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Tag der mündlichen Prüfung : ………………..

INTERSECTION SPACES AND TORIC VARIETIES

Betreuer : Prof. Dr. Markus Banagl

In memory of Pooran.

This thesis is dedicated to my mother, Maryam.
Without her endless encouragement, I would never have been able to complete my graduate studies.

ABSTRACT

In the first part of the work, we study the topological aspects of compact toric varieties. Considering toric varieties as pseudomanifolds, we investigate their standard stratification. We prove the triviality of the link bundles of toric varieties. On the other hand, we endow toric varieties with CW structure. We compute the homology groups of real 6-dimensional toric varieties with rational coefficients by using CW structure. We also determine the homology groups of the links of 4-, 6-, and 8-co-dimensional strata. At this point, we compare the topological description of toric varieties with the one from algebraic geometry. We show that we have similar characterizations of singular points in both pictures. At the end of part I, we introduce a class of pseudomanifolds called pseudo-toric varieties. We briefly study their link bundles and compute their Betti numbers. Finally, we generalize the notation of pseudomanifolds and introduce the so-called $\mathbb{Q}$-pseudomanifolds. Showing that a $\mathbb{Q}$-manifold with boundary satisfies the Lefschetz duality rationally gives us the necessary tools to generalize the theory of intersection spaces for $\mathbb{Q}$-pseudomanifolds with isolated singularities.

In the second part of the work, we study the theory of intersection spaces introduced by Banagl. We construct the intersection spaces of real 4-dimensional toric and pseudo-toric varieties and compute the associated Betti numbers. In the next step, we generalize the theory of intersection spaces to $\mathbb{Q}$-pseudomanifolds. An arbitrary real 6-dimensional toric variety has stratification depth 2. But we can also consider it as a $\mathbb{Q}$-pseudomanifold with isolated singularities. Applying the generalized construction of intersection spaces, we compute the associated Betti numbers of intersection spaces of 6-dimensional toric varieties. Comparing the Betti numbers of intersection spaces with the associated Betti numbers of intersection homology, we get the following results. Betti numbers of the intersection space are not combinatorial invariant. On the other hand, the intersection homology does not even determine the combinatorial data of the fan.

In the last part of the work, we study the theory of intersection space pairs introduced by M. AGUSTÍN and J. FERNÁNDEZ DE BOBADILLA. As the first example, we endow real 6-dimensional toric varieties with the standard stratification and construct the associated intersection space pairs. Comparing our results with the last part yields that the Betti numbers of the intersection space pairs determine only the combinatorial data of the fan. We then construct the intersection space pairs of a specific class of 6-dimensional pseudo-toric varieties, where the generalization of the theory of intersection spaces is not applicable.

Finally, we construct the intersection space of the link of an isolated singularity in an arbitrary 8-dimensional toric variety. Using the duality of the Betti numbers implies that, similar to real 6-dimensional toric varieties, we only have one non-combinatorial invariant parameter in the Betti numbers of the link.

ZUSAMMENFASSUNG

Im ersten Teil dieser Arbeit beschäftigen wir uns mit den topologischen Aspekten der kompakten torischen Varietäten. Wir betrachten die torische Varietäten als Pseudo-Mannigfaltigkeit und untersuchen deren übliche Stratifizierung. Zuerst beweisen wir, dass die Link-Bündel der torischen Varietäten trivial sind. Auf der anderen Seite statten wir die torische Varietäten mit CW-Struktur aus. Wir berechnen die Homologiegruppen der 6-dimensionalen torische Varietäten mit rationalen Koeffizienten anhand CW-Struktur. Außerdem bestimmen wir die Homologiegruppen der Links der 4-, 6-, und 8-co-dimensionalen Strata. An dieser Stelle vergleichen wir die topologische und algebraisch geometrische Singularitäten. Wir zeigen, dass in beiden Fällen die Beschreibung der Singularitäten identisch sind. Am Ende des ersten Teils führen wir die sogenannte "Pseudo-torische Varietätenëin. Wir untersuchen außerdem kurz deren Link-Bündels und berechnen deren Betti-Zahlen. Danach verallgemeinern wir die Notation der Pseudo-Mannigfaltigkeiten und führen die sogenannte $\mathbb{Q}$-Pseudo-Mannigfaltigkeiten ein. Eine $\mathbb{Q}$-Mannigfaltigkeit mit Rand erfüllt die Lefschetz-Dualität rational. Wir benutzen diese Tatsache, um die Theorie der Schnitträume auf $\mathbb{Q}$-Pseudo-Mannigfaltigkeiten mit isolierten Singularitäten zu verallgemeinern.

Im zweiten Teil untersuchen wir die Theorie der Schnitträume eingeführt von BANAGL. Wir bilden die Schnitträume der 4-dimensionalen torischen und Pseudo-torischen Varietäten und berechnen deren Betti-Zahlen. Im nächsten Schritt verallgemeinern wir die Theorie der Schnitträume auf $\mathbb{Q}$-Pseudo-Mannigfaltigkeiten. Eine beliebige 6-dimensionale torische Varietät hat Stratifizierungstiefe 2. Aber als $\mathbb{Q}$-Pseudo-Mannigfaltigkeit hat eine torische Varietät nur isolierte Singularitäten. Nach der Anwendung der verallgemeinerten Theorie der Schnitträume auf 6-dimensionale torische Varietäten bestimmen wir die assoziierte Betti-Zahlen. Bei der Vergleich der Betti-Zahlen der Schnitthomologie und des Schnittraumes findet man heraus, dass in diesem Fall die Betti-Zahlen der Schnitträume im Allgemeinen nicht kombinatorisch invariant sind. Auf der anderen Seite bestimmen die Betti-Zahlen der Schnitthomologie nicht mal die kombinatorische Daten der Fan.

Im letzten Teil beschäftigen wir uns mit der Theorie der Schnittraumpaare eingeführt von M. AGUSTÍN und J. FERNÁNDEZ DE BOBADILLA. Als erstes Beispiel betrachten wir die übliche Stratifizierung der 6-dimensionalen torische Varietäten und konstruieren die assoziierte Schnittraumpaare. Vergleichen mit dem vorherigen Teil liefert uns, dass die Betti-Zahlen der Schnittraumpaare in diesem Fall nur die kombinatorische Daten der Fan bestimmen können. Außerdem konstruieren wir die Schnittraumpaare der bestimmten Klasse von Pseudo-torischen Varietäten, wo die verallgemeinerte Theorie der Schnitträume nicht anwendbar ist.

Am Ende untersuchen wir das Aufbauen des Schnittraumes vom Link einer isolierten Singularität in einer beliebigen 8-dimensionalen torische Varietät. Anhand der Dulität

der Betti-Zahlen zeigt man, dass die Betti-Zahlen des Links nur ein kombinatorisch nicht invariantes Parameter besitzen.

ACKNOWLEDGMENTS

First of all, I would like to thank my supervisor, Prof. Dr. Markus Banagl, without whom I would not have been able to complete this research, and without whom I would not have made it through my Ph.D. degree!
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Part I

INTRODUCTION

In the first step, we give an introduction to this work. The main Idea of this part is to give readers a general overview of the main path chosen in this work. We briefly compare different methods tackling the same problem, namely establishing a theory to achieve Poincaré duality for singular spaces, specially pseudomanifolds. Later, we give a motivation for choosing the theory of intersection spaces as the main frame of this work. We also provide an insight into the recent development of the theory of intersection spaces which leads to the theory of intersection space pairs.

1 INTERSECTION SPACES AND INTERSECTION HOMOLOGY

For this introduction, we mainly used [3], [7], and [4].
The homology groups of a compact, oriented, connected, n-dimensional manifold X have a fundamental property called Poincaré duality. However, Poincaré duality fails in general for singular spaces. For example, let $\mathcal{S}^2 \vee \mathcal{S}^2$ be the one-point union of two 2-spheres. Then we have

$$\mathbf{H}_0(\mathcal{S}^2 \vee \mathcal{S}^2; \mathbb{Q}) = \mathbb{Q}$$
$$\mathbf{H}_2(\mathcal{S}^2 \vee \mathcal{S}^2; \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}.$$

Thus, we have

$$\mathrm{rk}(\mathbf{H}_0(\mathcal{S}^2 \vee \mathcal{S}^2; \mathbb{Q})) \neq \mathrm{rk}(\mathbf{H}_2(\mathcal{S}^2 \vee \mathcal{S}^2; \mathbb{Q})).$$

One can show that the union point in the above example is not locally Euclidean. We give yet another example of spaces containing points that are not locally Euclidean. Consider the suspended torus, $S\mathcal{T}^2$. The claim is that the two suspension points are not locally Euclidean. Note that each point has a neighborhood homeomorphic to the cone on the torus $\mathcal{C}(\mathcal{T}^2)$. We know that cones are contractible. Let ν be the cone point of $\mathcal{C}(\mathcal{T}^2)$. By using the long exact sequence of pair and homotopy invariance of homology, we get

$$\mathbf{H}_2(\mathcal{C}(\mathcal{T}^2), \mathcal{C}(\mathcal{T}^2) - \{\nu\}; \mathbb{Q}) \cong \mathbf{H}_1(\mathcal{C}(\mathcal{T}^2); \mathbb{Q}) \cong \mathbf{H}_1(\mathcal{T}^2; \mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}.$$

But now assume that ν would have a neighborhood homeomorphic to $\mathbb{R}^3$. Then using excision would yield

$$\mathbf{H}_2(\mathcal{C}(\mathcal{T}^2), \mathcal{C}(\mathcal{T}^2) - \{\nu\}; \mathbb{Q}) \cong \mathbf{H}_2(\mathbb{R}^3, \mathbb{R}^3 - \{\nu\}; \mathbb{Q}) \cong \mathbf{H}_1(\mathbb{R}^3 - \{\nu\}; \mathbb{Q}) \cong \mathbf{H}(\mathcal{S}^1; \mathbb{Q}) \cong \mathbb{Q}.$$

Hence, $S\mathcal{T}^2$ has two points that are not locally Euclidean. However, one can easily show that if we remove a sufficient small cone-like neighborhood of each singularity, the resulting space is a manifold. Many singular spaces have dense open subsets that are manifolds, such as the former example. A large class of examples comes by considering algebraic varieties and orbit spaces of manifolds and varieties by group actions. Note that generally, the singularities are not necessarily isolated points. But as we will see under specific assumptions, such singular spaces contain dense open manifold subsets. They will be filtered by closed subsets

$$X = X_n \supset X_{n-1} \supset \ldots X_0 \supset X_{-1} = \emptyset,$$

such that $X_k - X_{k-1}$ is either a manifold or empty. The connected components of the $X_k - X_{k-1}$ are called strata. For $k < n$, we call them singular strata. We require that a point

$x \in X_k - X_{k-1}$ should have a neighborhood $\mathcal{U}$ of the form $\mathcal{U} \cong \mathbb{R}^k \times \mathcal{C}(\mathcal{L})$, where $\mathcal{L}$ is a compact filtered space and such that the homeomorphism takes $\mathbb{R}^k \times \{\nu\}$ to a neighborhood of x in $X_k - X_{k-1}$, where ν is the cone point. Finally, we arrive at the notation of stratified pseudomanifolds, which we will define later. Given a pseudomanifold, one could ask about recovering the Poincaré duality, at least by some modifications. The theory of intersection homology introduced by Mark Goresky and Robert MacPherson first in [10] recovers the Poincaré duality for compact, connected, and oriented pseudomanifolds. It is crucial to notice that intersection homology is defined by modification of the definition of homology groups. The modification is being carried out on the chain level. Only chains satisfying certain extra geometric conditions are allowed. The so-called perversity function determines the specific form of these geometric conditions. Considering the above modifications, we have the perversity $\bar{p}$ intersection chains $\mathbf{I}^{\bar{p}}\mathbf{C}_*(X)$ and their homology groups $\mathbf{I}^{\bar{p}}\mathbf{H}_*(X)$. The duality is established between intersection homology groups with complementary perversities. For example, one should note that the intersection homology groups are not homotopy invariant. Note that intersection homology groups of a pseudomanifold are independent of the chosen stratification.

