Thermodynamics of permutation-invariant quantum many-body systems: A group-theoretical framework

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Quantum systems of indistinguishable particles are commonly described using the formalism of second quantization, which relies on the assumption that any admissible quantum state must be either symmetric or antisymmetric under particle permutations. Coherence-induced many-body effects such as superradiance, however, can arise even in systems whose constituents are not fundamentally indistinguishable as long as all relevant dynamical observables are permutation-invariant. Such systems are not confined to symmetric or antisymmetric states and therefore require a different theoretical approach. Focusing on noninteracting systems, here we combine tools from representation theory and thermodynamically consistent master equations to develop such a framework. We characterize the structure and properties of the steady states emerging in permutation-invariant ensembles of arbitrary multilevel systems that are collectively weakly coupled to a thermal environment. As an application of our general theory, we further explore how these states can in principle be used to enhance the performance of quantum thermal machines. Our group-theoretical framework thereby makes it possible to analyze various limiting cases that would not be accessible otherwise. In addition, it allows us to show that the properties of multilevel ensembles differ qualitatively from those of spin ensembles, which have been investigated earlier using the standard Clebsch-Gordan theory. Our results have a large scope for future generalizations and pave the way for systematic investigations of collective effects arising from permutation invariance in quantum thermodynamics.

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I. INTRODUCTION

Permutation symmetry plays a fundamental role in many-body quantum mechanics. According to the principle of *indistinguishability*, quantum states that differ only by interchanges of identical particles cannot be distinguished by any measurement [1,2]. As a result, any dynamical observable of a system of identical particles, including its Hamiltonian, must be invariant under particle permutations. The eigenstates of such observables can be divided into linearly independent sets according to their symmetry type under permutations; for two particles, the eigenstates of a permutation-invariant observable can be chosen as either symmetric or antisymmetric; for more than two particles, additional, partially symmetric types arise. These symmetry types are preserved under the time evolution of the system. Moreover, they give rise to superselection rules which forbid transitions between states of different types and imply that coherences between such states cannot be observed.

The *symmetrization postulate* is more restrictive than the principle of indistinguishability. It states that, depending on the sort of particle, only quantum states with one specific symmetry type exist: identical bosons are described by symmetric states, identical fermions by antisymmetric ones [2]. This restriction has profound consequences. It implies, for example, that the particles of an ideal quantum gas must be distributed in energy space according to either Bose-Einstein or Fermi-Dirac statistics, which lead to fundamentally different physical properties. On a mathematical level, the symmetrization postulate is elegantly accommodated by the formalism of second quantization, which inherently restricts the Hilbert space of a many-body system to its fully symmetric or antisymmetric subspace.

Here we focus on many-body systems that are not subject to the symmetrization postulate but whose relevant dynamical observables are still permutation-invariant. Specifically, we consider collections of noninteracting multilevel systems which are coupled to an environment that cannot distinguish between their constituents. The particles may be not fundamentally identical as an observer may still be able to tell them apart through an external degree of freedom; see Fig. 1. For clarity, we will refer to this type of many-body system as *permutation-invariant ensembles*. Such settings can be realized, for instance, with cold atoms [4,5], ions [6], or artificial atoms [7] and can host remarkable phenomena with *superradiance* being a prime example [3].

Such *collective effects* in many-body systems have recently received much attention in quantum thermodynamics, both

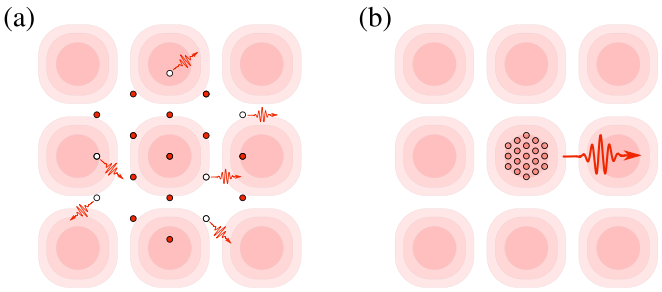

FIG. 1. Permutation-invariant ensembles. A collection of two-level atoms, which can be externally distinguished by their position in a lattice, is coupled to a thermal radiation field whose spatial intensity is represented by the red shaded areas. The system is initially prepared in a fully symmetric state, where all atoms are excited. (a) If the lattice spacing is larger than the typical wave length of the radiation field, the atoms decay independently under the spontaneous emission of photons. At long times, the ensemble settles to a Gibbs state that is a statistical mixture of many-body quantum states with all possible permutation symmetry types. (b) When its typical wave length is comparable to the lattice spacing, the atoms become indistinguishable to the radiation field. As a result, the permutation symmetry of the initial state must be preserved during the decay process, which leads to the collective emission of radiation, i.e., superradiance [3]. The ensemble thereby approaches a nonthermal steady state, which contains only fully symmetric quantum states.

in the context of fundamental questions [8–11] and on the more applied side, for instance, as a means of enhancing the performance of quantum heat engines and refrigerators [12–26] or isothermal machines [27–34]. Studies of devices featuring strong interparticle interactions [12,13,27–33] are usually based on specific models, since interacting many-body systems are notoriously hard to describe on a general level, especially when coupled to a thermal environment. Indistinguishability-induced collective thermodynamical effects can be observed even in noninteracting systems [14–25], which in many cases admit a comparatively simple theoretical description in terms of thermodynamically consistent quantum master equations [35,36]. This approach, however, has so far been mostly limited to spin ensembles [19–25], whose collective behavior is captured by standard Clebsch-Gordan theory. In developing a systematic generalization of this theory to arbitrary multilevel systems, our work provides a universal framework to further investigate the principal role of collective effects in quantum thermodynamical processes and to uncover their potential applications.

The quantum states of a permutation-invariant ensemble are generally not restricted to a specific symmetry type and span the entire many-body Hilbert space, which renders the formalism of second quantization inapplicable. Our first major aim is to show that such systems nevertheless admit an efficient theoretical description, which, among other results, provides explicit means to calculate the expectation values of permutation-invariant observables. At the heart of this framework lies *Schur-Weyl duality*, a powerful tool from representation theory, which makes it possible to endow the Hilbert space of a permutation-invariant ensemble with a universal structure [37,38]. Specifically, this structure consists of a series of invariant subspaces that are associated with both the permutation group and the symmetry group generated by the dynamical variables of the system. For spin systems, these dynamical variables correspond to angular momentum degrees of freedom, which generate the quantum rotation group SU(2). In this special case, which has been studied before [19–25] and will serve as a reference for our analysis, the Schur-Weyl decomposition reduces to the conventional Clebsch-Gordan series [2].

As a consequence of the superselection rules implied by the indistinguishability principle, permutation-invariant ensembles generally do not relax to a Gibbs state in a thermal environment. Instead, they settle to a nontrivial steady state that still depends on the initial state in which the ensemble was originally prepared; see Fig. 1. Our second major aim is to systematically characterize these steady states, determine their thermodynamical properties, analyze their internal structure, and explore how they can be utilized to enhance the performance of quantum engine cycles. Our mathematical framework does, however, have a wider scope, which extends beyond steady states. It thus opens an interesting perspective for future research.

A second potential area of application for our mathematical framework is quantum information theory, where permutation-invariant systems have been studied in the context of entanglement and reference frames [39], as well as for describing hidden degrees of freedom in quantum optics [40–42]. Similarly, bunching effects in quantum optics have been shown to have implications for energy transfer between light and a mechanical oscillator [43,44]. In general, there is potential for exploring a variety of quantum settings in which the same permutation-invariant structures arise.

The paper is structured as follows. In Sec. II we outline the mathematical background necessary for the analysis. Readers familiar with representation theory may skip this section. In Sec. III we describe the physical setting and weak-coupling model of thermalization in open systems, giving the general partially thermalized form of the steady state. We give computable formulas for basic thermodynamical quantities in Sec. IV and then put these to use in Sec. V by studying a model of a heat engine, the Otto cycle. Finally, in Sec. VI we explore the nonclassical implications of higher-order symmetry groups beyond spins, followed by broader perspectives in Sec. VII.

II. MATHEMATICAL TOOLS

In this section we introduce the mathematical tools that will be used in our analysis. We provide a brief guide to the Lie group SU(d), its Lie algebra $su(d)$, and their representations. We further discuss Schur-Weyl duality, Young diagrams, group characters, and Weyl's character formula. To illustrate these general concepts, we show how they appear in the common theory of angular momentum. Readers already familiar with these mathematical tools may skip to the next section.

A. The special unitary group

The special unitary group of degree d, denoted by SU(d), is the group of unitary $d \times d$ matrices with determinant 1.

Any element u of this group can be written in the form $u = e^{ih}$, where h is traceless Hermitian $d \times d$ matrix. These matrices form a vector space over the real numbers. Hence, upon choosing a basis $\{x_a\}$ in this space, we can express any $u \in \mathrm{SU}(d)$ as $u = \exp(i \sum_a \theta_a x_a)$ with $a = 1, \ldots, d^2 - 1$ and $\theta_a \in \mathbb{R}$. The matrices x_a are called *generators* of $\mathrm{SU}(d)$. They satisfy a set of characteristic commutation relations

$$[x_a, x_b] = i \sum_c f_c^{ab} x_c, \tag{1}$$

which define the Lie algebra $su(d)$ with the coefficients $f_c^{ab} = -f_c^{ba} \in \mathbb{R}$ being called *structure constants* (taking values depending on the chosen basis).

In practice, it is often convenient to chose a special non-Hermitian basis for the Lie algebra $su(d)$, which is known as the *Cartan basis* [45]. To construct this basis, we first define a set of $d - 1$ diagonal Hermitian generators d_i, which satisfy

$$[d_i, d_j] = 0. \tag{2}$$

These generators form the Cartan subalgebra of $su(d)$. The remaining $d(d-1)$ generators, which we denote by e_μ, can be chosen such that they obey the commutation relations

$$[d_i, e_\mu] = v_\mu^i e_\mu, \tag{3}$$

where the constants $v_\mu^i \in \mathbb{R}$ form the *root vector* $\mathbf{v}_\mu = (v_\mu^1, \ldots, v_\mu^{d-1})$ associated with the nondiagonal generator e_μ. For simplicity we will often refer to μ as a root.

The nondiagonal generators e_μ come in Hermitian conjugate pairs. As can be seen from Eq. (3), they obey the symmetry $e_\mu^\dagger = e_{-\mu}$ with $\mathbf{v}_{-\mu} = -\mathbf{v}_\mu$. In the Cartan basis, the exponential map that connects the group $\mathrm{SU}(d)$ with its Lie algebra $su(d)$ reads $u = \exp[i(\sum_i \alpha_i d_i + \sum_\mu \gamma_\mu e_\mu)]$, where $\alpha_i \in \mathbb{R}$ and $\gamma_\mu^* = \gamma_{-\mu} \in \mathbb{C}$. Using the chain rule for the commutator, one can now derive the remaining commutation relations

$$[e_\mu, e_\nu] = N_{\mu\nu}\, e_{\mu+\nu} \quad \text{for} \quad \nu \neq -\mu \quad \text{and} \tag{4}$$

$$[e_\mu, e_{-\mu}] = \sum_i M_{\mu i}\, d_i, \tag{5}$$

which, together with the relations (2) and (3), fully specify the Lie algebra $su(d)$ in the Cartan basis. Here $\mu + \nu$ denotes a root with corresponding root vector $\mathbf{v}_{\mu+\nu} := \mathbf{v}_\mu + \mathbf{v}_\nu$. The coefficients $N_{\mu\nu}, M_{\mu i} \in \mathbb{R}$ depend on the normalization of the generators, where $N_{\mu\nu}$ obey the symmetry $N_{\mu\nu} = -N_{\nu\mu}$ and vanish if there exists no generator with the root vector $\mathbf{v}_{\mu+\nu}$.

The $d(d-1)$ root vectors $\mathbf{v}_\mu$ are in one-to-one correspondence with the generators e_μ and span a $(d-1)$-dimensional vector space and therefore cannot all be linearly independent. In fact, one can always find a minimal generating set of roots $\hat{\mathbf{v}}_\mu$ such that any root vector can be decomposed as $\mathbf{v}_\mu = \sum_{\nu=1}^{d-1} n_{\mu\nu} \hat{\mathbf{v}}_\nu$, where the coefficients $n_{\mu\nu}$ are integers, which are either all nonpositive or nonnegative. The vectors $\hat{\mathbf{v}}_\mu$ are then referred to as *simple roots* [45]. A complete set of simple roots has $d - 1$ elements (and there is in general no unique choice).

In physics, the Hermitian generators of $\mathrm{SU}(d)$ are related to the dynamical variables of a d-level system. For example, a Hermitian basis of $su(2)$ is given by

$$s_x = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad s_y = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad s_z = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{6}$$

These matrices represent, up to a factor $\hbar$, the dynamical variables of a spin-$\frac{1}{2}$ system. Upon setting $x_1 = s_x$, $x_2 = s_y$, and $x_3 = s_z$, the generators satisfy the commutation relations (1) with the structure constants $f_c^{ab} = \varepsilon_{abc}$, where ε_{abc} denotes the Levi-Civita symbol. The Cartan basis of $su(2)$ is usually constructed by choosing the diagonal generator as $d_1 = s_z$. The nondiagonal generators are then given by the raising and lowering operators $e_\pm = s_x \pm i s_y$, which obey the commutation relations (3) with the one-dimensional root vectors $\mathbf{v}_\pm = \pm 1$; the coefficients appearing in Eq. (5) are given by $M_{\pm 1} = \pm 1/2$. Since $\mathbf{v}_- = -\mathbf{v}_+$, either $\mathbf{v}_+$ or $\mathbf{v}_-$ can be chosen as a simple root of $su(2)$.

As a second example, we consider $su(3)$, whose two diagonal generators can be identified with the Gell-Mann matrices

$$\Lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Lambda_8 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \tag{7}$$

i.e., $d_1 = \Lambda_3$ and $d_2 = \Lambda_8$. In relativistic particle physics, these generators are associated with the dynamical variables *Isospin* and *Hypercharge*. A complete set of simple roots for $su(3)$ is given by $\hat{\mathbf{v}}_{+1} = (2, 0)$, $\hat{\mathbf{v}}_{+2} = (1, \sqrt{3})$, and the corresponding nondiagonal generators have the matrix form

$$e_{+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_{+2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

B. Irreducible representations

A unitary representation of $\mathrm{SU}(d)$ on some Hilbert space $\mathcal{H}$ associates every $u \in \mathrm{SU}(d)$ with a unitary operator $U(u)$ such that $U(u)U(u') = U(uu')$. If $\mathcal{H}$ has no nontrivial subspace that is invariant under the action of all operators $U(u)$, the representation is called *irreducible*. Any reducible, i.e., not irreducible, representation can be decomposed into a direct sum of irreducible representations, or irreps. That is, there exists an orthonormal basis of $\mathcal{H}$, in which $U(u)$ takes the form

$$U(u) = \bigoplus_\lambda U^\lambda(u) \tag{9}$$

with the operators $U^\lambda(u)$ forming irreps of $\mathrm{SU}(d)$ on some orthogonal subspaces $\mathcal{H}^\lambda$ of $\mathcal{H}$. The label λ thereby represents an ordered partition of some integer n into d nonnegative integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_d$. The decomposition (9) may contain multiple terms with the same label λ. We note that the natural representation $U(u) = u$ of $\mathrm{SU}(d)$ on $\mathcal{H} = \mathbb{C}^d$ is always irreducible.

