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ORIGINAL PAPER

Towards a research base for implementation strategies addressing mathematics teachers and facilitators

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Abstract

Implementing innovations in mathematics education relies on scaling up professional development (PD) for mathematics teachers. Within the German Center of Mathematics Teacher Education (DZLM), the complexities involved in scaling-up processes are addressed with three implementation strategies: a material, a personnel, and a systemic strategy. For each of them, the research base is empirically established by design research and intervention studies. The *personnel strategy* focuses on preparing teachers for innovative teaching approaches and on preparing PD facilitators for conducting PDs on the content involved. The empirical research required for creating a research base for the personnel strategy investigates what teachers and facilitators need to know about their learners (students on the classroom level, teachers on the PD level) with regard to a specific (PD) content. The main focus of the *material strategy* is to provide support for teachers and facilitators by adaptable curriculum materials for classrooms and PD sessions. The empirical research on the material strategies focuses on how such material needs to be constructed to ensure both flexibility and fidelity for impactful quality implementation. The *systemic strategy* involves considering the systemic contexts on each level and cooperating with the respective stakeholders (e.g., school principals or regional PD authorities). The corresponding empirical research focuses on the interplay of processes and systemic conditions. In this paper, we present the implementation strategies on different levels, and typical research approaches and results for strengthening their empirical and theoretical bases.

1 Introduction: why implementation through scaling up professional development?

Many rich and promising approaches to mathematics teaching have been developed, qualitatively investigated with respect to the initiated learning processes (Cobb et al. 2003), and/or quantitatively studied under laboratory conditions of controlled trials to provide evidence for efficacy (Slavin and Lake 2008). However, the wider implementation of these teaching approaches is still a desideratum as emphasized in the introduction of this special issue (Koichu, Sánchez Aguilar and Misfeldt 2021; see also Burkhardt and Schoenfeld 2003).

For most authors (including us), implementation does not aim only at sustainably reaching *some* classrooms, but also at comprehensively *scaling up* (Maaß, Cobb, Krainer and Potari 2019). When the aim is to reach *many* classrooms for an *implementation at scale*, the necessary research cannot concentrate solely on the classroom level, but must take professional development (PD) of teachers and of PD facilitators into account (Roesken-Winter, Hoyles and Blömeke 2015). In this paper, we present three *implementation strategies* used in the German Center for Mathematics Teacher Education (DZLM). Particularly, we elaborate on content-related research questions and findings that are crucial for establishing a theoretical foundation and empirical base for these implementation strategies, and that are also specific to mathematics education. We consider this overview as a contribution to *content-related implementation research*, as it can provide orientation for future implementation research.

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Although PDs can be very diverse concerning mathematical content (e.g., early mathematics education, stochastics learning, digital tools), each PD research project in our context is realized in content-specific ways. Even if the content cannot be fully unfolded in this overview, we try to exemplify why content-specificity is crucial.

Section 1 briefly presents some general thoughts on implementation research and the DZLM-implementation strategies on three levels; Sects. 2 and 3 present typical content-related research studies from the DZLM research network that can contribute to underpinning these implementation strategies (Sect. 2 on the teacher PD level, Sect. 3 on the PD facilitator level). Both Sections also list relevant questions for future studies, as for the moment, we have more questions than answers.

2 The material, personnel, and systemic implementation strategies

2.1 Current discourse on implementation strategies and implementation research

Implementation research has received much attention in recent years, yielding different definitions. Describing the landscape of implementation research, Century and Cassata (2016) provided a working definition of implementation research as "systematic inquiry regarding innovations enacted in controlled settings or in ordinary practice, the factors that influence innovation enactment, and the relationships between innovations, influential factors, and outcomes" (p. 170). By 'innovations,' Century and Cassata (2016) understand "programs, interventions, […], approaches, methods, strategies, […], that involve a change (e.g., in behavior or practice) for the individuals […] enacting them" (p. 170). In the context of PD for mathematics teachers, these interventions are often driven by results from mathematics education research. In our case, we focus on research-based innovations, stemming from our own or colleagues' design research on the classroom level.

Regarding the *implementation* of innovations, Century and Cassata (2016) mainly consider as relevant the individual users, organizational and environmental factors, as well as the attributes of the innovation. Implementation support strategies such as providing PD or specific teaching resources help the user to bring the innovation into practice and to establish the innovation over time (Maaß et al. 2019).

With respect to research on such implementation strategies, Century and Cassata (2016) reminded us that "while one may assume that innovations designated as 'evidence-based' are indisputably of value, in reality, evidence is relative" (p. 202). Particularly, they call for embracing complexity when investigating implementations, and acknowledging the respective contexts and their characteristics features. For science education, Penuel et al. (2011) argued in the same way, when underlining that a specific kind of research, which they call design-based implementation research, is needed. Particularly, they regarded the systematic analysis of implementation processes, which help "to iteratively refine strategies for improving the implementation effectiveness of interventions" (p. 282) as decisive. Besides this qualitative perspective, Desimone (2009) emphasized the need to "identify the variables that mediate (explain) and moderate (interact to influence) professional development's effects" (p. 184).

In the following, we build upon these ideas and introduce three implementation strategies for the context of mathematics education and pay particular attention to the research required to inform these strategies. We explain what theoretical and empirical aspects need clarification before a successful implementation can be initiated. Following Penuel et al. (2011), we consider implementation research not only as research *about* implementation (e.g., by studying the effects in effectiveness studies after implementing the innovation), but also *for* implementation (e.g., by providing the descriptive and explanative knowledge on determining factors, learning processes of the involved stakeholders and conditions of success, also informing effectiveness studies).

