Intersections of loci of admissible covers with tautological classes

Johannes Schmitt[1] · Jason van Zelm[2]

© The Author(s) 2020

Abstract

For a finite group G, let $\overline{\mathcal{H}}_{g,G,\xi}$ be the stack of admissible G-covers $C \to D$ of stable curves with ramification data ξ, $g(C) = g$ and $g(D) = g'$. There are source and target morphisms $\phi\colon \overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{M}}_{g,r}$ and $\delta\colon \overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{M}}_{g',b}$, remembering the curves C and D together with the ramification or branch points of the cover respectively. In this paper we study admissible cover cycles, i.e. cycles of the form $\phi_*[\overline{\mathcal{H}}_{g,G,\xi}]$. Examples include the fundamental classes of the loci of hyperelliptic or bielliptic curves C with marked ramification points. The two main results of this paper are as follows: firstly, for the gluing morphism $\xi_A\colon \overline{\mathcal{M}}_A \to \overline{\mathcal{M}}_{g,r}$ associated to a stable graph A we give a combinatorial formula for the pullback $\xi_A^*\phi_*[\overline{\mathcal{H}}_{g,G,\xi}]$ in terms of spaces of admissible G-covers and ψ classes. This allows us to describe the intersection of the cycles $\phi_*[\overline{\mathcal{H}}_{g,G,\xi}]$ with tautological classes. Secondly, the pull–push $\delta_*\phi^*$ sends tautological classes to tautological classes and we give an explicit combinatorial description of this map. We show how to use the pullbacks to algorithmically compute tautological expressions for cycles of the form $\phi_*[\overline{\mathcal{H}}_{g,G,\xi}]$. In particular, we compute the classes $[\overline{\mathcal{H}}_5]$ and $[\overline{\mathcal{H}}_6]$ of the hyperelliptic loci in $\overline{\mathcal{M}}_5$ and $\overline{\mathcal{M}}_6$ and the class $[\overline{\mathcal{B}}_4]$ of the bielliptic locus in $\overline{\mathcal{M}}_4$.

✉ Johannes Schmitt
schmitt@math.uni-bonn.de

Jason van Zelm
jasonvanzelm@outlook.com

[1] Mathematical Institute, University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany

[2] Department of Mathematics, Humboldt-Universität zu Berlin, Rudower Chaussee 25, Room 1.415, 12489 Berlin, Germany

Ⓑ Birkhäuser

Contents

1 Introduction

1.1 Motivation: admissible cover cycles

A classical way of constructing closed subvarieties of the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves is by considering families of finite covers of curves; for example we can consider the closure of the locus of smooth curves $(C, p_1, ..., p_n)$ such that there exists a finite degree d cover $C \to D$ to a curve of fixed genus g' with fixed ramification profile at the marked points p_i of C. An example would be the loci $\overline{\mathcal{H}}_{g,n_1,2n_2}$ and $\overline{\mathcal{B}}_{g,n_1,2n_2}$ of hyperelliptic and bielliptic curves with n_1 marked fixed points of the corresponding involution and $2n_2$ marked points pairwise switched.

Given such a locus of covers it is natural to study its fundamental class in the Chow or cohomology ring[1] of $\overline{\mathcal{M}}_{g,n}$. The class of any locus of admissible covers of genus 0 curves with fixed degree d and fixed ramification profile is tautological (see [18]). However, not all classes coming from spaces of admissible covers are tautological (see for example [27,47]). In fact such classes are the prime example of explicit nontautological algebraic classes.

In the cases where the admissible cover cycles are tautological, there is a wide literature on expressing such cycles in terms of decorated stratum classes, which are a set of additive generators of the tautological ring.[2] In [15, Theorem 2.2] the class $[\mathrm{Wp}] \in A^1(\overline{\mathcal{M}}_{2,1})$ of the locus of pointed genus 2 curves where the marked point is one of the Weierstrass points of the curve is computed in terms of decorated stratum classes. Since every genus 2 curve is hyperelliptic and the Weierstrass points coincide with ramification points of the hyperelliptic map, the cycle $[\mathrm{Wp}]$ equals $[\overline{\mathcal{H}}_{2,1,0}]$, the class of the locus of hyperelliptic genus 2 curves with one point fixed by the involution. The computation of the class $[\overline{\mathcal{H}}_3] \in A^1(\overline{\mathcal{M}}_3)$ of the locus of genus 3 hyperelliptic curves is well known (see for example [30, Section 3.H]). The computation of $[\overline{\mathcal{H}}_4]$ in terms of decorated stratum classes is much harder and was first given in [18, Proposition 5]. More recently in [11] a generating series for the class of the locus $[\overline{\mathcal{H}}_{2,n_1,0}]$ is given for all $0 \leq n_1 \leq 5$ in terms of ψ classes and graphs. In [19] the class $[\overline{\mathcal{B}}_3] \in A^2(\overline{\mathcal{M}}_3)$ of the locus of bielliptic curves of genus 3 is calculated.

In this paper, we give a systematic way for computing the intersection of admissible cover cycles with decorated stratum classes and show how to use this information to compute the admissible cover cycles, in the cases where they are tautological.

1.2 The tautological ring

In the following we give a reminder about the tautological ring on the moduli space of curves. For a more detailed introduction see e.g. [3,17,37].

The moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves $(C, p_1, \ldots, p_n)$ has a stratification according to the topological type of C. This type is encoded in a stable graph A, whose vertices $V(A)$ correspond to the irreducible components of C and whose edges $E(A)$ correspond to nodes of C connecting these components. Further, the vertices of A carry legs $l_1, \ldots, l_n$ indicating on which component the marked points $p_1, \ldots, p_n$ are located.

Given a stable graph A we can parametrize the closure of the locus of stable curves $(C, p_1, \ldots, p_n)$ with type A by a map

$$\xi_A : \overline{\mathcal{M}}_A = \prod_{v \in V(A)} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_{g,n}$$

which glues a union of marked curves by identifying pairs of markings and forms the corresponding stable curve, prescribed by the data of the graph A.

---

[1] In this paper we restrict to the case of rational coefficients, both for Chow groups and for cohomology groups.

[2] See the following section for a reminder about the tautological ring.

On the moduli spaces $\overline{\mathcal{M}}_{g(v),n(v)}$ there are natural cycle classes given by the cotangent line bundles ψ_i and the Arbarello–Cornalba classes κ_a. We call a *decorated stratum class* the pushforward of a product of κ and ψ classes under the morphism ξ_A. In particular, pushing forward the fundamental class of $\overline{\mathcal{M}}_A$ gives (a multiple of) the class of the closure of the locus of curves of type A. The vector spaces $R^\bullet(\overline{\mathcal{M}}_{g,n}) \subset A^\bullet(\overline{\mathcal{M}}_{g,n})$ and $RH^\bullet(\overline{\mathcal{M}}_{g,n}) \subset H^\bullet(\overline{\mathcal{M}}_{g,n})$ spanned by decorated stratum classes turn out to be closed under the intersection product and are called the tautological rings. Alternatively (see [18]) these rings can be defined as the smallest system of $\mathbb{Q}$-subalgebras of $A^\bullet(\overline{\mathcal{M}}_{g,n})$ and $H^\bullet(\overline{\mathcal{M}}_{g,n})$ closed under pushforward along the gluing and forgetful morphisms.

The intersection between two decorated stratum classes has an explicit, combinatorially defined expression in terms of decorated stratum classes. This result was first written down in [27] and has since been implemented as a computer program, see [41,48].

Our primary goal in this paper is to derive a similar formula for the intersection of a locus of admissible covers with a decorated stratum class. In order to state our results, we need to establish some notation in the following section.

1.3 Stacks of admissible *G*-covers

A modular interpretation for the loci of covers can be given by stacks of admissible covers. The definition of admissible covers was first written down in [29] for the case of d-sheeted covers of genus 0 curves and later clarified and extended in [4,7,31]. See Definition 3.2 for a reminder.

In this paper, we are going to study loci of admissible G-covers for a fixed finite group G, that is, admissible covers $\varphi : C \to D$ of curves together with a G-action on C making φ the quotient map.

The reason why we restrict ourselves to G-covers instead of arbitrary covers of fixed degree is that it makes the moduli spaces of such covers behave like G-equivariant versions of moduli spaces of curves, allowing a simple description of the corresponding combinatorics. Since every degree 2 cover is automatically a $\mathbb{Z}/2\mathbb{Z}$-cover, the important cases of hyperelliptic and bielliptic loci are covered by our treatment. For a remark about the case of arbitrary covers see Sect. 1.6 below.

Given a G-cover $\varphi : C \to D$, the ramification points of φ are exactly those smooth points of C having a nontrivial G-stabilizer. The ramification behaviour of φ is encoded in a *monodromy datum* ξ, a tuple of elements of G generating the stabilizers at the various marked points in C.

Fix a genus $g \geq 0$, a finite group G and a monodromy datum $\xi = (h_1, \ldots, h_b) \in G^b$. Then we can define a stack $\overline{\mathcal{H}}_{g,G,\xi}$ whose objects are given by morphisms

$$G \curvearrowright (C, (p_{i,a})_{\substack{i=1,\ldots,b \\ a \in G/\langle h_i \rangle}}) \xrightarrow{\varphi} (D, (q_i)_{i=1,\ldots,b}) \tag{1}$$

where $(C, (p_{i,a})_{i,a})$ and $(D, (q_i)_i)$ are (connected) stable curves, with the genus of C being g. The map $\varphi : C \to D$ is an admissible cover such that the action of G on C is a principal G-bundle outside the preimages of markings and nodes of D. Further

we have $\varphi^{-1}(q_i) = \{p_{i,a} : a \in G/\langle h_i \rangle\}$ for all i. The monodromy of the G-action at the $p_{i,a}$ is given by $a\, h_i\, a^{-1}$ and the G-action permutes the points $p_{i,a}$ according to left-multiplication of G on $a \in G/\langle h_i \rangle$.

The space $\overline{\mathcal{H}}_{g,G,\xi}$ is a smooth proper Deligne-Mumford stack. It is closely related to the stacks of pointed admissible covers defined in [31]. For further discussion of stacks of admissible G-covers, in particular concerning the singularities and birational geometry of their coarse moduli spaces, see also the papers [22,23].

A crucial difference between our paper and [31] is that we require the curve C to be connected, while in [31] also disconnected admissible covers are considered. We ask C to be connected since it allows us to define maps

$$
\begin{array}{ccc}
\overline{\mathcal{H}}_{g,G,\xi} & \xrightarrow{\phi} & \overline{\mathcal{M}}_{g,r} \\
\downarrow{\scriptstyle\delta} & & \\
\overline{\mathcal{M}}_{g',b} & &
\end{array}
\tag{2}
$$

where r and b are the number of ramification and branch points of the admissible cover and where ϕ sends the admissible cover (1) to the stable curve $(C, (p_{i,a})_{i,a})$ and δ sends it to $(D, (q_i)_i)$. The map ϕ is representable, finite, unramified and a local complete intersection, while δ is flat, proper and quasi-finite, but in general not representable.

We obtain a cycle class $\phi_*[\overline{\mathcal{H}}_{g,G,\xi}] \in A^\bullet(\overline{\mathcal{M}}_{g,r})$. These cycles (and their pushforwards under forgetful maps $\pi : \overline{\mathcal{M}}_{g,r} \to \overline{\mathcal{M}}_{g,n}$) are the main objects of study in this paper. These include the cycles of loci $\overline{\mathcal{H}}_{g,n_1,2n_2}$ and $\overline{\mathcal{B}}_{g,n_1,2n_2}$ of hyperelliptic and bielliptic curves mentioned above.

1.4 Intersection results

Now we are ready to present our results about the intersection of $\phi_*[\overline{\mathcal{H}}_{g,G,\xi}]$ with a cycle $\xi_{A*}[\overline{\mathcal{M}}_A]$ coming from a boundary stratum for a stable graph A. In the result, we use that, similarly to $\overline{\mathcal{M}}_{g,n}$, the space $\overline{\mathcal{H}}_{g,G,\xi}$ has a stratification by topological type. Roughly, the strata are enumerated by the stable graphs Γ of the curves C together with a G-action on Γ induced by the G-action on the components, nodes and markings of C (see Sect. 3.4). Denoting the graph with G-action by (Γ, G), the corresponding locus in $\overline{\mathcal{H}}_{g,G,\xi}$ admits a parametrization by an equivariant gluing map

$$
\xi_{(\Gamma,G)} : \overline{\mathcal{H}}_{(\Gamma,G)} = \prod_{w \in V(\Gamma)/G} \overline{\mathcal{H}}_{g(w),G_w,\xi_w} \to \overline{\mathcal{H}}_{g,G,\xi}.
$$

The factors $\overline{\mathcal{H}}_{g(w),G_w,\xi_w}$ correspond to *orbits* of vertices in Γ under G. Indeed, for a curve in $\overline{\mathcal{H}}_{g,G,\xi}$ with stable graph (Γ, G), an orbit of a vertex w in Γ corresponds to an orbit of components of the curve. Since G acts by isomorphisms of the curve, all these components are isomorphic, and their isomorphism type is specified by an element of $\overline{\mathcal{H}}_{g(w),G_w,\xi_w}$.

One checks that the intersection of (the image of) $\phi : \overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{M}}_{g,r}$ with (the image of) ξ_A is given as the union of the spaces $\overline{\mathcal{H}}_{(\Gamma,G)}$ such that Γ is a specialization of the graph A. Technically speaking this means that we have a morphism $f : \Gamma \to A$ of stable graphs (also called an A-structure on Γ or a contraction of Γ) such that each G-orbit of edges of Γ contains at least one edge coming from A. For details see Definition 4.1. We call such an f a *generic A-structure on* (Γ, G). Denote by $\mathfrak{H}_{A;G,\xi}$ the set of isomorphism classes of generic A-structures (Γ, G, f) for a given space $\overline{\mathcal{H}}_{g,G,\xi}$.

In this situation, the composition of $\xi_{(\Gamma,G)}$ with the map $\phi : \overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{M}}_{g,r}$ factors through the gluing maps ξ_Γ and ξ_A via a diagram

$$
\phi_f \left(
\begin{array}{ccc}
\overline{\mathcal{H}}_{(\Gamma,G)} & \xrightarrow{\xi_{(\Gamma,G)}} & \overline{\mathcal{H}}_{g,G,\xi} \\
\downarrow{\phi_{(\Gamma,G)}} & & \downarrow{\phi} \\
\overline{\mathcal{M}}_\Gamma & \xrightarrow{\xi_\Gamma} & \overline{\mathcal{M}}_{g,r} \\
\downarrow & \nearrow{\xi_A} & \\
\overline{\mathcal{M}}_A & &
\end{array}
\right. \tag{3}
$$

Theorem 1 *see Theorem* 4.9. *We have*

$$
\xi_A^* \phi_*([\overline{\mathcal{H}}_{g,G,\xi}]) = \sum_{(\Gamma,G,f) \in \mathfrak{H}_{A;G,\xi}} c_{\mathrm{top}}(E_f) \cap \phi_{f*}[\overline{\mathcal{H}}_{(\Gamma,G)}].
$$

where $c_{\mathrm{top}}(E_f)$ is the top Chern class of an excess intersection bundle. It is the product of factors $-(\psi_h + \psi_{h'})$ for some half-edges h, h' of A, as described in Proposition 4.7.

By the projection formula this theorem allows us to express the intersection of $\phi_*[\overline{\mathcal{H}}_{g,G,\xi}]$ with a decorated stratum class for the graph A using tautological classes and the fundamental classes of the factors in $\overline{\mathcal{H}}_{(\Gamma,G)}$. One thing to note is that in the description of the map $\phi_{(\Gamma,G)}$, diagonal embeddings $\Delta : \overline{\mathcal{M}}_{g_i,n_i} \to (\overline{\mathcal{M}}_{g_i,n_i})^{\times m}$ naturally appear (see Sect. 3.4). Thus to give a complete description of the intersection of $\phi_*[\overline{\mathcal{H}}_{g,G,\xi}]$ with a decorated stratum class, we need to understand the class of the diagonal. While this class will not be tautological in general, it is tautological for many small (g, n) and in that case we can describe its Kunneth decomposition (see Application 2.27).

There are several variants of the above result. With a bit more care, it is possible to replace $\phi_*[\overline{\mathcal{H}}_{g,G,\xi}]$ by a pushforward $\pi_*\phi_*[\overline{\mathcal{H}}_{g,G,\xi}]$ from a moduli space with more markings (in other words, we forget some of the ramification points). This is described in Sect. 4.4.

Another interesting consequence of the proof of the theorem above is the following result.

Theorem 2 *see Theorem* 4.10 *and Corollary* 4.11 *For the maps* ϕ, δ *as in the diagram* (2), *the composition*

$$\delta_*\phi^*\colon A^\bullet(\overline{\mathcal{M}}_{g,r}) \to A^\bullet(\overline{\mathcal{M}}_{g',b})$$

sends tautological classes to tautological classes. More precisely, on decorated boundary strata, this map admits an explicit, combinatorial formula.

In particular, this theorem allows us to compute intersection numbers of cycles $\phi_*[\overline{\mathcal{H}}_{g,G,\xi}]$ with decorated boundary strata of the complementary dimension: these strata are mapped to tautological zero cycles via $\delta_*\phi^*$ and we can compute the degrees of those explicitly.

See the papers [12,13] for a different approach to boundaries of loci of curves with given automorphism groups from the perspective of hyperbolic surfaces.

1.5 Applications

One of the main applications of the results in the previous section is that they often allow us to compute cycles of admissible covers in terms of decorated stratum classes in the case that these cycles are tautological. Let $\alpha \in H^{2i}(\overline{\mathcal{M}}_{g,n})$ be an admissible cover cycle, i.e. a cycle of the form $\alpha = \pi_*\phi_*[\overline{\mathcal{H}}_{g,G,\xi}]$ for some forgetful map $\pi \colon \overline{\mathcal{M}}_{g,r} \to \overline{\mathcal{M}}_{g,n}$. Assume that α is tautological.

The first method to compute α is to calculate its intersection numbers with tautological cycles of the complementary cohomological degree $2(3g - 3 + n - i)$. If the pairing

$$RH^{2i}(\overline{\mathcal{M}}_{g,n}) \otimes RH^{2(3g-3+n-i)}(\overline{\mathcal{M}}_{g,n}) \to \mathbb{Q}$$

is perfect in the relevant degree (e.g. if all cohomology is tautological) then these intersection numbers completely determine α.

A second method is to compute the pullback of α via boundary maps $\xi_A \colon \overline{\mathcal{M}}_A \to \overline{\mathcal{M}}_{g,n}$ for various stable graphs A. Using a variant of Theorem 1 (see Theorem 4.10) we can express the result in terms of cycles of smaller-dimensional spaces $\overline{\mathcal{H}}_{g_i,G_i,\xi_i}$, which can be computed by recursion. Comparing the result with the pullbacks of a basis of $RH^{2i}(\overline{\mathcal{M}}_{g,n})$ we obtain a linear condition on $\alpha \in RH^{2i}(\overline{\mathcal{M}}_{g,n})$. Combining this information for different graphs A is often sufficient to compute α.

Since these computations quickly become untractable by hand, we have implemented the operations discussed in this section in a computer program [14], written in SAGE [44]. This program uses several functions already implemented in [41]. By now, the program has been expanded to allow more general computations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ and has found a range of other applications (see [14] for details).

Using the program we are able to greatly extend the list of such classes which can be computed (as well as repeat and verify most of the afformentioned calculations). In Figs. 1 and 2 we give lists of the values (g, n_1, n_2) for which we can calculate the cycles $[\overline{\mathcal{H}}_{g,n_1,2n_2}]$ and $[\overline{\mathcal{B}}_{g,n_1,2n_2}]$ of hyperelliptic and bielliptic curves as sums of decorated

g	0			1					2				3		4		5		6
n_1	0	1	2	0	1	2	3	4	0	1	2	3	0	1	0	1	0	1	0
n_2	4	3	2	2	2	1	1	0	2	0	0	0	1	0	0	0	0	0	0

Fig. 1 Values of g, n_1, n_2 for which we computed the cycle $[\overline{\mathcal{H}}_{g,n_1,2n_2}]$ explicitly

Fig. 2 Values of g, n_1, n_2 for which which we computed the cycle $[\overline{\mathcal{B}}_{g,n_1,2n_2}]$ explicitly

g	1	2			3		4
n_1	0	0	1	2	0	1	0
n_2	2	1	0	0	0	0	0

stratum classes. Note that in the hyperelliptic case, we always have $n_1 \leq 2g + 2$ and in the bielliptic case $n_1 \leq 2g - 2$.

As explicit examples, we give expressions for the cycles $[\overline{\mathcal{H}}_5] \in H^6(\overline{\mathcal{M}}_5)$ and $[\overline{\mathcal{B}}_4] \in H^6(\overline{\mathcal{M}}_4)$ in Theorems 5.5 and 5.7. We describe the computation of $[\overline{\mathcal{H}}_6] \in H^8(\overline{\mathcal{M}}_6)$ in Remark 5.6.

Together with the program [14], these explicit formulas can be used to further study properties of admissible cover cycles. For example, in Remark 5.14 we explain how the formulas for $[\overline{\mathcal{B}}_g]$ have been used to verify expressions for Hodge integrals on bielliptic loci shown in [43].

Our methods also allow us to go beyond the case of double covers. In Sect. 5.4 we give some examples of cycles of cyclic triple covers that we can currently compute, which can be used to verify of formulas from [35] of Hodge integrals against such loci.

1.6 Outlook

The results in our paper show that cycles of admissible covers behave very well under intersection with tautological classes. It is therefore natural to define an extension of the tautological ring obtained by adding all such cycles. As shown in [27,47], this eventually gives strictly more cycle classes.

However, to work with this extension we must also understand the intersection of different admissible cover cycles. We investigate a first example of this in Sect. 5.5, where we pull back the locus of bielliptic curves in $\overline{\mathcal{M}}_4$ to the space of hyperelliptic curves. As shown there, this intersection between the hyperelliptic and bielliptic locus again has a neat description in terms of admissible cover spaces and the Chern class of the excess bundle is tautological. We expect this to be true in general, which would allow us to write down an explicit additive set of generators for the extended tautological ring, together with a combinatorial description of their intersections.

Another direction is the generalization from admissible G-covers to arbitrary covers of a fixed degree d. As described in [4, Section 4.2], for every admissible cover $E \to D$ of degree d there is a S_d-admissible cover $C \to D$ such that $E = C/S_{d-1}$, where S_d

is the symmetric group. In other words we have a diagram

$$
\begin{array}{ccc}
 & C & \\
{\scriptstyle/S_{d-1}}\downarrow & & \searrow{\scriptstyle/S_d} \\
E & \longrightarrow & D
\end{array}
$$

This suggests, given a space $\overline{\mathcal{H}}_{g,G,\xi}$ and a subgroup $H \subset G$, to consider the map

$$\phi_H : \overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{M}}_{g'',b'},$$
$$\left[(G \curvearrowright (C, (p_{i,a})_{i,a}) \to (D, (q_i)_i)) \right] \mapsto (C/H, (\eta(p_{i,a}))_{i,a}),$$

where $\eta : C \to C/H$ is the quotient map. For the trivial subgroup $H = \{e_G\} \subset G$ we obtain our previous map ϕ, for $H = G$ we obtain the map δ. By the argument above, for $G = S_d$ the cycles $\phi_{S_{d-1}*}[\overline{\mathcal{H}}_{g,G,\xi}]$ allow us to describe all degree d admissible covers, and we expect that they behave similarly to the cycles $\phi_*[\overline{\mathcal{H}}_{g,G,\xi}]$ we saw above. We plan to investigate this in further papers.

Finally, it is possible to look at the intersection theory on the spaces $\overline{\mathcal{H}}_{g,G,\xi}$ themselves. There is a natural definition of a tautological ring $R^\bullet(\overline{\mathcal{H}}_{g,G,\xi}) \subset A^\bullet(\overline{\mathcal{H}}_{g,G,\xi})$ and thus of tautological relations in this ring. A first basic way to obtain such relations is to pull back usual tautological relations on $\overline{\mathcal{M}}_{g',b}$ via the target map $\delta : \overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{M}}_{g',b}$. Any tautological relation on $\overline{\mathcal{H}}_{g,G,\xi}$ pushes forward to a relation between cycles from admissible cover spaces under the source map $\phi : \overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{M}}_{g,r}$. Such relations can potentially be used to express one admissible cover cycle in terms of smaller-dimensional ones, enabling us to compute them recursively.

1.7 Outline of the paper

In Sect. 2 we recall the definition of decorated stratum classes and compute the intersection between such classes following [27]. This section mainly serves as a warm-up and to establish notation. In Sect. 3 we introduce the stack of admissible covers and state a number of facts for later. We compute the intersection between spaces of admissible covers and decorated stratum classes in Sect. 4. Finally in Sect. 5 we show how these theorems can be used to compute classes of admissible covers in terms of decorated stratum classes. We give a number of examples explicitly.

2 Intersections in the tautological ring

In this section we will compute the intersection between two decorated stratum classes in terms of decorated stratum classes. This was first done in [27]. The purpose of this section is to introduce notation and to serve as a warmup for later when we will compute the intersection between decorated stratum classes and stacks of pointed admissible G-covers. Readers familiar with tautological classes can likely skip to Sect. 3 and refer back to this section as necessary.

Definition 2.1 Recall that an *undirected finite graph* (or simply *graph*) is a triple

$$(V, E, s: E \to \mathrm{Sym}^2 V)$$

where V is a finite set of vertices, E is a finite set of edges and s sends each edge into the second symmetric power of V thus assigning two vertices to every edge.

Definition 2.2 We define a *stable graph* Γ to be the data

$$\Gamma := (V, H, L, g: V \to \mathbb{Z}_{\geq 0}, \iota: H \to H, a: H \to V, \zeta: L \to V)$$

which satisfies the following conditions:

  (i) ι is a fixed point free involution,
  (ii) let E be the set of orbits of ι and let $s: E \to \mathrm{Sym}^2 V$ be the function induced by a on E then (V, E, s) is a *connected graph*,
  (iii) for each vertex $v \in V$ the stability condition $2g(v) - 2 + |a^{-1}(v)| + |\zeta^{-1}(v)| > 0$ is satisfied.

Notation 2.3 We call the elements of H *half edges* and the elements of L *legs*. We call g the *genus function* and the genus of a vertex is defined to be $g(v)$. For a vertex $v \in V$ we set $n(v) = |a^{-1}(v)| + |\zeta^{-1}(v)|$ the number of *half edges plus legs incident to* v. The *genus of a stable graph* Γ is defined to be

$$g(\Gamma) := \sum_{v \in V} g(v) + h^1(\Gamma).$$

We will say that a stable graph Γ is *n-pointed* if $n = |L|$.

Example 2.4 Given a stable curve C over $\mathrm{Spec}\,\mathbb{C}$, we associate a stable graph to C as follows. Let V be the set of irreducible components of C, let H be the set of sections of the normalization of C corresponding to the nodes of C, L the set of marked points of C, g the function sending the irreducible components to the genus of their normalization, ι the involution identifying sections $h \in H$ mapping to the same node of C, a the map identifying sections with the irreducible component they lie on and ζ the map sending a marked point to its corresponding irreducible component. The graph $(V, H, L, g, \iota, a, \zeta)$ is stable and is called the *dual graph* of C (see Fig. 3 for an illustration).

The following definition formalizes the concept that a stable graph A is a contraction of a stable graph Γ. We follow the terminology in [27], see also [25,33] for related definitions.

Definition 2.5 Let A and Γ be genus g stable graphs with set of legs $L_\Gamma = L = L_A$. An A-structure on Γ is a triple

$$(\alpha: V_\Gamma \twoheadrightarrow V_A, \ \beta: H_A \hookrightarrow H_\Gamma, \ \gamma: H_\Gamma \setminus \mathrm{Im}\,\beta \to V_A)$$

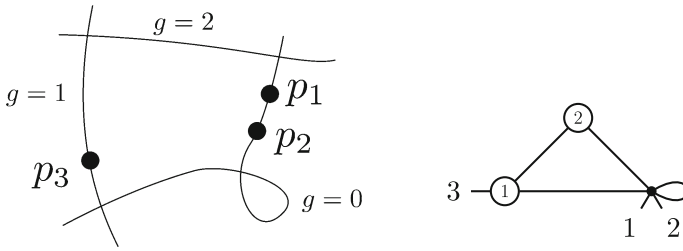which satisfies the following conditions:

Fig. 3 A curve and its dual graph. When the genus of a vertex is 0 we shall depict it as a black dot

(i) the map β commutes with the involutions, i.e. $\beta \circ \iota_A = \iota_\Gamma \circ \beta$,

(ii) the map α respects the leg assignments, i.e. $\alpha \circ \zeta_\Gamma = \zeta_A$,

(iii) if $h \in \operatorname{Im}\beta$ and $a_\Gamma(h) = v$ then $a_A(\beta^{-1}(h)) = \alpha(v)$,

(iv) if $h \in H_\Gamma \setminus \operatorname{Im}\beta$ and $a_\Gamma(h) = v$ then $\alpha(v) = \gamma(h)$,

(v) if $v \in V_A$ then

$$(\alpha^{-1}(v),\ \gamma^{-1}(v),\ \beta(a_A^{-1}(v)) \cup \zeta_A^{-1}(v),\ g_\Gamma, a_\Gamma, \iota_\Gamma, \zeta)$$

is a stable graph of genus $g(v)$ (where g_Γ, a_Γ and ι_Γ are restricted to the appropriate subsets and ζ is defined by ζ_Γ on $\zeta_A^{-1}(v)$ and by a_Γ on $\beta(a_A^{-1}(v))$).

