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Construction of Control Lyapunov Function with Region of Attraction Using Union Theorem in Sum-Of-Squares Optimization

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Abstract— **Control Lyapunov function (CLF) paves the way for designing a certified controller with a known stable region, which is the out-most importance in control systems. Sum-of-Squares (SOS) optimization is one method to construct the CLF with this stable region known as a region of attraction (ROA). However, existing methods yield quite conservative results. A new approach for constructing CLF overcoming existing limitations is proposed in this paper. The proposed method is based on the Union Theorem in sum-of-squares optimization, which enables the application of more than one variable size region generated by positive functions known as the Shape Function. Numerical simulations demonstrate the effectiveness of the proposed method, which outperforms the existing methods and provides a significantly enhanced ROA.**

I. INTRODUCTION

The analysis and design of controllers or policies for autonomous systems that are target-oriented are crucial in various fields, such as aerospace, autonomous driving, and robotics. For stability analysis and control design for nonlinear systems, the control Lyapunov function (CLF) is a well-established tool. The construction of a CLF is an important approach for the analysis and control of nonlinear systems, and it has been effective in stabilizing such systems. Nevertheless, practically implementing a CLF often involves determining the region of attraction (ROA), which is the set of initial conditions that result in the system remaining stable and converging to a fixed state over time. Constructing a CLF with ROA can be a challenging and computationally difficult task. Several numerical approaches, including Sum-of-Squares (SOS) optimization-based methods, offer a promising way to construct a CLF with ROA for nonlinear systems.

One of the early works in the development of CLFs is proposed in [1]. In this paper, the author proposed the concept of relaxed controls, which introduced a novel method for stabilizing nonlinear systems. Furthermore, the author showed that relaxed controls can be used to construct CLF that stabilize the system, even when the control law is not continuous. In [2], the author developed a universal construction of Artstein's theorem on nonlinear stabilization, which allowed for the construction of CLF for a broader class of nonlinear systems. The authors of [3] show that the CLF of a linear system can be obtained by solving the Riccati equation. The feedback linearization method, proposed in [4], can also be used to construct CLF for a class of nonlinear systems by transforming the system dynamics into a linear form that can be controlled using linear techniques. In [5] and [6], the authors propose a method to construct CLF for the linear system. In [7], the authors re-analyze the concept of CLF through the method of set-valued analysis and design a nonlinear controller based on it. However, all the mentioned methods provide CLF without ROA. To address it, various methods such as SOS optimization technique have been proposed to construct the CLF with ROA. In this paper, we propose a method using the SOS optimization technique to get a CLF with a larger estimation of the ROA.

An SOS polynomial is a non-negative polynomial, which can be expressed as a sum of squares of other polynomials. A Lyapunov function (LF) can be represented as an SOS polynomial. The fundamental theories of linear matrix inequality and SOS polynomial have been discussed in [8] and [9]. SOS is a relaxation procedure, and the conditions required to express a non-negative polynomial as a relaxed SOS polynomial have been discussed in [10] and [11]. Tutorials on the SOS optimization technique, including MATLAB code for various examples, have been presented in [12], [13], and [14]. In [15], the authors have presented an algorithm to solve the nonconvex SOS problems efficiently. Tools such as BiSOS [15], SOSOPT [16], and SOSTOOLS [17] can be used to solve SOS optimization-related problems.

The work of [18] illustrates the application of the SOS optimization technique to construct CLF alongside the estimation of ROA. The authors employed a method based on Shape Function (SF) to obtain a larger ROA. This method estimates a very large ROA. However, the method gives a conservative result if the actual ROA is non-symmetric or unbounded.

This paper presents a novel approach for constructing the CLF with ROA using Union theorem in the SOS optimization method for the polynomial dynamical system. The approach results in a CLF with a larger ROA estimation even if the actual ROA is non-symmetric. The effectiveness of the proposed method is demonstrated using a benchmark example.

The paper is structured as follows: in Section 2, the background information on CLF and SOS optimization is provided, and the problem is formulated. In Section 3, the proposed method for constructing the CLF with ROA using Union Theorem in SOS optimization is discussed. The

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numerical solution process is presented in Section 4. Section 5 explains the selection of the required SF. The simulation results are presented in Section 6. Finally, in Section 7, the paper concludes with final thoughts and potential future work.