2.2 DZLM implementation strategies on three levels

For grasping the complexity of scaling up implementation through PD, Prediger, Roesken-Winter and Leuders (2019a, b) introduced the three-tetrahedron-model (3T-Model) shown in Fig. 1, which connects the classroom level with the teacher PD level (TPD) and the facilitator PD level (FPD). In this section, we successively explain the 3T-Model with its three interconnected levels while using it as an introduction for our implementation strategies. That is, we explain the material and the systemic strategies for each of the three levels and we elaborate on the personnel strategy, which means providing PD for teachers and facilitators, on the two PD levels (see Fig. 1).

2.2.1 Implementation strategies on the classroom level

We start our explanations on the *classroom level*, the lowest level: Extending the classical didactical triangle, Ruthven (2012) and others have suggested describing mathematics classrooms by four components, namely, teacher, students, content, and classroom resources and materials. The model of the *didactical tetrahedron* on the classroom level can portray the complex inner relationships of the interacting four components. The circle around the tetrahedron has been added, here, to signify that what happens in the classrooms

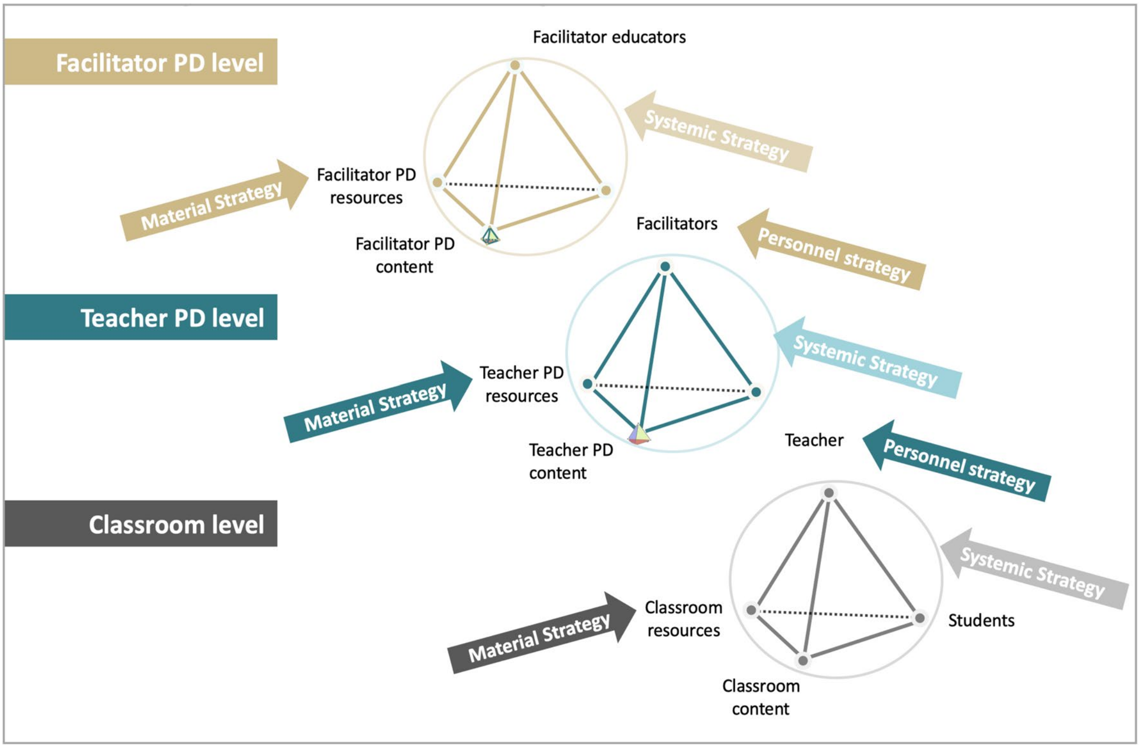

Fig. 1 Implementation strategies on three levels of the three-tetrahedron-model (Prediger, Roesken-Winter, and Leuders 2019a, b)

is always embedded in a systemic school context. When PD designers or policy makers intend to implement teaching approaches on the classroom level, they can adopt different strategies, as follows:

- *Material strategy on the classroom level:* They can design and spread well-designed curriculum materials that support teachers in realizing the chosen teaching approach for specific mathematics content (e.g., Remillard and Heck 2014; Potari, Psycharis, Sakonidis and Zachariades 2019). Usually, the design teams comprise not only researchers but also experienced teachers.
- *Systemic strategy on classroom/school level:* They can consider and improve the systemic conditions under which teachers can develop their instructional practices and cooperate with the responsible stakeholders (e.g., the school principles) to ensure sustainability of the innovations (e.g., Goos and Bennison 2018). For example, establishing professional learning communities (PLCs) of inquiry (Jaworski 2006) is a typical systemic strategy. Such PLCs are useful in themselves to promote teachers' cooperation. As a complementary element to TPD, they can also facilitate planning and reflecting on classroom practice.

