is monotonic decreasing. Then the inverse transform of $\mathscr{A}$ is given by

$$f(x) = \pm\frac{c_1K_1'(x)}{2\pi\mathrm{i}K_1(x)}\lim_{T\to\infty}\int_{\frac{\mu-c_2-\mathrm{i}T}{c_1}}^{\frac{\mu-c_2+\mathrm{i}T}{c_1}} K_1(x)^{-(c_1s+c_2)}\mathscr{A}\{f\}(s)\,\mathrm{d}s, \tag{3.12}$$

where the negative sign is used when K_1 is monotonic decreasing.

3.7 The differential operator $\mathscr{D}$

Now that conditions for the uniqueness of our transform have been presented, as well as an inversion formula and a convolution formula, an application of integral transforms with logarithmic separable kernels arises. Due to the introduction of our transform, a closed-form solution can now be derived for certain types of second-order differential equations which, to the author's knowledge, could not be solved previously.

As before, suppose that K is logarithmic separable, i.e. $K(x,s) = K_1(x)^{K_2(s)}$ and $K_1'(x)\neq 0$ for $x\in I$. We will assume $K_1(I)\subseteq(0,1)$. Define the linear differential operator $\mathscr{D}$ by

$$\mathscr{D}f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{K_1(x)}{K_1'(x)}f(x)\right]. \tag{3.13}$$

We claim that

$$\mathscr{A}\{\mathscr{D}f\}(s) = \left[K_1(x)^{K_2(s)}\frac{K_1(x)}{K_1'(x)}f(x)\right]_I - K_2(s)\mathscr{A}\{f\}(s). \tag{3.14}$$

Integrating by parts, we see that

$$\begin{aligned}\mathscr{A}\{\mathscr{D}f\}(s) &= \int_I K_1(x)^{K_2(s)}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{K_1(x)}{K_1'(x)}f(x)\right]\mathrm{d}x\\ &= \left[K_1(x)^{K_2(s)}\frac{K_1(x)}{K_1'(x)}f(x)\right]_I - \int_I\frac{K_1(x)}{K_1'(x)}f(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[K_1(x)^{K_2(s)}\right]\mathrm{d}x.\end{aligned}$$

Since we have
$$\frac{K_1(x)}{K_1'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left[K_1(x)^{K_2(s)}\right] = K_2(s)K_1(x)^{K_2(s)},$$

the equation (3.14) holds. Similarly, if $\mathscr{D}^2f = \mathscr{D}(\mathscr{D}f)$, integrating by parts and using (3.14), we obtain

$$\begin{aligned}\mathscr{A}\{\mathscr{D}^2f\}(s) = &\left[K_1(x)^{K_2(s)}\frac{K_1(x)}{K_1'(x)}\mathscr{D}f(x) - K_2(s)K_1(x)^{K_2(s)}\frac{K_1(x)}{K_1'(x)}f(x)\right]_I\\ &+[K_2(s)]^2\mathscr{A}\{f\}(s).\end{aligned} \tag{3.15}$$

We are now ready to solve a family of nonhomogeneous, second-order differential equations of the form

$$\alpha\mathscr{D}^2 f(x)+\beta\mathscr{D}f(x)+\gamma f(x)=g(x), \quad x\in I=(a,b), \tag{3.16}$$

where $\alpha,\beta,\gamma\in\mathbb{R}$ and $\alpha>0$ without loss of generality. Furthermore, we assume $g\in C(I)$. Suppose that

$$\begin{gathered}\left[K_1(x)^{K_2(s)}\frac{K_1(x)}{K_1'(x)}f(x)\right]_I=0, \\ \left[K_1(x)^{K_2(s)}\frac{K_1(x)}{K_1'(x)}\mathscr{D}f(x)-K_2(s)K_1(x)^{K_2(s)}\frac{K_1(x)}{K_1'(x)}f(x)\right]_I=0.\end{gathered} \tag{3.17}$$

Due to the generality of our framework, it is not straightforward to examine the asymptotic behaviour at the endpoints of I. A classic example is the Mellin transform where, for $\operatorname{Re}(s)>1$ the kernel goes to ∞ when $x\to\infty$. Similarly, for $\operatorname{Re}(s)<1$, the kernel goes to 0 as $x\to\infty$. The existence of the transform also depends on the behaviour of the underlying function, which increases the complexity of analysing a general underlying function at the endpoints of I. To ensure generality, we assume that (3.17) holds to avoid varying behaviour like what happens with the Mellin transform. Applying the $\mathscr{A}$-transform

$$\mathscr{A}\{f\}(s)=\int_I K_1(x)^{K_2(s)}f(x)\,\mathrm{d}x$$

to (3.16) gives

$$\alpha[K_2(s)]^2\hat{f}(s)-\beta K_2(s)\hat{f}(s)+\gamma\hat{f}(s)=\hat{g}(s), \quad \hat{f}(s)=\mathscr{A}\{f\}(s), \quad \hat{g}(s)=\mathscr{A}\{g\}(s),$$

which implies

$$\hat{f}(s)=\frac{\hat{g}(s)}{\alpha[K_2(s)]^2-\beta K_2(s)+\gamma}.$$

If we can find a function $G=G(x)$ ("Green's function") such that

$$\hat{G}(s)=\mathscr{A}\{G\}(s)=\frac{1}{\alpha[K_2(s)]^2-\beta K_2(s)+\gamma},$$

$\hat{f}(s)=\hat{G}(s)\hat{g}(s)$ and the convolution property would imply that the solution of (3.16) is

$$f(x)=(G*g)(x)=\int_I G(\psi(x,u))g(u)D_1\psi(x,u)\chi_{J_u}(x)\,\mathrm{d}u, \tag{3.18}$$

where ψ is such that $K_1(\psi(x,u))=K_1(x)/K_1(u)$ in accordance with (A1)'.

As we are free to choose K_2, provided that it is a linear function of s, let us assume that $K_2(s)=s$ for simplicity so as to be able to apply the inversion formula (3.12). The

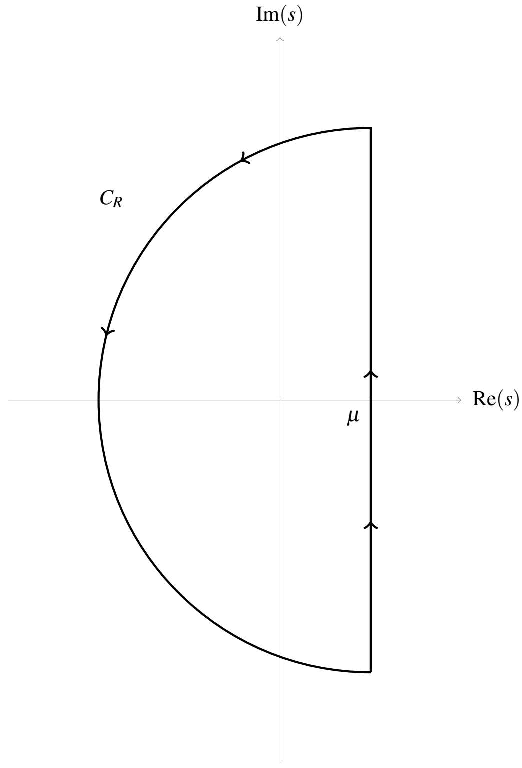

Figure 3.4: Visualisation of the contour used to evaluate the inverse of $\mathscr{A}\{f\}(s)$.

problem reduces to inverting

$$\hat{G}(s) = \frac{1}{\alpha s^2 - \beta s + \gamma}.$$

For definiteness, we assume that $K_1'(t) > 0$ and let us use the inversion formula (3.12) to find $G(x)$. Then ψ satisfies

$$K_1(\psi(x,u)) = K_1(x)/K_1(u) \quad \text{or} \quad \psi(x,u) = K_1^{-1}(K_1(x)/K_1(u)).$$

We need to evaluate the integral

$$G(x) = \frac{K_1'(x)}{2\pi \mathrm{i}} \int_{\mu-\mathrm{i}\infty}^{\mu+\mathrm{i}\infty} \frac{K_1(x)^{-s-1}}{\alpha s^2 - \beta s + \gamma} \,\mathrm{d}s = \frac{K_1'(x)}{2\alpha\pi \mathrm{i}} \int_{\mu-\mathrm{i}\infty}^{\mu+\mathrm{i}\infty} \frac{K_1(x)^{-s-1}}{(s-s_1)(s-s_2)} \,\mathrm{d}s,$$

where

$$s_1 = \frac{\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}, \quad s_2 = \frac{\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}.$$

The above contour integral can be expressed as

$$G(x) = \lim_{R\to\infty} \frac{K_1'(x)}{2\alpha\pi \mathrm{i}} \left[\int_{\Gamma_R} \frac{K_1(x)^{-s-1}}{(s-s_1)(s-s_2)} \,\mathrm{d}s - \int_{C_R} \frac{K_1(x)^{-s-1}}{(s-s_1)(s-s_2)} \,\mathrm{d}s \right], \tag{3.19}$$

where $\Gamma = [\mu - \mathrm{i}T, \mu + \mathrm{i}T] \cup C_R$ and $T = \sqrt{R^2 - \mu^2}$ (see Figure 3.4). Such techniques in solving integrals in the complex plane have been used in solid mechanics (see Boley & Chao [15]). Our aim is apply the Residue Theorem (see Conway [24]). The second term in (3.19) can be shown to tend to zero using the Estimation Lemma. The first term in (3.19) tends to $K_1'(x)/\alpha$ multiplied by the sum of the residues of

$$h(s) = \frac{K_1(x)^{-s-1}}{(s-s_1)(s-s_2)}.$$

Note that $\alpha(s_1 - s_2) = -\sqrt{\beta^2 - 4\alpha\gamma}$. Moreover,

$$\begin{aligned} &K_1(x)^{-s_1-1} - K_1(x)^{-s_2-1} \\ &\quad = K_1(x)^{-\beta/(2\alpha)-1}\Big\{K_1(x)^{\sqrt{\beta^2-4\alpha\gamma}/(2\alpha)} - K_1(x)^{-\sqrt{\beta^2-4\alpha\gamma}/(2\alpha)}\Big\} \\ &\quad = 2K_1(x)^{-\beta/(2\alpha)-1}\sinh\Big(\frac{\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}\log(K_1(x))\Big). \end{aligned}$$

1 We now need to consider the sign of the discriminant $\beta^2 - 4\alpha\gamma$.

2 3.7.1 $\beta^2 - 4\alpha\gamma > 0$

In this case s_1 and s_2 are simple poles of $h(s)$ and

$$\begin{aligned} \mathrm{Res}(h(s); s_1) &= \lim_{s\to s_1}(s-s_1)h(s) = \frac{K_1(x)^{-s_1-1}}{s_1 - s_2} \\ \mathrm{Res}(h(s); s_2) &= \lim_{s\to s_2}(s-s_2)h(s) = -\frac{K_1(x)^{-s_2-1}}{s_1 - s_2}. \end{aligned}$$

Then

$$\begin{aligned} G(x) &= \frac{K_1'(x)}{\alpha}\left\{\frac{K_1(x)^{-s_1-1}}{s_1-s_2} - \frac{K_1(x)^{-s_2-1}}{s_1-s_2}\right\} \\ &= -\frac{K_1'(x)}{\sqrt{\beta^2-4\alpha\gamma}}2K_1(x)^{-\beta/(2\alpha)-1}\sinh\Big(\frac{\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}\log(K_1(x))\Big) \\ &= -\frac{2}{\sqrt{\beta^2-4\alpha\gamma}}K_1'(x)K_1(x)^{-\beta/(2\alpha)-1}\sinh\Big(\frac{\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}\log(K_1(x))\Big). \end{aligned}$$

3 3.7.2 $\beta^2 - 4\alpha\gamma < 0$

The same argument works as in the previous case. Since $\sqrt{\beta^2 - 4\alpha\gamma} = \mathrm{i}\sqrt{4\alpha\gamma - \beta^2}$ and $\sinh(\mathrm{i}z) = \mathrm{i}\sin(z)$, we obtain

$$G(x) = -\frac{2}{\sqrt{4\alpha\gamma-\beta^2}}K_1'(x)K_1(x)^{-\beta/(2\alpha)-1}\sin\Big(\frac{\sqrt{4\alpha\gamma-\beta^2}}{2\alpha}\log(K_1(x))\Big).$$

3.7.3 $\beta^2 - 4\alpha\gamma = 0$

This time $s = s_1 = s_2$ is a pole of order 2 of $h(s)$; hence

$$\begin{aligned}\operatorname{Res}(h(s); s_1) &= \lim_{s\to s_1} \frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}s}\left[(s-s_1)^2 h(s)\right] \\ &= -\lim_{s\to s_1} K_1(x)^{-s-1}\log(K_1(x)) \\ &= -K_1(x)^{-\beta/(2\alpha)-1}\log(K_1(x)).\end{aligned}$$

Thus

$$G(x) = -\frac{K_1'(x)}{\alpha}K_1(x)^{-\beta/(2\alpha)-1}\log(K_1(x)).$$

It should be noted that we assumed K_1 was strictly monotonic increasing. If it is the case that K_1 is a strictly monotonic decreasing function, then we simply multiply the Green's function by -1. Furthermore, we highlight that we let K_2 be the identity function. This needs to be considered when examining numerical results.

3.8 Examples and numerical simulations

We present some concrete examples and compare our analytical solutions to the numerical solutions. Numerical calculations were implemented in MATLAB. More specifically, we used "ode45" to solve these differential equations numerically. The following examples are solutions to differential equations in which a unique solution can be shown to exist (see Zill [107]). After comparing these differential equations to those given in Polyanin & Zaitsev [79], it seems that analytical solutions to these differential equations have not been found. The coefficients α, β and γ in each example are chosen such that each case of the Green's function is used.

Example 59. Consider the case when the solution is defined on the interval $I = (1,\infty)$. Let $K_1(x) = \mathrm{e}^{1/x-x}$ and $J_u = (u,\infty)$. The general differential equation in (3.16) reduces to

$$\alpha\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^2}{x^2+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{x^2+1}f(x)\right)\right] - \beta\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{x^2+1}f(x)\right) + \gamma f(x) = g(x), \tag{3.20}$$

whose exact solution is given in (3.18), where

$$\psi(x,u) = \frac{1}{2}\left\{\sqrt{\left[\frac{1}{x} - x - \left(\frac{1}{u} - u\right)\right]^2 + 4} - \left[\frac{1}{x} - x - \left(\frac{1}{u} - u\right)\right]\right\}.$$

From Figure 3.5 it can be seen that there appears to be no significant difference between the numerical and analytical solutions. This provides evidence that, given the right conditions, the solution to the differential equation in (3.16) is the unique solution. Observe that for our choice of g, we have that $g \in L^1((1,\infty))$, so $g \in L^1((1,\infty), x \mapsto \mathrm{e}^{s(1/x-x)})$

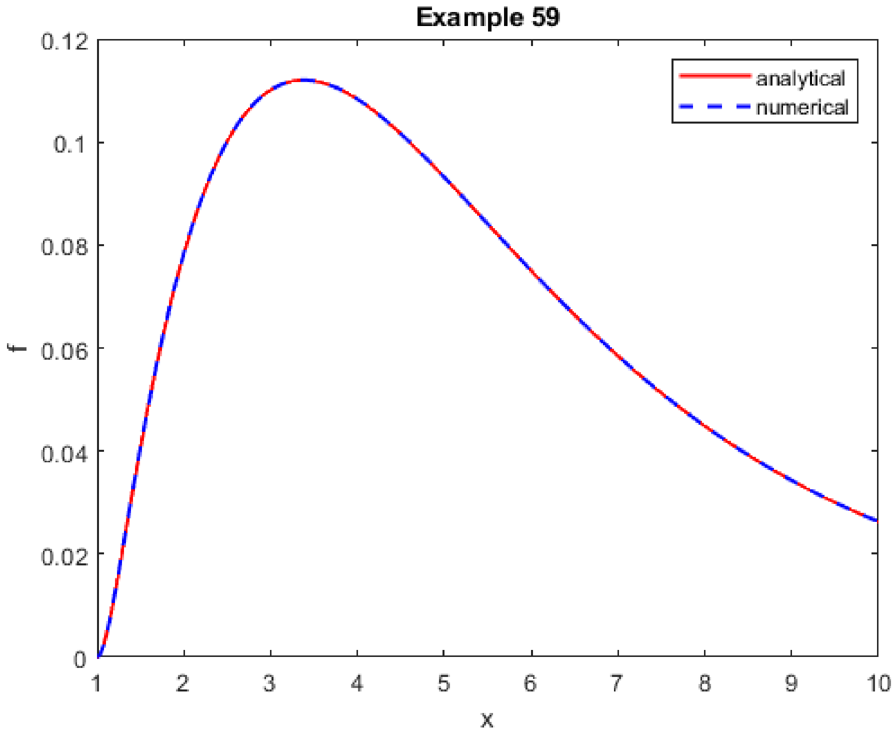

Figure 3.5: A plot of the solution to (3.20) when $\alpha = 2$, $\beta = -3$, $\gamma = 1$, $f(1) = 0$, $f'(1) = 0$, $g(x) = \dfrac{1}{1+x^2}$.

for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Due to the value of β being negative, it is to be expected that the solution to this differential equation vanishes as x approaches ∞. The numerical results appear to give a reasonable evidence for this belief. This gives an empirical justification for the solution to this differential equation belonging to the space $L^1((1,\infty), x \mapsto \mathrm{e}^{s(1/x-x)})$.

Example 60. Consider the case when the solution of the differential equation is defined on the interval $I = (1,\infty)$. Let $K_1(x) = \mathrm{e}^{-(x^2-1)}$. The differential equation in (3.16) simplifies to

$$\alpha \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2x}f(x)\right)\right] - \beta\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2x}f(x)\right) + \gamma f(x) = g(x), \tag{3.21}$$

where

$$\psi(x,u) = \sqrt{1+x^2-u^2}.$$

From Figure 3.6 it can once again be seen that the numerical solution appears to closely approximate the analytical solution. The discrepancy in the numerical and analytical solutions can be attributed to the oscillations which are not handled very well in MATLAB. We observe the oscillatory behaviour of the function, which is to be expected due to the presence of the sine function. We state for the sake of clarity that since K_1 is a monotonic decreasing function, in this case the Green's function is multiplied by -1. Despite the oscillatory behaviour of the solution, the maximum relative error of the an-

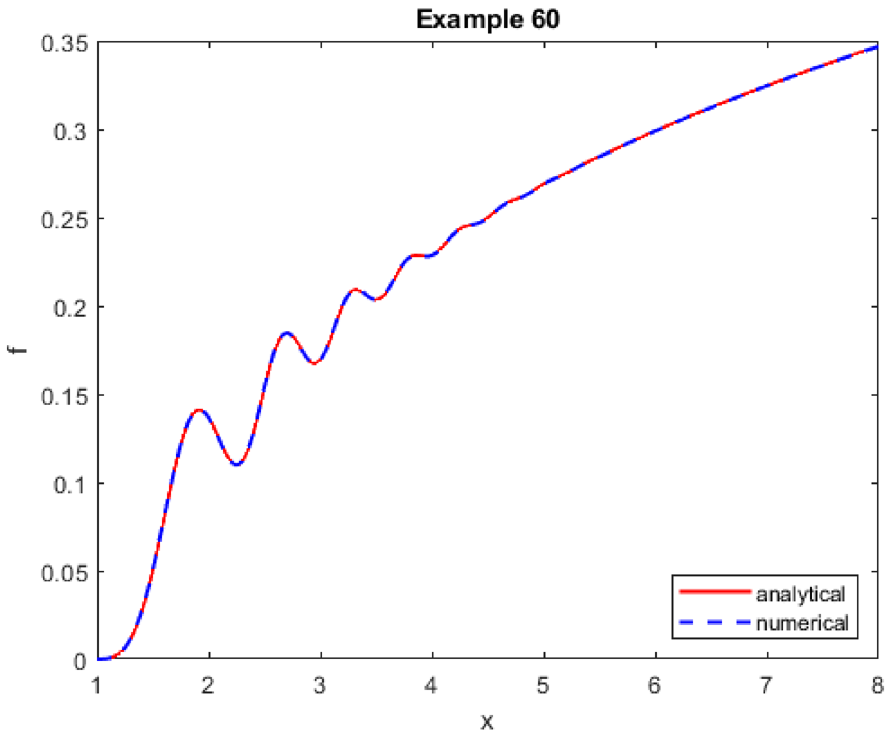

Figure 3.6: A plot of the solution to (3.21) when $\alpha = 2$, $\beta = -1$, $\gamma = 6$, $f(1) = 0$, $f'(1) = 0$, $g(x) = \log(x)$.

alytical solution and the numerical solution is approximately 0.0015. Furthermore, this occurs around $x = 1.09$. The relative errors at specified points can be found in Table 3.1. The relative error appears to be largest around the initial value for x. Based on the domain that we have restricted, the solution appears to diverge as its argument approaches infinity.

x	$f(x)$ (analytical)	$f(x)$ (numerical)	Relative error
1.045	3.2035×10^{-5}	3.1991×10^{-5}	0.0014
1.051	4.6925×10^{-5}	4.6873×10^{-5}	0.0015
1.942	0.14071859	0.14070198	1.1799×10^{-5}
2.249	0.11005336	0.11003224	1.9194×10^{-5}
2.476	0.14715522	0.14720036	3.0679×10^{-4}
3.011	0.17031870	0.17033485	9.4846×10^{-5}
3.91	0.22830278	0.22828523	7.6874×10^{-5}
4.293	0.24484085	0.24482939	4.6823×10^{-5}
5.377	0.28082541	0.28082523	6.3182×10^{-7}
6.554	0.31361967	0.31362256	9.2059×10^{-6}
7.739	0.34128029	0.34128010	5.7910×10^{-7}

Table 3.1: Tabulated results showing numerical values.

