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Geometry of CMC surfaces of finite index

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Abstract: Given $r_0 > 0$, $I \in \mathbb{N} \cup \{0\}$, and $K_0, H_0 \geq 0$, let X be a complete Riemannian 3-manifold with injectivity radius $\mathrm{Inj}(X) \geq r_0$ and with the supremum of absolute sectional curvature at most K_0, and let $M \looparrowright X$ be a complete immersed surface of constant mean curvature $H \in [0, H_0]$ and with index at most I. We will obtain geometric estimates for such an $M \looparrowright X$ as a consequence of the hierarchy structure theorem. The hierarchy structure theorem (Theorem 2.2) will be applied to understand global properties of $M \looparrowright X$, especially results related to the area and diameter of M. By item E of Theorem 2.2, the area of such a noncompact $M \looparrowright X$ is infinite. We will improve this area result by proving the following when M is connected; here, $g(M)$ denotes the genus of the orientable cover of M:

(1) There exists $C_1 = C_1(I, r_0, K_0, H_0) > 0$, such that $\mathrm{Area}(M) \geq C_1(g(M) + 1)$.
(2) There exist $C > 0$, $G(I) \in \mathbb{N}$ independent of r_0, K_0, H_0, and also C independent of I such that if $g(M) \geq G(I)$, then $\mathrm{Area}(M) \geq \frac{C}{\left(\max\left\{1, \frac{1}{r_0}, \sqrt{K_0}, H_0\right\}\right)^2}(g(M) + 1)$.
(3) If the scalar curvature ρ of X satisfies $3H^2 + \frac{1}{2}\rho \geq c$ in X for some $c > 0$, then there exist $A, D > 0$ depending on c, I, r_0, K_0, H_0 such that $\mathrm{Area}(M) \leq A$ and $\mathrm{Diameter}(M) \leq D$. Hence, M is compact and, by item 1, $g(M) \leq A/C_1 - 1$.

Keywords: constant mean curvature, finite index H-surfaces, area estimates for constant mean curvature surfaces, hierarchy structure theorem, Bishop-Cheeger-Gromov relative volume comparison theorem, area of hyperbolic annuli

MSC 2020: Primary 53A10, Secondary 49Q05, 53C42

1 Introduction

Throughout the article, X denotes a complete Riemannian 3-manifold with positive injectivity radius $\mathrm{Inj}(X)$ and bounded absolute sectional curvature. Let M be a complete immersed surface in X of constant mean curvature $H \geq 0$, which we call an *H-surface* in X. The Jacobi operator of M is the Schrödinger operator

$$L = \Delta + |A_M|^2 + \mathrm{Ric}(N),$$

where Δ is the Laplace-Beltrami operator on M, $|A_M|$ is the norm of its second fundamental form, and $\mathrm{Ric}(N)$ denotes the Ricci curvature of X in the direction of the unit normal vector N to M; the index of M is the index of L,

$$\mathrm{Index}(M) = \lim_{r\to\infty} \mathrm{Index}(B_M(p, r)),$$

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where $B_M(p, r)$ is the intrinsic metric ball in M of radius $r > 0$ centered at a point $p \in M$, and $\mathrm{Index}(B_M(p, r))$ is the number of negative eigenvalues of L on $B_M(p, r)$ with Dirichlet boundary conditions. Here, we have assumed that the immersion is two sided (this holds in particular if $H > 0$). In the case, $H = 0$ and the immersion is one sided, then the index is defined in a similar manner using compactly supported variations in the normal bundle; see Definition 2.3 for details.

The primary goal of this article is to apply the hierarchy structure theorem 2.2 (proven in [9]) to understand certain global properties of closed constant mean curvature surfaces in Riemannian 3-manifolds. Theorem 2.2 describes the geometric structure of complete immersed H-surfaces $F : M \looparrowright X$ (also called *H-immersions*), which have a fixed bound $I \in \mathbb{N} \cup \{0\}$ on their index and a fixed upper bound H_0 for their constant mean curvature $H \geq 0$, in certain small intrinsic neighborhoods of points with sufficiently large norm $|A_M|$ of their second fundamental forms.

Our main applications of Theorem 2.2 appear in Theorems 1.1 and 3.5; these two theorems provide lower bounds for the areas and intrinsic diameters of immersed closed H-surfaces M in X of finite index in terms of their genera, when the indices and the constant mean curvatures of the surfaces are bounded from above by fixed constants. Theorem 3.5 also provides upper bounds for the area of balls $B_M(x, r)$ in M for every $x \in M$ and $r > 0$, independently on whether M is compact but depending on upper bounds for H and the index of M.

In the case that M is nonorientable, the genus $g(M)$ of M is the genus of its oriented cover.

Theorem 1.1. (Area and diameter estimates) *For $r_0 > 0$, $K_0, H_0 \geq 0$, consider all complete Riemannian 3-manifolds X with injectivity radius* $\mathrm{Inj}(X) \geq r_0$ *and absolute sectional curvature bounded from above by K_0, and let $\lambda = \max\left\{1, \frac{1}{r_0}, \sqrt{K_0}, H_0\right\}$. Let M be a complete immersed H-surface in X with empty boundary, $H \in [0, H_0]$, index at most $I \in \mathbb{N} \cup \{0\}$ and genus $g(M)$, which in the language of Theorem 2.2 implies $M \in \Lambda = \Lambda(I, H_0, r_0, 1, K_0)$ with additional chosen constant $\tau = \pi/10$. Then:*

(0) *The area of M is greater than C_A/λ^2, where*

$$C_A := \pi\left(\frac{\pi}{4}\right)^2 e^{-\frac{\pi}{2}-1+\frac{\pi}{4}} \approx 0.325043,$$

and if M is compact, the extrinsic diameter of each component of M is greater than $\frac{\pi}{4\lambda}$.

(1) (Item 1 in the abstract). *There exists $C_1(I) > 0$ (independent of M, r_0, K_0, H_0) such that:*

$$\mathrm{Area}(M) \geq \frac{C_1(I)}{\lambda^2}(g(M) + 1). \tag{1.1}$$

(2) (Item 2 in the abstract). *Let $C_s \geq 2\pi$ be the universal curvature estimate for stable H-surfaces described in Theorem 3.6 and let $C = \pi/(3 + 4C_s + 4C_s^2)$. There exists a $G(I) \in \mathbb{N}$, so that whenever $g(M) \geq G(I)$, then:*

$$\mathrm{Area}(M) \geq \frac{C}{\lambda^2}(g(M) + 1). \tag{1.2}$$

(3) (Item 3 in the abstract). *Suppose that the scalar curvature ρ of X satisfies $3H^2 + \frac{1}{2}\rho \geq c$ for some $c > 0$. Then, if M is connected and two sided, then M is compact, and furthermore, there exists $A_2(I, c) > 0$ such that:*

$$\mathrm{Area}(M) \leq \frac{A_2(I, c)}{\lambda^2} \quad \mathrm{Diameter}(M) \leq \frac{4\pi(I+1)}{\lambda\sqrt{3c}}, \quad g(M) \leq \frac{A_2(I, c)}{C_1(I)} - 1. \tag{1.3}$$

In the proof of Theorem 1.1, the estimates for the constants $C_1(I)$, $G(I)$, and $A_2(I, c)$ will be given in terms of the related constants $A_1(I)$, $\delta(I)$ given in the hierarchy structure theorem (for the value $\tau = \frac{\pi}{10}$) for the space $\Lambda(I, 1, 1, 1, 1)$ described in Definition 2.1.

There are a number of recent results in the literature related to area estimates for connected, closed, embedded minimal and constant mean curvature surfaces of finite index in a closed three-dimensional

Riemannian manifold, some of which include results described in Theorem 1.1 under more restrictive geometric hypotheses on the surfaces and/or the ambient space. Some of these recent results, obtained independently, can be found in the previous articles [1–5,7,11,15]. We refer the interested reader to [3] for further references in this active research area and for the general historical background that motivates this subject material.

2 The hierarchy structure theorem

In the sequel, we will denote by $B_X(x, r)$ (resp. $\overline{B}_X(x, r)$) the open (resp. closed) metric ball centered at a point $x \in X$ of radius $r > 0$. For a Riemannian surface M with smooth compact boundary ∂M,

$$\kappa(M) = \int_{\partial M} \kappa_g$$

will stand for the total geodesic curvature of ∂M, where κ_g denotes the pointwise geodesic curvature of ∂M with respect to the inward pointing unit conormal vector of M along ∂M.

Definition 2.1. For every $I \in \mathbb{N} \cup \{0\}$, $\varepsilon_0 > 0$, and $H_0, A_0, K_0 \geq 0$, we denote by

$$\Lambda = \Lambda(I, H_0, \varepsilon_0, A_0, K_0)$$

the space of all H-immersions $F : M \looparrowright X$ satisfying the following conditions:

(A1) X is a complete Riemannian 3-manifold with injectivity radius $\mathrm{Inj}(X) \geq \varepsilon_0$ and absolute sectional curvature bounded from above by K_0.

(A2) M is a complete surface with smooth boundary (possibly empty), and when $\partial M \neq \varnothing$, there is at least one point in M of distance ε_0 from ∂M.

(A3) $H \in [0, H_0]$ and F has index at most I.

(A4) If $\partial M \neq \varnothing$, then for any $\varepsilon \in (0, \infty]$, we let $U(\partial M, \varepsilon) = \{x \in M | d_M(x, \partial M) < \varepsilon\}$ be the open intrinsic ε-neighborhood of ∂M. Then, $|A_M|$ is bounded from above by A_0 in $U(\partial M, \varepsilon_0)$.

Suppose that $(F : M \looparrowright X) \in \Lambda$ and $\partial M \neq \varnothing$. For any positive $\varepsilon_1 \leq \varepsilon_2 \in [0, \infty]$, let

$$U(\partial M, \varepsilon_1, \varepsilon_2) = U(\partial M, \varepsilon_2) \backslash \overline{U(\partial M, \varepsilon_1)}, \quad \overline{U}(\partial M, \varepsilon_1, \varepsilon_2) = \overline{U(\partial M, \varepsilon_2)} \backslash U(\partial M, \varepsilon_1).$$

When $\partial M = \varnothing$, we define $U(\partial M, \varepsilon_1, \infty) = \overline{U}(\partial M, \varepsilon_1, \infty)$ as M.

In the next result, we will make use of harmonic coordinates $\varphi_x : U \to B_X(x, r)$ defined on an open subset U of $\mathbb{R}^3$ containing the origin, taking values in a geodesic ball $B_X(x, r)$ centered at a point $x \in X$ of radius $r \in (0, \mathrm{Inj}_X(x))$ (here, $\mathrm{Inj}_X(x)$ stands for the injectivity radius of X at x) and with a $C^{1,\alpha}$ control of the ambient metric on X, see Definition 2.4 for details.