2.2.2 Implementation strategies on the teacher PD level

Many projects have shown that curriculum material and initiating cooperation alone are not enough (Ball and Cohen 1996). Teachers need to understand and be convinced of the innovative idea elaborated in the material. Rather than expecting any simple 'implementation with fidelity', the focus must lie on helping teachers to connect the new ideas to their own students' learning and their teaching practices, and to use the material in an intended way with well-informed adaptations, often termed as teachers' design capacities (Brown 2009). For developing the individual design capacity for the particular innovation in view, the implementation is usually enriched by the following:

- *Personnel strategy on the TPD level:* This strategy realizes PD courses for teachers to enhance their professional knowledge and instructional practices, also for developing their design capacities for well-informed adaptations (e.g., Maaß et al. 2019; Ball and Cohen 1996).

In Fig. 1, the specificity of the TPD level is as follows: In the TPD level tetrahedron the teachers are the learners, the PD facilitators are the educators supported by TPD resources, and the TPD content can comprise different aspects of the classroom tetrahedron. When PD designers or policy makers seek to implement research-based PD approaches on a large scale (not only in the PD they provide themselves), they can lift the material and systemic strategy to the TPD level, and adapt it to the specificities of this level.

- *Material strategy on TPD level:* They can design and spread well-designed curriculum and TPD materials that support PD facilitators in realizing the chosen PD approach for a specific PD content (e.g., video vignettes to be discussed, PD activity sheets etc., see Karsenty 2018), and sometimes also classroom materials designed to provide opportunities for professional growth by experimentation and reflection (Swan 2007). The design teams comprise not only researchers, but also experienced PD facilitators.
- *Systemic strategy on TPD level:* They can consider and improve the systemic context conditions in which the PD facilitators work (e.g., by creating PLCs for facilitators, see Schifter and Lester 2005) and the cooperation with responsible stakeholders (e.g., the regional authorities responsible for PD courses, see Cobb and Jackson 2015) to ensure sustainability of the innovations.

Scaling up TPD cannot count on only the material and systemic strategies; it also requires qualified personnel beyond the researchers/TPD designers themselves. This aspect of scaling has its limits, as many countries do not have enough TPD facilitators (Roesken-Winter et al. 2015). Therefore, qualifying new facilitators is the aim of the third level, the facilitator PD level.

2.2.3 Implementation strategies on the facilitator PD level

The structural analogy of the three levels (Fig. 1) calls for lifting and accommodating the strategies of the classroom and the TPD level to the tetrahedron on the FPD level:

- *Personnel strategy on the FPD level:* This strategy realizes FPD courses for PD facilitators in order to enhance their personal professional expertise and their PD practices.
- *Material strategy on the FPD level*: This strategy consists of designing and spreading well-designed curriculum and FPD materials that support facilitator educators in realizing the chosen FPD approach for a specific FPD content (Schifter and Lester 2005).
- *Systemic strategy on FPD level:* This strategy considers and improves the systemic context conditions in which the FPD takes place and the cooperation with responsible stakeholders, the regional authorities and ministries of education, for the purpose of taking into consideration the systemic conditions of facilitator PD.

These strategies have been successfully implemented in different contexts (also under other names), as exemplified by the studies mentioned in brackets. Whereas the strategies tended to be discussed separately, they work best in combination, as the following section indicates.

2.3 Example for combining implementation strategies on different levels in a long-term and large-scale implementation project

The interplay of implementation strategies can be illustrated by one DZLM implementation project, the project *Mastering Maths* (2006–2024) in which the strategies were successively combined on different levels for the purpose of increasingly scaling them up (Prediger, Fischer, Selter and Schöber 2019a, b). The goal of the project is to enable teachers to enhance the understanding of conceptual basics of middle school students at risk, i.e., students with limited access to mathematics. In spite of a very convincing research base on the learning needs of students at risk, and effective instructional approaches (Maccini, Mulcahy, and Wilson 2007; Moser Opitz et al. 2017), these had not sufficiently reached classrooms in German schools.

So, the first project phase invested six years in designing, testing and redesigning curriculum materials (based on the existing research base on student learning) to support teachers in formatively assessing and enhancing students' understanding of conceptual basics, such as the place value system, or meanings of multiplication and division. The curriculum materials support teachers in increasing students' access to these conceptual basics by engaging them in rich discussions of multiple representations, encouraging teachers to notice students' resources and challenges by formative assessment, and to work with manipulatives and scaffoldings, and progress to a successive fading out. The curriculum materials were co-developed in design teams comprising researchers and teachers and are based on empirical research, including both design research and randomized trials (Moser Opitz et al. 2017). They were optimized to allow for teachers' adaptations and are freely available as open educational resources *(material strategy on classroom level*).

In the second phase, three schools volunteered to cooperate in a first small-scale implementation. A TPD design team of researchers and PD facilitators successively refined the learning opportunities for teachers to develop the expertise necessary for enacting the innovation and for taking well-informed adaptation decisions. For example, PD activities were developed to enable teachers' identification of relevant conceptual aspects for the mathematics in view (e.g., which aspects are most relevant for place value understanding?) and to diagnose in students' utterances the next step for students' learning pathways. Based on research findings on the relevance of initiating in-school teacher cooperation (Jaworski 2006; Selter, Gräsel, Reinold, and Trempler 2015), we established PLCs in the schools that were working with the curriculum materials after having some *PD (combination of material and systemic strategy on the classroom level with personnel strategy on the TPD level*). After the first months, the systemic strategy was widened to the school principals' systemic support for institutionalizing the remediation courses and fixed meeting times of the PLCs in the schools' agenda (Prediger et al. 2019a, b).