Let us elaborate on the above discussion with an example. Let X be the pinched torus. X is a pseudomanifold in the above sense and has only one isolated singularity. Then for middle perversity, we have

$$\begin{aligned} &\mathbf{I}^{\bar{m}}\mathbf{H}_2(X;\mathbb{Q}) = \mathbb{Q} \\ &\mathbf{I}^{\bar{m}}\mathbf{H}_1(X;\mathbb{Q}) = 0 \\ &\mathbf{I}^{\bar{m}}\mathbf{H}_0(X;\mathbb{Q}) = \mathbb{Q}. \end{aligned}$$

On the other hand, for the homology groups, we have

$$\begin{aligned} &\mathbf{H}_2(X;\mathbb{Q}) = \mathbb{Q} \\ &\mathbf{H}_1(X;\mathbb{Q}) = \mathbb{Q} \\ &\mathbf{H}_0(X;\mathbb{Q}) = \mathbb{Q}. \end{aligned}$$

In other words, the modification implies that any 1-chain crossing the singularity is not allowed.

There is yet also another approach to the matter at hand. In contrast to intersection homology, one can implement the modifications on the spatial level. It is the main idea of the theory of intersection spaces. Thus, to a stratified pseudomanifold X, we associate spaces

$$I^{\bar{p}}X,$$

the intersection spaces of X, such that the ordinary homology $\widetilde{\mathbf{H}}_*(I^{\bar{p}}X;\mathbb{Q})$ satisfies generalized Poincaré duality when X is closed and oriented. In the absence of odd-co-dimensional strata and if $\bar{p}$ is the middle perversity $\bar{p} = \bar{m}$, then we are associating to a singular pseudomanifold a (rational) Poincaré complex. Let us consider the pinched torus again. In [4], the homology groups of associated intersection spaces have been computed. Let X be the pinched torus. Thus, we have

$$\widetilde{\mathbf{H}}_2(I^{\bar{m}}X;\mathbb{Q}) = 0$$
$$\widetilde{\mathbf{H}}_1(I^{\bar{m}}X;\mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$$
$$\widetilde{\mathbf{H}}_1(I^{\bar{m}}X;\mathbb{Q}) = 0.$$

In the middle degree comparing intersection homology, the homology of intersection spaces sees more structure. As we will see later, it is also the case for real 6-dimensional compact toric varieties. Hence, the resulting homology theory is not isomorphic to intersection homology. From the physical point of view, the above approach solves a problem in type II string theory related to the existence of massless D-branes, which is neither solved by ordinary homology nor by intersection homology. $\mathbf{I}^{\bar{m}}\mathbf{H}_*(X)$ delivers the correct theory for type IIA string theory (giving physically correct counts of massless particles). On the other hand, $\widetilde{\mathbf{H}}_*(I^{\bar{m}}X)$ provides the correct theory for type IIB string theory. In other words, the two theories $\mathbf{I}^{\bar{m}}\mathbf{H}_*(X)$ and $\widetilde{\mathbf{H}}_*(I^{\bar{m}}X)$ form a mirror pair in the sense of mirror symmetry in algebraic geometry. $\widetilde{\mathbf{H}}_*(I^{\bar{p}}X)$ satisfies generalized Poincaré duality across complementary perversities. However, it is too much to expect the full naturality of the assignment $X \rightsquigarrow I^{\bar{p}}X$. One can not achieve a corresponding property for intersection homology either. In spatial modification, a pseudomanifold is modified as little as possible. To be more precise, the modification is carried out only near singularities. The homotopy type away from the singularities is completely preserved. Another benefit of spatial modification is that cochain complexes automatically come equipped with internal multiplications, turning them into differential-graded algebras. The ordinary cochain complex $\mathbf{C}^*(I^{\bar{p}}X)$ of the intersection space $I^{\bar{p}}X$ is a differential graded algebra, simply by employing the ordinary cup product. Similarly, the cohomology of $I^{\bar{p}}X$ is, by default, endowed with internal cohomology operations. However, the intersection chain complexes $\mathbf{I}^{\bar{p}}\mathbf{C}_*(X)$ are not generally algebras, unless $\bar{p}$ is the zero perversity, which implies that $\mathbf{I}^{\bar{p}}\mathbf{C}_*(X)$ is the ordinary cochain complex of X. In [3], Banagl developed the theory of intersection spaces for pseudomanifolds with isolated singularities. The case of pseudomanifolds with stratification depth two and trivial link bundles has also been studied in [3]. One can develop the theory of intersection spaces beyond isolated singularities. It is the subject of [2]. Here, one should consider the intersection space pairs, and with some extra considerations, relative homology groups of the intersection space pairs satisfy the generalized Poincaré duality.

Part II

TORIC VARIETIES

Initially, we review some basic definitions. We give a topological description of compact toric varieties. Endowing toric varieties with a CW structure and studying the link bundle of stratified toric varieties gives us the needed tools for the next part of this work. Proving that toric varieties possess a stratification with only even dimensional strata and trivial link bundle makes the theory of intersection spaces, which was introduced first here [3] and improved here [2], applicable to toric varieties. We determine the homology groups of links for 4-dimensional, 6-dimensional and 8-dimensional toric varieties.

2 A BRIEF INTRODUCTION TO TORIC VARIETIES

Let us start with the definition of a rational polyhedral cone in $\mathbb{R}^n$.

Definition 2.1
We call $\sigma \in \mathbb{R}^n$ a rational polyhedral cone, if there exist finitely many vectors $v_1, \ldots, v_n \in \mathbb{Z}^n$ such that

$$\sigma = \{x_1 v_1 + \cdots + x_n v_n | x_1, \ldots x_n \in \mathbb{R}_{\geq 0}\}. \tag{2.1}$$

*σ is **proper** if it is not spanned by any proper subset of $\{v_1, \ldots, v_n\}$, and $v_1, \ldots, v_n$ are lie strictly on one side of some hyperplane in $\mathbb{R}^n$.*

The dimension of σ is defined to be the dimension of $span(\sigma) \subset \mathbb{R}^n$. Note also that σ is called simplicial if the generating vectors $v_1, \ldots, v_n \in \mathbb{Z}^n$ can be chosen linearly independent. This implies that $n = \dim(\sigma)$.

Definition 2.2
We call a cone τ a (proper) face of cone σ if it spans by a (proper) subset of $\{v_1, \ldots, v_n\}$ and contained in the topological boundary of σ and we write $\tau \preceq \sigma$ ($\tau \prec \sigma$ if τ is a proper face of σ).

Definition 2.3 (Complete rational polyhedral cone)
*A **complete rational polyhedral cone complex** (**complete fan**) is a set of proper rational polyhedral cones in $\mathbb{R}^n$ satisfying the following conditions:*

1. *Every cone in $\sigma \in \Sigma$ has a vertex at 0.*
2. *If τ is a face of a cone $\sigma \in \Sigma$, then $\tau \in \Sigma$.*
3. *If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face of both σ and σ'.*
4. *$\bigcup_{\sigma \in \Sigma} \sigma = \mathbb{R}^n$.*

Definition 2.4 (Dual polytope)
*Let $\Sigma \in \mathbb{R}^n$be a complete rational polyhedral cone complex. Reverse the inclusions and the grading in the face lattice of Σ. The Resulting abstract polyhedron can be realized in $\mathbb{R}^n$ as a regular (polyhedral) cell complex. We call this **the dual polytope** $\mathcal{P}$.*

Remark 2.5
The faces in $\mathcal{P}(\Sigma)$ are in one-to-one correspondence (in complementary dimension) with the cones in Σ. Thus, we define a bijection $\delta : \mathcal{P} \longrightarrow \Sigma$ and denote the dual cone to $\tau \in \mathcal{P}$ with $\delta(\tau) \in \Sigma$.

We define regular CW-complexes, as in [11].

Definition 2.6
*A CW-complex is called **regular** if its characteristic maps can be chosen to be embeddings.*

Definition 2.7
*The 0-dimensional faces of $\mathcal{P}$ are called **vertices**. The 1-dimensional faces are called **edges** and **facets** are faces with co-dimension 1.*
$\mathcal{P}^i = \{F : F \text{ is a face of } \mathcal{P}, \dim(F) \leq i\}$ *is the i-skeleton of* $\mathcal{P}$.

REGULAR CW STRUCTURE OF THE DUAL POLYTOPE $\mathcal{P}$
Note that $\mathcal{P}$ is homeomorphic to $\mathcal{D}^n$, where $\dim(\mathcal{P}) = n$. The structure of the underlying fan induces a regular CW structure on $\mathcal{P}$ as follows:
The $\{0\} \in \Sigma$ is dual to the interior of $\mathcal{P}$, represented by an n-dimensional cell in the CW structure. Each k-dimensional cone in Σ is dual to an (n-k)-dimensional face of $\mathcal{P}$, represented by an (n-k)-dimensional cell in the CW structure. Note also that the boundary maps are induced from the inclusion of faces in Σ (or dually in $\mathcal{P}$). This gives us a CW-complex on $\mathcal{P} \cong \mathcal{D}^n$, which we will use later. Although, we do not require the exact form of the boundary operators of $\mathcal{P}$ in our preceding discussion, we briefly describe the attaching maps. As usual, we start with a discrete set, X^0, whose points represent the vertices of $\mathcal{P}$. We glue each 1-dimensional cell to its topological boundary in $\mathcal{P}$, which consists of its neighboring 0-dimensional cells. Inductively, we attach each k-cell, representing a k-dimensional face of $\mathcal{P}$, to its lower-dimensional neighboring faces. Note that $|\mathcal{P}| - \text{int}(\mathcal{P}) \cong \mathcal{S}^{n-1}$. In the last step we attach $\text{int}(\mathcal{P})$, represented by the n-cell, to its topological boundary $|\mathcal{P}| - \text{int}(\mathcal{P})$. Note that due to the fact that each k-dimensional face of $\mathcal{P}$ is homeomorphic to $\mathcal{D}^k$, each characteristic map can be chosen to be an embedding. Thus, the above CW structure is a regular CW structure.