Theorem 2.2. (Structure theorem for finite index H-surfaces [9]) *Given $\varepsilon_0 > 0$, $K_0, H_0, A_0 \geq 0$, $I \in \mathbb{N} \cup \{0\}$, and $\tau \in (0, \pi/10]$, there exist $A_1 \in [A_0, \infty)$, $\delta_1, \delta \in (0, \varepsilon_0/2]$ with $\delta_1 \leq \delta/2$, such that the following hold:*

For any $(F : M \looparrowright X) \in \Lambda = \Lambda(I, H_0, \varepsilon_0, A_0, K_0)$, there exists a (possibly empty) *finite collection $\mathcal{P}_F = \{p_1, \dots, p_k\} \subset U(\partial M, \varepsilon_0, \infty)$ of points, $k \leq I$, and numbers $r_F(1), \dots, r_F(k) \in [\delta_1, \frac{\delta}{2}]$ with $r_F(1) > 4r_F(2) > \dots > 4^{k-1}r_F(k)$, satisfying the following:*

(1) *<u>Portions with concentrated curvature :</u> Given $i = 1, \dots, k$, let Δ_i be the component of $F^{-1}(\overline{B}_X(F(p_i), r_F(i)))$ containing p_i. Then:*
   - (a) *$\Delta_i \subset \overline{B}_M(p_i, \frac{5}{4}r_F(i))$ (in particular, Δ_i is compact).*
   - (b) *Δ_i has smooth boundary and $F(\partial \Delta_i) \subset \partial \overline{B}_X(F(p_i), r_F(i))$.*
   - (c) *$B_M(p_i, \frac{7}{5}r_F(i)) \cap B_M(p_j, \frac{7}{5}r_F(j)) = \varnothing$ for $i \neq j$. In particular, the intrinsic distance between Δ_i, Δ_j is greater than $\frac{3}{10}\delta_1$ for every $i \neq j$.*

(*d*) $|A_M|(p_i) = \max_{\Delta_i}|A_M| = \max\{|A_M|(p) : p \in M\setminus \cup_{j=1}^{i-1} B_M(p_j, \frac{5}{4}r_F(j))\} \geq A_1$, *see* Figure 1.

(*e*) *The index* $\mathrm{Index}(\Delta_i)$ *of* Δ_i *is positive.*

(2) Transition annuli : *For* $i = 1,\dots,k$ *fixed, let* $e(i) \in \mathbb{N}$ *be the number of boundary components of* Δ_i. *Then, there exist planar disks* $\mathbb{D}_1,\dots,\mathbb{D}_{e(i)} \subset T_{F(p_i)}X$ *of radius* $2r_F(i)$ *centered at the origin in* $T_{F(p_i)}X$, *such that if we denote by*

$$P_{i,h} = \varphi_{F(p_i)}(\mathbb{D}_h), \quad h \in \{1, \dots, e(i)\},$$

(*here,* $\varphi_{F(p_i)}$ *denotes a harmonic chart centered at* $F(p_i)$, *see Definition* 2.4), *then*

$$F(\Delta_i) \cap [\overline{B}_X(F(p_i), r_F(i))\setminus B_X(F(p_i), r_F(i)/2)]$$

consists of $e(i)$ *annular multi-graphs*[1] $G_{i,1},\dots,G_{i,e(i)}$ *over their projections to* $P_{i,1},\dots,P_{i,e(i)}$, *with multiplicities* $m_{i,1},\dots m_{i,e(i)} \in \mathbb{N}$, *respectively, and whose related graphing functions* u *satisfy*

$$\frac{|u(x)|}{|x|} + |\nabla u|(x) \leq \tau, \tag{2.1}$$

where we have taken coordinates x *in each of the* $P_{i,h}$ *and denoted by* $|x|$ *the extrinsic distance to* $F(p_i)$ *in the ambient metric of* X, *see* Figure 2.

(3) Region with uniformly bounded curvature : $|A_M| < A_1$ *on* $\widetilde{M} := M\setminus\bigcup_{i=i}^{k}\mathrm{Int}(\Delta_i)$.

Moreover, the following additional properties hold:

(*A*) $\sum_{i=1}^{k} I(\Delta_i) \leq I$, *where* $I(\Delta_i) = \mathrm{Index}(\Delta_i)$.

(*B*) Geometric and topological estimates : *Given* $i = 1,\dots,k$, *let* $m(i) := \sum_{h=1}^{e(i)} m_{i,h}$ *be the total spinning of the boundary of* Δ_i, *let* $g(\Delta_i)$ *denote the genus of* Δ_i (in the case Δ_i *is nonorientable,* $g(\Delta_i)$ *denotes the genus of its oriented cover*[2]). *Then,* $m(i) \geq 2$ *and the following upper estimates hold:*

(*a*) *If* $I(\Delta_i) = 1$, *then* Δ_i *is orientable,* $g(\Delta_i) = 0$, *and* $(e(i), m(i)) \in \{(2, 2), (1, 3)\}$.

(*b*) *If* $I(\Delta_i) \geq 2$ *and* Δ_i *is orientable, then* $m(i) \leq 3I(\Delta_i) - 1$, $e(i) \leq 3I(\Delta_i) - 2$, *and* $g(\Delta_i) \leq 3I(\Delta_i) - 4$.

(*c*) If Δ_i *is nonorientable, then* $I(\Delta_i) \geq 2$, $m(i) \leq 3I(\Delta_i) - 1$, $e(i) \leq 3I(\Delta_i) - 2$, *and* $g(\Delta_i) \leq 6I(\Delta_i) - 8$.

(*d*) $\chi(\Delta_i) \geq -6I(\Delta_i) + 2m(i) + e(i)$, *and thus,* $\chi(\cup_{i=1}^{k}\Delta_i) \geq -6I + 2S + e$, *where*

$$e = \sum_{i=1}^{k} e(i), \quad S = \sum_{i=1}^{k} m(i).$$

(*e*) $|\kappa(\Delta_i) - 2\pi m(i)| \leq \frac{\tau}{m(i)}$, *and so, the total geodesic curvature* $\kappa(\widetilde{M})$ *of* $\widetilde{M}$ *along* $\partial\widetilde{M}\setminus\partial M$ *satisfies* $|\kappa(\widetilde{M}) + 2\pi S| \leq \frac{\tau}{2}k$, *and so,*

$$2\pi S - \frac{\tau}{2}k \leq \sum_{i=1}^{k} \kappa(\Delta_i) \leq 2\pi S + \frac{\tau}{2}k. \tag{2.2}$$

(*f*) $-\int_{\Delta_i} K > 3\pi$, *and so,*

$$-\int_{\cup_{i=1}^{k}\Delta_i} K = -2\pi\chi(\cup_{i=1}^{k}\Delta_i) + \int_{\cup_{i=1}^{k}\partial\Delta_i} \kappa_g > 3k\pi. \tag{2.3}$$

(*C*) Genus estimate outside the concentration of curvature : *If* M *is orientable,* $k \geq 1$ *and the genus* $g(M)$ *of* M *is finite, then the genus* $g(\widetilde{M})$ *of* $\widetilde{M}$ *satisfies* $0 \leq g(M) - g(\widetilde{M}) \leq 3I - 2$.

(*D*) Area estimate outside the concentration of curvature : *If* $k \geq 1$, *then*

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1 See Definition 2.5 for this notion of multigraph.

2 If Σ is a compact nonorientable surface and $\widehat{\Sigma} \stackrel{2:1}{\to} \Sigma$ denotes the oriented cover of Σ, then the genus of $\widehat{\Sigma}$ plus 1 equals the number of cross-caps in Σ.

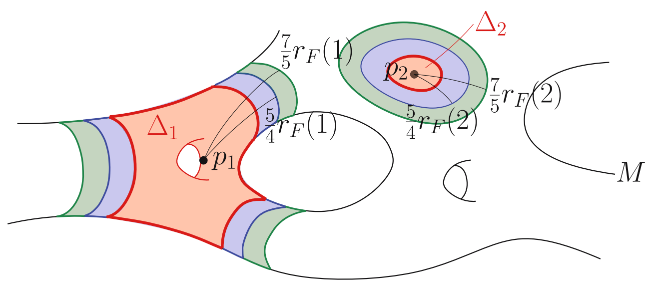


Figure 1: The second fundamental form concentrates inside the intrinsic compact regions Δ_i (in red), each of which is mapped through the immersion F to a surface inside the extrinsic ball in X centered at $F(p_i)$ of radius $r_F(i) > 0$, with $F(\partial\Delta_i) \subset \partial\overline{B}_X(F(p_i), r_F(i))$. Although the boundary $\partial\Delta_i$ might not be at constant intrinsic distance from the "center" p_i, Δ_i lies entirely inside the intrinsic ball centered at p_i of radius $\frac{5}{4}r_F(i)$. The intrinsic open balls $B_M(p_i, \frac{7}{5}r_F(i))$ are pairwise disjoint.

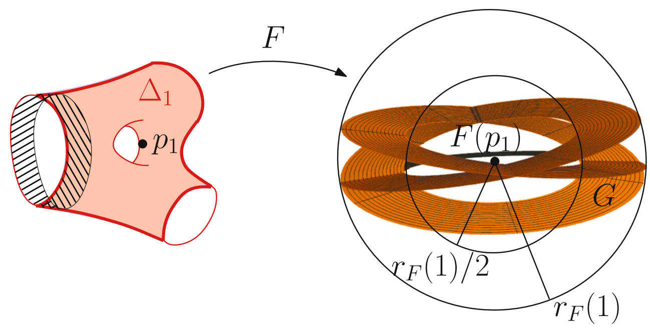


Figure 2: The transition annuli: On the right, one has the extrinsic representation in X of one of the annular multi-graphs G in $F(\Delta_1) \cap [\overline{B}_X(F(p_1), r_F(1)) \backslash B_X(F(p_1), r_F(1)/2)]$; in this case, the multiplicity of the multi-graph is 3. On the left, one has the intrinsic representation of the same annulus (shadowed); there is one such annular multi-graph for each boundary component of Δ_i.

$$\mathrm{Area}(\widetilde{M}) \geq 2\pi \sum_{i=1}^{k} m(i) r_F(i)^2 \geq \mathrm{Area}\left(\bigcup_{i=1}^{k} \Delta_i\right) \geq k\pi\delta_1^2.$$

(*E*) *There exists a $C > 0$, depending on ε_0, K_0, H_0, and independent of I, such that*

$$\mathrm{Area}(M) \geq \begin{cases} C \max\{1, \mathrm{Radius}(M)\} & \text{if } \partial M \neq \varnothing, \\ C \max\{1, \mathrm{Diameter}(M)\} & \text{if } \partial M = \varnothing, \end{cases} \tag{2.4}$$

where

$$\mathrm{Radius}(M) = \sup_{x \in M} d_M(x, \partial M) \in (0, \infty] \quad \text{if } \partial M \neq \varnothing,$$
$$\mathrm{Diameter}(M) = \sup_{x,y \in M} d_M(x, y) \quad \text{if } \partial M = \varnothing.$$

In particular, if M has infinite radius or if M has empty boundary and it is noncompact, then its area is infinite.