In the third phase, we intended to reach 40 schools; thus, the design researchers only could not provide the PD courses. In cooperation with five school districts, nine new PD facilitators from existing district TPD institutions were recruited and 40 schools with many at risk students were invited to participate (*systemic strategy on the TPD level*). Due to the *personnel strategy* on the FPD level, these PD facilitators were prepared for the new TPD content in newly developed FPD courses and by continuous FPD in monthly team meetings throughout the whole project. The team of nine PD facilitators and the design researchers together further developed the TPD materials for supporting a *material strategy on the TPD level*. The TPD materials were informed by qualitative research on typical teacher professionalization pathways when starting to work with the classroom materials, and when learning about students' needs in developing understanding for conceptual basics (Prediger 2020). In particular, several design experiments were required for designing mathematically productive PD activities to answer, for instance, questions such as the following: How must the PD activity of discussing students' utterances be enriched by some input on conceptual aspects of place value understanding so that the teachers' reflections can reach the necessary mathematical depth? What kind of support do the PD facilitators need to reach and maintain the mathematical depth in the discussion of the PD activities and to connect teachers' utterances to the target knowledge?

In the fourth phase, the implementation project reached out to four other federal states in Germany. For this scaling up step, the *material strategy on the TPD level* was strengthened by redesigning the TPD material into versions that could be published as open educational resources, i.e., into materials which were self-explanatory enough for supporting PD facilitators who had not been involved in the design and development. This redesign of the material was informed by research on typical facilitator adaptation processes when working with materials that were too implicit in their structures, goals, and content, in a parallel project with other PD content (Leufer et al. 2019). In particular, for reaching mathematical depth in their PD discussions, the facilitators need well-structured materials that give them the flexibility for well-informed adaptation decisions according to the teachers' pathways towards the mathematically relevant conceptual aspects.

The scaling up to more than 200 schools required also a further development of the *personnel strategy on the FPD level.* The project team thus started to apply a *material strategy on the FPD* level to support the increasing group of facilitator educators in conducting their FPD courses for more than 30 PD facilitators. Additionally, the *systemic strategy on the FPD level* urged the project team to adapt the FPD courses to differing systemic conditions in each of the four states. Collaborating with the regional authorities led to adapting the FPD for diverse preparedness of PD facilitators and diverse time budgets allotted for their qualification development, as well as modifying school cooperation contracts to the different regional conditions on the TPD level.

Even if implementation strategies have successively been extended to the three levels, the material strategy on the classroom level is still the major cornerstone of the project (see also Swan 2007). However, not only a research base on instructional approaches for students was required, but also insights into the learning processes of teachers, the facilitation processes of facilitators and their support by material and systemic strategies, as well as the learning processes of facilitators.

3 Creating a research-base for the three implementation strategies on the teacher PD level

With the three implementation strategies on different levels being formulated in Sect. 1.2 and the practical example of *Mastering Maths* in Sect. 1.3, we can now turn to the programmatic question of what research *for* implementation—supplementary to research *about* implementation—means in our context:

Implementation research in the DZLM scaling-up context has the aim of creating an empirical research base for each of the strategies on the three levels. The research base encompasses insights into learning processes (with typical pathways and obstacles) of teachers and facilitators, as well as conditions of success and effects of design elements to inform design decisions in each of the strategies.

As material and systemic strategies on the classroom level have been extensively studied (e.g., Burkhardt and Schoenfeld 2003; Pepin, Gueudet, and Trouche 2013), we focus on the FPD level (later in Sect. 3) and the TPD level (in this Sect. 2), and describe research for informing the personnel (Sect. 2.1), material (Sect. 2.2) and systemic strategy (Sect. 2.3).

3.1 Towards a research-base for the personnel strategy on the TPD level

The aim of the *personnel implementation strategy* on the TPD level is to provide effective TPD that helps teachers sustainably to develop the content-specific expertise necessary for implementing an innovation. To achieve this aim, research on typical and crucial content-related learning pathways as well as on essential design elements to foster teachers' learning is needed. In particular, three questions guide research necessary for the personnel strategy:

(RQ1) *TPD content:* What do teachers need to learn in a TPD course about specific PD content (e.g., which diagnostic categories are relevant for differentiating in geometry classes) for implementing a particular content-specific innovation in their classrooms?

(RQ2) *Teachers' learning pathways:* Which content-related learning pathways do teachers typically pass through during/after TPD and which crucial (content-related, affective, …) obstacles have to be overcome? Which adaptation practices for the classroom material can we observe, and what do they tell about the teachers' learning pathways towards achieving a well-founded adaptation capacity?

(RQ3) *TPD design:* How should TPD courses be designed to enable teachers to implement a particular content-related innovation in their classrooms effectively, and to make well-informed adaptations necessary for the mutual contexts?

Pursuing these three overarching research questions for content-related TPD is at the center of the DZLM research underpinning the personnel implementation strategy (Prediger et al. 2019a, b).

For all three research questions, mainly generic PD research has revealed important answers. As the TPD design has mainly been in the focus, five general design features have been identified in many research projects: collective participation, duration, active learning, coherence, and content focus (e.g., Yoon et al. 2007; Desimone 2009; Penuel et al. 2007). Particularly, the principle *content focus* acknowledges the significance and impact of subject-specific TPD. Although effects of content-intensive TPD on teacher learning could be ascertained (Desimone 2009), this focus of TPD does not seem to be sufficient to improve student learning. For instance, Garet et al. (2016) found effects of content-intensive TPD on teachers' knowledge and classroom practice, but student achievement did not improve. The authors drew the following conclusion: "Future research might also seek to identify other aspects of knowledge and practice to target with PD that are strongly related to improved student achievement" (p. 45).