Example 61. Consider the case when $K_1(x) = \tan(x - \pi/4)$ and the differential equation is defined on the finite interval $I = (0, \pi/4)$. Let $J_u = (u, \pi/4)$. Substituting in the appropriate values, the differential equation in (3.16) is seen to be equivalent to

$$\alpha \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\tan(x)}{\sec^2(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\tan(x)}{\sec^2(x)}f(x)\right)\right] - \beta\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\tan(x)}{\sec^2(x)}f(x)\right) + \gamma f(x) = g(x). \tag{3.22}$$

Based on our choice of K_1, we have

$$\psi(x,u) = \pi/4 - \arctan\left(\frac{\tan(\pi/4-x)}{\tan(\pi/4-u)}\right).$$

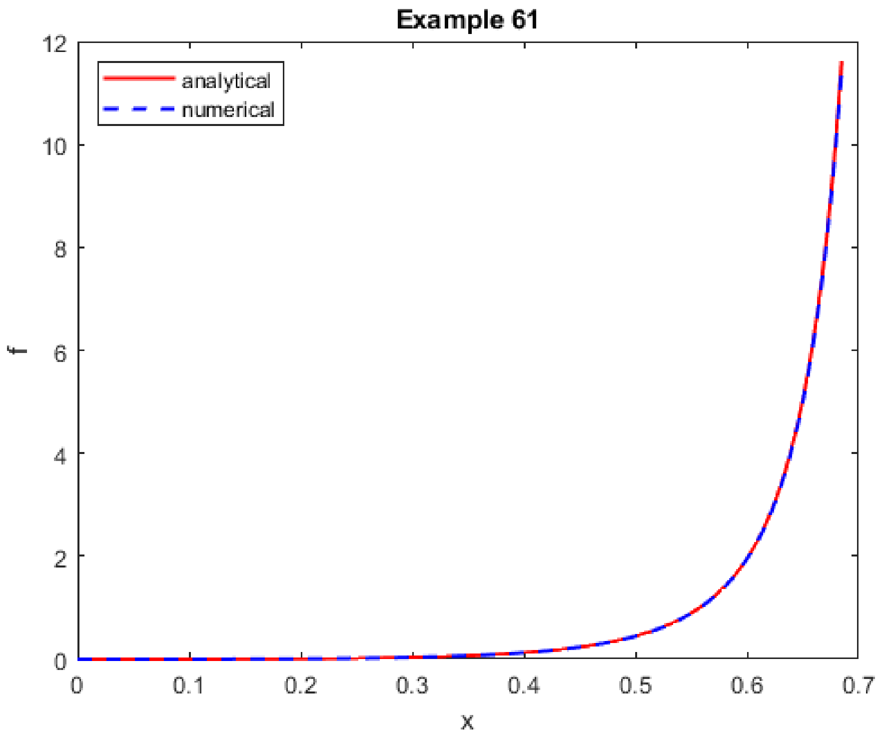

Figure 3.7: A plot of the solution to (61) when $\alpha = 1$, $\beta = 2$, $\gamma = 1$, $f(1) = 0$, $f'(1) = 0$, $g(x) = x$.

One thing to note about the solution is that it appears to approach ∞ as its argument approaches $\pi/4$. This is to be expected due to the presence of the reciprocal of the K_1 function which is present in the Green's function. The analytical and numerical solutions in Figure 3.7 are plotted on the interval $(0, \pi/4 - 0.1]$. The relative error of the analytical solution and the numerical solution at $x = \pi/4 - 0.1$ is approximately 0.0105. However, when the analytical and numerical solutions are calculated on the interval $(0, \pi/4 - 0.01]$, then the relative error at $x = \pi/4 - 0.001$ is approximately 0.084. This highlights the need for the analytical solution and gives evidence for the usefulness of our new class of integral transforms.

Chapter 4

Unifying discrete and integral transforms through the use of a Banach algebra

Recall that a weighted L^1 space was introduced in Chapter 2 as well as a class of discrete and integral transforms. Furthermore, the space J was restricted to two different kinds of connected regions in the complex plane. The weighted L^1 space is known to be a Banach space. The results in this chapter will be to show that, under a type of convolution product which is similar to the φ-convolution, the space $L^1(I,\mathscr{B},|\mu|)$ is a Banach algebra. We also show that the corresponding discrete and integral transforms defined on the aforementioned spaces are continuous homomorphisms between normed algebras.

4.1 Examples of our discrete and integral transforms

We start by introducing several transforms which are well known in the literature.

Example 62. Let K be a kernel which is of type (ii). Suppose $K_1(n) = -n$ and $w = 0$. The kernel becomes $K(n,s) = s^{-n}$. If we choose $I = \mathbb{N}_0$, our arbitrary transform reduces to the well-known Z-transform. Namely

$$\mathscr{A}\{f\}(s) = \sum_{n=0}^{\infty} s^{-n} f(n),$$

where we let $J = \mathbb{C}\backslash\overline{B_r(0)}$ for some $r > 1$.

Example 63. By choosing $I = (0,\infty)$, $K_1(x) = \mathrm{e}^{-x}$ and $w = 0$, we obtain the Laplace transform which is given by

$$\mathscr{A}\{f\}(s) = \int_0^{\infty} \mathrm{e}^{-sx} f(x)\,\mathrm{d}x,$$

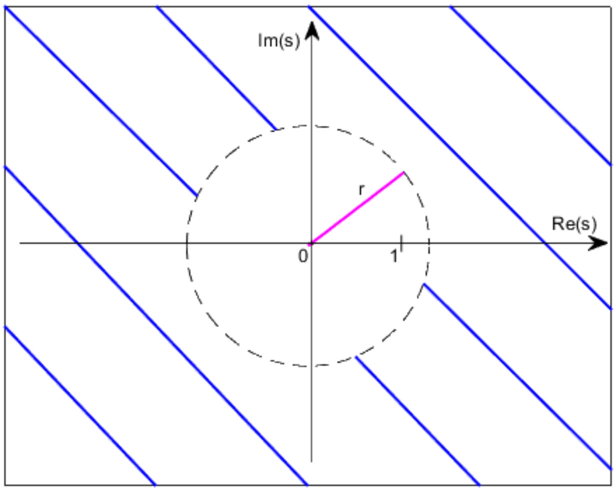

Figure 4.1: The region $J = \mathbb{C} \setminus \overline{B_r(0)}$.

where $J = \{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$. Note that the map $s \mapsto \mathrm{e}^{-sx}$ is continuous on J for every $x \in I$.

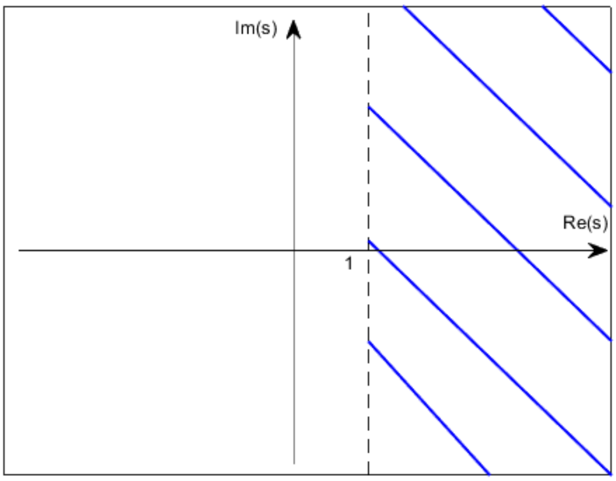

Figure 4.2: The region $J = \{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$.

It is highlighted that the constant function can be transformed in both Examples 62 and 63, which is the purpose of choosing the space J.

Example 64. We consider a kernel of the first kind where $K_1(n) = 1/n$ and $w = 0$. More specifically, we have $K(n,s) = n^{-s}$. Based on the way I is defined, the maximal domain for this potential transform is $I = \mathbb{N}$ given $K_1(n) > 0$. We also let J be any strip in the complex plane such that $\overline{J}$ lies on the right of the line $\mathrm{Re}(s) = 1$. By doing this, we see that the well-known Dirichlet series falls within our class of transforms. Namely, we have

$$\mathscr{A}\{f\}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} f(n).$$

If $f(n) = 1$ for every $n \in \mathbb{N}$ then $\mathscr{A}\{f\}(s) = \zeta(s)$, where ζ is the Riemann zeta function.

As stated in the introduction, the Dirichlet series has found applications to number theory. More specifically, the L-functions are related to the Bernoulli numbers which we will introduce for the reader's interest. A *Dirichlet character modulo m* is a function $\chi : \mathbb{Z} \to \mathbb{C} \setminus \{0\}$ which satisfies the following properties:

- $\chi(n+m) = \chi(n), \forall n \in \mathbb{Z}$;
- $\chi(nk) = \chi(n)\chi(k), \forall n,k \in \mathbb{Z}$;
- $\chi(n) \neq 0$ if and only if $\gcd(n,m) = 1$.

If χ is a Dirichlet character modulo m, the L-functions associated to χ are given by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \mathscr{A}\{\chi\}(s).$$

We now examine some lesser-known transforms which also fall in our class.

Example 65. Another discrete transform which falls into our class has a kernel of type (i) where $K_1(n) = \mathrm{e}^{-n}$ and $w = 0$. We can choose I to be any bounded subset of $\mathbb{N}_0$. However, we let $I = \mathbb{N}_0$. So we have

$$\mathscr{A}\{f\}(s) = \sum_{n=0}^{\infty} \mathrm{e}^{-sn} f(n).$$

One thing which should be noted is the large class of sequences whose transform exist when $K(n,s) = \mathrm{e}^{-sn}$ and $\overline{J}$ lies in the part in the complex plane where the real part is positive.

Example 66. We will present another example where I is an interval. Consider $I = (0,1)$, $K_1(x) = (1-x)/(1+x)$ and $w = 0$. This example was stated in Chapter 3, and in Futcher & Rodrigo [36]. The transform is of the form

$$\mathscr{A}\{f\}(s) = \int_0^1 \left(\frac{1-x}{1+x}\right)^s f(x)\,\mathrm{d}x.$$

Example 67. Another integral transform which falls in our proposed class of transforms is the one which transforms functions defined on the interval $I = (1,\infty)$, where $K_1(x) =$

$\mathrm{e}^{-(x^2-1)}$ and $w=0$. This gives us

$$\mathscr{A}\{f\}(s)=\int_1^\infty \mathrm{e}^{-s(x^2-1)}f(x)\,\mathrm{d}x.$$

This transform was analysed in Chapter 3. It was shown that this transform could be used to determine the closed-form solution to a second-order linear differential equation.

It is apparent from these examples that this class of transforms unifies the Laplace transform and the Dirichlet series. Another way in which these two concepts are linked is that they are both special cases of the Laplace-Stieltjes transform (see Apostol [7]). The Laplace-Stieltjes transform of f is precisely the integral of the function $x\mapsto \mathrm{e}^{-sx}$ with respect to the integrator function f over $[0,\infty)$. We highlight that there are transforms which can not be written as Laplace-Stieltjes transforms, however they fall in our class of transforms due to the generality of the kernel.

4.2 Continuity and the codomain of $\mathscr{A}$

We can now impose some restrictions for the codomain for $\mathscr{A}$ given an appropriate domain of functions.

Theorem 68. *Let $\mathscr{A}$ be either a discrete or integral transform, where K is logarithmic separable and is of either type (i) or type (ii). If $f\in L^1(I,\Omega,|\mu|)$, then $\mathscr{A}\{f\}\in C(J)$.*

Proof. It is noted that, for each fixed $x\in I$, the function

$$s\mapsto K(x,s)f(x)$$

is continuous. Furthermore, based on the assumptions on our class of transforms, we have

$$|f(x)||K(x,s)|\le |f(x)||K(x,\tilde{s})|=g(x)$$

for every $x\in I$ and $s\in J$. Here, we assume that $\tilde{s}\in\partial J$ has been chosen appropriately. We note that $g\in L^1(I,\Omega,\lambda)$, where λ is is the Lebesgue measure if the transform is an integral transform and the counting measure in the case when the transform is discrete. So ,if we let $\{s_j\}$ be a sequence in J which converges to some s_0, by Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned}\mathscr{A}\{f\}(s_0)&=\int_I f(x)\,\mathrm{d}\mu_{s_0}(x)\\&=\int_I K(x,s_0)f(x)\,\mathrm{d}\lambda(x)\\&=\lim_{j\to\infty}\int_I K(x,s_j)f(x)\,\mathrm{d}\lambda(x)\end{aligned}$$

$$= \lim_{j\to\infty} \mathscr{A}\{f\}(s_j).$$

As s_0 was an arbitrary point in J, we conclude that $\mathscr{A}\{f\}$ is continuous on J. □

The previous theorem shows that the codomain of $\mathscr{A}$ may be taken to be the set of continuous functions on J. We may now think of the operator $\mathscr{A}$ as the map $\mathscr{A}: L^1(I,\Omega,|\mu|) \to C(J)$. The previous proof was inspired by Bartle [11], who considered the integrand to be a general real-valued function of two variables. The variable in which the integration was taken over was an arbitrary set whereas the other variable belonged to a closed interval over the real line. We now establish some general properties of this class of transforms. We state for clarity that the norm on $BC(J)$ is defined by

$$||F||_{BC(J)} = \sup_{s\in J} |F(s)|,$$

where $BC(J)$ is the set of all complex-valued, bounded and continuous functions on J. Note that despite the fact that the transform of a function is only defined on an open set J, $\mathscr{A}\{f\}$ is continuous on $\overline{J}$ given it is defined by the formula in (2.23). This will help us with the following proposition.

Proposition 69. *Let $f \in L^1(I,\Omega,|\mu|)$ be defined such that $f \geq 0$ almost everywhere on I, then $||\mathscr{A}\{f\}||_{BC(J)} = ||f||_{\mu}$.*

Proof. Let $\{s_j\}$ be a sequence in J such that $s_j \to \tilde{s}$, where $\tilde{s} \in \partial J$ which has the following properties:

- For every $x \in I$, $K(x,\tilde{s})$ is real;
- For every $(x,s) \in I \times J$ we have $|K(x,s)| \leq K(x,\tilde{s})$.

By the way J is defined for each type of transform, such a sequence exists. Fix $f \in L^1(I,\Omega,|\mu|)$. Observe that for every $s \in J$,

$$\begin{aligned} |\mathscr{A}\{f\}(s)| &= \left| \int_I K(x,s) f(x)\,\mathrm{d}\lambda(x) \right| \\ &\leq \int_I K(x,\tilde{s})|f(x)|\,\mathrm{d}\lambda(x) \\ &= \max_{s\in\partial J} \int_I |f|\,\mathrm{d}|\mu_s|. \end{aligned} \tag{4.1}$$

Hence, for every $f \in L^1(I,\Omega,|\mu|)$, we have $||\mathscr{A}\{f\}||_{BC(J)} \leq ||f||_{\mu}$. Now, consider the sequence defined previously. An undemanding calculation yields

$$\max_{s\in\partial J} \int_I |f|\,\mathrm{d}|\mu_s| = \mathscr{A}\{f\}(\tilde{s}) = \lim_{j\to\infty} \mathscr{A}\{f\}(s_j),$$

since f is nonnegative almost everywhere. So, for every $\varepsilon > 0$, there exists a point s_N such that

$$\max_{s\in\partial J}\int_I |f|\,\mathrm{d}|\mu_s| < |\mathscr{A}\{f\}(s_N)| + \varepsilon$$
$$\leq \sup_{s\in J}|\mathscr{A}\{f\}(s)| + \varepsilon,$$

which implies

$$||f||_\mu \leq ||\mathscr{A}\{f\}||_{BC(J)}.$$

This completes the proof. □

The above proposition shows that each of our discrete and integral operators is an isometry on some subset of $L^1(I,\Omega,|\mu|)$. This also gives information about the continuity of $\mathscr{A}$ as a map between normed vector spaces.

Corollary 70. *The operator $\mathscr{A} : L^1(I,\Omega,|\mu|) \to BC(J)$ is continuous.*

Proof. Let $f \in L^1(I,\Omega,|\mu|)$ be such that $||f||_\mu = 1$. By the previous theorem we have that $||\mathscr{A}\{f\}||_{BC(J)} \leq 1$, thus $||\mathscr{A}|| \leq 1$. Since the operator $\mathscr{A}$ is bounded, it is therefore continuous. □

4.3 Convolution and shifting formulas for $\mathscr{A}$

Let $K : I \times \overline{J} \to \mathbb{C}$ be a logarithmic separable kernel of type (i) and let $K_1 \in C^1(I)$ if I is an interval . We define the convolution operation as follows

$$(f * g)(x) = \int_I f(\psi(x,u))g(u)\chi_{J_u}(x)(\overline{D}\psi)(x,u)\,\mathrm{d}\lambda(u), \tag{4.2}$$

where $J_u = K_1^{-1}(K_1(I)K_1(u)) = \left\{K_1^{-1}(K_1(x)K_1(u)) : x \in I\right\}$. Also, λ is the Lebesgue measure or the counting measure depending on the context. The function $\psi : J_u \times I \to I$ is defined by

$$\psi(x,u) = K_1^{-1}\left(\frac{K_1(x)}{K_1(u)}\right),$$

with $\overline{D}\psi = D_1\psi$, where $D_1\psi$ denotes the first partial derivative with respect to the first argument if I is an interval and $\overline{D}\psi \equiv 1$ if $I = \{r_i : i \in \mathbb{N}\}$. By the assumption that $K_1(I) \subseteq (0,1)$ or $K_1(I) \subseteq (1,\infty)$, combined with the value of $K_1(b)$, we guarantee that $J_u \subseteq I$ in the case where I is an interval. This follows from the following facts that K_1 is continuous and I is a connected subspace of $\mathbb{R}$, which implies that $K_1(I)$ is connected. We also have that the spaces $(0,1)$ and $(1,\infty)$ are closed under multiplication.

While these conditions on K_1 are sufficient for this to be true in the case where $\mathscr{A}$ is an integral transform, in the discrete case, we impose the further property that $K_1(I)K_1(u) \subseteq$

$K_1(I)$. Another convenient property is that $\psi(J_u,u)=I$, which is a consequence of the assumptions we have imposed.

Consider the case when the kernel is of type (ii). Here we let $\psi(x,u)=K_1^{-1}(K_1(x)-K_1(u))$ and $J_u=K_1^{-1}(K_1(I)+K_1(u))$. We further impose the condition that $K_1(u)+K_1(I)\subseteq K_1(I)$. We will now show that $\psi(J_u,u)=I$.

Fix $x\in\psi(J_u,u)$. By the definition of $\psi(J_u,u)$, there exists some $n\in J_u$ such that $x=K_1^{-1}(K_1(n)-K_1(u))$. Since $n\in J_u$, it holds that $n=K_1^{-1}(K_1(m)+K_1(u))$ for some $m\in I$. This is equivalent to $K_1(n)=K_1(m)+K_1(u)$. This gives us $x=K_1^{-1}(K_1(m)+K_1(u)-K_1(u))=m\in I$. Hence $\psi(J_u,u)\subseteq I$. Now, suppose $y\in I$. It follows that $K_1^{-1}(K_1(y)+K_1(u))\in J_u$. We then have

$$\begin{aligned}y&=K_1^{-1}(K_1(K_1^{-1}(K_1(y)+K_1(u)))-K_1(u))\\&=\psi(K_1^{-1}(K_1(y)+K_1(u)),u)\in\psi(J_u,u).\end{aligned}$$

Hence $I=\psi(J_u,u)$.

We observe that with how we have defined I and K_1, we guarantee that the set J_u is nonempty. It is helpful to derive the shifting property before we analyse the convolution property. Recall that $\mathscr{B}$ denotes the σ-algebra of all the Borel measurable sets on I. Note that in the case when I is countable, the Borel σ-algebra is exactly the power set of I.