The concept of representations naturally extends to the generators: any $U(u)$ can be expressed as $U(u) = \exp(i \sum_a \theta_a X_a)$, where $\theta_a \in \mathbb{C}$ and the X_a are traceless operators satisfying the same commutation relations as the

generators x_a,

$$[X_a, X_b] = i \sum_c f_c^{ab} X_c, \tag{10}$$

with the same structure constants f_c^{ab} as in Eq. (1). Hence, the operators X_a form a representation of the Lie algebra $su(d)$ on $\mathcal{H}$. These operators can be chosen to be Hermitian, in which case the coefficients θ_a must be real. In general, a representation can be constructed for any basis of $su(d)$. In particular, we may pick the Cartan basis, whose representation consists of $d-1$ Hermitian operators D_i and $d(d-1)$ operators E_μ that come in Hermitian conjugate pairs satisfying $E_\mu^\dagger = E_{-\mu}$. Since these operators obey the same commutation relations as the corresponding generators d_i and e_μ [cf. Eqs. (2)–(5)], there exists a basis of $\mathcal{H}$, in which D_i become diagonal and E_μ assume the roles of raising and lowering operators.

Any irrep of the group SU(d) must derive from an irrep of the Lie algebra $su(d)$. That is, if the operators $U(u)$ do not share a nontrivial invariant subspace, the same must be true for the corresponding representations X_a of the generators. The converse statement also holds: any irrep X_a of $su(d)$ gives rise to an irrep of SU(d) via the exponential map $U(u) = \exp(i \sum_a \theta_a X_a)$. Any reducible representation of SU(d) must therefore be associated with a reducible representation of $su(d)$, which can be decomposed into a direct sum of irreps. Specifically, in the basis of $\mathcal{H}$ where Eq. (9) holds, we also have

$$X_a = \bigoplus_\lambda X_a^\lambda \tag{11}$$

with the X_a^λ forming an irrep of $su(d)$ on the subspace $\mathcal{H}^\lambda$. Since changing the basis of $su(d)$ is a linear operation on both the generators and their representations, Eq. (11) holds for any basis of $su(d)$.

In the previous section, we have seen that the Hermitian generators of SU(d) can be identified with the fundamental dynamical variables of a physical system. This interpretation extends also to higher-dimensional representations of these generators. In particular, a Hermitian irrep of $su(2)$ on $\mathbb{C}^{2s+1}$ describes a spin-s system. For example, the three-dimensional analogs of the matrices (6),

$$S_x^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; S_y^1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix},$$
$$S_z^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{12}$$

form an irrep of $su(2)$ on $\mathbb{C}^3$ with $X_1 = S_x^1$, $X_2 = S_y^1$, and $X_3 = S_z^1$ and represent the dynamical variables of a spin-1 system.

C. Schur-Weyl duality

We now consider the representations of SU(d) on the product space $\mathcal{H}^{(n)} = (\mathbb{C}^d)^{\otimes n}$, which is the Hilbert space of an ensemble of n d-level systems. We will refer to the individual systems as *particles* in the following. Extending the natural representation of SU(d) to $\mathcal{H}^{(n)}$ yields the tensor-product representation $U^{(n)}(u) = u^{\otimes n}$, which is generally reducible. Hence, the operators $U^{(n)}(u)$ can be brought to the block-diagonal form of Eq. (9). A block with dimension $d_\lambda = \dim \mathcal{H}^\lambda$ can thereby appear multiple times, and we denote its multiplicity by m_λ. The basis of $\mathcal{H}^{(n)}$ that produces the decomposition (9) is called the *Schur basis*. Finding this basis is generally complicated. However, the dimensions d_λ and multiplicities m_λ of the individual irreps can be calculated without explicit knowledge of the Schur basis, as we will see in the next section. In this section, we first introduce the mathematical tool that makes these calculations possible.

Schur-Weyl duality asserts that the tensor-product representation $U^{(n)}(u)$ of SU(d) admits the decomposition

$$U^{(n)}(u) = \bigoplus_\lambda \mathbb{1}_{\mathcal{K}^\lambda} \otimes U^\lambda(u), \tag{13}$$

where λ stands for an ordered partition of n into d integers, $U^\lambda(u)$ is an irrep of SU(d) with dimension d_λ, and $\mathbb{1}_{\mathcal{K}^\lambda}$ is the identity operator on an m_λ-dimensional Hilbert space $\mathcal{K}^\lambda$ [37,38]. As will be explained in Sec. II D, each λ indicates a different symmetry type of the wave function.

More generally, Schur-Weyl duality states that the product space $\mathcal{H}^{(n)}$ decomposes as

$$\mathcal{H}^{(n)} = \bigoplus_\lambda \mathcal{K}^\lambda \otimes \mathcal{H}^\lambda, \tag{14}$$

where $\mathcal{K}^\lambda$ and $\mathcal{H}^\lambda$ carry irreps of the permutation group over n elements S_n and the unitary group $U(d)$, respectively [37,38]. In the following, we will therefore refer to $\mathcal{K}^\lambda$ as a *permutation subspace* and to $\mathcal{H}^\lambda$ as a *unitary subspace*. In practical terms, Schur-Weyl duality implies that any operator O on the product space $\mathcal{H}^{(n)}$ that is invariant under arbitrary particle permutations takes the form

$$O = \bigoplus_\lambda \mathbb{1}_{\mathcal{K}^\lambda} \otimes O^\lambda \tag{15}$$

in the Schur basis. The identity (13) then follows by noting that the operators $U^{(n)}(u)$ are permutation-invariant and that every irrep of $U(d)$ remains irreducible when being restricted to SU(d).

We now consider the Lie algebra $su(d)$. From a given basis x_a, we can construct the tensor-product representation $X_a^{(n)} = \sum_{k=0}^{n-1} \mathbb{1}_d^{\otimes k} \otimes x_a \otimes \mathbb{1}_d^{\otimes(n-1-k)}$ of $su(d)$ on $\mathcal{H}^{(n)}$, where $\mathbb{1}_d$ is the identity matrix of dimension d. Since operators $X_a^{(n)}$, which generate the tensor-product representation of SU(d), are permutation-invariant, they decompose in the same way as the operators $U^{(n)}(u)$. Hence, in the Schur basis, we have

$$X_a^{(n)} = \bigoplus_\lambda \mathbb{1}_{\mathcal{K}^\lambda} \otimes X_a^\lambda, \tag{16}$$

where the X_a^λ form an irrep of $su(d)$ with dimension d_λ.

Before moving on, it is again instructive to consider the special case of SU(2). The space $\mathcal{H}^{(n)}$ is then the joint Hilbert space of n spin-$\frac{1}{2}$ particles. The operators $X_a^{(n)} = S_{x,y,z}^{(n)} = \sum_{k=0}^{n-1} \mathbb{1}_2^{\otimes k} \otimes s_{x,y,z} \otimes \mathbb{1}_2^{\otimes(n-1-k)}$, which form a representation of the canonical basis of $su(2)$, correspond to the collective angular momentum operators of the system. Every ordered partition $\lambda = (\lambda_1, \lambda_2)$ of n can now be uniquely associated with an eigenvalue $[J(J+1)]^{\frac{1}{2}}$ of the

total angular momentum operator $S^{(n)} = [\mathbf{S}^{(n)} \cdot \mathbf{S}^{(n)}]^{\frac{1}{2}}$, where $\mathbf{S}^{(n)} = (S_x^{(n)}, S_y^{(n)}, S_z^{(n)})$, via the rule $J = (\lambda_1 - \lambda_2)/2$. Hence, the decomposition (16) becomes the usual Clebsch-Gordan series, where every irrep of $su(2)$ corresponds to a different value of J.

The Schur basis is given by the collective angular momentum basis $\{|J, m, p\rangle\}$, where m is an eigenvalue of $S_z^{(n)}$ and p is an index for the permutation subspace $\mathcal{K}^J$. From the theory of angular momenta, we know that m takes the values $-J, -J+1, \ldots, J$ for fixed J [2]. The irrep J thus has dimension $d_J = 2J + 1$. The multiplicities can be determined recursively from the usual rules for the addition of angular momenta. For $n = 1$, we trivially have $J = \frac{1}{2}$ and $m_{\frac{1}{2}} = 1$. For two spins, J takes the values 0 and 1 corresponding to singlet and triplet states; hence, we have $m_0 = m_1 = 1$. Adding a third spin turns the value 0 into $\frac{1}{2}$, while the value 1 branches into $\frac{3}{2}$ and $\frac{1}{2}$, which gives $m_{\frac{3}{2}} = 1$ and $m_{\frac{1}{2}} = 2$. Analogously, we find $m_0 = 2$, $m_1 = 3$, and $m_2 = 1$ for $n = 4$. This scheme can, in principle, be applied arbitrarily often, although it becomes cumbersome for large n. In the next section, we describe a more efficient method to calculate the multiplicities of individual irreps for SU(2) and SU(d) in general.

D. Young diagrams

An ordered partition $\lambda = (\lambda_1, \ldots, \lambda_d)$ of n can be graphically represented as a set of d left-justified rows made up of $\lambda_1, \ldots, \lambda_d$ boxes. This representation is called the Young diagram Y_λ of λ. For instance, the partition $\lambda = (3, 1)$ has the Young diagram

$$Y_\lambda = \begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1} \end{array}\,. \tag{17}$$

The boxes of a Young diagram can be considered as containers for indices that label either single-particle states or particles [37]. In the former case, a so-called semistandard Young tableau is obtained by filling the numbers $1, \ldots, d$ into the Young diagram Y_λ such that the sequence of numbers in every row is nondecreasing and the sequence in every column is strictly increasing. The Young diagram (17) thus admits three semistandard Young tableaux for $d = 2$:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1} \end{array}\,. \tag{18}$$

Each Young tableau that is constructed in this way represents a many-particle state with a specific permutation symmetry pattern, which depends only on λ. All of these states are linearly independent and states with different symmetry patterns; i.e., states belonging to different Young diagrams, are orthogonal to each other. Since the symmetry patterns are invariant under the action of the tensor-product operators $U^{(n)}(u)$ [37,46], the many-particle states derived from a given Young diagram Y_λ span an invariant subspace of the $U^{(n)}(u)$. Moreover, it can be shown that any invariant subspace of the $U^{(n)}(u)$ can be constructed in this way. It follows that the dimensions of these subspaces equal the dimensions d_λ of the corresponding irreps U^λ. Thus, d_λ can be determined by counting the admissible Young tableaux for a given Young diagram. This argument leads to the general formula [38]

$$d_\lambda = \frac{1}{(d-1)!(d-2)!\cdots 1!} \prod_{1 \leqslant i < j \leqslant d} (\tilde{\lambda}_i - \tilde{\lambda}_j) \quad \text{with}$$
$$\tilde{\lambda} = \lambda + \delta \quad \text{and} \quad \delta = (d-1, d-2, \ldots, 0). \tag{19}$$

For $d = 2$, we have $d_\lambda = \lambda_1 - \lambda_2 + 1$, and, upon identifying $J = (\lambda_1 - \lambda_2)/2$, we recover the result $d_J = 2J + 1$.

To determine the multiplicities of the irreps U_n^λ, we have to construct a different set of standard Young tableaux by filling the Young diagrams with the numbers $1, \ldots, n$, which label the particles of the ensemble, such that the sequence of numbers in every row and column is strictly increasing. For the Young diagram (17), this rule yields three possible standard Young tableaux given by

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}\,. \tag{20}$$

The number of possible Young tableaux that can be obtained in this way from a given Young diagram Y_λ gives the multiplicity of the corresponding irrep U_n^λ and can be determined by combinatorial arguments, which yield the closed-form expression [38]

$$m_\lambda = \frac{n!}{\tilde{\lambda}_1! \cdots \tilde{\lambda}_d!} \prod_{1 \leqslant i < j \leqslant d} (\tilde{\lambda}_i - \tilde{\lambda}_j). \tag{21}$$

For the special case $d = 2$, we thus have, upon substituting $\lambda_1 = n/2 + J$ and $\lambda_2 = n/2 - J$,

$$m_J = \frac{n!(2J+1)}{(n/2+J)!(n/2-J)!}. \tag{22}$$

It is easy to check that this formula reproduces the multiplicities that we found iteratively for $n \leqslant 4$ in the previous section.

E. Characters

The final tool that will become essential to our analysis is the character associated with an irrep of SU(d) [47]. In general, the character $\chi_U(u)$ is a function on the group that associates with each element u the trace of the corresponding representation:

$$\chi_U(u) = \mathrm{Tr} U(u). \tag{23}$$

Here we are specifically interested in the characters of the Cartan subgroup of SU(d), whose elements u_c can be expressed solely in terms of the diagonal generators d_i. That is, we want to calculate objects of the form

$$\chi_{U^\lambda}(u_c) = \mathrm{Tr} e^{i \sum_i \alpha_i D_i^\lambda} \tag{24}$$

for an irrep $U^\lambda(u)$ of SU(d) corresponding to the ordered partition λ of n.

A general expression for these characters is provided by Weyl's character formula [48], which does not require explicit knowledge of the representation and can be applied as follows. We first define the $d \times d$ matrices c_i, whose ith diagonal element is 1 while all other elements are 0 [49]. The diagonal generators d_i can now be expanded in these matrices

as $d_i = \sum_j q_{ij} c_j$ with $\sum_j q_{ij} = 0$, since the d_i are traceless. Once the coefficients q_{ij} have been determined, we can formally express the representations of the diagonal generators as $D_i^\lambda = \sum_j q_{ij} C_j^\lambda$ such that Eq. (24) becomes

$$\chi_{U^\lambda}(u_c) = \mathrm{Tr} e^{i \sum_j \tilde{\alpha}_j C_j^\lambda}, \tag{25}$$

with $\tilde{\alpha}_j = \sum_i \alpha_i q_{ij}$. This quantity can now be calculated by means of the formula [50]

$$\chi_{U^\lambda}(u_c) = \frac{\det \mathbb{A}(\lambda + \delta)}{\det \mathbb{A}(\delta)}, \tag{26}$$

where δ was introduced in Eq. (19) and $\mathbb{A}$ denotes a $d \times d$ matrix with elements

$$\left\{\mathbb{A}[(r_1, \ldots, r_d)]\right\}_{kl} = e^{i\tilde{\alpha}_k r_l}. \tag{27}$$

To show how this recipe works, we consider again the special case of SU(2). With $d_1 = s_z$ the coefficients q_{ij} become $q_{11} = 1/2$ and $q_{12} = -1/2$. Hence, we have $\tilde{\alpha}_1 = \alpha_1/2$ and $\tilde{\alpha}_2 = -\alpha_1/2$ and Eq. (26) gives

$$\begin{aligned}\chi_\lambda(u_c) &= \mathrm{Tr} e^{i\alpha_1 D_1^\lambda} \\ &= \det \begin{bmatrix} e^{i\alpha_1(\lambda_1+1)/2} & e^{i\alpha_1\lambda_2/2} \\ e^{-i\alpha_1(\lambda_1+1)/2} & e^{-i\alpha_1\lambda_2/2} \end{bmatrix} \det \begin{bmatrix} e^{i\alpha_1/2} & 1 \\ e^{-i\alpha_1/2} & 1 \end{bmatrix}^{-1} \\ &= \frac{\sin[\alpha_1(2J+1)/2]}{\sin(\alpha_1/2)}, \end{aligned} \tag{28}$$

where we have substituted $J = (\lambda_1 - \lambda_2)/2$ in the second line. We note that this result could have been obtained directly from Eq. (24) upon recalling that $D_1 = S_z^J$ represents the z component of a spin-J system and thus has the matrix form $S_z^J = \mathrm{diag}[J, J-1, \ldots, -J]$. For $d > 2$, however, such a direct approach is no longer feasible. The character formula (26) then becomes a valuable tool, as we shall see in the following.

III. SETUP

Permutation-invariant systems encompass all those in which the dynamics are generated by Hamiltonians that are unchanged under exchange of any subsystems. Here our main objects of interest are ensembles of n identical, noninteracting d-level systems, to which we refer as particles. In the following, we show how the dynamics and stationary states that emerge when such an ensemble is coupled to a thermal bath can be systematically described using the tools of the previous section.