II. Preliminaries and Problem Formulation

Let us consider the following affine polynomial dynamical system,

$$\dot{x} = f(x) + g(x)u, \quad f(0) = 0 \tag{1}$$

- $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}$
- $f(x), g(x)$ are smooth polynomial vector fields

Definition 1 (Control Lyapunov Function): A function V is a Control Lyapunov Function (CLF) for the system (1) if $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth, radially unbounded, and positive definite function such that, [18],

$$\inf_{u\in\mathbb{R}} \left\{ \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial x} gu \right\} < 0 \quad \forall\, x \neq 0 \tag{2}$$

The existence of such a V implies that (1) is globally asymptotically stabilizable at the origin.

Definition 2 (Positive Semi-definite Polynomial): A polynomial, $f(x)$, is called Positive Semi-definite Polynomial if

$$f(x) \geq 0 \text{ for all } x \in \mathbb{R}^n$$

Definition 3 (Sum-of-Squares (SOS) polynomial): A polynomial, $p(x) \in \mathcal{R}_n$ is a SOS polynomial if there exist polynomials $f(x)_i \in \mathcal{R}_n$ such that

$$p = \sum_{i=1}^{t} f_i^2,\ f_i \in \mathcal{R}_n, i = 1, ..., t$$

p is an SOS if there exists $Q \geq 0$ such that $p = Z^T Q Z$. The set of SOS polynomials in n variables is defined as

$$\sum\nolimits_n := \left\{ p \in \mathcal{R}_n \;\middle|\; p = \sum_{i=1}^{t} f_i^2,\ f_i \in \mathcal{R}_n, i = 1, ..., t \right\}$$

Now, always $p(x) \geq 0$ if $p(x) \in \sum_n \forall\, x \in \mathbb{R}^n$.

A CLF can be expressed as an SOS polynomial as CLF is a positive definite function.

A. SOS Optimization form of CLF

A CLF always satisfies Eq. (2). The LHS of the Eq. (2) can be written in the following way,

$$\inf_{u\in\mathbb{R}} \left\{ \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)u \right\} = \begin{cases} -\infty & \text{when } \frac{\partial V}{\partial x} g(x) \neq 0 \\ \frac{\partial V}{\partial x} f(x) & \text{when } \frac{\partial V}{\partial x} g(x) = 0 \end{cases} \tag{3}$$

The Eq. (2) can be made always true by choosing a large value of u with correct sign when $\frac{\partial V}{\partial x} g(x) \neq 0$ for a fixed x. Therefore, it is important to check the set of x such that $\frac{\partial V}{\partial x} g(x) = 0$. So, a CLF should follow the below condition,

$$\frac{\partial V}{\partial x} f(x) < 0 \ \forall x \ s.t. \ \frac{\partial V}{\partial x} g(x) = 0, \ \ x \neq 0 \tag{4}$$

All the system is not globally asymptotically stabilizable. If a candidate CLF, V fails to satisfy Eq. (2) then it can be considered that the system is not globally asymptotically stabilizable and some points x are there for that $\frac{\partial V}{\partial x} f(x) \geq 0$ and $\frac{\partial V}{\partial x} g(x) = 0$. Henceforth, we need to check the local stability of the system. This can be done by finding a level set that excludes the unstable points, [18].

Definition 4 (Level Set): Level set, $\Omega_\gamma := \{x \in \mathbb{R}^n : V(x) \leq \gamma\}$ s.t. $\forall x \in \Omega_\gamma \setminus \{0\}$ and $\frac{\partial V}{\partial x} g(x) = 0$

Ω_γ is a ROA as it follows the below criteria of ROA, [18],

$$\text{If } a = \frac{\partial V}{\partial x} f(x) \text{ and } b = \frac{\partial V}{\partial x} g(x) \text{ then}$$
$$\frac{\partial V}{\partial t} = \left. \begin{cases} -\sqrt{a^2 + b^4} < 0 & \text{when } b \neq 0 \\ a < 0 & \text{when } b = 0 \end{cases} \right\} \forall x \in \Omega_\gamma \setminus \{0\} \tag{5}$$

To get a large ROA it is required to optimize the level set. Using the Positivstellensatz theorem the level set can be converted in the following SOS optimization form, [18].

$$\max_{\mathbf{s_0}, \mathbf{s_{01}} \in \sum_\mathbf{n},\, V(x), s_{02} \in \mathcal{R}_n} \gamma$$

$$\text{Subject to:} \quad V - l_1 \in \sum\nolimits_n \tag{6a}$$

$$-\left[s_{01} \frac{\partial V}{\partial x} f + s_{02} \frac{\partial V}{\partial x} g + l_2 \right] + (V - \gamma) s_0 \in \sum\nolimits_n \tag{6b}$$

Here, l_1 and l_2 are SOS polynomials. The above optimization problem (6) provides a CLF. However, it estimates a very small ROA. To improve the ROA estimation, a new function called shape function has been introduced in [18] along with equation (6), and we will discuss this further in the next section.