Remark 2.6 If Γ has an A-structure $f = (\alpha, \beta, \gamma)$ and A has a B-structure $g = (\alpha', \beta', \gamma')$ it is easy to check that there exists a unique $\gamma'' : H_\Gamma \setminus \operatorname{Im}(\beta \circ \beta') \to V_B$ making $(\alpha' \circ \alpha, \beta \circ \beta', \gamma'')$ a B-structure on Γ. Indeed, γ'' is given as $\alpha' \circ \gamma$ on $H_\Gamma \setminus \operatorname{Im}\beta$ and as $\gamma' \circ \beta^{-1}$ on $\operatorname{Im}\beta \setminus \operatorname{Im}\beta \circ \beta'$. We can therefore define a *morphism of n pointed genus g stable graphs* $\Gamma \to A$ as an A-structure on Γ. An *isomorphism* of stable graphs $f : A \to B$ is thus a B-structure $f = (\alpha, \beta, \gamma)$ on A and an A-structure $g = (\alpha', \beta', \gamma')$ on B such that $g \circ f = (\operatorname{id}_{V_A}, \operatorname{id}_{H_B}, \operatorname{id}_\emptyset)$ and $f \circ g = (\operatorname{id}_{V_B}, \operatorname{id}_{H_A}, \operatorname{id}_\emptyset)$.

We have formed an essentially finite category of stable graphs of genus g with set of legs L.

Example 2.7 Let

$$\Gamma = \quad , \quad A = $$
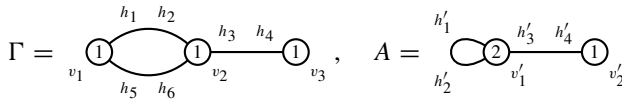
One A-structure on Γ is

$$\alpha : v_1, v_2 \mapsto v_1'$$
$$v_3 \mapsto v_2'$$
$$\beta : h_i' \mapsto h_i$$
$$\gamma : h_5, h_6 \mapsto v_1'.$$

In total there are 4 different A-structures which can be given to Γ. Indeed we can send h_1' to h_1, h_2, h_5, or h_6 and each of these choices completely determines an A-structure on Γ.

Notation 2.8 For a stable graph Γ we define

$$\overline{\mathcal{M}}_\Gamma := \prod_{v \in V} \overline{\mathcal{M}}_{g(v), n(v)}.$$

There is a natural *gluing morphism*

$$\xi_\Gamma \colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$$

defined by Γ. It glues the curve

$$\left(\coprod_{v \in V} C_v \to S \ ; \ (\sigma_h \colon S \to C_{a(h)})_{h \in H}, \ (\sigma'_l \colon S \to C_{\zeta(l)})_{l \in L} \right) \in \overline{\mathcal{M}}_\Gamma$$

over S together by gluing the section σ_h to $\sigma_{\iota(h)}$.

The morphism ξ_Γ is a representable morphism of Deligne-Mumford stacks (see [3, Proposition 10.25]) and since both its domain and codomain are smooth and complete it is an lci and proper morphism.

If we have a morphism of stable graphs $\Gamma \to A$ we can construct a corresponding map of moduli spaces

$$\xi_{\Gamma \to A} \colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_A$$

defined as a composition of gluing morphisms component wise.

Remark 2.9 The image of ξ_Γ in $\overline{\mathcal{M}}_{g,n}$ equals the closure of the locus of all curves in $\overline{\mathcal{M}}_{g,n}$ with dual graph isomorphic to Γ. The generic degree of ξ_Γ equals the order of the automorphism group of Γ.

Let A and B be n pointed genus g stable graphs. We will compute the intersection

$$\xi_{A*}([\overline{\mathcal{M}}_A]) \cdot \xi_{B*}([\overline{\mathcal{M}}_B])$$

as an explicit sum of classes of $\overline{\mathcal{M}}_{g,n}$. By the excess intersection formula (see for example [21, Proposition 17.4.1]) we have to identify the fiber product

$$
\begin{array}{ccc}
\mathcal{F}_{A,B} & \xrightarrow{p_2} & \overline{\mathcal{M}}_B \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \xi_B} \\
\overline{\mathcal{M}}_A & \xrightarrow{\xi_A} & \overline{\mathcal{M}}_{g,n}
\end{array}
$$

and the top Chern class of the excess intersection bundle $E = p_1^* N_{\xi_A} / N_{p_2}$. We will prove $\mathcal{F}_{A,B}$ is isomorphic to a disjoint union of stacks $\overline{\mathcal{M}}_\Gamma$ where Γ is a graph which appears as a specialization of both A and B.

Definition 2.10 We will say that Γ has an (A, B)-*structure* (f, g), or that Γ *is an* (A, B)-*graph*, if Γ has both an A-structure $f = (\alpha_A, \beta_A, \gamma_A)$ and a B-structure $g = (\alpha_B, \beta_B, \gamma_B)$. We say that (f, g) is *a generic* (A, B)-*structure* on Γ if every edge of Γ corresponds to an edge of A or an edge of B, i.e. if

$$\beta_A(H_A) \cup \beta_B(H_B) = H_\Gamma.$$

We say that two stable graphs Γ, Γ' with (A, B)-structures (f, g) and (f', g') are isomorphic (A, B)-graphs if there exists an isomorphism $\tau : \Gamma \to \Gamma'$ such that the following diagram commutes

$$
\begin{array}{ccc}
 & B & \\
{\scriptstyle g}\nearrow & & \uparrow{\scriptstyle g'} \\
\Gamma \xrightarrow{\;\tau\;} & \Gamma' & \cdot \\
{\scriptstyle f}\searrow & & \downarrow{\scriptstyle f'} \\
 & A &
\end{array}
$$

Example 2.11 Let

$$A = B = \underset{h_2}{\overset{h_1}{\textcircled{3}\,\bigcirc}} \; .$$

There are three stable graphs which admit a generic (A, B)-structure:

$$\Gamma_1 = \underset{h'_2}{\overset{h'_1}{\textcircled{3}\,\bigcirc}} \qquad\qquad \Gamma_2 = \underset{h'_3\; h'_4}{\overset{h'_1\; h'_2}{\textcircled{2}\,\subset\!\!\supset\,\textcircled{1}}} \qquad\qquad \Gamma_3 = \underset{h'_2}{\overset{h'_1}{\bigcirc\,\textcircled{2}\,\bigcirc}}\overset{h'_3}{\underset{h'_4}{}} \; .$$

The stable graph Γ_1 has two isomorphism classes of generic (A, B)-structures (f, g): Set $f = (\alpha_f, \beta_f, \gamma_f)$ and $g = (\alpha_g, \beta_g, \gamma_g)$. Up to isomorphism we can assume that $\beta_f(h_1) = h'_1$, there are then two possible nonisomorphic choices for the images of $\beta_g(h_1)$, namely h'_1 and h'_2.

The graph Γ_2 has four isomorphism classes of generic (A, B)-structures: There is always an isomorphism such that β_f sends the edge (h_1, h_2) to (h'_1, h'_2). We then have $\beta_f(h_1) = h'_1$ or $\beta_f(h_1) = h'_2$ and these choices are nonisomorphic. Since the (A, B)-structure is generic β_g must send (h_1, h_2) to (h'_3, h'_4) and we again have 2 choices.

The stable graph Γ_3 has only one isomorphism class of stable (A, B)-structures (f, g). Up to isomorphism we have $\beta_f(h_1) = h'_1$ and $\beta_g(h_1) = h'_3$.

Notation 2.12 Let A and B be stable n pointed genus g graphs. We will denote by $\mathfrak{G}_{A,B}$ a set of representatives of the set of all isomorphism classes of generic (A, B)-graphs. We set

$$\mathcal{X} = \coprod_{(\Gamma, f, g) \in \mathfrak{G}_{A,B}} \overline{\mathcal{M}}_\Gamma.$$

Proposition 2.13 (see Proposition 9, [27]) *There is a natural isomorphism $\mathcal{X} \xrightarrow{\sim} \mathcal{F}_{A,B}$.*

In the following we recall the proof of the above proposition, since our proofs in Sect. 4.1 will follow a similar strategy and we introduce some essential terminology on the way.

We start by giving a different modular interpretation of $\overline{\mathcal{M}}_\Gamma$ for any stable graph Γ (see [3, page 315] or [27, Section A2]):

Definition 2.14 Let $\Gamma = (V, H, L, g, \iota, a, \zeta)$ be an n-pointed stable graph of genus g and let C be an n-pointed stable curve

$$\pi : C \to S, \qquad s_i : S \to C \quad i = 1, ..., n$$

of genus g over a connected base S. A Γ-*marking on C* is the following additional data: (this is called a Γ-structure in [27])

(i) #E additional disjoint sections $\sigma_1, ..., \sigma_{e(\Gamma)}$ of π with image in the singular locus of C,
(ii) #H sections $\tilde{\sigma}_{1,1}, \tilde{\sigma}_{1,2}, \tilde{\sigma}_{2,1}, ..., \tilde{\sigma}_{e(\Gamma),2}$ of the normalization $\tilde{C}$ of C along the sections σ_i,
(iii) #V disjoint connected components C_v of $C \setminus \{\sigma_i\}$ whose union is $C \setminus \{\sigma_i\}$ and such that each C_v remains connected upon pullback along any morphism $S' \to S$ of base schemes (we shall call such components π-*relative components of $C \setminus \{\sigma_i\}$*),
(iv) a choice of isomorphism between Γ and the stable graph

$$(\{C_v\}, \{\tilde{\sigma}_{i,j}\}, \{s\}, g, \iota, \alpha, \zeta)$$

where $g(C_v)$ is the arithmetic genus of C_v, the involution ι is defined by $\iota(\tilde{\sigma}_{i,1}) = \tilde{\sigma}_{i,2}$, α maps $\tilde{\sigma}_{i,j}$ to the π-relative component corresponding to the component of $\tilde{C}$ it lies on and ζ maps s'_i to the π-relative component it lies on.

We will denote the curve C together with the data of a Γ-marking on C by C_Γ.

The data of a Γ-marking on a stable curve can be pulled back under any morphism of connected base schemes. We can therefore define a stack $\overline{\mathcal{M}}'_\Gamma$ whose objects consist of stable curves with Γ-marking and whose morphisms respects the Γ-marking under pullback.

Proposition 2.15 (See Proposition 8, [27]) *There exists a natural isomorphism between $\overline{\mathcal{M}}_\Gamma$ and $\overline{\mathcal{M}}'_\Gamma$.*

Proof We can construct a natural morphism from $\overline{\mathcal{M}}_\Gamma$ to $\overline{\mathcal{M}}'_\Gamma$ by assigning the canonical Γ-marking to the universal curve over $\overline{\mathcal{M}}_\Gamma$. In the other direction given a S valued point of $\overline{\mathcal{M}}'_\Gamma$ we naturally obtain a collection of $v(\Gamma)$ stable curves by analysing the π-relative components of C. Since we have a bijection between these curves and $v(\Gamma)$ and a bijection between the sections of the normalization of C and the sections of curves in $\overline{\mathcal{M}}_\Gamma$ we obtain a S valued point of $\overline{\mathcal{M}}_\Gamma$. It is straightforward to check that this correspondence induces a bijection on the space of morphisms between corresponding objects. $\qquad\square$

Proof of Proposition 2.13 In this proof we will always use the modular interpretation of curves with a Γ-structure for $\overline{\mathcal{M}}_\Gamma$ and write $\overline{\mathcal{M}}_\Gamma$ for $\overline{\mathcal{M}}'_\Gamma$ everywhere.

Let $u\colon \mathcal{X} \to \overline{\mathcal{M}}_A$ be the map defined as $\xi_{f\colon \Gamma \to A}\colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_A$ on the connected component $\overline{\mathcal{M}}_\Gamma$ of $\mathcal{X}$ indexed by (Γ, f, g). Similarly define $v\colon \mathcal{X} \to \overline{\mathcal{M}}_B$ to be the map $\xi_{g\colon \Gamma \to B}\colon \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_B$ on the connected component of $\mathcal{X}$ indexed by (Γ, f, g). An object (C_Γ, f, g) of $\mathcal{X}$, over a connected base scheme S, consists of a graph Γ with a generic (A, B)-structure (f, g) together with a stable curve C over S endowed with a Γ-marking. Let (C_Γ, f, g) be one such object of $\mathcal{X}$ over S. By definition we have $\xi_A(u(C_\Gamma, f, g)) = C = \xi_B(v(C_\Gamma, f, g))$ a natural isomorphism $\xi_A \circ u \Rightarrow \xi_B \circ v$ is therefore given by the identity. We have the following diagram:

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\;\;v\;\;} & \\
& {\scriptstyle q \searrow} & \\
& \mathcal{F}_{A,B} & \xrightarrow{\;p_2\;} \overline{\mathcal{M}}_B \\
{\scriptstyle u} & {\scriptstyle p_1}\big\downarrow \quad \Rightarrow & \big\downarrow{\scriptstyle \xi_B} \\
& \overline{\mathcal{M}}_A & \xrightarrow{\;\xi_A\;} \overline{\mathcal{M}}_{g,n}.
\end{array}
$$

where the map q is given by the strict universal property of the fiber product. It sends the object (C_Γ, f, g) over S to the object $(C_A, C_B, \mathrm{id}_C)$ over S and a morphism $C'_\Gamma \to C_\Gamma$ over $S' \to S$ to the induced pair of morphisms $(C'_A \to C_A, C'_B \to C_B)$.

We want to prove that q is an isomorphism. We will do so by defining a map $r\colon \mathcal{F}_{A,B} \to \mathcal{X}$ and showing that $r \circ q$ and $q \circ r$ are naturally isomorphic to the respective identities on $\mathcal{X}$ and on $\mathcal{F}_{A,B}$.

Let $(D_A, C_B, \alpha\colon D \xrightarrow{\sim} C)$ be an object of $\mathcal{F}_{A,B}$ over S. Following the notation of Definition 2.14 we will define an A-marking on C by passing through the isomorphism α:

  (i) if $\{\sigma_i\}$ are the $e(A)$ sections of $\varpi\colon D \to S$ in the singular locus of D defined by the A-marking, we get sections $\sigma'_i := \alpha \circ \sigma_i$ in the singular locus of C,

  (ii) the pullback of α along the partial normalization $\tilde{C} \to C$ defines a map $\tilde{\alpha}\colon \tilde{D} \to \tilde{C}$, in this way we obtain sections $\{\tilde{\sigma}'_{i,j}\} := \{\tilde{\alpha} \circ \tilde{\sigma}_{i,j}\}$ in the partial normalization $\tilde{C}$ of C at $\{\sigma'_i\}$,

  (iii) if $D_{v,A}$ are the ϖ' relative components of D then the π-relative components are given by $C_{v,A} := \alpha(D_{v,A})$,

  (iv) we obtain an isomorphism of stable graphs by the composition

$$
(\{C_{v,A}\}, \{\tilde{\sigma}'_{i,j}\}, \{\alpha \circ s_k\}, g', \iota', a', \zeta') \longrightarrow (\{D_{v,A}\}, \{\tilde{\sigma}_{i,j}\}, \{s_k\}, g, \iota, a, \zeta) \xrightarrow{\;\lambda\;} A
$$

where λ is the isomorphism of stable graphs defined by the A-structure on D.

Let τ_i be the sections on C defined by the B-structure, $C_{v,B}$ the π-relative components defined by the B-structure. The curve C now comes with the following structure:

  i a set of sections $E := \{\sigma'_i\} \cup \{\tau_i\}$ of π in the singular locus of C,

  ii a set of sections $H := \{\tilde{\sigma}'_{i,j}\} \cup \{\tilde{\tau}_{i,j}\}$ in the partial normalization of C at E,

  iii a set of π-relative components V of $C\backslash\{\sigma'_i\} \cup \{\tau_i\}$.

This data defines a stable graph

$$\Gamma := (V, H, \{\alpha(s_i)\}, g, \iota, a, \zeta)$$

as in Definition 2.14.iv (where the $\alpha(s_i)$ are the n sections of C outside of the singular locus corresponding to the marked points). This data defines a Γ-marking on C.
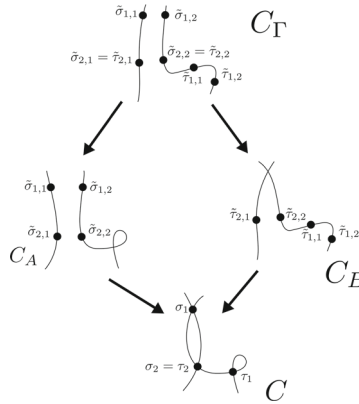
This Γ has an (A, B)-structure, indeed let $\alpha_A \colon V \to V_A$, be the map of π-relative components given by the inclusion $C \backslash \{\sigma_i'\} \cup \{\tau_i\} \hookrightarrow C \backslash \{\sigma_i'\}$, let $\beta_A \colon H_A \hookrightarrow H$ be the obvious inclusion of sections and let $\gamma_A \colon H \backslash \operatorname{Im} \beta \to H_A$ be the map that sends $\tilde{\tau}_{i,j}$ to the π-relative component $C_{v,A}$ in which τ_i lies. In this way we have constructed an A-structure $f = (\alpha_A, \beta_A, \gamma_A)$ on Γ and we can define a B-structure g on Γ in the same way. Clearly $H = \beta_A(H_A) \cup \beta_B(H_B)$, so the (A, B)-structure is generic. In other words we have defined an object (C_Γ, f, g) of $\mathcal{X}$ over S. This completes the definition of the functor r on the objects of $\mathcal{F}_{A,B}$.

Let $(\lambda_1 \colon D_A' \to D_A, \lambda_2 \colon C_B' \to C_B)$ be a morphism in $\mathcal{F}_{A,B}$ over $\lambda \colon S' \to S$. Let $C_{\Gamma'}'$ and C_Γ be the curves with Γ' and Γ-structure defined as above by D_A', C_B' and respectively D_A, C_B. The maps λ_1 and λ_2 together define an isomorphism of stable graph $\Gamma' \to \Gamma$ which commutes with the respective (A, B)-structures (f', g') and (f, g) of these graphs. In other words this defines an isomorphism of (A, B)-graphs and a map

$$(C_{\Gamma'}', f', g') \to (C_\Gamma, f, g).$$

This completes the definition of the functor r on the morphisms of $\mathcal{F}_{A,B}$.

It remains to check that r and q are inverses of each other. Let (C_Γ, f, g) be an object of $\mathcal{X}$. Then $q(C_\Gamma, f, g) = (C_A, C_B, \operatorname{id}_C)$. If we pass this through the above construction we see that $r(C_A, C_B, \operatorname{id}_C) = (C_{\Gamma'}', f', g')$ where Γ and Γ' are isomorphic (A, B)-graphs. In other words $r \circ q$ is the identity on $\mathcal{X}$.

In the other direction let $(D_A, C_B, \alpha \colon D \to C)$ be a S-point of $\mathcal{F}_{A,B}$. We have

$$q(r(D_A, C_B, \alpha)) = q((C_\Gamma, f, g)) = (C_A, C_B, \alpha).$$

An isomorphism between D_A and C_A is given by passing the A-structure through α as above when we define the functor r. It is clear that this defines an isomorphism of objects $(D_A, C_B, \alpha) \to (C_A, C_B, \alpha)$. It follows that $q \circ r$ is naturally isomorphic to the identity on $\mathcal{F}_{A,B}$. $\qquad \square$

Let A and B be n pointed genus g stable graphs. To identify the excess bundle $E := p_1^* N_{\xi_A}/N_{p_2}$ on $\mathcal{F}_{A,B}$ of the intersection between $\overline{\mathcal{M}}_A$ and $\overline{\mathcal{M}}_B$ we can compute the excess bundle on the connected components of $\mathcal{X}$. For any $(\Gamma, f, g) \in \mathfrak{G}_{A,B}$ consider the diagram

$$
\begin{array}{ccc}
\overline{\mathcal{M}}_\Gamma & \xrightarrow{\xi_{\Gamma \to B}} & \overline{\mathcal{M}}_B \\
{\scriptstyle \xi_{\Gamma \to A}} \downarrow & & \downarrow {\scriptstyle \xi_B} \\
\overline{\mathcal{M}}_A & \xrightarrow{\xi_A} & \overline{\mathcal{M}}_{g,n}
\end{array} \quad .
$$

We want to identify

$$
E_\Gamma := \xi_{\Gamma \to A}^* N_{\xi_A}/N_{\xi_{\Gamma \to B}}.
$$

and to compute the top Chern class $c_{\text{top}}(E_\Gamma)$.

Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful morphism, let ω_π be the dualizing sheaf and let σ_i be the sections $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ given by the n markings. Set

$$
\mathbb{L}_i = \sigma_i^*(\omega_\pi).
$$

Recall (see for example [3, Section 13.3]) that the normal bundle N_{ξ_A} can be identified with

$$
N_{\xi_A} = \bigoplus_{(h,h') \in E_A} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee,
$$

similarly

$$
N_{\xi_{\Gamma \to B}} = \bigoplus_{(h,h') \in E_B \setminus \operatorname{Im} \beta_B} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee.
$$

It follows that

$$
E_\Gamma = \bigoplus_{(h,h') \in \operatorname{Im} \beta_A \cap \operatorname{Im} \beta_B} \mathbb{L}_h^\vee \otimes \mathbb{L}_{h'}^\vee.
$$

Since $c_1(\mathbb{L}_h) = \psi_h$, the top Chern class of E is just the product over the relevant ψ-classes.

In conclusion we get:

Proposition 2.16 *Let A and B be stable graphs, then*

$$\xi_A^* \xi_{B*}([\overline{\mathcal{M}}_B]) = \sum_{\Gamma \in \mathfrak{G}_{A,B}} \xi_{\Gamma \to A*} \left(\prod_{(h,h') \in \mathrm{Im}\,\beta_A \cap \mathrm{Im}\,\beta_B} (-\psi_h - \psi_{h'}) \cdot [\overline{\mathcal{M}}_\Gamma] \right)$$

Proof This follows directly from the excess intersection formula (see [21, Proposition 17.4.1], Proposition 2.13 and the computation above. □

2.1 Decorated stratum classes

We will now define decorated stratum classes and compute the product of two such classes.

For a given space $\overline{\mathcal{M}}_{g,n}$ let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the map forgetting the marking $n + 1$ and stabilizing, with sections $\sigma_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ corresponding to the markings i for $i = 1, \ldots, n$. Let ω_π be the relative dualizing sheaf.

Recall the ψ-classes and (Arbarello–Cornalba) κ classes on $\overline{\mathcal{M}}_{g,n}$ are defined by

$$\psi_i = \sigma_i^* c_1(\omega_\pi) \in A^1(\overline{\mathcal{M}}_{g,n}), \qquad \kappa_j = \pi_*(\psi_{n+1})^{j+1} \in A^j(\overline{\mathcal{M}}_{g,n}).$$

Now let $A = (V, H, L, g, \iota, a, \zeta)$ be a stable graph and $v \in V$. Consider the projection map

$$p_v \colon \overline{\mathcal{M}}_A = \prod_{w \in V} \overline{\mathcal{M}}_{g(w),n(w)} \to \overline{\mathcal{M}}_{g(v),n(v)}.$$

We will set $\kappa_{v,i} := p_v^*(\kappa_i) \in A^i(\overline{\mathcal{M}}_A)$ and $\psi_{v,i} := p_v^*(\psi_i) \in A^1(\overline{\mathcal{M}}_A)$.

Definition 2.17 A *decorated stable graph* A_θ is a stable graph A together with a decoration

$$\theta = \prod_{v \in V} \left(\prod_{i \in a^{-1}(v) \cup \zeta^{-1}(v)} \psi_{v,i}^{a_i} \prod_{j=1}^m \kappa_{v,j}^{b_j} \right) \in A^\bullet(\overline{\mathcal{M}}_A).$$

Remark 2.18 Restricted to each vertex the decoration θ is just a monomial in ψ and κ classes. Given another decoration θ' the product $\theta \cdot \theta'$ in $A^\bullet(\overline{\mathcal{M}}_A)$ is given by

$$\theta \cdot \theta' = \prod_{v \in V} \left(\prod_{i \in a^{-1}(v) \cup \zeta^{-1}(v)} \psi_{v,i}^{a_i+a_i'} \prod_{j=1}^m \kappa_{v,j}^{b_j+b_j'} \right)$$

Notation 2.19 We set

$$[A_\theta] := \frac{1}{|\operatorname{Aut} A|} \xi_{A*}(\theta) \in A^\bullet(\overline{\mathcal{M}}_{g,n})$$

and will call $[A_\theta]$ a *decorated stratum class*. If $\theta = 1$ we will drop it from the notation.
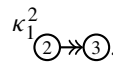
We will draw decorated stratum classes by attaching a number of arrowheads to each half edge or leg i equal to a_i and by attaching the monomial

$$\prod_{j=1}^{m} \kappa_{v,j}^{b_j}$$

to each vertex v.

Example 2.20 Let A be the graph ②—③.
If v_1 is the vertex of genus 2 and v_2 is the vertex of genus 3 and we have a decoration $\theta = \psi_{v_2,h}^2 \kappa_{v_1,1}^2$ we will draw the decorated graph A_θ as

$$\kappa_1^2 \quad \text{②} \twoheadrightarrow \text{③}.$$

Warning 2.21 There are two conflicting notations in the literature for decorated stratum classes. One is as given in 2.19 the other one is without dividing by the size of the automorphism group of A.

The advantage to our definition is that the class $[A]$ is the Poincaré dual of the closure of the locus of all stable curves with dual graph isomorphic to A. In particular $[A]$ corresponds to an actual closed integral substack of $\overline{\mathcal{M}}_{g,n}$. The advantage of not dividing by the order of the automorphism group is that it makes calculations slightly easier.

Remark 2.22 The codimension (or degree) of ψ_i in $A^\bullet(\overline{\mathcal{M}}_{g,n})$ is 1, the codimension of κ_j is j and the codimension of $[A] \in A^\bullet(\overline{\mathcal{M}}_{g,n})$ equals the number of edges of A. Therefore if A_θ is a decorated boundary graph with decoration

$$\theta = \prod_{v \in V} \left(\prod_{i \in a^{-1}(v) \cup \zeta^{-1}(v)} \psi_{v,i}^{a_i} \prod_{j=1}^{m} \kappa_{v,j}^{b_j} \right)$$

then

$$\mathrm{codim}[A_\theta] = \#E_A + \sum_{i \in H \cup L} a_i + \sum_{v \in V, j} j b_{v,j}.$$

To determine the product $[A_\theta] \cdot [B_\lambda]$ for two decorated stable graphs A_θ, B_λ we need to know the pullback of θ under $\xi_{\Gamma \to A}$. Since the pullback is a ring homeomorphism this can be done by pulling back the individual ψ and κ classes in θ.

Lemma 2.23 *Let $f = (\alpha, \beta, \gamma): \Gamma \to A$ be a map of stable graphs, then*

$$\xi_{f: \Gamma \to A}^*(\psi_{v,i}) = \psi_{a_\Gamma \circ \beta(i), \beta(i)},$$

$$\xi_{f: \Gamma \to A}^*(\kappa_{v,i}) = \sum_{w \in \alpha^{-1}(v)} \kappa_{w,i}.$$

Proof The first of these is trivial, the second is [3, Lemma 4.31]. $\qquad\square$

Corollary 2.24 *Let A and Γ be stable graphs, let f be an A-structure on Γ and let θ be a decoration on A. We have*

$$\xi_f^*(\theta) = \prod_{v \in V_A} \left(\prod_{i=1}^{n(v)} \psi_{a_\Gamma \circ \beta_f(i), \beta_f(i)}^{a_i} \prod_{j=1}^{m} \left(\sum_{w \in \alpha_f^{-1}(v)} \kappa_{w,j} \right)^{b_j} \right)$$

Theorem 2.25 *Let A_θ and B_λ be decorated stable n pointed genus g graphs. Then*

$$[A_\theta] \cdot [B_\lambda] = \frac{1}{|\operatorname{Aut} A| \cdot |\operatorname{Aut} B|} \sum_{(\Gamma, f, g) \in \mathfrak{G}_{A,B}} \xi_{\Gamma*}$$

$$\times \left(\xi_f^*(\theta) \cdot \xi_g^*(\lambda) \prod_{(h,h') \in \operatorname{Im} \beta_A \cap \operatorname{Im} \beta_B} (-\psi_h - \psi_{h'}) \right).$$

Proof This follows by pushing forward the expression of Proposition 2.16. $\qquad\square$

Remark 2.26 If A_θ and B_λ define classes in complementary degrees for $\overline{\mathcal{M}}_{g,n}$, i.e. $[A_\theta] \cdot [B_\lambda]$ is a zero cycle, we can compute the degree of this zero cycle using Theorem 2.25. Indeed, the formula above reduces this to the question of computing the intersection numbers of κ and ψ-classes on the factors $\overline{\mathcal{M}}_{g(v),n(v)}$ of the spaces M_Γ. These intersection numbers are governed by the KdV hierarchy as shown by Kontsevich [34] after a conjecture by Witten. Thus in principle, we are able to compute intersection numbers of decorated strata classes.