Thus, for implementing research-based innovations, general design principles need to be substantiated by content-related features of effective TPD. This leads to RQ1, namely specifying the TPD content. What Garet et al. (2016) label as "other aspects of knowledge and practice" for specific TPD content, requires insights into teacher learning processes within knowledge facets and the interplay with affective factors (Goldsmith et al. 2014), which is emphasized in RQ2.

An example from DZLM TPD research, for the PD content stochastics education, illustrates how the content-related focus bears an added value for informing the personnel strategy: The innovation to be implemented was teaching distributions with the support of digital tools for experiments and simulations (TPD content, RQ1). The empirical research by Griese, Binner and Rösken-Winter (2020) focused on investigating teachers' practices with a prepared digital learning environment, in teaching statistical distributions. Based on teachers' planning documents, classroom observations, and interviews, they found that teachers face content-specific challenges regarding the rich phenomena with unforeseen outcomes, when their students described, compared and explained the statistically relevant characteristics of distributions while using these digital experiments and simulations (teachers' learning pathways, RQ2). Particularly, they felt unsure because they could not control the exact outcomes. Based on this insight into a content-specific challenge, the TPD was re-designed to include discussions about different outcomes of statistical simulations and how to use them for students' learning (TPD design, RQ3).

A second example stems from a DZLM TPD research project with early childhood teachers: Gasteiger and Benz (2018) developed an empirically grounded model of teachers' professional mathematics knowledge, which grasps the complexity of necessary dispositions and skills to identify and exploit spontaneously occurring mathematical learning opportunities in natural learning situations (TPD content, RQ1). As a consequence, the TPD design was optimized towards situated case-based TPD activities (Bruns, Eichen, and Gasteiger 2017), enhancing kindergarten teachers' skills in identifying mathematical structures and patterns in natural learning situations, and their knowledge about children's mathematical abilities and the required adaptive mathematical learning activities for content such as geometric shapes or measuring (TPD design, RQ3). This example illustrates that is it necessary to choose the TPD content and methods on a sound empirical and theoretical foundation. Based on such a foundation, it is possible to present the TPD objectives transparently and to analyze the effectiveness of the implementation according to the theoretically and empirically elaborated criteria (here the dispositions and skills represented in the model of teachers professional mathematics knowledge).

A third example stems from an intervention study with a TPD course on teaching calculus with digital tools (Thurm and Barzel 2020). The authors report significant effects of the TPD on mathematics teachers' beliefs about teaching with technology, but no significant increase of frequency of their technology use in comparison to a control group. As an obstacle on their learning pathways, teachers' limited self-efficacy beliefs regarding using digital tools in calculus were identified. In the re-design of their TPD, the researchers strengthened opportunities for enhancing teachers' self-efficacy beliefs (identified as relevant TPD content, RQ1) by including micro-teaching activities, particularly for these content-related teaching practices with digital tools that were identified as most critical for teachers (TPD design, RQ 3).

The three examples illustrate that different research approaches are adopted to scrutinize conditions, typical learning pathways and obstacles for specific TPD content, and the TPD design.

3.2 Towards a research-base for the material strategy on the TPD level

The aim of the *material strategy on the TPD level* is to support PD facilitators with well-designed TPD materials and additional resources on the classroom level. The design of curriculum material on the classroom level and the TPD level can be informed by research on two main questions:

(RQ4) *TPD design and TPD content:* Which content-related design elements in the TPD material support the facilitators' realization of the TPD design principles and the enactment of the TPD content?

(RQ5) *Facilitators' adaptation processes:* How do facilitators adapt the materials while using them for their TPD courses?

For RQ4, the DZLM has agreed on a set of six design principles (Barzel and Selter, 2015) to design highly-specified TPD materials for different TPD content. First, the TPD materials state particular learning goals and activities designed to reach these goals. The research results for RQ1 and RQ2 (see Sect. 2.1) led to a strong focus on the quality and depth of the TPD activities that enable teachers to activate the gained knowledge for their practices (Prediger 2019; Thurm and Barzel 2020; Gasteiger and Benz 2018). In design research projects on the TPD level, these activities were tested in several design experiments and iteratively redesigned to lead teachers to the intended learning goals for particular PD content, for example, identifying mathematically relevant language demands in a PD on language-responsive teaching (Prediger 2019). The careful analysis of teachers' learning pathways (RQ2) also helps to scrutinize such typical learning pathways and how they can be supported in the TPD design and TPD content (RQ4). With the aim of scaling-up, the DZLM publishes these research-based TPD materials as open educational resources. The target group are facilitators who were not involved in the development themselves.

Regarding RQ5, the DZLM research on facilitators' adaptation practices is not conducted for optimizing "implementation fidelity" (as criticized by Jacobs et al. 2017), but for understanding the facilitators' reasoning for their adaptation processes (for which conceptual frameworks of research on teachers' adaptations can be lifted, e.g., from Remillard 2005; Sherin and Drake 2009) On the one hand, this can inform decisions on necessary FPD content (see later RQ8); on the other hand, we drew consequences for RQ4 on necessary design elements in TPD materials and manuals, allowing facilitators to make productive adaptations to the individual PD contexts.