Definition 2.3. Given a one-sided minimal immersion $F : M \looparrowright X$, let $\widetilde{M} \to M$ be the two-sided cover of M and let $\tau : \widetilde{M} \to \widetilde{M}$ be the associated deck transformation of order 2. Denote by $\widetilde{\Delta}$, $|\widetilde{A}|^2$ the Laplacian and squared norm of the second fundamental form of $\widetilde{M}$, and let $N : \widetilde{M} \to TX$ be a unitary normal vector field. The index of F is defined as the number of negative eigenvalues of the elliptic, self-adjoint operator $\widetilde{\Delta} + |\widetilde{A}|^2 + \mathrm{Ric}(N, N)$ defined over the space of compactly supported smooth functions $\phi : \widetilde{M} \to \mathbb{R}$ such that $\phi \circ \tau = -\phi$.

Definition 2.4. Given a (smooth) Riemannian manifold X, a local chart $(x_1, \dots x_n)$ defined on an open set U of X is called *harmonic* if $\Delta x_i = 0$ for all $i = 1, \dots n$.

Following Definition 5 in [6], we make the next definition. Given $Q > 1$ and $\alpha \in (0, 1)$, we define the $C^{1,\alpha}$-*harmonic radius* at a point $x_0 \in X$ as the largest number $r = r(Q, \alpha)(x_0)$ so that on the geodesic ball $B_X(x_0, r)$ of center x_0 and radius r, there is a harmonic coordinate chart such that the metric tensor g of X is $C^{1,\alpha}$-controlled in these coordinates. Namely, if g_{ij}, $i, j = 1, \dots, n$, are the components of g in these coordinates, then

(1) $Q^{-1}\delta_{ij} \leq g_{ij} \leq Q\delta_{ij}$ as bilinear forms,

(2) $\sum_{\beta=1}^{3} r \sup_y |\frac{\partial g_{ij}}{\partial x_\beta}(y)| + \sum_{\beta=1}^{3} r^{1+\alpha} \sup_{y \neq z} \frac{\left| \frac{\partial g_{ij}}{\partial x_\beta}(y) - \frac{\partial g_{ij}}{\partial x_\beta}(z) \right|}{d_X(y,z)^\alpha} \leq Q - 1.$

The $C^{1,\alpha}$-harmonic radius $r(Q, \alpha)(X)$ of X is now defined by

$$r(Q, \alpha)(X) = \inf_{x_0 \in X} r(Q, \alpha)(x_0).$$

If the absolute sectional curvature of X is bounded by some constant $K_0 > 0$ and $\text{Inj}(X) \geq r_0 > 0$, then Theorem 6 in [6] implies that given $Q > 1$ and $\alpha \in (0, 1)$, there exists $C = C(Q, \alpha, r_0, K_0)$ (observe that C does not depend on X) such that $r(Q, \alpha)(X) \geq C$.

Definition 2.5. Let $f : \Sigma \looparrowright \mathbb{R}^3$ be an immersed annulus, P is a plane passing through the origin, and $\Pi : \mathbb{R}^3 \to P$ the orthogonal projection. Given $m \in \mathbb{N}$, let $\sigma_m : P_m \to P^* = P \setminus \{\vec{0}\}$ be the m-sheeted covering space of P^*. We say that Σ is an *m-valued graph* over P if $\vec{0} \notin (\Pi \circ f)(\Sigma)$ and $\Pi \circ f : \Sigma \to P^*$ has a smooth injective lift $\widetilde{f} : \Sigma \to P_m$ through σ_m; in this case, we say that Σ has *degree m* as a multi-graph.

Given $Q > 1$ and $\alpha \in (0, 1)$, let X be a Riemannian 3-manifold and (x_1, x_2, x_3) a harmonic chart for X defined on $B_X(x_0, r)$, $x_0 \in X, r > 0$, where the metric tensor g of X is $C^{1,\alpha}$-controlled in the sense of Definition 2.4. Let $P \subset B_X(x_0, r)$ be the image by this harmonic chart of the intersection of a plane in $\mathbb{R}^3$ passing through the origin with the domain of the chart. In this setting, the notion of m-valued graph over P generalizes naturally to an immersed annulus $f : \Sigma \looparrowright B_X(x_0, r)$, where the projection Π refers to the harmonic coordinates. If $f : \Sigma \looparrowright B_X(x_0, r)$ is an m-valued graph over P and u is the corresponding graphing function that expresses $f(\Sigma)$, we can consider the gradient ∇u with respect to the metric on P induced by the ambient metric of X. Both u and $|\nabla u|$ depend on the choice of harmonic coordinates around x_0 (and they also depend on Q), but if $\frac{|u(x)|}{|x|} + |\nabla u| < \tau$ for some $\tau \in (0, \pi/10]$ and $Q > 1$ sufficiently close to 1, then $\frac{|u(x)|}{|x|} + |\nabla u| < 2\tau$ for any other choice of harmonic chart around x_0 with this restriction of Q.

3 The proof of Theorem 1.1

This section is dedicated to the proof of Theorem 1.1. Note the complete surfaces considered in this theorem have empty boundary. Let $F : M \looparrowright X$ be an immersion as in the statement of Theorem 1.1.

We will use the notation in Theorem 2.2 and fix $\tau = \pi/10$. Notice that as the boundary of M is empty, then we may consider the H-immersion $F : M \looparrowright X$ of index I described in Theorem 1.1 to be an element of $\Lambda(I, H_0, r_0, 1, K_0)$, where H_0, r_0, K_0 are given in the hypotheses of Theorem 1.1.

3.1 Normalizing the space Λ

After scaling the Riemannian metric of X by the square root of

$$\lambda = \max\left\{1, \frac{1}{r_0}, \sqrt{K_0}, H_0\right\}, \tag{3.1}$$

one obtains a new Riemannian manifold X'; note that this scaling of the metric scales arc length in X by the factor $\lambda \geq 1$, and that the metric of M induced by the isometric immersion F creates an associated isometric immersion $F' : M \looparrowright X'$ such that $F'(p) = F(p)$ for each $p \in M$. After this homothetic change of the metric, we can consider $F' : M \looparrowright X'$ to be an immersion satisfying the following properties:

(1) $\mathrm{Inj}(X') \geq 1$.
(2) The absolute sectional curvature of X' is less than or equal to 1.
(3) F' is an isometric immersion of constant mean curvature $H' \in [0, 1]$.
(4) $(F' : M \looparrowright X') \in \Lambda(I, 1, 1, 1, 1)$.
(5) $\mathrm{Area}(F) = \lambda^2\ \mathrm{Area}(F')$.
(6) $\mathrm{Diameter}(F) = \lambda\ \mathrm{Diameter}(F')$.

Items 5 and 6 allow us to easily convert estimates on the area of subdomains and lengths of curves in the domain of F to areas and lengths of the corresponding domains and curves in the domain of F', and thereby, these conversion formulae reduce the proofs of statements given in Theorem 1.1 for $F \in \Lambda(I, H_0, r_0, 1, K_0)$ to the corresponding estimates for F' in $\Lambda(I, 1, 1, 1, 1)$. Thus, for the remainder of the proof of Theorem 1.1, we will assume $F : M \looparrowright X$ lies in $\Lambda(I, 1, 1, 1, 1)$ and refer to Area(M) and Diameter(M) for those with respect to the induced metric by F.

3.2 Proof of item 0 of Theorem 1.1

Consider an element $(F : M \looparrowright X) \in \Lambda(I, 1, 1, 1, 1)$. If M is noncompact, then the last sentence in item E of Theorem 2.2 states that M has infinite area, which proves that the inequality Area(M) $\geq C_A$ in item 0 holds vacuously (for any choice of $C_A > 0$). If moreover M is connected, then there exists a geodesic ray in M, i.e., an embedded, length-minimizing unit-speed geodesic arc $\gamma : [0, \infty) \to M$; in particular, the diameter of M is infinite, and thus, the second statement in item 0 also holds vacuously.

For the remainder of this section, we will assume that M is compact.

Lemma 3.1. *Given $x_0 \in M$, let $M(x_0)$ be the component of M containing x_0. Then, $M(x_0)$ is not contained in the closed extrinsic ball $\overline{B}_X(x_0, \pi/4)$ (in particular, $\partial B_M(x_0, \pi/4)$ is not empty).*

Remark 3.2. Observe that if the lemma holds, then the extrinsic diameter of M is greater than $\pi/4$ (in particular, the intrinsic diameter has the same lower bound), which proves the second statement in item 0 of Theorem 1.1.

Proof of Lemma 3.1. Fix a point $x_0 \in M$ and let $r \in (0, \pi/4)$. Since the injectivity radius of X is at least 1, all the distance spheres $\partial B_X(x_0, r)$ with $r \in (0, 1)$ are geodesic spheres. By comparison results and since the absolute sectional curvature of X is bounded by 1, the second fundamental form of $\partial B_X(x_0, r)$ has normal curvatures greater than 1. Assume that $M(x_0)$ is contained in $\overline{B}_X(x_0, r)$. As $M(x_0)$ is compact, then there exists a largest $r_1 \in (0, r]$ such that $M(x_0) \subset \overline{B}_X(x_0, r_1)$, and there exists $x \in M(x_0) \cap \partial B_X(x_0, r_1)$. This implies that all the normal curvatures of M at x are greater than 1, which implies that the mean curvature of M is greater than 1, which contradicts that $F : M \looparrowright X$ lies in $\Lambda(I, 1, 1, 1, 1)$. This contradiction proves that $M(x_0)$ cannot be contained in $\overline{B}_X(x_0, r)$. Since this holds for every $r \in (0, \pi/4)$ and M is compact, we conclude that $M(x_0)$ cannot be contained in $B_X(x_0, \pi/4)$. In fact, $M(x_0)$ cannot be contained in $\overline{B}_X(x_0, \pi/4)$ (otherwise the maximum principle for the mean curvature operator would imply that $M(x_0) = \partial B_X(x_0, r)$, which contradicts that $x_0 \in M(x_0)$). Now the lemma is proved. □

Using [8, Proposition 2.5 and item 3 of Remark 2.5] with $R_1 = a = H_0 = 1$, for each $p \in \mathrm{Int}(M)$, we have

$$\text{Area}[B_M(p,r)] \geq E(r) := \pi r^2 e^{-2r-1+r\cot(r)} \quad \text{for every } r \in (0,\pi/4]. \tag{3.2}$$

Therefore, since $M \not\subset \overline{B}_X(p,\pi/4)$ by Lemma 3.1, the extrinsic diameter of M is greater than $\pi/4$ and $\text{Area}(M) > \text{Area}[B_M(p,\pi/4)] = E(\pi/4) \approx 0.325043$. This completes the proof of item 0 of Theorem 1.1.

3.3 Proof of item 1 of Theorem 1.1

Consider an element $(F : M \looparrowright X) \in \Lambda(I,1,1,1,1)$. If M is noncompact, then the last sentence in item E of Theorem 2.2 states that M has infinite area, which vacuously implies item 1 of the theorem holds (for any choice of $C_1(I)$). Henceforth, assume M is compact.