Application 2.27 As an application of the intersection numbers above, we remark that for some g, n it is possible to express the class of the diagonal $[\Delta] = [\Delta_{\overline{\mathcal{M}}_{g,n}}] \in H^{6g-6+2n}(\overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{g,n})$ in terms of tautological classes. Indeed, assume that $H^*(\overline{\mathcal{M}}_{g,n}) = RH^*(\overline{\mathcal{M}}_{g,n})$ and let $(e_i)_i$ be a basis of the tautological ring as a $\mathbb{Q}$-vector space. By Remark 2.26 we can compute the pairing matrix $\eta_{i,j} = \deg(e_i \cdot e_j)$. Let $(\eta^{i,j})_{i,j}$ be the inverse matrix. Then we have a Kunneth decomposition of the diagonal

$$[\Delta] = \sum_{i,j} \eta^{i,j} e_i \otimes e_j \in H^*(\overline{\mathcal{M}}_{g,n} \times \overline{\mathcal{M}}_{g,n}) = RH^*(\overline{\mathcal{M}}_{g,n}) \otimes RH^*(\overline{\mathcal{M}}_{g,n}).$$

The assumption that all cohomology is tautological is for instance satisfied for all spaces $\overline{\mathcal{M}}_{0,n}$ (see [32]), $\overline{\mathcal{M}}_{1,n}$ for $n \leq 10$ (see [27, Proposition 2] and [38]) and $\overline{\mathcal{M}}_{2,n}$ for $n \leq 8$ (see [39,40]).

We are going to see an application of these tautological Kunneth decompositions in Sect. 5.2.

3 Stacks of pointed admissible *G*-covers

In the following section we introduce the stacks $\overline{\mathcal{H}}_{g,G,\xi}$ parametrizing finite covers $\varphi : C \to D$ of (connected) curves together with a *G*-action on *C*, such that *C* has genus *g* and such that φ is the quotient map under the *G*-action. Further included is the data of a tuple of smooth points $p_{i,a} \in C$, containing all ramification points of φ, such that the behaviour of the *G*-action at these points is described by a monodromy datum ξ.

This problem has been studied in a number of contexts. In [29] Harris and Mumford first introduced the notion of an admissible cover, a technical condition we need to require if the curves *C*, *D* above become nodal. This allowed them to write down a compact moduli space for such covers. However, they did not require the data of a *G*-action on *C* but considered all covers φ of a given degree.

The paper [4] by Abramovich, Corti and Vistoli instead notes that if we take the quotient stack [*C/G*] instead of the quotient *D* = *C/G* in the realm of schemes, the data of the quotient morphism $\varphi : C \to [C/G]$ is equivalent to a stable map [*C/G*] → *BG* of the (stacky) curve [*C/G*] into the classifying stack *BG* of the finite group *G*. In [31], Jarvis, Kaufmann and Kimura take a variant of this construction. The crucial difference is that they consider the data of marked points on *C*, not on [*C/G*]. This is also the convention of our paper and it will be the case that our space $\overline{\mathcal{H}}_{g,G,\xi}$ is a union of connected components of their moduli space. The only further restriction we impose is that the domain *C* of the cover is a connected curve.

Finally, there is the book [7] by Bertin and Romagny, which focuses on the data of the *G*-action on the curve *C*. This action of course determines the data of the quotient map $\varphi : C \to C/G$. This perspective and their results will be useful, since later in our paper we are interested in the map $\phi : \overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{M}}_{g,r}$ which forgets the data of the group action and cover and just remembers the curve *C* with the markings $p_{i,a}$. Here is where we crucially use the connectedness of *C*, since *C* with the marked points $p_{i,a}$ is a stable curve.

Our goal in the following is to give a self-contained introduction to the theory of admissible *G*-covers and to define the stack $\overline{\mathcal{H}}_{g,G,\xi}$. We prove properties and functorialities of this stack which we will use later, often citing relevant results from the literature. We comment on the detailed relation between our convention and those in the references above in Remark 3.6.

3.1 Group actions on smooth curves

As a warm-up we start in the situation of a finite group *G* acting effectively on a smooth curve *C*, i.e. via an injective group homomorphism $G \hookrightarrow \mathrm{Aut}(C)$. Then the quotient *D* = *C/G* exists (as a scheme) and it is itself a smooth curve. The corresponding quotient map $\varphi : C \to D$ is a *Galois cover*, i.e. the group of automorphisms of *C* over *D* acts transitively on the fibres and is in fact equal to *G*.

A general point of *D* has exactly |*G*| preimages under φ, corresponding to the fact that the general point of *C* has trivial stabilizer in *G*. The ramification points *p* of φ

are exactly those points which have a nontrivial stabilizer $H = G_p$. We recall the following result about the local action of H around p.

Lemma 3.1 *Let the finite group G act on the smooth curve C effectively and let $p \in C$ be a point. Then the stabilizer $H = G_p$ of p is a cyclic group. The induced action of a generator of H on $T_pC \cong \mathbb{C}$ is given by multiplication with a primitive e-th root of unity, where $e = |H|$.*

It follows that there exists a distinguished generator h of H acting by the root $\zeta_e = \exp(2\pi i/e)$. We call h the *monodromy of G at p*.

Now note that for $q = \varphi(p)$ we have a bijection

$$G/H \rightarrow \varphi^{-1}(q), gH \mapsto gp.$$

If we had chosen a different point $p' = tp \in \varphi^{-1}(q)$, $t \in G$, we would have obtained a stabilizer $G_{p'} = tHt^{-1}$ with the distinguished generator $h' = tht^{-1}$. Overall we see that to a branch point q of φ we can uniquely associate a conjugacy class $[h]$ of G. If $q_1, \ldots, q_b \in D$ are the branch points of φ, we can write a formal sum

$$[h_1] + \cdots + [h_b]$$

of the corresponding conjugacy classes. In [7] this is called the Hurwitz datum associated to φ. By making this a formal sum and by having conjugacy classes $[h_i]$ instead of group elements, this is independent of a choice of ordering of branch points q_i as well as a distinguished point in the preimage $\varphi^{-1}(q_i)$.

For our purposes it will be more convenient to include this ordering and the choice of a preimage. For us a *monodromy datum* will be an ordered tuple

$$\xi = (h_1, \ldots, h_b)$$

of elements $h_i \in G$.

We immediately note that formally it makes sense to include entries of the form e_G in ξ, where $e_G \in G$ is the neutral element. This corresponds to marking a point $q \in D$ such that G acts freely on the fibre $\varphi^{-1}(q)$.

Knowing the full branching behaviour of φ it is easy to write down the Riemann–Hurwitz formula

$$2g - 2 = |G|(2g' - 2) + \sum_{i=1}^{b} \left(|G| - \frac{|G|}{\mathrm{ord}_G(h_i)} \right). \tag{4}$$

determining the genus g' of D.

3.2 Admissible covers

As a next step we need to understand what happens when the curves C and D degenerate, obtaining nodal singularities. In this case, we must require the map $\varphi : C \rightarrow D$

to be an *admissible cover*. The following definition is essentially the same as in [31, Definition 2.1], with the crucial difference that we require the curve C to be connected.

Definition 3.2 An *admissible G-cover* over a scheme S is a map $\varphi : C \to D$ between connected, nodal curves over S together with an action $G \curvearrowright C$ of G on C and markings $q_1, \dots, q_b : S \to D$, such that

- $(D, q_1, \dots, q_b)$ is a stable curve,
- φ is finite, mapping every node of C to a node of D,
- the action of G preserves φ and the restriction of φ over the set D_{gen} - of unmarked, smooth points of D-is a principal G-bundle,
- the local picture of $C \to D \to S$ at a point of C over a node of D is (analytically isomorphic to) that of

$$\mathrm{Spec}A[\zeta, \eta]/(\zeta\eta - a) \to \mathrm{Spec}A[x, y]/(xy - a^e) \to \mathrm{Spec}A,$$

  at $(\zeta, \eta) = (0, 0)$ for some $e \geq 1$, where $\varphi^*x = \zeta^e$ and $\varphi^*y = \eta^e$,
- the local picture of $C \to D \to S$ at a point of C over a marking of D is (analytically isomorphic to) that of

$$\mathrm{Spec}A[\zeta] \to \mathrm{Spec}A[x] \to \mathrm{Spec}A,$$

  at $\zeta = 0$ for some $e \geq 1$, where $\varphi^*x = \zeta^e$,
- for each geometric nodal point $P \in C$ with image point $s \in S$, the stabilizer G_P is cyclic and its action on the fiber C_s is *balanced*, i.e. the local picture around P looks like $\mathrm{Spec}A[\zeta, \eta]/(\zeta\eta - a)$ and for a generator h of G_P the action is given by

$$h.(\zeta, \eta) = (\mu\zeta, \mu^{-1}\eta)$$

  with a primitive root μ of unity of order equal to the order of h.

Given an admissible G-cover, it is natural to place markings on C at the preimages of the markings q_i of the curve D. While in [31] exactly one marking p_i is chosen in the preimage of every q_i, it will be more convenient for us to have individual markings at *all* preimage points of q_i. Note that these two data are equivalent: given any marking $p_i \in \varphi^{-1}(q_i)$ with stabilizer $\langle h_i \rangle$ under G, the remaining points in $\varphi^{-1}(q_i)$ are given by the G-orbit of p_i and naturally bijective to $G/\langle h_i \rangle$ via $a \mapsto ap_i$. Below we denote the markings ap_i by $p_{i,a}$.

Definition 3.3 A *pointed admissible G-cover* with *monodromy data* $\xi = (h_1, \dots, h_b) \in G^b$ over a scheme S is an admissible G-cover

$$\varphi : G \curvearrowright C \to (D, q_1, \dots, q_b) \to S$$

together with sections

$$p_{i,a} : S \to C, i = 1, \dots, b, a \in G/\langle h_i \rangle,$$

such that

– $\varphi^{-1}(q_i) = \{p_{i,a} : a \in G/\langle h_i \rangle\}$ for $i = 1, \ldots, b$ and for $h \in G$ we have $hp_{i,a} = p_{i,ha}$,
– the monodromy of the G-action at the marking $p_i = p_{i,[e_G]}$ corresponding to the neutral element $e_G \in G$ is given by h_i for all i, i.e. the generator h_i of the stabilizer G_{p_i} of p_i acts on $T_{p_i}C$ by multiplication with the root $\zeta_e = \exp(2\pi i/e)$ of unity, where $e = \mathrm{ord}_G(h_i)$.

The reason we mark the other preimages of q_i as well is the following.

Lemma 3.4 *Given a pointed admissible G-cover $\varphi : G \curvearrowright (C, (p_{i,a})_{i,a}) \to (D, q_1, \ldots, q_b)$, the curve $(C, (p_{i,a})_{i,a})$ is stable.*

Proof This is essentially clear: a rational component C' of C must map to a rational component D' of D. Then D' contains at least 3 special points and each has some preimage in C' which is again a special point, so C' is stable. □

Definition 3.5 For a genus $g \geq 0$, a finite group G and a monodromy datum $\xi = (h_1, \ldots, h_n)$ with elements $h_1, \ldots, h_n \in G$, define the stack $\overline{\mathcal{H}}_{g,G,\xi}$ whose objects over a scheme S are pointed admissible covers

$$\varphi : G \curvearrowright (C, (p_{i,a})_{\substack{i=1,\ldots,b \\ a \in G/\langle h_i \rangle}}) \to (D, q_1, \ldots, q_b) \to S. \tag{5}$$

Isomorphisms between such objects are G-equivariant isomorphisms of the curves C fixing the markings.

Define the stack $\mathcal{H}_{g,G,\xi}$ as the open substack of $\overline{\mathcal{H}}_{g,G,\xi}$ where the curve C (and D) is smooth.

In the future we will abbreviate the notation in (5) to $(C \to D, (p_{i,a})_{i,a}, (q_i)_i)$ or even $(C \to D)$, with the understanding that the data of the group action on C and the markings is implicit.

Remark 3.6 We want to outline here how the space $\overline{\mathcal{H}}_{g,G,\xi}$ compares to other constructions in the literature.

As remarked before, our definition is closest to the one from [31]. Given $g, G, \xi = (h_1, \ldots, h_b)$ denote by g' the genus of the curve D, which can be computed via the Riemann–Hurwitz formula (4). Then the space $\overline{\mathcal{H}}_{g,G,\xi}$ is (canonically isomorphic to) the union of the components of the space $\overline{\mathcal{M}}^G_{g',b}(h_1, \ldots, h_b)$ in [31] where the domain C of the admissible cover $\varphi : C \to D$ is connected.

On the other hand, in [7, Definition 6.2.3.] Bertin and Romagny also define a stack $\overline{\mathcal{H}}_{g,G,[h_1]+\cdots+[h_b]}$, which we denote by $\overline{\mathcal{H}}^{\mathrm{BR}}_{g,G,\xi}$ for clarity. Instead of pointed admissible G-covers, it just parametrizes admissible G-covers $\varphi : G \curvearrowright C \to (D, q_1, \ldots, q_b) \to S$ such that for all i some preimage p_i of q_i has monodromy given by h_i. In other words there is no distinguished preimage of q_i. By comparison of definitions, one sees that this has a relation to the space $\mathcal{A}^{\mathbf{d},\mathbf{e}}_{g',b}$ defined in [4, Appendix B.2.] as

$$\mathcal{A}^{\mathbf{d},\mathbf{e}}_{g',b} = \bigcup_\xi \overline{\mathcal{H}}^{\mathrm{BR}}_{g,G,\xi}.$$

Here the union on the right goes over all monodromy data ξ giving rise to the ramification multiplicities specified by the vectors $\mathbf{d}$, $\mathbf{e}$ of positive integers. By the results from the Appendix and Theorem 4.3.2. of [4] there exist natural isomorphisms and inclusions as open and closed substacks

$$\mathcal{A}_{g',b}^{\mathbf{d},\mathbf{e}} \cong \mathcal{A}dm_{g',b}^{d,e}(G) \subset \mathcal{A}dm_{g',b}(G) \cong \mathcal{B}_{g',b}^{\mathrm{bal}}(G) = \overline{\mathcal{M}}_{g',b}(BG).$$

The stack on the right is the stack of balanced twisted stable maps to the classifying stack BG of the finite group G. The basic idea how to go from $\overline{\mathcal{H}}_{g,G,\xi}^{\mathrm{BR}}$ to $\overline{\mathcal{M}}_{g',b}(BG)$ is that given a curve $C \to \mathrm{Spec}(\mathbb{C})$ with action by G we can take the quotient of G on both sides (acting trivially on $\mathrm{Spec}(\mathbb{C})$) and obtain a morphism $[C/G] \to BG$. The inverse operation is taking the fibre product via the universal torsor $\mathrm{Spec}(\mathbb{C}) \to BG$.

From the proof of [31, Theorem 2.4] it follows that the canonical map $\overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{H}}_{g,G,\xi}^{\mathrm{BR}}$, forgetting the distinguished markings $p_{i,a}$ in C and only remembering the markings q_i in D, is étale, since it is a union of components of an iterated fibre product of étale morphisms to $\overline{\mathcal{H}}_{g,G,\xi}^{\mathrm{BR}}$.

3.3 Properties of $\overline{\mathcal{H}}_{g,G,\xi}$

In this section we collect results about the stacks $\overline{\mathcal{H}}_{g,G,\xi}$ we are going to use later. We recall that $g \geq 0$ is the genus of the domain curve C of the admissible covers, G a finite group, $\xi = (h_1, \ldots, h_b)$ a monodromy datum of elements $h_i \in G$. We require that the number g' determined by the formula (4) is a nonnegative integer, giving the genus of the curve D in the admissible cover $\varphi : C \to D$. Let furthermore $n = \sum_{i=1}^{b} |G|/\mathrm{ord}_G(h_i)$ be the total number of markings $p_{i,a}$ in C.

Theorem 3.7 *The stack $\overline{\mathcal{H}}_{g,G,\xi}$ is a smooth proper Deligne-Mumford stack of dimension $3g' - 3 + b$ over $\mathbb{C}$.*

There exist natural maps

$$\begin{array}{c} \overline{\mathcal{H}}_{g,G,\xi} \xrightarrow{\phi} \overline{\mathcal{M}}_{g,r} \\ \downarrow{\scriptstyle\delta} \\ \overline{\mathcal{M}}_{g',b} \end{array} \tag{6}$$

defined by

$$\phi((C \to D, (p_{i,a})_{i,a}, (q_i)_i)) = (C, (p_{i,a})_{i,a}),$$
$$\delta((C \to D, (p_{i,a})_{i,a}, (q_i)_i)) = (D, (q_i)_i).$$

The morphism ϕ is representable, finite, unramified and a local complete intersection.

The morphism δ is flat, proper and quasi-finite, however not necessarily representable. Moreover, δ is étale over $\mathcal{M}_{g',b}$.

Proof Since $\overline{\mathcal{H}}_{g,G,\xi}$ is a union of components of $\overline{\mathcal{M}}_{g',b}^{G}(\xi)$ from the paper by Jarvis, Kaufmann and Kimura, the fact that $\overline{\mathcal{H}}_{g,G,\xi}$ is a smooth, proper Deligne-Mumford stack together with the properties of δ over $\overline{\mathcal{M}}_{g',b}$ follow from [31, Theorem 2.4]. The property that δ is étale over $\mathcal{M}_{g',b}$ is discussed in the proof of [7, Proposition 6.5.2.]. Also, that δ is not necessarily representable comes from the possibility of having G-equivariant automorphisms of $(C, (p_{i,a})_{i,a})$ over the identity map on D (see the discussion preceding Theorem 3.19 for details).

The proof of the properties of ϕ is exactly analogous to the proof of [7, Proposition 6.5.2.].

Indeed, assume we are given an S-point $S \rightarrow \overline{\mathcal{M}}_{g,r}$ for a scheme S, given by a family of stable curves $(C, p_1, \ldots, p_n) \rightarrow S$. Then we first want to show that the pullback of ϕ given as

$$\mathcal{F} := \overline{\mathcal{H}}_{g,G,\xi} \times_{\overline{\mathcal{M}}_{g,r}} S \rightarrow S$$

is finite and unramified, with $\mathcal{F}$ an algebraic space. For $T \rightarrow S$ a morphism of schemes, giving a T-point of $\mathcal{F}$ over S means specifying a suitable G-action on $C_T := C \times_S T$. Once this G-action is specified, we obtain the admissible cover as $C_T \mapsto D = C_T/G$. To express the G-action, let

$$\mathcal{A} = \{\eta \in \text{Aut}_S(C) : \eta(\{p_1, \ldots, p_n\}) \subset \{p_1, \ldots, p_n\}\}$$

be the group of automorphisms of C over S leaving invariant the *set* of marked points. Then $\mathcal{A} \rightarrow S$ is finite and unramified (corresponding to the fact that the quotient $\mathcal{M}_{g,r}/S_n$ is still a separated Deligne-Mumford stack).

We can specify a G-action on C_T by giving a T-point of the $|G|$-fold product

$$\mathcal{A}_G = \mathcal{A} \times_S \mathcal{A} \times_S \cdots \times_S \mathcal{A}$$

satisfying a number of closed conditions (defining a G-action, permuting the markings p_i in the correct way, with the correct local monodromies, etc.). Thus $\mathcal{F}$ is represented by a closed subscheme of $\mathcal{A}_G$ so it is finite and unramified over S. We conclude that ϕ is representable, finite and unramified. Since $\overline{\mathcal{H}}_{g,G,\xi}$ and $\overline{\mathcal{M}}_{g,r}$ are smooth, it is also a local complete intersection. $\square$

We call ϕ the *source morphism* and δ the *target morphism*, since they remember the source and target of the admissible cover, respectively.

Definition 3.8 We denote by $[\overline{\mathcal{H}}_{g,G,\xi}] \in A_{3g'-3+b}(\overline{\mathcal{M}}_{g,r})$ the pushforward of the fundamental class of $\overline{\mathcal{H}}_{g,G,\xi}$ under the map ϕ.

These cycles and their intersections with tautological classes are the main object of study of the present paper.

Over the stack $\overline{\mathcal{H}}_{g,G,\xi}$ we have universal curves $\mathcal{C}, \mathcal{D}$ with sections $p_{i,a}, q_i$ and a universal admissible G-cover

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\varphi} & \mathcal{D} \\
{\scriptstyle p_{i,a}} \uparrow \downarrow {\scriptstyle \pi} & \swarrow {\scriptstyle \tilde{\pi}} \nearrow {\scriptstyle q_i} & \\
\overline{\mathcal{H}}_{g,G,\xi} & &
\end{array}
\qquad . \tag{7}
$$

The pointed families of curves $\mathcal{C}, \mathcal{D}$ induce the maps ϕ, δ from Theorem 3.7. There is a natural isomorphism $\mathcal{C} \cong \overline{\mathcal{H}}_{g,G,(\xi,e_G)}$, see [7, Section 6.5.2].

Given one of the markings $p_{i,a}$ on $\overline{\mathcal{H}}_{g,G,\xi}$ we define $\psi_{p_{i,a}} \in A^1(\overline{\mathcal{H}}_{g,G,\xi})$ as the pullback of the corresponding ψ-class on $\overline{\mathcal{M}}_{g,r}$ under ϕ. This is equivalent to defining it as $c_1(p_{i,a}^*\omega_\pi)$ by the functoriality of the relative dualizing sheaf.

Similarly, given $\ell \geq 1$ we define $\kappa_\ell \in A^\ell(\overline{\mathcal{H}}_{g,G,\xi})$ as the pullback of $\kappa_\ell \in A^\ell(\overline{\mathcal{M}}_{g,r})$ under ϕ. For notational distinction we will write the ψ and κ classes on $\overline{\mathcal{M}}_{g',b}$ as ψ'_{q_i} and κ'_ℓ. Then we have the following convenient comparison result (see [28, Proposition 4.4] for a related result).

Lemma 3.9 *Assume the stabilizer of $p_{i,a}$ has order $e_i = \operatorname{ord}_G(h_i)$, then we have*

$$
\psi_{p_{i,a}} = \frac{1}{e_i}\delta^*(\psi'_{q_i}) \in A^1(\overline{\mathcal{H}}_{g,G,\xi}). \tag{8}
$$

On the other hand, for $\ell \geq 1$ we have

$$
\kappa_\ell = |G|\delta^*(\kappa'_\ell) \in A^\ell(\overline{\mathcal{H}}_{g,G,\xi}). \tag{9}
$$

Proof To prove (8) consider the universal admissible cover (7). Then we have

$$
\delta^*(\psi'_{q_i}) = q_i^*(c_1(\omega_{\tilde{\pi}})) = (\varphi \circ p_{i,a})^*(c_1(\omega_{\tilde{\pi}})) = p_{i,a}^*(\varphi^*c_1(\omega_{\tilde{\pi}}))
$$

Since φ is ramified along the sections $p_{j,a}$ with ramification index $e_j - 1$, we have

$$
\omega_\pi = \varphi^*\omega_{\tilde{\pi}} \otimes \left(\otimes_j \mathcal{O}_\mathcal{C}(R_j)^{\otimes e_j - 1}\right), \tag{10}
$$

where R_j is the union of the images of the sections $p_{j,a}$. The equation above just says that φ is ramified along the divisors R_j with ramification index $e_j - 1$. Inserting in our computation above and using the sections $p_{j,a'}$ are all disjoint from $p_{i,a}$, except $p_{i,a}$ itself, we obtain

$$
\delta^*(\psi'_{q_i}) = p_{i,a}^*\left(c_1(\omega_\pi) - (e_i - 1)(p_{i,a})_*[\overline{\mathcal{H}}_{g,G,\xi}]\right) = \psi_{q_{i,a}} + (e_i - 1)\psi_{q_{i,a}} = e_i\psi_{q_{i,a}}.
$$

The statement about κ classes is proved exactly as in [7, Théorème 10.3.4.]. □

3.4 Boundary strata of $\overline{\mathcal{H}}_{g,G,\xi}$ and admissible G-Graphs

The moduli space $\overline{\mathcal{M}}_{g,n}$ has a stratification indexed by stable graphs Γ. Similarly, the space $\overline{\mathcal{H}}_{g,G,\xi}$ has a stratification indexed by stable graphs Γ together with a G-action. This is described in detail in Chapter 7 of [7]. We summarize the results in our notation.

Given $(C \to D, (p_{i,a})_{i,a}, (q_i)_i) \in \overline{\mathcal{H}}_{g,G,\xi}$ let Γ be the dual graph of C. Since the action of G on C permutes the irreducible components and nodes of C, respectively, we obtain a group action $G \curvearrowright \Gamma$ on the stable graph. In particular, for each leg *or* half-edge l we can write down its stabilizer G_l, which is a cyclic group.

We need to record some finer data describing the action of G_l on C around the point corresponding to l, namely the monodromy of the G-action there. Let $v \in V(\Gamma)$ be the vertex adjacent to l, let C_v be the normalization of the component of C corresponding to v and let $x_l \in C_v$ be the point corresponding to the leg or half-edge l. If l is a leg this is just the position of the corresponding marking, if l is a half-edge this is the corresponding preimage of the node. Elements in the stabilizer G_l preserve the vertex v and correspondingly there is an action $G_l \curvearrowright C_v$ fixing x_l. For $e = |G_l|$ there is then a unique element $h_l \in G_l$ acting as multiplication by $\zeta_e = \exp(2\pi i/e)$ on the tangent space $T_{x_l} C_v$.

Definition 3.10 We call the data

$$(\Gamma, G) = (G \curvearrowright \Gamma, (h_l)_{l \in L(\Gamma) \cup H(\Gamma)}) \tag{11}$$

the *admissible G-graph* associated to $(C \to D, (p_{i,a})_{i,a}, (q_i)_i)$.

Note that from the balancing condition of the group action on C it follows that for two half-edges a, b forming an edge we have $h_a = h_b^{-1}$. Moreover, there is no element $t \in G$ preserving the edge but flipping the orientation, i.e. with $ta = b$. Such an element would correspond to a stabilizer of a node which exchanges the branches of the node, contradicting the local picture of the action around nodes from Definition 3.2.

We denote by $\mathcal{H}_{g,G,\xi}(\Gamma, G)$ the set of elements in $\overline{\mathcal{H}}_{g,G,\xi}$ with associated admissible G-graph (Γ, G) and by $\overline{\mathcal{H}}_{g,G,\xi}(\Gamma, G)$ its closure. This defines a stratification of $\overline{\mathcal{H}}_{g,G,\xi}$ similar to the stratification of $\overline{\mathcal{M}}_{g,n}$ by boundary strata. In the following we want to derive a parametrization of $\overline{\mathcal{H}}_{g,G,\xi}(\Gamma, G)$ by a generalization of the usual gluing maps associated to the graph Γ.

Since the general case of these equivariant gluing maps has a fairly technical description, let us first look at a concrete example.

Example 3.11 The space $\overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1,1)}$ parametrizes double covers $C \to E$ from genus 2 curves C to genus 1 curves E, ramified at two points $p_1, p_2 \in C$. One boundary stratum of this space is given by the locus of curves

$$(C, p_1, p_2) = (C_1, q) \cup_{q=q'} (C_2, q', p_1, p_2, q'') \cup_{q''=q} (C_1, q), \tag{12}$$

where C_1 has genus 1, C_2 has genus 0 and C_2 carries a $G = \mathbb{Z}/2\mathbb{Z}$-action leaving p_1, p_2 fixed and exchanging q', q''. This action extends to all of C, where the first component C_1 is exchanged with the second one.

The corresponding G-admissible stable graph (Γ, G) is given by
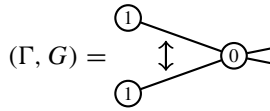
Then we see a point in the closure $\overline{\mathcal{H}}_{g,G,\xi}(\Gamma, G)$ can be specified by giving $(C_1, q) \in \overline{\mathcal{M}}_{1,1} = \overline{\mathcal{H}}_{1,\{0\},(0)}$ and $(C_1, p_1, p_2, q', q'') \in \overline{\mathcal{H}}_{0,G,(1,1,0)}$. This means that the corresponding equivariant gluing map

$$\xi_{(\Gamma,G)} : \overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{H}}_{0,G,(1,1,0)} \to \overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1,1)}$$

glues two copies of C_1 and one copy of C_2 to obtain the curve C from (12) together with the natural G-action on C.

Now we want to write down a similar map $\xi_{(\Gamma,G)}$ for a general admissible G-graph as in (11). As in the example, the domain of $\xi_{(\Gamma,G)}$ will need one factor $\overline{\mathcal{H}}_{g(v'),G_{v'},\xi_{v'}}$ for a representative v' of each orbit of the vertices $V(\Gamma)$ under G.

Because of this, it will be convenient to introduce the quotient graph $\tilde{\Gamma} = \Gamma/G$. Its sets of vertices, half-edges and legs are the quotient sets of the corresponding sets for Γ under the equivalence relation of being in the same G-orbit. Denote the quotient maps by

$$\pi_V : V(\Gamma) \to V(\tilde{\Gamma}), \pi_L : L(\Gamma) \to L(\tilde{\Gamma}), \pi_H : H(\Gamma) \to H(\tilde{\Gamma}). \qquad (13)$$

Since the group action respects the incidence of legs and half-edges with vertices and does not reverse the orientation of an edge, this is a well-defined stable graph. In fact, if (Γ, G) is the admissible G-graph associated to $(C \to D, (p_{i,a})_{i,a}, (q_i)_i)$, then $\tilde{\Gamma}$ is canonically identified with the dual graph of $(D, (q_i)_i)$.