Within the DZLM, Leufer et al. (2019) investigated facilitators' adaptation practices of curriculum material resources for PD courses on language-responsive mathematics instruction (Facilitators' adaptation processes, RQ5). Their results suggest that in order to ensure that adaptation practices improve quality, the structure of TPD courses must be very transparent for facilitators and should preserve a minimal adequate grain size (i.e., a whole thematic block with activities and theoretical backgrounds is preferable to single slides). Otherwise the adaptations risk not supporting teachers to engage in rich discussions and to connect the PD activities with the targeted PD content. As further consequence of these findings, the DZLM guidelines for designing TPD materials now emphasize the need for enriching the material with information for facilitators on crucial content and teacher learning pathways (TPD design and TPD content, RQ4).

3.3 Towards a research-base for the systemic strategy on the TPD level

The *systemic strategy on the TPD level* is to consider and improve the systemic context conditions in which the PD facilitators work (e.g., by creating PLCs for facilitators, in order to promote cooperation among them) and the cooperation with the responsible stakeholders, in order to ensure sustainability of the innovations on the TPD level.

In their model of implementing innovative teaching, Maaß et al. (2019) consider multiple systemic aspects, as for instance the context of innovations (e.g., size of country or state) or the cooperation between different stakeholders. Going beyond practical concerns (e.g., time/resources that teachers can allocate) at least two aspects require empirical research to establish a research base for the systemic strategy (see Cobb, Jackson and Dunlap Sharpe 2017):

(RQ6) *Facilitator PLCs:* How can facilitator PLCs be established for collectively planning and reflecting on TPD courses, and how can they be involved in the co-design of TPD material?

(RQ7) *Systemic conditions on the district level:* How can the cooperation with stakeholders on the district level support the implementation by 'organizing' the implementation project (e.g., in terms of constructively aligning school-district-contracts, increasing accountability for the school leaders, etc.)?

Whereas principles for RQ6 can be adapted from research on PLCs on the TPD level (e.g., Jaworski 2006), the systemic conditions have been an explicit part of the design research on the district level conducted by Cobb, Jackson and Dunlap Sharpe (2017). They underline that the constructive alignment of the implementation goals with the school goals (for which principles are accountable) and district goals and assessments, is crucial for the success of the implementation process.

Within the DZLM, the systemic strategy has been an important part of the implementation projects, and some practical experiences are reported (e.g., Prediger et al. 2020 for the project *Mastering Maths*), but our systematic research has not yet informed the practical decisions. In contrast, the systemic strategy on the classroom/school level has been underpinned by systematic DZLM TPD research which confirms the high relevance of the content focus, here on enhancing process-oriented competencies (such as modelling, problem solving, arguing) in primary mathematics teaching. For instance, Selter, Gräsel, Reinold and Trempler (2015) investigated the effectiveness of four TPD settings regarding teachers' acceptance of the PD and their beliefs about the new curriculum, as follows: (A) a TPD program with well-designed content on teaching process-oriented competences (i.e. a purely personnel strategy); (B) a TPD setting with only brief information on the new syllabus, requiring that process-oriented competences be taught and then building teacher PLCs (a purely systemic strategy); (C) a TPD program that combined both, the personnel and the systemic strategy; and (D) a control group. Acceptance for the innovation was found in settings A, B, and C, but the teachers' epistemological beliefs about mathematics teaching and learning developed significantly more in the intended direction in settings A and C than in B and D. The findings underline the relevance of a content-related personnel strategy, even in connection with a systemic strategy on the school level.

4 Creating a research-base for three implementation strategies on the facilitator PD level

4.1 Towards a research-base for the personnel strategy on the FPD level

Since Zaslavsky and Leikin (1999) called for paying more attention to facilitators of PD, research on facilitators and on their PD (FPD) has been successively established, as evidenced by two recent special issues (Tekkumru-Kisa and Stein 2017; Berry, Even, Krainer, and Park Rogers 2021). Lifting the research questions from the TPD level (see Sect. 2.1) results in the following questions for underpinning the personnel strategy on the FPD level:

(RQ8) *FPD content:* What do facilitators need to learn in a FPD course about specific TPD content (e.g., which diagnostic categories do teachers typically apply for differentiating when teaching fractions) for implementing a particular content-specific innovation in their TPD courses?

(RQ9) *Facilitators' learning pathways:* Which content-related learning pathways do facilitators typically pass through during/after FPD and which crucial (content-related, affective, …) obstacles have to be overcome (e.g., while guiding content-specific discussions, dealing with dual identities or overcoming attitude challenges)? Which adaptation practices for the TPD material can we observe, and what do they tell about the facilitators' learning pathways towards achieving a well-founded adaptation capacity?

(RQ10) *FPD design:* How should FPD courses be designed in order to enable facilitators effectively and sustainably to implement a particular content-specific innovation in their TPD, and to make well-informed adaptations for the mutual contexts?

In this emerging field of development and research, some more practical papers exist that present their programs and principles for *FPD courses*, addressing aspects of FPD content and FPD design principles (e.g., Borko et al. 2015; Rösken-Winter et al. 2015). The existing empirical studies with facilitators have mainly concentrated on informing the specification of *FPD content*; i.e., the knowledge and skills that facilitators need (e.g., Borko et al. 2014; Elliott et al. 2009; Lesseig et al. 2017). For this purpose, research on novice facilitators' evolving practices (e.g., Coles 2019; van Es et al. 2014; Bruns et al., 2020) also contributed to specifying the facilitators' learning needs.