Let $M = M_1 \cup \ldots \cup M_b$, $b \in \mathbb{N}$, be the decomposition of M in connected components. Assume inequality (1.1) holds for each M_i with respect to a constant $C_1 = C_1(I)$. Since the index of each M_i is at most I, then

$$\text{Area}(M) = \sum_{i=1}^{b} \text{Area}(M_i) \geq \sum_{i=1}^{b} C_1(g(M_i)+1) = C_1(g(M)+b) \geq C_1(g(M)+1), \tag{3.3}$$

where $g(M_i)$ is the genus of M_i. Hence, it suffices to prove that (1.1) holds under the additional assumption that M is connected, which we will assume henceforth.

The region $\widetilde{M} \subset M$ defined in item 3 of Theorem 2.2 for the space $\Lambda(I,1,1,1,1)$ produces a uniform bound $A_1 = A_1(I) \geq 1$ from above on the norm the second fundamental form of $\widetilde{M}$. Let us define

$$K_1 = K_1(I) := -1 - \frac{1}{2}A_1^2. \tag{3.4}$$

Since $A_1 \geq 1$, then $K_1 \leq -\frac{3}{2}$. The Gauss equation gives

$$K = K_X(TM) + \det(A_M), \tag{3.5}$$

where K denotes the Gaussian curvature of M and $K_X(TM)$ is the sectional curvature of X for the tangent plane to M. Since the absolute sectional curvature of X is bounded by 1, $H^2 \geq \det(A)$ and $H \in [0,1]$, we have the following upper and lower estimates for K in $\widetilde{M}$:

$$K_1 \leq -1 - \frac{1}{2}|A_M|^2 \leq -1 + \det(A_M) \leq K \leq 1 + \det(A_M) \leq 1 + H^2 \leq 2. \tag{3.6}$$

3.3.1 Item 1 holds when $k = 0$

We first show that item 1 of Theorem 1.1 holds in the special case that the integer k defined in Theorem 2.2 is zero. To see this, observe that $\widetilde{M} = M$, and thus, (3.6) ensures that M has Gaussian curvature bounded from below by K_1 and from above by 2. Let $\widehat{M}$ be the orientable cover of M.

Suppose $g(M) = 0$ (recall that M was assumed to be compact and connected). By applying to $\widehat{M}$ the Gauss-Bonnet theorem, we have

$$2 \cdot \text{Area}(\widehat{M}) \geq \int_{\widehat{M}} K = 4\pi.$$

If M is nonorientable, then $\text{Area}(M) = \frac{1}{2}\text{Area}(\widehat{M}) \geq \pi$. This inequality also holds in the case M is orientable (in fact, $M = \widehat{M}$ and so, $\text{Area}(M) = \text{Area}(\widehat{M}) \geq 2\pi$). Therefore, inequality (1.1) holds with $C_1(I) = \pi$ if $g(M) = 0$ and $k = 0$.

Suppose now that $g(M) \geq 2$. Hence, Gauss-Bonnet applied to $\widehat{M}$ gives

$$-K_1 \cdot \text{Area}(\widehat{M}) \geq -\int_{\widehat{M}} K = -2\pi\chi(\widehat{M}) = 4\pi(g(\widehat{M})-1) = 4\pi(g(M)-1) \geq \frac{4\pi}{3}(g(M)+1).$$

If M is nonorientable, then $\mathrm{Area}(M) = \frac{1}{2}\mathrm{Area}(\widehat{M}) \geq \frac{2\pi}{3|K_1|}(g(M)+1)$. This inequality also holds in the case M is orientable (in fact, $M = \widehat{M}$ and thus, $\mathrm{Area}(M) = \mathrm{Area}(\widehat{M}) \geq \frac{4\pi}{3|K_1|}(g(M)+1)$). Therefore, inequality (1.1) holds with $C_1(I) = \frac{2\pi}{3|K_1|}$ if $g(M) \geq 2$ and $k = 0$.

By the already proven item 0 of Theorem 1.1, the area of M is at least C_A. In particular, if $g(M) = 1$ (i.e., M is a torus or a Klein bottle), then one can still obtain a lower bound estimate for the area of M by

$$\mathrm{Area}(M) \geq C_A = \frac{C_A}{2}(g(M)+1).$$

Therefore, inequality (1.1) holds with $C_1(I) = \frac{C_A}{2}$ if $g(M) = 1$ and $k = 0$.

Finally, we consider the minimum of the constants $\pi, \frac{2\pi}{3|K_1|}, \frac{C_A}{2}$ obtained in the three cases mentioned earlier. As observed previously, $|K_1| \geq \frac{3}{2}$, and so,

$$C_3 = C_3(I) = \min\left\{\pi, \frac{2\pi}{3|K_1|}, \frac{C_A}{2}\right\} = \min\left\{\frac{2\pi}{3|K_1|}, \frac{C_A}{2}\right\},$$

we deduce that (1.1) holds with C_3, instead of C_1, for connected compact M when $k = 0$.

3.3.2 Item 1 holds when $k \geq 1$

Assume that $k \geq 1$ (in particular, $I \geq 1$), and we will obtain a constant $C_4 = C_4(I) \in (0, \pi\delta_1^2)$ that satisfies

$$\mathrm{Area}(M) \geq C_4(g(M)+1), \tag{3.7}$$

which will complete the proof of item 1 of Theorem 1.1 after setting $C_1(I) = \min\{C_3(I), C_4(I)\}$. We will need the following two claims.

Claim 3.3. If $g(M) \geq 12I - 3$, then inequality (3.7) holds with constant $C_4'(I) = \frac{\pi}{|K_1(I)|}$.

Claim 3.4. If $g(M) < 12I - 3$. then inequality (3.7) holds with $C_4''(I) = \frac{C_A}{12I-3}$.

Proof of Claim 3.3. We start applying the Gauss-Bonnet formula and (3.6):

$$|K_1|\cdot\mathrm{Area}(\widetilde{M}) \geq \left|\int_{\widetilde{M}} K\right| = \left|\int_M K - \int_{\cup_{i=1}^k \Delta_i} K\right|. \tag{3.8}$$

On the other hand, calling $g = g(M)$,

$$\begin{aligned}\int_M K - \int_{\cup_{i=1}^k \Delta_i} K &= 2\pi\chi(M) - 2\pi\chi(\cup_{i=1}^k \Delta_i) + \int_{\partial(\cup_{i=1}^k \Delta_i)} \kappa_g \\ &\leq 2\pi(1-g) + 2\pi(6I - 2S - e) + 2\pi S + \tau k,\end{aligned} \tag{3.9}$$

where in the last inequality, we have used item $B(d)$ of Theorem 2.2 and (2.2). Since $\tau \leq \pi/10$ in Theorem 2.2 and $S + e \geq 4k$ (this last inequality follows since $e(i) \geq 1$ and $m(i) \geq 2$, and if $e(i) = 1$, then $m(i) \geq 3$), and we can bound (3.9) from above by $2\pi(1 - g + 6I - 4k + \frac{k}{20})$, which in turn is at most $2\pi(-g + 6I - 2)$ because $k \geq 1$. Therefore,

$$\int_M K - \int_{\cup_{i=1}^k \Delta_i} K \leq -\pi(2g - 12I + 4). \tag{3.10}$$

Since $g \geq 12I - 3$ by hypothesis, then the right-hand side (RHS) of (3.10) is at most $-\pi(g+1)$, and thus, we conclude that

$$\int_M K - \int_{\cup_{i=1}^k \Delta_i} K \leq -\pi(g+1). \tag{3.11}$$

Now, (3.11) and (3.8) give

$$|K_1|\cdot \mathrm{Area}(\widetilde{M}) \geq \pi(g+1), \tag{3.12}$$

from where Claim 3.3 follows. □

Proof of Claim 3.4. By the already proven item 0 of Theorem 1.1, we have $\mathrm{Area}(M) \geq C_A$, which is $\geq \frac{C_A}{12I-3}(g(M)+1)$ because $g(M) < 12I - 3$. This finishes the proof of Claim 3.4. □

Once Claims 3.3 and 3.4 are proved, we will conclude that inequality (3.7) holds in all cases with $k \geq 1$ with $C_4(I) = \min\{C_4'(I), C_4''(I)\}$. This completes the proof of item 1 of Theorem 1.1.

3.4 A preliminary result on area estimates of balls in *M* and its diameter

We temporarily pause the proof of Theorem 1.1 to state and prove the next auxiliary result, which gives general upper estimates on the areas of balls of radius r in M and general upper estimates on the diameter of M in terms of constants described in the hierarchy structure theorem 2.2. The next theorem will be crucial in the proofs of the remaining items 2 and 3 of Theorem 1.1. The proof of Theorem 3.5 will be given in Sections 3.5–3.7.

Theorem 3.5. (Area estimates for intrinsic balls and diameter estimates for M) *For $r_0 > 0$, $K_0, H_0 \geq 0$, consider all complete Riemannian 3-manifolds X with injectivity radius $\mathrm{Inj}(X) \geq r_0$ and absolute sectional curvature bounded from above by K_0, and let $\lambda = \max\left\{1, \frac{1}{r_0}, \sqrt{K_0}, H_0\right\}$. Let M be a complete immersed H-surface in X with empty boundary, $H \in [0, H_0]$, index at most $I \in \mathbb{N} \cup \{0\}$ and genus $g(M)$, which in the language of Theorem 2.2 implies $M \in \Lambda = \Lambda(I, H_0, r_0, 1, K_0)$ with additional chosen constant $\tau = \pi/10$. Then:*

(1) *Suppose that one of the following two conditions holds:*
  (*i*) *$I = 0$, i.e., M is stable.*
  (*ii*) *$I \geq 1$ and $k = 0$ with the notation of Theorem 2.2 (in particular, $M = \widetilde{M}$).*
  Depending on whether condition (i) or (ii) holds, we introduce the following constant $K_1 = K_1(I)$. If condition (i) holds, let $C_s \geq 2\pi$ be the universal curvature estimate for stable H-surfaces described in Theorem 3.6, and let $K_1 := -1 - \frac{1}{2}C_s^2$. If condition (ii) holds, let $K_1 = K_1(I) = -1 - \frac{1}{2}A_1^2$, where $A_1 = A_1(I) \geq 1$ is the constant given by Theorem 2.2 for the space $\Lambda(I, 1, 1, 1, 1)$.
  For all $x \in M$ and $r > 0$,

$$\mathrm{Area}(B_M(x,r)) \leq \frac{2\pi}{-K_1\lambda^2}[\cosh(\lambda\sqrt{-K_1}\,r) - 1], \tag{3.13}$$

  and if M is connected, then

$$\mathrm{Diameter}(M) \geq \frac{1}{\sqrt{K_1}\,\lambda}\operatorname{arccosh}\left[\frac{-K_1 C_1(I)}{2\pi}(g(M)+1)+1\right]. \tag{3.14}$$