B. Shape Function

The Shape function (SF) represents a function with an enclosed region that is circumscribed by the CLF. It prompts the CLF to grow by enlarging the size of the SF itself in each iteration until a portion of the CLF makes contact with the unstable region. Put simply, the SF establishes the concluding form and scale of the CLF.

Definition 5 (Shape Function): Shape function is a positive definite function. The variable size region under it, is defined as, [18],

$$P_\beta := \{x : p(x) \leq \beta\} \subseteq \Omega_\gamma \tag{7}$$

If we introduce the SF in the Eq. (6) then we get the following modified optimization form of Eq. (6), [18],

$$\max_{\mathbf{s_0}, \mathbf{s_{01}}, \mathbf{s_1} \in \sum_\mathbf{n},\, V(x), s_{02} \in \mathcal{R}_n,\, \gamma \in \mathbb{R}^+} \beta$$

$$\text{Subject to:} \quad V - l_1 \in \sum\nolimits_n \tag{8a}$$

$$-\left[s_{01} \frac{\partial V}{\partial x} f + s_{02} \frac{\partial V}{\partial x} g + l_2 \right] + (V - \gamma) s_0 \in \sum\nolimits_n \tag{8b}$$

$$-(V - \gamma) + (p - \beta) s_1 \in \sum\nolimits_n \tag{8c}$$

The process of maximizing the variable β is demonstrated by equation (8). However, this approach also yields the optimal values of V and γ. It is crucial to note that the addition of an SF does not change the existing equation expressed by (6); instead, it adds an extra constraint. The usage of a single SF combined with equation (6) provides superior results compared to utilizing equation (6) alone. Nevertheless, equation (8) only estimates a portion of the actual ROA, particularly for systems with non-symmetric ROAs. In the subsequent section, we present a new method that employs the Union Theorem in SOS optimization to address this issue.

III. CLF Construction Based on Union Theorem

The interpretation of equation (8c) is that the area enclosed by the SF must remain within the region defined by the CLF and its value γ. The SOSOPT program raises the value of β during each iteration to encompass additional regions under the SF. It then produces a new CLF, which encloses the SF's area, for a fixed γ. This iterative technique is known as V-S iteration and will be discussed in a subsequent section.

The V-S iteration algorithm is utilized in SOS optimization to create a CLF with ROA based on a single SF as expressed by (8). This algorithm enlarges the ROA estimation by increasing the number of iterations. However, this method may not be effective in all cases, such as early converged optimization and/or numerical infeasibility, especially for systems with irregular ROAs. Additionally, the location of the SF's geometric center at the origin, whether fixed or adaptive, limits the estimation, particularly for non-symmetric ROAs. After several iterations, the SF may come very close to the unstable region in some directions of the actual non-symmetric ROA, and further increments of β will not provide any new surrounding CLF because it is on the verge of touching the unstable region. Although the CLF captures the ultimate region in this specific direction, there may still be a significant untapped area between the estimated ROA based on (8) and the actual ROA in other directions, which cannot be captured using the single SF-based approach. Therefore, we suggest a new method based on the Union Theorem outlined in paper [20].