Theorem 71. *Let K be a logarithmic separable kernel. Let $\mathscr{A}$ be in our proposed class of transforms. If $f\in L^1(I,\mathscr{B},|\mu|)$, then*

$$\mathscr{A}\{x\mapsto f(\psi(x,u))\chi_{J_u}(x)(\overline{D}\psi)(x,u)\}(s)=K(u,s)\mathscr{A}\{f\}(s). \tag{4.3}$$

Proof. We will first consider the case where $\mathscr{A}$ is an integral transform. First we need to show that

$$x\mapsto f(\psi(x,u))\chi_{J_u}(x)(\overline{D}\psi)(x,u) \tag{4.4}$$

is Borel measurable. As $K_1:I\to\mathbb{R}$ is a continuous injective map, by the Invariance of Domain Theorem, $K_1(I)$ is open and $K_1:I\to K_1(I)$ is a homeomorphism (see Munkres [71]). This implies that J_u is open, as such the Borel σ-algebra on J_u is a subset of $\mathscr{B}$. To show (4.4) is measurable, let $U\subseteq\mathbb{R}$ be an open set. The preimage of U under the function defined in (4.4) is

$$\{x\in J_u:f(\psi(x,u))(\overline{D}\psi)(x,u)\in U\}\bigcup\{x\in I\setminus J_u:0\in U\}. \tag{4.5}$$

The set of all values in the intersection of J_u and the set defined in (4.5) is in $\mathscr{B}$. This is evident due to the fact that the composition of Borel measurable functions is Borel measurable (see Rudin [84]) combined with the fact that products of measurable functions are measurable. Similarly, the intersection of $I\setminus J_u$ and the set in (4.5) is either the empty set or the the set $I \setminus J_u$ itself. As both of these are closed, we see that the function in (4.4) is measurable. Taking the transform of (4.4) gives us

$$\int_{J_u} f(\psi(x,u))(D_1\psi)(x,u)K_1(x)^{s-w}\,\mathrm{d}\lambda(x).$$

Note that since we stated that $\mathscr{A}$ is an integral transform, the kernel is of the form $K_1(x)^{s-w}$. Making the substitution $t = K_1^{-1}(K_1(x)/K_1(u))$, we have that the above integral is given by

$$\int_{K_1^{-1}(K_1(J_u)/K_1(u))} f(t)K_1(K_1^{-1}(K_1(t)K_1(u)))^{s-w}\,\mathrm{d}\lambda(t).$$

Observe that $I = K_1^{-1}(K_1(J_u)/K_1(u))$. So the above integral is given by

$$\int_I f(t)K_1(u)^{s-w}K_1(t)^{s-w}\,\mathrm{d}\lambda(t).$$

Writing the integral in terms of the integral transform gives us $K(u,s)\mathscr{A}\{f\}(s)$.

We will now assume I is countable. As $\mathscr{B}$ is the power set, the function in (4.4) is Borel measurable. Once again taking the transform of the function in (4.4) gives us

$$\sum_{x\in J_u} f(\psi(x,u))K(x,s).$$

Here the kernel can be of either type (i) or type (ii). We observe that $\overline{D}\psi \equiv 1$. Making the substitution $m = \psi(x,u)$ changes the above expression to

$$\sum_{m\in\psi(J_u,u)} f(m)K(\phi(m,u),s),$$

where $\phi(\cdot,u)$ is the inverse of the function $\psi(\cdot,u)$. It is straightforward to show that $\phi(\cdot,u)$ exists as $\psi(\cdot,u)$ is a composition of invertible functions. The sum above is over the set I. One can show that if the kernel is of either type (i) or (ii), we have $K(\phi(m,u),s) = K(m,s)K(u,s)$. So the $\mathscr{A}$-transform of the function $x \mapsto f(\psi(x,u))\chi_{J_u}(x)(\overline{D}\psi)(x,u)$ can be simplified to

$$\sum_{m\in I} f(m)K(m,s)K(u,s).$$

It is easy to see the above expression can be written as $K(u,s)\mathscr{A}\{f\}(s)$. This completes the proof. $\square$

There are well-known examples of integral transforms which have a so-called shifting property, such as the Fourier transform and the Mellin transform. While our class of transforms do not include these, we attain more generality by including discrete transforms. The Fourier and Mellin transforms were considered in [36] as well as in Chapter 3. Other examples of integral transforms with a shifting property include the Fourier sine and cosine transforms. Furthermore, integral transforms which have a shifting property and are logarithmic separable include the Fourier transform and the Mellin transform. We will now give some examples of some discrete transforms which have a shifting property.

Example 72. We see that the transform in Example 65 has a shifting property. Since $K_1(n) = \mathrm{e}^{-n}$, the image of $\mathbb{N}_0$ under K_1 is given by $K_1(\mathbb{N}_0) = \{\mathrm{e}^{-n} : n \in \mathbb{N}_0\}$. For every $K_1(m)$, it is evident that

$$\begin{aligned} K_1(m)K_1(\mathbb{N}_0) &= \{\mathrm{e}^{-(m+n)} : n \in \mathbb{N}_0\} \\ &\subseteq \{\mathrm{e}^{-n} : n \in \mathbb{N}_0\} = K_1(\mathbb{N}_0). \end{aligned}$$

Hence $\psi(J_u,u) \subset \mathbb{N}_0$ and a formula which satisfies the shifting property exists for this transform. Observe that $J_m = \{n+m : n \in \mathbb{N}_0\}$ and, substituting in the appropriate functions, the formula which satisfies the shifting property is defined on $n \in \mathbb{N}_0$ and given by

$$n \mapsto f(n-m)\chi_{J_m}(n). \tag{4.6}$$

Note that the function f is defined on nonnegative integers. We highlight the term $f(n-m)$ is undefined for $n < m$. However, by abuse of notation, we ensure the above expression is zero when $n < m$.

Example 73. We now present the shifting property for the Z-transform. The following function gives us the well-known shifting formula for the Z-transform

$$n \mapsto f(n-m)\chi_{m+\mathbb{N}_0}(n), \tag{4.7}$$

where $m+\mathbb{N}_0 = \{m+n : n \in \mathbb{N}_0\}$.

The convolution operation is an operation on some class of functions such that the formula $\mathscr{A}\{f*g\} = \mathscr{A}\{f\}\mathscr{A}\{g\}$ holds. It is straightforward to show this holds by making some formal calculations. We will now show the formula in (4.2) exists almost everywhere on I.

Theorem 74. *Suppose that $f,g \in L^1(I,\mathscr{B},|\mu|)$. Then $f*g$ exists almost everywhere on I and $f*g \in L^1(I,\mathscr{B},|\mu|)$.*

Proof. Fix $s \in \overline{J}$ such that $K(x,s) \neq 0$ for every $x \in I$. The case when I is an interval has already been proven in Futcher & Rodrigo [36]. However, we will modify the proof slightly to ensure further generality. Consider the measure space $(I\times I, \mathscr{B}\otimes\mathscr{B}, \lambda\times\lambda)$, where λ is the Lebesgue measure. We only need to show that the function defined by

$$(x,u) \mapsto K_1(x)^{s-w} f(\psi(x,u))g(u)\chi_{J_u}(x)(\overline{D}\psi)(x,u) \tag{4.8}$$

is $\mathscr{B}\otimes\mathscr{B}$-measurable. Fix $U\subseteq\mathbb{R}$ to be open. The preimage of the function in (4.8) is the union of the two following sets

$$\begin{gathered}\{(x,u)\in J_u\times I : K_1(x)^{s-w}f(\psi(x,u))g(u)(\overline{D}\psi)(x,u)\in U\}, \qquad (4.9)\\ \{(x,u)\in (I\times I)\setminus(J_u\times I) : 0\in U\}.\end{gathered}$$

Since the function $(x,u)\mapsto\psi(x,u)$ is continuous, it is a Borel function on $J_u\times I$, similarly for the function $(x,u)\mapsto(\overline{D}\psi)(x,u)$. Using the fact that the composition of Borel maps is Borel, it follows that the function in (4.8) is Borel measurable on $J_u\times I$. So the intersection of the preimage of the function in (4.8) with the set $J_u\times I$, which we will call S, is in the Borel σ-algebra on $J_u\times I$. As $J_u\times I\subseteq I\times I$ is open, the set S is in the Borel σ-algebra on $I\times I$. Because I is separable, the Borel σ-algebra on $I\times I$ is equal to $\mathscr{B}\otimes\mathscr{B}$ (see Folland [32]). From here it is straightforward to see that the union of the sets in (4.9) is in $\mathscr{B}\otimes\mathscr{B}$. As U is an arbitrary open set, we conclude that the function in (4.8) is $\mathscr{B}\otimes\mathscr{B}$-measurable. From here the proof is the same as that in Futcher & Rodrigo [36].

We will now present the case when I is countable. Fix $s\in\overline{J}$, we will be working with the measure space $(I\times I,\mathscr{B}\otimes\mathscr{B},\lambda\times\lambda)$, where λ is the counting measure on I. As $\mathscr{B}\otimes\mathscr{B}$ is the smallest σ-algebra containing all the measurable rectangles, it follows that $\mathscr{B}\otimes\mathscr{B}=\mathscr{P}(I\times I)$. As such the function defined by

$$(n,m)\mapsto K(n,s)f(\psi(n,m))g(m)\chi_{J_m}(n)$$

is $\mathscr{B}\otimes\mathscr{B}$-measurable. Now, consider the iterated integral

$$\begin{aligned}&\int_I\int_I |K(n,s)f(\psi(n,m))g(m)\chi_{J_m}(n)|\,\mathrm{d}\lambda(n)\,\mathrm{d}\lambda(m) &(4.10)\\ &=\sum_{m\in I}\sum_{n\in I}|K(n,s)f(\psi(n,m))g(m)|\chi_{J_m}(n). &(4.11)\end{aligned}$$

Making the same substitution that was used in Theorem 71, that is, letting $j=\psi(n,m)$, and by renaming variables, the above expression is simplified to

$$\begin{aligned}\sum_{m\in I}|K(m,s)||g(m)|\sum_{n\in I}|K(n,s)||f(n)| &= \int_I |g|\,\mathrm{d}|\mu_s|\int_I |f|\,\mathrm{d}|\mu_s|\\ &\leq ||g||_\mu\cdot||f||_\mu. \qquad (4.12)\end{aligned}$$

Hence we deduce that the iterated integral in (4.10) is finite. By Fubini's Theorem (see Hewitt & Stromberg [51]) the function $f*g$ defined by

$$(f*g)(n)=\sum_{m\in I}f(\psi(n,m))g(m)\chi_{J_m}(n)$$

exists almost everywhere on I due to the fact that $K(n,s) \neq 0$ for every $n \in I$. Furthermore, since the inequality in (4.12) holds for every $s \in \overline{J}$, it follows that $f*g \in L^1(I,\mathscr{B},|\mu|)$. □

We note that it is possible to take the transform of $f*g$ due to the fact that if $f,g \in L^1(I,\mathscr{B},|\mu|)$, then so too is $f*g$. We observe that the proof in Futcher & Rodrigo [36] shows that $f*g$ exists almost everywhere on I and that $f*g \in L^1(I,\mathscr{M},|\mu|)$ where $\mathscr{M}$ is the collection of all Lebesgue measurable sets which are subsets of I. The proof assumes that f is continuous, but it suffices to assume f is a Borel function given we restrict the σ-algebra to Borel sets on I. Also, it is evident by Fubini's Theorem (see Hewitt & Stromberg [51]) that if both f and g are Borel maps, then $f*g$ is also a Borel map. Using this and the previous theorem, the convolution operation is a binary operation $*: L^1(I,\mathscr{B},|\mu|) \times L^1(I,\mathscr{B},|\mu|) \to L^1(I,\mathscr{B},|\mu|)$ defined by (4.2).

Here, we will look at some examples where the convolution operation is commutative.

Example 75. Consider the transform in Example 64. The convolution formula is given by

$$(f*g)(n) = \sum_{m=1}^{\infty} f\left(\frac{n}{m}\right) g(m)\chi_{m\mathbb{N}}(n),$$

where $m\mathbb{N} = \{mn : n \in \mathbb{N}\}$. If we make the substitution $r = n/m$, the sum is equivalent to

$$(f*g)(n) = \sum_{n/r \in \mathbb{N}} f(r)g\left(\frac{n}{r}\right)\chi_{\mathbb{N}}(r).$$

If we let $S_n = \{n/k : k \in \mathbb{N}\}$, then the convolution formula is given by

$$(f*g)(n) = \sum_{r \in S_n \cap \mathbb{N}} f(r)g\left(\frac{n}{r}\right) = \sum_{r \in \mathbb{N}} f(r)g\left(\frac{n}{r}\right)\chi_{S_n}(r).$$

As we have $\chi_{S_n}(r) = \chi_{r\mathbb{N}}(n)$, it is apparent that

$$(f*g)(n) = \sum_{r=1}^{\infty} g\left(\frac{n}{r}\right) f(r)\chi_{r\mathbb{N}}(n) = (g*f)(n).$$

Note that the transform in Example 64 can be considered the discrete Mellin transform. While the function K_1 is defined in a similar way despite their respective domains, we highlight that a characteristic function appears in the this convolution formula. It should also be noted that the term $1/m$ does not appear in the above sum which is taken over m. A fact of significant importance is that this agrees with the convolution for the Dirichlet series stated in Apostol [9]. So we may write the sum present in the convolution formula as a sum over all the values of m which divide n. Namely

$$(f*g)(n) = \sum_{m|n} f\left(\frac{n}{m}\right) g(m).$$

Example 76. We will now analyse the convolution formula for the transform described in Example 65. The formula is given by

$$(f*g)(n)=\sum_{m=0}^{\infty} f(n-m)g(m)\chi_{m+\mathbb{N}_0}(n).$$

Now we make the substitution $r=n-m$, which gives us

$$(f*g)(n)=\sum_{n-r\in\mathbb{N}_0} f(r)g(n-r)\chi_{\mathbb{N}_0}(r).$$

Define $S_n=\{n-k:k\in\mathbb{N}_0\}$, the convolution formula is given by

$$\begin{aligned}(f*g)(n)&=\sum_{r\in S_n\cap\mathbb{N}_0} g(n-r)f(r)\\&=\sum_{r\in\mathbb{N}_0} g(n-r)f(r)\chi_{S_n}(r).\end{aligned}$$

Since we have $\chi_{S_n}(r)=\chi_{r+\mathbb{N}_0}(n)$, we deduce that $(f*g)(n)=(g*f)(n)$.

We will soon see that under further conditions the convolution operator is both commutative and associative. We now impose a further condition. If I is an interval, let $K_1(a)=1$. Before we do this we will introduce a new property.

4.4 Proof of the injectivity of $\mathscr{A}$

Note that when I is an interval, the kernel is of the form $K_1(x)^{s-w}$. Consider the case when $\mathrm{Re}(s-w)>0$ for every $s\in\overline{J}$. Since the function $1\in L^1(I,\Omega,|\mu|)$, we must have $K_1(a)=1$ and $K_1(b)=0$, where

$$K_1(a)=\lim_{x\to a^+} K_1(x).$$

This is due to the fact that the kernel must be bounded in $I\times\overline{J}$.

We see that if J is defined such that $\mathrm{Re}(s-w)<0$ for every $s\in\overline{J}$, then $K_1(a)=1$ and $K_1(b)=\infty$. It has been shown in Futcher and Rodrigo [36] that if $\mathscr{A}\{f\}\equiv 0$ on the space $J=\{s\in\mathbb{C}:\mathrm{Re}(s)>c\}$ for some fixed $c>\mathrm{Re}(w)$, and if $f\in L^1(I,\Omega,|\mu|)\cap C(I)$, then $f\equiv 0$. Because the case when $\mathscr{A}$ is an integral transform has already been considered in [36], we will focus on the case when $\mathscr{A}$ is discrete.

Theorem 77. *Let $f\in L^1(I,\Omega,|\mu|)$, where $\mathscr{A}$ is a discrete transform and the kernel is of type (i). If $\mathscr{A}\{f\}(s)=0$ for every $s\in J$ where $J=\{s\in\mathbb{C}:\mathrm{Re}(s)>c\}$ for some fixed $c>\mathrm{Re}(w)$, then $f\equiv 0$.*

Proof. We first highlight that with the way the kernel and J are defined, K_1 is monotonic decreasing. We will use mathematical induction to show that if $\mathscr{A}\{f\} \equiv 0$ then $f \equiv 0$. Let $\{s_j\}$ be a sequence of points in J such that the real part $\mathrm{Re}(s_j)$ approaches ∞. Now, consider the functions g_j, where $g_j(n) = f(n)K(n, s_j)$ and g, where g is defined by

$$g(n) = \lim_{j\to\infty} g_j(n).$$

Since $|g_j| \leq |f(\cdot)||K(\cdot,\tilde{s})|$, for some fixed $\tilde{s} \in \partial J$, and $|f(\cdot)||K(\cdot,\tilde{s})|$ is integrable, we apply Lebesgue's Dominated Convergence Theorem to the sequence of functions defined by $K_1(r_1)^{-s_j+w} g_j(\cdot)$, where r_1 is the smallest element in the set I. We deduce that

$$\begin{aligned} 0 &= \lim_{n\to\infty} K_1(r_1)^{-s_j+w} \mathscr{A}\{f\}(s_j) \\ &= \lim_{j\to\infty} \int_I \left[\frac{K_1(n)}{K_1(r_1)}\right]^{s_j-w} f(n)\,\mathrm{d}\lambda(n) \\ &= \lim_{j\to\infty} \int_I K_1(r_1)^{-s_j+w} g_j(n)\,\mathrm{d}\lambda(n), \end{aligned}$$

where λ is the counting measure. Moving the limit inside the integral gives us

$$0 = \int_I \lim_{j\to\infty} K_1(r_1)^{-s_j+w} g_j(n)\,\mathrm{d}\lambda(n).$$

Note that the integrand above is given by $(f(r_1), 0, 0, \ldots)$. Therefore $f(r_1) = 0$. Now, assume $f(r_i) = 0$ for every $1 \leq i \leq k$. Using this, we will show that $f(r_{k+1}) = 0$. By a similar argument shown previously, applying Lebesgue's Dominated Convergence Theorem to $K_1(r_{k+1})^{-s_j+w} g_j(\cdot)$ gives us

$$0 = \int_I \lim_{j\to\infty} K_1(r_{k+1})^{-s_j+w} g_j(n)\,\mathrm{d}\lambda(n).$$

1 The above integral is equal to $f(r_{k+1})$. Hence $f \equiv 0$. $\square$

The previous proof assumed the kernel was of type (i). A similar proof of the injectivity of the transform $\mathscr{A}$ holds if $K(t,s) = K_1(t)^{s-w}$ where K_1 is strictly monotonic increasing on the domain I and we choose $J = \{s \in \mathbb{C} : \mathrm{Re}(s) < d\}$ for some fixed $d < \mathrm{Re}(w)$. From now on, when we work with kernels of type (i), we let $J = \{s \in \mathbb{C} : \mathrm{Re}(s) > c\}$ or $J = \{s \in \mathbb{C} : \mathrm{Re}(s) < d\}$. When the kernel is of type (ii), and if $\mathscr{A}\{f\} \equiv 0$ on J for some $f \in L^1(I, \Omega, |\mu|)$, then we have

$$\sum_{i=1}^{\infty} f(r_i)(s-w)^{K_1(r_i)} = 0.$$

since $f \in L^1(I, \Omega, |\mu|)$. The above series converges absolutely on J. Suppose that $C : [0, 2\pi] \to J$ is the function defined by $C(t) = w + \varepsilon \mathrm{e}^{\mathrm{i}t}$. Fix $j \in \mathbb{N}$. Multiplying the above function by the function $s \mapsto (s-w)^{-K_1(r_j)-1}$ and then integrating the new function over the closed contour C gives us

$$\sum_{i=1}^{\infty} \int_C \frac{f(r_i)}{(s-w)^{K_1(r_j)-K_1(r_i)+1}}\,\mathrm{d}s = 0.$$

Recall that

$$\int_C \frac{1}{(s-w)^{K_1(r_j)-K_1(r_i)+1}}\,\mathrm{d}s = \begin{cases} 2\pi\mathrm{i} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The above computation shows that $f(r_j) = 0$. As j is arbitrary, we conclude that $f \equiv 0$. From here, a simple proof of an inversion formula for transforms of the form

$$\mathscr{A}\{f\}(s) = \sum_{x \in I} (s-w)^{K_1(x)} f(x), \tag{4.13}$$

where K_1 is an integer-valued function which does not change sign. As the sum in (4.13) converges uniformly on compact subsets of J, we see that the function $\mathscr{A}\{f\}$ is holomorphic when $f \in L^1(I,\mathscr{B},|\mu|)$. Let C be a contour in J whose interior contains w. A straightforward computation yields

$$\int_C \frac{\mathscr{A}\{f\}(s)}{(s-w)^{K_1(x)+1}}\,\mathrm{d}s = \int_C \sum_{t \in I} (s-w)^{K_1(t)-K_1(x)-1} f(t)\,\mathrm{d}s.$$

Since the series in (4.13) converges uniformly, we may interchange the summation and the integral, which gives us

$$f(x) = \frac{1}{2\pi\mathrm{i}} \int_C \frac{\mathscr{A}\{f\}(s)}{(s-w)^{K_1(x)+1}}\,\mathrm{d}s. \tag{4.14}$$

By letting $\mathscr{A}$ be a discrete transform, and using the injectivity of $\mathscr{A}$, it is straightforward to show certain properties of the convolution operation. These properties are associativity, commutativity, and distributivity of the convolution operation with respect to addition.