(2) *Suppose $k \geq 1$ (in particular, $I \geq 1$). Let $\delta = \delta(I) \in (0, \frac{1}{2}]$ be the constant described in Theorem 2.2 for the space $\Lambda(I, 1, 1, 1, 1)$, and let $\Delta_1, \dots, \Delta_k$, $k \leq I$, be the smooth compact domains associated to M introduced in item 1 of Theorem 2.2. There exists $A_3(I) \geq 6$, independent of M, r_0, K_0, H_0, such that for all $x \in M$ and all $r > 0$, the following estimates hold:*

$$\mathrm{Area}\left(B_M(x,r)\setminus \bigcup_{i=1}^{k}\Delta_i\right) \leq \frac{2(6\pi+1)}{\lambda^2 A_3(I)} I\left[2[\cosh(\lambda\sqrt{A_3(I)}\,r)-1]+\frac{\sinh(\lambda\sqrt{A_3(I)}\,r)}{A_3(I)^{1/2}}\right], \tag{3.15}$$

$$\mathrm{Area}(B_M(x,r)) \leq \frac{I}{\lambda^2}\left[\frac{2(6\pi+1)}{A_3(I)}\left(2[\cosh(\lambda\sqrt{A_3(I)}\,r)-1]+\frac{\sinh(\lambda\sqrt{A_3(I)}\,r)}{A_3(I)^{1/2}}\right)+\frac{3\pi}{8}\right], \tag{3.16}$$

and if M is connected, then

$$\mathrm{Diameter}(M) \geq \frac{1}{\lambda\sqrt{A_3(I)}}\operatorname{arccosh}\left[\frac{C_1(I)}{20I}(g(M)+1)\right]. \tag{3.17}$$

3.5 The proof of item 1 of Theorem 3.5

It is worth recalling some aspects related to curvature estimates for complete stable H-surfaces Σ (possibly with boundary) in complete Riemannian 3-manifolds of absolute sectional curvature at most 1 and injectivity radius at most 1; such curvature estimates are independent on the value of the (constant) mean curvature. Rosenberg et al. [14, Main Theorem] proved that there exists a universal constant $C_s' > 0$ such that if Σ is two-sided, then for any point $p \in \Sigma$ of distance at least 1 from $\partial\Sigma$, then $|A_M|(p) \leq C_s'$. In [9], we generalized this curvature estimate to include the case where M is not necessarily two-sided. Namely, we proved the following statement.

Theorem 3.6. (Curvature estimate for stable H-surfaces [9]) *There exists $C_s'' \geq 2\pi$ such that given $K_0 > 0$ and a complete Riemannian 3-manifold (Y, g) of bounded sectional curvature $|K| \leq K_0$, then for any immersed one-sided stable minimal surface $M \looparrowright Y$ and for any $p \in M$,*

$$|A_M|(p) \leq \frac{C_s''}{\min\{\mathrm{Inj}_Y(p),\, d_M(p,\partial M),\, \frac{\pi}{2\sqrt{K_0}}\}}. \tag{3.18}$$

Let $C_s := \max\{C_s', C_s''\}$. Given $\varepsilon_0 > 0$, $K_0 \geq 0$, if X is a complete Riemannian 3-manifold with injectivity radius at least ε_0 and bounded sectional curvature $|K| \leq K_0$, and $F : M \looparrowright X$ is a stable H-immersion, then

$$|A_M|(p) \leq \frac{C_s}{\min\{\varepsilon_0,\, d_M(p,\partial M),\, \frac{\pi}{2\sqrt{K_0}}\}}. \tag{3.19}$$

Consider an element $(F : M \looparrowright X) \in \Lambda(I = 0, 1, 1, 1, 1)$, in particular, M is stable. Particularizing (3.19) to the case $\varepsilon_0 = 1$, $\partial M = \varnothing$, $K_0 = 1$, we obtain that $|A_M| \leq C_s$ in M. By the same argument using the Gauss equation as in (3.4) and (3.6), we deduce that the Gaussian curvature K of M satisfies $K \geq K_1 := -1 - \frac{1}{2}C_s^2$ in M. In this setting, the Bishop-Cheeger-Gromov relative volume comparison theorem [12, Lemma 36] implies that for every $x \in M$ and $r > 0$,

$$\mathrm{Area}(B_M(x,r)) \leq \mathrm{Area}\left(\mathbb{B}_{K_1}(r)\right) = \frac{2\pi}{-K_1}[\cosh(\sqrt{-K_1}\,r)-1], \tag{3.20}$$

where $\mathbb{B}_{K_1}(r)$ denotes the metric ball of radius r in the hyperbolic plane of curvature $K_1 < 0$. This finishes the proof of inequality (3.13) provided that $I = 0$ (i.e., assuming condition 1(i) in Theorem 3.5 holds).

Now suppose $(F : M \looparrowright X) \in \Lambda(I, 1, 1, 1, 1)$, $I \geq 1$. To prove (3.13) provided that condition 1(ii) in Theorem 3.5 holds, observe that in this case $M = \widetilde{M}$. By (3.6), $K \geq K_1$, where $K_1 = -1 - \frac{1}{2}A_1^2$ and $A_1 = A_1(I) \geq 1$ is given by Theorem 2.2 for the space $\Lambda(I, 1, 1, 1, 1)$. By applying the aforementioned arguments to this new choice of the constant K_1, we obtain that (3.20) holds, which proves (3.13) provided that 1(ii) in Theorem 3.5 holds.

To show (3.14) (regardless of whether condition 1(i) or 1(ii) in Theorem 3.5 holds), assume M is connected. Observe that we can assume that M is compact (otherwise its diameter is infinite by the argument in the first paragraph of Section 3.2 and (3.14) holds vacuously). Choose a point $x \in M$. By taking $r = \text{Diameter}(M) := D$ in (3.20) and using the already proven inequality (1.1), we obtain

$$C_1(I)(g(M)+1) \leq \text{Area}(M) = \text{Area}(B_M(x, D)) \overset{(3.20)}{\leq} \frac{2\pi}{-K_1}[\cosh(\sqrt{-K_1}D) - 1],$$

or equivalently,

$$D \geq \frac{1}{\sqrt{K_1}} \operatorname{arccosh}\left[\frac{-K_1 C_1(I)}{2\pi}(g(M)+1) + 1\right],$$

which proves inequality (3.14), and so, finishes the proof of item 1 of Theorem 3.5.

3.6 Area growth of collar neighborhoods of $\widetilde{M}$ if $k \geq 1$

Definition 3.7. For a complete surface Σ with boundary $\partial\Sigma$ and for any $r > 0$, let

$$\Sigma(r) = \{x \in \Sigma | d_\Sigma(x, \partial\Sigma) \leq r\}$$

be the collar neighborhood of $\partial\Sigma$ in Σ of radius r.

Consider an element $(F : M \looparrowright X) \in \Lambda(I, 1, 1, 1, 1)$. Assume $k \geq 1$ with the notation of Theorem 2.2. Hence, $\widetilde{M}$ is a surface with smooth boundary. For later uses, next we will give an upper estimate for the area growth of the collar neighborhood $\widetilde{M}(r)$ of $\widetilde{M}$, $r > 0$.

Proposition 3.8. *Let $c_1, \dots, c_e$ be the set of components of $\partial\widetilde{M}$. Choose for each $i \in \{1, \dots, e\}$ a parametrization by arc length $\gamma_i : [0, L_i] \to c_i$ with associated geodesic curvature function $\kappa_i(t)$ with respect to the inward pointing unit conormal vector $\eta = \eta(t)$ of $\widetilde{M}$ along $\partial\widetilde{M}$. Then, for each $i \in \{1, \dots, e\}$:*
(1) *$\kappa_i(t)$ is negative in $[0, L_i]$.*
(2) *There exists a complete annulus Σ_i with boundary, with constant Gaussian curvature K_1 (this constant is defined in (3.4)), whose boundary is parameterized by the arc length by $\hat{\gamma}_i = \hat{\gamma}_i(t) : [0, L_i] \to \partial\Sigma_i$, and such that the geodesic curvature function $\hat{\kappa}_i$ of $\partial\Sigma_i$ with respect to the inward pointing unit conormal vector of Σ_i along $\partial\Sigma_i$ satisfies $\hat{\kappa}_i(t) = \kappa_i(t)$ for all $t \in [0, L_i]$. Furthermore, Σ_i is unique up to isometry.*
(3) *For each $r > 0$, we have*

$$\text{Area}(\widetilde{M}(r)) \leq \sum_{i=1}^{e} \text{Area}(\Sigma_i(r)) = \kappa(\widetilde{M}) \frac{1 - \cosh(\sqrt{-K_1}r)}{-K_1} + L \frac{\sinh(\sqrt{-K_1}r)}{(-K_1)^{3/2}}, \tag{3.21}$$

where $\kappa_g(t) = \sum_{i=1}^{e} \kappa_i(t)$ and $\kappa(\widetilde{M}) = \int_{\partial\widetilde{M}} \kappa_g$ is the total geodesic curvature of ∂M with respect to the inward pointing unit conormal vector of $\widetilde{M}$ along $\partial\widetilde{M}$, and $L = \sum_{i=1}^{e} L_i$ is the length of $\partial\widetilde{M}$.

Proof. Item 1 follows from the proof of the hierarchy structure theorem 2.2 in [9]; specifically, see Lemma 6.4 in [9].

Item 2 follows directly from the following two facts. First, given $\ell > 0$ and a smooth function $\kappa : [0, \ell] \to (-\infty, 0)$, standard geometry of curves in the hyperbolic plane $\mathbb{H}^2(K_1)$ with constant Gaussian curvature K_1 ensures that there exists a smooth unit speed curve $\alpha : [0, \ell] \to \mathbb{H}^2(-K_1)$ such that $\kappa(t)$ is the geodesic curvature of α at $\alpha(t)$, for all $t \in [0, \ell]$; furthermore, α is unique up to isometries of $\mathbb{H}^2(K_1)$. Second, if $n : [0, \ell] \to U\mathbb{H}^2(K_1)$ is the unit normal vector to α pointing to its nonconvex side (here $U\mathbb{H}^2(K_1)$ denotes the unit tangent bundle to $\mathbb{H}^2(K_1)$), then the map $\phi : [0, \ell] \times [0, \infty) \to \mathbb{H}^2(K_1)$

$$\phi(t, r) = \exp_{\alpha(t)}(rn(t)), \quad (t, r) \in [0, \ell] \times [0, \infty) \tag{3.22}$$

is a submersion, where $\exp: T\mathbb{H}^2(K_1) \to \mathbb{H}^2(K_1)$ is the exponential map. ϕ induces a hyperbolic metric g_h on $[0, L] \times [0, \infty)$ so that for each $t_0 \in [0, \ell]$, the curve of the form $r \in [0, \infty) \mapsto \phi(t_0, r)$ is a unitary geodesic orthogonal to the arc $\alpha([0, L])$ at $\alpha(t_0)$; in particular, after identifying the two geodesic arcs $\phi(\{0\} \times [0, \infty))$ and $\phi(\{\ell\} \times [0, \infty))$ by a hyperbolic isometry, we obtain a quotient hyperbolic annulus (Σ_α, g_h) with the properties desired in item 2, in the special case that $\ell = L_i$ and $\kappa = \kappa_i$.