Theorem 1 (Union Theorem): Let us consider that the given polynomials V, p_1, p_2,...,p_n define sets A_V, A_1, A_2,...,A_n such that,

$$\begin{aligned}
A_V &= \{x \in \mathbb{R}^n : V - \gamma \leq 0\} \\
A_1 &= \{x^1 \in \mathbb{R}^n : p_1 - \beta_1 \leq 0\} \\
A_2 &= \{x^2 \in \mathbb{R}^n : p_2 - \beta_2 \leq 0\} \\
&\vdots \\
A_n &= \{x^n \in \mathbb{R}^n : p_n - \beta_n \leq 0\}
\end{aligned}$$

Here, $\left\{\gamma, \beta_1, \beta_2, ...\beta_n\right\} \in \mathbb{R}^+$. Now if there exit polynomials $s_1 \in \sum_n$, $s_2 \in \sum_n$,...,$s_n \in \sum_n$, such that,

$$\begin{aligned}
&-(V-\gamma) + s_1(p_1 - \beta_1) \in \sum\nolimits_n \\
&-(V-\gamma) + s_2(p_2 - \beta_2) \in \sum\nolimits_n \\
&\quad\vdots \\
&-(V-\gamma) + s_n(p_n - \beta_n) \in \sum\nolimits_n
\end{aligned}$$

Then, $A_1 \cup A_2 \cup ... \cup A_n \subseteq A_V$.

The Union Theorem is one kind of S-procedure which is a relaxation form of the Positivstellensatz theorem, [20]. The Union Theorem tells us that if we introduce multiple SFs in different places then we can get a CLF, which will encircle all of them. If we incorporate the Union Theorem with the Eq. (8), then the modified form of the Eq. (8) can be expressed in the following way

$$\max_{\mathbf{s_0}, \mathbf{s_{01}}, \mathbf{s_1}, ..., \mathbf{s_n} \in \sum_\mathbf{n},\, V(x), s_{02} \in \mathcal{R}_n,\, \gamma \in \mathbb{R}^+} \beta_1, \beta_2, ..., \beta_n$$

$$\text{Subject to:} \qquad V - l_1 \in \sum\nolimits_n \tag{9a}$$

$$-\left[s_{01}\frac{\partial V}{\partial x}f + s_{02}\frac{\partial V}{\partial x}g + l_2\right] + (V-\gamma)s_0 \in \sum\nolimits_n \tag{9b}$$

$$-(V-\gamma) + s_1(p_1 - \beta_1) \in \sum\nolimits_n \tag{9c}$$

$$-(V-\gamma) + s_2(p_2 - \beta_2) \in \sum\nolimits_n \tag{9d}$$

$$\vdots$$

$$-(V-\gamma) + s_n(p_n - \beta_n) \in \sum\nolimits_n \tag{9n}$$

IV. Numerical Analysis

Equation (9) represents a bi-linear optimization problem since the decision variables V and s_0 (SOS polynomial) are linked, and both are linear in the equations. A two-way iterative search algorithm, known as $V - s$ iteration, is utilized to solve the problem. Further details about this algorithm can be found in [18]. The method employs three steps, namely the γ-step, β-step, and V-step, to produce the final result. We modified the original V-s algorithm to incorporate multiple SFs in the optimization process based on equations (9c)-(9n). The algorithm is outlined as follows:

V-s Iteration Algorithm:

1. γ-step: Solve for s_0, s_{01},s_{02} and γ^* for a given V and fixed l_2 :

$$\gamma^* = \max_{\mathbf{s_0}, \mathbf{s_{01}} \in \sum_n, s_{02} \in \mathcal{R}_n} \gamma \;\; \text{s.t. :}$$

$$-\left[s_{01}\frac{\partial V}{\partial x}f + s_{02}\frac{\partial V}{\partial x}g + l_2\right] + (V-\gamma)s_0 \in \sum\nolimits_n \tag{10}$$

2. β-step: Solve for $s_1, ..., s_n$ and $\beta_1^*, ..., \beta_n^*$ for a given V, p and using obtained γ^* from γ-step:

$$\left[\beta_1^*, , ..., \beta_n^*\right] = \max_{\mathbf{s_1}, ..., \mathbf{s_n} \in \sum_\mathbf{n}, \mathbf{s_{02}} \in \mathcal{R}_\mathbf{n}} \beta_1, ..., \beta_n \;\; \text{s.t. :}$$

$$-(V-\gamma^*) + (p_1 - \beta_1)s_1 \in \sum\nolimits_n \tag{11a}$$

$$\vdots$$

$$-(V-\gamma^*) + (p_n - \beta_n)s_n \in \sum\nolimits_n \tag{11n}$$

3. V-step: Using the obtained γ^*, $s_0, s_{01}, s_{02}, s_1, ..., s_n$ and $\beta_1^*, ..., \beta_n^*$ from previous two steps, solve for a new V which satisfies the following :