Now choose a section of π_V, i.e. a set $V' \subset V(\Gamma)$ of representatives $v \in V(\Gamma)$ for each $[v] \in V(\tilde{\Gamma})$. For each $v' \in V'$ choose representatives $L'_{v'}$, $H'_{v'}$ of the legs and half-edges in Γ incident to v' up to the action of $G_{v'}$. With these choices made, there are natural G-equivariant identifications

$$\pi_V^{-1}([v']) \cong G/G_{v'} \ (v' \in V'),$$
$$\pi_L^{-1}([l]) \cong G/G_l \ (l \in L'_{v'}), \quad \pi_H^{-1}([h]) \cong G/G_h \ (h \in H'_{v'}).$$

For every $v' \in V'$ let

$$\xi_{v'} = (h_l : l \in L'_{v'} \cup H'_{v'}) \qquad (14)$$

be a monodromy datum. We will construct a natural morphism

$$\xi_{(\Gamma,G)} : \overline{\mathcal{H}}_{(\Gamma,G)} = \prod_{v' \in V'} \overline{\mathcal{H}}_{g(v'),G_{v'},\xi_{v'}} \to \overline{\mathcal{H}}_{g,G,\xi}, \tag{15}$$

whose image is $\overline{\mathcal{H}}_{g,G,\xi}(\Gamma, G)$.

The idea is the following: for a curve C with group action by G and admissible G-graph Γ, look at the orbit of a vertex $v' \in V'$. This orbit corresponds to an orbit of the component $C_{v'}$ of C associated to v'. Since G acts on C by isomorphisms, all components in the G-orbit of $C_{v'}$ must be isomorphic. On the other hand, the stabilizer $G_{v'}$ of v' acts on C_v with monodromy datum $\xi_{v'}$. The idea of the map $\xi_{(\Gamma,G)}$ is to take all these restricted actions $G_{v'} \curvearrowright C_v$ (together with the marked points from $\xi_{v'}$) and to reconstruct the curve C from this.

The first step is to go from $C_{v'}$ to the orbit of components of C, which were all isomorphic to $C_{v'}$. So for each $v' \in V'$ and for an element $(G_{v'} \curvearrowright C_{v'}, (p_{i,a}^{v'})_{i,a}) \in \overline{\mathcal{H}}_{g(v'),G_{v'},\xi_{v'}}$ there is a (usually disconnected) curve

$$\widehat{C}_{v'} = \bigcup_{[g] \in G/G_{v'}} C_{v'}$$

together with a G-action on $\widehat{C}_{v'}$ such that the induced action on the connected components is the left-multiplication of G on $G/G_{v'}$ and such that on the component $C_{v'} \subset \widehat{C}_{v'}$ corresponding to the trivial element $[e_G]$ the induced action of $G_{v'}$ is the given one.

Note that for the action of G on $\widehat{C}_{v'}$ there is a choice involved. This is the same kind of ambivalence that appears when defining the induced representation of $G_{v'}$ in G by choosing a set of representatives of $G/G_{v'}$. In the end, the resulting map $\xi_{(\Gamma,G)}$ does not depend on this choice.

The map $\xi_{(\Gamma,G)}$ is defined by taking the disjoint union of all curves $\widehat{C}_{v'}$, $v' \in V'$, and glueing them together to a curve C according to the graph Γ. Since the actions of G on the curves $\widehat{C}_v$ are compatible with this gluing, the new curve C inherits a G-action. Taking the quotient map $C \to D = C/G$ gives the desired admissible G-cover. This ends the construction of $\xi_{(\Gamma,G)}$.

From the construction above it follows that the composition of $\xi_{(\Gamma,G)}$ with the map ϕ forgetting the G-action factors through the gluing map ξ_Γ associated to the stable graph Γ as

$$\begin{array}{ccc}
\overline{\mathcal{H}}_{(\Gamma,G)} = \prod_{v' \in V'} \overline{\mathcal{H}}_{g(v),G_v,\xi_v} & \xrightarrow{\xi_{(\Gamma,G)}} & \overline{\mathcal{H}}_{g,G,\xi} \\
\downarrow{\phi_{(\Gamma,G)}} & & \downarrow{\phi} \\
\overline{\mathcal{M}}_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} & \xrightarrow{\xi_\Gamma} & \overline{\mathcal{M}}_{g,r}
\end{array} \tag{16}$$

For the map $\phi_{(\Gamma,G)}$ note that the set $V(\Gamma)$ is a disjoint union of the G-orbits of the elements $v' \in V'$. Then $\phi_{(\Gamma,G)}$ is a product of maps ϕ followed by diagonals $\Delta : \overline{\mathcal{M}}_{g(v'),n(v')} \to \prod_{v \in Gv'} \overline{\mathcal{M}}_{g(v'),n(v')}$. These diagonal maps arise since in the

construction above, when defining $\widehat{C}_v$ we take the disjoint union of identical curves C_v, indexed by G/G_v, which is canonically bijective to the orbit of v.

Putting everything together, we see that the map $\phi_{(\Gamma,G)}$ factors as

$$
\phi_{(\Gamma,G)}\colon \underbrace{\prod_{v'\in V'} \overline{\mathcal{H}}_{g(v),G_v,\xi_v}}_{=\overline{\mathcal{H}}_{(\Gamma,G)}} \xrightarrow{\prod \phi} \prod_{v'\in V'} \overline{\mathcal{M}}_{g(v'),n(v')} \xrightarrow{\prod \Delta} \underbrace{\prod_{v'\in V'}\prod_{v\in Gv'} \overline{\mathcal{M}}_{g(v'),n(v')}}_{=\overline{\mathcal{M}}_\Gamma} .
$$

$$(17)$$

Likewise we have a map

$$
\delta_{(\Gamma,G)}\colon \overline{\mathcal{H}}_{(\Gamma,G)} \to \prod_{v'\in V'} \overline{\mathcal{M}}_{g'(v'),b(v')} := \overline{\mathcal{M}}_{\Gamma/G}, \tag{18}
$$

given by taking the target morphism $\delta\colon \overline{\mathcal{H}}_{g(v'),G_{v'},\xi_{v'}} \to \overline{\mathcal{M}}_{g'(v'),b(v')}$ for each $v' \in V' \cong V(\Gamma/G)$.

Definition 3.12 A *morphism of admissible G-graphs* $f : (\Gamma', G) \to (\Gamma, G)$ is a usual morphism of stable graphs $f : \Gamma' \to \Gamma$ (see Definition 2.5) which is equivariant with respect to the G-action and such that for each half-edge l of Γ mapping to a half-edge l' of Γ' we have $h_l = h'_{l'}$, where h, h' are the monodromy data of (Γ, G) and (Γ', G), respectively.

For an admissible G-graph (Γ, G) the set of automorphisms of (Γ, G) is denoted $\mathrm{Aut}(\Gamma, G) \subset \mathrm{Aut}(\Gamma)$.

Proposition 3.13 (See [7], Proposition 7.4.1.) *The morphism $\xi_{(\Gamma,G)}$ has as image the closure $\overline{\mathcal{H}}_{g,G,\xi}(\Gamma, G)$ of the set of elements in $\overline{\mathcal{H}}_{g,G,\xi}$ with associated admissible G-graph (Γ, G). It is generically of degree $|\mathrm{Aut}(\Gamma, G)|$ onto its image.*

One quickly checks that the domain of $\xi_{(\Gamma,G)}$ (and thus $\overline{\mathcal{H}}_{g,G,\xi}(\Gamma, G)$) is of the same dimension as the space $\overline{\mathcal{M}}_{\Gamma/G}$ associated to the quotient graph Γ/G. In particular, the boundary divisors of $\overline{\mathcal{H}}_{g,G,\xi}$ are indexed by the admissible G-graphs (Γ, G) such that Γ/G has exactly one edge. In other words, Γ has exactly one G-orbit of edges.

3.5 Abstract characterization of admissible *G*-Graphs

In Definition 3.10 we defined the admissible G-graph associated to a given pointed admissible G-cover $(C \to D, (p_{i,a})_{i,a}, (q_i)_i)$. However, for computations it will be necessary to give a purely combinatorial definition of an admissible G-graph.

It turns out that not every graph Γ with a group action by G and monodromy data at the half-edges and legs can occur as the admissible G-graph of an admissible G-cover. This is related to the fact that not every tuple (g, G, ξ) leads to a nonempty space $\overline{\mathcal{H}}_{g,G,\xi}$. The question whether $\overline{\mathcal{H}}_{g,G,\xi}$ is empty (and more generally the degree of the map $\delta : \overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{M}}_{g',b}$) can be answered in terms of the group theory of G.

We begin by defining a group action with stabilizer data on a stable graph.

Definition 3.14 Let G be a finite group and $\Gamma = (V, H, L, g, \iota, a, \zeta)$ an n-pointed stable graph. A *G-action on Γ* is an action of G on the sets V, H, L such that $g :$ $V \to \mathbb{Z}_{\geq 0}$ is G-invariant and such that ι, a, ζ are G-equivariant.

A *G-action with stabilizer datum* on Γ is a G-action together with a collection $(h_l)_{l \in H \cup L}$ of elements of G such that the stabilizer of each leg or half-edge l of Γ is cyclic with generator h_l and such that for each $l \in H$ we have $h_{\iota(l)} = h_l^{-1}$ (In other words, for (l, l') forming an edge of Γ we have $h_{l'} = h_l^{-1}$) and $h_{gl} = g h_l g^{-1}$ for $g \in G$.

A *pre-admissible G-graph* (Γ, G) is a stable graph Γ together with a G-action and stabilizer datum such that the induced action of ι on the quotient H/G is fixed point free, i.e. for an edge (l, l') there is no element $t \in G$ with $tl = l'$.

One verifies easily that the admissible G-graph associated to a given admissible G-cover $(C \to D, (p_{i,a})_{i,a}, (q_i)_i)$ gives a pre-admissible G-graph in the sense above. However, not every pre-admissible G-graph will correspond to some (nonempty) stratum of a suitable space of pointed admissible G-covers.

Definition 3.15 Let (Γ, G) be a pre-admissible G-graph. With respect to a choice $l_1, \ldots, l_b \in L$ of representatives of the G-action on L, the *monodromy datum of* (Γ, G) is the collection $(h_{l_1}, \ldots, h_{l_b}) \in G^b$.

For each vertex v of Γ with stabilizer G_v under G let L_v', H_v' be representatives of the legs and half-edges of Γ incident to v up to the action of G_v. Then the tuple $(h_l)_{l \in L_v' \cup H_v'}$ is called the *local monodromy datum at v*.

Definition 3.16 An *admissible G-graph* is a pre-admissible G-graph (Γ, G) such that for each vertex v of Γ, the space $\overline{\mathcal{H}}_{g(v), G_v, (h_l)_{l \in L_v' \cup H_v'}}$ is nonempty (for some choice of representatives L_v', H_v' as above[3]).

Note that the space $\overline{\mathcal{H}}_{g(v), G_v, (h_l)_{l \in L_v' \cup H_v'}}$ is nonempty if and only if its interior part $\mathcal{H}_{g(v), G_v, (h_l)_{l \in L_v' \cup H_v'}}$ is nonempty. Indeed, the balancing condition of admissible covers was chosen in such a way that a nodal admissible cover can always be smoothed in a family (see [4]).

Using the definition of the G-equivariant gluing maps $\xi_{(\Gamma, G)}$ above, the following is immediate.

Proposition 3.17 *A pre-admissible G-graph (Γ, G) with monodromy datum ξ corresponds to a nonempty stratum of the space $\overline{\mathcal{H}}_{g, G, \xi}$ if and only if (Γ, G) is an admissible G-graph.*

Proof If (Γ, G) is admissible, the domain of the equivariant gluing map $\xi_{(\Gamma, G)}$ is nonempty and thus a general element of its image will be in the stratum $\mathcal{H}_{g, G, \xi}(\Gamma, G)$. On the other hand, if the stratum corresponding to a pre-admissible G graph (Γ, G) is nonempty, then suitable components of the normalization of a general element $C \to D$ in this stratum (as described above) will give elements in the spaces $\mathcal{H}_{g(v), G_v, (h_l)_{l \in L_v' \cup H_v'}}$, which are thus nonempty, hence (Γ, G) is admissible. □

---

[3] The nonemptyness of the space of pointed admissible G-covers is independent of these choices.

Using this result and Definition 3.16 we can reduce the classification of the strata in our space of pointed admissible G-covers to the question for which (g, G, ξ) the space $\overline{\mathcal{H}}_{g,G,\xi}$ is nonempty.

A first obvious criterion is that the Riemann–Hurwitz formula needs to be satisfied, in the sense that for $\xi = (h_1, \ldots, h_b)$ the number

$$g' = \frac{1}{2|G|} \left(2g - 2 - |G| \sum_{i=1}^{b} (1 - \frac{1}{\mathrm{ord}_G(h_i)}) \right) + 1 \qquad (19)$$

is a nonnegative integer.

If this is satisfied, it makes sense to ask what the degree of the map $\delta : \overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{M}}_{g',b}$ is. In other words, given a smooth curve D of genus g' with markings $q_1, \ldots, q_b$, how many pointed admissible G-covers $(C \to D, (p_{i,a})_{i,a}, (q_i)_i)$ lie above it (counted in a stacky sense, i.e. dividing by automorphisms)? This can be answered in terms of classical Hurwitz theory (see e.g. [20] or [9] for a modern treatment). The following discussion is related to results in [7, Chapter 2.3] concerning the number of connected components of the spaces $\overline{\mathcal{H}}_{g,G,\xi}$.

Let the fundamental group Ξ of D punctured at $q_1, \ldots, q_b$—with respect to some base point $y \in D^* = D \setminus \{q_1, \ldots, q_b\}$—be given by

$$\Xi = \langle A_1, \ldots, A_{g'}, B_1, \ldots, B_{g'}, \gamma_1, \ldots, \gamma_b \rangle / \langle \left(\prod_{j=1}^{g'} [A_j, B_j] \right) \gamma_1 \cdots \gamma_b \rangle. \qquad (20)$$

Here the A_j, B_j are the usual generators of the fundamental group of D and the γ_k are loops around the punctures at q_k.

Given a G-cover $\varphi : C \to D$ whose branch points are contained in $\{q_1, \ldots, q_b\}$, it is étale over D^*. For a loop γ in D^* based at y, we get a permutation of $\varphi^{-1}(y)$ by lifting γ under φ and sending the start point of the lifted path to the end point. Choose a base point $x \in \varphi^{-1}(y)$, then lifting the path γ starting at x there exists a unique $g_\gamma \in G$ such that the lifted path ends at $g_\gamma x$. The map

$$\psi : \Xi \to G, \gamma \mapsto g_\gamma \qquad (21)$$

is a group homomorphism.[4]

If we had chosen a different base point $x' = \eta x \in \varphi^{-1}(y)$ for $\eta \in G$, we would have obtained $g'_\gamma = \eta g_\gamma \eta^{-1}$. Thus ψ is only unique up to conjugation in G.

From the presentation in (20) it is clear that ψ is determined by the images $a_j, b_j, \sigma_k \in G$ of the generators A_j, B_j, γ_k of Ξ. Conversely, for any choice of such $a_j, b_j, \sigma_k \in G$ satisfying $(\prod_j [a_j, b_j])\sigma_1 \cdots \sigma_b = e_G$ we obtain a group homomorphism ψ. Thus, at least in principle, we can understand the set of group homomorphisms $\mathrm{Hom}(\Xi, G)$ very concretely as a subset of the finite set $G^{2g'+b}$.

---

[4] If composition in Ξ is defined as usual by concatenation of paths, we actually have to replace Ξ with the opposite group Ξ^{op}. Since these groups are isomorphic by $g \mapsto g^{-1}$ this does not change any of the following conclusions, so to keep notation simple we just write Ξ.

Due to the way we constructed ψ from a G-cover $C \to D$, it is contained in a special subset $\mathrm{Hom}_\xi(\Xi, G)$ of the set of all homomorphisms $\mathrm{Hom}(\Xi, G)$.

Definition 3.18 We define $\mathrm{Hom}_\xi(\Xi, G)$ to be the set of group homomorphisms $\psi : \Xi \to G$ satisfying

(1) The homomorphism ψ is surjective.
(2) The elements $\sigma_k = \psi(\gamma_k)$ describing the monodromy around the points $q_k \in D$ are contained in the conjugacy class $[h_k]$ of the elements h_k from the monodromy datum ξ.

Since these properties are invariant under conjugating ψ with an element of G, denote by $\mathrm{Hom}_\xi(\Xi, G)/G$ the quotient of $\mathrm{Hom}_\xi(\Xi, G)$ under this conjugation action.

We claim that the morphism ψ from (21) is contained in $\mathrm{Hom}_\xi(\Xi, G)$. Property (1) follows from the fact that the cover C is connected. Indeed, this is the case iff we can connect our base point $x \in C$ to any other point tx in the same fibre by some path $\widehat{\gamma}$, for $t \in G$. In this case we can choose $\widehat{\gamma}$ such that it lies completely over D^*. But then it is the unique lift of the path $\gamma = \pi \circ \widehat{\gamma} \in \Xi$ and so $\psi(\gamma) = g_\gamma = t$, hence every element $t \in G$ lies in the image of ψ. Finally property (2) comes from the definition of the monodromy datum of a cover.

On the other hand, given $\psi \in \mathrm{Hom}_\xi(\Xi, G)$ we can reverse the construction (by Riemann's existence theorem) and obtain a G-cover $\varphi : C \to D$ with ramification behaviour described by ξ. Moreover, up to isomorphism, the cover φ does not change if we conjugate ψ by an element of G.

Note however that for the markings $(p_{i,a})_{i,a}$ in C we can have some additional choices: in the preimage of the marking $q_i \in D$ there is at least one marking p_i with mondoromy h_i (i.e. with stabilizer generated by h_i such that h_i acts by multiplication with $\exp(2\pi i / \mathrm{ord}_G(h_i))$ on $T_{p_i} C$). For a different preimage ηp_i of q_i, the corresponding monodromy is given by $\eta h_i \eta^{-1}$. Thus the set of all points in the fibre with monodromy h_i is given by the quotient of the centralizer $C_G(h_i) = \{\eta \in G : \eta h_i \eta^{-1} = h_i\}$ by the subgroup generated by h_i.

A priori it would seem that the choice of markings is parametrized by the product $\prod_{j=i}^b C_G(h_i)/\langle h_i \rangle$. However, we also need to consider the automorphisms of the G-cover $C \to D$. First, any isomorphism $\rho : C \to C$ covering the identity of $D = C/G$ is of the form $p \mapsto \eta p$ for some $\eta \in G$, since $C \to D$ is a G-Galois cover. This isomorphism is G-equivariant iff $\rho(bp) = b\rho(p)$ for all $b \in G$, i.e. η commutes with all elements $b \in G$. Conversely, for such a $\eta \in C(G)$ the map $\rho(p) = \eta p$ is a G-equivariant isomorphism of C over the identity of D. Thus the set of choices of markings is actually parametrized by $(\prod_{i=1}^b C_G(h_i)/\langle h_i \rangle)/C(G)$.

We have now assembled all relevant ingredients to write down the degree of δ.

Theorem 3.19 *Let $g \geq 0$, G a finite group and $\xi = (h_1, \ldots, h_b)$ a monodromy datum. Then the space $\overline{\mathcal{H}}_{g,G,\xi}$ is empty unless the number g' from (19) is a nonnegative integer. If this number is a nonnegative integer, the degree of the map $\delta : \overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{M}}_{g',b}$ is given by*

$$\deg(\delta) = \left| \mathrm{Hom}_\xi(\Xi, G)/G \right| \cdot \left(\prod_{i=1}^b \frac{|C_G(h_i)|}{\mathrm{ord}_G(h_i)} \right) / |C(G)|, \tag{22}$$

for $\text{Hom}_\xi(\Xi, G)/G$ *as in Definition* 3.18.

In particular, the space $\overline{\mathcal{H}}_{g,G,\xi}$ is empty iff $\text{Hom}_\xi(\Xi, G) = \emptyset$.

Proof In the section above we have shown that the preimage of a general $\mathbb{C}$-point $(C, q_1, \ldots, q_n) \in \overline{\mathcal{M}}_{g',b}$ is isomorphic to the quotient of

$$\left(\text{Hom}_\xi(\Xi, G)/G\right) \times \prod_{i=1}^{b} C_G(h_i)/\langle h_i \rangle$$

by the action of $C(G)$. This gives the multiplicity from above. $\qquad\square$

For general finite groups G and monodromy data ξ it can be difficult to compute this degree. However, for a cyclic group $G = \mathbb{Z}/m\mathbb{Z}$ it is possible to write it very explicitly.

Proposition 3.20 *Let* $m \geq 1$ *and let* $G = \mathbb{Z}/m\mathbb{Z}$. *Let* $g \geq 0$, $\xi = (h_1, \ldots, h_b)$ *a monodromy datum such that the number* g' *from* (19) *is a nonnegative integer. Let* $m_0 = \gcd(m, h_1, \ldots, h_b)$ *and let* $p_1, \ldots, p_s$ *be the distinct prime factors of* m_0.

Then the space $\overline{\mathcal{H}}_{g,G,\xi}$ *is empty unless* $h_1 + \cdots + h_b = 0 \in \mathbb{Z}/m\mathbb{Z}$. *If this sum vanishes, the degree of* δ *satisfies*

$$\deg(\delta) = m^{2g'-1} \prod_{j=1}^{s} \left(1 - \frac{1}{(p_j)^{2g'}}\right) \cdot \prod_{i=1}^{b} \gcd(m, h_i), \tag{23}$$

where $\gcd(m, 0) = m$. *In particular, assuming* $h_1 + \cdots + h_b = 0 \in \mathbb{Z}/m\mathbb{Z}$, *the space* $\overline{\mathcal{H}}_{g,G,\xi}$ *is always nonempty for* $g' \geq 1$ *and is nonempty for* $g' = 0$ *iff* $m_0 = 1$.

Proof First we compute $\text{Hom}_\xi(\Xi, G)$. As we saw, a group homomorphism $\psi : \Xi \to G$ is uniquely determined by the images a_j, b_j, σ_k of the generators A_j, B_j, γ_k of Ξ. The second condition of Definition 3.18 forces σ_k to lie in the conjugacy class $[h_k]$. Since G is abelian we have $\sigma_k = h_k$ for all k. On the other hand, the homomorphism ψ is surjective onto G iff the elements a_j, b_j, σ_k generate G. In other words if and only if

$$1 = \gcd(m, a_1, \ldots, a_{g'}, b_1, \ldots b_{g'}, \sigma_1, \ldots, \sigma_b) = \gcd(a_1, \ldots, a_{g'}, b_1, \ldots b_{g'}, m_0). \tag{24}$$

Each a_j, b_j can be chosen from $0, 1, \ldots, m-1$, so there are $m^{2g'}$ choices in total. We need to subtract those where all the numbers a_j, b_j and m_0 share a common factor. For each number q dividing m there are m/q numbers among $0, \ldots, m-1$ divisible by q, namely $0, q, \ldots, q(m/q - 1)$. Thus there are $(m/q)^{2g'}$ choices of a_j, b_j with common factor q.

We see that from the set of all $m^{2g'}$ choices of a_j, b_j we must subtract

$$\bigcup_{j=1}^{s} \left\{(a_1, \ldots, b_{g'}) : p_j \,|\, \gcd(a_1, \ldots, b_{g'}, m_0)\right\}, \tag{25}$$

which is exactly the set of choices not satisfying (24). Applying the inclusion-exclusion principle for computing the cardinality of (25) we obtain

$$
\begin{aligned}
|\operatorname{Hom}_\xi(\Xi, G)| &= m^{2g'} - \sum_{\emptyset \neq S \subset \{1,\dots,s\}} (-1)^{|S|+1} \left(\frac{m}{\prod_{j \in S} p_j} \right)^{2g'} \\
&= m^{2g'} \sum_{S \subset \{1,\dots,s\}} (-1)^{|S|} \frac{1}{\prod_{j \in S} (p_j)^{2g'}} \\
&= m^{2g'} \sum_{S \subset \{1,\dots,s\}} \prod_{j \in S} \left(-\frac{1}{(p_j)^{2g'}} \right) \\
&= m^{2g'} \prod_{j=1}^{s} \left(1 - \frac{1}{(p_j)^{2g'}} \right)
\end{aligned}
$$

To compute the remaining factors from (22) coming from the choice of markings and from automorphisms, we note that $C_G(h_i) = C(G) = G$ since G is abelian. Note also that $|G|/\operatorname{ord}_G(h_i) = \gcd(m, h_i)$. This explains the remaining terms in (23) and finishes the proof. $\qquad\square$

4 Pullbacks by boundary morphisms

Let A be an n pointed graph of genus g and let G be a finite group. In this section we will compute the intersection between $[\overline{\mathcal{M}}_A]$ and $[\overline{\mathcal{H}}_{g,G,\xi}]$ using the excess intersection formula (cf. [21, Proposition 17.4.1]). For this we need to identify the fiber product $\overline{\mathcal{M}}_A \times_{\overline{\mathcal{M}}_{g,r}} \overline{\mathcal{H}}_{g,G,\xi}$ and compute the top Chern class of the resulting excess bundle.

4.1 The fibre product

Recall that $\overline{\mathcal{H}}_{g,G,\xi}$ is stratified according to admissible G-graphs (Γ, G). The stratum corresponding to (Γ, G) is contained in the image of $\overline{\mathcal{M}}_A$ iff Γ has an A-structure. This leads to the following definition.

Definition 4.1 Let (Γ, G) be an admissible G-graph and A a stable graph. An A-structure f on (Γ, G) is an A-structure $f = (\alpha, \beta, \gamma)$ on Γ. We say that the A-structure on (Γ, G) is generic if

$$
G \operatorname{Im} \beta = H_\Gamma.
$$

An isomorphism of A-structures $(\Gamma', G, f') \to (\Gamma, G, f)$ is an isomorphism of admissible G-graphs $(\Gamma', G) \to (\Gamma, G)$ such that the induced diagram of stable graphs

$$
\begin{array}{ccc}
 & A & \\
{\scriptstyle f'}\nearrow & & \uparrow{\scriptstyle f} \\
\Gamma' & \longrightarrow & \Gamma
\end{array}
$$

commutes.

Notation 4.2 For a space $\overline{\mathcal{H}}_{g,G,\xi}$, such that the number of markings $p_{i,a}$ is n, and a fixed stable graph A of genus g with n legs we will denote by $\mathfrak{H}_{A;G,\xi}$ a set of representatives of isomorphism classes of generic A-structures (Γ, G, f) on admissible G-graphs (Γ, G) for $\overline{\mathcal{H}}_{g,G,\xi}$.

For a generic A-structure (Γ, G, f) there exists a natural map

$$\phi_{(\Gamma,G,f)} : \overline{\mathcal{H}}_{(\Gamma,G)} \xrightarrow{\phi_{(\Gamma,G)}} \overline{\mathcal{M}}_{\Gamma} \xrightarrow{\xi_{f:\Gamma \to A}} \overline{\mathcal{M}}_A,$$

where the first map was defined above and the second map is the partial gluing map induced by the A-structure on Γ.

Proposition 4.3 *There is a natural Cartesian diagram*

$$
\begin{array}{ccc}
\coprod_{(\Gamma,G,f)\in\mathfrak{H}_{A;G,\xi}} \overline{\mathcal{H}}_{(\Gamma,G)} & \xrightarrow{\coprod \xi_{(\Gamma,G)}} & \overline{\mathcal{H}}_{g,G,\xi} \\
\coprod \phi_{(\Gamma,G,f)} \downarrow & & \downarrow \phi \\
\overline{\mathcal{M}}_A & \xrightarrow{\xi_A} & \overline{\mathcal{M}}_{g,r}
\end{array} \quad .
$$

Example 4.4 Let $G = \mathbb{Z}/2\mathbb{Z}$ and let $\xi = (1, ..., 1) = (1^8)$ be the monodromy datum of length 8 prescribing 8 fixed points under G. For the stable graph

$$A = \quad$$
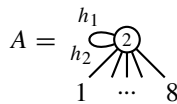
we will compute the set $\mathfrak{H}_{A;G,\xi}$. In other words we want to know which admissible G-graphs (Γ, G) admit a generic A structure. We claim that all such admissible G-graphs have the following form:
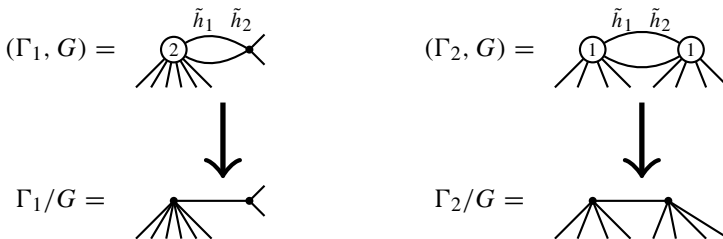
where the generator $\tau = 1$ of G acts as the involution fixing all the legs and all the vertices and switching the edges.

To see this first note that by definition for any admissible G-graph (Γ, G) with monodromy datum ξ, the genus of Γ/G must equal 0. It immediately follows that Γ cannot have any loops. Since A has only one edge, the cardinality of G is 2 and the A

structure is generic Γ has at most 2 edges. The possible admissible G-graphs are now completely determined by the condition that for each vertex v the involution τ must fix exactly $2g(v) + 2$ legs or half-edges incident to v.

There are in total $\binom{8}{6}$ admissible G-graphs of the form (Γ_1, G) given by the different distributions of the legs. Likewise there are $\frac{1}{2}\binom{8}{4}$ admissible G-graphs of the form (Γ_2, G).

For each A-structure $f = (\alpha, \beta, \gamma)$ on (Γ_1, G) we can compose f with an isomorphism of (Γ_1, G) such that the edge (h_1, h_2) is mapped to $(\tilde{h}_1, \tilde{h}_2)$. There are two possible choices, either $\beta(h_1) = \tilde{h}_1$ or $\beta(h_1) = \tilde{h}_2$. After these choices the A-structure f is completely determined so there are 2 isomorphism classes of generic A-structure on (Γ_1, G). The same argument holds for (Γ_2, τ_2) and there are 2 isomorphism classes of A-structures on (Γ_2, G).