Let $\mathcal{T}^n = \mathbb{R}^n/\mathbb{Z}^n \cong \overbrace{\mathcal{S}^1 \times \cdots \times \mathcal{S}^1}^{n}$ be an n-torus. The projection map $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n$ maps a **rational** k-dimensional subspace $\mathbf{V} \subset \mathbb{R}^n$ to a compact subtorus $\pi(\mathbf{V}) \subset \mathcal{T}^n$. The **rationality** here means that $\mathbf{V}$ has a basis in $\mathbb{Z}^n$. Now each affine k-plane parallel to $\mathbf{V}$ in $\mathbb{R}^n$ determines a subtorus "parallel" to $\pi(\mathbf{V})$. Collapsing each of this parallel subtori to a point will gives us a compact subspace $\mathcal{T}^n/\pi(\mathbf{V}) \subset \mathcal{T}^n$, which is obviously homeomorphic to $\mathcal{T}^{n-k}$.

TORIC VARIETY
In this thesis, we mainly study compact toric varieties, viewed as topological spaces. Thus, our description of toric varieties should adjust this point of view. Hence, as we will see, we restrict ourselves to toric varieties associated with complete fans. It ensures that the resulting toric variety is compact. Compact toric varieties are even dimensional topological pseudomanifolds with only even co-dimensional strata. We give the same description of toric varieties as in [15].
Topologically, we can construct the toric variety associated with a **complete rational cone complex** Σ (**complete fan**) as follows:

Let $\mathcal{P}$ be the dual polytope of Σ and $\overline{X} = \mathcal{P} \times \mathcal{T}^n$. Each σ_D, a k-dimensional face of $\mathcal{P}$, is dual to an (n-k)-dimensional cone σ in Σ. Let $\mathbf{V}_\sigma$ be the linear span of σ in $\mathbb{R}^n$. Now if $x \in \text{int}(\sigma_D)$ we collapse $\{x\} \times \mathcal{T}^n$ to $\{x\} \times \mathcal{T}^n / \pi(\mathbf{V}_\sigma)$. This procedure can be done for each face of $\mathcal{P}$ and the resulting space $X_\mathcal{P}$ or (X_Σ) is called the toric variety associated with Σ (or $\mathcal{P}$).
Note that for each $x \in \mathcal{P}$ there is a unique σ_D such that $x \in \text{int}(\sigma_D)$ thus the above construction is well-defined.

Remark 2.8
Note that the above construction gives us the projection maps $\overline{X} \xrightarrow{\overline{p}} X_\mathcal{P}$ *and* $X_\mathcal{P} \xrightarrow{p} \mathcal{P}$.
From the construction, it is clear that toric varieties are even-dimensional topological spaces.

In the following, we take a look at some simple examples.

Example 2.9 (n=1)
There is only one complete cone complex in $\mathbb{R}$ *which is the following cone complex and it is shown in (2.1).*

$$\Sigma_1 = \{\{0\}, \sigma_+, \sigma_-\}, \textit{ where}$$
$$\sigma_i = \{ix \mid x \in \mathbb{R}_{\geq 0}\} \textit{ for } i = +1, -1.$$

Figure 2.1: The complete 1-dimensional fan Σ_1 and the dual polytope $\mathcal{P} \cong \mathcal{I}$.

The dual polytope of Σ_1 *is simply the unit interval* $\mathcal{I}$ *and thus* $\overline{X} = \mathcal{I} \times \mathcal{S}^1$. X_{Σ_1} *is simply the result of collapsing each end of the cylinder* $\overline{X}$ *to a point.*
Hence, we have $X = \mathcal{S}^2 \cong \mathbb{P}^1$, *where* $\mathbb{P}^1$ *denotes the complex projective plane.*

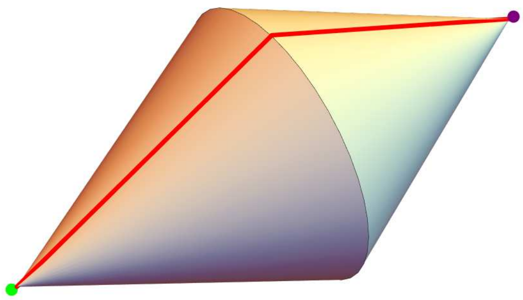

Figure 2.2: The associated toric variety to Σ_1 which is homeomorphic to $\mathcal{S}^2$

Note that, as mentioned earlier, $\mathcal{I} \cong \mathcal{D}^1$. The fan Σ_1 induces a CW structure on $\mathcal{I}$ with two 0-cells and one 1-cell. $\mathcal{S}^2$ can also be endowed with a CW structure in a similar manner, which we will discuss later on.

Example 2.10 (n=2)
Let $n, m \in \mathbb{N}_{\geq 0}$, $(n, m) \neq 0$ and n and m be relative prime. Consider the following fan:

$$\begin{aligned}
\Sigma_2 &= \{\{0\}, \tau_1, \tau_2, \tau_3, \tau_4, \sigma_{12}, \sigma_{23}, \sigma_{34}, \sigma_{41}\} \\
&\quad \textit{where} \\
\tau_1 &= \{x(n, m) \mid x \in \mathbb{R}_{\geq 0}\}, \\
\tau_2 &= \{x(-m, n) \mid x \in \mathbb{R}_{\geq 0}\}, \\
\tau_3 &= \{x(-n, -m) \mid x \in \mathbb{R}_{\geq 0}\}, \\
\tau_4 &= \{x(m, -n) \mid x \in \mathbb{R}_{\geq 0}\}, \\
\sigma_{12} &= \{x(n, m) + y(-m, n) \mid x, y \in \mathbb{R}_{\geq 0}\}.
\end{aligned}$$

σ_{23}, σ_{34} and σ_{41} are defined similarly. The dual polytope is homeomorphic to an 2-dimensional convex polygon with four 1-dimensional faces or simply a square. Thus we can write $\overline{X} = \mathcal{I} \times \mathcal{I} \times \mathcal{T}^2$. $\mathcal{T}^2$ can be written as $\pi(\mathbf{V}_{\tau_1}) \times \pi(\mathbf{V}_{\tau_2})$ hence $\overline{X}$ can be rewritten as $(\mathcal{I} \times \pi(\mathbf{V}_{\tau_1})) \times (\mathcal{I} \times \pi(\mathbf{V}_{\tau_2}))$. Note that due to the structure of Σ_2 applying the topological construction of toric varieties yields that either $\pi(\mathbf{V}_{\tau_1})$ or $\pi(\mathbf{V}_{\tau_2})$ is subject to collapses on 1-dimensional faces of $\mathcal{P}(\Sigma_2)$. Based on this observation it is easy to show that $X_{\Sigma_2} \cong X_{\Sigma_1} \times X_{\Sigma_1} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

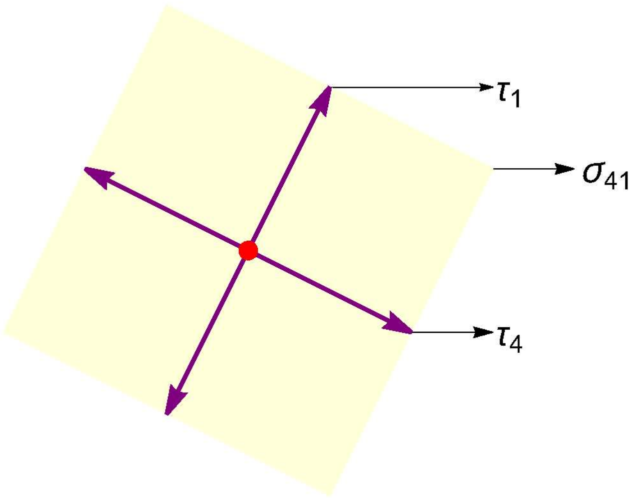

Figure 2.3: The 2-dimensional complete fan Σ_2.

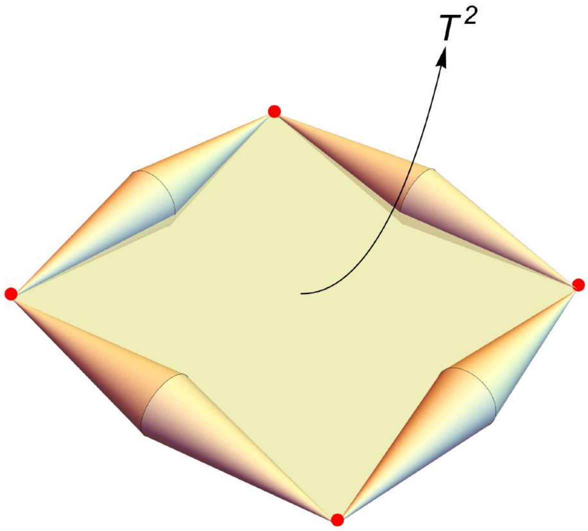

Figure 2.4: The associated toric variety to Σ_2 which is homeomorphic to $\mathcal{S}^2 \times \mathcal{S}^2$.

Remark 2.11
Note that the above example can be inductively generalized to any dimension. Thus we have found a description of $(\mathbb{P}^1)^n$ *as a toric variety.*

We can generalize what we have seen in the previous remark as follows. The reader can find the proof in [9] as an exercise in section 1.4.

Proposition 2.12 (Product of toric varieties)
Let Σ be a fan in $\mathbb{Z}^n$ or in other words a rational fan in $\mathbb{R}^n$. Let Σ' be a fan in $\mathbb{Z}^m$. The set of products $\sigma \times \sigma'$, $\sigma \in \Sigma$, $\sigma' \in \Sigma$, forms a fan $\Sigma \times \Sigma'$ in $\mathbb{Z}^n \oplus \mathbb{Z}^m$ and we have

$$X_{\Sigma \times \Sigma'} = X_\Sigma \times X_{\Sigma'}.$$

There is yet another generalization of the above proposition given in [5] in Theorem 2.4.7.