$$V - l_1 \in \sum\nolimits_n \tag{12a}$$

$$-\Big[s_{01}\frac{\partial V}{\partial x}f + s_{02}\frac{\partial V}{\partial x}g + l_2\Big] + (V-\gamma)s_0 \in \sum\nolimits_n \tag{12b}$$

$$-(V-\gamma^*) + (p_1 - \beta_1^*)s_1 \in \sum\nolimits_n \tag{12c}$$

$$\vdots$$

$$-(V-\gamma^*) + (p_n - \beta_n^*)s_n \in \sum\nolimits_n \tag{12n}$$

4. Scale V: Replace V with V/γ^* after each V-step. This scaling process roughly normalizes V and tends to keep the γ^* computed in the next step (γ-step) close to unity.

5. Repeat all the steps from $1-4$ using the scaled V from step 4 to the γ-step as an input. Continue the whole process until the final feasible CLF V is obtained.

Stopping Criteria of the Algorithm: The algorithm can be stopped in the following ways

A. Numerical Infeasibility: This criterion stops the algorithm by indicating that it is not possible to obtain any new CLF that satisfies all the constraints of step 3 (V-step).

B. Negligible increment of β: Reaching this condition terminates the algorithm if any two consecutive estimations of all β_i are less than a pre-defined tolerance value. This criteria can be used if the SF is fixed.

C. Fixed Iteration Number: The algorithm comes to an end after running through all the specified iteration numbers.

The algorithm may also stop for reasons other than those mentioned above, e.g., if it does not get the value of any β_i and related s_i in the β-step. This problem can be solved by changing the SF or placing the center of the SF in a new location.

Some discussions about the V-s algorithm are as follows:

Remark 1. The Union Theorem allows using multiple SFs to construct the CLF, as demonstrated in Eqs. (9c)-(9n), which also states that the CLF must encircle all the SFs, $p_1, ..., p_n$. In each iteration, the value of β for all SFs increases, and the newly generated CLF encompasses more regions by enclosing all of them. Although the algorithm may terminate if one SF touches the unstable region in a specific direction, by that time, the CLF has already captured a substantial area in other directions due to the presence of multiple SFs and their expansion in those directions.

Remark 2. To utilize the $V-s$ algorithm in the γ-step, an initial CLF V_0 is required, which can be obtained using different methods, including the Riccati equation [5]. Several rounds of iterations, as suggested by [19], can improve the estimation of the ROA. After running the algorithm, a final CLF V_{1R} and a larger value of β of the SF p_{1R} are obtained. However, even with multiple SFs, V_{1R} may not cover the entire ROA. To expand the area, the algorithm is rerun with V_{1R} as the initial CLF for the second round of iteration. It is important to note that the CLF may become saturated after some rounds of iteration and cannot capture more area, even with the additional round of iterations. It may also be the case that no new V that satisfies constraint (12) can be found, despite there being a significant stable region outside the captured region of V.

Remark 3. The following degrees are required to satisfy to get a feasible solution of (10)-(12), ([18] and [19]),

$$\deg(V) \geq \deg(l_1) \tag{13a}$$

$$\deg(p_i s_i) \geq \deg(V), \;\; [i = 1, ..., n] \tag{13b}$$

$$\deg(V s_0) \geq \deg(V f s_{01}) - 1 \tag{13c}$$

$$\deg(V s_0) \geq \deg(l_2) \tag{13d}$$

V. Selection of Shape Function

The SF is crucial for getting a better estimation of the ROA with CLF. The total iteration time also depends on the SF. Different types of SFs will give different CLFs with varying ROAs and total elapsed times. According to the definition, any positive definite function can be used as an SF. The below SF with a shifting center x^*, proposed in [19],

$$p(x) = (x - x^*)^T N (x - x^*) \tag{14}$$

The shape matrix N is used in this study with different values to obtain all the results. Although this method is straightforward, it yields satisfactory outcomes. It is important to note that using only one fixed value for N may not produce good results. Therefore, in each iteration round, we need to experiment with different values of N and compare the estimated ROA to choose the best one. Additionally, we use a method proposed in [19] to obtain the sifting center x^* of the SF. This method suggests that we can calculate x^* by solving the equation of a straight line that passes through the center and the bounded CLF. We have given the mathematical expression to choose the shifting center for the 2-D case only. However, the method can be applied for 3-D case also. Let us consider the equations of bounded CLF and 2-D straight line are,