4.5 The Banach algebra $L^1(I,\mathscr{B},|\mu|)$

We will consider the case when $\mathscr{A}$ is a discrete transform and combine the three properties stated previously into one theorem. While the three properties hold when $\mathscr{A}$ is an integral transform, the proof is more complicated and will be shown later.

Theorem 78. *Let $\mathscr{A} : L^1(I,\mathscr{B},|\mu|) \to BC(J)$ be a discrete transform, where J is defined* *such that $\mathscr{A}$ is injective. Then for every $f,g,h\in L^1(I,\mathscr{B},|\mu|)$, we have*

$$f*g=g*f, \qquad f*(g*h)=(f*g)*h, \qquad f*(g+h)=f*g+f*g.$$

Proof. By Theorem 74 we can convolve any functions in $L^1(I,\mathscr{B},|\mu|)$. We will use the commutativity, associativity and distributivity properties of the product for complex-valued functions. A simple calculation yields

$$\mathscr{A}\{f*g\}=\mathscr{A}\{f\}\mathscr{A}\{g\}=\mathscr{A}\{g\}\mathscr{A}\{f\}=\mathscr{A}\{g*f\}.$$

Which used the commutativity of complex-valued functions. We also have

$$\begin{aligned}\mathscr{A}\{f*(g*h)\}&=\mathscr{A}\{f\}\mathscr{A}\{g*h\}=\mathscr{A}\{f\}\mathscr{A}\{g\}\mathscr{A}\{h\}\\ \mathscr{A}\{(f*g)*h\}&=\mathscr{A}\{f*g\}\mathscr{A}\{h\}=\mathscr{A}\{f\}\mathscr{A}\{g\}\mathscr{A}\{h\},\end{aligned}$$

which used the fact that multiplication of complex-valued functions is associative. Furthermore, the distributative properties of complex functions gives us

$$\begin{aligned}\mathscr{A}\{f*(g+h)\}&=\mathscr{A}\{f\}\mathscr{A}\{g\}+\mathscr{A}\{f\}\mathscr{A}\{h\}\\ &=\mathscr{A}\{f*g\}+\mathscr{A}\{f*h\}=\mathscr{A}\{f*g+f*h\}.\end{aligned}$$

Since $\mathscr{A}\{f*g\}=\mathscr{A}\{g*f\}$, $\mathscr{A}\{f*(g*h)\}=\mathscr{A}\{(f*g)*h\}$ and $\mathscr{A}\{f*(g+h)\}=\mathscr{A}\{f*g+f*h\}$, applying the injectivity of $\mathscr{A}$ gives us the desired results. □

We note that the previous proof would not work if $\mathscr{A}$ is an integral transform as we require the underlying functions to be continuous. We will now show these three properties are true when $\mathscr{A}$ is an integral transform. For convenience, we will only present the case when K_1 is monotonic increasing, and $K_1(a)=1$, $K_1(b)=\infty$.

Proposition 79. *Let $f,g\in L^1(I,\mathscr{B},|\mu|)$. Given I is an interval and K_1 satisfies the properties stated previously, then $f*g=g*f$.*

Proof. Observe that since $\chi_{J_u}(x)=\chi_{(a,x)}(u)$, we have

$$(f*g)(x)=\int_{(a,x)}f(\psi(x,u))g(u)D_1\psi(x,u)\,\mathrm{d}u. \tag{4.15}$$

If $y=\psi(x,u)$, then $u=\psi(x,y)$ and

$$\frac{\partial y}{\partial u}=-\frac{D_1\psi(x,u)}{D_1\psi(x,y)}.$$

Making this substitution gives us

$$(f*g)(x)=\int_{(a,x)} f(y)g(\psi(x,y))D_1\psi(x,y)\,\mathrm{d}y=(g*f)(x).$$

1 $\square$

2 **Theorem 80.** *The space $L^1(I,\mathscr{B},|\mu|)$ is associative under $*$.*

Proof. Let x be a value such that $(f*(g*h))$, $((f*g)*h)$ and $(|f|*(|g|*|h|))$ are defined at x. It should be noted that the set of all x where at least one of the following previously defined functions does not exist is a null set. We have

$$\begin{aligned}(f*(g*h))(x)&=\int_I f(\psi(x,u))\chi_{(u,b)}(x)D_1\psi(x,u)\\&\qquad\times\int_I g(\psi(u,y))\chi_{(y,b)}(u)D_1\psi(u,y)h(y)\,\mathrm{d}y\,\mathrm{d}u\\&=\int_I\int_I f(\psi(x,u))\chi_{(u,b)}(x)D_1\psi(x,u)g(\psi(u,y))\chi_{(y,b)}(u)\\&\qquad\times D_1\psi(u,y)h(y)\,\mathrm{d}u\,\mathrm{d}y\\&=\int_{(a,x)}\int_{(y,x)} f(\psi(x,u))D_1\psi(x,u)g(\psi(u,y))D_1\psi(u,y)h(y)\,\mathrm{d}u\,\mathrm{d}y.\end{aligned}$$

For the sake of clarity we highlight that we can reverse the order of integration due to the fact that $(|f|*(|g|*|h|))(x)$ is finite. If we let $z=\psi(u,y)$, we have

$$\frac{\partial z}{\partial u}=D_1\psi(u,y),$$

which implies

$$\begin{aligned}(f*(g*h))(x)=\int_a^x\int_a^{\psi(x,y)} &f(\psi(\psi(x,y),z))g(z)h(y)\\&\times D_1\psi(x,K_1^{-1}(K_1(y)K_1(z)))\,\mathrm{d}z\,\mathrm{d}y.\end{aligned}$$

For the purpose of being thorough we repeat the definition that

$$\psi(x,u)=K_1^{-1}\left(K_1(x)/K_1(u)\right).$$

In the above expression for $f*(g*h)$, we used the fact that $\psi(x,K_1^{-1}(K_1(y)K_1(z)))=\psi(\psi(x,z),y)$. Because the identity

$$D_1\psi(x,K_1^{-1}(K_1(y)K_1(z)))=D_1\psi(\psi(x,y),z)D_1\psi(x,y)$$

holds, the function $f*(g*h)$ evaluated at x can be manipulated as follows

$$\begin{aligned}(f*(g*h))(x) &= \int_a^x \int_a^{\psi(x,y)} f(\psi(\psi(x,y),z))g(z)h(y)\\ &\qquad \times D_1\psi(\psi(x,y),z)D_1\psi(x,y)\,\mathrm{d}z\,\mathrm{d}y\\ &= \int_a^x \int_a^{\psi(x,y)} f(\psi(\psi(x,y),z))g(z)D_1\psi(\psi(x,y),z)\,\mathrm{d}z\\ &\qquad \times h(y)D_1\psi(x,y)\,\mathrm{d}y\\ &= \int_a^x (f*g)(\psi(x,y))h(y)D_1\psi(x,y)\,\mathrm{d}y\\ &= ((f*g)*h)(x).\end{aligned}$$

As this is true for almost all $x \in I$, it follows that $(f*(g*h))$ and $((f*g)*h)$ are in the same equivalence class. Hence the convolution operation is associative. □

Proposition 81. *Let $f,g,h \in L^1(I,\mathscr{B},|\mu|)$, then $f*(g+h) = f*g+f*h$.*

Proof. Fix $x \in I$ such that $(f*g)(x)$ and $(f*h)(x)$ exist. By definition, we have $(f*g+f*h)(x) = (f*g)(x)+(f*h)(x)$, which gives us

$$\begin{aligned}(f*g+f*h)(x) &= \int_I f(\psi(x,u))g(u)\chi_{J_u}(x)D_1\psi(x,u)\,\mathrm{d}u\\ &\qquad + \int_I f(\psi(x,u))h(u)\chi_{J_u}(x)D_1\psi(x,u)\,\mathrm{d}u\\ &= \int_I f(\psi(x,u))(g+h)(u)\chi_{J_u}(x)D_1\psi(x,u)\,\mathrm{d}u\\ &= (f*(g+h))(x).\end{aligned}$$

This is true almost everywhere on I. Therefore $f*(g+h) = f*g+f*h$. □

We note that since the convolution operation $*$ is commutative, the following equation $(f+g)*h = f*h+g*h$ holds. It is straightforward to show that for any $\alpha \in \mathbb{C}$ that $(\alpha f)*g = f*(\alpha g) = \alpha(f*g)$. Because of this, we only need to show one other property to conclude $L^1(I,\mathscr{B},|\mu|)$ is a Banach algebra.

Theorem 82. *Let $f,g \in L^1(I,\mathscr{B},|\mu|)$. Then $||f*g||_\mu \leq ||f||_\mu \cdot ||g||_\mu$.*

Proof. We will start by assuming I is countable and prove the case when I is an interval later. The following holds for every $s \in \overline{J}$

$$\begin{aligned}\int_I |f*g|\,\mathrm{d}|\mu_s| &\leq \int_I\int_I |K(x,s)||f(\psi(x,m))||g(m)|\chi_{J_m}(x)\,\mathrm{d}\lambda(x)\,\mathrm{d}\lambda(m)\\ &= \sum_{m\in I}\sum_{x\in J_m} |K(x,s)||f(\psi(x,m))||g(m)|,\end{aligned}$$

where λ is the counting measure. By introducing the substitution

$$n = K_1^{-1}(K_1(x)K_1(m)),$$

and using the fact that $K(K_1^{-1}(K_1(x)K_1(m)),s) = K(n,s)K(m,s)$, we have

$$\begin{aligned}\int_I |f*g|\,\mathrm{d}|\mu_s| &\le \sum_{m\in I}|K(m,s)|\sum_{n\in I}|K(n,s)||f(n)||g(m)|\\ &= \int_I |g|\,\mathrm{d}|\mu_s|\int_I |f|\,\mathrm{d}|\mu_s|\\ &\le ||f||_\mu\cdot||g||_\mu.\end{aligned}$$

As this is true for any $s\in\overline{J}$, the following inequality holds

$$\begin{aligned}||\mathscr{A}\{|f*g|\}||_{BC(J)} &= \sup_{s\in J}\int_I |f*g|\,\mathrm{d}|\mu_s|\\ &\le ||f||_\mu\cdot||g||_\mu.\end{aligned}$$

By Proposition 69, the inequality $||f*g||_\mu \le ||f||_\mu\cdot||g||_\mu$ holds for every $f,g\in L^1(I,\mathscr{B},|\mu|)$.

Now, suppose I is an interval. The computation is similar to that in the countable case. We see that

$$\begin{aligned}\int_I |f*g|\,\mathrm{d}|\mu_s| &\le \int_I\int_{J_u}|K(x,s)||f(\psi(x,u))||g(u)|D_1\psi(x,u)\,\mathrm{d}\lambda(x)\,\mathrm{d}\lambda(u)\\ &\le \int_I\int_{J_u}K(x,\tilde{s})|f(\psi(x,u))||g(u)|D_1\psi(x,u)\,\mathrm{d}\lambda(x)\,\mathrm{d}\lambda(u)\\ &= \int_I \mathscr{A}\{|f(\psi(\cdot,u))|D_1\psi(\cdot,u)\}(\tilde{s})|g(u)|\,\mathrm{d}\lambda(u).\end{aligned}$$

An application of the shifting property gives us

$$\begin{aligned}\int_I |f*g|\,\mathrm{d}|\mu_s| &= \int_I |K(t,\tilde{s})||f(t)|\,\mathrm{d}\lambda(t)\int_I |K(u,s)||g(u)|\,\mathrm{d}\lambda(u)\\ &\le ||f||_\mu\cdot||g||_\mu.\end{aligned}$$

This holds true for every $s\in\overline{J}$. Using Proposition 69 once again shows $||f*g||_\mu \le ||f||_\mu\cdot||g||_\mu$. $\square$

We have thus shown that $L^1(I,\mathscr{B},|\mu|)$ is a Banach algebra using the new convolution formula we have defined. When I is an interval, this space shares some properties with the typical Banach algebra $L^1(\lambda)$ under the standard addition of functions and Fourier convolution for the product. One other similarity which arises is the fact that our Banach algebra contains no unit when I is an interval.

Proposition 83. *Let I be an interval, then the space $L^1(I, \mathscr{B}, |\mu|)$ contains no unit.*

Proof. Suppose the space contains a unit v and we will derive a contradiction. Let $f \in L^1(I, \mathscr{B}, |\mu|)$ be defined such that it is positive almost everywhere on I. We know such a function exists because the constant function 1 is in $L^1(I, \mathscr{B}, |\mu|)$. Observe that

$$v * f = f \text{ which implies } \mathscr{A}\{v\}\mathscr{A}\{f\} = \mathscr{A}\{f\}.$$

This yields

$$(\mathscr{A}\{v\} - 1)\mathscr{A}\{f\} = 0.$$

Let z be a value such that $K(x,z)$ is positive for every $x \in I$. Suppose J is extended to a half of the plane. More specifically, let $J = \{s \in \mathbb{C} : \mathrm{Re}(s) > c\}$. We introduce the set

$$\hat{J} = \{s \in \mathbb{C} : s = z + t \text{ for some } t \in [0, \infty)\}.$$

It is evident that $\mathscr{A}\{f\}(s) > 0$ for every $s \in \hat{J}$. Hence $\mathscr{A}\{v\} \equiv 1$ on $\hat{J}$. Now, let $\{s_j\} \subseteq \hat{J}$ be a sequence of points such that $\mathrm{Re}(s_j) \to \infty$. As $\mathscr{A}\{v\}(s_j)$ is finite for every j, and $v \in L^1(I, \mathscr{B}, |\mu|)$, we can apply Lebesgue's Dominated Convergence Theorem which gives us

$$\begin{aligned} 0 &= \int_I \lim_{j\to\infty} K_1(x)^{s_j - w} v(x)\,\mathrm{d}x \\ &= \lim_{j\to\infty} \int_I K_1(x)^{s_j - w} v(x)\,\mathrm{d}x \\ &= \lim_{j\to\infty} \mathscr{A}\{v\}(s_j) = 1. \end{aligned}$$

Which is a contradiction. Hence $L^1(I, \mathscr{B}, |\mu|)$ has no unit when I is an interval. □

The above proposition does not hold when $\mathscr{A}$ is a discrete transform. This is due to the fact that there are examples where $L^1(I, \mathscr{B}, |\mu|)$ has a unit.

Example 84. Let $\mathscr{A}$ be the Z-transform. Consider the function $\delta_0 : \mathbb{N}_0 \to \mathbb{R}$ defined by $\delta_0(0) = 1$ and $\delta_0(n) = 0$ for $n \in \mathbb{N}$. For every $f \in L^1(I, \mathscr{B}, |\mu|)$, it holds that

$$(\delta_0 * f)(n) = \sum_{m=0}^{\infty} \delta_0(m-n) f(m) \chi_{m+\mathbb{N}_0}(n).$$

Using the fact that $\delta_0(m-n) = 0$ for $m \neq n$ gives us

$$(\delta_0 * f)(n) = \delta(n-n) f(n) \chi_{n+\mathbb{N}_0}(n) = f(n).$$

Hence $\delta_0 * f = f$ for every $f \in L^1(I, \mathscr{B}, |\mu|)$.

We would still like to give some conditions which guarantee that $L^1(I,\mathscr{B},|\mu|)$ is a Banach algebra without a unit when $\mathscr{A}$ is a discrete transform. More specifically, when K is a kernel of type (ii). This proof was suggested by an anonymous referee.

Proposition 85. *Let K be a kernel of type (ii). That is to say $K(x,s)=(s-w)^{K_1(x)}$, where K_1 is an integer-valued function which does not change sign. If $K_1(x)\neq 0$ on I, then $L^1(I,\mathscr{B},|\mu|)$ is nonunital.*

Proof. Let $f\in L^1(I,\mathscr{B},|\mu|)$ be any nonzero function and suppose $L^1(I,\mathscr{B},|\mu|)$ contains a unit v. Since

$$v*f=f,$$

it is straightforward to show that

$$(\mathscr{A}\{v\}(s)-1)\mathscr{A}\{f\}(s)=0$$

for every $s\in J$. Since f is not zero and $\mathscr{A}$ is injective, $\mathscr{A}\{f\}$ is not zero. By the identity theorem, it follows that $\mathscr{A}\{f\}(s)$ is nonzero for almost all $s\in J$. This implies that $\mathscr{A}\{v\}(s)=1$ almost everywhere on J.

Now, let C be a closed contour in J which contains w. Using the formula in (4.14), we have

$$\begin{aligned} v(x) &= \frac{1}{2\pi \mathrm{i}}\int_C \frac{\mathscr{A}\{v\}(s)}{(s-w)^{K_1(x)+1}}\,\mathrm{d}s \\ &= \frac{1}{2\pi \mathrm{i}}\int_C \frac{1}{(s-w)^{K_1(x)+1}}\,\mathrm{d}s. \end{aligned}$$

Here we have used the fact that $\mathscr{A}\{v\}(s)=1$ almost everywhere on J. Since C is homotopic to a circle, an application of the Cauchy integral formula yields $v(x)=0$, which is a contradiction. □

Chapter 5

Generalised convolutions on distributions and measures

In the previous chapter it was shown that $L^1(I, \mathscr{B}, |\mu|)$ is a nonunital Banach algebra. We introduce several concepts in this chapter. Several miscellaneous theorems which did not fit in Chapter 3 or Chapter 4 are included in this chapter. Furthermore, we extend the convolution product to distributions and measures. Before we do this, we change some of the assumptions which were present in the previous chapters. One such assumption is that, given $K_1 : I \to \mathbb{R}$ is defined such that $K_1(I) = (0,1)$ or $K_1(I) = (1,\infty)$, we assume that the domain of K_1 can be extended continuously such that $K_1(I) = (0,1]$ or $K_1(I) = [1,\infty)$. As such, we will no longer require I to be an open interval. Instead we let $I \backslash \{K_1^{-1}(1)\}$ be open and $I = [a,b)$ or $I = (a,b]$. We still require I to be a subset of $\mathbb{R}$. For the sake of simplifying the notation, we impose

$$\hat{I} = I \backslash \{K_1^{-1}(1)\}.$$

Recall that $J_u = K_1^{-1}(K_1(I)K_1(u))$. Based on the assumptions we have made, this gives two possible options for J_u. Namely:

$$J_u = (a,u], \qquad\qquad J_u = [u,b).$$

Previously we assumed that $K_1(I) \subseteq (0,1)$ or $K_1(I) \subseteq (1,\infty)$. Hence we have placed further restrictions on our class of transforms. Furthermore, we will focus on integral transforms, while making few comments on discrete transforms.

Despite the restrictions we have imposed, there are plenty of transforms which fall in our class.

It appears that some abuse of notation needs to be clarified. More specifically, in the case

$$K_1(x) = \frac{\sin(x)}{x}.$$

$K_1(x)$	I	$\hat{I}$	J
e^{-x}	$[0,\infty)$	$(0,\infty)$	$\{s\in\mathbb{C}:\mathrm{Re}(s)>1\}$
$1/(1+x^2)$	$[0,\infty)$	$(0,\infty)$	$\{s\in\mathbb{C}:\mathrm{Re}(s)>1\}$
$(1-x)/(1+x)$	$[0,1)$	$(0,1)$	$\{s\in\mathbb{C}:\mathrm{Re}(s)>0\}$
$\tan(x)$	$(0,\pi/4]$	$(0,\pi/4)$	$\{s\in\mathbb{C}:\mathrm{Re}(s)>0\}$
$1/(c+x)$	$[1-c,\infty)$	$(1-c,\infty)$	$\{s\in\mathbb{C}:\mathrm{Re}(s)>2\}$
$e^{-(x^2-1)}$	$[1,\infty)$	$(1,\infty)$	$\{s\in\mathbb{C}:\mathrm{Re}(s)>1\}$
x/e^{x-1}	$[0,\infty)$	$(0,\infty)$	$\{s\in\mathbb{C}:\mathrm{Re}(s)>1\}$
$1/\cosh(x)$	$[0,\infty)$	$(0,\infty)$	$\{s\in\mathbb{C}:\mathrm{Re}(s)>1\}$
$e^{1/x-x}$	$[1,\infty)$	$(1,\infty)$	$\{s\in\mathbb{C}:\mathrm{Re}(s)>1\}$
x	$(0,1]$	$(0,1)$	$\{s\in\mathbb{C}:\mathrm{Re}(s)>0\}$
$\sin(x)/x$	$[0,\pi)$	$(0,\pi)$	$\{s\in\mathbb{C}:\mathrm{Re}(s)>0\}$
$e^{1-1/[1-(x-c)^2]}$	$[c,c+1)$	$(c,c+1)$	$\{s\in\mathbb{C}:\mathrm{Re}(s)>0\}$
$x^n+1/x^{n+1}+1$	$[1,\infty)$	$(1,\infty)$	$\{s\in\mathbb{C}:\mathrm{Re}(s)>1\}$
$\log(x)$	$(1,e]$	$(1,e)$	$\{s\in\mathbb{C}:\mathrm{Re}(s)>0\}$
x^{x^x}	$(0,1]$	$(0,1)$	$\{s\in\mathbb{C}:\mathrm{Re}(s)>0\}$

Table 5.1: Examples of transforms in our proposed class.