Example 2.13 (n=3)
Consider the following fan (Σ_3)

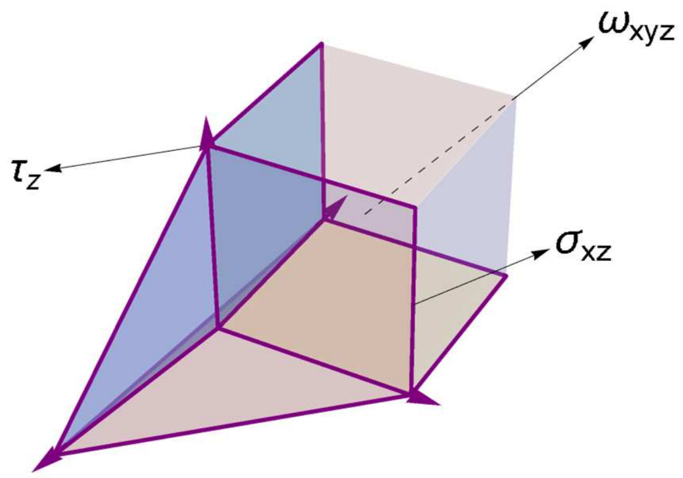

Figure 2.5: The complete fan Σ_3 in $\mathbb{R}^3$.

where τ_z is generated by $\hat{i}_z = (0,0,1)$. τ_x and τ_y are generated by $\hat{i}_x$ and $\hat{i}_y$, respectively. The forth 1-dimensional cone of Σ_3 is generated by $i_4 = (-1,-1,-1)$. The generators of the 2-dimensional cone σ_{xz} are $\hat{i}_x$ and $\hat{i}_z$ and the five remaining 2-dimensional cones are constructed in a similar manner. Finally, ω_{xyz} is generated by $\hat{i}_x$, $\hat{i}_y$ and $\hat{i}_z$. The three remaining 3-dimensional cones are constructed similarly. It is easy to show that the dual polytope is a triangular pyramid. Applying the topological construction of toric varieties on Σ_3 yields a toric variety which is homeomorphic to $\mathbb{P}^3$.

Algebraic descriptions of all the above examples have been given in [9].

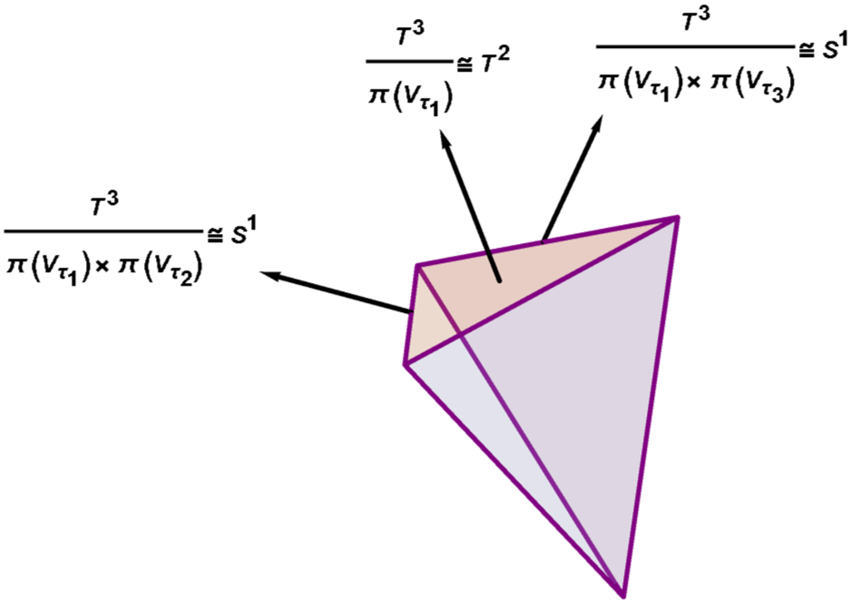

Figure 2.6: The toric variety X_{Σ_3} which is homeomorphic to $\mathbb{P}^3$.

3 TOPOLOGY OF TORIC VARIETIES

In this chapter, we investigate the stratification of toric varieties. First of all, we go through the definition of topological pseudomanifolds. Based on the given definitions, we endow toric varieties with topological stratification.

3.1 TORIC VARIETIES AS PSEUDOMANIFOLDS

In this section, we study toric varieties from the perspective of pseudomanifolds. First of all, we will show that toric varieties are indeed pseudomanifolds. In the next step, we will exhibit that the link bundles of toric varieties, viewed as pseudomanifolds, are trivial. An immediate consequence is that toric varieties are normal pseudomanifolds. In other words, the link of the top stratum is always homeomorphic to a $\mathcal{S}^1$.

We use the definition of topologically stratified spaces, which is given in [12].

Definition 3.1
We defined a ***topologically stratified space*** *inductively on dimension. A 0-dimensional topologically stratified space X is a countable set with the discrete topology. For $m > 0$ an m-****dimensional topologically stratified space*** *is a para-compact Hausdorff topological space X equipped with a filtration*

$$X = X_m \supseteq X_{m-1} \supseteq \cdots \supseteq X_1 \supseteq X_0$$

of X by closed subsets X_j such that if $x \in X_j - X_{j-1}$ there exits a neighborhood $\mathcal{N}_x$ of x in X, a compact $(m-j-1)$-dimensional topologically stratified space $\mathcal{L}$ with filtration

$$\mathcal{L} = \mathcal{L}_{m-j-1} \supseteq \cdots \supseteq \mathcal{L}_1 \supseteq \mathcal{L}_0,$$

and a homeomorphism

$$\phi : \mathcal{N}_x \longrightarrow \mathbb{R}^j \times C(\mathcal{L}),$$

where $C(\mathcal{L})$ is the open cone on $\mathcal{L}$, such that ϕ takes $\mathcal{N}_x \cap X_{j+i+1}$ homeomorphically onto

$$\mathbb{R}^j \times C(\mathcal{L}_i) \subseteq \mathbb{R}^j \times C(\mathcal{L})$$

for $m - j - 1 \geq i \geq 0$, and ϕ takes $\mathcal{N}_x \cap X_j$ homeomorphically onto

$$\mathbb{R}^j \times \{\textit{vertex of } C(\mathcal{L})\}.$$

Remark 3.2
*Straightforwardly, we can show that $X_j - X_{j-1}$ is a $j-$dimensional topological manifold. We call the connected components of these manifolds the **strata** of X. $\mathcal{L}$ depends on the stratum in which the point x lies. It is referred to as the **link** of the stratum.*

Definition 3.3
*An m-**dimensional topological pseudomanifold** is a para-compact Hausdorff topological space X which possesses a topological stratification such that*

$$X_{m-1} = X_{m-2}$$

and $X - X_{m-1}$ is dense in X.

Definition 3.4
*We call a stratum **singular** if its link is not a homological sphere. A stratum is **rationally singular** if its link is not a rational homological sphere.*

Now, we show that there is a natural stratification of toric varieties. We also give an algebraic description of the situation.
Consider the map $X \xrightarrow{p} \mathcal{P}$ introduced earlier. $X_{2i} = p^{-1}(\mathcal{P}^i)$ is an 2i-dimensional topological space. We claim that the filtration

$$X_{\mathcal{P}} = X_{2m} \supset X_{2(m-1)} \supset \cdots \supset X_2 \supset X_0, \tag{3.1}$$

with local homeomorphisms, which we will define later, is a stratification of the toric variety $X_{\mathcal{P}}$. But first we need to give a topological description of links.

CONSTRUCTION OF LINKS
Let $\mathcal{M}_\tau$ be an abstract polytope, geometrically realized as a subspace of $\mathcal{P}$, and is defined as follows:
Let $\mathcal{S}_\tau = \{\sigma | \sigma \in \mathcal{P},\ \sigma \cap \tau \neq \emptyset \text{ and } \dim(\sigma) > \dim(\tau)\}$ be the set of all higher dimensional neighboring faces of τ in $\mathcal{P}$.
Now, we define $\mathcal{M}_\tau$ be an abstract polytope with the following properties:

1. $\forall \sigma \in \mathcal{S}_\tau : \exists!\ \gamma_\sigma \in \mathcal{M}_\tau$ such that $\text{int}(\gamma_\sigma) \cap \text{int}(\sigma) \neq \emptyset$ with $\dim(\gamma_\sigma) = \dim(\sigma) - (1 + \dim(\tau))$ and $\text{int}(\gamma_\sigma) \cap \text{int}(\sigma') = \emptyset$ if $\sigma' \in \mathcal{P}$ and $\sigma' \neq \sigma$.

2. If $\omega, \sigma \in \mathcal{S}_\tau$ with $\omega \prec \sigma$. Then, we demand that $\gamma_\omega \prec \gamma_\sigma$ and $\gamma_\sigma \cap \omega = \gamma_\omega$

Note that $\text{int}(\sigma) \cong \text{int}(\mathcal{D}^{\dim(\sigma)})$. So, the first requirement can be satisfied by embedding $\text{int}(\mathcal{D}^{\dim(\gamma_\sigma)})$ in $\text{int}(\mathcal{D}^{\dim(\sigma)})$. Yet the second condition describes how the boundary of $\mathcal{D}^{\dim(\gamma_\sigma)}$ meets the boundary of $\mathcal{D}^{\dim(\sigma)}$. We refer to $\mathcal{M}_\tau$ as **base** of the link of $x \in p^{-1}(\text{int}(\tau)) \subset X_{\mathcal{P}}$.

Remark 3.5
Consider $\mathcal{M}_\tau$ and $\mathcal{M}'_\tau$ such that they both satisfy the above conditions. Then for each $\sigma \in \mathcal{S}_\tau$ and $\gamma_\sigma \in \mathcal{M}_\tau$ there exists a $\gamma'_\sigma \in \mathcal{M}'_\tau$ with $\dim(\gamma_\sigma) = \dim(\gamma'_\sigma)$. Note that the construction of $\mathcal{S}_\tau$ and the uniqueness in the first condition implies that $n(\mathcal{M}^i_\tau - \mathcal{M}^{i-1}_\tau)$ or simply the number of i-dimensional faces of $\mathcal{M}_\tau$ is equal to $n({\mathcal{M}'}^i_\tau - {\mathcal{M}'}^{i-1}_\tau)$. Thus there is a bijection between $\mathcal{M}_\tau$ and $\mathcal{M}'_\tau$.

Now, let $|\mathcal{M}_\tau| \subset |\mathcal{P}|$ be a geometrical realization of $\mathcal{M}_\tau$ and $n = \dim(|\mathcal{P}|)$. Recall the bijection $\delta : \mathcal{P} \longrightarrow \Sigma_{\mathcal{P}}$ that we introduced earlier. For each $\gamma \in \mathcal{M}_\tau$ choose $\sigma_\gamma \in \mathcal{S}_\tau$ such that $\gamma \cap \sigma_\gamma \neq \emptyset$. Note $\tau \prec \sigma_\gamma$ and thus $\delta(\sigma_\gamma) \prec \delta(\tau)$ in $\Sigma_{\mathcal{P}}$. This implies that $\pi(\delta(\sigma_\gamma)) \subset \pi(\delta(\tau))$ in $\mathcal{T}^n$. We construct $\mathcal{L}_\tau$, the link of a point $x \in \mathrm{int}(\tau)$, by means of the following map:

$$\begin{aligned} &\mathcal{M}_\tau \times \mathcal{T}^n \longrightarrow \mathcal{L}_\tau \\ &\{y\} \times \mathcal{T}^n \longmapsto \{y\} \times \ \mathcal{T}^n \Big/ \Big(\pi(\delta(\sigma_\gamma)) \times \big(\mathcal{T}^n / \pi(\delta(\tau)) \big) \Big), \end{aligned} \tag{3.2}$$

where $y \in \mathrm{int}(\gamma)$. Note that similar to the topological construction of toric varieties $\mathcal{T}^n \Big/ \Big(\pi(\delta(\sigma_\gamma)) \times \big(\mathcal{T}^n / \pi(\delta(\tau)) \big) \Big)$ is defined by collapsing $\pi(\delta(\sigma_\gamma))$ and each parallel torus to $\pi(\delta(\sigma_\gamma))$ in $\mathcal{T}^n$ to a point and then collapsing each parallel tori to $\mathcal{T}^n/\pi(\delta(\tau))$ in $\mathcal{T}^n/\pi(\delta(\sigma_\gamma))$ to a point. Due to the previous consideration, $\mathcal{T}^n \Big/ \Big(\pi(\delta(\sigma_\gamma)) \times \big(\mathcal{T}^n / \pi(\delta(\tau)) \big) \Big)$ is well-defined.