To prove item 3 of the proposition, first observe that since $K \geq -K_1$ on $\widetilde{M}(r)$ by (3.6), we can use relative volume comparison arguments [12, Lemma 36] to deduce that

$$\operatorname{Area}(\widetilde{M}(r)) \leq \sum_{i=1}^{e} \operatorname{Area}(\Sigma_i(r)).$$

It remains to prove that for all $i = 1, \dots, e$ and $r > 0$, the following holds

$$\operatorname{Area}(\Sigma_i(r)) = \frac{1}{-K_1}\left[(1 - \cosh(\sqrt{-K_1}\, r))\int_0^{L_i} \kappa(s)\mathrm{d}s + \frac{L_i}{\sqrt{-K_1}} \sinh(\sqrt{-K_1}\, r)\right]. \tag{3.23}$$

Claim 3.9. Let $\alpha : [0, \ell] \to \mathbb{H}^2(-1)$ a smooth arc parameterized by arc length, with negative geodesic curvature function $\kappa = \kappa(s)$. Consider the complete hyperbolic annulus with boundary (Σ_α, g_h) constructed in (3.22) in terms of α with $K_1 = -1$. Given $r > 0$, let $\alpha_r : [0, \ell] \to \Sigma_\alpha$ be the equidistant arc to α at distance r on the nonconvex side of α. Then:

(1) The geodesic curvature function κ_r of α_r is given by

$$\kappa_r(s) = \frac{\kappa(s) - \tanh(r)}{1 - \tanh(r)\kappa(s)}, \quad \forall s \in [0, \ell]. \tag{3.24}$$

(2) Let $\Sigma_\alpha(r) \subset \Sigma_\alpha$ be the domain enclosed by α_r and the two geodesics of Σ_α that join the extrema of α_r (so that these geodesics are orthogonal to both α_r at their extrema). Then,

$$\operatorname{Area}(\Sigma_\alpha(r)) = (1 - \cosh(r))\int_0^{\ell} \kappa(s)\mathrm{d}s + \ell \sinh(r), \tag{3.25}$$

where $\ell = L(\alpha)$ is the length of α.

(3) If we replace $\mathbb{H}^2(-1)$ by $\mathbb{H}^2(K_1)$, then (3.25) becomes

$$\operatorname{Area}(\Sigma_\alpha(r)) = \frac{1}{-K_1}\left[(1 - \cosh(\sqrt{-K_1}\, r))\int_0^{\ell} \kappa(s)\mathrm{d}s + \frac{\ell}{\sqrt{-K_1}} \sinh(\sqrt{-K_1}\, r)\right]. \tag{3.26}$$

Proof of the claim. Recall that Σ_α submerses into $\mathbb{H}^2(-1)$ through the map ϕ given in (3.22). In particular, $\Sigma_\alpha \setminus \partial\Sigma_\alpha$ is locally isometric to $\mathbb{H}^2(-1)$. This property clearly allows us to prove the claim assuming that $\Sigma_\alpha(r)$ embeds into $\mathbb{H}^2(-1)$: for item 1 of the claim, this is obvious, while for items 2 and 3, we can divide $[0, \ell]$ into a partition $0 = s_0 < s_1 < \ldots < s_n = \ell$ such that if we denote by $\alpha_i = \alpha\,|_{[s_{i-1}, s_i]}$, $i = 1, \dots, n$ and we apply the same procedure as with Σ_α to construct n “rectangles” $\Sigma_{\alpha_i}(r)$, then each $\Sigma_{\alpha_i}(r)$ embeds into $\mathbb{H}^2(-1)$. In this way, both equations (3.25) and (3.26) will follow by adding up the corresponding equalities over the rectangles $\Sigma_{\alpha_1}(r), \dots, \Sigma_{\alpha_n}(r)$, which only intersect along geodesics in their boundaries. Therefore, for the remainder of this proof, we will assume that $\Sigma_\alpha(r)$ is embedded in $\mathbb{H}^2(-1)$.

We will use the model of $\mathbb{H}^2(-1)$ as the upper sheet of a hyperboloid in the Lorentz-Minkowski space $\mathbb{L}^3 = (\mathbb{R}^3, \langle,\rangle_L = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 - \mathrm{d}x_3^2)$. In this model, $\mathbb{H}^2(-1) = \{x \in \mathbb{L}^3 | \langle x, x\rangle_L = -1, x_3 > 0\}$, and the induced metric by $\langle,\rangle_L$ on $\mathbb{H}^2(-1)$ is positive definite and has constant Gaussian curvature -1. Given $x \in \mathbb{H}^2(-1)$, the tangent plane $T_x\mathbb{H}^3$ identifies to $\langle x\rangle^\perp \subset \mathbb{L}^3$. Given $x \in \mathbb{H}^2(-1)$ and $v \in \langle x\rangle^\perp$, the unique geodesic in $\mathbb{H}^2(-1)$ with initial conditions $\gamma(0) = x$, $\gamma'(0) = v$ is

$$\gamma(t) = \gamma(t, x, v) = \cosh(|v|t)x + \frac{\sinh(|v|t)}{|v|}v, \tag{3.27}$$

and the parallel transport along $\gamma(\cdot, x, v)$ from 0 to t is given by

$$\tau_0^t : T_x\mathbb{H}^2(-1) = \langle x\rangle^\perp \to T_{\gamma(t)}\mathbb{H}^2 = \langle \gamma(t)\rangle^\perp, \quad \tau_0^t(w) = w + \frac{\langle v, w\rangle_L}{|v|^2}(\dot{\gamma}(t) - v), \tag{3.28}$$

where $\dot{\gamma}(t) = \frac{d\gamma}{dt}$.

Given $s \in [0, \ell]$, the equidistant curve α_r is

$$\alpha_r(s) = \exp_{\alpha(s)}(r\mathcal{R}\alpha'(s)) = \gamma(r, \alpha(s), \mathcal{R}\alpha'(s)) = \cosh(r)\alpha(s) + \sinh(r)\mathcal{R}\alpha'(s),$$

where $\mathcal{R}$ is the rotation of angle $\pi/2$ in each tangent plane to $\mathbb{H}^2(-1)$ so that $\mathcal{R}\alpha'$ points to the nonconvex side of α. $\alpha_r'(s) = J_s(r)$ is the value at $t = r$ of the unique Jacobi field $J_s = J_s(t)$ along the geodesic $t \mapsto \gamma(t, \alpha(s), J\alpha'(s))$ with initial conditions

$$J_s(0) = \alpha'(s), \quad \frac{DJ_s}{dt}(0) = \frac{D(\mathcal{R}\alpha')}{ds}(s) = -\kappa(s)\alpha'(s), \tag{3.29}$$

where $\{\alpha', n_\alpha = \mathcal{R}\alpha'\}$ is the Frenet dihedron for α. Since both $J_s(0)$ and $\frac{DJ_s}{dt}(0)$ are orthogonal to $\dot{\gamma}(0)$, we deduce that $J_s(t)$ is everywhere orthogonal to $\dot{\gamma}(t)$. In particular, $J_s(t) = f_s(t)\tau_0^t(\alpha'(s))$, where $f_s(t)$ is a solution of the ordinary differential equation $\ddot{f} - f = 0$. Imposing (3.29), we have $f_s(0) = 1$, $\dot{f}_s(0) = -\kappa(s)$, and thus,

$$f_s(t) = \cosh(t) - \kappa(s)\sinh(t). \tag{3.30}$$

Therefore, an orthogonal basis of the tangent and normal line to α_r is

$$\alpha_r'(s) = J_s(r) = f_s(r)\tau_0^r(\alpha'(s)) \overset{(3.28)}{=} f_s(r)\alpha'(s), \quad \text{(not necessarily unitary)}$$
$$n_{\alpha_r}(s) = \tau_0^r(n_\alpha(s)) \overset{(3.28)}{=} n_\alpha(s) = \mathcal{R}\dot{\gamma}(r) = \sinh(r)\alpha(s) + \cosh(r)\mathcal{R}\alpha'(s), \quad \text{(unitary)}$$

Hence, by the Frenet equations for α_r, we will obtain the negative of the geodesic curvature $\kappa_r(s)$ of α_r by taking the derivative with respect to s to $n_{\alpha_r}(s)$ and dividing by $|\alpha_r'(s)| = f_s(r)$:

$$-\kappa_r(s) = \frac{\frac{d}{ds}(n_{\alpha_r}(s))}{f_s(r)} = \frac{\sinh(r) - \cosh(r)\kappa(s)}{\cosh(r) - \kappa(s)\sinh(r)},$$

which proves the first item of the claim.

As for item 2, observe that the interior of $\Sigma_\alpha(r)$ is topologically a disk, and that $\partial\Sigma_\alpha(r)$ contains four cusps, in each of which the exterior angle to $\Sigma_\alpha(r)$ along its boundary is $\pi/2$. By applying the Gauss-Bonnet theorem to $\Sigma_\alpha(r)$, we obtain

$$0 = -\text{Area}(\Sigma_\alpha(r)) + \int_\alpha \kappa + \int_{\alpha_r} \kappa_r = -\text{Area}(\Sigma_\alpha(r)) + \int_0^\ell \kappa(s)ds - \int_0^\ell \kappa_r(s)ds. \tag{3.31}$$

Using (3.24),

$$\begin{aligned} \int_0^\ell \kappa_r(s)ds &= \int_0^\ell \frac{\kappa(s) - \tanh(r)}{1 - \tanh(r)\kappa(s)}|\alpha_r'(s)|ds \\ &\overset{(3.30)}{=} \int_0^\ell (\cosh(r)\kappa(s) - \sinh(r))ds \\ &= \cosh(r)\int_0^\ell \kappa(s)ds - \sinh(r)\ell. \end{aligned} \tag{3.32}$$

(3.31) and (3.32) give (3.25), which finishes the proof of item 2 of the claim. Item 3 follows from (3.25) after an elementary rescaling argument. ☐

Equation (3.23) follows directly from (3.26) with the obvious change of notation $\Sigma_i(r) = \Sigma_\alpha(r)$, $L_i = \ell$. This finishes the proof of Proposition 3.8. ☐

3.7 Proof of item 2 of Theorem 3.5

Consider an element $(F : M \looparrowright X) \in \Lambda(I, 1, 1, 1, 1)$. Assume $k \geq 1$ (hence $I \geq 1$) with the notation of Theorem 2.2. By (3.21), we have for each $r > 0$

$$\text{Area}(\widetilde{M}(r)) \leq f(r), \tag{3.33}$$

where f is the increasing function

$$f(r) = \frac{\kappa(\widetilde{M})}{K_1}[\cosh(\sqrt{-K_1}r) - 1] + L\frac{\sinh(\sqrt{-K_1}r)}{(-K_1)^{3/2}}.$$

Lemma 3.10. *Given $x \in M$ and $r > 0$, we have*

$$\text{Area}\left[B_M(x, r)\backslash\left(\bigcup_{i=1}^{k}\Delta_i\right)\right] \leq f(2r).$$

Proof. Suppose first that $x \in \cup_{i=1}^k \Delta_i$. Then, $B_M(x, r) \subset \widetilde{M}(r) \cup (\cup_{i=1}^k \Delta_i)$, and thus, $B_M(x, r)\backslash(\cup_{i=1}^k \Delta_i) \subset \widetilde{M}(r)$. Hence,

$$\text{Area}\,[B_M(x, r)\backslash(\cup_{i=1}^k \Delta_i)] \leq \text{Area}\,(\widetilde{M}(r)) \overset{(3.33)}{\leq} f(r) < f(2r).$$

Now suppose $x \in \text{Int}(\widetilde{M})$ and let $d > 0$ be the distance from x to $\cup_{i=1}^k \Delta_i$. We distinguish two cases, depending on whether $r \leq d$.