A. Multifrequency systems

1. Dynamics

We assume that the Hamiltonian of a single particle has the form

$$h = \hbar\Omega \sum_i a_i d_i, \tag{29}$$

where Ω sets the overall energy scale, the $a_i \in \mathbb{R}$ are dimensionless constants and the d_i are the diagonal generators of SU(d). The Hamiltonian of the ensemble is thus given by

$$H = \hbar\Omega \sum_i a_i D_i, \tag{30}$$

with the D_i being the diagonal generators of the tensor-product representation of SU(d) on the Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$. Note that, from here onward, we drop the index (n) that was used in Sec. II C to denote the tensor-product representation unless it is required for clarity.

The ensemble is coupled to a thermal bath at inverse temperature $\beta = 1/k_\mathrm{B}T$, where k_B denotes Boltzmann's constant. We assume that the bath cannot distinguish between the particles of the ensemble so that the system-bath coupling can be described in terms of the collective interaction Hamiltonian

$$H_I = \hbar\Delta \sum_\mu E_\mu \otimes B_\mu. \tag{31}$$

Here the E_μ are the nondiagonal generators of the tensor-product representation of SU(d), the Hermitian operators $B_\mu = B_{-\mu}$ correspond to observables of the bath, and Δ sets the coupling strength. From the commutation relation $[D_i, E_\mu] = v_\mu^i E_\mu$, we can now determine the relevant Bohr frequencies ω_μ of the system (i.e., the gaps in its spectrum), which are defined by the relation

$$[H, E_\mu] = \hbar\Omega \sum_i a_i v_\mu^i E_\mu = \hbar\omega_\mu E_\mu. \tag{32}$$

Thus, we have $\omega_\mu = \Omega \sum_i a_i v_\mu^i$, where v_μ^i is the element i of the root vector associated with the generator e_μ.

To describe the dynamics of the ensemble, we apply the standard weak-coupling, Born-Markov, and secular approximations. These approximations require that the timescale of the system-bath interaction, which is determined to be the inverse coupling strength, is much larger than the relaxation time of the bath and the timescale of the bare system, which is determined by its Bohr frequencies [51]. Under these conditions, one can derive the collective weak-coupling master equation

$$\begin{aligned}\partial_t \rho_t = &-\frac{i}{\hbar}[H + H_{LS}, \rho_t] \\ &+ \sum_{\langle\mu,\nu\rangle} \frac{\Gamma_\omega^{\mu\nu}}{2}\{[E_\nu, \rho_t E_\mu^\dagger] + [E_\nu \rho_t, E_\mu^\dagger]\}, \end{aligned} \tag{33}$$

where ρ_t denotes the state of the ensemble, E_ν and $E_\mu^\dagger$ play the role of jump operators, and the Lamb shift is given by $H_{LS} = \hbar\Delta \sum_{\langle\mu,\nu\rangle} s_\omega^{\mu\nu} E_\mu^\dagger E_\nu$, where $s_\omega^{\mu\nu}$ is the anti-Hermitian part of the bath correlation matrix [51]. H_{LS} commutes with the system Hamiltonian H and does not enter into the steady state. The sum in Eq. (33) runs over all pairs of indices $\langle\mu, \nu\rangle$ for which $\omega_\mu = \omega_\nu = \omega$. The complex coefficients $\Gamma_\omega^{\mu\nu}$ are determined by the bath-correlation functions. They form a positive-semidefinite Hermitian matrix and obey the detailed balance condition $\Gamma_{-\omega}^{\mu\nu} = e^{\beta\hbar\omega}\Gamma_\omega^{\nu\mu}$ [51]. Here we further assume that the bath couples to all collective modes of the system independently so that the matrix $\Gamma_\omega^{\mu\nu}$ has full rank for every $\omega \neq 0$.

Schur-Weyl duality now makes it possible to simplify the dynamics of the system as follows. We first recall that any operator O that is invariant under arbitrary particle permutations takes the block-diagonal form

$$O = \bigoplus_\lambda \mathbb{1}_{\mathcal{K}^\lambda} \otimes O^\lambda \tag{34}$$

in the Schur basis, where the index λ corresponds to an ordered partition of n into d integers, $\mathbb{1}_{\mathcal{K}^\lambda}$ is the identity operator on the permutation subspace $\mathcal{K}^\lambda$, and the operator O^λ acts on the unitary subspace $\mathcal{H}^\lambda$. This result implies that any permutation-invariant state of the ensemble can be written as

$$\rho = \bigoplus_\lambda p^\lambda \frac{\mathbb{1}_{\mathcal{K}^\lambda}}{m_\lambda} \otimes \rho^\lambda. \tag{35}$$

Here $m_\lambda = \dim \mathcal{K}^\lambda$ is the multiplicity of the irrep λ, and we have applied the normalization condition $\mathrm{Tr}\rho_t^\lambda = 1$ so that the $p^\lambda \geqslant 0$ add up to 1 and can thus be regarded as the probabilities for the system to occupy the diagonal block λ. Since the operators H, H_{LS}, and E_μ, which enter the collective master equation (33), all decompose according to Eq. (34), the block structure of the state (35) is preserved at any later time. Thus, if the system is initially in a permutation-invariant state, the block-occupation probabilities $p_t^\lambda = p^\lambda$ are conserved and the substates ρ_t^λ follow the master equation (33) with H, H_{LS}, and E_μ replaced by H^λ, H_{LS}^λ, and E_μ^λ, respectively.

It is now convenient to introduce the reduced state

$$\tilde{\rho}_t = \bigoplus_\lambda p^\lambda \rho_t^\lambda, \tag{36}$$

which is obtained by tracing out the permutation subspaces $\mathcal{K}^\lambda$. This state lives on the *reduced Hilbert space* $\tilde{\mathcal{H}} = \bigoplus_\lambda \mathcal{H}^\lambda$, which contains all degrees of freedom that are available to an observer who has access only to permutation-invariant observables; the expectation values of such observables are given by $\mathrm{Tr}O\rho_t = \sum_\lambda p^\lambda \mathrm{Tr}O^\lambda \rho_t^\lambda$. As we will see in Sec. IV, the reduced state fully determines the operationally accessible thermodynamical properties of the system. Also note that for any general state ρ_t, the reduced state $\tilde{\rho}_t$ must always be block-diagonal. That is, there is a superselection rule preventing the existence of superpositions of different λ on this space.

2. Steady states

In the long-time limit $t \to \infty$, the ensemble settles to a steady state ρ_∞. For $n > 1$, this state is not unique. It does, however, admit a universal structure. In Appendix A we prove a general theorem, which shows that, as long as the dissipative part of the master equation contains a set of simple roots, the full-rank condition on the matrix $\Gamma_\omega^{\mu\nu}$ and the detailed-balance condition $\Gamma_{-\omega}^{\mu\nu} = e^{\beta\hbar\omega}\Gamma_\omega^{\mu\nu}$ imply that all steady states take the block diagonal form

$$\rho_\infty = \bigoplus_\lambda p^\lambda \sigma^\lambda \otimes \gamma_\beta^\lambda \tag{37}$$

in the Schur basis. Here the operators σ^λ, which have trace 1, and the block-occupation probabilities p^λ depend on the initial state [52] and

$$\gamma_\beta^\lambda = \frac{e^{-\beta H^\lambda}}{Z_\beta^\lambda} \tag{38}$$

is a Gibbs state with respect to the irrep H^λ of H, where the partial partition function Z_β^λ is fixed by the condition $\mathrm{Tr}\gamma_\beta^\lambda = 1$. That is, every block thermalizes independently, subject to λ being an effective conserved quantity. We stress that this result holds for any initial state, regardless of whether it has the block structure (35). Moreover, our theorem holds more generally for any permutation-invariant system Hamiltonian, which may in principle also contain interactions between particles, and even if the detailed-balance condition does not hold, in which case the Gibbs state γ_β^λ has to be replaced with some general unique state ρ^λ; for details see Appendix A.

A natural choice for the initial state of the ensemble, on which we will focus in the following, is given by the Gibbs state

$$\gamma_{\beta_0} = \frac{e^{-\beta_0 H}}{Z_{\beta_0}} = \bigoplus_\lambda \frac{m_\lambda Z_{\beta_0}^\lambda}{Z_{\beta_0}} \frac{\mathbb{1}_{\mathcal{K}^\lambda}}{m_\lambda} \otimes \gamma_{\beta_0}^\lambda. \tag{39}$$

Here $\beta_0 = 1/k_\mathrm{B}T_0$ is the inverse temperature of some environment, in which the ensemble has initially thermalized, and $Z_{\beta_0} = z_{\beta_0}^n$, where $z_{\beta_0} = \mathrm{Tr}e^{-\beta_0 h}$ is the single-particle partition function. The block-occupation probabilities are now given by $p^\lambda = p_{\beta_0}^\lambda = m_\lambda Z_{\beta_0}^\lambda / Z_{\beta_0}$. The steady state thus becomes

$$\rho_\infty = \rho_{\beta,\beta_0} := \bigoplus_\lambda p_{\beta_0}^\lambda \frac{\mathbb{1}_{\mathcal{K}^\lambda}}{m_\lambda} \otimes \gamma_\beta^\lambda. \tag{40}$$

Finally, tracing out the permutation subspaces in Eqs. (39) and (40), respectively, gives the reduced Gibbs and steady states

$$\tilde{\gamma}_{\beta_0} = \bigoplus_\lambda p_{\beta_0}^\lambda \gamma_{\beta_0}^\lambda, \quad \tilde{\rho}_{\beta,\beta_0} = \bigoplus_\lambda p_{\beta_0}^\lambda \gamma_\beta^\lambda. \tag{41}$$

We note that, for $\beta = \beta_0$, the Gibbs state γ_{β_0} is a stationary solution of the master equation (33), which implies that the ensemble remains in thermal equilibrium, i.e., $\tilde{\rho}_{\beta,\beta} = \tilde{\gamma}_\beta$. However, for $\beta \neq \beta_0$ the properties of the steady state can deviate substantially from those of a thermal state, as we will show in Sec. IV.

B. Spin systems

As a reference for our results, we will consider ensembles of noninteracting spin-s particles, which have been analyzed in detail in Ref. [20]. The dynamical variables of a single spin-s system, $S_{x,y,z}^s$, form an irrep of the Lie algebra $su(2)$ on the Hilbert space $\mathbb{C}^{2s+1}$. The corresponding tensor-product representation on the ensemble Hilbert space $\mathcal{H} = (\mathbb{C}^{2s+1})^{\otimes n}$ is given by

$$S_{x,y,z}^{s(n)} = \sum_{k=0}^{n-1} \mathbb{1}_{2s+1}^{\otimes k} \otimes S_{x,y,z}^s \otimes \mathbb{1}_{2s+1}^{\otimes(n-1-k)}. \tag{42}$$

Hence, with the single-particle Hamiltonian $h = \hbar\Omega S_z^s$, the ensemble Hamiltonian is $H = \hbar\Omega D_1$ with $D_1 = S_z^{s(n)}$. Choosing the system and bath to couple instead via a collective spin,

$$H_I = \hbar\Delta\, S_x^{s(n)} \otimes B, \tag{43}$$

the Lamb shift and the jump operators in the master equation (33) are then given by $H_{LS} = \hbar\Delta(s_\Omega^{++}E_-E_+ + s_{-\Omega}^{--}E_+E_-)$ and $E_\pm = S_x^{s(n)} \pm iS_y^{s(n)}$. Note that, in contrast to general d-level systems, spin systems are characterized by a single Bohr frequency $\omega_\pm = \pm\Omega$.

For $s > 1/2$, the irrep $S_{x,y,z}^s$ is no longer the natural representation of $su(2)$. Therefore, the Schur-Weyl duality does not apply in this case. Nevertheless, the operators D_1 and $E_\pm$

still take the block-diagonal form (34) in the collective angular momentum basis $|J, m\rangle$, where J is the total angular momentum quantum number and m is the magnetic quantum number. However, the blocks corresponding to each J are subdivisions of those implied by Schur-Weyl duality. Consequently, the results of Sec. III A equally apply to spin-s particles, where the irrep index λ has to be replaced with J. The crucial difference is that we can no longer use the formulas (19) and (22) to calculate the dimensions and multiplicities of the individual irreps. Instead, the dimension of an irrep J is now given by $d_J = 2J + 1$, and the multiplicities have to be determined from the standard rules for the addition of angular momenta, which lead to the recursion relation

$$m_J(n+1) = \sum_{\substack{J':\,|J'-s|\leqslant J\leqslant J'+s,\\ J'+s-J\in\mathbb{Z}}} m_{J'}(n), \tag{44}$$

where n is the number of particles. The theorem in Appendix A also holds for spin systems, giving rise to steady states of the same form as Eq. (37) but with λ replaced by J. This also settles a conjecture from Ref. [20] that the off-diagonal blocks vanish in the steady state.

IV. THERMODYNAMICAL QUANTITIES

In this section we show how the steady-state energy, reduced entropy, and reduced nonequilibrium free energy can be calculated for the setup laid out in the last section. To illustrate our results, we compare an ensemble of spin-1 particles with an ensemble of three-level particles with SU(3) symmetry. We further derive explicit expressions for our thermodynamical quantities of interest in certain limiting cases as functions of s and d for spin-s particles and general d-level particles, respectively.

A. General expressions

For a general state ρ, and some fixed inverse temperature β, the steady-state energy, reduced entropy, and reduced nonequilibrium free energy are defined according to the block structure (36) as

$$E(\rho) = \mathrm{Tr}\rho H = \sum_\lambda p^\lambda \mathrm{Tr}\rho^\lambda H^\lambda = \sum_\lambda p^\lambda E(\rho^\lambda), \tag{45}$$

$$\tilde{S}(\rho) = \sum_\lambda p^\lambda S(\rho^\lambda), \tag{46}$$

$$\tilde{F}(\rho) = E(\rho) - \tilde{S}(\rho)/\beta = \sum_\lambda p^\lambda [E(\rho^\lambda) - S(\rho^\lambda)/\beta], \tag{47}$$

where $S(\rho) = -\mathrm{Tr}\rho\ln(\rho)$ denotes the von Neumann entropy. The reduced quantities $\tilde{S}(\rho)$ and $\tilde{F}(\rho)$ have been constructed so that they do not contain entropic contributions from the degeneracy spaces $\mathcal{K}^\lambda$. This approach is motivated by the assumption that all degrees of freedom that interact with the experimenter's apparatus or the environment are represented by permutation-invariant observables, which do not give access to any information stored in the subspaces $\mathcal{K}^\lambda$. As a result, the quantities $\tilde{S}(\rho)$ and $\tilde{F}(\rho)$ can, like $E(\rho)$, be expressed as averages over their respective counterparts on the individual irreps λ with respect to the probability distribution p^λ. Note that, for steady states of the form $\bigoplus_\lambda p^\lambda \sigma^\lambda \otimes \rho^\lambda$, we can write the reduced entropy as a difference $\tilde{S}(\rho) = S(\rho) - S(\check{\rho})$, where $\check{\rho} = \bigoplus_\lambda p^\lambda \sigma^\lambda$ is obtained from ρ by tracing out the unitary subspaces.

For the steady state (40), the thermodynamical quantities (45)–(47) become

$$E(\rho_{\beta,\beta_0}) = E_{\beta,\beta_0} = -\sum_\lambda p^\lambda_{\beta_0} \partial_\beta \ln(Z^\lambda_\beta), \tag{48}$$

$$\tilde{S}(\rho_{\beta,\beta_0}) = \tilde{S}_{\beta,\beta_0} = -\sum_\lambda p^\lambda_{\beta_0} [\beta\partial_\beta \ln(Z^\lambda_\beta) - \ln(Z^\lambda_\beta)], \tag{49}$$

$$\tilde{F}(\rho_{\beta,\beta_0}) = \tilde{F}_{\beta,\beta_0} = -\frac{1}{\beta}\sum_\lambda p^\lambda_{\beta_0} \ln\left(Z^\lambda_\beta\right) \tag{50}$$

with $p^\lambda_{\beta_0} = m_\lambda Z^\lambda_{\beta_0}/Z_{\beta_0}$. To evaluate these expressions, we have to calculate the partial partition functions Z^λ_β, the total partition function Z_β, and the multiplicities of the individual irreps m_λ.