We see that in this case, K_1 is defined on $\hat{I}$ but not on I. We have implicitly defined K_1 such that

$$K_1(0)=\lim_{x\to 0^+}\frac{\sin(x)}{x}.$$

It is apparent that the last example is a contrived example. While it appears interesting it is not obvious that the function is monotonic on $(0,1)$. We will demonstrate this now.

The function $K_1:(0,1)\to\mathbb{R}$ defined by $K_1(x)=x^{x^x}$ is positive on $(0,1)$. A consequence of this is that $\log(K_1(x))=x^x\log(x)$ is defined for every $x\in(0,1)$. We deduce that

$$\frac{K_1'(x)}{K_1(x)}=x^{x-1}+\log(x)x^x[\log(x)+1].$$

The following inequality is true for all $x\in(0,1)$

$$\frac{K_1'(x)}{K_1(x)}>x^x\left\{1+\log(x)[\log(x)+1]\right\}.$$

Observe that $x^x>0$ on $(0,1)$ and $\log(x)^2+\log(x)+1=0$ if and only if

$$x\in\left\{e^{-1/2-i\sqrt{3}/2},e^{-1/2+i\sqrt{3}/2}\right\},$$

so $\log(x)^2+\log(x)+1$ does not change sign on $(0,1)$. As $\log(e^{-1})^2+\log(e^{-1})+1>0$, this implies that K_1 is strictly monotonic increasing on $I=(0,1)$. It is obvious that $K_1(1)=1$, we will also show that $K_1(0+)=0$. Fix $\varepsilon\in(0,1)$. It is a fact that the only stationary point of the function $x\mapsto x^x$ is at $x=e^{-1}$ and is a minimum. It is obvious that

$0^0 = 1^1 = 1$. From here, we see that $x^x \in (0,1)$ given $x \in (0,1)$. Now let $x < \varepsilon^2$

$$x^x > 0 \implies x^x \log(\varepsilon^2) > x^x \log(x). \tag{5.1}$$

Now, recall that

$$2 < \mathrm{e} < 4 \text{ which implies } \frac{1}{4} < \mathrm{e}^{-1} < \frac{1}{2}.$$

This gives us

$$\frac{1}{2}\log\left(\frac{1}{4}\right) < \mathrm{e}^{-1}\log(\mathrm{e}^{-1}).$$

Thus $(\mathrm{e}^{-1})^{\mathrm{e}^{-1}} > 1/2$ and thus $x^x > 1/2$ for every $x \in (0,1)$. Using this, we can find an upper bound on the inequality in (5.1)

$$x^x \log(x) < \frac{1}{2}\log(\varepsilon^2).$$

Which implies $0 < x^{x^x} < \varepsilon$. What we have shown is for every $\varepsilon \in (0,1)$,

$$0 < x < \varepsilon^2 \text{ implies } x^{x^x} < \varepsilon.$$

Therefore $K_1(0+) = 0$.

Now that some examples have been presented, we show that the standard convolution formula we have been working with in Chapters 3 and 4 is a binary operation on $L^1(I, \mathscr{B}, |\mu|)$. We need to take some care with how we have defined I.

Proposition 86. *Let $f, g \in L^1(I, \mathscr{B}, |\mu|)$. Then $f * g$ exists almost everywhere on I.*

Proof. It suffices to show that the function $h : I \times I \to \mathbb{R}$ defined by

$$h(x,u) = f(\psi(x,u))g(u)D_1\psi(x,u)\chi_{J_u}(x)$$

is measurable. The following proof works for for $I = [a,b)$, which implies $J_u = [u,b)$. A similar proof holds when $I = (a,b]$. The set $(I \times I)\backslash(J_u \times I)$ is $[a,u) \times I$ since $J_u = [u,b)$. It is straightforward to show that $J_u \times I \in \mathscr{B} \otimes \mathscr{B}$. Now, let $U \subseteq \mathbb{R}$ be open. Observe that

$$\begin{aligned} h^{-1}(U) = \{&(x,u) \in J_u \times I : x \neq u,\ u \neq a \text{ and } f(\psi(x,u))g(u)D_1\psi(x,u) \in U\} \\ &\cup \{(x,u) \in J_u \times I : u = a \text{ and } f(\psi(x,u))g(u)D_1\psi(x,u) \in U\} \\ &\cup \{(x,u) \in J_u \times I : x = u \text{ and } f(\psi(x,u))g(u)D_1\psi(x,u) \in U\} \\ &\cup \{(x,u) \in (I \times I)\backslash(J_u \times I) : 0 \in U\}. \end{aligned} \tag{5.2}$$

Using the fact that the set of all $(x,u) \in J_u \times I$ such that $x \neq u$ and $u \neq a$ is an open subset of $\hat{I} \times \hat{I}$, it can be seen that the following sets are in $\mathscr{B} \otimes \mathscr{B}$

$$\{(x,u) \in J_u \times I : x \neq u,\ u \neq a \text{ and } f(\psi(x,u))g(u)D_1\psi(x,u) \in U\}$$
$$\{(x,u) \in (I \times I) \backslash (J_u \times I) : 0 \in U\}.$$

Now, consider the function $\pi_2 : I \times I \to I$ where $\pi_2(x,u) = u$, $u \neq a$ and $p : I \to \mathbb{R}$ is defined by

$$p(u) = f(\psi(u,u))g(u)D_1\psi(u,u).$$

Observe that the set in (5.2) consisting of all points in $J_u \times I$ where $x = u$ is equivalent to the set

$$\{(x,u) \in I \times I : x = u\} \cap \{(x,u) \in I \times I : p(\pi_2(x,u)) \in U\}.$$

The line $x = u$ is clearly in $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}$ being a rotation of the real line, so the above set is $B \otimes B$-measurable. A similar process shows that the set

$$\{(x,u) \in J_u \times I : f(\psi(a,u))g(u)D_1\psi(a,u) \in U\} \cap (\{a\} \times I)$$

is $\mathscr{B} \otimes \mathscr{B}$-measurable. We conclude that h is measurable on $I \times I$. From here, the proof that the convolution formula exists is similar to that in Theorem 49. $\square$

We are now ready to analyse further properties of the convolution formula.

5.1 Continuity of the convolution of two functions

In this section, we present sufficient conditions in which the convolution (4.2) of two functions is continuous. While a general statement has been proven in Futcher & Rodrigo [36], this can be hard to check if f, g and K_1 meet all the requirements. Here, we give easily verifiable conditions which guarantee the convolution is continuous. First, we give some conditions on the underlying space where the convolution of two functions does not vanish on.

Lemma 87. *Suppose that the function $f * g$ exists on I. Let S be the closure of the set*

$$\{K_1^{-1}(K_1(x)K_1(u)) \in I : x \in \operatorname{supp}(f), u \in \operatorname{supp}(g)\}.$$

Then $\operatorname{supp}(f * g) \subseteq S$.

Proof. Fix $x \in I \backslash S$. If $u \in \operatorname{supp}(g)$ then there are two possible options. Given $x \notin J_u$, then

$$f(\psi(x,y))g(u)D_1\psi(x,u)\chi_{J_u}(x) = 0.$$

Now, if $x \in J_u$, then it follows that $\psi(x,u) \notin \operatorname{supp}(f)$. If $\psi(x,u) \in \operatorname{supp}(f)$ and $u \in \operatorname{supp}(g)$, then $x \in S$ which contradicts our assumption that $x \notin S$. Since $\psi(x,u) \notin \operatorname{supp}(f)$, it is evident that

$$f(\psi(x,u))g(u)D_1\psi(x,u)\chi_{J_u}(x) = 0.$$

Now, if $u \notin \operatorname{supp}(g)$, then it is straightforward to show that the integrand of the convolution formula is zero. This implies $(f * g)(x) = 0$. Hence $x \notin \operatorname{supp}(f * g)$. This completes the proof. $\square$

Example 88. Consider when $K_1(x) = 1/\cosh(x)$ and let f and g be the functions defined by

$$f(x) = 3(x-4)\chi_{[4,6)}(x) + 3(8-x)\chi_{[6,8]}(x), \qquad g(x) = \sqrt{1-(x-6)^2}\chi_{[5,7]}(x).$$

Observe that

$$S = [\operatorname{arccosh}(\cosh(4)\cosh(5)), \operatorname{arccosh}(\cosh(7)\cosh(8))].$$

It is clear that f and g have compact support. The results in Figure 5.1 give a visual representation of what to expect from Lemma 87. The numerical simulation shows that the convolution of f and g when considering this kernel is similar to that of a Gaussian function, which appears to be a coincidence.

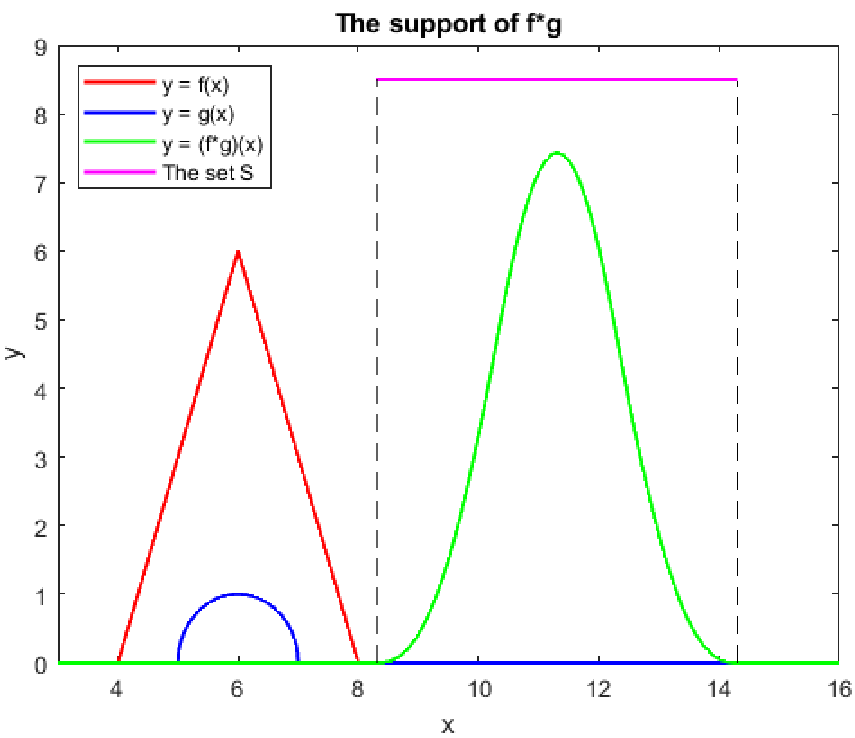

Figure 5.1: A visualisation of $\operatorname{supp}(f * g) \subseteq S$ where $K_1(x) = 1/\cosh(x)$.

Example 89. Consider when $f(x) = \chi_{(2,4)}(x)$ and $g(x) = \chi_{(6,8)}(x)$. It is obvious that f and g have compact support. When we let $K_1(x) = \mathrm{e}^{-(x^2-1)}$, it can be shown that $S = [\sqrt{39}, \sqrt{79}]$. Once again, the illustration in Figure 5.2 shows that the support of $f * g$ lies within the set S.

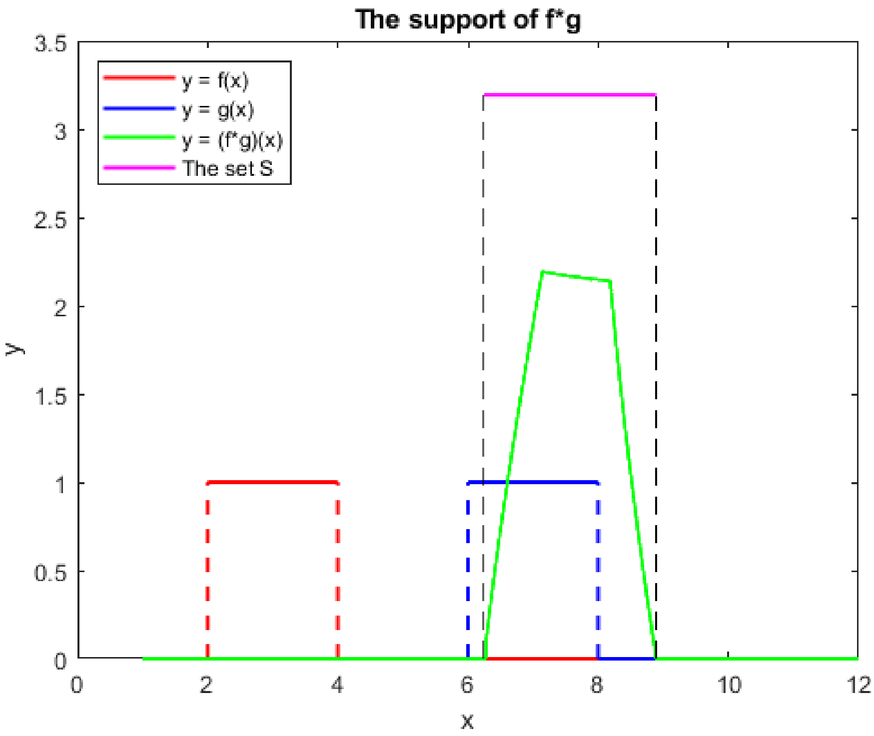

Figure 5.2: A visualisation of $\operatorname{supp}(f * g) \subseteq S$ where $K_1(x) = \mathrm{e}^{-(x^2-1)}$.

It is easily shown that if $f, g \in C_c(I;\mathbb{C})$, then $f * g$ exists almost everywhere on I. It was defined in Chapter 2 that a function vanishes at infinity if for every $\varepsilon > 0$, the set

$$\{x \in I : |f(x)| \geq \varepsilon\}$$

is compact. Recall the space

$$C_0(I;\mathbb{C}) = \{f : I \to \mathbb{C} | f \text{ vanishes at infinity}\}.$$

A consequence of the next theorem is that the space $C_c(I;\mathbb{C})$ is closed under convolution assuming some further conditions on ψ.

Theorem 90. *Suppose both ψ and $D_1\psi$ are uniformly continuous on every bounded subset of $J_u \times I$, $D_1\psi$ is bounded on $J_u \times I$ and let p and q be conjugate exponents. If $f \in L^p(I, \mathscr{B}, \lambda)$ and $g \in L^q(I, \mathscr{B}, \lambda)$, then $f * g \in C_0(I;\mathbb{C})$.*

Proof. For the sake of concreteness, we state that $|D_1\psi(x,u)| \leq M$ for every $(x,u) \in J_u \times I$.

Suppose $f,g \in C_c(I;\mathbb{C})$. We know that $f*g$ exists almost everywhere on I. We note that

$$|(f*g)(x)| \leq \int_{\hat{J}_x} |f(\psi(x,u))||g(u)||D_1\psi(x,u)|\,\mathrm{d}\lambda(u),$$

where

$$\hat{J}_x = \psi(x,I) = \left\{ K_1^{-1}\left(\frac{K_1(x)}{K_1(y)}\right) : y \in I \right\}.$$

An application of Hölder's inequality gives us

$$\begin{aligned}|(f*g)(x)| &\leq \left(\int_{\hat{J}_x} |f(\psi(x,u))|^p |D_1\psi(x,u)|\,\mathrm{d}\lambda(u)\right)^{1/p} \\ &\quad \times \left(\int_{\hat{J}_x} |g(u)|^q |D_1\psi(x,u)|\,\mathrm{d}\lambda(u)\right)^{1/q} \\ &\leq M^{1/q}\left(\int_I |f(u)|^p\,\mathrm{d}\lambda(u)\right)^{1/p}\left(\int_I |g(u)|^q\,\mathrm{d}\lambda(u)\right)^{1/q}.\end{aligned}$$

By our assumptions, we therefore have that $f*g$ is defined everywhere on I. The integral in the convolution then reduces to an integral over the support of g. As $f,g \in C_c(I;\mathbb{C})$, the proof that $f*g$ is continuous follows similarly to that in Futcher & Rodrigo [36]. Due to the slight differences, we will present a brief proof here.

Consider the case where $J_u = [u,b)$. Fix $c \in I$. Observe that each of the functions f, g, ψ and $D_1\psi$ are uniformly continuous on bounded subset of $J_u \times I$. Suppose $x \geq c$. As D_1, f and g, are bounded, the function $(x,u) \mapsto f(\psi(x,u))g(u)D_1\psi(x,u)$ is bounded, say

$$|f(\psi(x,u))g(u)D_1\psi(x,u)| \leq M.$$

For every $\varepsilon > 0$, there exists $\hat{\delta} > 0$ such that

$$|x-c| < \hat{\delta} \implies |f(\psi(x,u))g(u)D_1\psi(x,u) - f(\psi(c,u))g(u)D_1\psi(c,u)| < \frac{\varepsilon}{4(c-a)},$$

for every $u \in [a,c]$. Now, let $\delta = \min\{\hat{\delta}, \varepsilon/4M\}$, we obtain an estimate on the distance between $(f*g)(x)$ and $(f*g)(c)$ as follows

$$\begin{aligned}|(f*g)(x)-(f*g)(c)| &\leq \int_c^x |f(\psi(x,u))g(u)D_1\psi(x,u)|\,\mathrm{d}u \\ &\quad + \int_a^c |f(\psi(x,u))g(u)D_1\psi(x,u) \\ &\qquad - f(\psi(c,u))g(u)D_1\psi(c,u)|\,\mathrm{d}u \\ &\leq \frac{\varepsilon}{4M}(x-c) + \frac{\varepsilon}{4} < \varepsilon.\end{aligned}$$

So $(f*g)$ is continuous. Since $(x,y) \mapsto K_1^{-1}(K_1(x)K_1(y))$ is continuous, the set S as defined in Lemma 87 is compact given $f,g \in C_c(I;\mathbb{C})$. By Lemma 87, it is evident that

$f*g \in C_c(I;\mathbb{C})$. Now, suppose $f \in L^p(I,\mathscr{B},\lambda)$ and $g \in L^q(I,\mathscr{B},\lambda)$. As $C_c(I)$ is dense in both of these spaces, there exists sequences $\{f_n\}$, $\{g_n\} \subset C_c(I)$ such that $f_n \to f$ and $g_n \to g$. Note that the kernel of the associated transforms is bounded, as such it follows that $L^p(I,\mathscr{B},\lambda) \subseteq L^p(I,\mathscr{B},|\mu|)$. Recall that $|\mu_{\tilde{s}}|$ is a finite measure, where $\tilde{s} \in \partial J$ such that $|\mu_s|(I)$ is maximised, so $L^p(I,\mathscr{B},|\mu|) \subseteq L^1(I,\mathscr{B},|\mu|)$. As such all functions in the following calculation are elements of the Banach algebra $L^1(I,\mathscr{B},|\mu|)$. We deduce that

$$||(f_n*g_n)-(f*g)||_\infty \leq ||f_n*(g_n-g)||_u + ||(f_n-f)*g||_u.$$

The previous calculation involving Hölder's inequality gives us

$$||(f_n*g_n)-(f*g)||_\infty \leq M||f_n||_{L^p(\lambda)}||g_n-g||_{L^q(\lambda)} + M||f_n-f||_{L^p(\lambda)}||g||_{L^q(\lambda)}.$$

From here, we can see that $f_n*g_n \to f*g$ uniformly on I, which implies that $f*g \in C_0(I;\mathbb{C})$. □

We note that this proof focuses on when I is an interval. Some comments are now highlighted when I is a countable set which satisfies the assumptions stated in Futcher & Rodrigo [37]. We note some other interesting properties. Using a similar method as in Folland [32], it can be shown that $L^p(I,\mathscr{B},|\mu|) \subseteq L^q(I,\mathscr{B},|\mu|)$, where $1 \leq p < q \leq \infty$. Since $|\mu|$ is also a finite measure, it is clear that $L^q(I,\mathscr{B},|\mu|) \subseteq L^p(I,\mathscr{B},|\mu|)$. Because of this, this leads to interesting consequences such as the fact that the members of $L^p(I,\mathscr{B},|\mu|)$ are exactly the set of bounded functions on I. Also, the set of bounded functions on I is a Hilbert space. We now give examples of transforms and their corresponding convolution formulas which are continuous due to satisfying Theorem 90. We will show the convolution formula vanishes at infinity explicitly.

Example 91. Consider the Laplace transform. That is, let $I=[0,\infty)$, $J=\{s\in\mathbb{C} : \mathrm{Re}(s) > \varepsilon > 0\}$, $K_1(x)=\mathrm{e}^{-x}$, $p=q=2$ and $w=0$. It is apparent that $J_u=[u,\infty)$ and

$$(f*g)(x)=\int_0^x f(x-u)g(u)\,\mathrm{d}u, \qquad \psi(x,u)=x-u, \qquad D_1\psi(x,u)=1.$$

Let $f(x)=g(x)=\mathrm{e}^{-x}$. A straightforward calculation yields

$$(f*g)(x)=x\mathrm{e}^{-x}$$

which tends to zero as x tends to ∞. Hence, $f*g$ vanishes at infinity.