Granted Proposition 4.3 it now follows that we have a fibered diagram.

$$
\begin{array}{ccc}
(\overline{\mathcal{H}}_{(\Gamma_1,G)})^{\sqcup 2\binom{8}{6}} \coprod (\overline{\mathcal{H}}_{(\Gamma_2,G)})^{\sqcup \binom{8}{4}} & \longrightarrow & \overline{\mathcal{H}}_{3,G,(1^8)} \\
\downarrow & & \downarrow \phi \\
\overline{\mathcal{M}}_A & \xrightarrow{\ \ \xi_A\ \ } & \overline{\mathcal{M}}_{3,8}
\end{array}
\ .
$$

An easy dimension count shows that there is no excess of intersection. We have therefore computed the intersection between $[\phi(\overline{\mathcal{H}}_{3,G,(1^8)})]$ and $[\xi_A(\overline{\mathcal{M}}_A)]$ in $A^\bullet(\overline{\mathcal{M}}_{3,8})$.

For the proof of Proposition 4.3, we will proceed in the same way as in the proof of Proposition 2.13.

Definition 4.5 Let $C \to D$ be an admissible cover of $\overline{\mathcal{H}}_{g,G,\xi}$ over a connected base scheme S. A (Γ, G)-marking on $C \to D$ is a Γ-marking on C (see Definition 2.14) such that the action of G on Γ is induced by the action of G on C. We will denote an admissible cover with (Γ, G) marking by $(C \to D)_{(\Gamma,G)}$.

A (Γ, G)-marking on $C \to D$ defines a Γ-marking on C and a Γ/G-marking on C/G. The data of a (Γ, G)-marking can be pulled back along morphisms of base schemes $S' \to S$ (by definition of the pullback of an admissible cover along such a morphism and the pullback of the Γ-structure on C). We define a stack $\overline{\mathcal{H}}'_{(\Gamma,G)}$ of admissible covers with a (Γ, G)-marking.

Proposition 4.6 *There is a natural isomorphism $\overline{\mathcal{H}}'_{(\Gamma,G)} \simeq \overline{\mathcal{H}}_{(\Gamma,G)}$.*

Proof The proof is completely analogous to that of Proposition 2.15.   □

Proof of Proposition 4.3 We use the modular interpretation of $\overline{\mathcal{H}}_{(\Gamma,G)}$ given by admissible covers with a (Γ, G)-marking from Proposition 4.6 and the modular interpretation of $\overline{\mathcal{M}}_A$ in terms of curves with an A-marking (Definition 2.14).

An object $((C \to D)_{(\Gamma,G)}, f)$ of $\coprod_{(\Gamma,G,f) \in \mathfrak{H}_{A;G,\xi}} \overline{\mathcal{H}}_{(\Gamma,G)}$ over a (connected) scheme S consists of an admissible G-graph (Γ, G) with a generic A-structure f and a Hurwitz cover $C \to D$ with a (Γ, G)-marking. By definition we have

$$
\xi_A \circ \phi_{(\Gamma,G,f)}((C \to D)_{(\Gamma,G)}, f) = C = \phi \circ \xi_{(\Gamma,G)}((C \to D)_{(\Gamma,G)}, f).
$$

We have a natural isomorphism $\xi_A \circ \phi_{(\Gamma,G,f)} \Rightarrow \phi \circ \xi_{(\Gamma,G)}$ given by the identity, so the strict universal property of fibre products (of stacks) gives us a morphism

$$q: \coprod_{(\Gamma,G,f)\in\mathfrak{H}_{A;G,\xi}} \overline{\mathcal{H}}_{(\Gamma,G)} \to \overline{\mathcal{M}}_A \times_{\overline{\mathcal{M}}_{g,r}} \overline{\mathcal{H}}_{g,G,\xi}.$$

In other words we have a commutative diagram

We will construct an inverse functor r to q.

Let $(C'_A, C \to D, \alpha: C' \xrightarrow{\sim} C)$ be an object of $\overline{\mathcal{M}}_A \times_{\overline{\mathcal{M}}_{g,r}} \overline{\mathcal{H}}_{g,G,\xi}$ over a base scheme S and let G be the group associated to the admissible cover $C \to D$. As explained in the proof of Proposition 2.13 the A-structure on C' again passes through α to define an A-structure on the curve C. We now have the following data on $C \to D$:

(i) a set of sections

$$E := \bigcup_{a \in G} \{a(\sigma_i)\}_i$$

in the singular locus of C where σ_i are the sections defined by the A-structure on C',

(ii) a set of sections

$$H := \bigcup_{a \in G} \{a(\tilde\sigma_{i,j})\}_{i,j}$$

in the normalization $\tilde{C}$ of C,

(iii) a set V of π-relative components of $C \backslash E$

This data defines a stable graph

$$\Gamma := (V, H, L, g, \iota, a, \zeta)$$

where g is the genus function that assigns the geometric genus to each element of V, ι is the involution defined by $\iota(a(\tilde\sigma_{i,1})) = a(\tilde\sigma_{i,2})$ for all $a \in G$, and $a: H \to V$ is the function that sends a section to the π-relative component it lies on.
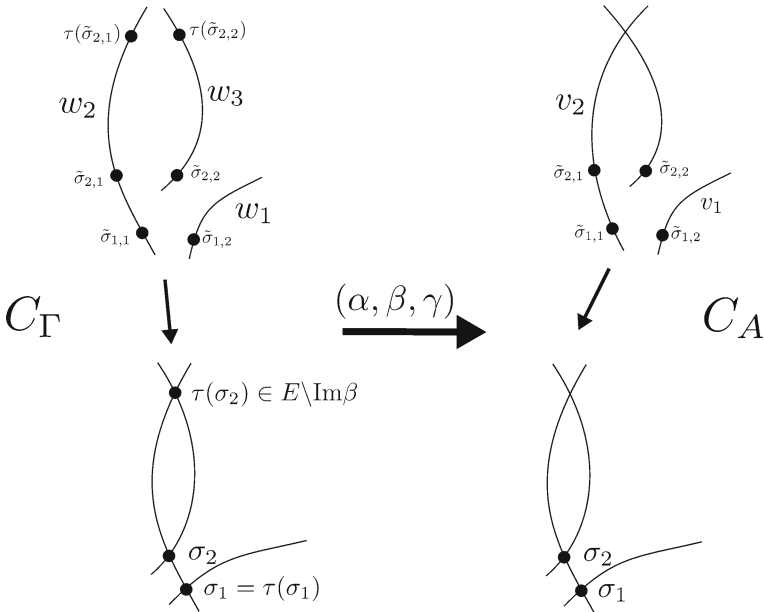
Fig. 4 The curve C with A structure and Γ structure when $G \simeq \mathbb{Z}/2\mathbb{Z}$ with generator τ

The action of G on C (and on the normalization $\tilde{C}$) induces a G-action (with stabilizer datum) on Γ. Since $C \to D$ is an object of $\overline{\mathcal{H}}_{g,G,\xi}$, it follows that (Γ, G) is an admissible G-graph.

Let $\beta \colon H_A \hookrightarrow H$ be the obvious inclusion. Let $\alpha \colon V \to V_A$ be the map sending a π relative component of D_Γ to the π relative component of D_A it is mapped to under the partial normalization defined by the sections $E \backslash \operatorname{Im} \beta$. Let $\gamma \colon H \backslash \operatorname{Im} \beta \to V_A$ be the map which sends a section $a(\sigma_{i,j})$, where a is not the trivial element, to the π-relative component of C_A it is mapped to under the partial normalization map. This defines an A-structure f on (Γ, G). This A-structure is generic since $G\beta(H_A) = H$ by definition of H (where H_A is the set of half edges of A) (Fig. 4).

Note that f is unique up to isomorphisms of A-structures on (Γ, G). We therefore have a well defined object of $\coprod_{(\Gamma,G,f)\in\mathfrak{H}_{A;G,\xi}} \overline{\mathcal{H}}_{(\Gamma,G)}$. This defines the functor r on the objects of $\overline{\mathcal{M}}_A \times_{\overline{\mathcal{M}}_{g,r}} \overline{\mathcal{H}}_{g,G,\xi}$.

Let $(\lambda \colon C_1' \to C_2', \eta \colon (C_1 \to D_1) \to (C_2 \to D_2))$ be a morphism in $\overline{\mathcal{M}}_A \times_{\overline{\mathcal{M}}_{g,r}} \overline{\mathcal{H}}_{g,G,\xi}$ (note that $\xi_A(\lambda) = \phi(\eta)$). By passing through the above construction we see that we get an isomorphism of admissible G-graphs $(\Gamma_1, G) \to (\Gamma_2, G)$ which commutes with the A-structures f_1 and f_2 on Γ_1, Γ_2. We therefore obtain a morphism

$$((C_1 \to D_1)_{(\Gamma_1,G)}, f_1) \to ((C_2 \to D_2)_{(\Gamma_2,G)}, f_2).$$

This defines a morphism in $\coprod_{(\Gamma,G,f)\in\mathfrak{H}_{A;G,\xi}} \overline{\mathcal{H}}_{(\Gamma,G)}$. This completes the definition of the functor

$$r\colon \overline{\mathcal{M}}_A \times_{\overline{\mathcal{M}}_{g,r}} \overline{\mathcal{H}}_{g,G,\xi} \rightarrow \coprod_{(\Gamma,G,f)\in\mathfrak{H}_{A;G,\xi}} \overline{\mathcal{H}}_{(\Gamma,G)}.$$

It remains to check that $r \circ q$ and $q \circ r$ are naturally isomorphic to the respective identity maps. Let $((C \rightarrow D)_{(\Gamma,G)}, f)$ be an object of $\coprod_{(\Gamma,G,f)\in\mathfrak{H}_{A;G,\xi}} \overline{\mathcal{H}}_{(\Gamma,G)}$. Since f is a generic A-structure on (Γ, G) it is easy to verify that

$$r \circ q((C \rightarrow D)_{(\Gamma,G)}, f) = r(C_A, (C \rightarrow D), \mathrm{id}_C)$$
$$= ((C \rightarrow D)_{(\Gamma,G)}, f).$$

If $(C'_A, (C \rightarrow D), \alpha)$ is an object of $\overline{\mathcal{M}}_A \times_{\overline{\mathcal{M}}_{g,r}} \overline{\mathcal{H}}_{g,G,\xi}$, then

$$q \circ r(C'_A, (C \rightarrow D), \alpha) = q((C \rightarrow D)_{(\Gamma,G)}, f)$$
$$= (C_A, (C \rightarrow D), \mathrm{id}_C).$$

The isomorphism α induces an isomorphism of curves with A structure $\alpha^{-1}\colon C'_A \rightarrow C_A$ so $(C_A, (C \rightarrow D), \alpha)$ and $(C_A, (C \rightarrow D), \mathrm{id}_S)$ are isomorphic by $(\alpha^{-1}, \mathrm{id}_{C\rightarrow D})$. We thus have a natural isomorphism of functors $q \circ r \overset{\sim}{\Rightarrow} \mathrm{id}$. $\qquad\square$

4.2 The excess bundle

Let A be a stable n pointed genus g graph. We have shown in Proposition 4.3 that the diagram

$$\begin{array}{ccc}
\coprod_{(\Gamma,G,f)\in\mathfrak{H}_{A,G,\xi}} \overline{\mathcal{H}}_{(\Gamma,G)} & \overset{\coprod \xi_{(\Gamma,G)}}{\longrightarrow} & \overline{\mathcal{H}}_{g,G,\xi} \\
{\scriptstyle \coprod \phi_{(\Gamma,G,f)}}\downarrow & & \downarrow{\scriptstyle \phi} \\
\overline{\mathcal{M}}_A & \overset{\xi_A}{\longrightarrow} & \overline{\mathcal{M}}_{g,r}.
\end{array} \qquad (26)$$

is Cartesian. We can compute the excess bundle separately on each component $\overline{\mathcal{H}}_{(\Gamma,G)}$ of the disjoint union above. Each component $\overline{\mathcal{H}}_{(\Gamma,G)}$ comes with an A-structure $f = (\alpha, \beta, \gamma)$ on Γ so we have to compute

$$E_f = \phi_f^* N_{\xi_A}/N_{\xi_{(\Gamma,G)}},$$

where we abbreviate $\phi_{(\Gamma,G,f)}$ as ϕ_f from now on.

Let us slightly abuse notation and also denote by β the induced map of edges $E_A \rightarrow E_\Gamma$. Let N be a set of representatives for the orbits of the action of G on E such that $N \subset \mathrm{Im}\,\beta$ (this is possible since the A-structure f is generic). Then:

Proposition 4.7 *With the above notation we have on $\overline{\mathcal{H}}_{(\Gamma,G)}$ that*

$$c_{top}E_f = \prod_{(h,h')\in\mathrm{Im}\,\beta\backslash N} -\psi_h - \psi_{h'}. \tag{27}$$

Some remarks concerning this formula: given a graph (Γ, G) and a half-edge h, for any element $t \in G$ we have $\psi_{th} = \psi_h$, since the corresponding cotangent spaces are related by the action via t. This shows that the above formula is independent of the choice of representatives N made before.

In fact, we can interpret this formula as follows: for any orbit of an edge $e = (h, h')$ in Γ let $k = |\mathrm{Im}\,\beta \cap (Ge)|$ be the number of edges in this orbit coming from the graph A. By the assumption that the A-structure on (Γ, G) is generic, we have $k \geq 1$. Then the orbit of e contributes a factor $(-\psi_h - \psi_{h'})^{k-1}$ to the Chern class $c_{top}E_f$ of the excess bundle. In particular, there is excess if and only if there is an orbit of edges containing at least two edges coming from A.

For the proof of Proposition 4.7 we will use the following fact:

Lemma 4.8 *Recall that fibre products (of stacks) commute are compatible with composition. That is, if we have maps $f_1 \colon X_1 \to Z$, $f_2 \colon X_2 \to X_1$ and $g \colon Y \to Z$ then there is an isomorphism $X_2 \times_{X_1} (X_1 \times_Z Y) \simeq X_2 \times_Z Y$. Moreover if the morphisms f_1 and f_2 are lci, then in the resulting diagram*

$$
\begin{array}{ccccc}
X_2 \times_{X_1} (X_1 \times_Z Y) \simeq X_2 \times_Z Y & \xrightarrow{h_2} & X_1 \times_Z Y & \xrightarrow{h_1} & Y \\
\downarrow{g_2} & & \downarrow{g_1} & & \downarrow{g} \\
X_2 & \xrightarrow{f_2} & X_1 & \xrightarrow{f_1} & Z
\end{array}
$$

we have

$$c_{top}(g_2^* N_{f_1\circ f_2}/N_{h_1\circ h_2}) = c_{top}(g_2^* N_{f_2}/N_{h_2}) \cdot h_2^* c_{top}(g_1^* N_{f_1}/N_{h_1}).$$

Proof of Proposition 4.7 We will argue by induction on the edges of A. We will show that whenever multiple edges of A land in the same G orbit of edges of Γ there is a contribution to the excess bundle and then identify this contribution for each such edge of A.

We can decompose the map ξ_A into a sequence of gluing morphisms ξ_i each gluing a single edge. In this way we obtain a sequence of fibered diagrams

$$
\begin{array}{ccccccccc}
F & \longrightarrow & \cdots & \longrightarrow & F_i & \xrightarrow{\eta_i} & F_{i-1} & \longrightarrow & \cdots & \longrightarrow & \overline{\mathcal{H}}_{g,G,\xi} \\
\downarrow{\amalg\phi_f} & & & & \downarrow{\phi_i} & & \downarrow{\phi_{i-1}} & & & & \downarrow{\phi} \\
\overline{\mathcal{M}}_A & \longrightarrow & \cdots & \longrightarrow & \overline{\mathcal{M}}_{A_i} & \xrightarrow{\xi_i} & \overline{\mathcal{M}}_{A_{i-1}} & \longrightarrow & \cdots & \longrightarrow & \overline{\mathcal{M}}_{g,r}
\end{array}. \tag{28}
$$

Abusing notation we then have

$$c_{top}(\phi_f^* N_{\xi_A}/N_{\xi_{(\Gamma,G)}}) = \prod_{i=1}^{\#E_A} c_{top}(\phi_i^* N_{\xi_i}/N_{\eta_i}).$$

It therefore suffices to determine $c_{\text{top}}(\phi_i^* N_{\xi_{A_i}}/N_{\eta_i})$ for each i.

By Proposition 4.3 the fibre products F_i of Diagram (28) take the form

$$F_i = \coprod_{(\Gamma_i, G, f_i) \in \mathfrak{H}_{A_i, G, \xi}} \overline{\mathcal{H}}_{(\Gamma_i, G)}$$

Let $\overline{\mathcal{H}}_{(\Gamma', G)} \subset F_i$ be one of the components. The map

$$\phi_{(\Gamma_i, G, f_i)} \colon \overline{\mathcal{H}}_{(\Gamma_i, G)} \to \overline{\mathcal{M}}_{A_i}$$

factors as $\xi_{\Gamma_i \to A_i} \circ \phi_{\Gamma_i}$ where $\phi_{\Gamma_i} \colon \overline{\mathcal{H}}_{\Gamma_i, G} \hookrightarrow \overline{\mathcal{M}}_{\Gamma_i}$ is the source map and $\xi_{\Gamma_i \to A_i} \colon \overline{\mathcal{M}}_{\Gamma_i} \to \overline{\mathcal{M}}_{A_i}$ is the gluing morphism.

We can then form the fibered diagram

$$
\begin{array}{ccc}
\coprod_{B \in \mathfrak{G}_{A_{i+1}, \Gamma_i}} \coprod_{(\Gamma_{i+1}, G, f_{i+1}) \in \mathfrak{H}_{B, G, \xi}} \overline{\mathcal{H}}_{(\Gamma_{i+1}, G)} \xrightarrow{\coprod \xi'_{i+1}} \overline{\mathcal{H}}_{(\Gamma_i, G)} \\
\downarrow{\scriptstyle \coprod \phi_f} \qquad\qquad\qquad \downarrow{\scriptstyle \phi_{\Gamma_i}} \\
\coprod_{B \in \mathfrak{G}_{A_{i+1}, \Gamma_i}} \overline{\mathcal{M}}_B \longrightarrow \overline{\mathcal{M}}_{\Gamma_i} \\
\downarrow \qquad\qquad \downarrow{\scriptstyle \xi_{\Gamma_i \to A_i}} \\
\overline{\mathcal{M}}_{A_{i+1}} \xrightarrow{\quad \xi_{i+1} \quad} \overline{\mathcal{M}}_{A_i}
\end{array}
\qquad (29)
$$

where the lower square is Cartesian by Proposition 2.13 and the upper square is Cartesian by Proposition 4.3.

Since $\text{codim}_{\overline{\mathcal{M}}_{A_i}} \text{Im}(\xi_{i+1}) = 1$ the codimension of $\xi'_{i+1}(\overline{\mathcal{H}}_{(\Gamma_{i+1}, G)})$ in $\overline{\mathcal{H}}_{(\Gamma_i, G)}$ is either 1 or 0. If it is 1 then there is no excess on this irreducible component.

If the codimension is 0 then $\overline{\mathcal{H}}_{(\Gamma_{i+1}, G)} = \overline{\mathcal{H}}_{(\Gamma_i, G)}$, i.e. $(\Gamma_{i+1}, G) = (\Gamma_i, G)$. Restricting Diagram (29) we get

$$
\begin{array}{ccc}
\overline{\mathcal{H}}_{(\Gamma_{i+1}, G)} \xrightarrow{\text{id}_{\overline{\mathcal{H}}}} \overline{\mathcal{H}}_{(\Gamma_i, G)} \\
\downarrow{\scriptstyle \phi_{\Gamma_{i+1}}} \qquad \downarrow{\scriptstyle \phi_{\Gamma_i}} \\
\overline{\mathcal{M}}_{\Gamma_i} \xrightarrow{\text{id}_{\overline{\mathcal{M}}}} \overline{\mathcal{M}}_{\Gamma_i} \\
\downarrow{\scriptstyle \xi_{\Gamma_i \to A_{i+1}}} \qquad \downarrow \\
\overline{\mathcal{M}}_{A_{i+1}} \xrightarrow{\xi_{i+1}} \overline{\mathcal{M}}_{A_i}
\end{array}
\qquad .
$$

We deduce that

$$\phi_{\Gamma_{i+1}}^* \xi_{\Gamma_i \to A_{i+1}}^* N_{\xi_{i+1}}/N_{\text{id}_{\overline{\mathcal{H}}}} = \phi_{\Gamma_{i+1}}^* \left(\xi_{\Gamma_i \to A_{i+1}} N_{\xi_{i+1}}/N_{\text{id}_{\overline{\mathcal{M}}}} \right) = \phi_{\Gamma_{i+1}}^* (\mathbb{L}_h \otimes \mathbb{L}_{h'})$$

where (h, h') is the edge of Γ corresponding to the edge of A_{i+1} glued together by the morphism ξ_{i+1} and where the second equality is due to.

Now note that this situation only occurs when A_{i+1} is a specialization of Γ_i. In other words this situation occurs exactly if and only if the node glued together by ξ_{i+1} is already a node of Γ, which can only happen if and only if the edge glued by ξ_{i+1} is in the G orbit of one of the edges glued by $\overline{\mathcal{M}}_{A_i} \to \overline{\mathcal{M}}_{g,r}$.   □

From the excess intersection formula, Propositions 4.3 and 4.7 we now obtain:

Theorem 4.9 *We have*

$$\xi_A^* \phi_*([\overline{\mathcal{H}}_{g,G,\xi}]) = \sum_{(\Gamma,G,f) \in \mathfrak{H}_{A;G,\xi}} \phi_{f*}(c_{\text{top}} E_f \cap [\overline{\mathcal{H}}_{(\Gamma,G)}]).$$

where $c_{top}(E_f)$ is as in Proposition 4.7.

Note that for an A-structure $f = (\alpha, \beta, \gamma)$ appearing in the above sum and a factor $(-\psi_h - \psi_{h'})$ from $c_{\text{top}}(E_f)$ (where (h, h') is an edge of Γ in Im β), we have, by definition of ψ classes on spaces of admissible covers

$$-\psi_h - \psi_{h'} = \phi_f^*(-\psi_{\beta^{-1}(h)} - \psi_{\beta^{-1}(h')}).$$

In other words the class $c_{\text{top}}(E_f)$ is a pullback of a suitable combination of ψ-classes on $\overline{\mathcal{M}}_A$. By the projection formula and slight abuse of notation, we can therefore also write

$$\xi_A^* \phi_*([\overline{\mathcal{H}}_{g,G,\xi}]) = \sum_{(\Gamma,G,f) \in \mathfrak{H}_{A;G,\xi}} c_{\text{top}} E_f \cap \phi_{f*}[\overline{\mathcal{H}}_{(\Gamma,G)}].$$

4.3 Pushing forward by the target morphism

Let

$$\begin{array}{ccc} \overline{\mathcal{H}}_{g,G,\xi} & \xrightarrow{\phi} & \overline{\mathcal{M}}_{g,r} \\ \downarrow{\scriptstyle \delta} & & \\ \overline{\mathcal{M}}_{g',b} & & \end{array} \qquad (30)$$

be the diagram from Theorem 3.7 and let $[A_\theta]$ be a decorated stratum class on $\overline{\mathcal{M}}_{g,r}$. In certain situations it is useful to compute the pull–push

$$\delta_* \phi^* [A_\theta]$$

in terms of decorated stratum classes on $\overline{\mathcal{M}}_{g',b}$. We can now do this by forming the diagram

$$
\begin{array}{ccc}
& \coprod_{(\Gamma,G,f)\in\mathfrak{H}_{A,G,\xi}} \overline{\mathcal{H}}_{(\Gamma,G)} & \xrightarrow{\coprod\phi_f} \overline{\mathcal{M}}_A \\
\scriptstyle{\coprod\delta_{(\Gamma,G)}} \nearrow & \scriptstyle{\coprod\xi_{(\Gamma,G)}}\downarrow & \downarrow\scriptstyle{\xi_A} \\
& \overline{\mathcal{H}}_{g,G,\xi} \xrightarrow{\quad\phi\quad} \overline{\mathcal{M}}_{g,r} \\
& \downarrow\scriptstyle{\delta} & \\
\coprod_{(\Gamma,G,f)\in\mathfrak{H}_{A,G,\xi}} \overline{\mathcal{M}}_{\Gamma/G} \xrightarrow[\coprod\xi_{\Gamma/G}]{} & \overline{\mathcal{M}}_{g',b} &
\end{array}
$$

To express $\delta_*\phi^*[A_\theta]$ we first observe that

$$
\phi^*[A_\theta] = \phi^*\xi_{A*}\theta = \sum_{(\Gamma,G,f)\in\mathfrak{H}_{A,G,\xi}} \xi_{(\Gamma,G)*}\big(c_{\text{top}}(E_f)\cdot\phi_f^*\theta\big).
$$

Here we use that the upper right square is Cartesian by Proposition 4.3 and the excess bundle of the pullback along ϕ is given, for each $(\Gamma, G, f) \in \mathfrak{H}_{A,G,\xi}$, in terms of a pullback of ψ classes on $\overline{\mathcal{M}}_\Gamma$, as $c_{\text{top}}(E_f)$ in Proposition 4.7.

As a next step, the commutativity of the left part of the diagram above implies that

$$
\delta_*\phi^*[A_\theta] = \sum_{(\Gamma,G,f)} \delta_*\xi_{(\Gamma,G)*}\big(c_{\text{top}}(E_f)\cdot\phi_f^*\theta\big) = \sum_{(\Gamma,G,f)} \xi_{\Gamma/G*}\delta_{(\Gamma,G)*}\big(c_{\text{top}}(E_f)\cdot\phi_f^*\theta\big).
$$

The term $c_{\text{top}}(E_f) \cdot \phi_f^*\theta$ is a linear combination of monomials α_j in κ and ψ-classes on $\overline{\mathcal{H}}_{(\Gamma,G)}$. Using (30) and Lemma 3.9 we can find for any such monomial α_j a monomial term[5] $\tilde\alpha_j$ in κ and ψ-classes on $\overline{\mathcal{M}}_{\Gamma/G}$ such that $\alpha_j = \delta_{(\Gamma,G)}^*\tilde\alpha_j$. Let F be the $\mathbb{Q}$-linear extension of the map $\alpha_j \mapsto \tilde\alpha_j$, such that $\alpha = \delta_{(\Gamma,G)}^* F(\alpha)$ for any polynomial α in κ and ψ-classes on $\overline{\mathcal{M}}_{(\Gamma,G)}$. Then we have

$$
\delta_{(\Gamma,G)*}c_{\text{top}}(E_f)\cdot\phi_f^*\theta = \delta_{(\Gamma,G)*}\delta_{(\Gamma,G)}^* F\big(c_{\text{top}}(E_f)\cdot\phi_f^*\theta\big) = \deg\delta_{(\Gamma,G)}\cdot F\big(c_{\text{top}}(E_f)\cdot\phi_f^*\theta\big)
$$

again by the projection formula.

Combining these results we get the following combinatorial description of $\delta_*\phi^*[A_\theta]$ by the excess intersection formula and the projection formula:

Theorem 4.10 *With the above notation we have*

$$
\delta_*\phi^*\xi_{A*}([\theta]) = \sum_{(\Gamma,G,f)\in\mathfrak{H}_{A;G,\xi}} \deg\delta_{(\Gamma,G)}\ \xi_{\Gamma/G*}\big(F(c_{\text{top}}E_f \cdot \phi_f^*(\theta)) \cap [\overline{\mathcal{M}}_{\Gamma/G}]\big).
$$

Note that $\deg\delta_{(\Gamma,G)}$ is given componentwise in Theorem 3.19 (and more explicitly for G cyclic in Proposition 3.20).

Corollary 4.11 *If $\overline{\mathcal{H}}_{g,G,\xi}$ is a stack of pointed admissible G-covers with source and target morphisms ϕ and δ, then in particular Theorem 4.10 implies that the pull–push*

$$
\delta_*\phi^*\colon A^\bullet(\overline{\mathcal{M}}_{g,r}) \to A^\bullet(\overline{\mathcal{M}}_{g',b})
$$

---

[5] That is, a rational coefficient times a monomial in κ and ψ-classes.

sends tautological classes to tautological classes.

4.4 Forgetting points

In practice we are usually interested in the image of $\overline{\mathcal{H}}_{g,G,\xi}$ under the composition

$$\phi_{(h_{k_1},\dots,h_{k_j})}\colon \overline{\mathcal{H}}_{g,G,\xi} \xrightarrow{\phi} \overline{\mathcal{M}}_{g,r} \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n} \tag{31}$$

where π is the map forgetting the markings associated to $h_{k_1}, \dots, h_{k_j}$ and their G-orbits, and stabilizing the resulting curve. For example the hyperelliptic locus $\overline{\mathcal{H}}_g \subset \overline{\mathcal{M}}_g$ can be defined in our terms as $\phi_{(1^{2g+2})}(\overline{\mathcal{H}}_{g,\mathbb{Z}/2\mathbb{Z},(1^{2g+2})})$.