$$V(x_1, x_2) = \gamma^* \tag{15}$$

$$x_2 = \tan\theta x_1 \tag{16}$$

Here, θ is the pre-defined inclination angle of the line, and for every shifting center, there is a fixed θ. Consider the solution point or intersection point of the above equations to be (x_{1I}^*, x_{2I}^*). The distance from the center to the point (x_{1I}^*, x_{2I}^*) is given by $\rho_\alpha = \sqrt{x_{1I}^* + x_{2I}^*}$. So, the shifting centre is,

$$x_1^* = \sigma\rho_\alpha \cos\theta \tag{17}$$

$$x_2^* = \sigma\rho_\alpha \sin\theta \tag{18}$$

Where, $\sigma \in (0, 1)$. More discussion about the selection of the value of σ can be found on [19] and [20].

VI. Simulation Results

Extensive simulations are conducted using the SOSOPT Toolbox [16] to evaluate the effectiveness of the suggested approach. To carry out the simulations, an initial CLF is required. It should be noted that a new initial/base CLF is required for each round of iteration. For the first round of iteration, the CLF is obtained by solving the Riccati equation for linearized dynamics. For subsequent rounds, the estimated CLF from the previous round is used as the initial/base CLF for the new round. The center of the SF, as represented by Eqs. (17) and (18), is necessary to initiate the simulation for each new round of iteration, as it changes with each round. To determine the center, x^*, equations (15) and (16) are solved with a pre-defined angle θ. The angles and the SF matrix, N, are selected based on error and trial methods. Moreover, we selected $deg(V) \rightarrow [min(2), max(2)]$, $deg(s_0) \rightarrow [min(2), max(4)]$, $deg(s_{01}) \rightarrow [min(0), max(2)]$, $deg(s_{02}) \rightarrow [min(1), max(4)]$, $deg(s_{1,\ldots,n}) \rightarrow [min(0), max(2)]$, $l_1 = l_2 = 10^{-4}(x^T x)$ and $\sigma = 0.8$ to start the simulation. The simulation is terminated by simultaneously employing the stopping criteria A and B. Additionally, we compared the results generated by the proposed method in this paper to those generated by another method, such as an origin-centered shape function.

Example: Consider the following nonlinear system [18]

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= -x_1 + \frac{1}{6}x_1^3 - u \end{aligned}$$

The results for this example have been obtained by conducting three rounds of iterations with one SF in the first and two SFs in the second and third rounds. The parameters used for this process are given in Table I.

TABLE I: Simulation Parameters

Round	Initial LF (V_0)	Angle ($\theta°$)	Shape Matrix (N)
1st(1R)	V_0 = From LD	Not Applicable	$N1_{1R} = N_0$
2nd(2R)	$V_0 = V_{1R}$	$\theta1_{2R}$=135	$N1_{2R} = N_1$
		$\theta2_{2R}$=325	$N2_{2R} = N_1$
3rd(3R)	$V_0 = V_{2R}$	$\theta1_{3R}$=135	$N1_{3R} = N_1$
		$\theta2_{3R}$=325	$N2_{3R} = N_1$

Where, $N_0 = \begin{bmatrix} 1/6 & 1/12 \\ 1/12 & 1/12 \end{bmatrix}$; $N_1 = \begin{bmatrix} 0.0159 & 0.005102 \\ 0.005102 & 0.0076278 \end{bmatrix}$

The final obtained CLF is $V(x) = 0.00077667x_1^2 + 0.0012246x_1x_2 + 0.00062324x_2^2$ with $\gamma = 0.9594$. CLFs and their corresponding SFs obtained at the 1st, 2nd, and 3rd rounds of iterations are presented in Figs. 1a, 1b and 1c correspondingly. The obtained CLF corresponding to the different rounds are compared in Fig.1d. It can be noted that the second and third rounds capture almost the same area, which indicates that further rounds were not required and the algorithm converged. The final result obtained from 3rd round is presented in Fig.1f. Comparison of results obtained by the proposed method and the results from [18] given in Fig.1e manifests that the proposed method estimates a significantly larger ROA. The total computational time for this simulation is 157 seconds, including 19, 6, and 2 iterations in the 1st, 2nd, and 3rd rounds of iteration, respectively.