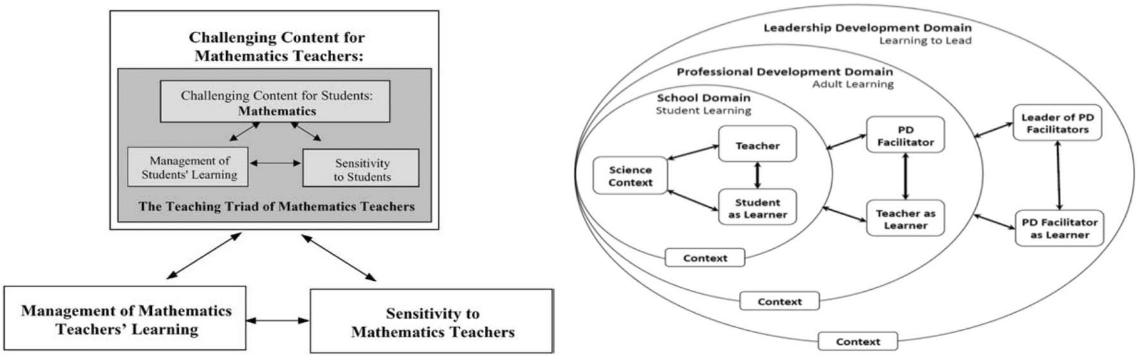

Fig. 2 Nested didactical triangles (left, Zavlasky and Leikin 2004; right, Luft and Hewson 2014)

These different kinds of studies have explored different models of facilitator's professional expertise. In particular, the nested structure of FPD and facilitator's professional knowledge has often been emphasized and modeled; e.g., by nested didactical triangles (Zaslavsky and Leikin 2004; Luft and Hewson 2014; Wood and Turner 2015; Ball 2012, see Fig. 2) or nested tetrahedrons (Prediger et al. 2019a, b, see Fig. 1). Within the DZLM, a framework describing facilitators' competence was developed that takes this nested structure of domains into account, while also considering affective factors such as self-concept, identity and others (Peters-Dasdemir, Holzäpfel, Barzel, and Leuders 2021).

Although most research is situated in a particular PD setting with particular PD content (e.g., the problem solving cycle in the case of Borko et al. 2015), most of the early research projects aimed at formulating generic findings in terms of general structures or facilitation moves (van Es et al. 2014; González, Deal, and Skultety 2016), without yet making the specificities of the PD content explicit.

Beyond these generic perspectives, the DZLM research has the aim of providing a research base for content-specific aspects of the personnel strategy on the FPD level. For example, Wilhelm et al. (2019) conducted a longitudinal interview study on facilitators' learning pathways in an FPD supporting students with learning difficulties in mathematics (Facilitators' learning pathways, RQ9). They disentangled the knowledge domains that facilitators activated for preparing their first TPDs and revealed that in the beginning, they mainly rely on generic pedagogical knowledge domains. During the FPD course, the facilitators increasingly drew on specialized PD content knowledge (about promoting students with mathematical difficulties). In particular, the study showed that different knowledge domains were successively better connected. That is, some facilitators increasingly used content knowledge of arithmetic and pedagogical content knowledge of teaching arithmetic for decisions about the TPD content goals, but not all of them unpacked the PD content sufficiently down to its mathematical core to address it in depth during the PD. As a consequence for FPD designs, the researchers suggest that FPD should systematically initiate the situated drawing of connections between different knowledge domains (FPD design, RQ10).

Pöhler et al. (2021) investigated novice PD facilitators' practices in their first year of facilitating TPD on language-responsive classrooms. Based on observation data from the PDs conducted by the novice facilitators and interview data, they found novice facilitators mostly pursuing atmospheric and process-oriented goals ("all teachers should take part in the conversation") in the beginning, whereas they rarely worked towards the TPD content goals. The PD content goals (distinguishing technical, school academic and everyday language, and situating those language varieties though connecting them to the construct of multiple representations) were hastily pushed into the background as teachers indicated that they had already considered them. During the interviews based on video clips from the PD, the facilitator and the researcher worked out together, ways that the teachers could use all representations (symbolic, verbal, graphic), but mostly how they instead connected them to procedural aspects and not to conceptual understanding. Based on these findings on RQ9, the research team identified an explicit focus on the TPD content goals as a crucial category for noticing and guiding teachers' discussions, thus as an important part of FPD content (FPD content, RQ8). The more the facilitators became aware of the TPD content goals and their interconnections through guided and explicit reflections, the more they were able to connect the participating teachers' ideas to the TPD content goals (Facilitators' learning pathways RQ9).

Both studies' empirical findings on facilitators' learning pathways (RQ9) also contributed to establishing a research base for both what facilitators need to learn in FPD (RQ8) and how they can connect their knowledge to their facilitation practices by mutual FPD design elements (RQ10). However, much more content-related research is needed in the future.

4.2 Towards a research-base for the material strategy on the FPD level

The aim of the *material strategy on the FPD level* is to support facilitator educators with well-designed FPD materials. As most PD facilitator courses are still conducted by the researchers themselves, the material strategy for FPD is not yet widely spread. Nevertheless, a scaling-up-process has to take into account that facilitator educators (in addition to the researchers themselves) are responsible for the FPD and need specific material. Of course, providing well-designed TPD material can support facilitators' experimentation which is also a contribution to the FPD level, but material for the FPD itself is still underdeveloped.

For supporting this (still rare) strategy, one main research question can be lifted from the TPD to the FPD level:

(RQ11) *FPD design and FPD content:* Which content-related design elements in the FPD material can support the facilitator educators' realization of the FPD design principles and the enactment of the FPD content (e.g., documents from teachers' and/or facilitators' learning pathways and reflection activities for facilitators, or background literature)?