Example 92. For $K_1(x)=1/(1+x)$ and $I=[0,\infty)$. We consider the functions $f(x)=x\chi_{[0,1]}(x)$ and $g(x)=\mathrm{e}^{-x}$. Let $p=q=2$ and $w=0$. We have that $J_u=[u,\infty)$. Substituting all of this into the appropriate formulas gives

$$(f*g)(x) = \int_0^x f\left(\frac{x-u}{1+u}\right)\frac{g(u)}{1+u}\,\mathrm{d}u, \qquad \psi(x,u) = \frac{x-u}{1+u}, \qquad D_1\psi(x,u) = K_1(u).$$

Fix $x > 1$. The convolution formula simplifies to

$$(f*g)(x) = \int_{\psi(x,1)}^x \frac{x-u}{(1+u)^2}\mathrm{e}^{-u}\,\mathrm{d}u.$$

As the function $u \mapsto 1/(1+u)^2$ is positive on $[\psi(x,1),x)$ and is less than 1, we deuce that

$$(f*g)(x) \leq \int_{\psi(x,1)}^x (x-u)\mathrm{e}^{-u}\,\mathrm{d}u \leq \int_{\psi(x,1)}^x x\mathrm{e}^{-u}\,\mathrm{d}u.$$

A simple computation gives us

$$0 \leq (f*g)(x) \leq \frac{x}{\mathrm{e}^{\psi(x,1)}} - \frac{x}{\mathrm{e}^x}.$$

Using the fact that $\psi(x,1) = (x-1)/2$, it is evident that $(f*g)(x) \to 0$ as $x \to \infty$.

Before we extend our convolution formula to distributions, we further restrict the codomain of $\mathscr{A}$, where $\mathscr{A}$ is in our class of transforms. We show that $\mathscr{A}$ maps $L^1(I,\mathscr{B},|\mu|)$ to the space of holomorphic functions on J, which we denote by $H(J)$.

Proposition 93. *If $\mathscr{A}$ is either a discrete or integral transform, and $f \in L^1(I,\mathscr{B},|\mu|)$, then $\mathscr{A}\{f\} \in H(J)$.*

Proof. Fix $f \in L^1(I,\mathscr{B},|\mu|)$. Let $\gamma : [c,d] \to J$ be a straight line segment between two points in J. We have

$$\int_c^d \int_I |K(x,\gamma(t))||f(x)||\gamma'(t)|\,\mathrm{d}\lambda(x)\,\mathrm{d}t \leq \int_c^d \int_I |K(x,\tilde{s})||f(x)||\gamma'(t)|\,\mathrm{d}\lambda(x)\,\mathrm{d}t,$$

where $\tilde{s}$ is a point on the boundary of J which maximises $K(x,\cdot)$ for each $x \in I$. Furthermore, choose $\tilde{s}$ such that $K(x,\tilde{s})$ is real for each $x \in I$. This gives us

$$\int_c^d \int_I |K(x,\gamma(t))||f(x)||\gamma'(t)|\,\mathrm{d}\lambda(x)\,\mathrm{d}t \leq \mathscr{A}\{|f|\}(\tilde{s})\int_c^d |\gamma'(t)|\,\mathrm{d}t < \infty.$$

So, we may apply Fubini's theorem given γ is a straight line segment in J. This also holds given $\gamma([c,d])$ is a finite union of straight line segments in J . Now, let Δ be a triangular path in some open ball $B(s_0,\varepsilon) \subset J$. It follows that

$$\begin{aligned}\int_\Delta \mathscr{A}\{f\}(s)\,\mathrm{d}s &= \int_I \int_\Delta K(x,s)f(x)\,\mathrm{d}s\,\mathrm{d}\lambda(x)\\ &= \int_I f(x)\int_\Delta K(x,s)\,\mathrm{d}s\,\mathrm{d}\lambda(x).\end{aligned}$$

For every $x \in I$, it is evident that $s \mapsto K(x,s)$ is holomorphic on J. This gives us

$$\int_{\Delta} K(x,s)\,\mathrm{d}s.$$

Now, since $\mathscr{A}\{f\}$ is continuous on $B(s_0,\varepsilon)$ by Theorem 68 and satisfies the following

$$\int_{\Delta} \mathscr{A}\{f\}(s)\,\mathrm{d}s = 0,$$

we have by Morera's theorem (see Conway [24]) that $\mathscr{A}\{f\} \in H(B(s_0,\varepsilon))$. As $B(s_0,\varepsilon)$ was an arbitrary ball in J, we conclude that $\mathscr{A}\{f\} \in H(J)$. □

Proposition 94. *Let $\mathscr{A}$ be either a discrete or integral transform and suppose $x \mapsto \log(K_1(x))f(x) \in L^1(I,\mathscr{B},|\mu|)$. If the kernel is of type (i) then*

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{A}\{f\}(s) = \mathscr{A}\{x \mapsto \log(K_1(x))f(x)\}(s),$$

where differentiation is occurring in the complex plane.

Proof. Since the function $\mathscr{A}\{f\}$ is holomorphic on J, it is differentiable on $\mathbb{R}\cap J$. Now, the function $t \mapsto K_1(x)^{t-w}f(x)$ is differentiable for almost all $x \in I$. A straightforward computation yields

$$\begin{aligned}\left|\frac{\partial}{\partial t}[K_1(x)^{t-w}f(x)]\right| &= |\log(K_1(x))K_1(x)^{t-w}f(x)| \\ &\leq |\log(K_1(x))K_1(x)^{\tilde{s}-w}f(x)|.\end{aligned}$$

Using the fact that $x \mapsto \log(K_1(x))f(x) \in L^1(I,\mathscr{B},|\mu|)$, it follows that $\mathscr{A}\{f\}$ is differentiable, in the real sense, on $\mathbb{R}\cap J$. Moving the derivative inside the integral shows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{A}\{f\}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_I K_1(x)^{t-w}f(x)\,\mathrm{d}\lambda(x) \\ &= \int_I \frac{\partial}{\partial t}[K_1(x)^{t-w}f(x)]\,\mathrm{d}\lambda(x) \\ &= \int_I \log(K_1(x))K_1(x)^{t-w}f(x)\,\mathrm{d}\lambda(x) \\ &= \mathscr{A}\{x \mapsto \log(K_1(x))f(x)\}(t)\end{aligned}$$

We know that $\mathscr{A}\{f\}$ is holomorphic on J, so the limit

$$\lim_{s\to s_0}\frac{\mathscr{A}\{f\}(s)-\mathscr{A}\{f\}(s_0)}{s-s_0}$$

exists for every $s_0 \in J$. As such we may take any path from s to s_0 to compute this limit.

Thus

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{A}\{f\}(s) = \mathscr{A}\{x \mapsto \log(K_1(x))f(x)\}(s)$$

for all $s \in \mathbb{R} \cap J$, where differentiation is in the complex sense. Now, observe that the functions defined pointwise by

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{A}\{f\}(s), \qquad \mathscr{A}\{x \mapsto \log(K_1(x))f(x)\}(s)$$

are holomorphic, so the function defined by

$$s \mapsto \frac{\mathrm{d}}{\mathrm{d}s}\mathscr{A}\{f\}(s) - \mathscr{A}\{x \mapsto \log(K_1(x))f(x)\}(s) \tag{5.3}$$

is holomorphic on J. Since the function in (5.3) is identically zero on $\mathbb{R} \cap J$, and $\mathbb{R} \cap J$ contains a limit point in J it follows that the function in (5.3) is identically zero on J. This completes the proof. □

The above result can be formally obtained by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{A}\{f\}(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\int_I K_1(x)^{s-w} f(x)\,\mathrm{d}\lambda(x)\\ &= \int_I \frac{\mathrm{d}}{\mathrm{d}s} K_1(x)^{s-w} f(x)\,\mathrm{d}\lambda(x).\end{aligned}$$

However, we feel that it is important to provide conditions such that such a computation is valid.

5.2 Convolution of distributions

It was discussed in Futcher & Rodrigo [37] that $L^1(I, \mathscr{B}, |\mu|)$ is a Banach algebra without a unit. We will introduce a distribution δ_t such that, when we define the convolution which includes distributions, the distribution δ_t satisfies $\delta_t * f = f$ for every $f \in C_c^\infty(I;\mathbb{R})$. Recall the discussion in Chapter 2 regarding Fréchet spaces. The preliminary results become relevant here.

Before we consider the function $f * g$, where $f \in \mathscr{D}'(I;\mathbb{R})$ and $g \in C_c^\infty(I;\mathbb{R})$, we will first examine the following set

$$V_g = \{x \in \hat{I} : \psi(x,u) \in \hat{I} \text{ for every } u \in \operatorname{supp}(g)\}. \tag{5.4}$$

Proposition 95. *Suppose that g has compact support. Then the set V_g defined in* (5.4) *is open.*

Proof. Fix $x \in V_g$. As $\operatorname{supp}(g)$ is compact, we have that $\psi(x,y_i) \in \hat{I}$ and $\psi(x,y_s) \in \hat{I}$ where $y_i = \inf(\operatorname{supp}(g))$ and $y_s = \sup(\operatorname{supp}(g))$. We will consider the case when K_1 is monotonic increasing. As $\hat{I}$ is an open interval, we have that $\hat{J}_y = K_1^{-1}(K_1(y)K_1(\hat{I}))$ is open and $x \in \hat{J}_y$ for every $y \in \operatorname{supp}(g)$. As such, there exist $\varepsilon_{y_i}, \varepsilon_{y_s} > 0$ such that $(x - \varepsilon_{y_i}, x + \varepsilon_{y_i}) \subseteq \hat{J}_{y_i}$ and $(x - \varepsilon_{y_s}, x + \varepsilon_{y_s}) \subseteq \hat{J}_{y_s}$. We choose $\varepsilon = \min\{\varepsilon_{y_i}, \varepsilon_{y_s}\}$. Note that K_1 is monotonic increasing, so

$$K_1(x - \varepsilon_{y_s}) < K_1(x - \varepsilon) < K_1(x + \varepsilon) < K_1(x + \varepsilon_{y_s}).$$

A similar inequality holds when ε_{y_s} is replaced by ε_{y_i}. This implies

$$\begin{gathered}(\psi(x - \varepsilon, y_i), \psi(x + \varepsilon, y_i)) \subseteq \hat{I} \\ (\psi(x - \varepsilon, y_s), \psi(x + \varepsilon, y_s)) \subseteq \hat{I}.\end{gathered}$$

Observe that

$$\frac{1}{K_1(y_s)} \leq \frac{1}{K_1(y)} \leq \frac{1}{K_1(y_i)},$$

for every $y \in \operatorname{supp}(g)$. From here, we see that $\psi(x - \varepsilon, y_s) \leq \psi(x - \varepsilon, y)$ and $\psi(x + \varepsilon, y) \leq \psi(x + \varepsilon, y_i)$ for every $y \in \operatorname{supp}(g)$. As $\hat{I}$ is an interval, we have

$$(\psi(x - \varepsilon, y), \psi(x + \varepsilon, y)) \subseteq (\psi(x - \varepsilon, y_s), \psi(x + \varepsilon, y_i)) \subseteq \hat{I}$$

for every $y \in \operatorname{supp}(g)$. This implies that $(x - \varepsilon, x + \varepsilon) \subseteq V_g$ and completes the proof. □

5.2.1 Convolution as a function

We state that $J_u \times I$ is no longer an open set. For convenience, we impose the fact that ψ is $C^\infty(J_u \times I)$, where the partial derivatives of ψ on the boundary of $J_u \times I$ are interpreted to be left or right side derivatives where appropriate.

Before we attempt to justify the convolution of distributions we make one more observation. Let $\tilde{J}_x = \psi^{-1}(x, \hat{I}) = \{y \in I : \psi(x, y) \in \hat{I}\}$. If $x \in V_g$ then, by the definition of V_g, we have that $u \in \tilde{J}_x$ for every $u \in \operatorname{supp}(g)$. Hence $\operatorname{supp}(g) \subseteq \tilde{J}_x$.

By considering the convolution of two functions to be defined on V_g, where g has compact support, the formula which satisfies the convolution property is simplified. If $x \in V_g$, then we guarantee that $x \in J_y$ for every $y \in \operatorname{supp}(g)$. If $f \in L^1_{\mathrm{loc}}(I;\mathbb{R})$ and $g \in C_c^\infty(I;\mathbb{R})$, then we have

$$\begin{aligned}|(f * g)(x)| &\leq \int_I |f(\psi(x,u)) D_1 \psi(x,u) \chi_{J_u}(x) g(u)| \, \mathrm{d}u \\ &= \int_{\operatorname{supp}(g)} |f(\psi(x,u)) D_1 \psi(x,u) g(u)| \, \mathrm{d}u.\end{aligned}$$

Recall that $D_1 \psi(x, \cdot)$ is continuous on $\operatorname{supp}(g)$ and $g \in C_c^\infty(I;\mathbb{R})$, As such, g and $D_1 \psi$ are bounded by M_1 and M_2 respectively. This gives us the following inequality

$$|(f*g)(x)| \leq M_1 M_2 \int_{\operatorname{supp}(g)} |f(\psi(x,u))|\,\mathrm{d}u.$$

We also have that K_1 never vanishes and is strictly increasing. This implies that $|D_2(x,\cdot)|$ is bounded below on $\operatorname{supp}(g)$ by some $1/M_3$. We see that

$$\begin{aligned}|(f*g)(x)| &\leq M_1 M_2 M_3 \int_{\operatorname{supp}(g)} |f(\psi(x,u))||D_2\psi(x,u)|\,\mathrm{d}u \\ &= M_1 M_2 M_3 \int_{\tilde{J}_x} |f(\psi(x,u))||D_2\psi(x,u)|\chi_{\operatorname{supp}(g)}(u)\,\mathrm{d}u.\end{aligned}$$

Note that $\tilde{J}_x$ is an open set. From here, we make the substitution $z = \psi(x,u)$ which gives us

$$\begin{aligned}|(f*g)(x)| &\leq M_1 M_2 M_3 \int_{\hat{I}} |f(z)|\chi_{\operatorname{supp}(g)}(\psi(x,z))\,\mathrm{d}z \\ &= M_1 M_2 M_3 \int_{\psi(x,\operatorname{supp}(g))} |f(z)|\,\mathrm{d}z.\end{aligned}$$

As $\psi(x,\operatorname{supp}(g))$ is compact, we see that $|(f*g)(x)| < \infty$. That is, the convolution is defined on V_g. We see that by making a substitution, we have

$$\begin{aligned}(f*g)(x) &= \int_{\tilde{J}_x} f(\psi(x,u))D_1\psi(x,u)g(u)\chi_{\operatorname{supp}(g)}(u)\,\mathrm{d}u \\ &= \int_{\psi(x,\operatorname{supp}(g))} f(u)D_1\psi(x,u)g(\psi(x,u))\,\mathrm{d}u.\end{aligned}$$

Observe that if $u \in \psi(x,\operatorname{supp}(g))$ if and only if $\psi(x,u) \in \operatorname{supp}(g)$, then the following formula holds

$$(f*g)(x) = \int_I f(u)D_1\psi(x,u)g(\psi(x,u))\,\mathrm{d}u. \tag{5.5}$$

We will now show that $u \in \psi(x,\operatorname{supp}(g))$ if and only if $\psi(x,u) \in \operatorname{supp}(g)$.

Fix $u \in \psi(x,\operatorname{supp}(g))$. By the definition of $\psi(x,\operatorname{supp}(g))$, we have

$$K_1(u) \in \left\{\frac{K_1(x)}{K_1(y)} : y \in \operatorname{supp}(g)\right\}.$$

So $K(u) = K_1(x)/K_1(z)$ for some $z \in \operatorname{supp}(g)$, this gives us $z = \psi(x,u)$. Hence $\psi(x,u) \in \operatorname{supp}(g)$.

Now, suppose $\psi(x,u) \in \operatorname{supp}(g)$. This implies $K_1(x)/K_1(u) \in K_1(\operatorname{supp}(g))$. This means

$$\frac{K_1(x)}{K_1(u)} = K_1(z).$$

Rearranging this equation gives $u = \psi(x,z) \in \psi(x,\operatorname{supp}(g))$. We may now define the convolution of a distribution and a test function.

Definition 96. Let $f \in \mathscr{D}'(I;\mathbb{R})$ and $g \in C_c^\infty(I;\mathbb{R})$. We define the convolution $f * g : V_g \to \mathbb{R}$ by the formula

$$(f*g)(x) = f[D_1\psi(x,\cdot)g(\psi(x,\cdot))]. \tag{5.6}$$

Consider the distribution $\delta_t : C_c^\infty(I;\mathbb{R}) \to \mathbb{R}$, where $\delta_t[g] = g(t)$. We now see that since $1 \in K_1(I)$, then for $t = K_1^{-1}(1)$, we have

$$\begin{aligned}(\delta_t * g)(x) &= \delta_t[D_1\psi(x,\cdot)g(\psi(x,\cdot))]\\ &= D_1\psi(x,K_1^{-1}(1))g(\psi(x,K_1^{-1}(1)))\\ &= g(x).\end{aligned}$$

That is, $\delta_t * g \equiv g$. So by extending the convolution formula to distributions, the convolution of some distribution with any function in $C_c^\infty(I;\mathbb{R})$ is the function itself. We now examine the properties of this newly defined function.

Theorem 97. *Let $f \in \mathscr{D}'(I;\mathbb{R})$ and $g \in C_c^\infty(I;\mathbb{R})$, the function $f * g : V_g \to \mathbb{R}$ is continuous.*

Proof. Fix $\{x_n\} \subset V_g$, where $x_n \to x_0 \in V_g$, and fix $\varepsilon > 0$. We wish to show that the sequence of functions $D_1\psi(x_n,\cdot)g(\psi(x_n,\cdot))$ converges to $D_1\psi(x_0,\cdot)g(\psi(x_0,\cdot))$ in the set $C_c^\infty(I,F;\mathbb{R})$ for some compact set F. Note that there exists a $\delta > 0$ such that $[x_0-\delta, x_0+\delta] \subset V_g$, since V_g is an open set. Now, as $x_n \to x_0$, for this $\delta > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in [x_0-\delta, x_0+\delta]$. This guarantees that all but a finite amount of terms of the sequence $\{x_n\}$ are elements of the set $[x_0-\delta, x_0+\delta]$. Observe that $\psi(x_n,u) \in \operatorname{supp}(g)$ if and only if $u \in \psi(x_n, \operatorname{supp}(g))$. We deduce that for $n \geq N$, the functions $g(\psi(x_n,\cdot))$, and as a consequence $D_1\psi(x_n,\cdot)g(\psi(x_n,\cdot))$, vanish outside the set $\psi([x_0-\delta, x_0+\delta]\times\operatorname{supp}(g))$. Now, let

$$F = \bigcup_{n=1}^{N-1} \psi(\{x_n\}\times\operatorname{supp}(g)) \cup \psi([x_0-\delta,x_0+\delta]\times\operatorname{supp}(g)), \tag{5.7}$$

We see that F is a compact set and the functions $g(\psi(x_n,\cdot))$ vanish outside F for every $n \in \mathbb{N}_0$. As such, for every $n \in \mathbb{N}_0$, the function $D_1\psi(x_n,\cdot)g(\psi(x_n,\cdot))$ and its derivatives of all orders vanish outside F. That is to say

$$\{D_1\psi(x_n,\cdot)g(\psi(x_n,\cdot))\}_{n=0}^\infty \subseteq C_c^\infty(I,F;\mathbb{R}).$$

Let r_j be defined such that $r_j(x_1,x_2) = x_j$ for $j = 1,2$. We now wish to show that

$$\frac{\partial^k}{\partial r_2^k}[D_1\psi(x_n,\cdot)g(\psi(x_n,\cdot))] \to \frac{\partial^k}{\partial r_2^k}[D_1\psi(x_0,\cdot)g(\psi(x_0,\cdot))] \tag{5.8}$$

for every $k \in \mathbb{N}_0$. We note that we are using the same notation for partial derivatives as that in Tu [98]. Fix $\varepsilon > 0$, $k \in \mathbb{N}_0$ and $u \in F$. Observe the function defined by

$$\frac{\partial^{k+1}}{\partial r_1 \partial r_2^k}[D_1\psi(g \circ \psi)]$$

is continuous on the compact set $[x_0 - \delta, x_0 + \delta] \times F$ and as such is bounded by some value M. Choose a natural number $\tilde{N} \geq N$ which guarantees that $|x_n - x_0| < \min\{\delta, \varepsilon/M\}$ when $n \geq \tilde{N}$. For a fixed $n \geq \tilde{N}$, we have by the Mean Value Theorem

$$\begin{aligned} &\left| \frac{\partial^k}{\partial r_2^k}[D_1\psi(x_n,u)g(\psi(x_n,u))] - \frac{\partial^k}{\partial r_1^k}[D_1\psi(x_0,u)g(\psi(x_0,u))] \right| \\ &\qquad = \left| \frac{\partial^{k+1}}{\partial r_1 \partial r_2^k}[D_1\psi(c_n,u)(g\circ\psi)(c_n,u)] \right| |x_n - x_0|, \end{aligned} \tag{5.9}$$

where c_n is some value in the open interval whose endpoints are x_0 and x_n. As $c_n \in [x_0 - \delta, x_0 + \delta]$, we have that the $(k+1)$-order derivative in equation (5.9) is bounded by M for every $u \in F$. This gives us

$$\begin{aligned} \left| \frac{\partial^k}{\partial r_2^k}[D_1\psi(x_n,u)g(\psi(x_n,u))] - \frac{\partial^k}{\partial r_2^k}[D_1\psi(x_0,u)g(\psi(x_0,u))] \right| &\leq M|x - x_0| \\ &< \varepsilon, \end{aligned}$$

for every $u \in F$. What has been shown is for every $\varepsilon > 0$, there exists $\tilde{N} \in \mathbb{N}$ such that $n \geq \tilde{N}$ implies the function $D_1\psi(x_n,\cdot)g(\psi(x_n,\cdot))$ is in the set

$$\left\{ y \in C_c^\infty(I,F;\mathbb{R}) : ||y - D_1\psi(x_0,\cdot)g(\psi(x_0,\cdot))||_{(k,m)} < \varepsilon \right\}.$$

As such, the expression in (5.8) holds. As k was arbitrary, we have that $x_n \to x_0$ implies

$$D_1\psi(x_n,\cdot)g(\psi(x_n,\cdot)) \to D_1\psi(x_0,\cdot)g(\psi(x_0,\cdot)) \tag{5.10}$$

in $C_c^\infty(I,F;\mathbb{R})$. As $f \in \mathscr{D}'(I;\mathbb{R})$, if $x_n \to x_0$ then

$$f[D_1\psi(x_n,\cdot)g(\psi(x_n,\cdot))] \to f[D_1\psi(x_0,\cdot)g(\psi(x_0,\cdot))],$$

which implies

$$(f * g)(x_n) \to (f * g)(x_0).$$

This completes the proof. □

We now introduce a lemma which will help us show the convolution is infinitely differentiable on V_g.