We are therefore interested in computing the intersection between such a locus and a decorated stratum class. If A is an n-pointed stable graph we can form the commutative diagram

$$\begin{array}{ccc}
\coprod_{B\in\mathfrak{A}_A}\overline{\mathcal{M}}_B & \xrightarrow{\coprod \xi_B} & \overline{\mathcal{M}}_{g,r} \\
\Big\downarrow{\scriptstyle\coprod \pi_B} & & \Big\downarrow{\scriptstyle\pi} \\
\overline{\mathcal{M}}_A & \xrightarrow{\xi_A} & \overline{\mathcal{M}}_{g,n}
\end{array} \tag{32}$$

where $\mathfrak{A}_A$ is the set of all isomorphism classes of r-pointed graphs B such that after forgetting $r-n$ of the legs of B we get A (recall here that if $f = (\alpha, \beta, \gamma)\colon A' \to A$ is an isomorphism of stable graphs then the leg assignment must commute with α, i.e. $\alpha \circ \zeta_{A'} = \zeta_A$, in particular a permutation of the set of legs of a stable graph A does not neccesarily induce an isomorphism of A). We define the map $\pi_B\colon \overline{\mathcal{M}}_B \to \overline{\mathcal{M}}_A$ as the map forgetting the relevant points (and stabilizing) componentwise.

The diagram (32) is not Cartesian. Indeed $\overline{\mathcal{M}}_{g,r}$ has connected fibers over $\overline{\mathcal{M}}_{g,n}$ but $\coprod_{B\in\mathfrak{A}_A}\overline{\mathcal{M}}_B \to \overline{\mathcal{M}}_A$ does not have connected fibers. However the map

$$\xi\colon \coprod_{B\in\mathfrak{A}_A} \overline{\mathcal{M}}_B \to \overline{\mathcal{M}}_A \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,r}$$

is proper and generically 1-to-1 (unwrapping definitions we see that ξ is surjective and $\coprod \xi_B$ is generically injective so ξ is as well). It follows that $\pi^*([A]) = \sum_{B\in\mathfrak{A}_A}[B]$.

To summarize, we have the following diagram.

$$\begin{array}{ccc}
\coprod_{B\in\mathfrak{A}_A}\coprod_{(\Gamma,G,f)\in\mathfrak{H}_{B,G,\xi}}\overline{\mathcal{H}}_{(\Gamma,G)} & \xrightarrow{\coprod \xi_{(\Gamma,G)}} & \overline{\mathcal{H}}_{g,G,\xi} \\
{\scriptstyle\coprod \phi_f}\Big\downarrow & & \Big\downarrow{\scriptstyle\phi} \\
\coprod_{B\in\mathfrak{A}_A}\overline{\mathcal{M}}_B & \xrightarrow{\coprod \xi_B} & \overline{\mathcal{M}}_{g,r} \\
{\scriptstyle\coprod \pi_B}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi} \\
\overline{\mathcal{M}}_A & \xrightarrow{\xi_A} & \overline{\mathcal{M}}_{g,n}
\end{array}$$

Then putting everything together, we get the following generalization of Theorem 4.9.

Theorem 4.12 *With notation as above we have*

$$\xi_A^* \pi_* \phi_* ([\overline{\mathcal{H}}_{g,G,\xi}]) = \sum_{B \in \mathfrak{A}_A} \sum_{(\Gamma,G,f) \in \mathfrak{H}_{B;G,\xi}} (\pi_B)_* c_{\text{top}} E_f \cap \phi_{f*}([\overline{\mathcal{H}}_{(\Gamma,G)}]). \quad (33)$$

Next we want to generalize Theorem 4.10. That is, looking at the diagram

$$\begin{array}{ccc}
\overline{\mathcal{H}}_{g,G,\xi} \xrightarrow{\ \phi\ } \overline{\mathcal{M}}_{g,r} \xrightarrow{\ \pi\ } \overline{\mathcal{M}}_{g,n} \\
\ \downarrow{\scriptstyle\delta} \\
\overline{\mathcal{M}}_{g',b}
\end{array} \quad (34)$$

we want to compute the pull–push $\delta_* \phi^* \pi^* [A_\theta]$ for decorated stratum classes $[A_\theta]$ on $\overline{\mathcal{M}}_{g,n}$.

From the diagram (32) it follows that

$$\pi^* [A_\theta] = \pi^* \xi_{A*} \theta = \sum_{B \in \mathfrak{A}_A} \xi_{B*} \pi_B^* \theta. \quad (35)$$

The maps π_B are just products of the usual forgetful maps between moduli spaces of stable curves. The pullback of monomials θ in κ and ψ-classes is determined by the following well known result

Proposition 4.13 *Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the map forgetting marking $n+1$. Then we have*

$$\psi_i = \pi^*(\psi_i) + D_{i,n+1}$$
$$\kappa_a = \pi^*(\kappa_a) - \psi_{n+1}^a$$

where $D_{i,n+1}$ is the divisor which has the markings i and $n + 1$ on an irreducible component R of arithmetic genus 0 and all other markings on a component of genus g not containing R.

Again, putting things together we can first use Eq. (35) and Proposition 4.13 to compute the pullback $\pi^* [A_\theta]$ in terms of decorated stratum classes $[B_{\theta'}]$ on $\overline{\mathcal{M}}_{g,r}$ and then apply Theorem 4.10 to each of these $[B_{\theta'}]$ and push the result forward through δ.

4.5 Example computations

In the following we apply the techniques from the last sections to compute some examples of pullbacks for hyperelliptic and bielliptic cycles.

Notation 4.14 We will denote by $\overline{\mathcal{H}}_{g,n_1,2n_2}, \overline{\mathcal{B}}_{g,n_1,2n_2} \subset \overline{\mathcal{M}}_{g,n_1+2n_2}$ the spaces

$$\phi_{(1^{2g+2-n_1})}(\overline{\mathcal{H}}_{g,\mathbb{Z}/2\mathbb{Z},(1^{2g+2},0^{n_2})}),$$
$$\phi_{(1^{2g-2-n_1})}(\overline{\mathcal{H}}_{g,\mathbb{Z}/2\mathbb{Z},(1^{2g-2},0^{n_2})}),$$

respectively. Note that on the smooth locus these are the hyperelliptic and bielliptic curves of genus g with n_1 points fixed by the covering involution and n_2 pairs of points conjugate under the covering involution. We will drop n_1 and n_2 from the notation when they are both zero.

Example 4.15 Let A be the stable graph

$$\overset{\displaystyle v_1 \qquad v_2}{\textcircled{2}\!\!-\!\!\textcircled{2}}$$
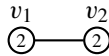
we will use Theorem 4.9 to compute $\xi_A^*([\overline{\mathcal{B}}_4]) = \xi_A^*[\phi_{(1^6)}(\overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^6)})]$. We form the diagram

$$
\begin{array}{ccc}
\coprod_{B\in\mathfrak{A}_A}\coprod_{(\Gamma,G,f)\in\mathfrak{H}_{(B,G,(1^6))}}\overline{\mathcal{H}}_{(\Gamma,G)} & \longrightarrow & \overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^6)} \\
\downarrow & & \downarrow \\
\coprod_{B\in\mathfrak{A}_A}\overline{\mathcal{M}}_B & \longrightarrow & \overline{\mathcal{M}}_{4,6} \\
\downarrow & & \downarrow \\
\overline{\mathcal{M}}_A & \longrightarrow & \overline{\mathcal{M}}_4
\end{array}
$$

where the upper square is Cartesian. The set $\mathfrak{A}_A$ consists of all possible distributions of legs over the vertices of A. We claim that from the nonemptyness condition of admissible G graphs it follows that any graph B in $\mathfrak{A}_A$, such that there exists an admissible G graph (Γ, G) with generic B structure, will have one leg attached to one of the vertices and all other legs to the other vertex. Indeed since A has a separating edge, any $B \in \mathfrak{A}_A$ has a seperating edge. It follows that for any (Γ, G) with a generic B structure the graph Γ and thus Γ/G must also have a seperating edge. The graph Γ/G without the legs is therefore of the form

$$\overset{\displaystyle v_1 \qquad v_2}{\textcircled{1}\!\!-\!\!\textcircled{0}}.$$
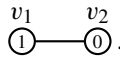
From Riemann–Hurwitz it now follows that to satisfy the nonemptyness condion the vertex of genus 1 must have 2 legs and/or half edges and the vertex of genus 0 must have 6 legs and/or half edges attached to it. This proves the claim. The set of graphs in $\mathfrak{A}_A$ having nonzero intersection with $\phi(\overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^6)})$ thus consists of graphs of the following form:

$$B_1 = \overset{\displaystyle v_1 \qquad v_2}{\textcircled{2}\!\!-\!\!\textcircled{2}}\!\!\!< \qquad\qquad B_2 = >\!\!\!\overset{\displaystyle v_1 \qquad v_2}{\textcircled{2}\!\!-\!\!\textcircled{2}}$$
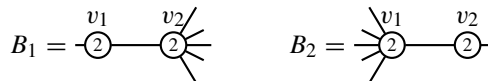
The corresponding admissible G-graphs in $\mathfrak{H}_{B_i,\mathbb{Z}/2\mathbb{Z},(1^6)}$ are, respectively, of the form

$$(\Gamma_1, G) = \quad \overset{\overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^2)}}{\textcircled{2}} \rule{1cm}{0.4pt} \overset{\overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^6)}}{\textcircled{2}}\!\!\!\!\prec \qquad (\Gamma_2, G) = \quad \succ\!\!\!\!\overset{\overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^6)}}{\textcircled{2}} \rule{1cm}{0.4pt} \overset{\overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^2)}}{\textcircled{2}}$$
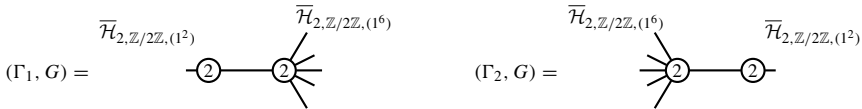
with the obvious B_i-structure on (Γ_i, G). Note that the top Chern class of the excess bundle is trivial. We therefore have

$$\xi_A^* \phi_{(1^6)*}[\overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^6)}] = \binom{6}{1}\bigg(\phi_{(1)*}[\overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^2)}] \otimes \phi_{(1^5)*}[\overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^6)}]$$
$$+ \phi_{(1^5)*}[\overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^6)}] \otimes \phi_{(1)*}[\overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^2)}] \bigg).$$

For smooth bielliptic curves of genus 4 the bielliptic involution is unique and generically such curves have no further automorphisms. Therefore the degree of $\phi_{(1^6)*}\colon \overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^6)} \to \overline{\mathcal{M}}_4$ onto its image is 6!, the number of possible orderings of the 6 ramification points of the bielliptic map. Similarly the degree of $\phi_{(1)*}\colon \overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^2)} \to \overline{\mathcal{M}}_{2,1}$ to its image is 1 and the degree of $\phi_{(1^5)*}\colon \overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^2)} \to \overline{\mathcal{M}}_{2,1}$ is 5!. Thus

$$\xi_A^*[\overline{\mathcal{B}}_4] = \frac{1}{6!} \xi_A^* \phi_{(1^6)*}[\overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^6)}]$$
$$= \frac{\binom{6}{1}}{6!}\big(\phi_{(1)*}[\overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^2)}] \otimes \phi_{(1^5)*}[\overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^2)}]$$
$$+ \phi_{(1^5)*}[\overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^6)}] \otimes \phi_{(1)*}[\overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^2)}]\big)$$
$$= \frac{1}{5!}\Big([\overline{\mathcal{B}}_{2,1}] \otimes 5![\overline{\mathcal{H}}_{2,1}] + 5![\overline{\mathcal{H}}_{2,1}] \otimes [\overline{\mathcal{B}}_{2,1}] \Big)$$
$$= [\overline{\mathcal{B}}_{2,1}] \otimes [\overline{\mathcal{H}}_{2,1}] + [\overline{\mathcal{H}}_{2,1}] \otimes [\overline{\mathcal{B}}_{2,1}] \in A^3(\overline{\mathcal{M}}_{2,1} \times \overline{\mathcal{M}}_{2,1}).$$

This computation will turn out useful later when we compute $[\overline{\mathcal{B}}_4]$ in Theorem 5.7.

Example 4.16 Let

$$A = \;\; \text{\textcircled{1}}$$

and consider the maps $\phi_{(1^8)}\colon \overline{\mathcal{H}}_{3,\mathbb{Z}/2\mathbb{Z},(1^8)} \to \overline{\mathcal{M}}_3$, $\delta\colon \overline{\mathcal{H}}_{3,\mathbb{Z}/2\mathbb{Z},(1^8)} \to \overline{\mathcal{M}}_{0,8}$. We will compute $\delta_* \phi_{(1^8)}^*([A])$ using Theorem 4.10. We will go step by step through the different parts of the equation

- First note that $\# \operatorname{Aut} A = 8$ so $[A] = \frac{1}{8}\xi_{A*}([\overline{\mathcal{M}}_A])$.
- The set $\mathfrak{A}$, introduced in (32), consists of all possible distributions of the legs $\{1, \ldots, 8\}$ over the vertices of A. Since A has only one vertex, the set $\mathfrak{A}_A$ consist

of a single element.

$$\mathfrak{A} = \left\{ B := \begin{array}{c} i_1 \quad i_3 \\ i_2 \overbrace{} i_4 \\ 1 \quad \cdots \quad 8 \end{array} \right\}$$

– There are two possible types of admissible $\mathbb{Z}/2\mathbb{Z}$-graphs with a generic B-structure.
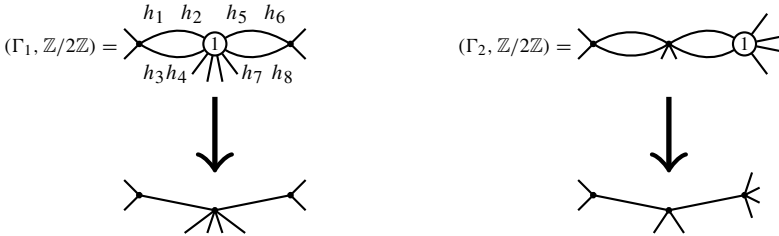
There are $\frac{1}{2}\binom{8}{2,4,2}$ possible nonisomorphic distributions of the legs for the graph Γ_1 and $\binom{8}{2,2,4}$ for Γ_2.

– There are 8 isomorphism classes of B-structures on $(\Gamma_1, \mathbb{Z}/2\mathbb{Z})$. For any generic B-structure f' on $(\Gamma_1, \mathbb{Z}/2\mathbb{Z})$ there exists an automorphism of B-structures

$$(\Gamma_1, \mathbb{Z}/2\mathbb{Z}, f') \to (\Gamma_1, \mathbb{Z}/2\mathbb{Z}, f)$$

with $f = (\alpha, \beta, \gamma)$ such that the edges of B are mapped to the edges (h_1, h_2) and (h_5, h_6) under β. There are 4 possible choices for the image of the half edge i_1 and given this choices there are 2 possible images for the halfedge i_3. This completely determines a B-structure on $(\Gamma_1, \mathbb{Z}/2\mathbb{Z})$. A similar argument shows that there are 8 isomorphism classes of generic B-structures for $(\Gamma_2, \mathbb{Z}/2\mathbb{Z})$.

– We have $\deg \delta_{(\Gamma_1, \mathbb{Z}/2\mathbb{Z})} = \deg \delta_{(\Gamma_2, \mathbb{Z}/2\mathbb{Z})} = 2$ by Proposition 3.20.
– For all generic B-structures f on Γ_1 and Γ_2, the set $\text{Im } \beta_f \cap G(\text{Im } \beta_f)$ is empty, so the top Chern class of the excess bundle is 1.
– The graph A is undecorated, in other words $\theta = 1$. Therefore we have

$$\delta_{(\Gamma_i, \mathbb{Z}/2\mathbb{Z})*} \phi_{f_i}^* \pi_B^*(\theta) = \deg \delta_{(\Gamma_i, \mathbb{Z}/2\mathbb{Z})} = 2$$

.

– The image of $\delta_* \phi_{(1^8)}^* : A^\bullet(\overline{\mathcal{M}}_3) \to A^\bullet(\overline{\mathcal{M}}_{0,8})$ lies in the $\mathfrak{S}_8$ invariant part $A^\bullet(\overline{\mathcal{M}}_{0,8})^{\mathfrak{S}_8}$ of $A^\bullet(\overline{\mathcal{M}}_{0,8})$. Denote by $d_{2,4,2}, d_{4,2,2} \in A^\bullet(\overline{\mathcal{M}}_{0,8})^{\mathfrak{S}_8}$ the classes given by taking the sum over all nonisomorphic dual graphs of the form (respectively):

Note that there are $\frac{1}{2}\binom{8}{2,4,2}$ irreducible boundary components in $d_{2,4,2}$ and $\binom{8}{2,2,4}$ in $d_{4,2,2}$.

Putting everything together we have

$$
\delta_* \phi^*_{(1^8)}([A]) = \frac{1}{8} \cdot 1 \left(\frac{1}{2}\binom{8}{2,4,2} \cdot 16 \cdot 2 \binom{8}{2,4,2}^{-1} d_{2,4,2} \right.
$$

$$
+ \binom{8}{2,2,4} \cdot 16 \cdot \binom{8}{2,2,4}^{-1} d_{2,4,2} \Bigg)
$$

$$
= 2d_{2,4,2} + 2d_{4,2,2} \in R^2(\overline{\mathcal{M}}_{0,8})
$$

More examples are given in [46, Section 2.3].

5 Computing tautological expressions for admissible cover cycles

Let $\phi_{\tilde{\xi}} \colon \overline{\mathcal{H}}_{g,G,\xi} \to \overline{\mathcal{M}}_{g,n}$ be the composition $\pi \circ \phi$ as in (31), let $\overline{\mathcal{H}} := \operatorname{Im} \phi_{\tilde{\xi}}$ and assume that the class $[\overline{\mathcal{H}}]$ is tautological. We can then try to compute $[\overline{\mathcal{H}}]$ in terms of decorated stratum classes. In this section we will outline a few possible methods using Theorems 4.9 and 4.10 to do this.

5.1 Perfect pairing

The intersection pairing $H^k(\overline{\mathcal{M}}_{g,n}) \otimes H^{2\dim \overline{\mathcal{M}}_{g,n} - k}(\overline{\mathcal{M}}_{g,n}) \to \mathbb{C}$ is perfect since $\overline{\mathcal{M}}_{g,n}$ is a smooth complete Deligne-Mumford stack. The Gorenstein conjecture asks whether this pairing restricted to the tautological rings $RH^{2k}(\overline{\mathcal{M}}_{g,n}) \otimes RH^{2\dim \overline{\mathcal{M}}_{g,n} - 2k}(\overline{\mathcal{M}}_{g,n})$ is perfect as well. As was first shown in [42] the Gorenstein conjecture is false in general. However for many low values of g, n and k it is known that $H^{2k}(\overline{\mathcal{M}}_{g,n}) = RH^{2k}(\overline{\mathcal{M}}_{g,n})$ and $H^{2\dim \overline{\mathcal{M}}_{g,n} - 2k}(\overline{\mathcal{M}}_{g,n}) = RH^{2\dim \overline{\mathcal{M}}_{g,n} - 2k}(\overline{\mathcal{M}}_{g,n})$ so that the pairing in this degree is also perfect when restricted to the tautological ring. In particular this is the case

- for $g = 0$ [32] and $g = 1$ [38] and arbitrary n as well as $g = 2, n \le 19$ [40],
- for $k = 0, 1$ [1],
- for $(g, n) = (3, 0), (3, 1), (3, 2), (4, 0)$. Here the ranks of the cohomology groups $H^{2k}(\overline{\mathcal{M}}_{g,n})$ have been computed [5,8,24,26] and agree with the ranks of the intersection pairing on $RH^{2k}(\overline{\mathcal{M}}_{g,n})$ computed in [48].

Assume $[\overline{\mathcal{H}}] \in H^{2k}(\overline{\mathcal{M}}_{g,n})$ is tautological and that the intersection pairing of $\overline{\mathcal{M}}_{g,n}$ restricted to the tautological ring is perfect in degree $2k$. Then we can compute this class in the following way. Suppose we have fixed bases $\{D_i\}$ and $\{\hat{D}_j\}$ for $RH^{2k}(\overline{\mathcal{M}}_{g,n})$ and for $RH^{2\dim \overline{\mathcal{M}}_{g,n} - 2k}(\overline{\mathcal{M}}_{g,n})$ where D_i and $\hat{D}_j$ are decorated stratum classes. As described in Remark 2.26 we can compute the intersection matrix

$$
M = (D_i \cdot \hat{D}_j)_{ij}.
$$

By assumption, there is a unique expression $[\overline{\mathcal{H}}] = \sum a_i D_i$, which we want to compute. It must satisfy

$$([\overline{\mathcal{H}}] \cdot \hat{D}_j)_j = \left(\sum_i a_i D_i \cdot \hat{D}_j \right)_j.$$

We can compute the numbers on the left hand side using Theorem 4.10. Indeed, for any decorated stratum class $\hat{D}_j = [A_\theta]$, the theorem allows us to express the zero cycle $\delta_* \phi_{\tilde{\xi}}^*[A_\theta]$ in terms of tautological classes. Again using Remark 2.26 we can compute the degree of this tautological zero cycle.

Finally, since M is an invertible matrix by assumption, we can solve this expression for the a_i.

Example 5.1 We can use this method to compute the class of the hyperelliptic locus in the genus 3 case explicitly.

A basis for $A^1(\overline{\mathcal{M}}_3)$ is given by λ, δ_0 and δ_1 (or κ_1, δ_0 and δ_1, note that $12\lambda = \kappa_1 + \delta_0 + \delta_1$). A basis in terms of decorated stratum classes for $A^5(\overline{\mathcal{M}}_3)$ is given by (see [16])

$$\hat{D}_1 = \left[\begin{array}{c} \circ\!\!-\!\!\circ\!\!\infty \end{array} \right] \qquad \hat{D}_2 = \left[\begin{array}{c} \circ\!\!-\!\!\delta\!\!-\!\!\circ \end{array} \right] \qquad \hat{D}_3 = \left[\begin{array}{c} \textcircled{1}\!\!-\!\!\circ\!\!-\!\!\circ\!\!\infty \end{array} \right]$$

The intersection numbers between the bases have been computed in [16, pg. 418] (or use Theorem 2.25):

$$
\begin{array}{lll}
\lambda \cdot \hat{D}_1 = 0 & \lambda \cdot \hat{D}_2 = 0 & \lambda \cdot \hat{D}_3 = \frac{1}{96} \\
\delta_0 \cdot \hat{D}_1 = -\frac{1}{4} & \delta_0 \cdot \hat{D}_2 = 0 & \delta_0 \cdot \hat{D}_3 = \frac{1}{8} \\
\delta_1 \cdot \hat{D}_1 = \frac{1}{8} & \delta_1 \cdot \hat{D}_2 = -\frac{1}{16} & \delta_1 \cdot \hat{D}_3 = -\frac{1}{96}
\end{array}
$$

Recall that we denote by $[\overline{\mathcal{H}}_3]$ the class $\frac{1}{8!}\phi_{(1^6)*}[\overline{\mathcal{H}}_{3,\mathbb{Z}/2\mathbb{Z},(1^6)}]$. We leave it as an exercise to the reader to compute the intersection numbers

$$[\overline{\mathcal{H}}_3] \cdot \hat{D}_1 = -\frac{1}{8} \qquad [\overline{\mathcal{H}}_3] \cdot \hat{D}_2 = \frac{3}{16} \qquad [\overline{\mathcal{H}}_3] \cdot \hat{D}_3 = 0$$

using Theorems 4.9 or 4.10. Solving the resulting system of equations we get the well known expression

$$[\overline{\mathcal{H}}_3] = (9\lambda - \delta_0 - 3\delta_1) \in A^1(\overline{\mathcal{M}}_3)$$

Remark 5.2 The Sage program [14] has this technique implemented as a function. However it is computationally not the most efficient method. It suffices to compute $[\overline{\mathcal{B}}_{2,2,0}]$ in a short amount of time but most classes which are more complicated require too much computing time and memory. The problem is that in calculating the intersection of $[\text{Im } \phi_{\tilde{\xi}*}]$ with a decorated stratum class $[A_\theta]$ we first have to pull back $[A_\theta]$ through the forgetful map $\pi : \overline{\mathcal{M}}_{g,r} \to \overline{\mathcal{M}}_{g,n}$. If the number $r - n$ is very high then the pullback $\pi^*[A_\theta]$ in terms of decorated stratum classes involves a large number of different classes which leads to a large increase in computation time. Moreover,

many of these classes live on complicated stable graphs. The classification of all G-admissible graphs with large codimension is another computationally intensive operation.

5.2 Repeated pullback to lower genus

A second method for studying cohomology classes such as $\alpha = \phi_{\tilde{\xi}*}[\overline{\mathcal{H}}_{g,G,\xi}]$ on $\overline{\mathcal{M}}_{g,n}$ is to pull them back via the boundary gluing maps ξ_A for stable graphs A. As the following result of Arbarello and Cornalba shows, knowing all the pullbacks $\xi_A^*\alpha$ is in many cases sufficient to determine α uniquely.

Lemma 5.3 (Lemma 2.5, 2.6, [2]) *Let $\mathcal{A}$ be the set of stable graphs A of genus g with n markings having exactly one edge. Then the pullback*

$$H^k(\overline{\mathcal{M}}_{g,n}) \xrightarrow{\Pi \xi_A^*} \prod_{A \in \mathcal{A}} H^k(\overline{\mathcal{M}}_A)$$

via all boundary gluing maps $\xi_A : \overline{\mathcal{M}}_A \to \overline{\mathcal{M}}_{g,n}$ is injective for $k \le d(g,n)$ with

$$d(g,n) = \begin{cases} n-4 & \text{if } g = 0, \\ 2g-2 & \text{if } n = 0, \\ 2g-3+n & \text{if } g > 0, n > 0. \end{cases}$$

Remark 5.4 In particular note that $\dim_{\mathbb{C}} \overline{\mathcal{H}}_g = 2g - 1$ and $\dim_{\mathbb{C}} \overline{\mathcal{B}}_g = 2g - 2$ and so, since the cohomological degree is computed in terms of the real codimension, we have $[\overline{\mathcal{H}}_g] \in H^{2g-4}(\overline{\mathcal{M}}_g)$ and $[\overline{\mathcal{B}}_g] \in H^{2g-2}(\overline{\mathcal{M}}_g)$. Hence the classes of the (unmarked) hyperelliptic and the bielliptic loci are always in the range of Lemma 5.3.

In a lot of ways, computing the pullback by ξ_A gives us finer information than the method of Sect. 5.1, since knowing the pullback $\xi_A^*\alpha$ determines the intersection number of α with any decorated stratum class A_θ on the graph A. The only intersection numbers we cannot obtain this way are those with pure polynomials in κ and ψ-classes on $\overline{\mathcal{M}}_{g,n}$.

So for every stable graph A we form the diagram

$$
\begin{array}{ccc}
\coprod_{B \in \mathfrak{A}} \coprod_{(\Gamma,G,f) \in \mathfrak{H}_{B,G,\xi}} \overline{\mathcal{H}}_{(\Gamma,G)} & \xrightarrow{\coprod \xi_{(\Gamma,G)}} & \overline{\mathcal{H}}_{g,G,\xi} \\
{\scriptstyle \coprod \phi_f} \downarrow & & \downarrow {\scriptstyle \phi} \\
\coprod_{B \in \mathfrak{A}} \overline{\mathcal{M}}_B & \xrightarrow{\coprod \xi_B} & \overline{\mathcal{M}}_{g,r} \\
{\scriptstyle \coprod \pi_B} \downarrow & & \downarrow {\scriptstyle \pi} \\
\overline{\mathcal{M}}_A & \xrightarrow{\xi_A} & \overline{\mathcal{M}}_{g,n}
\end{array} \quad .
$$

Using Theorem 4.9 we can express the pullback $\xi_A^* \pi_* \phi_* ([\overline{\mathcal{H}}_{g,G,\xi}])$ as

$$\xi_A^* \pi_* \phi_* ([\overline{\mathcal{H}}_{g,G,\xi}]) = \sum_{B \in \mathfrak{A}} \sum_{(\Gamma,G,f) \in \mathfrak{H}_{B;G,\xi}} (\pi_B)_* c_{\text{top}} E_f \cap \phi_{f*} ([\overline{\mathcal{H}}_{(\Gamma,G)}]). \qquad (36)$$

Our goal is to express the right hand side of (36) entirely in terms of decorated strata. As shown in Proposition 4.7, the cycle $c_{\text{top}} E_f$ is a combination of ψ-classes, which are tautological. Also we know how the pushforward $(\pi_B)_*$ acts on tautological classes. Now observe that the map ϕ_f is the composition $\xi_f \circ \phi_{(\Gamma,G)}$ and we also know how the partial gluing map ξ_f acts on tautological classes. Thus effectively, we have to be able to express terms of the form $(\phi_{(\Gamma,G)})_* [\overline{\mathcal{H}}_{(\Gamma,G)}]$.

The space $\overline{\mathcal{H}}_{(\Gamma,G)}$ itself was a product of moduli spaces $\overline{\mathcal{H}}_{g(v),G_v,\xi_v}$ for a set of representatives v of the G-action on $V(\Gamma)$. Since these spaces have smaller dimension than $\overline{\mathcal{H}}_{g,G,\xi}$, we assume that the corresponding cycles $\phi_* [\overline{\mathcal{H}}_{g(v),G_v,\xi_v}] \in H^*(\overline{\mathcal{M}}_{g(v),n(v)})$ are tautological and that we have computed them in terms of the generators of the tautological ring.