VII. Conclusion

This study proposes a method that employs the Union Theorem in SOS optimization to construct a CLF with ROA. Through extensive simulations on a locally stabilizable system, the proposed technique has proven to be effective. The main advantage of this approach is that it constructs a CLF with a large ROA. However, the method has certain limitations at present. The outcome depends on the type (N) and number of SF, and the selection of SF centers can affect the outcome. This study manually selects SFs and angles related to their center positioning. Selecting these components in a more structured manner would potentially improve the results. The current research is investigating a more automatic (structured) approach to selecting SFs, optimizing the center, constructing the CLF, and designing a certified controller with ROA using the constructed CLF.

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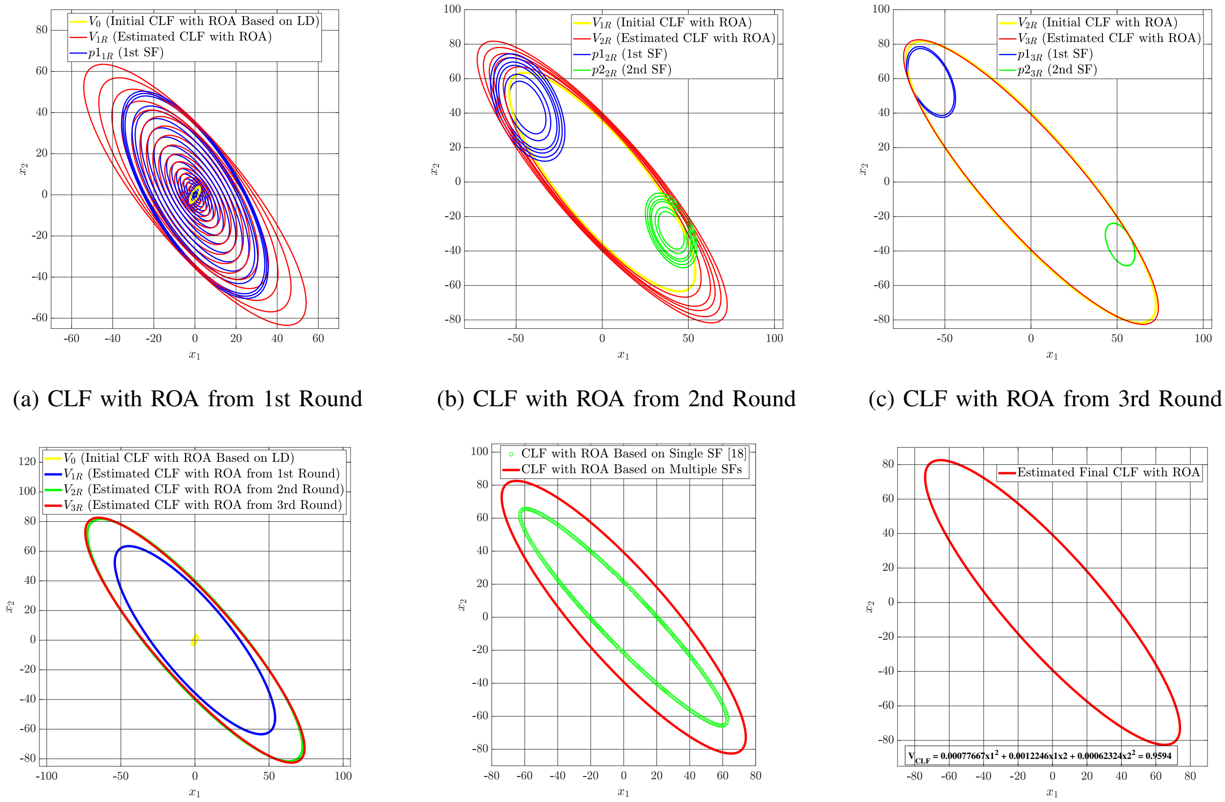

(a) CLF with ROA from 1st Round

(b) CLF with ROA from 2nd Round

(c) CLF with ROA from 3rd Round

(d) Comparison Among Results from all the Rounds

(e) Comparison Among Estimated CLFs with ROA based on Different Methods

(f) Final Estimated ROA (Obtained From 3rd Round)

Fig. 1: CLF with Estimated ROA Using Union Theorem

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Construction of control Lyapunov function with region of attraction using union theorem in sum-of-squares optimization

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