Remark 3.6
Note that for each $y \in int(\gamma)$, $\mathcal{T}^n \Big/ \Big(\pi(\delta(\sigma_\gamma)) \times \big(\mathcal{T}^n / \pi(\delta(\tau)) \big) \Big) \subseteq \mathcal{T}^n/\pi(\delta(\sigma_\gamma))$ and hence $\mathcal{L}_\tau \subset X$, where $n = \dim(|\mathcal{P}|)$.

Remark 3.7
Note that $\dim(|\mathcal{M}_\tau|) = \dim(|\mathcal{P}|) - (1 + \dim(\tau))$. This comes from the fact that $int(|\mathcal{P}|)$ considered as a face in $\mathcal{P}$, which is dual to $\{0\} \in \Sigma_{\mathcal{P}}$, is a higher dimensional neighboring face of each $\tau \in \mathcal{P}$. So $\dim(\mathcal{L}_\tau) = \dim(|\mathcal{P}|) - (1 + \dim(\tau)) + \dim(\pi(\delta(\tau))$. Note that $dim(\tau) = \dim(|\mathcal{P}|) - \dim(\delta(\tau))$. Thus, we have

$$\dim(\mathcal{L}_\tau) = \dim(X_{\mathcal{P}}) - (1 + 2\dim(\tau)).$$

Remark 3.8
Let $\Sigma_{\mathcal{P}}$ be a complete fan and $\mathcal{P}$ be the dual polytope associated with $\Sigma_{\mathcal{P}}$. Let $\delta : \mathcal{P} \longrightarrow \Sigma_{\mathcal{P}}$ be the bijection defined in 2.5. We can construct the link of a connected component of a stratum of $X_{\Sigma_{\mathcal{P}}}$, the associated toric variety to $\Sigma_{\mathcal{P}}$, dually in $\Sigma_{\mathcal{P}}$. Let $\Sigma_{\mathcal{S}_\tau}$ to be the set of all lower dimensional neighboring cones of $\tau \in \mathcal{P}$ in $\Sigma_{\mathcal{P}}$. Similar to the previous construction, we define the dual fan associated with $\mathcal{M}_\tau$ be a fan in $\mathbb{R}^n$ with the following property:

- *$\forall \delta(\sigma) \in \Sigma_{\mathcal{S}_\tau} : \exists! \delta(\gamma_\sigma) \in \Sigma_{\mathcal{M}_\tau}$ such that $\delta(\gamma_\sigma) \subset \delta(\tau)$ as a cone. In other words, $\delta(\gamma_\sigma)$ lies in the topological boundary of the cone $\delta(\tau)$.*

Then $\Sigma_{\mathcal{M}_\tau}$ is a set of cones where we have

$$0 \leq \dim(\delta(\gamma_\sigma)) < \dim(\delta(\tau))$$

Hence, $\Sigma_{\mathcal{S}_\tau}$ is an $(\dim(\delta(\tau)) - 1)$-dimensional cone-complex, embedded in $\mathbb{R}^{\dim(\delta(\tau))}$, a rational subspace of $\mathbb{R}^{\dim(\Sigma_{\mathbb{P}})}$. By cone-complex, we mean that $\Sigma_{\mathcal{M}_\tau}$ satisfies the three first axioms of 2.2. However, instead of the fourth axiom, we have

$$\bigcup_{\delta(\gamma_\sigma)\in\Sigma_{\mathcal{M}_\tau}} \cong \mathbb{R}^{\dim(\delta(\tau))-1}.$$

But the crucial point to bear in mind is the following. Reversing the inclusions and grading in $\Sigma_{\mathcal{M}_\tau}$ will still result in an abstract polytope, which we can geometrically realize in $\mathbb{R}^{\dim(\delta(\tau)-1)}$. The above polytope is $\mathcal{M}_\tau$, and we can embed it in $\mathcal{P}$, as described earlier. Having $\mathcal{M}_\tau$ as a convex polytope embedded in $\mathcal{P}$, the rest of the construction goes along similar lines as described above. It yields that we can always geometrically realize $\mathcal{M}_\tau$, as a ***convex polytope*** *embedded in $\mathcal{P}$. Finally, one can easily deduce that $\Sigma_{\mathcal{S}_\tau} = \Sigma_{\mathcal{M}_\tau}$, which is simply the topological boundary of $\delta(\tau)$.*

Remark 3.9
There is a natural projection $\mathcal{L}_\tau \xrightarrow{p_\mathcal{L}} \mathcal{M}_\tau$. Similarly, we define $\left(\mathcal{L}_\tau\right)_{2i+1} = p_\mathcal{L}^{-1}(\mathcal{M}_\tau^i)$.

At this point, we want to show that there is a natural stratification of toric varieties. We will also give an algebraic description of the strata. We claim that the filtration in 3.1 is a stratification in the sense of the definition 3.1.
For a given point $x \in \mathrm{int}(\tau)$, where $\dim(\tau) \geq 1$, let $\mathcal{V} \cong \mathbb{R}^{\dim(|\mathcal{M}_\tau|)+1}$ be an affine subspace of $\mathbb{R}^{\dim(|\mathcal{P}|)}$ which is orthogonal to τ in $\mathbb{R}^{\dim(|\mathcal{P}|)}$ and $x \in \mathcal{V}$. Note that $\dim(|\mathcal{M}_\tau|) = \dim(|\mathcal{P}|) - (1 + \dim(\tau))$, hence, such $\mathcal{V}$ can always be found. Thus, we have $\sigma_\gamma \cap \mathcal{V} \neq \emptyset$ for each $\gamma \in \mathcal{M}_\tau$, where $\sigma_\gamma \in \mathcal{S}_\tau$ with $\mathrm{int}(\gamma) \cap \mathrm{int}(\sigma_\gamma) \neq \emptyset$. Now, we choose a geometrical realization of γ such that $\gamma \subset \sigma_\gamma \cap \mathcal{V}$. Note that $\dim(\sigma_\gamma \cap \mathcal{V}) = dim(\gamma) + 1$. Thus, we can find such a geometrical realization consistently. Now, let $\mathcal{C}(|\mathcal{M}_\tau|) = (|\mathcal{M}_\tau| \times \mathcal{I})/|\mathcal{M}_\tau| \times \{1\}$ be the cone of $|\mathcal{M}_\tau|$. We embed $\mathcal{C}(|\mathcal{M}_\tau|)$ into $|\mathcal{P}|$ as follows:
Let $v \in \mathcal{C}(|\mathcal{M}_\tau|)$ be the vertex of $\mathcal{C}(|\mathcal{M}_\tau|)$ and $\theta : \mathcal{C}(|\mathcal{M}_\tau|) \longrightarrow |\mathcal{P}|$ a map such that for each $\gamma \in \mathcal{M}_\tau$

$$\theta\big(\mathrm{int}(\gamma) \times [0,1)\big) \subset \mathrm{int}(\sigma_\gamma) \cap \mathcal{V}, \tag{3.3}$$

$$\theta\big(\gamma \times [0,1)\big) \subset \Big(\mathrm{int}(\sigma_\gamma) \cup \Big(\bigcup_{\substack{\omega_\gamma \in \mathcal{S}_\tau \\ \omega_\gamma \prec \sigma_\gamma}} \mathrm{int}(\omega_\gamma)\Big)\Big) \bigcap \mathcal{V} \tag{3.4}$$

$$\theta\big(\gamma \times \{0\}\big) \cong |\mathcal{M}_\tau| \cap \mathrm{int}(\sigma_\gamma) \cong \mathrm{int}(\gamma). \tag{3.5}$$

Note that for each $\eta, \gamma \in \mathcal{M}_\tau$ if $\gamma \prec \eta$ then we have $\sigma_\gamma \prec \sigma_\eta$ in $\mathcal{S}_\tau$. We require that

$$\theta\big(\eta \times [0,1)\big) \cap \sigma_\gamma = \theta\big(\gamma \times [0,1)\big). \tag{3.6}$$

As the last requirement, we want $\theta\big(\mathrm{int}(\gamma) \times [0,1)\big) \hookrightarrow \mathrm{int}(\sigma_\gamma)$ to be a topological embedding. Note that this can always be fulfilled because $\gamma \cong \mathcal{D}^{\dim(\gamma)}$ and $\sigma_\gamma \cong \mathcal{D}^{\dim(\sigma_\gamma)}$. At last, one should bear in mind that 3.6 ensures that θ is also an embedding on the topological boundary of $\sigma_\gamma \cap \mathcal{V}$ for each $\sigma_\gamma \in \mathcal{S}_\tau$. We set $\theta(v) = x$. Note that because of $\gamma \subset \sigma_\gamma \cap \mathcal{V}$ and the fact that we can choose $\theta\big(\gamma \times [0,1)\big) \hookrightarrow \mathrm{int}(\sigma_\gamma)$ as an embedding, it is possible to choose θ continuous on $\mathcal{C}(|\mathcal{M}_\tau|)$ and thus an embedding of $\mathcal{C}(|\mathcal{M}_\tau|)$ into $|\mathcal{P}|$. Now let $\mathcal{V}^\perp \subset \mathbb{R}^{\dim(|\mathcal{P}|)}$ be an orthogonal affine subspace to $\mathcal{V}$ such that $\tau \subset \mathcal{V}^\perp$. Choose $\mathcal{U}$ a neighborhood of x in $|\mathcal{P}|$ such that $\mathcal{U} \cap \mathcal{V}^\perp \cong \mathrm{int}(\mathcal{D}^{\dim(\tau)}) \cong \mathbb{R}^{\dim(\tau)} \cong \mathrm{int}(\tau)$ and $\mathcal{U} \cap \mathcal{V} \cong \mathcal{C}(|\mathcal{M}_\tau|)$. Thus, we have

$$\mathcal{U} \cong \mathbb{R}^{\dim(\tau)} \times \mathcal{C}(|\mathcal{M}_\tau|). \tag{3.7}$$