From the definition of $\phi_{(\Gamma,G)}$ we saw in (17) that it factors as a product of the maps ϕ for the spaces $\overline{\mathcal{H}}_{g(v),G_v,\xi_v}$ composed with diagonal morphisms $\Delta : \overline{\mathcal{M}}_{g(v),n(v)} \to \prod_{w \in Gv} \overline{\mathcal{M}}_{g(v),n(v)}$, where Gv is the orbit of $v \in V(\Gamma)$ under the action of G. By Application 2.27, we can express the class of the diagonal in terms of (tensor products of) tautological classes in the case that the entire cohomology $H^*(\overline{\mathcal{M}}_{g(v),n(v)})$ is tautological. As discussed in Application 2.27, this is the case for many small $(g(v), n(v))$.

Suppose that the cycles $\phi_* [\overline{\mathcal{H}}_{g(v),G_v,\xi_v}]$ and the classes of the diagonals appearing in (36) are tautological. Then we can determine $\xi_A^* \pi_* \phi_* ([\overline{\mathcal{H}}_{g,G,\xi}])$ in terms of decorated stratum classes. Using Theorem 2.25 we can on the other hand compute $\xi_A^* : R^\bullet(\overline{\mathcal{M}}_{g,n}) \to R^\bullet(\overline{\mathcal{M}}_A)$ explicitly as a linear map in terms of (generating sets of) decorated stratum classes. Thus knowing $\xi_A^* \pi_* \phi_* ([\overline{\mathcal{H}}_{g,G,\xi}])$ determines the class $\pi_* \phi_* ([\overline{\mathcal{H}}_{g,G,\xi}])$ in terms of decorated stratum classes up to the kernel of ξ_A^*.

Note that in order to compute the kernel of ξ_A^*, we need to identify all relations between decorated stratum classes on $\overline{\mathcal{M}}_{g,n}$ and on the factors of $\overline{\mathcal{M}}_A$. While in principle these relations can be determined from the intersection form (in case that all cohomology in the corresponding degrees is tautological), this is too computationally intensive in most cases. Instead, we use the implementation of the generalized Faber-Zagier relations from [41], which are known to be a complete set of relations in many cases.

If we can do this for a number of graphs A_i and if

$$\bigcap \ker \xi_{A_i}^* = \{0\}$$

then we can completely determine the class $\pi_* \phi_* ([\overline{\mathcal{H}}_{g,G,\xi}])$. In some cases where the intersection of kernels is not trivial, we can obtain some further linear restrictions on $\pi_* \phi_* ([\overline{\mathcal{H}}_{g,G,\xi}])$ by computing intersection numbers with polynomials in κ and ψ-classes on $\overline{\mathcal{M}}_{g,n}$ as described in Sect. 5.1.

As an example we will now compute the class of the hyperelliptic locus in genus 5.

Theorem 5.5 *In $R^3(\overline{\mathcal{M}}_5)$ we have*

$$
[\overline{\mathcal{H}}_5] = \tfrac{13307}{360}\kappa_3 \quad -\tfrac{1583}{288}\kappa_2\kappa_1 \quad +\tfrac{37}{144}\kappa_1^3 \quad -\tfrac{1943}{288}\Big[\underset{\kappa_2}{\circ\!-\!\circ}\Big] \quad +\tfrac{5}{72}\Big[\underset{\kappa_1^2}{\circ\!-\!\circ}\Big] \quad +\tfrac{407}{96}\Big[\underset{\kappa_1}{\circ\!\!\leftarrow\!\!-\!\circ}\Big]
$$

(remaining terms: a large sum of decorated stratum classes with coefficients)

$$
-\tfrac{28807}{1440}\Big[\cdots\Big] +\tfrac{11}{4}\Big[\cdots\Big] -\tfrac{11}{12}\Big[\cdots\Big] -\tfrac{23}{144}\Big[\cdots\Big] +\tfrac{89}{144}\Big[\cdots\Big] +\tfrac{307}{144}\Big[\cdots\Big]
$$

$$
-\tfrac{274397}{288}\Big[\cdots\Big] +\tfrac{34355}{288}\Big[\cdots\Big] +\tfrac{135}{32}\Big[\cdots\Big] -\tfrac{8219}{96}\Big[\cdots\Big] +\tfrac{21923}{1440}\Big[\cdots\Big] -\tfrac{31163}{1440}\Big[\cdots\Big]
$$

$$
+\tfrac{208729}{720}\Big[\cdots\Big] -\tfrac{79}{144}\Big[\cdots\Big] +\tfrac{1975}{288}\Big[\cdots\Big] -\tfrac{11}{12}\Big[\cdots\Big] +\tfrac{1}{16}\Big[\cdots\Big] +\tfrac{11}{12}\Big[\cdots\Big]
$$

$$
-\tfrac{23}{18}\Big[\cdots\Big] -\tfrac{4057}{32}\Big[\cdots\Big] +\tfrac{199}{32}\Big[\cdots\Big] +\tfrac{34147}{96}\Big[\cdots\Big] +\tfrac{509}{24}\Big[\cdots\Big] -\tfrac{23717}{288}\Big[\cdots\Big]
$$

$$
+\tfrac{10315}{96}\Big[\cdots\Big] +\tfrac{13163}{96}\Big[\cdots\Big] -\tfrac{1909}{16}\Big[\cdots\Big] -\tfrac{13}{36}\Big[\cdots\Big] -\tfrac{53}{36}\Big[\cdots\Big] +\tfrac{425}{36}\Big[\cdots\Big]
$$

$$
+\tfrac{273}{4}\Big[\cdots\Big] -\tfrac{1141}{288}\Big[\cdots\Big] +\tfrac{26357}{1440}\Big[\cdots\Big] -\tfrac{2063}{288}\Big[\cdots\Big] -\tfrac{35713}{144}\Big[\cdots\Big] -\tfrac{35}{576}\Big[\cdots\Big]
$$

$$
-\tfrac{1}{36}\Big[\cdots\Big] +\tfrac{97}{288}\Big[\cdots\Big] -\tfrac{469}{480}\Big[\cdots\Big] -\tfrac{61}{160}\Big[\cdots\Big] -\tfrac{71}{576}\Big[\cdots\Big] -\tfrac{5}{96}\Big[\cdots\Big]
$$

$$
+\tfrac{181}{288}\Big[\cdots\Big] +\tfrac{19}{192}\Big[\cdots\Big] -\tfrac{1}{8}\Big[\cdots\Big] +\tfrac{5}{16}\Big[\cdots\Big] +\tfrac{259}{288}\Big[\cdots\Big] -\tfrac{305}{288}\Big[\cdots\Big]
$$

$$
-\tfrac{13}{288}\Big[\cdots\Big] -\tfrac{2063}{192}\Big[\cdots\Big] +\tfrac{13285}{288}\Big[\cdots\Big] +\tfrac{11}{12}\Big[\cdots\Big] +\tfrac{1}{12}\Big[\cdots\Big] -\tfrac{365}{288}\Big[\cdots\Big]
$$

$$
-\tfrac{7}{288}\Big[\cdots\Big] -\tfrac{1}{16}\Big[\cdots\Big] +\tfrac{17}{48}\Big[\cdots\Big] -\tfrac{19}{48}\Big[\cdots\Big] +\tfrac{7}{576}\Big[\cdots\Big]
$$

Proof of Theorem 5.5 Consider the following two graphs

$$A_1 = \; \circled{2}\!\!\bigcirc\!\!\circled{2} \qquad\qquad A_2 = \circled{3}\!-\!\circled{2}.$$

Using Theorem 2.25 (and [14]) the pullbacks

$$
\xi_{A_1}^* : A^3(\overline{\mathcal{M}}_5) \to A^3(\overline{\mathcal{M}}_{2,2} \times \overline{\mathcal{M}}_{2,2})
$$
$$
\xi_{A_2}^* : A^3(\overline{\mathcal{M}}_5) \to A^3(\overline{\mathcal{M}}_{3,1} \times \overline{\mathcal{M}}_{2,1})
$$

can be computed in terms of decorated stratum classes. It turns out that $\ker \xi_{A_1}^* \cap \xi_{A_2}^* = \{0\}$. By Theorem 4.9 we have

$$
\xi_{A_1}^*[\overline{\mathcal{H}}_5] = -\Big[\;\overset{\overline{\mathcal{H}}_{2,0,2}\quad\overline{\mathcal{H}}_{2,0,2}}{\circled{2}\!\!\bigcirc\!\!\updownarrow\!\!\bigcirc\!\!\circled{2}}\;\Big] - \Big[\;\overset{\overline{\mathcal{H}}_{2,0,2}\quad\overline{\mathcal{H}}_{2,0,2}}{\circled{2}\!\!\bigcirc\!\!\updownarrow\!\!\bigcirc\!\!\circled{2}}\;\Big]
$$
$$
= -\psi_1 \cap [\overline{\mathcal{H}}_{2,0,2}] \otimes [\overline{\mathcal{H}}_{2,0,2}] - [\overline{\mathcal{H}}_{2,0,2}] \otimes \psi_1 \cap [\overline{\mathcal{H}}_{2,0,2}] \in R^\bullet(\overline{\mathcal{M}}_{2,2} \times \overline{\mathcal{M}}_{2,2}).
$$

The class $[\overline{\mathcal{H}}_{2,0,2}] \in A^1(\overline{\mathcal{M}}_{2,2})$ was originally computed in [6, Lemma 6]. We have:

$$
[\overline{\mathcal{H}}_{2,0,2}] = \Big[\underset{2}{\circled{2}\!\!\diagup^1}\Big] + \Big[\underset{2}{\circled{2}\!\!\diagdown_{\diagup}^1}\Big] - 3\Big[\underset{2}{\circled{2}\!\!-\!\!\diagup^{\;1}}\Big] - \tfrac{6}{5}\Big[\underset{2}{\circled{1}\!\!-\!\!\circled{1}\!\!\diagup^{\;1}}\Big] - \tfrac{1}{5}\Big[\overset{2}{\circled{1}}\!\!\diagup\!\!\overset{1}{\circled{1}}\Big] - \tfrac{1}{10}\Big[\underset{2}{\bigcirc\!\circled{1}\!\!\diagup^{\;1}}\Big].
$$

This immediately gives $\psi_1 \cap [\overline{\mathcal{H}}_{2,0,2}]$ as a class in terms of decorated stratum classes. We thus know $\xi_{A_1}^*([\overline{\mathcal{H}}_5])$ in terms of decorated stratum classes.

By Theorem 4.9 we have

$$
\xi_{A_2}^*([\overline{\mathcal{H}}_5]) = [\overline{\mathcal{H}}_{3,1,0} \times \overline{\mathcal{H}}_{2,1,0}] \in H^6(\overline{\mathcal{M}}_{3,1} \times \overline{\mathcal{M}}_{2,1})
$$

The class $[\overline{\mathcal{H}}_{2,1,0}]$ has originally been computed in [15, Theorem 2.2]. We have

$$[\overline{\mathcal{H}}_{2,1,0}] = 3\psi_1 - \frac{6}{5}\left[\,\overset{\downarrow}{①}\!—\!①\,\right] - \frac{1}{10}\left[\,①①\!—\,\right].$$

The class $[\overline{\mathcal{H}}_{3,1,0}]$ was first computed in [10] using an expression for $[\overline{\mathcal{H}}_4]$ in terms of decorated stratum classes proven in [18, Proposition 5]. Indeed, for the graph

$$B = ③\!—\!①.$$

it is true that

$$\xi_B^*([\overline{\mathcal{H}}_4]) = [\overline{\mathcal{H}}_{3,1,0}] \otimes [\overline{\mathcal{H}}_{1,1,0}] \in A^2(\overline{\mathcal{M}}_{3,1}) \otimes A^0(\overline{\mathcal{M}}_{1,1})$$

From this, Chen and Tarasca computed the cycle $[\overline{\mathcal{H}}_{3,1,0}]$ in [10, Theorem 4.1]. In our preferred basis, it has the expression

$$
\begin{aligned}
[\overline{\mathcal{H}}_{3,1,0}] = \ &6\left[\,\overset{\downarrow}{③}\,\right] \quad -\tfrac{24}{7}\left[\,①\!—\!\overset{\downarrow}{②}\,\right] -\tfrac{1}{7}\left[\,①\!\rightarrow\!\overset{\downarrow}{②}\,\right] -\tfrac{1}{10}\left[\,\overset{\downarrow}{①}\!\rightarrow\!②\,\right] -\tfrac{53}{7}\left[\,②\!—\!\overset{\downarrow}{①}\,\right] +\tfrac{48}{35}\left[\,①\!—\!\overset{\downarrow}{①}\!—\!①\,\right] \\
&+\tfrac{54}{35}\left[\,\overset{\downarrow}{①}\!—\!①\!—\!①\,\right] -\tfrac{2}{7}\left[\,①②\!\leftarrow\,\right] -\tfrac{6}{7}\left[\,②\!\rhd\!—\,\right] +\tfrac{1}{84}\left[\,\multimap\!②\!—\,\right] -\tfrac{53}{84}\left[\,②\!—\!\overset{\downarrow}{\circ}\,\right] \quad +\tfrac{4}{35}\left[\,①\!—\!\overset{\downarrow}{①①}\,\right] \\
&+\tfrac{9}{70}\left[\,\overset{\downarrow}{①}\!—\!①①\,\right] -\tfrac{1}{35}\left[\,\overset{\downarrow}{①}①①\,\right] +\tfrac{1}{105}\left[\,\overset{\downarrow}{\aleph}\!—\,\right].
\end{aligned}
$$

$\square$

Remark 5.6 In the same way we can find an expression for $[\overline{\mathcal{H}}_6] \in R^4(\overline{\mathcal{M}}_6)$ in terms of decorated stratum classes. Let

$$A_1 = ②\!\subset\!③ \qquad\qquad A_2 = ④\!—\!②.$$

With the help of a computer one verifies that, restricted to $R^4(\overline{\mathcal{M}}_6)$, we have

$$\ker \xi_{A_1}^* \cap \ker \xi_{A_2}^* = \{0\}.$$

We have

$$\xi_{A_1}^*[\overline{\mathcal{H}}_6] = -\left[\,\overset{\overline{\mathcal{H}}_{3,0,2}}{③}\!\updownarrow\!\overset{\overline{\mathcal{H}}_{2,0,2}}{②}\,\right] - \left[\,\overset{\overline{\mathcal{H}}_{3,0,2}}{③}\!\updownarrow\!\overset{\overline{\mathcal{H}}_{2,0,2}}{②}\,\right]$$

and

$$\xi_{A_2}^*([\overline{\mathcal{H}}_6]) = [\overline{\mathcal{H}}_{4,1,0} \times \overline{\mathcal{H}}_{2,1,0}]$$

the classes $[\overline{\mathcal{H}}_{3,0,2}]$ and $[\overline{\mathcal{H}}_{4,1,0}]$ can readily be computed from our expression for $[\overline{\mathcal{H}}_5]$. We thus obtain an expression for $[\overline{\mathcal{H}}_6]$. The set of decorated stratum classes in $R^4(\overline{\mathcal{M}}_6)$ up to Pixton's relations is of cardinality 376 (so conjecturally $\dim R^4(\overline{\mathcal{M}}_6) = 376$). To save ink we shall therefore not write down an explicit expression for $[\overline{\mathcal{H}}_6]$ here, but it is available upon request.

The number of decorated stratum classes in $R^5(\overline{\mathcal{M}}_7)$ is 7078. Our computer is not powerful enough to compute all the Pixton relations between these decorated stratum classes. Although most likely with more serious computing power computing these relations and computing $\overline{\mathcal{H}}_7$ is still possible. In general, with our method the main difficulty for computing $[\overline{\mathcal{H}}_g] \in R^{g-2}(\overline{\mathcal{M}}_g)$ seems to be the computation of Pixton's relations. That is, in all cases where we can compute these relations - both on $\overline{\mathcal{M}}_g$ and on the spaces $\overline{\mathcal{M}}_{g_i,n_i}$ appearing in the boundary divisors - we can also compute $[\overline{\mathcal{H}}_g]$.

5.3 Example: the bielliptic locus in genus 4

As an application of the methods of the previous section, we now compute the class $[\overline{\mathcal{B}}_4] \in H^6(\overline{\mathcal{M}}_4)$ of the locus of bielliptic curves in genus 4.

The dimension of $H^6(\overline{\mathcal{M}}_4)$ is 32 by [8, Theorem 1]. By [48, Page 11] the dimension of $RH^6(\overline{\mathcal{M}}_4)$ is also 32, hence the cohomology group is completely tautological. A basis for $H^6(\overline{\mathcal{M}}_4)$ in terms of decorated stratum classes is given by taking the last 32 generators of the relation given in [48, Proposition 2].

Theorem 5.7 *In $H^6(\overline{\mathcal{M}}_4)$ we have*

Remark 5.8 We only know that the equality of Theorem 5.7 holds in cohomology. Because the entire cohomology ring is tautological we know that the cohomology class $[\overline{\mathcal{B}}_4]$ is tautological. However the authors are unaware of any argument that the corresponding Chow class is tautological.

By Riemann–Hurwitz, a bielliptic involution on a genus 4 curve has 6 fixed points, so the natural space describing such curves is $\overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^6)}$. As a first step, we need to compare the fundamental class $[\overline{\mathcal{B}}_4]$ with the pushforward $\phi_{(1^6)*}[\overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^6)}]$ supported on the same locus.

Next we want to show that on the generic bielliptic curve C of genus 4, the bielliptic involution is unique. Assume otherwise, i.e. that C has two distinct bielliptic involutions τ_1, τ_2. Then inside $\mathrm{Aut}(C)$ they generate a (finite) subgroup. Since $\tau_1^2 = \mathrm{id}_C = \tau_2^2$, this subgroup must be a dihedral group of the form D_{2n}, where the additional relation $(\tau_1\tau_2)^n = \mathrm{id}_C$ holds.

We now consider the Riemann–Hurwitz formula for the quotient map $C \to C/D_{2n}$. Let g' be the genus of C/D_{2n}, then since there exists a map $C/\langle\tau_1\rangle \to C/D_{2n}$, given by taking a further quotient, we have $g' \leq 1$, since by assumption $C/\langle\tau_1\rangle$ has genus

1. Let b be the number of branch points of $C \to C/D_{2n}$ and let $h_1, \ldots, h_b \in D_{2n}$ be generators of the stabilizers of some point over these b branch points. Then Riemann–Hurwitz tells us

$$6 = 2 \cdot 4 - 2 = (2n) \cdot (2g' - 2) + \sum_{i=1}^{b} \left(2n - \frac{2n}{\operatorname{ord} h_i} \right).$$

Now $\operatorname{ord} h_i \geq 2$, so $2n - \frac{2n}{\operatorname{ord} h_i} \geq 2n - n = n$ and we conclude

$$6 \geq 2n(2g' - 2) + bn \implies \frac{6}{n} + 2(2 - 2g') \geq b.$$

Since τ_1, τ_2 were distinct by assumption, we have $n \geq 2$, which implies $b \leq 7$ in the case $g' = 0$ and $b \leq 3$ in the case $g' = 1$. Correspondingly, the moduli spaces $\overline{\mathcal{M}}_{g',b}$ parametrizing the curve C/D_{2n} with marked branch points have dimensions at most 4 and 3, respectively. Since this dimension is the same as the corresponding space $\overline{\mathcal{H}}_{4, D_{2n}, \xi'}$ parametrizing the curve C with D_{2n}-action, the general bielliptic curve of genus 4 cannot have such an action of a dihedral group D_{2n}, since the space of bielliptic curves is of dimension 6. Thus the bielliptic involution on the general bielliptic curve is unique.

We conclude that the degree of the map $\phi_{(1^6)} : \overline{\mathcal{H}}_{4, \mathbb{Z}/2\mathbb{Z}, (1^6)} \to \overline{\mathcal{B}}_4 \subset \overline{\mathcal{M}}_4$ to its image is given by the number of ways to order the 6 fixed points of the bielliptic involution, which is $|\mathfrak{S}_6| = 6!$. To summarize we have

$$[\overline{\mathcal{B}}_4] = \frac{1}{6!} \phi_{(1^6)*}[\overline{\mathcal{H}}_{4, \mathbb{Z}/2\mathbb{Z}, (1^6)}].$$

We determine this class by pullback to three distinct boundary strata. Let

$$A = \underset{2}{\bigcirc}\hspace{-0.3em}\underset{1}{\bigcirc} \qquad\qquad B = \underset{2}{\bigcirc}\!-\!\underset{2}{\bigcirc} \qquad\qquad C = \underset{3}{\bigcirc}\!-\!\underset{1}{\bigcirc}.$$

The pullback maps

$$\xi_A^* : H^6(\overline{\mathcal{M}}_4) \to H^6(\overline{\mathcal{M}}_A), \quad \xi_B^* : H^6(\overline{\mathcal{M}}_4) \to H^6(\overline{\mathcal{M}}_B),$$
$$\text{and} \quad \xi_C^* : H^6(\overline{\mathcal{M}}_4) \to H^6(\overline{\mathcal{M}}_C)$$

can be computed in terms of bases of decorated stratum classes by Theorem 2.25, using the generalized Faber-Zagier relations calculated by [41]. Inside the vector space $H^6(\overline{\mathcal{M}}_4) = RH^6(\overline{\mathcal{M}}_4)$ of dimension 32, we have

$$\dim \ker \xi_A^* = 4, \quad \dim \ker \xi_B^* = 14, \quad \dim \ker \xi_C^* = 3$$

and it holds that

$$(\ker \xi_A^*) \cap (\ker \xi_B^*) \cap (\ker \xi_C^*) = \{0\}.$$

Thus the pullbacks of $[\overline{\mathcal{B}}_4]$ via ξ_A, ξ_B, ξ_C uniquely determine $[\overline{\mathcal{B}}_4]$. Furthermore, looking at the dimensions of the kernels of $\xi_A^*, \xi_B^*, \xi_C^*$ we see that the linear system of equations for $[\overline{\mathcal{B}}_4]$ which they provide is significantly overdetermined. This provides a check for the computations below: an error in the formulas for $\xi_A^*[\overline{\mathcal{B}}_4], \xi_B^*[\overline{\mathcal{B}}_4]$ or $\xi_C^*[\overline{\mathcal{B}}_4]$, which form the inhomogeneous term of the corresponding linear equation, would in general cause this equation to have no solution.

In the following we compute the pullbacks of $[\overline{\mathcal{B}}_4]$ and point out for various classes that appear in the expression for these pullbacks how to express them as tautological classes.

Lemma 5.9 *The pullback of $[\overline{\mathcal{B}}_4]$ along ξ_A is given by:*

$$= -\psi_1[\overline{\mathcal{B}}_{2,0,2}] \otimes [\overline{\mathcal{H}}_{1,0,2}] - [\overline{\mathcal{B}}_{2,0,2}] \otimes \psi_1[\overline{\mathcal{H}}_{1,0,2}] - \psi_1[\overline{\mathcal{H}}_{2,0,2}] \otimes [\overline{\mathcal{B}}_{1,0,2}]$$
$$- [\overline{\mathcal{H}}_{2,0,2}] \otimes \psi_1[\overline{\mathcal{B}}_{1,0,2}] + [\overline{\mathcal{H}}_{2,2,0}] \otimes [\overline{\mathcal{H}}_{1,0,2}] + 2[\overline{\mathcal{H}}_{1,0,2} \times \Delta_{\overline{\mathcal{M}}_{1,2}}].$$

where $\Delta_{\overline{\mathcal{M}}_{1,2}}$ is as in Application 2.27.

Proof This is an application of Theorem 4.9. Let $\mathfrak{A}$ be the set consisting of all possible distributions of 6 legs on A. Graphs in $\mathfrak{A}$ having nonzero intersection with $\phi(\overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^6)})$ are of the form:
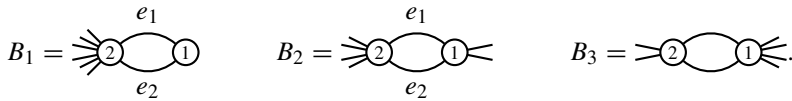
The following admissible G-graphs admit a B_1-structure:
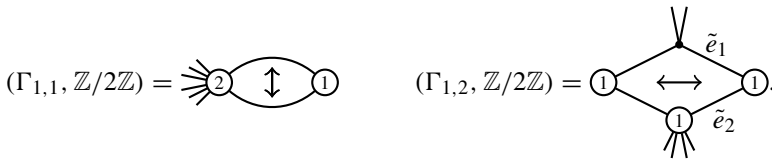
There is up to isomorphism one B_1-structures f on $(\Gamma_{1,1}, \mathbb{Z}/2\mathbb{Z})$. The top Chern class of the excess bundle E_f is given by a ψ class on each side. In other words

There are 2 isomorphism classes of B_1-structures on $(\Gamma_{1,2}, \mathbb{Z}/2\mathbb{Z})$. Indeed a B_1-structure $f = (\alpha, \beta, \gamma)$ on $(\Gamma_{1,2}, \mathbb{Z}/2\mathbb{Z})$ can be given either by sending e_1 to $\tilde{e}_1$ or to $\tilde{e}_2$ under β. It is easy to see they are not isomorphic.

The admissible bielliptic pairs with a B_2-structure are given by

Again there is up to isomorphism only one B_2-structure f on $(\Gamma_{2,1}, \mathbb{Z}/2\mathbb{Z})$ and the excess bundle is given by

There are 2 isomorphism classes of B_2-structures on $(\Gamma_{2,2}, \mathbb{Z}/2\mathbb{Z})$. We can send the edge e_1 to either $\tilde{e}_1$ or $\tilde{e}_2$ and this completely determines the B_2-structure.

There is only one admissible pair $(\Gamma_{3,1}, \mathbb{Z}/2\mathbb{Z})$ with a B_3-structure. It is given by

There is only one isomorphism class of B_3-structures and the top Chern class of the excess bundle is trivial.

We have determined the sum of the classes in the expression of Theorem 4.9. Pushing all of these classes forward through the morphisms $\pi_B \circ \phi_f$ we obtain the desired expression. $\qquad\square$

Remark 5.10 The classes $[\overline{\mathcal{B}}_{1,0,2}]$ and $[\overline{\mathcal{B}}_{2,0,2}]$ are relatively easy to compute in terms of decorated stratum classes (see [46, Remark 3.4.2, Proposition 3.2.10]). We have $[\overline{\mathcal{H}}_{1,0,2}] = [\overline{\mathcal{M}}_{1,0,2}]$. The class $[\overline{\mathcal{H}}_{2,0,2}]$ is computed in [6, Lemma 6]. The class $[\overline{\mathcal{H}}_{2,2,0}]$ is given as $[\overline{\mathcal{DR}}_2(2)]$ in [45, Theorem 0.1]. The class $[\overline{\mathcal{H}}_{1,0,2}]$ is given as $[\bar{A}_2]$ in [36, Theorem 3.33]. The class of the diagonal $[\Delta_{\overline{\mathcal{M}}_{1,2}}]$ can easily be determined by Application 2.27. We have thus completely determined the class $\xi_A^*([\overline{\mathcal{B}}_4])$ in terms of decorated stratum classes.

The pullback of $[\overline{\mathcal{B}}_4]$ under ξ_B and ξ_C now works in a similar way.

Lemma 5.11 *We have*

$$\xi_B^*([\overline{\mathcal{B}}_4]) = [\overline{\mathcal{B}}_{2,1,0} \times \overline{\mathcal{H}}_{2,1,0}] + [\overline{\mathcal{H}}_{2,1,0} \times \overline{\mathcal{B}}_{2,1,0}] \in H^6(\overline{\mathcal{M}}_{2,1} \times \overline{\mathcal{M}}_{2,1}).$$

Proof This is Example 4.15. $\qquad\square$

Lemma 5.12 *In $H^6(\overline{\mathcal{M}}_{3,1} \times \overline{\mathcal{M}}_{1,1})$ we have*

$$
\xi_C^*([\overline{\mathcal{B}}_4]) =
\left[
\begin{array}{c}
\overline{\mathcal{B}}_{3,1,0} \qquad \overline{\mathcal{H}}_{1,1,0} \\
\textcircled{3}\!\!-\!\!-\!\!-\!\!-\!\!\textcircled{1}
\end{array}
\right]
+ 2
\left[
\begin{array}{c}
\textcircled{1} \qquad \Delta_{\overline{\mathcal{M}}_{1,1}} \\
\overline{\mathcal{H}}_{2,0,2} \searrow \\
\textcircled{2}\!\!-\!\!-\!\!\textcircled{1}
\end{array}
\right]
$$

$$
= [\overline{\mathcal{B}}_{3,1,0} \times \overline{\mathcal{H}}_{1,1,0}] + 2[\overline{\mathcal{H}}_{2,0,2} \times \Delta_{\overline{\mathcal{M}}_{1,1}}]
$$

The class of $\overline{\mathcal{B}}_{3,1,0} \subset \overline{\mathcal{M}}_{3,1}$ has not been computed before but it can be determined by further pullbacks to the boundary.

Proof of Theorem 5.7 We have determined the three linear maps ξ_A^*, ξ_B^* and ξ_C^* in terms of a basis of decorated stratum classes for $H^6(\overline{\mathcal{M}}_4)$.