Lemma 98. *Let $x \in V_g$ and let $\{h_n\}$ be a sequence such that $\{x+h_n\} \subseteq V_g$ and h_n converges to* 0*. For every $k \in \mathbb{N} \cup \{0\}$, we have*

$$\frac{1}{h_n}\left\{\frac{\partial^k}{\partial r_1^k}[D_1\psi(x+h_n,\cdot)g(\psi(x+h_n,\cdot))] - \frac{\partial^k}{\partial r_1^k}[D_1\psi(x,\cdot)g(\psi(x,\cdot))]\right\}$$
$$\to \frac{\partial^{k+1}}{\partial r_1^{k+1}}[D_1\psi(x,\cdot)g(\psi(x,\cdot))]$$

in $C_c^\infty(I;\mathbb{R})$.

Proof. Fix $x \in V_g$ and let $\{h_n\}$ be a sequence defined in the statement of the lemma. Fix $\varepsilon > 0$ and $u \in I$. Furthermore, let $k,l \in \mathbb{N}_0$. By the Mean Value Theorem, we have that

$$\frac{\partial^l}{\partial r_2^l}\frac{1}{h_n}\left[\frac{\partial^k}{\partial r_1^k}D_1\psi(x+h_n,u)g(\psi(x+h_n,u)) - \frac{\partial^k}{\partial r_1^k}D_1\psi(x,u)g(\psi(x,u))\right]$$
$$= \frac{\partial^{k+l+1}}{\partial r_2^l \partial r_1^{k+1}}[D_1\psi(x_n,u)g(\psi(x_n,u))] \tag{5.11}$$

for some x_n where $|x_n - x| < |h_n|$ and $x_n \to x$ as $h_n \to 0$. We note that for some fixed $\delta > 0$, we have $[x-\delta, x+\delta] \subseteq V_g$ and for some $N \in \mathbb{N}$, it holds that $x_n \in [x-\delta, x+\delta]$ for every $n \geq N$. We introduce the compact set

$$F = \bigcup_{n=1}^{N-1} \psi(\{x_n\} \times \operatorname{supp}(g)) \cup \psi([x-\delta, x+\delta] \times \operatorname{supp}(g)).$$

Note that if $u \in I \setminus F$, then $D_1\psi(x_n,u)g(\psi(x_n,u)) = 0$ for every $n \in \mathbb{N} \cup \{0\}$. Because of this, we may assume $u \in F$. It is evident that the function

$$\frac{\partial^{k+l+1}}{\partial r_2^l \partial r_1^{k+1}}[D_1\psi(g \circ \psi)]$$

is continuous on $[x-\delta, x+\delta] \times F$ and as such bounded by some constant M. So for a fixed $\varepsilon > 0$, there exists $\tilde{N} \in \mathbb{N}$ such that $n \geq \tilde{N} \geq N$ implies

$$\left|\frac{\partial^{k+l+1}}{\partial r_2^l \partial r_1^{k+1}}[D_1\psi(x_n,u)g(\psi(x_n,u))] - \frac{\partial^{k+l+1}}{\partial r_2^l \partial r_1^{k+1}}[D_1\psi(x,u)g(\psi(x,u))]\right|$$
$$\leq M|x_n - x| < \varepsilon. \tag{5.12}$$

As $u \in F$ was arbitrary and the inequality in (5.12) holds for every $u \in F$, for a fixed k, the sequence in the statement of the lemma converges in $C_c^\infty(I;\mathbb{R})$. As k was arbitrary, this completes the proof. □

From here, it is easy to see that the convolution is infinitely differentiable.

Theorem 99. *Let $f \in \mathscr{D}'(I;\mathbb{R})$ and $g \in C_c^\infty(I;\mathbb{R})$. Then $f * g \in C^\infty(V_g;\mathbb{R})$.*

Proof. Fix $x \in V_g$. We will show by induction that for every $k \in \mathbb{N}$,

$$(f*g)^{(k)}(x) = f\left[\frac{\partial^k}{\partial r_1^k}[D_1\psi(x,\cdot)g(\psi(x,\cdot))]\right]. \tag{5.13}$$

Let $k=1$, and let $\{h_n\}$ be a sequence which satisfies the properties stated in Lemma 98. Then

$$\frac{D_1\psi(x+h_n,u)g(\psi(x+h_n,u)) - D_1\psi(x,u)g(\psi(x,u))}{h_n} \to \frac{\partial}{\partial r_1}[D_1\psi(x,\cdot)g(\psi(x,\cdot))].$$

As we have

$$\begin{aligned}&\frac{(f*g)(x+h_n)-(f*g)(x)}{h_n}\\ &\qquad = f\left[\frac{D_1\psi(x+h_n,\cdot)g(\psi(x+h_n,\cdot)) - D_1\psi(x,\cdot)g(\psi(x,\cdot))}{h_n}\right],\end{aligned}$$

letting $h_n \to 0$ yields

$$(f*g)'(x) = f\left[\frac{\partial}{\partial r_1}[D_1\psi(x,\cdot)g(\psi(x,\cdot))]\right]. \tag{5.14}$$

Next, assuming this holds for all natural numbers less than or equal to k, we will show the formula holds for the $(k+1)$-order derivative. Observe that

$$\begin{aligned}&\frac{(f*g)^{(k)}(x+h_n)-(f*g)^{(k)}(x)}{h_n}\\ &\quad = \frac{1}{h_n}\left\{f\left[\frac{\partial^k}{\partial r_1^k}[D_1\psi(x+h_n,\cdot)g(\psi(x+h_n,\cdot))]\right]\right.\\ &\qquad\qquad \left. - f\left[\frac{\partial^k}{\partial r_1^k}[D_1\psi(x,\cdot)g(\psi(x,\cdot))]\right]\right\}\\ &\quad = f\left[\frac{1}{h_n}\left\{\frac{\partial^k}{\partial r_1^k}[D_1\psi(x+h_n,\cdot)g(\psi(x+h_n,\cdot))] - \frac{\partial^k}{\partial r_1^k}[D_1\psi(x,\cdot)g(\psi(x,\cdot))]\right\}\right].\end{aligned}$$

Once again, applying Lemma 98 and letting $h_n \to 0$ gives us that the formula in (5.13) holds for $k+1$. This completes the proof. ☐

5.2.2 Transform of a distribution

We start by defining an appropriate notion of convergence in $\mathscr{A}(C_c^\infty(I;\mathbb{C}))$. It is highlighted that due to the assumptions of functions in $C_c^\infty(I;\mathbb{C})$, for every $f \in C_c^\infty(I;\mathbb{C})$ we have that f satisfies the assumptions in Theorem 58. As such $\mathscr{A} : C_c^\infty(I;\mathbb{C}) \to \mathscr{A}(C_c^\infty(I;\mathbb{C}))$

is a bijection. So, we define convergence in $\mathscr{A}(C_c^\infty(I;\mathbb{C}))$ as follows.

Let $\{f_n\}$ be a sequence in $\mathscr{A}(C_c^\infty(I;\mathbb{C}))$. The sequence $\{f_n\}$ converges to f if $\mathscr{A}^{-1}\{f_n\} \in C_c^\infty(F;\mathbb{C})$ and $\mathscr{A}^{-1}\{f\} \in C_c^\infty(F;\mathbb{C})$ for some compact set $F \subseteq I$ and $\mathscr{A}^{-1}\{f_n\} \to \mathscr{A}^{-1}\{f\}$ in $C_c^\infty(I;\mathbb{C})$. An *$\mathscr{A}$-distribution* on I is a continuous linear functional on $\mathscr{A}(C_c^\infty(I;\mathbb{C}))$, which we denote by $\mathscr{D}'_{\mathscr{A}}(I;\mathbb{C})$. We consider the weak* topology on $\mathscr{D}'_{\mathscr{A}}(I;\mathbb{C})$.

Consider the $\mathscr{A}$-transform where the value of w present in the kernel is 0. Furthermore, suppose $I \subseteq J$. Now, consider the $\mathscr{A}^*$-transform defined by

$$\mathscr{A}^*\{f\}(y) = \int_I K_1(y)^x f(x)\,\mathrm{d}\lambda(x), \tag{5.15}$$

where $f \in C_c^\infty(I;\mathbb{C})$. Let $f, g \in C_c^\infty(I;\mathbb{C})$, a straightforward computation shows that

$$\begin{aligned}\int_I f(x)\mathscr{A}\{g\}(x)\,\mathrm{d}\lambda(x) &= \int_I f(x)\int_I K_1(y)^x g(y)\,\mathrm{d}\lambda(y)\,\mathrm{d}\lambda(x)\\ &= \int_I g(y)\int_I K_1(y)^x f(x)\,\mathrm{d}\lambda(x)\,\mathrm{d}\lambda(y)\\ &= \int_I g(y)\mathscr{A}^*\{f\}(y)\,\mathrm{d}\lambda(y).\end{aligned}$$

We see from this calculation the reason for denoting the operator in (5.15) by $\mathscr{A}^*$. The operator $\mathscr{A}^*$ is the adjoint of $\mathscr{A}$ as it satisfies

$$\langle f, \mathscr{A}\{g\}\rangle_{L^2(\lambda)} = \int_I f(x)\mathscr{A}\{g\}(x)\,\mathrm{d}\lambda(x) = \langle \mathscr{A}^*\{f\}, g\rangle_{L^2(\lambda)}.$$

We can now extend the $\mathscr{A}^*$-transform to distributions.

Definition 100. Let $f \in \mathscr{D}'_{\mathscr{A}}(I;\mathbb{C})$ and $\phi \in C_c^\infty(I;\mathbb{C})$. The *$\mathscr{A}$-transform of f* is defined by

$$\mathscr{A}^*\{f\}[\phi] = f[\mathscr{A}\{\phi\}]. \tag{5.16}$$

Theorem 101. *The map $\mathscr{A}^* : \mathscr{D}'_{\mathscr{A}}(I;\mathbb{C}) \to \mathscr{D}'(I;\mathbb{C})$ is a homeomorphism.*

Proof. We will first show that $\mathscr{A}$ is a bijection. Consider the map $\overline{\mathscr{A}} : \mathscr{D}'(I;\mathbb{C}) \to \mathscr{D}'_{\mathscr{A}}(I;\mathbb{C})$ which is defined by

$$\overline{\mathscr{A}}\{f\}[\phi] = f[\mathscr{A}^{-1}\{\phi\}].$$

We have

$$\begin{aligned}\overline{\mathscr{A}}\{\mathscr{A}^*\{f\}\}[\phi] &= \mathscr{A}^*\{f\}[\mathscr{A}^{-1}\{\phi\}] = f[\mathscr{A}\mathscr{A}^{-1}\{\phi\}] = f[\phi], \qquad \forall\phi \in \mathscr{A}(C_c^\infty(I;\mathbb{C})),\\ \mathscr{A}^*\{\overline{\mathscr{A}}\{f\}\}[\phi] &= \overline{\mathscr{A}}\{f\}[\mathscr{A}\{\phi\}] = f[\mathscr{A}^{-1}\mathscr{A}\{\phi\}] = f[\phi], \qquad \forall\phi \in C_c^\infty(I;\mathbb{C}).\end{aligned}$$

Hence, $\mathscr{A}^*$ is a bijection, and $\overline{\mathscr{A}}$ is the inverse $\mathscr{A}^*$-transform.

Now, let $\{f_a\}_{a\in A}$ be a net in $\mathscr{D}'_{\mathscr{A}}(I;\mathbb{C})$ such that $f_a \to f$. We wish to show that $\mathscr{A}\{f_a\} \to \mathscr{A}\{f\}$, where convergence is pointwise. Fix $\phi \in C_c^\infty(I;\mathbb{C})$. We have

$$\mathscr{A}\{f_a\}[\phi] = f_a[\mathscr{A}\{\phi\}] \to f[\mathscr{A}\{\phi\}] = \mathscr{A}\{f\}[\phi].$$

Hence $\mathscr{A}^*$ is continuous. A similar argument shows that $\mathscr{A}^{-1} = \overline{\mathscr{A}}$ is continuous. Hence $\mathscr{A}^* : \mathscr{D}'_{\mathscr{A}}(I;\mathbb{C}) \to \mathscr{D}'(I;\mathbb{C})$ is a homeomorphism. □

5.3 Convolution of measures

In this section, we introduce the convolution of two measures and show some basic properties. In Chapter 2, we introduced the space of all complex Borel measures on I, namely $M(I)$. Recall that we imposed that $K_1(I) = (0,1]$ or $K_1(I) = [1,\infty)$.

Definition 102. Let $\eta,\, \nu \in M(I)$. We define the *convolution* $\eta * \nu \in M(I)$ by the following formula

$$(\eta * \nu)(E) = \int_I \int_I \chi_E(K_1^{-1}(K_1(x)K_1(y)))\,\mathrm{d}\eta(x)\,\mathrm{d}\nu(y).$$

It is obvious that $(\eta * \nu)(\emptyset) = 0$. By splitting the integral up into its real and imaginary components, then the positive and negative components, an application of the Monotone Convergence Theorem shows that the function is countably additive on $\mathscr{B}$. Hence $\eta * \nu$ is a measure on I. By a classical argument involving the Dominated Convergence Theorem, it is evident that for every bounded Borel measurable function h,

$$\int_I h\,\mathrm{d}(\eta * \nu) = \int_I \int_I h(K_1^{-1}(K_1(x)K_1(y)))\,\mathrm{d}\eta(x)\,\mathrm{d}\nu(y). \tag{5.17}$$

We now prove that the map which takes the equivalence class $f \in L^1(I,\mathscr{B},|\mu|)$ to the measure defined by $\mathrm{d}\eta = f\,\mathrm{d}|\mu|$ is a homomorphism.

Theorem 103. *Let* $\mathrm{d}\eta = f\,\mathrm{d}|\mu|$ *and* $\mathrm{d}\nu = g\,\mathrm{d}|\mu|$. *Then* $\mathrm{d}(\eta * \nu) = (f * g)\,\mathrm{d}|\mu|$.

Proof. By the definition of the convolution of two measures, we deduce that

$$\begin{aligned}(\eta * \nu)(E) &= \int_I \int_I \chi_E(K_1^{-1}(K_1(x)K_1(y)))\,\mathrm{d}\eta(x)\,\mathrm{d}\nu(y)\\ &= \int_I \int_I \chi_E(K_1^{-1}(K_1(x)K_1(y)))f(x)g(y)[K_1(x)K_1(y)]^{\tilde{s}-w}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

which comes from the fact that

$$\eta(E) = \int_E f(x)K_1(x)^{\tilde{s}-w}\,\mathrm{d}\lambda(x), \qquad \nu(E) = \int_E g(y)K_1(y)^{\tilde{s}-w}\,\mathrm{d}\lambda(y).$$

By making the substitution $K_1(z) = K_1(x)K_1(y)$, we have

$$\begin{aligned}(\eta * \nu)(E) &= \int_I \int_{J_y} \chi_E(z) f(\psi(z,y)) g(y) D_1 \psi(z,y) K_1(z)^{\tilde{s}-w} \,\mathrm{d}z \,\mathrm{d}y \\ &= \int_I \int_I \chi_E(z) f(\psi(z,y)) g(y) D_1 \psi(z,y) \chi_{J_y}(z) \,\mathrm{d}|\mu|(z) \,\mathrm{d}y.\end{aligned}$$

Recall that $\psi(z,y) = K_1^{-1}(K_1(z)/K_1(y))$. An application of Fubini's Theorem gives us

$$\begin{aligned}(\eta * \nu)(E) &= \int_I \chi_E(z) \int_I f(\psi(z,y)) g(y) D_1 \psi(z,y) \chi_{J_y}(z) \,\mathrm{d}y \,\mathrm{d}|\mu|(z) \\ &= \int_E (f * g)(z) \,\mathrm{d}|\mu|(z).\end{aligned}$$

□

We note that one can embed $L^1(\lambda)$ into $M(\mathbb{R})$ by mapping $f \in L^1(\lambda)$ to η, where $\mathrm{d}\eta = f\,\mathrm{d}\lambda$ and λ is the Lebesgue measure. Suppose that the convolution of two measures is given by $(\eta * \nu)(E) = (\eta \times \nu)(s^{-1}(E))$, where $s(x,y) = x + y$. Given $\mathrm{d}\eta = f\,\mathrm{d}\lambda$ and $\mathrm{d}\nu = g\,\mathrm{d}\lambda$ then one has $\mathrm{d}(\eta * \nu) = (f * g)\,\mathrm{d}\lambda$, where $f * g$ is the Fourier convolution of f and g. The proof of Theorem 103 shows that we can embed $L^1(I, \mathscr{B}, |\mu|)$ into $M(I)$. We now show that $M(I)$ has a unit.

Proposition 104. *Let δ_t be the Dirac measure, where $t = K_1^{-1}(1)$. Then we have $\delta_t * \eta = \eta$ for every $\eta \in M(I)$.*

Proof. A straightforward computation shows that

$$\begin{aligned}(\delta_t * \eta)(E) &= \int_I \int_I \chi_E(K_1^{-1}(K_1(x)K_1(y))) \,\mathrm{d}\delta_t(x) \,\mathrm{d}\eta(y) \\ &= \int_I \chi_E(K_1^{-1}(K_1(t)K_1(y))) \,\mathrm{d}\eta(y).\end{aligned}$$

From here, substituting $t = K_1^{-1}(1)$ yields $\delta_t * \eta \equiv \eta$. □

We can define a transform of measures which belong to $M(I)$.

Definition 105. Let $\eta \in M(I)$. The *$\mathscr{A}$-transform* of η is given by

$$\mathscr{A}\{\eta\}(s) = \int_I K_1(x)^{s-w} \,\mathrm{d}\eta(x).$$

Using (5.17), it is easy to show the homomorphism property which is to be expected of the convolution operation. Taking the transform of the measure $(\eta * \nu)$ gives us

$$\begin{aligned}\mathscr{A}\{\eta * \nu\}(s) &= \int_I K_1(x)^{s-w} \,\mathrm{d}(\eta * \nu)(x) \\ &= \int_I \int_I [K_1(x)K_1(y)]^{s-w} \,\mathrm{d}\eta(x) \,\mathrm{d}\nu(y)\end{aligned}$$

$$= \int_I K_1(x)^{s-w}\,\mathrm{d}\eta(x) \int_I K_1(y)^{s-w}\,\mathrm{d}\nu(y).$$

Therefore $\mathscr{A}\{\eta * \nu\} \equiv \mathscr{A}\{\eta\}\mathscr{A}\{\nu\}$. By defining $\mathscr{A}_s\{\eta\} = \mathscr{A}\{\eta\}(s)$, the following inequality holds

$$|\mathscr{A}_s\{\eta\}| \leq \int_I |K_1(x)^{s-w}|\,\mathrm{d}|\eta|(x).$$

Recall that we consider kernels where K is bounded in $I \times \overline{J}$, say $|K_1(x)^{s-w}| \leq C$ on $I \times \overline{J}$. So for every $||\eta|| \leq 1$, we have

$$|\mathscr{A}_s\{\eta\}| \leq C \int_I 1\,\mathrm{d}|\eta| = C|\eta|(I) = C||\eta|| \leq C.$$

This shows that $\mathscr{A}_s$ is a bounded linear map. This guarantees the existence of a homomorphism between $M(I)$ and $\mathbb{C}$ with respect to our convolution, given that $M(I)$ is a normed algebra. It has already been shown that $L^1(I, \mathscr{B}, |\mu|)$ is a Banach algebra. We will now show that $M(I)$ is a normed algebra.