To this end, we first observe that Z^λ_β can be written as

$$Z^\lambda_\beta = \mathrm{Tr}e^{-\beta\sum_i a_i D^\lambda_i} = \mathrm{Tr}e^{i\sum_i \alpha_i D^\lambda_i}, \tag{51}$$

where we have replaced a_i by $-i\alpha_i/\beta_0$ in the second expression. Since the D^λ_i are diagonal generators of an irrep of SU(d), this quantity can be calculated by means of analytic continuation of Weyl's character formula, which we have discussed in Sec. II E. For $d = 3$, we find [50]

$$Z^\lambda_\beta = e^{\beta a_2(2x_1+x_2)/3} \times \sum_{k=0}^{x_1}\sum_{l=0}^{x_2} e^{-\beta a_2(k+l)} \frac{\sinh[\beta a_1(k-l+x_2+1)/2]}{\sinh(\beta a_1/2)}, \tag{52}$$

where $x_j = \lambda_j - \lambda_{j+1}$ and λ denotes the ordered partition $(\lambda_1, \lambda_2, \lambda_3)$. Analogous formulas can in principle be derived for any larger d. The single-particle partition function z_β is obtained as a special case for $\lambda = (1, 0, \ldots, 0)$. The total partition function is then given by $Z_\beta = (z_\beta)^n$. For an ensembles of spin-s systems the partial partition function over the irrep J is given by

$$Z^J_\beta = \frac{\sinh[\beta(2J+1)/2]}{\sinh(\beta/2)}, \tag{53}$$

as can be easily verified by direct computation.

If the individual particles are described in terms of the natural representation of SU(d), the multiplicities m_λ can be obtained from the combinatorial formula (21), which yields

$$m_\lambda = \frac{n!(\tilde{\lambda}_1 - \tilde{\lambda}_2)(\tilde{\lambda}_1 - \tilde{\lambda}_3)(\tilde{\lambda}_2 - \tilde{\lambda}_3)}{\tilde{\lambda}_1!\tilde{\lambda}_2!\tilde{\lambda}_3!} \tag{54}$$

for $d = 3$ with $\tilde{\lambda}_j$ being defined in Eq. (19). For spin systems with $s > 1/2$, the multiplicities can be obtained from the recursion relation (44). With these prerequisites, the quantities (48)–(50) are computationally accessible.

B. SU(2) vs SU(3)

As a first application of our theory, we investigate how the transition from SU(2) to SU(3) changes the thermodynamical properties of the steady state. For convenience, we now set Boltzmann's constant to 1 and rescale all energies

and temperatures with $\hbar\Omega$ so that β, β_0 and h, H, H^λ become dimensionless from here onward.

As the arguably simplest setting in which SU(2) and SU(3) can be compared, we consider ensembles of three-level systems with excited, intermediate, and ground states $|+\rangle$, $|0\rangle$, and $|-\rangle$. To isolate the effect of the symmetry group, we pick the single-particle Hamiltonian

$$h = |+\rangle\langle+| - |-\rangle\langle-|. \tag{55}$$

In an SU(2) description, this Hamiltonian corresponds to a spin-1 system with $h = d_1 = S_z^1$, where S_z^1 was defined in Eq. (12). In terms of SU(3), the Hamiltonian (55) describes a ladder system with $h = (d_1 + \sqrt{3}d_2)/2 = (\Lambda_3 + \sqrt{3}\Lambda_8)/2$, where we have identified the diagonal generators d_1, d_2 with the Gell-Mann matrices from Eq. (7).

Although the single-particle Hamiltonian is identical in both cases, the steady state (41) is not the same for SU(2) and SU(3), since different system-bath couplings [Eqs. (43) and (31)] lead to different dissipation mechanisms. For SU(2), the ensemble relaxes via a single dissipation channel described by the Lindblad operators $E_\pm$, which represent the nondiagonal generators

$$e_+ = e_-^\dagger = \frac{1}{\sqrt{2}}(|+\rangle\langle 0| + |0\rangle\langle -|). \tag{56}$$

By contrast, for SU(3), relaxation to the steady state occurs via three dissipation channels, whose Lindblad operators $E_{1\pm}$, $E_{2\pm}$, and $E_{3\pm}$ represent the generators

$$e_{1+} = e_{1-}^\dagger = |+\rangle\langle 0|, \quad e_{2+} = e_{2-}^\dagger = |+\rangle\langle -|, \tag{57}$$
$$e_{3+} = e_{3-}^\dagger = |0\rangle\langle -|.$$

Hence, in the SU(2) case the Lindblad operators induce coherent superpositions of jumps between intermediate and excited and ground and intermediate states, while for SU(3) all possible transitions in the single-particle system are addressed separately by the bath.

This difference changes the structure of the steady state and thus its thermodynamical properties. In Fig. 2 we plot the internal energy E_{β,β_0} and changes in reduced entropy and reduced free energy during the relaxation process, which are given by

$$\Delta\tilde{S}_{\beta,\beta_0} = \tilde{S}_{\beta,\beta_0} - \tilde{S}_{\beta_0,\beta_0},$$
$$\Delta\tilde{F}_{\beta,\beta_0} = \tilde{F}_{\beta,\beta_0} - \tilde{F}_{\beta_0,\beta_0}. \tag{58}$$

From these plots, we immediately note two features. First, as expected, all three quantities deviate from their thermal equilibrium references in the steady state (40). Second, although we use the same single-particle Hamiltonian for both symmetry groups, these deviations are typically stronger for SU(2) than for SU(3). This result suggest that the three-channel dissipation mechanism, which applies to systems with SU(3) symmetry, allows the system to come closer to thermal equilibrium than the single-channel mechanism applying to spin systems.

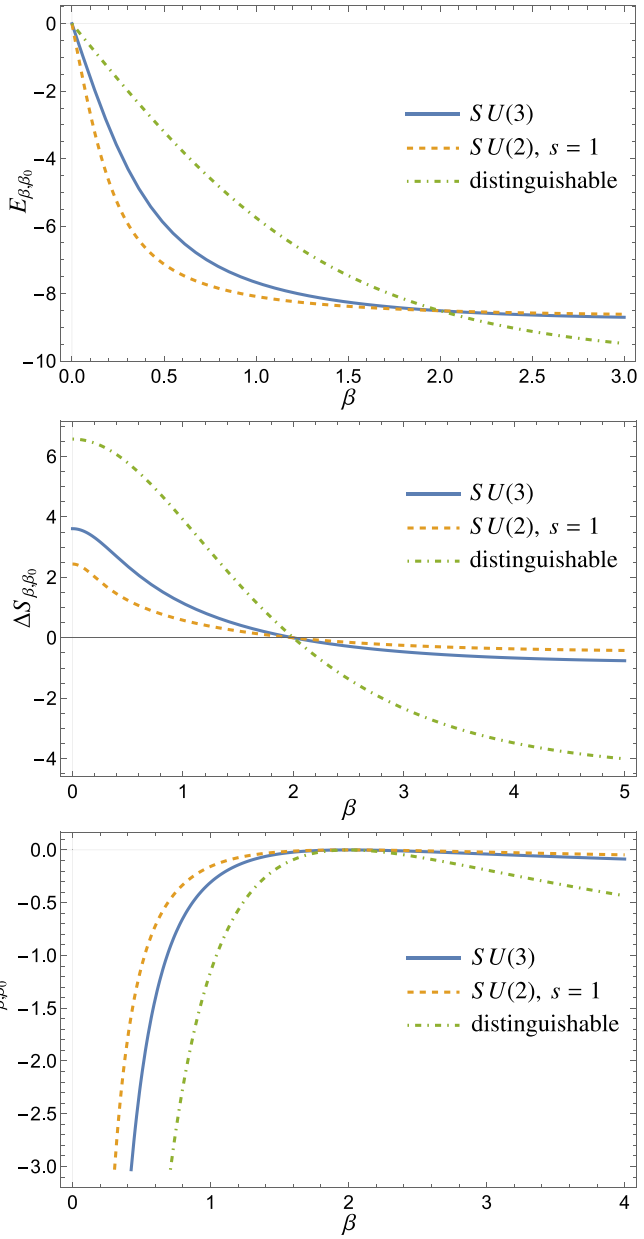

FIG. 2. Steady-state energy (48), the reduced entropy change (49), and the change in the reduced free energy (58) (top to bottom) as functions of the inverse bath temperature β for ensembles of three-level systems with SU(3) and SU(2) symmetry, starting from the thermal state (39) at β_0. For comparison, the dot-dashed lines labeled "distinguishable" show the fully thermal state energy $E_{\beta,\beta}$ (top) and the changes in entropy (middle) and free energy (bottom) during a full equilibration process of distinguishable particles from β_0 to β. For all plots we have chosen the single-particle Hamiltonian (55) and set $n = 10$ and $\beta_0 = 2$. Note that, for $\beta = \beta_0$, the initial Gibbs state γ_{β_0} is a stationary state of the master equation (33). Therefore, all curves intersect at $\beta = 2$, and all entropy and free energy differences are 0 at $\beta = 2$ in the lower two plots.

C. Limiting cases

We now return to the general case of ensembles with SU(d) symmetry. In the following, we derive explicit expressions for the thermodynamical quantities (48)–(50) as functions of d in special limits to show how our theory can be applied in practice. For comparison, we also analyze spin-s ensembles.

1. Low initial and high bath temperature

In the limit $\beta_0 \gg 1$, the ensemble will almost exclusively occupy the trivial symmetric subspace, which is spanned by permutation-invariant many-particle states. To understand this effect, we consider the probability $p^g_{\beta_0} = \langle g^n|\gamma_{\beta_0}|g^n\rangle$ of finding the ensemble in its nondegenerate ground state $|g^n\rangle = \otimes_{k=1}^n |g\rangle^{(k)}$, where $|g\rangle^{(k)}$ is the ground state of the particle k. Since the state $|g^n\rangle$ belongs to the symmetric subspace, the probability of finding the system in this subspace is subject to the bounds $1 \geqslant p^{\mathrm{sym}}_{\beta_0} \geqslant p^g_{\beta_0}$. Taking the low-temperature limit $\beta_0 \to \infty$ gives $p^g_{\beta_0} = 1$ and thus $p^{\mathrm{sym}}_{\beta_0} = 1$. Hence, by continuity, we have $p^{\mathrm{sym}}_{\beta_0} \simeq 1$ for $\beta_0 \gg 1$. Since the block structure (39) of the state γ_{β_0} is conserved by the master equation (33), so is the probability $p^{\mathrm{sym}}_{\beta_0}$. Consequently, the steady state of the ensemble is also nearly confined to the symmetric subspace, and the thermodynamical quantities (48)–(50) are dominated by contributions from the symmetric irrep $\lambda_{\mathrm{sym}} = (n, 0, \ldots, 0)$. This observation makes it possible to derive asymptotically exact expressions for these quantities in the limit $\beta \ll 1$. That is, we consider a situation where the system is initially thermalized in a low-temperature environment and then brought into contact with a high-temperature bath.

For an SU(d) ensemble with single-particle Hamiltonian $h = \mathrm{diag}[\varepsilon_1, \ldots, \varepsilon_d]$, the partial partition function Z^{sym}_β of the symmetric subspaces is given by

$$Z^{\mathrm{sym}}_\beta = d_{\mathrm{sym}}\left(1 + \beta^2 \frac{n(n+d)}{2(d+1)}\langle\langle\varepsilon^2\rangle\rangle\right) + O(\beta^3) \tag{59}$$

as we prove in Appendix B. Here

$$d_{\mathrm{sym}} = \binom{n+d-1}{n} \tag{60}$$

is the dimension of the symmetric subspace, which can be obtained from Eq. (19), and

$$\langle\langle\varepsilon^2\rangle\rangle = \frac{1}{d}\sum_{i=1}^{d}\varepsilon_i^2 \tag{61}$$

is the variance of the single-particle energy at infinite temperature. Note that we have $\langle\varepsilon\rangle = \frac{1}{d}\sum_{i=1}^d \varepsilon_i = 0$, since the single-particle Hamiltonian is traceless by assumption. Using the expansions (59) and (66), we find that the quantities (48)–(50) become

$$E_{\beta,\infty} = -2\beta q^d_n + O(\beta^2), \tag{62}$$

$$\tilde{S}_{\beta,\infty} = \ln(d_{\mathrm{sym}}) - \beta^2 q^d_n + O(\beta^3), \tag{63}$$

$$\tilde{F}_{\beta,\infty} = -\ln(d_{\mathrm{sym}})/\beta - \beta q^d_n + O(\beta^2) \tag{64}$$

in the limit $\beta_0 \to \infty$ with

$$q^d_n = \frac{n(n+d)\langle\langle\varepsilon^2\rangle\rangle}{2(d+1)}. \tag{65}$$

We now consider an ensemble of spin-s particles. By a similar argument to above, since the ground state belongs to the subspace of maximal angular momentum $J = ns$, the thermal state of such an ensemble will almost exclusively occupy this subspace in the limit $\beta_0 \gg 1$. The dimension of this subspace is given by $d_{\max} = 2ns + 1$, which is strictly smaller than the dimension of the symmetric subspace for $s > 1/2$ and $n > 1$. With the single-particle Hamiltonian $h = S^s_z$, the corresponding partial partition function becomes [20]

$$Z^{ns}_\beta = (1+2ns)\left(1 + \beta^2 \frac{ns(1+ns)}{6}\right) + O(\beta^3) \tag{66}$$

as can be seen by expanding Eq. (53). The asymptotic expressions for the internal energy, reduced entropy, and reduced free energy can now be obtained by replacing d_{sym} with $d_{\max}$ and q^d_n with

$$Q^s_n = \frac{ns(1+ns)}{6} = \frac{n(d-1)[2+n(d-1)]}{24} \tag{67}$$

in Eqs. (62)–(64). Here we have expressed $s = (d-1)/2$ in terms of the dimension d of the single-particle Hilbert space to facilitate the comparison between SU(d) and spin-s ensembles.

The expressions (65) and (67) show that, for the SU(d) ensemble as well as for the spin-s ensemble, the leading-order corrections in β to the internal energy, reduced entropy, and reduced free energy of the steady state all feature a quadratic dependence on n for all d. For a quantitative comparison, we choose a ladder Hamiltonian with eigenvalues $\varepsilon_i = -(d+1)/2 + i$ for the SU(d) ensemble, which yields $\langle\langle\varepsilon^2\rangle\rangle = (d-1)(d+1)/12$. We then have

$$q^d_n \sim n^2(d-1)/24 \quad \text{and} \quad Q^s_n \sim n^2(d-1)^2/24 \tag{68}$$

for $n \gg 1$. Hence, in the limit of many particles, the magnitudes of the first-order corrections in Eqs. (62)–(64) are suppressed by a factor $(d-1)$ in SU(d) ensemble compared to the spin-s ensemble. We stress that this effect arises solely from the different dimensions of the primarily occupied subspaces, which have multiplicity 1 in both cases.