We have also determined the the pullbacks $\xi_A^*[\overline{\mathcal{B}}_4]$, $\xi_B^*[\overline{\mathcal{B}}_4]$ and $\xi_C^*[\overline{\mathcal{B}}_4]$ in terms of decorated stratum classes. Since the kernels of $\xi_A^*, \xi_B^*, \xi_C^*$ have trivial intersection, this completely determines the class $[\overline{\mathcal{B}}_4]$. $\qquad\square$

Remark 5.13 The class $[\overline{\mathcal{B}}_4]$ has first been computed in [46] using the combined information of $\xi_A^*[\overline{\mathcal{B}}_4]$, $\xi_B^*[\overline{\mathcal{B}}_4]$ and the pull–push $\delta_*\phi_{(1^{10})}^*[\overline{\mathcal{B}}_4]$ through the hyperelliptic curves

$$
\begin{array}{ccc}
\overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^{10})} & \xrightarrow{\phi_{(1^{10})}} & \overline{\mathcal{M}}_4 \\
\downarrow{\scriptstyle \delta} & & \\
\overline{\mathcal{M}}_{0,10} & &
\end{array}
\quad .
$$

This last operation is not covered by our previous results but will be described in Sect. 5.5. It should be seen as proof of concept that the intersection of two loci of admissible covers can again be described in terms of tautological classes and further spaces of admissible covers. It turns out that the result for $\delta_*\phi_{(1^{10})}^*[\overline{\mathcal{B}}_4]$ is consistent with the formula in Theorem 5.7, providing a further check of the previous computations.

5.4 Further examples and applications

Using our implementation of the methods in the previous sections, we can compute many more examples of admissible cover cycles.

Apart from the hyperelliptic and bielliptic cycles listed in Figs. 1 and 2 in the introduction, we can compute a few other loci of admissible covers.

– $[\overline{\mathcal{H}}_{3,\mathbb{Z}/2\mathbb{Z},()}] \in H^6(\overline{\mathcal{M}}_3)$, the cycle of genus 3 curves C admitting an unramified double cover of a genus 2 curve,

– $[\overline{\mathcal{H}}_{3,\mathbb{Z}/3\mathbb{Z},(0)}] \in H^4(\overline{\mathcal{M}}_{1,3})$, the cycle of elliptic curves (E, p_0, p_1, p_2) such that there exists an unramified triple[6] cover $E \to E'$ to an elliptic curve E' such that p_0, p_1, p_2 form a fibre,

– $\pi_*[\overline{\mathcal{H}}_{3,\mathbb{Z}/3\mathbb{Z},(1,2)}] \in H^8(\overline{\mathcal{M}}_3)$, the cycle of genus 3 curves C admitting a cyclic triple cover of a genus 1 curve, with full ramification at 2 points.

In particular, our methods allow us to go beyond the case of double covers. As explained in [14, Section 4.4.3], the explicit expressions for such cycles of triple covers can be used to verify formulas for Hodge integrals against these cycles, proven in [35].

Note that, as described before in Remark 5.6, one of the main bottlenecks in the above computations is the enumeration of all tautological relations in various spaces $RH^{2k}(\overline{\mathcal{M}}_{g,n})$, which is done using the program [41]. Since cycles of admissible covers tend to have high codimension k, it takes a long time to enumerate all decorated stratum classes of this degree and to compute the coefficients of the relations between them. Using a more refined approach to express tautological classes in a basis of $RH^{2k}(\overline{\mathcal{M}}_{g,n})$, e.g. by computing intersection numbers of cycles in opposite degree, could potentially allow us to extend the above list of examples.

Remark 5.14 As an application of our computations, we can verify a formula for Hodge-integrals on bielliptic loci from [43]. More precisely, [43, Theorem 4] shows an equality between generating series of equivariant Gromov-Witten invariants of $\text{Hilb}^n(\mathbb{C}^2)$ and $\text{Sym}^n(\mathbb{C}^2)$, after suitable coordinate change. For $n = 2$ and $g = 1$ the series can be explicitly computed on the Hilbert scheme side, thus yielding a prediction for $\text{Sym}^2(\mathbb{C}^2)$. Using localization techniques, this in turn can be expressed in terms of integrals of bielliptic cycles against products of λ-classes. To state the corresponding result, let $\lambda_i \in H^{2i}(\overline{\mathcal{M}}_{n+1})$ be the ith Chern class of the Hodge bundle and let

$$\phi : \overline{\mathcal{H}}_{n+1,\mathbb{Z}/2\mathbb{Z},(1^{2n})} \to \overline{\mathcal{M}}_{n+1}$$

be the map remembering the bielliptic curve, without markings. Consider the generating series of integrals

$$F(u) = \sum_{n=1}^{\infty} \frac{u^{2n-1}}{(2n-1)!} \int_{\overline{\mathcal{H}}_{n+1,\mathbb{Z}/2\mathbb{Z},(1^{2n})}} \phi^*\left(\lambda_{n+1}\lambda_{n-1}\right).$$

Then a computation in [43, Section 0.6] shows that

$$F(u) = \frac{i}{24}\left(\frac{1 - e^{iu}}{1 + e^{iu}}\right) = \frac{1}{48}u + \frac{1}{576}u^3 + \frac{1}{5760}u^5 + \cdots. \tag{37}$$

Now, up to a factor, the pushforward $\phi_*[\overline{\mathcal{H}}_{n+1,\mathbb{Z}/2\mathbb{Z},(1^{2n})}]$ is exactly the class $[\overline{\mathcal{B}}_{n+1}]$ of the bielliptic locus and by the projection formula we have

$$\int_{\overline{\mathcal{H}}_{n+1,\mathbb{Z}/2\mathbb{Z},(1^{2n})}} \phi^*\left(\lambda_{n+1}\lambda_{n-1}\right) = \int_{\overline{\mathcal{M}}_{n+1}} \phi_*[\overline{\mathcal{H}}_{n+1,\mathbb{Z}/2\mathbb{Z},(1^{2n})}] \cdot \lambda_{n+1}\lambda_{n-1}$$

---

[6] Such a cover must necessarily be a group homomorphism. It follows that it is a cyclic cover and in fact for the elliptic curve (E, p_0) the point p_1 is a 3-torsion point and $p_2 = 2 \cdot p_1$.

For $n = 1, 2, 3$ we can compute the class of the bielliptic locus in terms of decorated stratum classes and thus the corresponding intersection numbers with $\lambda_{n+1}\lambda_{n-1}$. The results agree with the coefficients in $F(u)$ predicted by the formula (37) above. This provides a further nontrivial check for our computations.

5.5 Intersecting with the hyperelliptic locus

In [19] the authors calculate the class $[\overline{\mathcal{B}}_3]$ of the bielliptic locus in genus 3 by pulling it back to the hyperelliptic locus (and evaluating it on a small number of test surfaces). In other words they compute the pull–push of $[\overline{\mathcal{B}}_3]$ along the diagram

$$
\begin{array}{ccc}
\overline{\mathcal{H}}_{3,\mathbb{Z}/2\mathbb{Z},(1^8)} & \xrightarrow{\phi_{(1^8)}} & \overline{\mathcal{M}}_3 \\
\downarrow{\scriptstyle \delta} & & \\
\overline{\mathcal{M}}_{0,8} & &
\end{array}
$$

The space $\overline{\mathcal{M}}_{0,8}$ is relatively easy to understand; the inverse image of $\overline{\mathcal{B}}_3$ along $\phi_{(1^8)}$ can be determined by set theoretic arguments.[7] This allows the authors to compute the class $I := \delta_* \phi^*_{(1^8)}([\overline{\mathcal{B}}_3])$ up to a constant. The authors then determine the composition $\delta_* \circ \phi^*_{(1^8)} : A^2(\overline{\mathcal{M}}_3) \to A^2(\overline{\mathcal{H}}_{3,\mathbb{Z}/2\mathbb{Z},(1^8)}) \to A^2(\overline{\mathcal{M}}_{0,8})$, in terms of a basis for $A^2(\overline{\mathcal{M}}_3)$ consisting of products of divisors and κ_2, by computing the pull–push $\delta_* \phi^*_{(1^8)}(D)$ for all boundary divisors $D \in A^1(\overline{\mathcal{M}}_3)$ and multiplying out on both sides, and by directly computing $\delta_* \phi^*_{(1^8)}(\kappa_2)$. Since $\dim A^2(\overline{\mathcal{M}}_3) = 7$, $\dim A^2(\overline{\mathcal{M}}_{0,8}) = 6$ and $\delta_* \phi^*$ is surjective this determines 5 out of the 7 coefficients of the class $[\overline{\mathcal{B}}_3]$, where one coefficient is missing because the pullback of $[\overline{\mathcal{B}}_3]$ to the hyperelliptic locus is only determined up to a scalar.

Using Theorem 4.10 this approach becomes a bit more systematic. Indeed we have seen that the composition

$$
\delta_* \phi^*_{\xi} : R^k(\overline{\mathcal{M}}_{g,n}) \to R^k(\overline{\mathcal{H}}_{g,G,\xi}) \to R^k(\overline{\mathcal{M}}_{g',b})
$$

can be computed algorithmically. If we want to compute the class $[\phi_{\tilde{\xi}'} \overline{\mathcal{H}}_{g,G',\xi'}]$ of another space of admissible covers then we still have to determine the intersection

$$
\phi^*_{\tilde{\xi}}[\phi_{\tilde{\xi}'} \overline{\mathcal{H}}_{g,G',\xi'}]. \tag{38}
$$

---

[7] In the original published version of the paper the authors make a mistake in this part of the argument (see [46, Remark 3.4.2] for details).

Currently such pullbacks have not been worked out in full generality, but the expectation is that the fibered diagram

$$
\begin{array}{ccc}
F & \longrightarrow & \overline{\mathcal{H}}_{g,G',\xi'} \\
\downarrow & & \downarrow{\phi_{\tilde{\xi}'}} \\
\overline{\mathcal{H}}_{g,G,\xi} & \xrightarrow{\phi_{\tilde{\xi}}} & \overline{\mathcal{M}}_{g,n}
\end{array}
$$

and its excess bundle can be analysed in a way similar to that of Proposition 4.3 and 4.7. At the moment we just have set theoretic arguments to analyse the pullback (38) in particular cases.

Pulling Back the Bielliptic Locus in Genus 4 As an example we will now sketch the computation of the pullback of $[\overline{\mathcal{B}}_4]$ to the stack $\overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^{10})}$ parametrizing hyperelliptic curves of genus 4.

$$
\begin{array}{ccc}
\overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^{10})} & \xrightarrow{\phi_{(1^{10})}} & \overline{\mathcal{M}}_4 \\
\downarrow{\delta} & & \\
\overline{\mathcal{M}}_{0,10} & &
\end{array}
\;.
$$

We start by giving the set theoretic intersection of the bielliptic and hyperelliptic locus.

Proposition 5.15 *The inverse image of $\overline{\mathcal{B}}_4$ under the map $\phi_{(1^{10})} \colon \overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^{10})} \to \overline{\mathcal{M}}_4$ is given by two strata:*

(i) *the locus $\mathcal{A} \subset \overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^{10})}$ of admissible hyperelliptic covers $C \to D$ where C is a curve with two irreducible components C_1 and C_2 of genus 1 and 2 respectively and two nodes between them; and where C_2 admits a bielliptic involution switching the nodes,*

(ii) *the locus $\mathcal{B} \subset \overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^{10})}$ of admissible hyperelliptic covers $C \to D$ where C is a curve with two irreducible components C_1 and C_2 of genus 1 and 2 and two nodes between them; and where C_1 admits a bielliptic involution switching the nodes.*
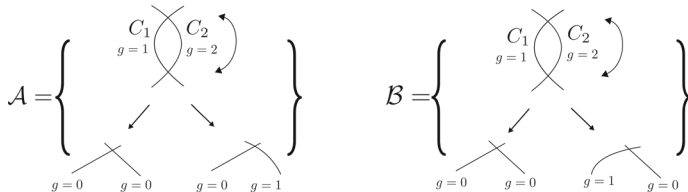
Proof Since $\mathcal{A}$ and $\mathcal{B}$ are subsets of the stack $\overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^{10})}$, one sees easily that the hyperelliptic involution exchanges in both cases the nodes connecting C_1 and C_2. On the other hand, we also have a bielliptic involution in both cases: for $\mathcal{A}$ it is given by the required bielliptic involution on C_2 and by the hyperelliptic involution of C_1.

On the other hand, for $\mathcal{B}$ the bielliptic involution is given by the required bielliptic involution on C_1 and by the hyperelliptic involution on C_2.

That these are all loci follows from a combinatorial exhaustion in the spirit of [19, Proposition 3] (in particular from the Castelnuovo–Severi inequality it follows that there are no smooth curves of genus ≥ 4 with both a hyperelliptic and a bielliptic involution). $\qquad\square$

In the following proposition, we show that the loci $\mathcal{A}$ and $\mathcal{B}$ are again parametrized by spaces of admissible G-covers. Since the hyperelliptic and bielliptic involution commute, they naturally induce an action of the group $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which appears in the formulas.

Proposition 5.16 *Let* $\phi_{\mathbb{Z}/2\mathbb{Z},((0,1)^2)}$ *denote the composition*

$$\phi_{\mathbb{Z}/2\mathbb{Z},((0,1)^2)} \colon \overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},((1,0)^3,(0,1),(1,1))} \to \overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^6,0,0)} \to \overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^6,0)},$$

where the first map forgets the second $\mathbb{Z}/2\mathbb{Z}$ action and the second forgets the pair of markings that previously had stabilizer $(0,1)$. Let $\phi_{\mathbb{Z}/2\mathbb{Z}}$ denote the map

$$\phi_{\mathbb{Z}/2\mathbb{Z}} \colon \overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},((1,0)^2,(1,1)^2)} \to \overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^4,0^2)} \to \overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^4,0)}.$$

forgetting the second $\mathbb{Z}/2\mathbb{Z}$ action and one of the pair of points that previously had stabilizer $(1,1)$. Let $\xi \colon \overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^6,0)} \times \overline{\mathcal{H}}_{1,\mathbb{Z}/2\mathbb{Z},(1^4,0)} \to \overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^{10})}$ be the map gluing the admissible hyperelliptic covers together in the points not fixed by the involution. We have

$$\mathcal{A} = \xi\big(\operatorname{Im}\phi_{\mathbb{Z}/2\mathbb{Z},((0,1)^2)}, \overline{\mathcal{H}}_{1,\mathbb{Z}/2\mathbb{Z},(1^4,0)}\big)$$
$$\mathcal{B} = \xi\big(\overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^6,0)}, \operatorname{Im}\phi_{\mathbb{Z}/2\mathbb{Z}}\big).$$

Proof We adopt the notation of Proposition 5.15. It is obvious that both $\mathcal{A}$ and $\mathcal{B}$ are contained in the image of ξ.

We start by considering $\mathcal{A}$. There are no conditions on the curve C_1. Let $f \colon C_2 \to D_2$ be the hyperelliptic map on C_2 and let σ be the corresponding hyperelliptic involution. The curve C_2 also admits a bielliptic involution τ, the involutions σ and τ commute and the involution $\sigma \circ \tau$ is a bielliptic involution. The involution τ induces an involution on D_2. Modding out by this involution gives a map $D_2 \to R$ where R is a stable genus 0 curve. By the Riemann–Hurwitz formula, the ramification divisor of the composition $g \colon C_2 \to D_2 \to R$ has degree 10. The only ramification points of g are the fixed points of σ, τ and $\sigma \circ \tau$. Therefore the fixed points of σ and of τ and of $\sigma \circ \tau$ are distinct.

The situation is summarized by the following diagram. Where the upper middle map is the hyperelliptic map. The lower middle map is the map $D_2 \to R$ and the upper maps on the right and left are the two bielliptic maps.
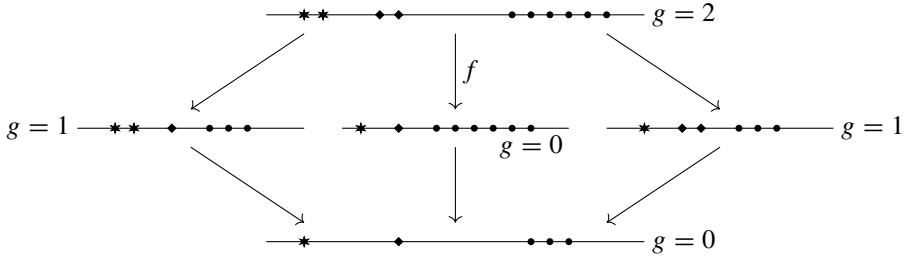
Identifying σ with $(1, 0)$ and τ with $(0, 1)$ it follows that cover C_2 lies in the image of $\phi_{\mathbb{Z}/2\mathbb{Z},((0,1)^2)}$.

The case of $\mathcal{B}$ is analogous. $\square$

Let us denote by

$$\overline{\mathcal{H}}_{0,1,2m}^{s} := \bigcup_{\sigma \in \mathfrak{S}_{2m}} \sigma(\overline{\mathcal{H}}_{0,1,2m}) \subset \overline{\mathcal{M}}_{0,1+2m}$$

the image under the action of the symmetric group $\mathfrak{S}_{2m}$ acting on the $2m$-markings permuted by the hyperelliptic involution. We can draw a conclusion from Proposition 5.16 in terms of loci on the target space as follows

Proposition 5.17 *Let* $\tilde{\xi} \colon \overline{\mathcal{M}}_{0,7} \times \overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{0,10}$ *be the gluing morphism gluing the curves together in the last points. Let* $\delta \colon \overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^{10})} \to \overline{\mathcal{M}}_{0,10}$ *be the target map. We have*

$$\delta(\mathcal{A}) = \tilde{\xi}(\overline{\mathcal{H}}_{0,1,6}^{s} \times \overline{\mathcal{M}}_{0,5})$$
$$\delta(\mathcal{B}) = \tilde{\xi}(\overline{\mathcal{M}}_{0,7} \times \overline{\mathcal{H}}_{0,1,4}^{s}).$$

Proof This follows immediately from applying the target map to the loci of Proposition 5.16. In particular note that δ induces a quotient map on the loci of Proposition 5.16 given by the $\mathbb{Z}/2\mathbb{Z}$ action associated to the hyperelliptic structure. $\square$

Consider now the commutative diagram (which we know to be Cartesian on the level of sets)

$$\begin{array}{ccc} \mathcal{A} \cup \mathcal{B} & \xrightarrow{\ g\ } & \overline{\mathcal{B}}_4 \\ \downarrow{\scriptstyle i} & & \downarrow \\ \overline{\mathcal{H}}_{4,\mathbb{Z}/2\mathbb{Z},(1^{10})} & \xrightarrow{\ \phi_{(1^{10})}\ } & \overline{\mathcal{M}}_4 \end{array} \qquad . \tag{39}$$

We have $\mathrm{codim}_{\overline{\mathcal{M}}_4}\overline{\mathcal{B}}_4 = 3$. Since $\mathrm{codim}_{\overline{\mathcal{M}}_{0,10}}\tilde{\xi}(\overline{\mathcal{H}}_{0,1,6}^{s} \times \overline{\mathcal{M}}_{0,5}) = 3$ the excess bundle $E = i^* N_{\phi_{(1^{10})}}/N_g$ is trivial on $\mathcal{A}$. We have $\mathrm{codim}_{\overline{\mathcal{M}}_{0,10}}\tilde{\xi}(\overline{\mathcal{M}}_{0,5} \times \overline{\mathcal{H}}_{0,1,4}^{s}) = 2$. Let $\phi_{(1^6)} \colon \overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^6,0)} \to \overline{\mathcal{M}}_{2,2}$ be the source map and let $p \colon \mathcal{B} \to \overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^6,0)}$ be the projection onto the second factor. The excess bundle restricted to $\mathcal{B}$ is given by

$$p^* N_{\phi_{(1^6)}}.$$

From the excess intersection formula we get the following.

Proposition 5.18 *Let $a, b \in \mathbb{Q}_{>0}$ be the multiplicity of the loci $\mathcal{A}, \mathcal{B}$ in the (stack-theoretic) fibre product (39). Let $\tilde{\delta} \colon \overline{\mathcal{H}}_{2,\mathbb{Z}/2\mathbb{Z},(1^6,0)} \to \overline{\mathcal{M}}_{0,7}$ be the target map. With the above notation and that of Proposition 5.17, we have,*

$$\delta_* \phi^*_{(1^{10})}([\overline{\mathcal{B}}_4]) = \tilde{\xi}_*(a[\overline{\mathcal{H}}^s_{0,1,6}] \otimes [\overline{\mathcal{M}}_{0,5}] + b(\tilde{\delta}_* c_1(N_{\phi_{(1^6)}}) \cap [\overline{\mathcal{M}}_{0,7}]) \otimes [\overline{\mathcal{H}}^s_{0,1,4}]). \quad (40)$$

The classes $[\overline{\mathcal{H}}^s_{0,1,4}]$ and $[\overline{\mathcal{H}}^s_{0,1,6}]$ can be computed in terms of decorated stratum classes as outlined in the rest of this paper. On the other hand we have

$$\tilde{\delta}_* c_1(N_{\phi_{(1^6)}}) = \tilde{\delta}_* \phi^*_{(1^6)}([\overline{\mathcal{H}}_{2,0,2}])$$

the class $[\overline{\mathcal{H}}_{2,0,2}]$ was first computed in terms of decorated stratum classes in [6, Lemma 6] and is again easy to compute with the methods of this paper. The pull–push map

$$\tilde{\delta}_* \phi^*_{(1^6)} \colon A^1(\overline{\mathcal{M}}_{2,2}) \to A^1(\overline{\mathcal{M}}_{0,7})$$

can be computed in terms of bases of decorated stratum classes using Theorem 4.10. Together this gives an expression in terms of decorated stratum classes for $\tilde{\delta}_* c_1(N_{\phi_{(1^6)}})$.

Details of these computations can be found in [46]. We end up with an expression in terms of decorated stratum classes for

(40).

The pull–push

$$\delta_* \phi^*_{(1^{10})} \colon A^3(\overline{\mathcal{M}}_4) \to A^3(\overline{\mathcal{M}}_{0,10})$$

can also be determined in terms of bases of decorated stratum classes. Now $\dim A^3(\overline{\mathcal{M}}_4) = 32$, $\dim A^3(\overline{\mathcal{M}}_{0,10}) = 21$ and $\delta_* \phi^*$ turns out to be surjective. Above, we determined the image of $[\overline{\mathcal{B}}_4]$ under $\delta_* \phi^*$ up to 2 coefficients a, b. This gives $19 = 21 - 2$ out of the 32 coefficients necessary to compute $[\overline{\mathcal{B}}_4]$. In [46] this computation was necessary for the computation of $[\overline{\mathcal{B}}_4]$.

References

1. Arbarello, E., Cornalba, M.: The Picard groups of the moduli spaces of curves. Topology **26**(2), 153–171 (1987)
2. Arbarello, E., Cornalba, M.: Calculating cohomology groups of moduli spaces of curves via algebraic geometry. Inst. Hautes Études Sci. Publ. Math. **88**(97–127), 1998 (1999)
3. Arbarello, E., Cornalba, M., Griffiths, P.: Geometry of Algebraic Curves, vol. II. Springer, Berlin (2011)
4. Abramovich, D., Corti, A., Vistoli, A.: Twisted bundles and admissible covers. Commun. Algebra **31**(8), 3547–3618 (2003)
5. Bergström, J.: Cohomology of moduli spaces of curves of genus three via point counts. J. Reine Angew. Math. **622**, 155–187 (2008)
6. Belorousski, P., Pandharipande, R.: A descendent relation in genus 2. Ann. Sc. Norm. Super. Pisa, Cl. Sci. IV Ser. **29**(1), 171–191 (2000)
7. Bertin, J., Romagny, M.: Champs de Hurwitz. Mém. Soc. Math. Fr. (N.S.) **1**(125–126), 219 (2011)
8. Bergström, J., Tommasi, O.: The rational cohomology of $\overline{\mathcal{M}}_4$. Math. Ann. **338**(1), 207–239 (2007)
9. Cavalieri, R., Miles, E.: Riemann surfaces and algebraic curves, volume 87 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge (2016). A first course in Hurwitz theory
10. Chen, D., Tarasca, N.: Loci of curves with subcanonical points in low genus. Math. Z. **284**(3–4), 683–714 (2016)
11. Cavalieri, R., Tarasca, N.: Classes of Weierstrass points on genus 2 curves. Trans. Am. Math. Soc. **372**(4), 2467–2492 (2019)
12. Díaz, R., González-Aguilera, V.: Boundary of the pyramidal equisymmetric locus of $\mathcal{M}_n$. Manuscr. Math. **163**, 553–567 (2020). https://doi.org/10.1007/s00229-019-01175-0
13. Díaz, R., González-Aguilera, V.: Limit points of the branch locus of $\mathcal{M}_g$. Adv. Geom. **19**(4), 505–526 (2019)
14. Delecroix, V., Schmitt, J., van Zelm, J.: Admcycles: a Sage package for calculations in the tautological ring of the moduli space of stable curves. arXiv:2002.01709 (2020)
15. Eisenbud, D., Harris, J.: The Kodaira dimension of the moduli space of curves of genus ≥ 23. Invent. Math. **90**(2), 359–387 (1987)
16. Faber, C.: Chow rings of moduli spaces of curves i: the Chow ring of $\overline{\mathcal{M}}_3$. Ann. Math. **132**(2), 331 (1990)
17. Faber, C., Pandharipande, R.: Logarithmic series and Hodge integrals in the tautological ring. With an appendix by Don Zagier. Mich. Math. J. **48**(1), 215–252 (2000)
18. Faber, C., Pandharipande, R.: Relative maps and tautological classes. JEMS **7**, 13–49 (2005)
19. Faber, C., Pagani, N.: The class of the bielliptic locus in genus 3. Int. Math. Res. Not. **2015**(12), 3943–3961 (2015)
20. Fulton, W.: Hurwitz schemes and irreducibility of moduli of algebraic curves. Ann. Math. **2**(90), 542–575 (1969)
21. Fulton, W.: Intersection Theory. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, Berlin (1984)
22. Galeotti, M.: Birational geometry of moduli of curves with an S_3-cover. arXiv:1905.03487 (2019)
23. Galeotti, M.: Moduli of G-covers of curves: geometry and singularities. arXiv:1905.02889 (2019)
24. Getzler, E.: Topological recursion relations in genus 2. arXiv:math/9801003 (1998)
25. Getzler, E., Kapranov, M.M.: Modular operads. Compos. Math. **110**(1), 65–126 (1998)
26. Getzler, E., Looijenga, E.: The Hodge polynomial of $\overline{M}_{3,1}$. ArXiv Mathematics e-prints, October (1999)
27. Graber, T., Pandharipande, R.: Constructions of nontautological classes of moduli spaces of curves. Mich. Math. J. **51**(1), 93–110 (2003)

28. Hain, R., Looijenga, E.: Mapping class groups and moduli spaces of curves. In: Algebraic Geometry—Santa Cruz 1995. Proceedings of the Symposium in Pure Mathematics, vol. 62, pp. 97–142. American Mathematical Society, Providence (1997)
29. Harris, J., Mumford, D.: On the Kodaira dimension of the moduli space of curves. Invent. Math. **67**(1), 23–86 (1982)
30. Harris, J., Morrison, I.: Moduli of Curves. Graduate Texts in Mathematics. Springer, New York (1998)
31. Jarvis, T.J., Kaufmann, R., Kimura, T.: Pointed admissible G-covers and G-equivariant cohomological field theories. Compos. Math. **141**(4), 926–978 (2005)
32. Keel, S.: Intersection theory of moduli space of stable n-pointed curves of genus zero. Trans. Am. Math. Soc. **330**(2), 545–574 (1992)
33. Kontsevich, M., Manin, Yu.: Gromov–Witten classes, quantum cohomology, and enumerative geometry. Commun. Math. Phys. **164**(3), 525–562 (1994)
34. Kontsevich, M.: Intersection theory on the moduli space of curves and the matrix airy function. Commun. Math. Phys. **147**(1), 1–23 (1992)
35. Owens, B., Somerstep, S.: Boundary expression for Chern classes of the Hodge bundle on spaces of cyclic covers. arXiv e-prints (2019)
36. Pagani, N.: Chen–Ruan Cohomology of $\mathcal{M}_{1,n}$ and $\overline{\mathcal{M}}_{1,n}$. Annales de l'Institut Fourier **63**(4), 1469–1509 (2013)
37. Pandharipande, R.: A calculus for the moduli space of curves. Proc. Symp. Pure Math. **97**(1), 459–488 (2018)
38. Petersen, D.: The structure of the tautological ring in genus one. Duke Math. J. **163**(4), 777–793 (2014)
39. Petersen, D.: Cohomology of local systems on the moduli of principally polarized abelian surfaces. Pac. J. Math. **275**(1), 39–61 (2015)
40. Petersen, D.: Tautological rings of spaces of pointed genus two curves of compact type. Compos. Math. **152**(7), 1398–1420 (2016)
41. Pixton, A.: Sage program. Private circulation. http://www-personal.umich.edu/~pixton/research.html. Accessed 03 Nov 2020
42. Petersen, D., Tommasi, O.: The Gorenstein conjecture fails for the tautological ring of $\overline{\mathcal{M}}_{2,n}$. Invent. Math. **196**(1), 139–161 (2014)
43. Pandharipande, R., Tseng, H.-H.: Higher genus Gromov-Witten theory of $\mathrm{Hilb}^n(\mathbb{C}^2)$ and CohFTs associated to local curves. Forum Math. Pi **7**:e4, 63 (2019)
44. The Sage Developers. SageMath, the Sage Mathematics Software System (Version 7.3). http://www.sagemath.org (2017). Accessed 03 Nov 2020
45. Tarasca, N.: Double total ramifications for curves of genus 2. Int. Math. Res. Not. **2015**(19), 9569–9593 (2015)
46. van Zelm, J.: The Enumerative Geometry of Double Covers of Curves. Ph.D. thesis, University of Liverpool, 5 (2018)
47. van Zelm, J.: Nontautological bielliptic cycles. Pac. J. Math. **294**(2), 495–504 (2018)
48. Yang, S.: Calculating intersection numbers on moduli spaces of pointed curves. arXiv:0808.1974 (2008)