Remark 3.10

To be more specific, we can construct $\mathcal{U}$ as follows. We embed $\mathcal{C}(|\mathcal{M}_\tau|)$ in $\mathcal{V}$ as above. Now, we embed $\mathcal{D}^{\dim(\tau)} \times \mathcal{C}(|\mathcal{M}_\tau|)$ in $\mathcal{V} \times \mathcal{V}^\perp \cong \mathbb{R}^{\dim(|\mathcal{P}|)}$ such that

$$\big(\mathcal{D}^{\dim(\tau)} \times \mathcal{C}(|\mathcal{M}_\tau|)\big) \cap \mathcal{V}^\perp = \mathcal{D}^{\dim(\tau)}.$$

Finally, we set

$$\mathcal{U} \cong \mathcal{D}^{\dim(\tau)} \times \mathcal{C}(|\mathcal{M}_\tau|) \cong \mathbb{R}^{\dim(\tau)} \times \mathcal{C}(|\mathcal{M}_\tau|).$$

For each $\sigma_\gamma \in \mathcal{S}_\tau$, we have $\mathcal{U} \cap \mathrm{int}(\sigma_\gamma) \cong \big(\mathbb{R}^{\dim(\tau)} \cap \mathrm{int}(\sigma_\gamma)\big) \times \big(\mathcal{C}(|\mathcal{M}_\tau|) \cap \mathrm{int}(\sigma_\gamma)\big)$. Bear also in mind that $\big(\bigcup_{\gamma \in \mathcal{M}_\tau} \mathrm{int}(\sigma_\gamma)\big) \cup \mathrm{int}(\tau)$ is an open cover of $\mathcal{U}$. Thus

$$p^{-1}(\mathcal{U}) = \bigcup_{\gamma \in \mathcal{M}_\tau} \Big(\big(\mathrm{int}(\sigma_\gamma) \cap \mathcal{U}\big) \times \mathcal{T}^n/\pi(\delta(\sigma_\gamma))\Big) \cup \Big(\big(\mathrm{int}(\tau) \cap \mathcal{U}\big) \times \mathcal{T}^n/\pi(\delta(\tau))\Big).$$

Recall that $\tau \prec \sigma_\gamma$ in $\mathcal{P}$, hence $\delta(\sigma_\gamma) \prec \delta(\tau)$ which implies $\pi(\delta(\sigma_\gamma)) \subset \pi(\delta(\tau))$ and $\mathcal{T}^n/\pi(\delta(\tau)) \subset \mathcal{T}^n/\pi(\delta(\sigma_\gamma))$. Now, define $\mathcal{T}_\tau = \mathcal{T}^n/\pi(\delta(\tau))$. Then, we have

$$p^{-1}(\mathcal{U}) = \mathcal{T}_\tau \times \left(\bigcup_{\gamma \in \mathcal{M}_\tau} \Big(\big(\mathrm{int}(\sigma_\gamma) \cap \mathcal{U}\big) \times \mathcal{T}^n/(\pi(\delta(\sigma_\gamma)) \times \mathcal{T}_\tau)\Big) \cup \big(\mathrm{int}(\tau) \cap \mathcal{U}\big)\right).$$

Using 3.7, we arrive at

$$p^{-1}(\mathcal{U}) = \big(\mathcal{T}_\tau \times \mathbb{R}^{\dim(\tau)}\big) \times \left(\bigcup_{\gamma \in \mathcal{M}_\tau} \Big(\theta\big(\mathrm{int}(\gamma) \times [0,1)\big) \times \mathcal{T}^n/(\pi(\delta(\sigma_\gamma)) \times \mathcal{T}_\tau)\Big) \cup v\right).$$

Using the fact that θ is topological embedding gives us

$$p^{-1}(\mathcal{U}) \cong \big(\mathcal{T}_\tau \times \mathbb{R}^{\dim(\tau)}\big) \times \left([0,1) \times \bigcup_{\gamma \in \mathcal{M}_\tau} \Big(\mathrm{int}(\gamma) \times \mathcal{T}^n/(\pi(\delta(\sigma_\gamma)) \times \mathcal{T}_\tau)\Big) \cup v\right).$$

Thus, we have

$$p^{-1}(\mathcal{U}) \cong (\mathcal{T}_\tau \times \mathbb{R}^{\dim(\tau)}) \times \mathcal{C}\Big(\bigcup_{\gamma \in \mathcal{M}_\tau} \operatorname{int}(\gamma) \times \mathcal{T}^n / (\pi(\delta(\sigma_\gamma)) \times \mathcal{T}_\tau) \Big). \tag{3.8}$$

3.8 gives us more than just the requiered local triviality. Recall that $X_{2j} - X_{2(j-1)} = p^{-1}(\mathcal{P}_j) - p^{-1}(\mathcal{P}_{j-1})$ is simply the disjoint union of preimages of all $j-$dimensional faces of $\mathcal{P}$. Hence, we have

$$\begin{aligned} X_{2j} - X_{2(j-1)} &= \bigsqcup_{\substack{\tau \in \mathcal{P} \\ \dim(\tau)=j}} p^{-1}(\operatorname{int}(\tau)) \\ &= \bigsqcup_{\substack{\tau \in \mathcal{P} \\ \dim(\tau)=j}} \operatorname{int}(\tau) \times \mathcal{T}^n / \pi(\delta(\tau)) \\ &\cong \bigsqcup_{\substack{\tau \in \mathcal{P} \\ \dim(\tau)=j}} \mathbb{R}^{\dim(\tau)} \times \mathcal{T}_\tau. \end{aligned}$$

This means that $\mathbb{R}^{\dim(\tau)} \times \mathcal{T}_\tau$ is a connected component of $X_{2j} - X_{2(j-1)}$. Thus, if we show that the given filtration endows $X_\mathcal{P}$ with a stratification then the link bundle is trivial. Now, consider the filtration of $\mathcal{L}_\tau$ that we introduced earlier

$$\mathcal{L} = \mathcal{L}_{2m+1} \supset \mathcal{L}_{2(m-1)+1} \supset \cdots \supset \mathcal{L}_1.$$

Let $x \in X_j - X_{j-1}$ and $x \in \operatorname{int}(\tau)$. Thus, we can write $j = 2\dim(\tau)$. Choose $\mathcal{U}$ as described above. We want to investigate the intersection of $p^{-1}(\mathcal{U})$ with $X_{2\dim(\tau)+i+1}$. Note that in the above filtration of toric varieties we have only even-dimensional[1] topological spaces X. Thus, we can write $i + 1 = 2l$ with $l \in \mathbb{N}_{>0}$. Hence, we have

$$\begin{aligned} & p^{-1}(\mathcal{U}) \cap X_{2(\dim(\tau)+l))} = \\ & \Big(\bigcup_{\substack{\sigma \in \mathcal{P} \\ \dim(\sigma) \leq \dim(\tau)+l}} \big(\operatorname{int}(\sigma) \times \mathcal{T}^n / \pi(\delta(\sigma))\big)\Big) \bigcap \\ & \Big[(\mathbb{R}^{\dim(\tau)} \times \mathcal{T}_\tau) \times \Big(([0,1) \times \bigcup_{\gamma \in \mathcal{M}_\tau} \operatorname{int}(\gamma) \times \mathcal{T}^n / (\pi(\delta(\sigma_\gamma)) \times \mathcal{T}_\tau) \ ) \cup v \Big) \Big] = \\ & (\mathbb{R}^{\dim(\tau)} \times \mathcal{T}_\tau) \times \Big[[0,1) \times \Big((\bigcup_{\substack{\sigma \in \mathcal{P} \\ \dim(\sigma) \leq \dim(\tau)+l}} \operatorname{int}(\sigma) \times \mathcal{T}^n / \pi(\delta(\sigma)) \ ) \\ & \bigcap (\bigcup_{\gamma \in \mathcal{M}_\tau} \operatorname{int}(\gamma) \times \mathcal{T}^n / (\pi(\delta(\sigma_\gamma)) \times \mathcal{T}_\tau) \ ) \bigcup v \Big) \Big] = \end{aligned}$$

1 Note that X_{2j} can be identified with X_{2j+1}. This means that $X_{2j+1} - X_{2j} = \emptyset$ and thus the required conditions in (3.1) are trivially fulfilled.

$$\big(\mathbb{R}^{\dim(\tau)} \times \mathcal{T}_\tau\big) \times \Big[[0,1) \times \Big(\bigcup_{\substack{\gamma \in \mathcal{M}_\tau \\ \dim(\gamma) \leq l-1}} \operatorname{int}(\gamma) \times \mathcal{T}^n / (\pi(\delta(\sigma_\gamma)) \times \mathcal{T}_\tau) \Big) \bigcup v \Big)\Big] =$$

$$\big(\mathbb{R}^{\dim(\tau)} \times \mathcal{T}_\tau\big) \times \mathcal{C}\big((\mathcal{L}_\tau)_{\underbrace{2l-1}_{=i}}\big).$$

Note that if $l = 0$ then we have

$$p^{-1}(\mathcal{U}) \cap X_{2\dim(\tau)} = \big(\mathbb{R}^{\dim(\tau)} \times \mathcal{T}_\tau\big) \times v.$$

Consider that $|\mathcal{P}| - |\mathcal{P}^{n-1}|$ is the interior of $|\mathcal{P}|$ and so dense in $|\mathcal{P}|$. This implies that $X_\mathcal{P} - X_{2(n-1)} \cong \mathcal{T}^n \times \operatorname{int}(\mathcal{P})$ is dense in $X_\mathcal{P}$. Now, if $\dim(\tau) = 0$ then $\mathcal{V} \cong \mathbb{R}^{\dim(|\mathcal{P}|)}$ and $\mathcal{U}$ can be chosen as $\mathcal{C}(|\mathcal{M}_\tau|)$. This yields

$$p^{-1}(\mathcal{U}) \cong \mathcal{C}\Big(\bigcup_{\gamma \in \mathcal{M}_\tau} \operatorname{int}(\gamma) \times \mathcal{T}^n / \pi(\delta(\sigma_\gamma)) \Big).$$

We have shown the following proposition.

Proposition 3.11
Let Σ be a complete fan and $\mathcal{P}$ the associated dual polytope. Then $X_\mathcal{P}$, the associated toric varieties with $\mathcal{P}$, is a topological pseudomanifold with a trivial link bundle.

Remark 3.12
Note that $\mathcal{S}^1 \times \mathbb{R} \cong \mathbb{C}^*$. So $p^{-1}(int(\tau)) = (\mathbb{C}^*)^{\dim(\tau)}$ and specially $p^{-1}(int(\mathcal{P})) = (\mathbb{C}^*)^{\dim(|\mathcal{P}|)}$. This implies that there is an algebraic action of $(\mathbb{C}^*)^n$ on $X_\mathcal{P}$ with finitely many orbits. Bear in mind that $p^{-1}(int(\mathcal{P}))$ is dense in $X_\mathcal{P}$, as mentioned earlier. One should also note that orbits are in one-to-one correspondence with the faces of $\mathcal{P}$ and each stratum can be written as disjoint union of finitely many orbits.

Remark 3.13
Given $\tau, \eta \in \mathcal{P}$ with $\dim(\tau) = \dim(\eta)$. Generally, $\mathcal{L}_\tau$ and $\mathcal{L}_\eta$ are not necessarily even homotopically equivalent.