If $r \leq d$, then since $K \geq K_1$ in $\widetilde{M}$ and $B_M(x, r) \subset \widehat{M}$, the Bishop-Cheeger-Gromov relative volume comparison theorem implies

$$\text{Area}\,(B_M(x, r)) \leq \text{Area}\left(\mathbb{B}_{K_1}(r)\right) = \frac{2\pi}{-K_1}[\cosh(\sqrt{-K_1}r) - 1] \overset{(\star)}{<} f(r) < f(2r),$$

where $\mathbb{B}_{K_1}(r)$ denotes the metric ball of radius r in the hyperbolic plane of curvature K_1, and in $(\star)$ we have used that $|\kappa(\widetilde{M})| > 2\pi$.

If $r > d$, then the triangle inequality ensures that $B_M(x, r)\backslash(\cup_{i=1}^k \Delta_i) \subset \widetilde{M}(2r)$, and thus,

$$\text{Area}\left[B_M(x, r)\backslash\left(\bigcup_{i=1}^{k}\Delta_i\right)\right] \leq \text{Area}\,(\widetilde{M}(2r)) \overset{(3.33)}{\leq} f(2r),$$

which finishes the proof of the lemma. □

Now we are ready to prove inequality (3.15). First, observe that (2.2) implies that

$$-2\pi S - \tau I \leq \kappa(\widetilde{M}) \leq -2\pi S + \tau I,$$

and so,

$$0 < \frac{\kappa(\widetilde{M})}{K_1} \leq \frac{2\pi S + \tau I}{-K_1} \overset{(\star)}{\leq} \frac{6\pi + \tau}{-K_1}I \leq \frac{6\pi + 1}{-K_1}I, \tag{3.34}$$

where in $(\star)$ we have used that $S \leq 3I$ (this follows from item B of Theorem 2.2).

Second, we can estimate from above the length L of $\partial\widetilde{M}$ as follows: each component c_j of $\partial\widetilde{M}$ is contained in the boundary of a certain compact set Δ_i, and the length $L(\partial\Delta_i)$ of $\partial\Delta_i$ can be estimated from above using [9, item (C1) of Lemma 6.1] (also see [9, Remark 6.2]) as follows:

$$L(\partial\Delta_i) \leq (2\pi m(i) + 1)r_F(i) \leq \frac{2\pi m(i) + 1}{2}\delta, \tag{3.35}$$

where $m(i)$ is the total spinning of the boundary of Δ_i ($m(i)$ was introduced in item B of Theorem 2.2) and $\delta \leq \varepsilon_0/2 = 1/2$, $\delta_1 \leq \frac{\delta}{2}$, and $r_F(i) \in [\delta_1, \delta/2]$ were introduced in the main statement of Theorem 2.2. By adding up (3.35) in the set $\{\Delta_1, \dots, \Delta_k\}$, we obtain

$$L \le \frac{2\pi S + k}{2}\delta \le \frac{6\pi I + I}{2}\delta = \frac{(6\pi + 1)\delta}{2} I \le \frac{6\pi + 1}{4} I. \tag{3.36}$$

Finally, Lemma 3.10, (3.34), and (3.36) give

$$\begin{aligned}
\text{Area}[B_M(x,r)\setminus(\cup_{i=1}^k \Delta_i)] &\overset{\text{(Lemma 3.10)}}{\le} f(2r) = \frac{\kappa(\widetilde{M})}{K_1}[\cosh(2\sqrt{-K_1}r) - 1] + L\frac{\sinh(2\sqrt{-K_1}r)}{(-K_1)^{3/2}} \\
&\overset{(3.34),(3.36)}{\le} \frac{6\pi + 1}{-K_1} I[\cosh(2\sqrt{-K_1}r) - 1] + \frac{6\pi + 1}{4} I \frac{\sinh(2\sqrt{-K_1}r)}{(-K_1)^{3/2}} \\
&= \frac{2(6\pi + 1)}{A_3(I)} I \left[2[\cosh(\sqrt{A_3(I)}r) - 1] + \frac{\sinh(\sqrt{A_3(I)}r)}{A_3(I)^{1/2}} \right],
\end{aligned}$$

where we have defined $A_3(I) = -4K_1(I) \ge 6$. This proves (3.15).

To see that inequality (3.16) holds, just observe that by item D of Theorem 2.2,

$$\text{Area}\left(\bigcup_{i=1}^k \Delta_i \right) \le 2\pi \sum_{i=1}^k m(i) r_F^2(i) \le 2\pi S \frac{\delta^2}{4} \le \frac{\pi S}{8} \le \frac{3\pi}{8} I, \tag{3.37}$$

and hence,

$$\begin{aligned}
\text{Area}[B_M(x,r)] &\le \text{Area}[B_M(x,r)\setminus(\cup_{i=1}^k \Delta_i)] + \text{Area}(\cup_{i=1}^k \Delta_i) \\
&\le \frac{2(6\pi + 1)}{A_3(I)} I \left[2[\cosh(\sqrt{A_3(I)}r) - 1] + \frac{\sinh(\sqrt{A_3(I)}r)}{A_3(I)^{1/2}} \right] + \frac{3\pi}{8} I \\
&= I \left[\frac{2(6\pi + 1)}{A_3(I)} \left(2[\cosh(\sqrt{A_3(I)}r) - 1] + \frac{\sinh(\sqrt{A_3(I)}r)}{A_3(I)^{1/2}} \right) + \frac{3\pi}{8} \right],
\end{aligned} \tag{3.38}$$

from where one deduces (3.16).

To finish this section, we prove (3.17). Let us denote by $h(r)$ the RHS of (3.38). Then, taking $r = D := \text{Diameter}(M)$ we have $B_M(x, D) = M$ for any $x \in M$, and so,

$$C_1(I)(g + 1) \overset{(1.1)}{\le} \text{Area}(M) = \text{Area}[B_M(x, D)] \le h(D). \tag{3.39}$$

Since $\sinh(t) \le \cosh(t)$,

$$\begin{aligned}
h(D) &\le I \left[\frac{2(6\pi + 1)}{A_3(I)} \left(2[\cosh(\sqrt{A_3(I)}D) - 1] + \frac{\cosh(\sqrt{A_3(I)}D)}{A_3(I)^{1/2}} \right) + \frac{3\pi}{8} \right] \\
&\overset{(A_3(I)\ge 6)}{\le} I \left[\frac{6\pi + 1}{3} \left(2[\cosh(\sqrt{A_3(I)}D) - 1] + \frac{\cosh(\sqrt{A_3(I)}D)}{\sqrt{6}} \right) + \frac{3\pi}{8} \right] \\
&= I \left[\frac{6\pi + 1}{3} \left(\left(2 + \frac{1}{\sqrt{6}} \right) \cosh(\sqrt{A_3(I)}D) - 2 \right) + \frac{3\pi}{8} \right] \\
&< I \frac{6\pi + 1}{3} \left(2 + \frac{1}{\sqrt{6}} \right) \cosh(\sqrt{A_3(I)}D).
\end{aligned} \tag{3.40}$$

(3.39) and (3.40) give

$$C_1(I)(g + 1) \le I \frac{6\pi + 1}{3} \left(2 + \frac{1}{\sqrt{6}} \right) \cosh(\sqrt{A_3(I)}D),$$

from where inequality (3.17) follows directly. This completes the proof of Theorem 3.5.

3.8 Proof of item 2 of the Theorem 1.1

Consider an element $(F : M \looparrowright X) \in \Lambda(I, 1, 1, 1, 1)$. As in previous sections, we may assume that M is compact and connected, and let $g = g(M)$ be the genus of M.

Claim 3.11. If $I = 0$, then (1.2) holds with $G(0) = 0$.

Proof. Since M is stable, we have $|A_M| \le C_s$ in M by (3.19). Thus, the Gauss equation implies that the Gaussian curvature K of M satisfies $K \ge -1 - \frac{1}{2}C_s^2$. By using the Gauss-Bonnet theorem, we obtain

$$\left(1 + \frac{1}{2}C_s^2\right)\mathrm{Area}(M) \ge -\int_M K = -2\pi\chi(M) \ge 2\pi(g-1).$$

Hence,

$$\mathrm{Area}(M) \ge \frac{2\pi}{1 + \frac{1}{2}C_s^2}(g-1),$$

which is strictly bigger than $\frac{\pi}{3+4C_s+4C_2^2}(g+1)$ when $g \ge 2$. Consequently, (1.2) holds whenever $g \ge 2$. To finish the proof of the claim, it remains to check that (1.2) holds for $g = 0, 1$, which we do next.

$$\mathrm{Area}(M) \overset{\text{(item 0)}}{\ge} C_A \overset{\text{(a)}}{>} \frac{2\pi}{3+4C_s+4C_2^2} \overset{\text{(b)}}{\ge} \frac{2\pi}{3+4C_s+4C_2^2}(g+1)$$

where in (a) we have used that $C_s \ge 2\pi$, and in (b) that $g \le 1$. Now the claim is proved. □

By Claim 3.11, it remains to prove item 2 of Theorem 1.1 assuming $I \ge 1$. The additional assumption $g \ge 12I - 3$ guarantees, by (3.11), that

$$\int_{\widetilde{M}} K = \int_M K - \int_{\cup_{i=1}^k \Delta_i} K \le -\pi(g+1). \tag{3.41}$$

By Lemma 7.1 in [9], there exists a positive constant $\widehat{C}_s(1)$, which in our setting is $1 + 2C_s$, such that if $\sup|A_M| > \widehat{C}_s(1)$, then there exists a nonempty finite subset $\{q_1, \dots, q_n\} \subset M$ with $1 \le n \le I$, such that

(1) $|A_M|$ achieves its maximum in M at q_1, and for $i = 2, \dots, n$, $|A_M|$ achieves its maximum in $M \setminus [B_M(q_1, 1) \cup \dots \cup B_M(q_{i-1}, 1)]$ at q_i.
(2) For each $i = 1, \dots, n$, $|A_M|(q_i) > \widehat{C}_s(1)$ and the intrinsic balls $B_M(q_i, 1/2)$ are pairwise disjoint and unstable.
(3) $|A_M| \le \widehat{C}_s(1)$ in $M \setminus [B_M(q_1, 1) \cup \dots \cup B_M(q_n, 1)]$.