Lemma 106. *The convolution operation is associative, commutative and satisfies* $||\eta * \nu|| \leq ||\eta|| \cdot ||\nu||$.

Proof. For every $\eta, \nu, \xi \in M(I)$, the following equation holds by the definition of the convolution product

$$((\eta * \nu) * \xi)(E) = \int_I \int_I \chi_E(K_1^{-1}(K_1(x)K_1(z)))\,\mathrm{d}(\eta * \nu)(x)\,\mathrm{d}\xi(z).$$

From (5.17) we see that

$$((\eta * \nu) * \xi)(E) = \int_I \int_I \int_I \chi_E(K_1^{-1}(K_1(x)K_1(y)K_1(z)))\,\mathrm{d}\eta(x)\,\mathrm{d}\nu(y)\,\mathrm{d}\xi(z).$$

Applying the definition of the convolution of two measures gives us

$$(\eta * (\nu * \xi))(E) = \int_I \int_I \chi_E(K_1^{-1}(K_1(x)K_1(z)))\,\mathrm{d}\eta(x)\,\mathrm{d}(\nu * \xi)(z).$$

Another application of equation (5.17) shows that $((\eta * \nu) * \xi)(E) = (\eta * (\nu * \xi))(E)$. Showing the convolution is commutative is trivial. Now, let f be a Borel map such that $|f| \leq 1$ on I. We have

$$\begin{aligned} \left| \int_I f\,\mathrm{d}(\eta * \nu) \right| &= \left| \int_I \int_I f(K_1^{-1}(K_1(x)K_1(y)))\,\mathrm{d}\eta(x)\,\mathrm{d}\nu(y) \right| \\ &\leq \int_I \int_I |f(K_1^{-1}(K_1(x)K_1(y)))|\,\mathrm{d}|\eta|(x)\,\mathrm{d}|\nu|(y). \end{aligned}$$

Using the fact that f is an arbitrary Borel map such that $|f| \leq 1$ on I, we have

$$\left| \int_I f \, \mathrm{d}(\eta * \nu) \right| \leq ||\eta|| \cdot ||\nu||. \tag{5.18}$$

An equivalent formula for $||\eta * \nu||$ is the supremum of the set of quantities on the left side of (5.18) for every $|f| \leq 1$ on I, as shown in Theorem 28. Therefore $||\eta * \nu|| \leq ||\eta|| \cdot ||\nu||$. □

We now introduce the convolution of functions in $L^1(I, \mathscr{B}, |\mu|)$ with Borel measures.

Definition 107. Let $f \in L^1(I, \mathscr{B}, |\mu|)$ and $\eta \in M(I)$. Define the function $f * \eta : I \to \mathbb{C}$ by

$$(f * \eta)(x) = \int_I K_1(y)^{-(\tilde{s}-w)} f(\psi(x,y)) \chi_{J_y}(x) D_1 \psi(x,y) \, \mathrm{d}\eta(y),$$

where $\tilde{s}$ is chosen such that $\tilde{s} \in \partial J$, $K_1(y)^{-(\tilde{s}-w)} \in \mathbb{R}$ and such that $|\mu_s|(I)$ is maximised.

Lemma 108. *Suppose that f and η are defined as in Definition 107. Then $f * \eta$ is defined almost everywhere on I and $f * \eta \in L^1(I, \mathscr{B}, |\mu|)$.*

Proof. We start by assuming f and η are nonnegative. By the usual arguments which involve splitting the integral up into its real and imaginary parts, then the positive and negative parts, one can show this holds for complex f and η. As $f * \eta$ is nonnegative, we apply Tonelli's Theorem which gives us

$$\begin{aligned}
&\int_I (f * \eta)(x) \, \mathrm{d}|\mu|(x) \\
&\quad = \int_I \int_I K_1(y)^{-(\tilde{s}-w)} f(\psi(x,y)) \chi_{J_y}(x) D_1 \psi(x,y) \, \mathrm{d}\eta(y) \, \mathrm{d}|\mu|(x) \\
&\quad = \int_I \int_{J_y} K_1(y)^{-(\tilde{s}-w)} f(\psi(x,y)) \chi_{J_y}(x) D_1 \psi(x,y) K_1(x)^{(\tilde{s}-w)} \, \mathrm{d}x \, \mathrm{d}\eta(y) \\
&\quad = \int_I \int_I f(z) K_1(z)^{(\tilde{s}-w)} \, \mathrm{d}z \, \mathrm{d}\eta(y).
\end{aligned}$$

Hence $||f * \eta||_\mu \leq ||f||_\mu \cdot ||\eta||$. □

So we have that $M(I)$ is a unital normed algebra, where I is defined such that $K_1^{-1}(1) \in I$. We have that $L^1(I, \mathscr{B}, |\mu|)$ is a subalgebra of $M(I)$ with respect to this new convolution product. Due to the introduction of the convolution of a function with a measure, we can prove a stronger result about $L^1(I, \mathscr{B}, |\mu|)$.

Proposition 109. *The subalgebra $L^1(I, \mathscr{B}, |\mu|)$ is an ideal of $M(I)$.*

Proof. Let $\mathrm{d}\eta = f \, \mathrm{d}|\mu|$ and $\nu \in M(I)$. Manipulating the measure $(f * \nu) \, \mathrm{d}|\mu|$ gives

$$\int_E (f * \nu)(x) \, \mathrm{d}|\mu|(x)$$

$$
\begin{aligned}
&= \int_I \int_I \chi_E(x) K_1(y)^{-(\tilde{s}-w)} f(\psi(x,y)) \chi_{J_y}(x) D_1\psi(x,y)\, \mathrm{d}\nu(y)\, \mathrm{d}|\mu|(x) \\
&= \int_I \int_I \chi_E(x) K_1(y)^{-(\tilde{s}-w)} f(\psi(x,y)) \chi_{J_y}(x) D_1\psi(x,y)\, \mathrm{d}|\mu|(x)\, \mathrm{d}\nu(y) \\
&= \int_I \int_I \chi_E(K_1^{-1}(K_1(z)K_1(y))) f(z)\, \mathrm{d}|\mu|(z)\, \mathrm{d}\nu(y) \\
&= \int_I \int_I \chi_E(K_1^{-1}(K_1(z)K_1(y)))\, \mathrm{d}\eta\, \mathrm{d}\nu.
\end{aligned}
$$

This in turn gives us $\mathrm{d}(\eta * \nu) = (f * \nu)\, \mathrm{d}|\mu|$. This completes the proof. □

Recall from Theorem 34 that $M(I)$ is complete, so $M(I)$ is a commutative unital Banach algebra and $L^1(I, \mathscr{B}, |\mu|)$ is a nonunital subalgebra of $M(I)$. However, it should be noted that $L^1(I, \mathscr{B}, |\mu|)$ is not a maximal ideal of $M(I)$.

Proposition 110. *The ideal $L^1(I, \mathscr{B}, |\mu)$ is not a maximal ideal in $M(I)$.*

Proof. Let $T : M(I) \to \mathbb{C}$ be a multiplicative functional. That is, an algebra homomorphism from $M(I)$ to $\mathbb{C}$ which satisfies $T(\delta_t) = 1$, where $t = K_1^{-1}(1)$. We know such a functional exists, namely $\mathscr{A}_s$. It was shown in Folland [35] that there is a one to one correspondence between multiplicative functionals on $M(I)$ and maximal ideals of $M(I)$. More specifically, every maximal ideal of $M(I)$ is the kernel of some multiplicative functional. We will now show if $L^1(I, \mathscr{B}, |\mu|) \subseteq \ker(T)$, then this inclusion is a proper inclusion.

Let $x \in K_1^{-1}(\hat{I})$. It is straightforward to show that $\delta_x \neq \delta_t$, where δ_x is the Dirac point measure at x. Since $\delta_x \notin L^1(I, \mathscr{B}, |\mu)$, if $T(\delta_x) = 0$, then $\delta_x \in \ker(T)$ and we are done. Now suppose $T(\delta_x) \neq 0$. Consider the Borel measure $T(\delta_x)\delta_t - \delta_x$. If $T(\delta_x)\delta_t - \delta_x \in L^1(I, \mathscr{B}, |\mu|)$, then there exists $f \in L^1(I, \mathscr{B}, |\mu|)$ such that

$$(T(\delta_x)\delta_t - \delta_x)(E) = \int_E f(x)\, \mathrm{d}|\mu|(x).$$

Letting E be the singleton set $\{K_1^{-1}(1)\}$, we see that

$$T(\delta_x) = 0,$$

which is a contradiction. So we have $T(\delta_x)\delta_t - \delta_x \notin L^1(I, \mathscr{B}, |\mu|)$. Observe that $T(T(\delta_x)\delta_t - \delta_x) = T(\delta_x)T(\delta_t) - T(\delta_x) = 0$. So $L^1(I, \mathscr{B}, |\mu|) \subsetneq \ker(T)$. This completes the proof. □

Chapter 6

Discussion and future work

A new class of discrete and integral transforms was introduced in Chapter 2. In Chapter 3, a class of integral transforms was analysed which encapsulates the integral transforms defined in Chapter 2. This class of transforms includes the Fourier, Laplace and Mellin transforms. Both the shifting and convolution properties were introduced for this class of transforms. Sufficient conditions were given for the existence of the convolution property. Further assumptions were placed on the convolution formula which guarantee the convolution of two functions is continuous. A subclass of integral transforms, each of which has a logarithmic separable kernel was shown to be injective. Following this, an inversion formula was presented for a subclass of continuously differentiable functions. The results in this chapter have found applications in second-order linear differential equations. More specifically, analytical solutions for a class of differential equations were derived which, to the author's knowledge, have not been derived previously. Due to the complex nature of these solutions, the analytical and numerical results were compared for the purpose of determining the accuracy of our analytical solutions. Once a justification was made, a comparison between the analytical and numerical solutions was made to highlight the need for the analytical solution for a specific differential equation. This, in turn, gives justification for our new class of integral transforms and our convolution formula.

Another potential application for the results in this thesis is to solve integral equations of the form

$$h(x) = \int_I f(\psi(x,u))g(u)D_1\psi(x,u)\chi_{J_u}(x)\,\mathrm{d}u,$$

where f and h are given, and g is to be determined. It follows that, under appropriate hypotheses, the solution g is found by applying the integral transform corresponding to that specific convolution formula, namely

$$g(x) = \int_I \frac{h(\psi(x,u))}{f(u)} D_1\psi(x,u)\chi_{J_u}(x)\,\mathrm{d}u.$$

Other types of integral equations include

$$f(x) = h(x) + \int_I g(\psi(x,u)) f(u) D_1 \psi(x,u) \chi_{J_u}(x) \, \mathrm{d}u,$$

where g and h are specified functions and f is unknown. For the case where $\mathscr{A}$ is the Laplace transform, these types of integral equations are the well-known Volterra integral equations, as $\psi(x,u) = x - u$. Assuming that g, h and K_1 satisfy all the necessary assumptions, we have

$$\mathscr{A}\{f\} = \mathscr{A}\{h\} + \mathscr{A}\{g\}\mathscr{A}\{f\}.$$

Rearranging this equation gives

$$\mathscr{A}\{f\} = \frac{\mathscr{A}\{h\}}{1 - \mathscr{A}\{g\}}.$$

Applying the inversion formula then yields

$$f(x) = \frac{c_1 K_1'(x)}{2\pi \mathrm{i} K_1(x)} \lim_{T \to \infty} \int_{\frac{\mu - c_2 - \mathrm{i}T}{c_1}}^{\frac{\mu - c_2 + \mathrm{i}T}{c_1}} K_1(x)^{-(c_1 s + c_2)} \frac{\mathscr{A}\{h\}(s)}{1 - \mathscr{A}\{g\}(s)} \, \mathrm{d}s.$$

In Chapter 4, the class of discrete and integral transforms introduced in Chapter 2 was analysed. Some examples of transforms which fall in our class were presented. We subsequently proved some elementary properties of discrete and integral transforms which fall in our class. These include the image of the integral transform being a subset of some bounded and continuous functions, as well as the transform being a continuous operator between normed vector spaces. The shifting and convolution properties for this class of transforms was introduced and sufficient conditions for the existence of these properties was given. Some examples of convolution formulas which are associated with discrete transforms were given and these formulas were shown to be commutative. It was then proven that the class of discrete transforms is injective on the space $L^1(I, \mathscr{B}, |\mu|)$. This allows for a simple proof that the convolution product is commutative, associative and distributes over addition. A proof which shows the convolution product is continuous on the product space $L^1(I, \mathscr{B}, |\mu|) \times L^1(I, \mathscr{B}, |\mu|)$ is given. Collecting the results in this chapter shows that the space $L^1(I, \mathscr{B}, |\mu|)$ is a Banach algebra and $\mathscr{A}$ is a continuous homomorphism between two Banach algebras. The chapter was concluded by showing that when I is an interval, the space $L^1(I, \mathscr{B}, |\mu|)$ has no unit. In the case where I is countable, the space $L^1(I, \mathscr{B}, |\mu|)$ may have a unit.

In Chapter 5, further properties of the convolution product were shown. This includes the fact that when an L^p function is convolved with an L^q function, the resulting function is a continuous function which vanishes at infinity. The problem of there being no unit in the Banach algebra $L^1(I, \mathscr{B}, |\mu|)$ was addressed. Distributions were analysed in this chapter. Before the convolution of a distribution and a test function was introduced, an appropriate space V_g, in which the new convolution was defined on, was studied. It was shown that the convolution of a distribution and a smooth function is smooth on the space V_g.

In the case of the Fourier convolution, it is possible to define the convolution of a distribution and a bump function either as a distribution or as a smooth function. It may not be possible to define the convolution of a distribution and a smooth function as a distribution in a natural way for our class of transforms. Suppose $f \in L^1_{\mathrm{loc}}(I;\mathbb{R})$ and $g \in C_c^\infty(I;\mathbb{R})$. Assume ϕ is some sufficiently well-behaved function. By making some formal calculations, we have

$$\begin{aligned}\int_I (f*g)(x)\phi(x)\,\mathrm{d}\lambda(x) &= \int_I\int_I f(u)g(\psi(x,u))D_1\psi(x,u)\chi_{J_u}(x)\phi(x)\,\mathrm{d}\lambda(u)\,\mathrm{d}\lambda(x)\\ &= \int_I f(u)\int_I g(\psi(x,u))D_1\psi(x,u)\chi_{J_u}(x)\phi(x)\,\mathrm{d}\lambda(x)\,\mathrm{d}\lambda(u).\end{aligned}$$

As such, it seems appropriate to define the following

$$(f*g)[\phi] = f\left[u\mapsto \int_I g(\psi(x,u))D_1\psi(x,u)\chi_{J_u}(x)\phi(x)\,\mathrm{d}\lambda(x)\right].$$

The issue with the above definition is that it has not been shown that the function defined by

$$u\mapsto \int_I g(\psi(x,u))D_1\psi(x,u)\chi_{J_u}(x)\phi(x)\,\mathrm{d}\lambda(x)$$

is a smooth function on I with compact support. However, suppose we have that the function K_1 satisfies $K_1(I) = (0,\infty)$, and here we assume I is an open interval. Then it is possible to extend a subclass of the integral transforms introduced in Chapter 3 to distributions.

Suppose I is an open interval and $K(x,s) = K_1(x)^{s-w}$. Let ψ and $D_1\psi$ be defined as in Chapter 4 and Chapter 5. Also, let V_g be defined as in (5.4). We highlight for $f,g\in C_c^\infty(I;\mathbb{R})$ that $f*g$ exists by Theorem 49. Since $J_u = I$ for every $u\in I$, we have that ψ is defined on $I\times I$. Now, if we wish to define $f*g$ as a function, then we let

$$(f*g)(x) = f[g(\psi(x,\cdot))D_1\psi(x,\cdot)].$$

Now, if we define $f*g$ as a distribution, then for every $\phi\in C_c^\infty(I,V_g;\mathbb{R})$, we let

$$(f*g)[\phi] = f\left[u\mapsto \int_I g(\psi(x,u))D_1\psi(x,u)\phi(x)\,\mathrm{d}\lambda(x)\right].$$

We note that such transforms do not fall in the class of discrete and integral transforms introduced in Chapter 2. However, the corresponding convolution product is a special case of the φ-convolution. We note that this seems like an appropriate definition for the convolution of f and g as a distribution. This is because the function

$$x \mapsto f[g(\psi(x,\cdot))D_1\psi(x,\cdot)]$$

is locally integrable on I. As such it defines a distribution, which is equivalent on V_g. That is

$$(f*g)[\phi] = \int_I f[g(\psi(x,\cdot))D_1\psi(x,\cdot)]\phi(x)\,\mathrm{d}\lambda(x).$$

This can be shown by making some formal calculations as well as some abuse of notation. We have that

$$\begin{aligned}
(f*g)[\phi] &= f\left[u \mapsto \int_I g(\psi(x,u))D_1\psi(x,u)\phi(x)\,\mathrm{d}\lambda(x)\right] \\
&= f\left[\lim_{N\to\infty}\sum_{i=1}^{N} g(\psi(t_i,\cdot))D_1\psi(t_i,\cdot)\phi(t_i)(x_i - x_{i-1})\right] \\
&= \lim_{N\to\infty}\sum_{i=1}^{N} f\left[g(\psi(t_i,\cdot))D_1\psi(t_i,\cdot)\right]\phi(t_i)(x_i - x_{i-1}) \\
&= \int_I f[g(\psi(x,\cdot))D_1\psi(x,\cdot)]\phi(x)\,\mathrm{d}\lambda(x).
\end{aligned}$$

It was mentioned in Chapter 2 that the Fourier transform can be defined on tempered distributions due to the underlying functions being Schwartz functions, and the fact that the Fourier transform is an isomorphism on the Schwartz functions. The $\mathscr{A}$-transform is a bijection between two spaces, which allows us to define a transform $\mathscr{A}^*$ on some class of distributions, where the $\mathscr{A}^*$-transform is related to the $\mathscr{A}$-transform. An alternative approach to extend our class of integral transforms with distributions is to define the transform of a distribution f by $f[K_1(\cdot)^{s-w}]$ where the distributions are defined on an appropriate set of functions.

We recall that, for a given transform $\mathscr{A}$, we were able to define the adjoint of $\mathscr{A}$ on some class of distributions. There is potential for our class of transforms to be defined on distributions. If it can be shown that the $\mathscr{A}^*$-transform is a bijection between two classes of functions, then it may be possible to extend the $\mathscr{A}$-transform to distributions formally as follows

$$\mathscr{A}\{f\}[\phi] = f[\mathscr{A}^*\{\phi\}].$$

The convolution was extended to complex Borel measures, which contains the set of probability measures on I. The Fourier convolution of measures is fundamental when studying independent random variables. More specifically, the Fourier transform of a probability measure is the characteristic function. The concept of vague convergence, which is weak* convergence in $M(I) = C_0(I;\mathbb{C})^*$, plays an important role in probability theory. If $\{\mu_n\}$ is a sequence of probability measures in $M(I)$ which converges to μ in the weak* topology on $M(I)$, then the sequence of distribution functions $\mu_n((-\infty,t])$ will converge to $\mu((-\infty,t])$ at every point where $t \mapsto \mu((-\infty,t])$ is continuous. In fact, these two properties are equivalent. That is, if $\mu_n((-\infty,t]) \to \mu((-\infty,t])$ at every point where $t \mapsto \mu((-\infty,t])$ is continuous, then $\mu_n \to \mu$ vaguely given μ_n and μ are probability measures. So $\mu_n \to \mu$ vaguely if and only if $\mu_n((-\infty,t]) \to \mu((-\infty,t])$ at each point $\mu((-\infty,\cdot])$ is continuous if and only if the corresponding sequence of characteristic functions converge pointwise to $\mathscr{F}\{\mu\}$. The utility of this can be found in the Central Limit Theorem, which states that a sequence of independent identically distributed random variables $\{X_i\}$ whose mean is μ and variance is σ, then the expression

$$\sum_{i=1}^{n} \frac{X_i - \mu}{\sigma/\sqrt{n}}$$

converges vaguely to the standard normal distribution. It is expected that the $\mathscr{A}$-transform of measures has a similar convergence property which may have applications to probability theory.

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