Let us continue with some examples. We start with rather an easy example.

Example 3.14 (n=1)
Consider the complete fan introduced in 2.9. Recall that $|\mathcal{P}| \cong \mathcal{I}$. We want to determine the link of $\{0\} \in \mathcal{I}$. There is only one neighboring higher dimensional face of $\{0\}$, which is the interior of $\mathcal{I}$ considered as the 1-dimensional face of $\mathcal{P}$. Thus, $\mathcal{M}_{\{0\}}$ consists of only one element, a point in $(0,1)$. $p^{-1}(0) = *$, where p again is the projection from the toric variety to its underlying polytope. Hence, $\mathcal{L}_{\{0\}} \cong \mathcal{S}^1$.

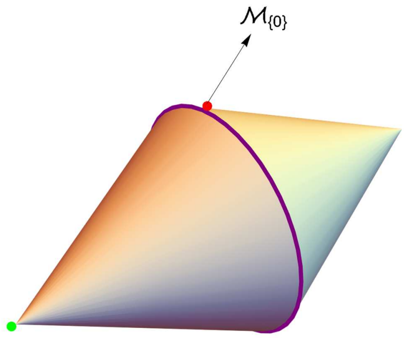

Figure 3.1: Link of the green point which is the product of $|\mathcal{M}_{\{0\}}| \cong *$ with the purple circle.

Example 3.15 (n=2)
Consider the complete fan introduced in 2.10. Let $\tau = \delta^{-1}(\tau_1)$ be the 1-dimensional dual face to τ_1 in $\mathcal{P}$. Consequently, $\mathcal{M}_\tau = \{\{0\}\}$ and $|\mathcal{M}_\tau| \cong *$. Note that $|\mathcal{M}_\tau| \subset int(\mathcal{P})$. Using 3.1 gives us

$$\begin{aligned} \mathcal{M}_\tau \times \mathcal{T}^n &\longrightarrow \mathcal{L}_\tau \\ * \times \mathcal{T}^2 &\longmapsto * \times \ \mathcal{T}^2/\mathcal{S}^1. \end{aligned}$$

Hence, we get $\mathcal{L}_\tau \cong \mathcal{S}^1$.
Now, let $\nu_{12} = \delta^{-1}(\sigma_{12})$ be the dual 0-dimensional face to σ_{12} in $\mathcal{P}$. $\mathcal{M}_{\nu_{12}}$ has 3 elements and $|\mathcal{M}_{\nu_{12}}| \cong \mathcal{I}$ with $int(\mathcal{M}_{\nu_{12}}) \subset int(\mathcal{P})$. We embed $\{0\} \in |\mathcal{M}_\tau|$ in $|\mathcal{P}|$ such that $\{0\} \subset int(\delta^{-1}(\tau_1))$ and similarly $\{1\} \subset int(\delta^{-1}(\tau_2))$. Accordingly, we can describe $\mathcal{L}_{\nu_{12}}$ as below

$$\mathcal{L}_{\nu_{12}} \cong \big(int(\mathcal{I}) \times \mathcal{T}^2\big) \bigcup \big(\{0\} \times \mathcal{T}^2/\pi(\tau_1)\big) \bigcup \big(\{1\} \times \mathcal{T}^2/\pi(\tau_2)\big).$$

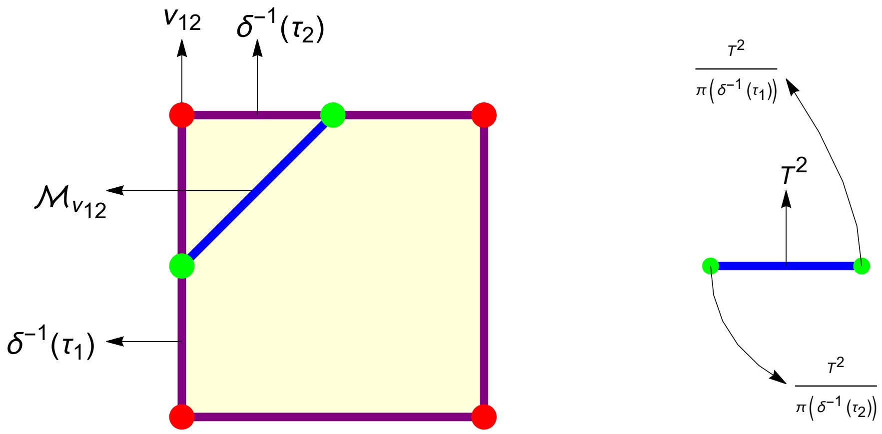

Figure 3.2: Link of ν_{12}.

Remark 3.16
Note that $\mathcal{L}_{\nu_{12}} \ncong \mathcal{S}^1 \times \mathcal{S}^2$. This can be easily deduced from the structure of σ_{12}. However, we can generalize our observation in the following form:

Let ν be a vertex of an 2-dimensional rational convex polytope $\mathcal{P}$ with the dual fan $\Sigma_{\mathcal{P}}$. Then ν has only 3 higher dimensional neighboring faces namely two one-dimensional faces τ and η and the two-dimensional face, $int(\mathcal{P})$. Thus, we have
$$\mathcal{L}_{\nu_{12}} \cong \big(int(\mathcal{I}) \times \mathcal{T}^2\big) \bigcup \big(\{0\} \times \mathcal{T}^2/\pi(\delta(\eta))\big) \bigcup \big(\{1\} \times \mathcal{T}^2/\pi(\delta(\tau))\big).$$
and $\mathcal{L}_{\nu_{12}} \ncong \mathcal{S}^1 \times \mathcal{S}^2$, which comes from the fact that we consider only complete proper fans.

Remark 3.17
Let $\mathcal{P}$ be an n-dimensional rational convex polytope and τ an (n-1)-dimensional face of $\mathcal{P}$. So $\mathcal{M}_\tau = \big\{\{0\}\big\}$, because $int(\mathcal{P})$ is the only higher dimensional neighboring face of τ. This implies that

$$\mathcal{L}_\tau \cong \mathcal{T}^n/\mathcal{T}^{n-1} \cong \mathcal{S}^1.$$

Later, we will use this observation and conclude that:

Let $X_{\mathcal{P}}$ be the toric variety associated to $\mathcal{P}$. Then $X_{\mathcal{P}}$ can not have a singular stratum with co-dimension 2.

Example 3.18 (n=3)
Consider the complete fan shown in 3.3. We define the 1-dimensional cones as follows:

$$\tau_1 = \{x(1,0,1) \mid x \in \mathbb{R}_{\geq 0}\},$$
$$\tau_2 = \{x(0,1,1) \mid x \in \mathbb{R}_{\geq 0}\},$$
$$\tau_3 = \{x(-1,0,1) \mid x \in \mathbb{R}_{\geq 0}\},$$
$$\tau_4 = \{x(0,-1,1) \mid x \in \mathbb{R}_{\geq 0}\},$$
$$\tau_5 = \{x(0,0,-1) \mid x \in \mathbb{R}_{\geq 0}\}.$$

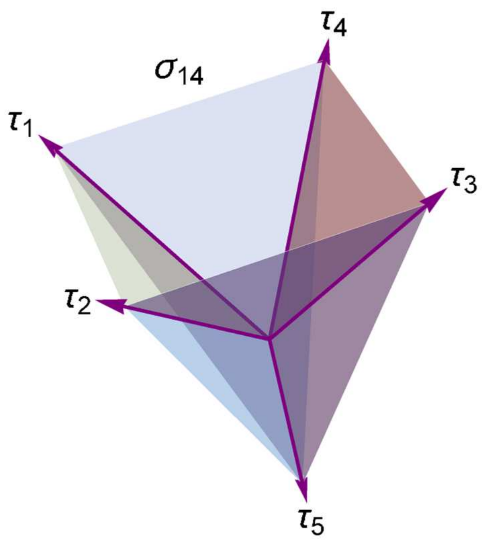

Figure 3.3: A complete fan Σ shown in $\mathbb{R}^3$.

Let σ_{14}, illustrated in the above figure, be the 2-dimensional cone generated by the generators of τ_1 and τ_4. We define the rest of 2-dimensional cones similarly. At last, let ω_{1234} be the 3-dimensional cone which is generated by the generators of τ_1, τ_2, τ_3, and τ_4. ω_{125}, ω_{235}, ω_{345} and ω_{145} are defined similarly. Consequently, we have

$$\Sigma = \{\{0\}, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \sigma_{12}, \sigma_{23}, \sigma_{34}, \sigma_{14}, \omega_{1234}, \omega_{125}, \omega_{235}, \omega_{345}, \omega_{145}\}.$$

The dual polytope to Σ *is then the pyramid shown in* 3.5.

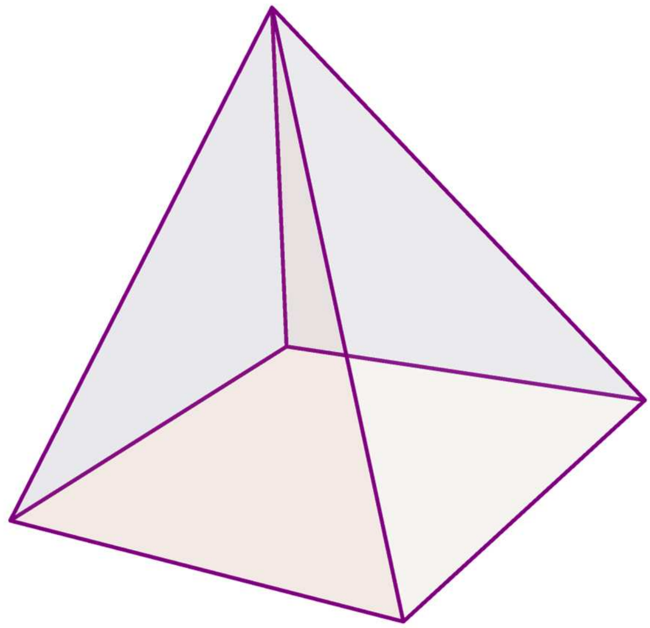

Figure 3.4: The dual polytope of Σ illustrated in 3.3

As we discussed earlier, the link of a 2-dimensional face of $\mathcal{P}_\Sigma$ *is homeomorphic to* $\mathcal{S}^1$. *Now, let* $\eta_{12} = \delta^{-1}(\sigma_{12})$ *be the 1-dimensional face of* $\mathcal{P}$ *which is dual to* σ_{12}. *It is easy to see that* $\mathcal{M}_{\eta_{12}} = \{\gamma_{int(\mathcal{P})}, \gamma_{\delta^{-1}(\tau_1)}, \gamma_{\delta^{-1}(\tau_2)}\}$. *As in the 2-dimensional case we have* $|\mathcal{M}_{\eta_{12}}| \cong \mathcal{I}$, *where* $int(\mathcal{M}_{\eta_{12}}) \subset int(\mathcal{P})$, *and* $\{0\} \subset int(\delta^{-1}(\tau_1))$ *considered as a 0-dimensional face of* $\mathcal{I}$. *Similarly, we have* $\{1\} \subset int(\delta^{-1}(\tau_2))$. *This gives us*

$$\mathcal{L}_{\eta_{12}} \cong \left(int(\mathcal{I}) \times \mathcal{T}^3/(\mathcal{T}^3/\pi(\sigma_{12}))\right) \bigcup \left(\{0\} \times \mathcal{T}^3/(\mathcal{T}^3/\pi(\sigma_{12}) \times \pi(\tau_1))\right) \bigcup \left(\{1\} \times \mathcal{T}^3/(\mathcal{T}^3/\pi(\sigma_{12}) \times \pi(\tau_2))\right).$$