2. High initial temperature and many particles

A second interesting limit is realized when the ensemble is initially prepared in a high-temperature state, that is, $\beta_0 \to 0$. In the following, we calculate the steady-state energy and reduced entropy in this limit, respectively, for thermalization with a low- and a high-temperature bath. To this end, we first observe that the probability distribution $p^\lambda_{\beta_0}$ tends to

$$p^\lambda_{\beta_0} \to \frac{m_\lambda Z^\lambda_{\beta_0}}{Z_{\beta_0}} = \frac{m_\lambda d_\lambda}{d^n} \tag{69}$$

for $\beta_0 \to 0$. This quantity is known as a *Plancherel-type measure* [53]. In the limit of many particles, this measure can be determined explicitly through methods of asymptotic representation theory. Specifically, upon changing variables from λ_i to $\zeta_i = (\lambda_i - n/d)/\sqrt{n}$, the limiting measure for $n \to \infty$ becomes the function [54,55]

$$\phi_d(\zeta) = \frac{d^{\frac{d(d-1)+1}{2}}\left(\frac{d}{2\pi}\right)^{\frac{d-1}{2}}}{1!2!\cdots(d-1)!}\prod_{i<j}(\zeta_i - \zeta_j)^2 e^{-\frac{d}{2}\sum_k \zeta_k^2}. \tag{70}$$

One interesting question is how far the mean energy of an initially hot ensemble can be reduced by thermalizing with a cold bath. When the bath temperature is low, i.e., if $\beta \to \infty$, the partial thermal state γ^λ_β becomes a projector on the ground state within the unitary subspace $\mathcal{H}^\lambda$. As we show

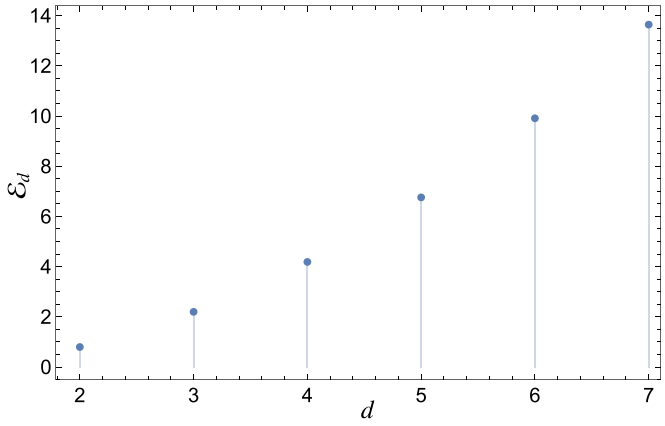

FIG. 3. Limiting energy coefficient $\mathcal{E}_d$ as defined in Eq. (72) as a function of d.

in Appendix C, each subspace has a unique ground state with energy $\sum_{i=1}^{d} \lambda_i \varepsilon_i$, where ε_i are the single-particle energies ordered increasingly. Here we assume that ε_1 is nondegenerate. Putting this observation together with the limiting distribution (70), we obtain the mean energy

$$E_{\infty,0} \to \int d\zeta \; \phi_d(\zeta) \left[\sum_{i=1}^{d} \lambda(\zeta)_i \varepsilon_i \right] \tag{71}$$

for $n \to \infty$. The integral in this expression runs over all $\zeta_1, \ldots, \zeta_d \in \mathbb{R}$ subject to the constraints $\zeta_1 \geqslant \zeta_2 \geqslant \cdots \geqslant \zeta_d$ and $\sum_{i=1}^{d} \zeta_i = 0$.

If we choose a ladder Hamiltonian with energy levels $\varepsilon_i = -(d+1)/2 + i$ for the single-particle system, this mean energy takes the form

$$E_{\infty,0} \to -\mathcal{E}_d \sqrt{n} \tag{72}$$

for $n \to \infty$. Notably, this result shows that the steady-state energy of the ensemble is subextensive in the particle number n. Hence, thermalization is strongly inhibited by the symmetry constraints of the system; if the system were to instead fully thermalize to its ground state, its steady-state mean energy would be trivially given by $E_{\infty,\infty} = n\varepsilon_1 = -n(d-1)/2$.

For $d = 2$ and $d = 3$, the constant $\mathcal{E}_d$ in Eq. (72) takes the values $\mathcal{E}_2 = \sqrt{2/\pi}$ and $\mathcal{E}_3 = 9\sqrt{3/16\pi}$; see Appendix D for details. For larger dimensions, $\mathcal{E}_d$ must be determined numerically by solving the integral in Eq. (71). These results are plotted in Fig. 3, which shows a nonlinear increase of $\mathcal{E}_d$ with d for $d \leqslant 7$. For completeness, we may again compare the SU(3) ensemble with a spin-1 ensemble, for which we find $E_{\infty,0} \to -\sqrt{16/3\pi}\sqrt{n}$; see Appendix E. Thus, low-temperature thermalization through the three-channel mechanism of the SU(3) case leads to a lower steady-state energy than single-channel mechanism of the spin-1 case by a factor of $27/16 \approx 1.69$. In line with our earlier results, this observation quite naturally suggests that a larger number of dissipation channels allows the system to relax closer to a thermal state. Note, however, that the scaling of the steady-state energy with n is insensitive to the dissipation mechanism.

We now assume that the bath temperature is also high, i.e., $\beta \to 0$. The ensemble then remains in a high-temperature equilibrium state, whose reduced entropy is given by

$$\tilde{S}_{0,0} = \sum_{\lambda} p_0^{\lambda} \ln(d_\lambda). \tag{73}$$

By using the distribution (70) for the many-particle limit, we prove in Appendix D that

$$\tilde{S}_{0,0} \to \int d\zeta \; \phi_d(\zeta) \, \ln(d_{\lambda(\zeta)}) \simeq \frac{d(d-1)}{4} \ln(n) \tag{74}$$

to leading order in n for $n \to \infty$. Thus, for $d = 3$, we find $\tilde{S}_{0,0} \to (3/2)\ln(n)$. In comparison, for a spin-1 ensemble, the reduced entropy is suppressed by a factor 3, that is, we have $\tilde{S}_{0,0} \to (1/2)\ln(n)$ to leading order in n. More generally, for a spin-s ensemble, it is clear that $\tilde{S}_{0,0}$ is upper bounded by the contribution $S(\gamma_\beta^{ns}) = \ln(2ns+1) \simeq \ln(n)$ from the maximal angular momentum subspace corresponding to $J = ns$; cf. Eq. (49). It follows that, compared to spin ensembles, the reduced entropy $\tilde{S}_{0,0}$ of SU(d) ensembles is asymptotically enhanced by a factor that is at least quadratic in the dimension of the single-particle Hilbert space.

V. ENGINE CYCLES

So far we have analyzed how steady-state thermodynamical quantities are affected by the symmetries of permutation-invariant ensembles of n indistinguishable and noninteracting particles. As a next step, we explore in this section what role these symmetries play in thermodynamical processes. To this end, we consider a thermodynamical engine cycle that uses a collective working medium and investigate how the properties of this medium affect its performance.

A. Protocol and output

For simplicity, we focus on the standard quantum Otto cycle [56,57]. We begin by stating the 0+4 strokes that the system undergoes during this cycle. In the zeroth stroke, the ensemble is initialized in the Gibbs state at the inverse temperature β_0, which determines the occupation probabilities $p_{\beta_0}^{\lambda}$ of the individual irreps; cf. Eq. (39). The ensemble then cyclically undergoes the four strokes of the Otto cycle between a hot and cold bath at inverse temperatures β_h and β_c:

(1) Equilibration with the hot bath
(2) Instantaneous change of the Hamiltonian $H \to H'$
(3) Equilibration with the cold bath
(4) Instantaneous change of the Hamiltonian $H' \to H$.

The net extracted work per cycle W, which is our main quantity of interest, is given by the sum of the work contributions from the two instantaneous strokes,

$$W = -\mathrm{Tr}[\rho_{\beta_\mathrm{h},\beta_0}(H' - H)] - \mathrm{Tr}[\rho'_{\beta_\mathrm{c},\beta_0}(H - H')]. \tag{75}$$

Here $\rho_{\beta_\mathrm{h},\beta_0}$ and $\rho'_{\beta_\mathrm{c},\beta_0}$ are the steady states of the system with Hamiltonian H and H', respectively; see Eq. (40). We now assume that the instantaneous strokes are simple compression, that is, $H' = \kappa H$, where the compression factor κ must obey the inequalities $\beta_\mathrm{h}/\beta_\mathrm{c} \leqslant \kappa \leqslant 1$ to ensure that a positive

amount of work is generated. Under these conditions, the net work extraction with a collective medium becomes

$$W^{\rm col} = (1-\kappa){\rm Tr}H(\rho_{\beta_{\rm h},\beta_0} - \rho_{\kappa\beta_{\rm c},\beta_0}) = (1-\kappa)(E_{\beta_{\rm h},\beta_0} - E_{\kappa\beta_{\rm c},\beta_0}) \quad (76)$$

with the internal energies given by Eq. (48). To uncover the role of collective effects, we compare this quantity with the work generated in the same cycle with an ensemble of distinguishable particles, which fully thermalizes during the isochoric strokes,

$$W^{\rm dis} = (1-\kappa){\rm Tr}H(\gamma_{\beta_{\rm h}} - \gamma_{\kappa\beta_{\rm c}}) = (1-\kappa)(E_{\beta_{\rm h},\beta_{\rm h}} - E_{\kappa\beta_{\rm c},\kappa\beta_{\rm c}}). \quad (77)$$

B. Collective work enhancement

Two natural questions arise at this point. First, is it possible to extract more work from the collective medium than from the distinguishable one, that is, can the ratio $W^{\rm col}/W^{\rm dis}$ become larger than 1? Second, is there an advantage in moving from spin ensembles to ensembles with higher-order symmetry?

Both of these questions can be answered with the help of Fig. 4, where we plot the ratio $W^{\rm col}/W^{\rm dis}$ against $\beta_{\rm h}$ for SU(2)- and SU(3)-symmetric ensembles. To enable a quantitative comparison, we have chosen the same single-particle Hamiltonian $h = {\rm diag}[1, 0, -1]$ for both cases. We find that, first, there is indeed a temperature range for which $W^{\rm col}/W^{\rm dis} > 1$, meaning that more work can be extracted with the collective medium than with the distinguishable one; a similar result was found in Ref. [20] for spin ensembles. Second, Fig. 4(b) shows a regime where $W^{\rm col}/W^{\rm dis} > 1$, while the extracted work is larger for the SU(3)- than for the spin ensemble. An advantage of SU(3) over SU(2) is also seen in Fig. 4(c), where $\kappa\beta_{\rm c} \simeq \beta_0$. Here, however, $W^{\rm col}/W^{\rm dis} < 1$, so the higher-order symmetry group rather mitigates the disadvantageous collective effects.

The first observation can be understood in the limiting regime where the initial temperature is low and both bath temperatures are high, that is, $\beta_0 \gg 1$ and $\beta_{\rm h}, \kappa\beta_{\rm c} \ll 1$. Under these conditions, we can use the asymptotic result (62) for the steady-state internal energy of an SU(d) ensemble to evaluate the work extraction from the collective medium. Using the formulas (76) and (77), we find

$$\frac{W^{\rm col}}{W^{\rm dis}} \simeq \frac{n+d}{d+1}. \quad (78)$$

This result shows that the work advantage grows monotonically in the number of particles n, but decreases monotonically with the order of the symmetry group d for $n > 1$. For the special case $n = 10$ and $d = 3$, we recover the limiting value $W^{\rm col}/W^{\rm dis} \simeq 3.25$ seen in Fig 4(a). For comparison, the analogous result for spin-s ensembles derived in Ref. [20] is

$$\frac{W^{\rm col}}{W^{\rm dis}} \simeq \frac{ns+1}{s+1} = \frac{n(d-1)+2}{d+1}. \quad (79)$$

C. Optimization

Higher-dimensional symmetry groups offer larger freedom in modeling the single-particle Hamiltonian as additional

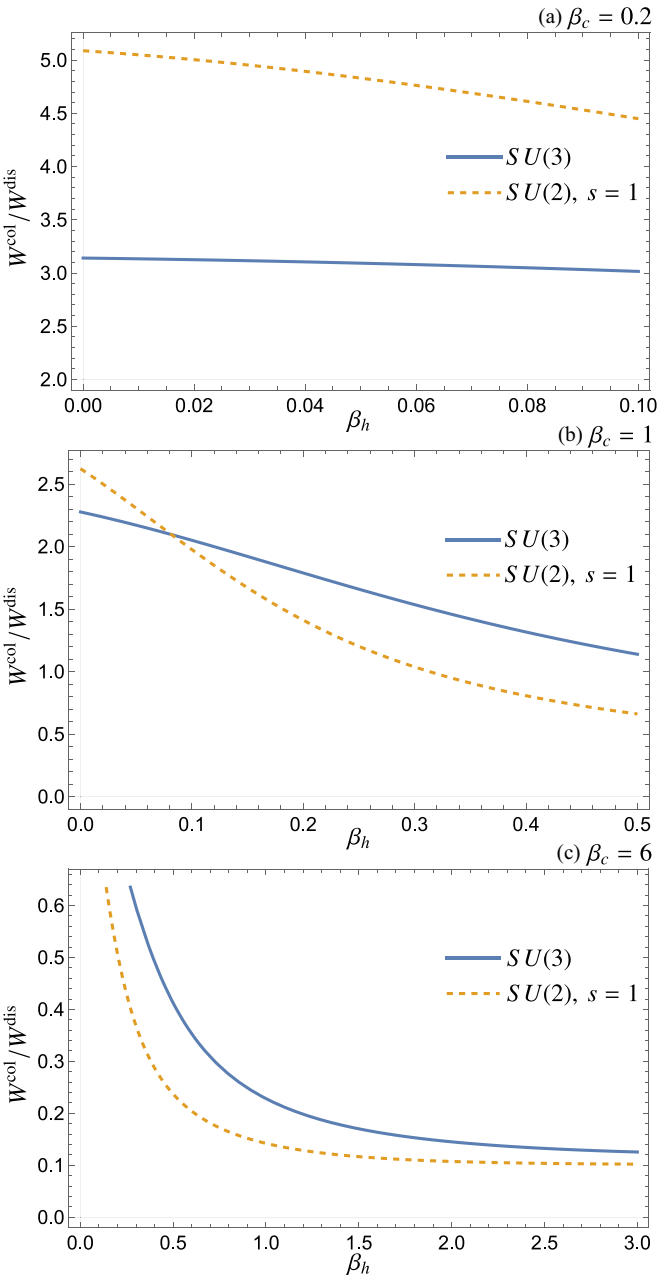

FIG. 4. Work advantage $W^{\rm col}/W^{\rm dis}$ as a function of $\beta_{\rm h}$ for different values of $\beta_{\rm c}$. Dashed lines correspond to spin-1 ensembles and solid lines to SU(3) ensembles. For all plots we have set $n = 10$. Bath temperatures adhere to the range $\beta_{\rm h}/\beta_{\rm c} \leqslant \kappa \leqslant 1$, where the compression factor is $\kappa = 1/2$ and the initial inverse temperature of the ensembles is $\beta_0 = 5$.

diagonal generators become available in the corresponding Lie algebra; cf. Eq. (29). We will now explore how this freedom can be exploited to optimize the work output of a collective quantum Otto cycle. To this end, we let the single-particle energies ε_i vary within a bounded range, which we choose as $\varepsilon_{\rm max} - \varepsilon_{\rm min} = d - 1$ to match a ladder Hamiltonian with unit spacing.