We next define a partition of the surface $\widetilde{M}$ that appears in item 3 of Theorem 2.2.

- If $\sup|A_M| \le \widehat{C}_s(1)$, let $\widetilde{M}_1 = \varnothing$ and $\widetilde{M}_2 = \widetilde{M}$.
- Otherwise, let $\widetilde{M}_1 = \widetilde{M} \cap [\cup_{i=1}^n B_M(q_i, 1)]$ and $\widetilde{M}_2 = \widetilde{M} \setminus [\cup_{i=1}^n B_M(q_i, 1)]$.

In particular, the second fundamental form of the surface $\widetilde{M}_2$ satisfies $|A_{\widetilde{M}_2}| \le \widehat{C}_s(1)$.

By the discussion around inequality (3.6), the Gaussian curvature function of $\widetilde{M}_1$ satisfies $K_{\widetilde{M}_1} \ge K_1$ (where $K_1 = K_1(I) \le -3/2$ is defined in (3.4)), and the Gaussian curvature function of $\widetilde{M}_2$ satisfies $K_{\widetilde{M}_2} \ge -1 - \frac{1}{2}\widehat{C}_s(1)^2$. Also, by inequalities (3.13) and (3.16), there exists an explicit function $h : \mathbb{N} \to (0, \infty)$ such that

$$\mathrm{Area}[\widetilde{M} \cap [\cup_{i=1}^n B_M(q_i, 1)]] \le \sum_{i=1}^n \mathrm{Area}\, B_M(q_i, 1) \le h(I). \tag{3.42}$$

Therefore,

$$K_1(I)h(I) - \left[1 + \frac{1}{2}\widehat{C}_s(1)^2\right]\mathrm{Area}(\widetilde{M}_2) \overset{(3.42)}{\le} \int_{\widetilde{M}_1} K_{\widetilde{M}_1} + \int_{\widetilde{M}_2} K_{\widetilde{M}_2} = \int_{\widetilde{M}} K \overset{(3.41)}{\le} -\pi(g+1). \tag{3.43}$$

Solving for the area of $\widetilde{M}_2$, we have

$$\mathrm{Area}(\widetilde{M}_2) \ge \frac{\pi(g+1) + K_1(I)h(I)}{1 + \frac{1}{2}\widehat{C}_s(1)^2}. \tag{3.44}$$

After setting $C := \frac{\pi/2}{1+\frac{1}{2}\widehat{C}_s(1)^2} = \frac{\pi}{3+4C_s+4C_s^2}$, we obtain the estimate

$$\mathrm{Area}(M) \ge \mathrm{Area}(\widetilde{M}) \ge \mathrm{Area}(\widetilde{M}_2) \ge C(g+1) + \frac{\frac{\pi}{2}(g+1) + K_1(I)h(I)}{1 + \frac{1}{2}\widehat{C}_s(1)^2}. \tag{3.45}$$

Define

$$G(I) := \max\left\{12I - 3, \left\lceil \frac{-2K_1(I)h(I)}{\pi} \right\rceil - 1\right\} \in \mathbb{N},$$

where for a real number x, we denote by $\lceil x \rceil$ the smallest integer that is not smaller than x (also known as the *ceiling function* at x). Then, whenever $g(M) \ge G(I)$, we have that the second term in the RHS of (3.45) is nonnegative, which completes the proof of item 2 of Theorem 1.1.

3.9 Proof of item 3 of Theorem 1.1

Recall that in Section 3.1, we normalized the space Λ, passing from an H-immersion $(F : M \looparrowright X) \in \Lambda(I, H_0, r_0, 1, K_0)$ to the immersion $(F' : M \looparrowright X') \in \Lambda(I, 1, 1, 1, 1)$, where λ is given by (3.1) and X' is the Riemannian manifold obtained after scaling the original metric of X by $\sqrt{\lambda}$. Observe that F' has mean curvature $H' = H/\lambda$ and X' has scalar curvature $\rho' = \rho/\lambda^2$.

Suppose that the scalar curvature ρ of X satisfies $3H^2 + \frac{1}{2}\rho \ge c$ in X for some $c > 0$, where H is the mean curvature of an immersion $(F : M \looparrowright X) \in \Lambda(I, H_0, r_0, 1, K_0)$. In this setting, Rosenberg [13], see also [10, Theorem 2.12] proved that every stable, two-sided subdomain $\Omega \subset M$ satisfies

$$d_M(x, \partial\Omega) \le \frac{2\pi}{\sqrt{3c}} := R_c, \tag{3.46}$$

for all $x \in \Omega$. Since $3(H')^2 + \frac{1}{2}\rho' \ge c' := c/\lambda^2$, the estimate (3.46) applied to the same stable subdomain Ω viewed inside the domain of F' gives that the intrinsic distance in the metric induced by F' from any $x \in \Omega$ to $\partial\Omega$ is at most $\frac{2\pi}{\sqrt{3c'}} = \frac{2\pi\lambda}{\sqrt{3c}}$. This linear scaling on the upper bound for the intrinsic radius of stable subdomains allows us to reduce item 3 of Theorem 1.1 to the following statement.

Proposition 3.12. *Let $F : M \looparrowright X$ be an H-immersion in $\Lambda(I, 1, 1, 1, 1)$, where M is connected and two-sided. Let $C_1(I)$ be the constant described in item 1 of Theorem 1.1 and R_c be as defined in (3.46). If the scalar curvature ρ of X satisfies $3H^2 + \frac{1}{2}\rho \ge c$ in X for some $c > 0$, then M is compact, and there exists $A_2(I, c) > 0$ such that*

$$\mathrm{Area}(M) \le A_2(I, c) \quad \mathrm{Diameter}(M) \le 2(I+1)R_c, \quad g(M) \le \frac{A_2(I, c)}{C_1(I)} - 1. \tag{3.47}$$

Proof. We first show that M is compact. Arguing by contradiction, suppose M is noncompact. Since M is complete, there is a geodesic ray in M, i.e., an embedded, length-minimizing unit-speed geodesic arc $\gamma : [0, \infty) \to M$. Consider the infinite collection

$$C(n) = \{B_M(\gamma(2jn), n) | j \in \mathbb{N} \cup \{0\}\}$$

of pairwise disjoint open intrinsic balls in M. Since the index of M is at most I, then the subcollection of unstable balls in $C(n)$ is finite. This implies that M contains stable balls of arbitrarily large radius, a property which contradicts that for $r > R_c$, $B_M(x, r)$ cannot be stable as follows from (3.46). Therefore, M is compact.

We will divide the proof of (3.47) into two claims.

Claim 3.13. Diameter$(M) \leq 2(I + 1)R_c$ (i.e., the second inequality in (3.47) holds). □

Proof of Claim 3.13. Arguing by contradiction, suppose that there exist points $p, q \in M$ at intrinsic distance $L := d_M(p, q) > 2(I + 1)R_c$. Let $\Gamma : [0, L] \to M$ be a geodesic arc parameterized by arc length, such that $\Gamma(0) = p$ and $\Gamma(L) = q$. Choose $R > R_c$ such that $2(I + 1)R \leq L$. Consider the following collection of $I + 1$ pairwise disjoint open intrinsic balls in M (Figure 3).

$$C'(n) = \{B_M(\Gamma((2j - 1)R), R) \mid j = 1, \dots, I + 1\}.$$

Since the index of M is at most I and the $I + 1$ balls in $C'(n)$ are pairwise disjoint, we deduce that at least one of these balls is stable. This contradicts that $R > R_c$ and (3.46). This contradiction proves the claim. □

Claim 3.14. Let $\widetilde{h} = \widetilde{h}(I, r) : (\mathbb{N} \cup \{0\}) \times (0, \infty) \to (0, \infty)$ be the maximum of the right-hand sides of (3.13) and (3.16). Then, the first and third inequalities in (3.47) hold for $A_2(I, c) = \widetilde{h}(I, 2(I + 1)R_c)$.

Proof of Claim 3.14. Observe that $r \mapsto \widetilde{h}(I, r)$ is increasing. Take $x \in M$. By the already proven inequalities (3.13) and (3.16), we have $\text{Area}(B_M(x, r)) \leq \widetilde{h}(I, r)$ for all $r > 0$. By applying this estimate to the choice $r = D :=$ Diameter$(M) < \infty$ (observe that $M = B_M(x, D)$) and using that $\widetilde{h}(I, r)$ is increasing in r, we obtain

$$\text{Area}(M) \leq \widetilde{h}(I, D) \overset{\text{(Claim 3.3)}}{\leq} \widetilde{h}(I, 2(I + 1)R_c) = A_2(I, c). \tag{3.48}$$

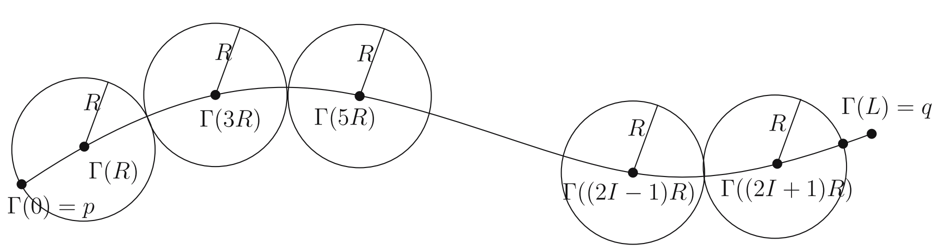
and thus, the first inequality in (3.47) holds. As for the third one, it is clearly equivalent to proving that $C_1(I)(g(M) + 1) \leq A_2(I, c)$. By applying (1.1), we have

$$C_1(I)(g(M) + 1) \leq \text{Area}(M) \overset{(3.48)}{\leq} A_2(I, c),$$

and the proof of the claim is complete. □

Claims 3.13 and 3.14 prove (3.47), which finishes the proof of Proposition 3.12, and consequently, item 3 of Theorem 1.1 is also proved.

Remark 3.15. Since the function $\widetilde{h} = \widetilde{h}(I, r)$ appearing in the proof of Claim 3.14 is increasing in r, and R_c is decreasing in c, we deduce that $A_2(I, c) = \widetilde{h}(I, 2(I + 1)R_c)$ is decreasing in c. This indicates that if we relax the hypothesis $3H^2 + \frac{1}{2}\rho \geq c$ in Proposition 3.12 by taking $c \to 0^+$, then the estimates for the area, diameter, and genus of M in Proposition 3.12 get worse (in fact, $\lim_{c \to 0^+} A_2(I, c) = \lim_{c \to 0^+} R_c = \infty$).



[h]

Figure 3: The pairwise disjoint collection of metric balls in $C'(n)$. Observe that the boundary of the last ball in the chain, $\partial B_M(\Gamma(2I + 1)R, R)$, intersects the image of Γ at the points $\Gamma(2IR)$, $\Gamma(2(I + 1)R)$, and that $2(I + 1)R \leq L$ by construction.

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