Although not much research has been conducted in this respect so far, some findings can inform the design: The set of six DZLM design principles from the TPD level (Barzel and Selter, 2015) has been lifted to the FPD level. Due to the results on facilitators' learning pathways (see Sect. 3.1), the set is now systematically complemented by a seventh principle: "connect the classroom and TPD level". This design principle was introduced due to the result that connecting different knowledge domains is crucial for the novice facilitators' learning pathways (Wilhelm et al. 2019, Pöhler et al. 2021).

Similar to the TPD level, the design of the content-related FPD activities seems to be crucial for developing facilitators' expertise (Kuzle and Biehler 2015). That is, for the DZLM PD on the topic 'teaching data analysis', Kuzle and Biehler (2015) found facilitators conducting the PD on their own for the first time to include examples only from their own teaching. The FPD was restructured to include vivid examples from other teachers' classrooms for this specific topic. The general design principle of case-based learning is thus specified with respect to the research findings.

These experiences resonate with those of Lesseig et al. (2017), who demonstrated how facilitators' noticing in TPD can be enhanced by video-cases from TPD, and ultimately how such FPD material can be designed. By a research-based fine-tuning of the prompts and tasks of the video-based activities, Lesseig et al. (2017) were able to help facilitators to focus more on the mathematical aspects in the video cases, to link them to the PD content goals, and to reason about the facilitator's work in advancing towards these learning goals. The research findings thus help facilitator educators to understand the background of the video-based learning material for facilitators, and to consider the relevant implementation aspects.

One of the DZLM design research projects on the FPD level drew upon the approaches by Lesseig et al. (2017) for designing case-based work with video cases from TPD: Schüler and Roesken-Winter (2018) investigated video cases to support facilitator educators in enhancing facilitator's noticing of productive teacher learning in stochastics education. For instance, the video case 'fitting representations' showed the facilitators helping teachers to work towards the following learning goals: knowing a variety of representations for statistical data, and assigning correct representations to the characteristics of a specific data set. By an expert validation, the potential of different videos was disentangled, and criteria for selecting appropriate content-specific cases were specified. That is, the video cases needed to show incidents of SCK, PD-PCK, and facilitation moves related to teacher learning, and enough variance to reflect upon the three content categories during facilitator PD. Furthermore, the experts suggested a set of noticing prompts to guide facilitators discussions, such as the following: For what mathematical reason did the facilitator not interrupt the lengthy discussion of the teachers? Why do you think the facilitator has decided to have the teachers solve the task for the students themselves? Again, the unpacking of different aspects of the PD content in view turned out to be crucial for the redesign of the case-based FPD activity in order to achieve the FPD content goals.

Beyond these studies, a huge research gap exists for developing a research base concentrating on facilitators' learning and the FPD material to support it.

4.3 Towards a research-base for the systemic strategy on the FPD level

The systemic strategy on the FPD level is to consider and improve the systemic context conditions in which the FPD takes place, and the cooperation with its responsible stakeholders.

(RQ12) *Systemic conditions on the state level:* How can the cooperation with stakeholders on the regional or state district level support the implementation by 'organizing' the implementation project (e.g., in terms of constructively aligning the state steering activities, increasing accountability for the districts, etc.)?

Educational policy research emphasizes the importance of cooperation with regional structures as well as sufficient alignment between official (state) standards and innovation goals for a successful and sustainable implementation (Datnow 2005). For mathematics education specifically, the research base for the systemic strategy is still limited. Potari and colleagues (2019) report on contradictions between teachers, researchers, and policy makers when jointly developing a reform-oriented national mathematics curriculum for compulsory education in Greece. Among other issues, they found that members of the team who acted as brokers and facilitated the interactions between the different groups were decisive for overcoming these contradictions. Similarly, Cobb and Jackson (2015) documented and investigated how collaboration with district stakeholders was implemented and successively improved in the context of disseminating the products of classroom design studies. They emphasized that these products need to be designed keeping in mind both the large-scale implementation and the "current instructional practices and the capacity of educational systems" (p. 1027).

These insights are only a first step towards a research base that can inform the systemic strategy at the FPD level. Further research is needed that pays particular attention to such content-specific as well as state-specific conditions and obstacles that hinder a successful and sustainable innovation. While we have collected practical experiences with different states and districts as well as different innovation content within the DZLM, we have not yet systematically investigated implementation processes at this level. However, we are well aware of the importance of a research base for the systemic strategy on the FPD level, and further efforts will be dedicated to investigating such implementation processes of innovations more thoroughly.

5 Outlook

In this paper, we elaborated on the material, personnel, and systemic strategies in the context of implementing research-based innovations through scaling up PD. Particularly, we pointed out what creating a research base for these three strategies on the teacher and the facilitator PD level can mean in the case of larger PD programs, using FPD to scale up innovations. That is, we emphasized how existing general findings on design principles and design elements need to be complemented by content-related findings. By listing twelve research questions, we delineated what research is needed to inform the three implementation strategies. We also exemplified various content-specific findings and how they can be used to underpin design decisions in each of the strategies. Distinguishing different strategies on the teacher and the facilitator levels thus helps to systematically classify existing and missing research for implementation.

Our approach is limited to some extent, as we focused on one type of scaling up. Other types such as establishing networks or co-teaching might need other or additional guiding research questions for content-related implementation research.

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