By using the expansion (62), we can find the optimal spectrum in the limit of low initial temperatures, $\beta_0 \to \infty$, and high bath temperatures, $\beta_{\rm h}, \kappa\beta_{\rm c} \ll 1$. As shown in

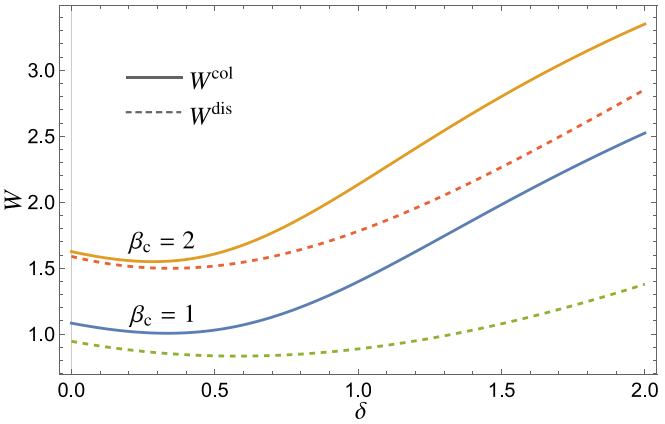

FIG. 5. Work output of a quantum Otto cycle using a working medium with single-particle Hamiltonian (82) as a function of the control parameter δ. Solid lines show the SU(3) collective ensemble, and dashed lines correspond to the distinguishable ensemble, which fully thermalizes in the equilibration strokes. We have set $n = 7$, $\beta_0 = 3$, $\beta_h = 0.1$, $\kappa = 1/2$ and two different values $\beta_c = 1,\ 2$.

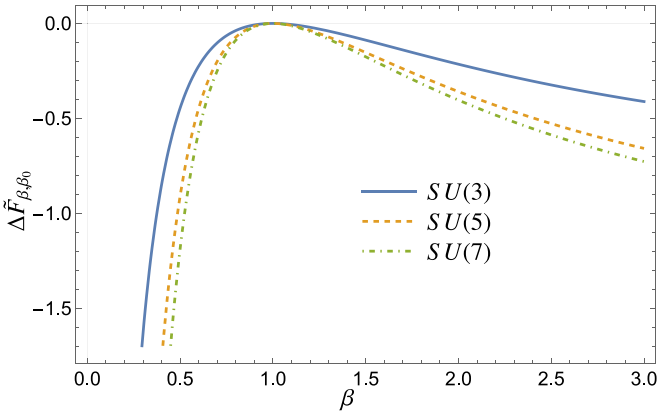

FIG. 6. Free energy change (58) as a function of the inverse bath temperature β for an ensemble of five particles and initial temperature $\beta_0 = 1$ for SU(d) ensembles with $d = 3, 5, 7$.

Appendix B, for even d, the optimal spectrum has two levels, which are $d/2$-fold degenerate with the maximal gap of $d-1$; that is, we have $\varepsilon_1, \ldots \varepsilon_{d/2} = -(d-1)/2$ and $\varepsilon_{d/2+1}, \ldots, \varepsilon_d = (d-1)/2$. For this spectrum, the optimal work output of a collective cycle and the work output generated by an ensemble of distinguishable particles are

$$W^{\mathrm{col}*} \simeq (1-\kappa)(\kappa\beta_c - \beta_h)\frac{n(n+d)(d-1)^2}{4(d+1)}, \tag{80}$$

$$W^{\mathrm{dis}*} \simeq \frac{d+1}{n+d} W^{\mathrm{col}*}. \tag{81}$$

Hence, the work-advantage ratio is still given by Eq. (78). For odd d, there is a small correction, since the remaining energy level can be either at 0 or $d-1$.

For arbitrary temperatures, we approach the optimization problem numerically for SU(3) ensembles. Normalizing the difference between ground and excited single-particle energies to 2, we can parameterize the single-particle Hamiltonian by the gap δ between the ground and first excited levels. That is, we make the ansatz

$$h_\delta = \frac{2}{3}\mathrm{diag}\left[2 - \frac{\delta}{2}, \delta - 1, -1 - \frac{\delta}{2}\right] \tag{82}$$

$$= 2(2-\delta)d_1 + \frac{2+\delta}{2\sqrt{3}}d_2, \qquad \delta \in [0, 2].$$

Here we have identified the diagonal generators with the Gell-Mann matrices (7), i.e., $d_1 = \Lambda_3$ and $d_2 = \Lambda_8$. As shown in Fig. 5, the maximum work output is attained with a doubly degenerate excited level, corresponding to $\delta = 2$. Notably, the collective medium outperforms the distinguishable one for any value of δ.

Our results show that the larger freedom in the choice of the single-particle Hamiltonian, which comes with higher-order symmetry groups, can indeed be used to optimize the performance of many-body Otto cycles, in addition to the boost coming from a collective dissipation mechanism. This can be done, at least in principle, even outside the regime of extreme initial and bath temperatures. Furthermore, our analysis seems to suggest that the optimal single-particle Hamiltonian for collective quantum Otto cycles should feature two levels with approximately balanced degeneracy. Corroborating this presumption with a more systematic investigation is beyond the scope of our present work, but provides an attractive problem for future research.

VI. HIGHER-ORDER SYMMETRIES: A CLOSER LOOK

As we have seen in the previous sections, the general tools developed in this paper can be used to calculate thermodynamical quantities for noninteracting spin-s ensembles and ensembles with arbitrary SU(d) symmetry. Beyond these applications, our framework also makes it possible to analyze the anatomy of the steady states of such ensembles on the level of individual irreps.

A. Free energy and dimensions

To illustrate this approach, which can provide further insights into the role of higher-order symmetries, we first consider the change in the reduced nonequilibrium free energy as defined in Eq. (58). For a quantitative comparison between the different symmetry groups, we choose the coefficients a_i in the single-particle Hamiltonian $h = \sum_i a_i d_i$ to reproduce the ladder spectrum $\varepsilon_i = -(d+1)/2 + i$. With this choice we first plot the nonequilibrium free energy change as a function of the inverse bath temperature β in Fig. 6. We see that, for all bath temperatures, $|\Delta\tilde{F}|$ increases with the order d of the applied symmetry group.

To understand this effect better, we now focus on the role of the individual irreps. In Fig. 7 we decompose $\Delta\tilde{F}_{\beta,\beta_0} = \sum_\lambda p^\lambda_{\beta_0} \Delta\tilde{F}^\lambda_{\beta,\beta_0}$ into contributions from the individual irreps λ; the size of each dot in these plots indicates the relative probability of occupying a particular irrep. For spin ensembles, the maximal angular momentum irrep $J = ns$, which has the largest dimension $d_{\max} = 2ns + 1$, is the predominantly occupied irrep in the limit $\beta_0 \to \infty$. For SU(d), the symmetric irrep, which does not generally have the largest dimension, is most occupied for large β_0. Thus, for spin ensembles, the dominant contribution to the free energy change must shift

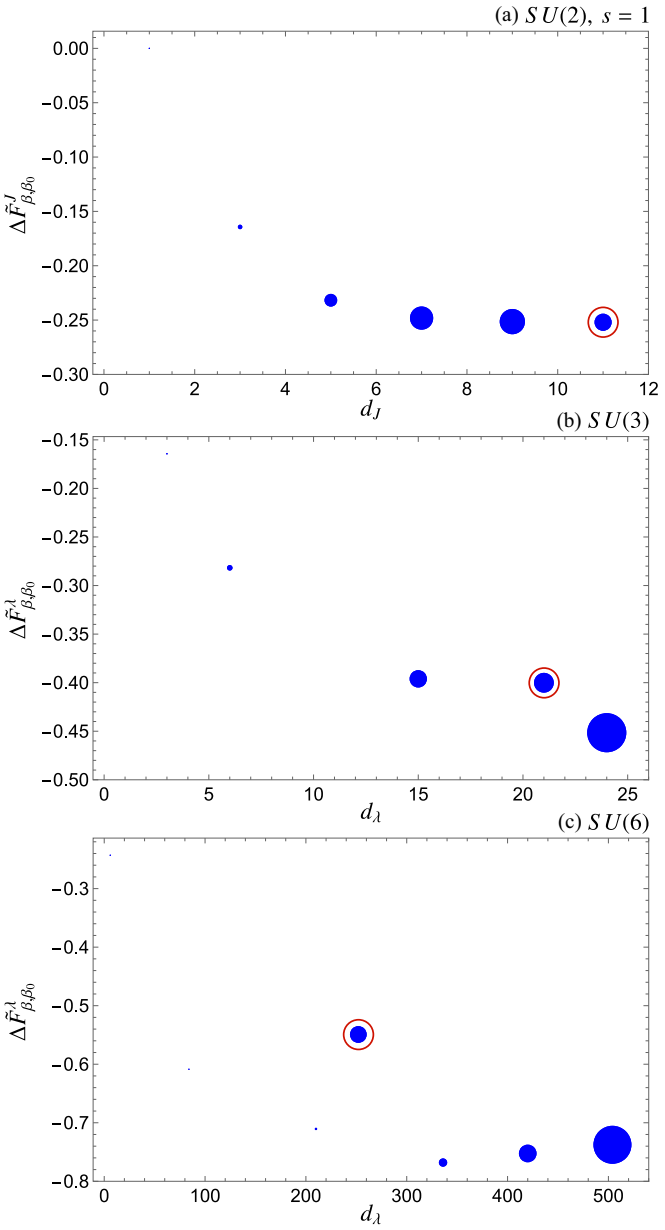

FIG. 7. Contributions to the free energy change $\Delta\tilde{F}^{\lambda}_{\beta,\beta_0}$ for each irrep λ, such that $\Delta\tilde{F}_{\beta,\beta_0} = \sum_\lambda p^{\lambda}_{\beta_0}\Delta\tilde{F}^{\lambda}_{\beta,\beta_0}$, as functions of the irrep dimension d_λ. The radius of each dot is scaled in proportion with the corresponding occupation probability $p^{\lambda}_{\beta_0}$. In the first plot, which corresponds to an SU(2) ensemble with spin $s = 1$, the maximal J irrep is circled. In the middle and bottom plots, which correspond to SU(3) and SU(6) ensembles, the circle indicates the symmetric irrep. The system parameters are $n = 5$, $\beta_0 = 1$, $\beta = 3$.

to a lower-dimensional irrep as β_0 decreases to a moderate value; see Fig. 7. By contrast, for SU(d) ensembles with $d > 2$, one observes a shift towards a higher-dimensional irrep.

B. Degeneracies

An explanation for the link between thermodynamical properties and irrep dimension can be given in terms of energy level degeneracies. Here one finds a notable difference between SU(2) and higher-order groups. SU(2) irreps are special in having nondegenerate energy levels—that is, for each J, the energy basis $|J, M\rangle$ varying over M is nondegenerate. For SU(3) and above, we observe two sources of degeneracy. The first is that different sets of occupation numbers n_i of the single-particle energy levels ε_i may have the same total energy. This can happen only when the ε_i are rationally dependent. For example, with the three-level ladder where $(\varepsilon_i)_i = (-1, 0, 1)$, the occupation numbers (0,2,1) and (1,0,2) represent three-particle configurations with the same energy of $+1$. The ability of the higher-order group dynamics to mix between such configurations results from the greater number of dissipation channels, compared with a single channel for SU(2).

The second type of degeneracy occurs *within* a given set of occupation numbers and can be seen by considering Young tableaux. Referring to the construction described in Sec. II D of a basis for the SU(d) irreps, we consider the example $d = 3$, $n = 3$ and the irrep for $\lambda = (2, 1, 0)$. Two of the possible Young tableaux associated with this irrep are

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\,. \tag{83}$$

These Young tableaux describe two linearly independent, though nonorthogonal, states in the subspace $\mathcal{H}^{\lambda}$. By orthogonalizing, one obtains a pair of basis states with the same occupation numbers $(n_i)_i = (1, 1, 1)$ but which are still distinguishable according to the permutation-invariant system dynamics. It is not hard to see that such situations can occur only with $n > 2$ and $d > 2$. Moreover, only symmetries other than fully symmetric and antisymmetric can support this degeneracy: with only a single row or column in the Young diagram, the filling order is fixed. This is why (as in Fig. 7) the symmetric irrep is not the one with highest dimension, other than for SU(2).

This second type of degeneracy is intrinsically nonclassical in the sense that states of classical indistinguishable particles are labeled only by their occupation numbers. Thus, for more than two particles and dynamics generating a higher-order symmetry group, the effective state space contains degrees of freedom that do not exist in ensembles of classical indistinguishable particles. Notably, the same mechanism underlies the quantum modification recently found in the well-known Gibbs paradox [11].

Are these additional degeneracies useful? Whereas the ground and fully excited states are nondegenerate (see Appendix C), the degeneracies tend to increase in the middle of the spectrum. The work output of the Otto cycle in Sec. V is dictated by the heat capacity $C = V(E)/T^2$, where $V(E)$ is the variance of the energy [21]. This can be seen when $\kappa\beta_c$ is close to β_h by relating the energy difference $E_{\beta_h,\beta_0} - E_{\kappa\beta_c,\beta_0}$ to its derivative with respect to bath temperature, using $C = \partial E/\partial T$. Thus the work output is proportional to the energy variance. For high bath temperatures T_c and T_h, the populations are roughly uniform over the energy spectrum, and so the advantage gained by the SU(2) system is due to its suppression of degeneracies in the middle of the spectrum, thus raising the energy variance. In contrast, for low bath

temperatures, most of the population is in the ground state, so raising the degeneracy of higher energy states will increase the variance. Therefore distinguishable particles will perform better in this case. For an intermediate temperature range, one expects SU(3) to perform best, since it has degeneracies at higher energies, though not as many as for distinguishable particles. These observations thus qualitatively explain the behavior in Fig. 4.

It remains to be explored whether the additional nonclassical degrees of freedom can be further exploited to gain an additional advantage in some setting. For example, one could engineer a Hamiltonian that breaks these degeneracies in order to extract work from them. Unlike superradiance, which has been argued to exist in classical models [58,59], this effect would have no classical analog.

VII. PERSPECTIVES

In this paper we have combined methods from representation theory to build a comprehensive theoretical framework for the thermodynamical description of noninteracting quantum many-body systems with permutation-invariant observables. As our main application of this formalism, we have investigated the structure and properties of steady states that emerge when such systems are weakly coupled to a thermal environment. We have further shown that permutation invariance induces collective effects that can in principle be used to enhance the performance of quantum thermal machines. At every step of our analysis, we have demonstrated that altering the constituents of a permutation-invariant ensemble from spin systems, which are covered by the conventional Clebsch-Gordan theory, to more general multilevel systems, for which we have developed a systematic description in this paper, can lead to qualitative changes in the thermodynamical properties of these ensembles.

Two possible extensions of our approach appear to be promising perspectives for future research. First, it would be interesting to include oscillating driving fields. For fast driving, this extension can be achieved by replacing steady-state master equations with thermodynamically consistent Floquet-Lindblad equations [35,60], which have been used earlier to study permutation-invariant spin systems [19]. In the slow-driving regime, one can instead employ adiabatic master equations, whose generators are obtained by replacing time-independent system parameters with external control protocols [36,61–63]. In this way, it would in particular be possible to explore the role of permutation invariance in the context of thermodynamic geometry, a topic that is currently attracting much interest in quantum thermodynamics [64–71].

From a mathematical perspective, progress in these directions would require an extension of our theorem on the structure of steady states arising from autonomous permutation-invariant master equations to periodic limit cycles. Since the generators of Floquet-Lindblad equations become time independent in a rotating basis, this generalization should be straightforward in the fast-driving regime. For the adiabatic limit, a counterpart of Spohn's theorem [72] for steady states that provides uniqueness conditions for periodic limit cycles is available [73]. This result will, however, still have to be generalized to permutation-invariant many-body systems. Similarly, it is important to generalize the setting to regimes beyond weak coupling, involving techniques such as stochastic Liouville equations [74] and the thermal leads approach [75]. Permutation symmetry can in principle be included in such settings, so the main theoretical challenge is to find simple models under which the steady state is the same partially thermalized state analyzed here.

A second key problem is to systematically investigate the role of interactions. Most of our results, in particular our theorem on steady states, depend only on permutation invariance and should therefore be applicable also to interacting systems. However, since interactions typically lead to a dense energy spectrum, they render the rotating wave approximation that underpins the conventional weak-coupling master equation invalid. It would thus be necessary to either identify a relevant class of interacting systems that still feature a spectrum with well-separated Bohr frequencies, or to employ new types of recently derived quantum master equations [76–78], whose thermodynamical consistency is, however, not yet settled. One class of interacting Hamiltonians which may be most tractable is those with a product state energy basis, such as pairwise $d_i d_j$ interactions, since the eigenvectors of H^λ can be taken as the Schur basis.

From an information-theoretical viewpoint, one extension could involve studying the full work statistics of the Otto cycle rather than just the mean; due to the conservation law relating to the symmetry type, each block has its own associated probability and work output, so one is forced to consider the work as a fluctuating random variable. The thermodynamical quantities considered here would then need to be replaced by single-shot versions [79]. For example: what is the minimal work output that is guaranteed per cycle with some small probability of failure? Another line of questioning is what our results have to say about the thermodynamical processing of ensembles beyond the "independent and identically distributed" (i.i.d.) regime, where one typically considers many independent copies of a state subject to arbitrary operations. This setting has been used to collapse the many quantum "second laws" [80] down to just the standard nonequilibrium free energy [81]. Perhaps collective operations as defined here may give rise to different laws.

All plotted data were generated from the equations discussed in the text. No further data were created by the research presented in this paper.

ACKNOWLEDGMENTS

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APPENDIX A: STEADY STATES OF NONINTERACTING ENSEMBLES WITH SU(d) SYMMETRY

The main result of this Appendix is Theorem 1. This theorem specifies the structure of the steady states of a class of collective quantum master equations, which is more general than the one discussed in the main text.

We consider a permutation-invariant ensemble of n d-level particles embedded in a large, not necessarily thermal, bath. The bare ensemble is described by a permutation-invariant Hamiltonian H, which may in principle also contain interactions between particles. We further assume that the system-bath interaction is described by the Hamiltonian $H_I = \hbar\Delta \sum_\mu E_\mu \otimes B_\mu$, where the E_μ are the nondiagonal generators of the tensor-product representation of SU(d) on the ensemble Hilbert space and the Hermitian operators $B_\mu = B_{-\mu}$ correspond to bath observables. The operators E_μ can now be decomposed into Bohr-frequency components with respect to the ensemble Hamiltonian H, that is, $E_\mu = \sum_\omega E_\mu(\omega)$ with $[H, E_\mu(\omega)] = \hbar\omega E_\mu(\omega)$. Provided that the conventional weak-coupling, Born-Markov, and rotating wave approximations are applicable, the time evolution of the state ρ of the ensemble can then be described in terms of the master equation $\partial_t \rho_t = \mathcal{L}(\rho_t)$ with the Lindblad generator $\mathcal{L}$ being defined in Eq. (A4).

Before we can move on to the proof of our main theorem, we need a lemma that characterizes those operators that commute with all E_μ. Note that no information is needed about the dimensions of multiplicity spaces $\mathcal{K}^\lambda$, so that the results below apply not just when Schur-Weyl duality is valid, but also to spin systems upon replacing λ by J.

Lemma 1. Let $\{\mu\}$ be a set of simple roots of SU(d). Then for an operator X it holds that

$$[X, E_\mu] = [X, E_{-\mu}] = 0 \ \forall \mu \quad \text{if and only if}$$
$$X = \bigoplus_\lambda X^\lambda \otimes \mathbb{1}_{\mathcal{H}^\lambda}, \tag{A1}$$

where X^λ is an arbitrary operator on the multiplicity space $\mathcal{K}^\lambda$.

Proof. The "if" part is obvious. For the "only if" part, we make use of the fact that the simple roots plus their negatives generate the whole root system [see Ref. [82], Eq. (21.20)]. In other words, for any root ν, there exists a sequence of simple roots or their negatives, $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)}$ (allowing for repetitions), such that

$$E_\nu \propto [E_{\mu^{(k)}}, [\ldots, [E_{\mu^{(2)}}, E_{\mu^{(1)}}]\ldots]]. \tag{A2}$$

If $[X, E_{\mu^{(i)}}] = 0 \ \forall i$, we therefore see that $[X, E_\nu] = 0$, as E_ν is constructed from products of operators commuting with X.

For the remaining diagonal generators, we make use of the commutation relation $[E_\mu, E_{-\mu}] = \sum_i M_{\mu i} D_i$. There is a choice of normalization such that $M_{\mu i} = v_\mu^i$ [45, Chap. VI.3]. Then, for any simple root μ in the set,

$$\begin{aligned}\sum_i v_\mu^i [X, D_i] &= [X, [E_\mu, E_{-\mu}]] \\ &= [E_\mu, [E_{-\mu}, X]] + [E_{-\mu}, [X, E_\mu]] \\ &= 0,\end{aligned} \tag{A3}$$

having used the Jacobi identity in the second line. Since the $d-1$ simple roots are a linearly independent set, (A3) can hold for all μ only if all commutators $[X, D_i]$ vanish. Therefore we have found that X must commute with the representation of SU(d).

Finally, we decompose the unitary representation as $U = \bigoplus_\lambda \mathbb{1}_{\mathcal{K}^\lambda} \otimes U^\lambda$. Considering any block of $X^{\lambda,\lambda'}$ with respect to the irrep structure, commutation implies $(\mathbb{1}_{\mathcal{K}^\lambda} \otimes U^\lambda) X^{\lambda,\lambda'} (\mathbb{1}_{\mathcal{K}^{\lambda'}} \otimes U^{\lambda'\dagger}) = X^{\lambda,\lambda'}$. Schur's lemma then tells us that the off-diagonal blocks vanish, while the diagonal ones are of the form $X^{\lambda,\lambda} = X^\lambda \otimes \mathbb{1}_{\mathcal{H}^\lambda}$. ■

Theorem 1. Consider the Lindblad generator

$$\begin{aligned}\mathcal{L}(\rho) = &-\frac{i}{\hbar}[H, \rho] + \sum_{\mu,\nu} \Gamma_\omega^{\mu\nu} \Big[E_\mu(\omega) \rho E_\nu(\omega)^\dagger \\ &- \frac{1}{2}\{E_\nu(\omega)^\dagger E_\mu(\omega), \rho\}\Big],\end{aligned} \tag{A4}$$

where the E_μ are the collective representations of the nondiagonal generators of the Lie algebra $su(d)$ in the Cartan basis, H is any permutation-invariant Hamiltonian and the coefficients $\Gamma_\omega^{\mu\nu}$ form a Hermitian matrix for each Bohr frequency ω. Further assume that there exists a set of simple roots R such that, for each ω, the coefficients $\Gamma_\omega^{\mu\nu}$ form a matrix that has full rank over those roots $\mu \in R$ for which the frequency components $E_\mu(\omega)$ do not vanish. Then the steady states of the generator $\mathcal{L}$ are all of the form

$$\rho_\infty = \bigoplus_\lambda p^\lambda \sigma^\lambda \otimes \rho^\lambda, \tag{A5}$$

where the probabilities p^λ and the operators σ^λ are arbitrary, while the ρ^λ are unique on each irrep λ.

Proof. We adapt the original proof by Spohn that guarantees a single steady state [72], more closely following the presentation of this proof in Ref. [83]. First, we show that a set of ρ^λ exists such that all states of the form (A5) are steady states. Due to the permutation symmetry of the dynamics, any such state must evolve in time as $\rho_t = \bigoplus_\lambda p^\lambda \sigma^\lambda \otimes \rho_t^\lambda$. Since the irreps $\mathcal{H}^\lambda$ are finite-dimensional, each contains at least one steady state ρ^λ.

Next, we still have to prove that this family covers all the steady states. Initially, we relabel the set $\{E_\mu(\omega)\}_{\mu,\omega}$ as $\{E_i\}_i$ such that i corresponds to a pair (μ_i, ω_i). Next we put the generator into diagonal form by transforming to $F_i = \sum_j u_{ji} E_j$, where u_{ji} form a unitary matrix. Then we have $E_i = \sum_j u_{ij}^* F_j$, so

$$\begin{aligned}\mathcal{L}(\rho) &= -\frac{i}{\hbar}[H, \rho] + \sum_{i,j} \delta_{\omega_i,\omega_j} \Gamma_{\omega_i}^{\mu_i\mu_j} \left(E_i \rho E_j^\dagger - \frac{1}{2}\{E_j^\dagger E_i, \rho\}\right) \\ &= -\frac{i}{\hbar}[H, \rho] + \sum_{k,l} A_{kl} \left(F_k \rho F_l^\dagger - \frac{1}{2}\{F_j^\dagger F_i, \rho\}\right),\end{aligned} \tag{A6}$$

with $A_{kl} = \sum_{i,j} u_{ik}^* \delta_{\omega_i,\omega_j} \Gamma_{\omega_i}^{\mu_i\mu_j} u_{jl}$. Choosing u to diagonalize the Hermitian matrix A, we obtain

$$\mathcal{L}(\rho) = -\frac{i}{\hbar}[H, \rho] + \sum_k a_k \left(F_k \rho F_k^\dagger - \frac{1}{2}\{F_k^\dagger F_k, \rho\}\right). \tag{A7}$$

We would like to guarantee that $a_k > 0$. To this end, we note that A is unitarily equivalent to the matrix

$$\Gamma = \bigoplus_{\omega} \Gamma_\omega := \bigoplus_{\omega} \sum_{\substack{\mu,\nu: \\ E_\mu(\omega),E_\nu(\omega)\neq 0}} \Gamma_\omega^{\mu\nu} |\omega, \mu\rangle\langle\omega, \nu|. \tag{A8}$$

Note that we use Dirac notation here for convenience, although the vectors $|\omega, \mu\rangle$ do not represent quantum states. The aim of this construction is to include only those terms for which E_μ has a nonzero frequency component $E_\mu(\omega)$; otherwise, the full-rank condition would not generally be possible to satisfy. Thus we see that the positivity of A is guaranteed by assuming that all Γ_ω have full rank.

We then make a second transformation by expanding $F_i = \sum_{i=1}^p f_{ik} G_k$ in some operator basis such that G_k are Hermitian and f_{ik} are complex. G_k are chosen to form a basis of the subspace $V = \mathrm{span}\{F_i\}_i = \mathrm{span}\{E_\mu\}_\mu$ with some dimension p. This substitution results in

$$\mathcal{L}(\rho) := -\frac{i}{\hbar}[H, \rho] + \sum_{k,l=1}^{p} B_{kl}\left(G_k \rho G_l - \frac{1}{2}\{G_l G_k, \rho\}\right), \tag{A9}$$

where $B_{kl} = \sum_i a_i f_{ik} f_{ik}^*$. Evidently $B = \sum_i a_i \boldsymbol{f}_i \boldsymbol{f}_i^\dagger > 0$, which means we can find some $b > 0$ smaller than the smallest eigenvalue of B such that $B - b\mathbb{1}_p > 0$. Using this result, we divide $\mathcal{L}$ into two parts:

$$\mathcal{L} := \mathcal{L}_1 + \mathcal{L}_2, \tag{A10}$$

$$\mathcal{L}_2(\rho) := b \sum_{k=1}^{p} G_k \rho G_k - \frac{1}{2}\{G_k^2, \rho\}, \tag{A11}$$

such that both parts are valid Lindblad generators.

We first analyze the spectral properties of $\mathcal{L}_2$. For any Hermitian operator X, we calculate the superoperator expectation value

$$\begin{aligned}\langle \mathcal{L}_2\rangle_X &= \mathrm{Tr}[X\mathcal{L}_2(X)] \\ &= b\sum_k \mathrm{Tr}\left[XG_kXG_k - \frac{1}{2}XG_k^2X - \frac{1}{2}X^2G_k^2\right] \\ &= \frac{b}{2}\sum_k \mathrm{Tr}[[G_k, X]^2] \\ &= -\frac{b}{2}\sum_k \|[G_k, X]\|_2^2. \end{aligned} \tag{A12}$$

This quantity is generally negative and vanishes if and only if $[G_k, X] = 0\ \forall i = 1, \ldots, p$. This condition is equivalent to $[E_\mu, X] = 0\ \forall \mu$, and by Lemma 1 also equivalent to $X = \bigoplus_\lambda X^\lambda \otimes \mathbb{1}_{\mathcal{H}^\lambda}$. Let us denote the subspace of such operators by $S = \{\bigoplus_\lambda X^\lambda \otimes \mathbb{1}_{\mathcal{H}^\lambda} \mid X^{\lambda\dagger} = X^\lambda\}$; this space has dimension $M = \sum_\lambda m_\lambda(m_\lambda + 1)/2$. We have already found a space of steady states with this dimension, so our goal is to show that $\mathcal{L}$ has precisely M linearly independent zero eigenvectors.

Next we divide the vector space of operators into two orthogonal parts $S \oplus S'$. There is then a block decomposition of $\mathcal{L}$ on a suitable basis respecting this structure: representing the superoperator as a matrix,

$$\mathcal{L} = \begin{pmatrix} * & * \\ * & \mathcal{L}' \end{pmatrix}, \tag{A13}$$

and similarly for $\mathcal{L}_{1,2}$, where $*$ denotes an unspecified block and $\mathcal{L}'$ is the projection onto S'. From (A12) we see that $\langle \mathcal{L}_2\rangle_X < 0$ for all $X \in S'$, so the eigenvalues of $\mathcal{L}_2'$ all have a strictly negative real part. Similarly, for all $X \in S'$,

$$\langle \mathcal{L}'\rangle_X = \langle \mathcal{L}_1'\rangle_X + \langle \mathcal{L}_2'\rangle_X < 0, \tag{A14}$$

since $\mathcal{L}_1'$ must have eigenvalues with a nonpositive real part in order to be a valid Lindblad generator. Therefore $\mathcal{L}'$ also has eigenvalues with a strictly negative real part.

Finally, to learn about the eigenvalues of $\mathcal{L}$, we use Theorem 1.4.10 of Ref. [84]. This theorem tells us that if an eigenvalue λ of some $n \times n$ matrix has geometric multiplicity of at least $k + 1$, then every $(n - k) \times (n - k)$ principal submatrix also has λ as an eigenvalue. Suppose for a contradiction that $\mathcal{L}$ has more than M zero eigenvectors. So we apply this theorem with $\lambda = 0$, $k = M$ and n as the dimension of the full operator space. This observation implies that $\mathcal{L}'$ has a zero eigenvector, which contradicts what we have just determined about the eigenvalues of $\mathcal{L}'$. Therefore $\mathcal{L}$ has at most M zero eigenvectors, and we have already found this many. ■

With a thermal environment, one can assume a standard detailed-balance relation on the bath correlation function and thereby obtain the following result guaranteeing thermalization within each block.

Corollary 1. With the same assumptions as in Theorem 1, together with $\Gamma_{-\omega}^{\mu\nu} = e^{\beta\hbar\omega}\Gamma_\omega^{\nu\mu}$, the steady states are all those of the form

$$\rho_\infty = \bigoplus_\lambda p^\lambda \sigma^\lambda \otimes \gamma^\lambda, \tag{A15}$$

where $\gamma^\lambda = e^{-\beta H^\lambda}/\mathrm{tr}[e^{-\beta H^\lambda}]$ is the thermal state on each irrep, according to the Hamiltonian $H = \bigoplus_\lambda \mathbb{1}_{\mathcal{K}^\lambda} \otimes H^\lambda$.

Proof. This result is an immediate consequence of Theorem 1 followed by applying, to each diagonal block, Spohn's result [72] that the condition $\Gamma_{-\omega}^{\mu\nu} = e^{\beta\hbar\omega}\Gamma_\omega^{\nu\mu}$ requires the steady state ρ^λ to be the thermal state γ^λ. ■

APPENDIX B: SYMMETRIC SUBSPACE ENERGIES

Here we study the mean energy for the symmetric subspace at high bath temperature to first order in β, for general n and d. We consider an arbitrary set of single-particle energies ε_i, $i = 1, \ldots, d$. A natural basis for the symmetric subspace is given by symmetrizing $\bigotimes_{i=1}^d |i\rangle^{\otimes n_i}$ for each configuration $\boldsymbol{n} = (n_1, n_2, \ldots, n_d)$, $\sum_i n_i = n$. These are energy eigenstates with energy $\varepsilon_{\boldsymbol{n}} = \sum_i \varepsilon_i n_i$.

The mean energy when confined to the symmetric subspace is then

$$E_\beta^{\mathrm{sym}} = \frac{\sum_{\boldsymbol{n}} \varepsilon_{\boldsymbol{n}} e^{-\beta\varepsilon_{\boldsymbol{n}}}}{\sum_{\boldsymbol{n}} e^{-\beta\varepsilon_{\boldsymbol{n}}}}, \tag{B1}$$

which is, to first order at high temperature,

$$E_\beta^{\mathrm{sym}} \approx \frac{1}{d_{\mathrm{sym}}}\sum_{\boldsymbol{n}} \varepsilon_{\boldsymbol{n}} - \beta\left[\frac{1}{d_{\mathrm{sym}}}\sum_{\boldsymbol{n}} \varepsilon_{\boldsymbol{n}}^2 - \left(\frac{1}{d_{\mathrm{sym}}}\sum_{\boldsymbol{n}} \varepsilon_{\boldsymbol{n}}\right)^2\right] = \langle\varepsilon\rangle_n - \beta\langle\langle\varepsilon^2\rangle\rangle_n, \tag{B2}$$

where $d_{\mathrm{sym}} = \binom{n+d-1}{n}$, and $\langle\varepsilon\rangle_n$, $\langle\langle\varepsilon^2\rangle\rangle_n$ denote the average and variance of energies, respectively, in a uniform mixture for n particles. We will express these in terms of their values for $n = 1$.

The mean energy is

$$\begin{aligned}\langle\varepsilon\rangle_n &= \sum_i \langle n_i\rangle_n \varepsilon_i \\ &= \langle n_1\rangle_n \sum_i \varepsilon_i \\ &= \frac{n}{d}\sum_i \varepsilon_i \\ &= n\langle\varepsilon\rangle_1,\end{aligned} \tag{B3}$$

having used the fact that the uniform distribution of energies is symmetric with respect to permutations of the energy levels. Similarly,

$$\begin{aligned}\langle\varepsilon^2\rangle_n &= \sum_{i,j}\langle n_i n_j\rangle_n \varepsilon_i\varepsilon_j \\ &= \sum_i \langle n_i^2\rangle_n \varepsilon_i^2 + \sum_{i\neq j}\langle n_i n_j\rangle_n \varepsilon_i\varepsilon_j \\ &= \langle n_1^2\rangle_n \sum_i \varepsilon_i^2 + \langle n_1 n_2\rangle_n\left(\sum_{i,j}\varepsilon_i\varepsilon_j - \sum_i \varepsilon_i^2\right) \\ &= \langle n_1^2\rangle_n d\langle\varepsilon^2\rangle_1 + \langle n_1 n_2\rangle_n\left(d^2\langle\varepsilon\rangle_1^2 - d\langle\varepsilon^2\rangle_1\right).\end{aligned} \tag{B4}$$

For simplicity, we can take $\langle\varepsilon\rangle_1 = 0$ without loss of generality; then we have

$$\langle\langle\varepsilon^2\rangle\rangle_n = \langle\varepsilon^2\rangle_n = \langle\varepsilon^2\rangle_1 d\left(\langle n_1^2\rangle_n - \langle n_1 n_2\rangle_n\right). \tag{B5}$$

Given that $\sum_i n_i = n$, we have

$$\begin{aligned}n^2 &= \sum_{i,j}\langle n_i n_j\rangle_n \\ &= \sum_i \langle n_i^2\rangle_n + \sum_{i\neq j}\langle n_i n_j\rangle_n \\ &= d\langle n_1^2\rangle_n + d(d-1)\langle n_1 n_2\rangle_n.\end{aligned} \tag{B6}$$

Substituting into Eq. (B5), we have

$$\frac{\langle\langle\varepsilon^2\rangle\rangle_n}{\langle\langle\varepsilon^2\rangle\rangle_1} = \frac{d^2\langle n_1^2\rangle_n}{d-1} - \frac{n^2}{d-1}. \tag{B7}$$

The probability distribution over n_1 is found by counting the number of ways of distributing the remaining $n - n_1$ particles over the other $d - 1$ energy levels:

$$p(n_1) = \frac{\binom{n-n_1+d-2}{d-2}}{\binom{n+d-1}{d-1}}, \tag{B8}$$

from which we get

$$\langle n_1^2\rangle_n = \sum_{n_1=0}^{n} n_1^2\, p(n_1) = \frac{n(2n+d-1)}{d(d+1)}. \tag{B9}$$

Equation (B7) then gives

$$\frac{\langle\langle\varepsilon^2\rangle\rangle_n}{\langle\langle\varepsilon^2\rangle\rangle_1} = \frac{n(n+d)}{d+1}. \tag{B10}$$

This is to be compared with the full (infinite temperature) thermal state of n particles, which has energy variance $n\langle\langle\varepsilon^2\rangle\rangle_1$.

We can now ask what set of single-particle energies gives the maximal variance; due to Eq. (B10), we need to consider only the case of a single particle. In order to make a well-posed question, we constrain the ε_i such that there is at least one at the minimum energy $\varepsilon_1 = 0$ and at least one at the maximum $\varepsilon_d = d - 1$. (These values are chosen to match the spread of the ladder Hamiltonian; note that the overall shift can be arbitrary.) The variance must always satisfy

$$\langle\langle\varepsilon^2\rangle\rangle_1 \leqslant \frac{(\varepsilon_d - \varepsilon_1)^2}{4} = \frac{(d-1)^2}{4}. \tag{B11}$$

For even d, this can be saturated by taking half the levels at 0 and half at $d - 1$. For odd d, the optimum is with either $\frac{d-1}{2}$ at 0 and $\frac{d+1}{2}$ at $d - 1$, or the opposite. Hence,

$$\max_{\{\varepsilon_i\},\,\varepsilon_1=0,\,\varepsilon_d=d-1}\langle\langle\varepsilon^2\rangle\rangle_1 = \begin{cases}\frac{(d-1)^2}{4}, & d \text{ even},\\ \frac{(d-1)^2}{4}\left(1-\frac{1}{d^2}\right), & d \text{ odd}.\end{cases} \tag{B12}$$

For the evenly spaced ladder, one instead has $\langle\langle\varepsilon^2\rangle\rangle_1 = \frac{d^2-1}{12}$– so the level optimization roughly gains a factor of three.

APPENDIX C: GROUND STATE IN EACH SUBSPACE

Here we argue that each term H^λ of the Hamiltonian has a unique ground state (on the reduced state space). This is seen by examining the construction of the irreps of SU(d) described in Sec. II D. Given a single-particle basis $|k\rangle$—which we take as the single-particle energy eigenbasis—one can construct a (generally nonorthogonal) basis of the irrep corresponding to a Young diagram λ by specifying all ways of filling the boxes of λ according to certain rules. Each box must be filled with a label k in a way that is weakly increasing (i.e., two neighboring elements can be the same) in each row and strictly increasing in each column. For example, one allowed tableau with $n = 8,\ d = 3$ is

1	1	2	3
2	3	3	
3			

. (C1)

Each allowed tableau corresponds to a vector constructed by taking a product state of each $|k\rangle$ per box and applying the Young symmetrizer, which symmetrizes over each row and then antisymmetrizes over each column. Each such state is an energy eigenstate with eigenvalue $\sum_{i=1}^n \varepsilon_{k_i}$, where k_i is the label of box i in the tableau. It is clear that there is only one way to construct a state of minimal energy: fill each column of length l with the labels $1, 2, \ldots, l$ from top to bottom. For

the above diagram, this gives

1	1	1	1
2	2	2	
3			

(C2)

The unique ground state in the irrep λ has energy

$$E_\lambda = \sum_{k=1}^{d} \lambda_k \varepsilon_k. \tag{C3}$$

Note that this argument depends crucially on the Hamiltonian being noninteracting.

APPENDIX D: ASYMPTOTIC ENERGIES AND ENTROPIES FOR SU(d)

Here we find the mean energy of a state with $\beta_0 = 0$ in the limit of large particle number $n \to \infty$. We use the ladder Hamiltonian with energies

$$\varepsilon_k = -\frac{(d+1)}{2} + k, \quad k = 1, \ldots, d, \tag{D1}$$

and the limiting probability distribution $\phi_d(\zeta)$ in Eq. (70) with $\zeta_i = \frac{\lambda_i - n/d}{\sqrt{n}}$. From Eq. (C3) together with $\sum_k \zeta_k = 0$ and $\sum_k \varepsilon_k = 0$, we have the ground-state energy in the block λ,

$$\begin{aligned} E_\lambda &= \sum_{k=1}^{d} \left(\sqrt{n}\zeta_k + \frac{n}{d}\right)\varepsilon_k \\ &= \sqrt{n}\sum_{k=1}^{d} \zeta_k \left(k - \frac{d+1}{2}\right) \\ &= \sqrt{n}\sum_{k=1}^{d} k\zeta_k \\ &= \sqrt{n}\left(\sum_{k=1}^{d-1} k\zeta_k + d\zeta_d\right) \\ &= \sqrt{n}\sum_{k=1}^{d-1} (k-d)\zeta_k. \end{aligned} \tag{D2}$$

We have chosen to eliminate ζ_d because of the constraint $\sum_k \zeta_k = 0$, leaving only the remaining $(d-1)$ variables independent. The limiting mean energy is then

$$E_{\infty,0} = \sqrt{n} \int d^{d-1}\zeta \; \phi_d(\zeta) \sum_{k=1}^{d-1} (k-d)\zeta_k \tag{D3}$$

$$=: -\sqrt{n}\mathcal{E}_d, \tag{D4}$$

where the integral is constrained to $\zeta_k \geqslant \zeta_{k+1} \; \forall k$. We find the coefficients

$$\mathcal{E}_2 = \sqrt{\frac{2}{\pi}}, \quad \mathcal{E}_3 = \frac{9}{4}\sqrt{\frac{3}{\pi}}. \tag{D5}$$

Higher dimensions are only numerically tractable; we find

$$\begin{aligned} &\mathcal{E}_2 \approx 0.798, \; \mathcal{E}_3 \approx 2.20, \; \mathcal{E}_4 \approx 4.19, \; \mathcal{E}_5 \approx 6.76, \\ &\mathcal{E}_6 \approx 9.91, \; \mathcal{E}_7 \approx 13.6. \end{aligned} \tag{D6}$$

For the entropy $\tilde{S}_{0,0}$, we need to find the average of $\ln d_\lambda$. Given that $\lambda_i - \lambda_j = \sqrt{n}(\zeta_i - \zeta_j)$, one can see from Eq. (19) that

$$d_\lambda \propto \prod_{i<j} \left[\sqrt{n}(\zeta_i - \zeta_j) + (i-j)\right]. \tag{D7}$$

Since the distribution $\phi_d(\zeta)$ is independent of n, the function to be averaged contains $d(d-1)/2$ terms of the order $\ln\sqrt{n}$. We therefore have, to leading order in n,

$$\tilde{S}_{0,0} = \langle \ln d_\lambda \rangle \approx \frac{d(d-1)}{4} \ln n. \tag{D8}$$

APPENDIX E: ASYMPTOTICS FOR SU(2), SPIN-1

Here we consider the same limit as above, with an initially high temperature $\beta_0 = 0$ in the limit of large n, for $d = 3$ and SU(2)-type coupling to the bath. We will find the limiting distribution p^J. For n particles with general spin s, the thermal state γ_0 on the full Hilbert space is maximally mixed, so one sees that

$$p^J = \frac{(2J+1)m_J}{(2s+1)^n}. \tag{E1}$$

We use the recursion relation (44) for m_J to derive a similar relation for p^J. Writing the probability explicitly as a function of n, we have

$$p(n+1, J) = \sum_{\substack{J' :\, |J'-s| \leqslant J \leqslant J'+s, \\ J'+s-J \in \mathbb{Z}}} \frac{(2J+1)}{(2s+1)(2J'+1)} p(n, J'). \tag{E2}$$

From now on assuming $s = 1$, we have

$$p(n+1, J) = \begin{cases} \frac{p(n,1)}{9}, & J = 0 \\ \frac{2J+1}{3}\left[\frac{p(n,J-1)}{2J-1} + \frac{p(n,J)}{2J+1} + \frac{p(n,J+1)}{2J+3}\right], & J = 1, 2, \ldots. \end{cases} \tag{E3}$$

with only integer values of J appearing. In the limit of large n, we make the ansatz that $p(n, J)$ takes the form

$$p(n, J) \approx \frac{1}{\sqrt{n}} f\left(\frac{J}{\sqrt{n}}\right), \tag{E4}$$

where f is some smooth function normalized such that $\int_0^\infty f(x)\,dx = 1$. This guess will be justified by showing that it approximately satisfies the recursion relation (E3), and we will determine f. For any $J \geqslant 1$, (E3) reduces to

$$\begin{aligned} &\frac{1}{\sqrt{n+1}} f\left(\frac{J}{\sqrt{n+1}}\right) \\ &\quad = \frac{1}{3\sqrt{n}}\left[f\left(\frac{J}{\sqrt{n}}\right) + \left(\frac{2J+1}{2J-1}\right) f\left(\frac{J-1}{\sqrt{n}}\right)\right. \\ &\qquad \left. + \left(\frac{2J+1}{2J+3}\right) f\left(\frac{J+1}{\sqrt{n}}\right)\right]. \end{aligned} \tag{E5}$$

We write $\delta = 1/\sqrt{n}$, $x = J\delta$, giving

$$\frac{1}{\sqrt{1+\delta^2}} f\left(\frac{x}{\sqrt{1+\delta^2}}\right) = \frac{1}{3}\left[f(x) + \left(\frac{2x+\delta}{2x-\delta}\right) f(x-\delta) + \left(\frac{2x+\delta}{2x+3\delta}\right) f(x+\delta)\right]. \tag{E6}$$

Since $\delta \ll 1$, we expand to lowest nonvanishing order in δ:

$$0 = \frac{\delta^2}{3}\left[-f''(x) + \left(\frac{2}{x} - \frac{3x}{2}\right) f'(x) - \left(\frac{2}{x^2} + \frac{3}{2}\right) f(x)\right] + O(\delta^3). \tag{E7}$$

Hence we obtain the second-order linear differential equation

$$2f''(x) + \left(3x - \frac{4}{x}\right) f'(x) + \left(\frac{4}{x^2} + 3\right) f(x) = 0. \tag{E8}$$

The only normalizable solution is found to be $f(x) \propto x^2 e^{-3x^2/4}$, resulting in

$$p(n, J) \approx \frac{3\sqrt{3}}{2\sqrt{\pi}} \frac{J^2}{n^{\frac{3}{2}}} e^{-\frac{3J^2}{4n}}. \tag{E9}$$

Since the ground-state energy of the irrep J is $-J$, we find the limiting mean energy with $\beta = \infty$ as

$$\begin{aligned} E_{\infty,0} &\approx -\int_0^\infty dJ\, J\, p(n, J) \\ &\approx -\sqrt{n} \int_0^\infty dx\, x f(x) \\ &= -\frac{4}{\sqrt{3\pi}} \sqrt{n}. \end{aligned} \tag{E10}$$

Similarly, the $\beta = 0$ reduced entropy is

$$\begin{aligned} \tilde{S}_{0,0} &\approx \int_0^\infty dx\, \ln(2\sqrt{n}x + 1) f(x) \\ &\approx \frac{1}{2} \ln n. \end{aligned} \tag{E11}$$

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