Note that $\mathcal{T}^3/(\mathcal{T}^3/\pi(\sigma_{12})) \cong \mathcal{T}^2$. *Hence, we have*

$$\mathcal{L}_{\eta_{12}} \cong \left(int(\mathcal{I}) \times \mathcal{T}^2\right) \bigcup \left(\{0\} \times \mathcal{T}^2/\pi(\tau_1)\right) \bigcup \left(\{1\} \times \mathcal{T}^2/\pi(\tau_2)\right).$$

With the same argument as in the 2-dimensional case, we can show that $\mathcal{L}_{\eta_{12}} \ncong \mathcal{S}^1 \times \mathcal{S}^2$. *Forthwith, we want to describe link of the point* v *at the apex of the pyramid.* M_v *is a 2-dimensional abstract polytope with four 1-dimensional faces and hence four 0-dimensional faces. Thus, we have*

$$\mathcal{L}_v \cong \left(int(\mathcal{M}_v) \times \mathcal{T}^3\right) \bigcup_{\substack{\gamma_{\tau_i} \in \mathcal{M}_v \\ \dim(\gamma_{\tau_i})=1}} \left(int(\gamma_{\tau_i}) \times \mathcal{T}^3/\pi(\delta(\tau_i))\right) \bigcup_{\substack{\gamma_{\sigma_i} \in \mathcal{M}_v \\ \dim(\gamma_{\sigma_i})=0}} \left(int(\gamma_{\sigma_i}) \times \mathcal{T}^3/\pi(\delta(\sigma_i))\right).$$

Recall that $\mathcal{T}^3/_{\pi(\delta(\sigma_i))} \cong \mathcal{S}^1$ *if* $\dim(\sigma_i) = 1$. *However, it is easy to show that we can not factor out any* $\mathcal{S}^1$ *in* $\mathcal{L}_v$. *The situation has been shown in the figure below:*

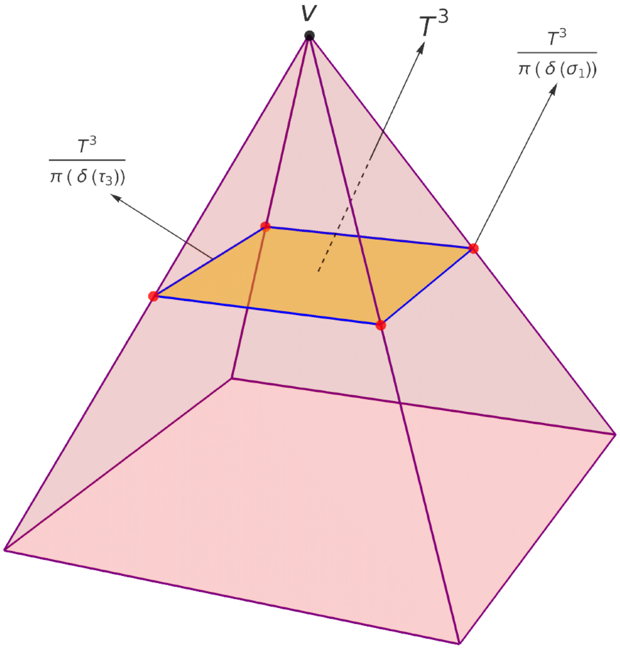

Figure 3.5: Link of the rationally singular point v in X_Σ.

3.2 CW STRUCTURE ON TORIC VARIETIES

In this section, we endow toric varieties with CW structures. Here [6] it has been shown that there is a CW-cell decomposition of toric varieties. However, our primary focus is on real 4- and 6-dimensional toric varieties. The crucial idea is to ensure that each collapse $\mathcal{T}^n/_{\pi(\tau)} \longrightarrow \mathcal{T}^n/_{\pi(\sigma)}$ for $\tau \prec \sigma$ with $\sigma, \tau \in \Sigma$, an n-dimensional complete rational fan, is cellular. We will see that in fact, the collapses are automatically cellular for any 4-dimensional toric varieties. However, a slight modification is needed for 6-dimensional toric varieties. We will in addition give an idea of how this procedure can inductively be used for an arbitrary dimension.

Bear in mind that $\mathcal{T}^n \cong \mathbb{R}^n/_{\mathbb{Z}^n}$. Note also that there is the natural projection $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n/_{\mathbb{Z}^n}$. Let $\sigma \in \Sigma$ be an k-dimensional cone. There is, in addition, the natural projection $\psi : \mathbb{R}^n \longrightarrow \sigma^\perp$, where $\sigma^\perp = \{x \in \mathbb{R}^d | x.y = 0 \ \forall y \in \sigma\}$. Hence, we can write $\mathcal{T}^n/_{\pi(\sigma)} \cong \sigma^\perp/_{\psi(\mathbb{Z}^n)}$. Thus, choosing a periodic CW structure on $\sigma^\perp$ with respect to $\psi(\mathbb{Z}^n)$ induces a finite CW structure on $\mathcal{T}^n/_{\pi(\sigma)}$.

3.2.1 *Real 4-dimensional toric varieties*

Let us start with real 4-dimensional toric varieties. We endow $\mathcal{T}^2$ with the common CW structure with two 1-cells and one 2-cell as follows:

$$\mathcal{T}^2 = e^0 \bigcup_{f=0} \left(e^1_{\mathcal{T}^2_x} \cup e^1_{\mathcal{T}^2_y}\right) \bigcup_{f=0} e^2_{\mathcal{T}^2}$$

induced by the following finite-dimensional CW structure on $\mathbb{R}^2$:
For $\mathbb{R}$ each interval $[n, n+1]$ is considered as a 1-cell and each point $(n) \in \mathbb{R}$ is viewed as a 0-cell, where $n \in \mathbb{Z}$. $\mathbb{R}^2$ is then equipped with the CW structure of $\mathbb{R} \times \mathbb{R}$. Now, let τ be a 1-dimensional cone in Σ, which is generated by $\begin{pmatrix} n \\ m \end{pmatrix}$ where n and m are relative prime.

Note that $\mathbb{R}^2 = \tau \oplus \tau^{\perp}$. Thus, we have then the following decomposition of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (-m) \begin{pmatrix} \frac{-m}{m^2+n^2} \\ \frac{n}{m^2+n^2} \end{pmatrix} + (n) \begin{pmatrix} \frac{n}{m^2+n^2} \\ \frac{m}{m^2+n^2} \end{pmatrix}.$$

Now, consider the following CW structure on $\tau^{\perp}$:
A 1-cell starts at the point $i \cdot \begin{pmatrix} \frac{-m}{m^2+n^2} \\ \frac{n}{m^2+n^2} \end{pmatrix}$ and ends at $(i+1) \cdot \begin{pmatrix} \frac{-m}{m^2+n^2} \\ \frac{n}{m^2+n^2} \end{pmatrix}$ for $i \in \mathbb{Z}$. Each point $i \cdot \begin{pmatrix} \frac{-m}{m^2+n^2} \\ \frac{n}{m^2+n^2} \end{pmatrix}$ considered to be a 0-cell. This CW-cell decomposition of $\tau^{\perp}$ induces a CW structure on $\mathcal{T}^2/\pi(\tau)$. For each $\tau \in \Sigma$ equip $\mathcal{T}^2/\pi(\tau)$ with the above CW structure, which is $\mathcal{T}^2/\pi(\tau) = e^0_\tau \cup e^1_\tau$. Consequently, we have the following chain complex for the toric variety X_Σ:

$$\begin{aligned}
\mathcal{C}_4(X) &= \mathbb{Q}\left\langle e^2_{\mathcal{T}^2} \times e^2_{\mathcal{P}} \right\rangle \\
\mathcal{C}_3(X) &= \mathbb{Q}\left\langle e^1_{\mathcal{T}^2_x} \times e^2_{\mathcal{P}} \right\rangle \bigoplus \mathbb{Q}\left\langle e^1_{\mathcal{T}^2_y} \times e^2_{\mathcal{P}} \right\rangle \\
\mathcal{C}_2(X) &= \mathbb{Q}\left\langle e^0_{\mathcal{T}^2} \times e^2_{\mathcal{P}} \right\rangle \bigoplus_{\substack{\tau_i \in \Sigma \\ \dim(\tau_i)=1}} \mathbb{Q}\left\langle e^1_{\tau_i} \times e^1_{\mathcal{P}_i} \right\rangle \\
\mathcal{C}_1(X) &= \bigoplus_{\substack{\tau_i \in \Sigma \\ \dim(\tau_i)=1}} \mathbb{Q}\left\langle e^0_{\tau_i} \times e^1_{\mathcal{P}_i} \right\rangle \\
\mathcal{C}_0(X) &= \bigoplus_{\substack{\sigma_i \in \Sigma \\ \dim(\sigma_i)=2}} \mathbb{Q}\left\langle e^0_{\sigma_i} \times e^0_{\mathcal{P}_i} \right\rangle .
\end{aligned}$$

Bear in mind that $p^{-1}(\mathrm{int}(\delta^{-1}(\tau_i))) \cong \mathcal{T}^2/\pi(\tau_i) \times \mathrm{int}(\delta^{-1}(\tau_i))$ and $\mathrm{int}(\tau_i)$ represents the interior of an 1-cell, denoted by $e^1_{\mathcal{P}_i}$ in the above CW-cell decomposition.
It remains to determine the boundary operators in the above CW structure. Similarly, we can write $\begin{pmatrix}0\\1\end{pmatrix}$ in the following form

$$\begin{pmatrix}0\\1\end{pmatrix} = (n)\begin{pmatrix}\frac{-m}{m^2+n^2}\\ \frac{n}{m^2+n^2}\end{pmatrix} + (m)\begin{pmatrix}\frac{n}{m^2+n^2}\\ \frac{m}{m^2+n^2}\end{pmatrix}.$$

Let τ_i be an 1-dimensional cone in Σ with $\begin{pmatrix}m_i\\n_i\end{pmatrix}$ as the generator. In consequence, we have

$$\partial(e^1_{\mathcal{T}^2_x} \times e^2_{\mathcal{P}}) = \sum_i n_i(e^1_{\tau_i} \times e^1_{\mathcal{P}_i})$$
$$\partial(e^1_{\mathcal{T}^2_y} \times e^2_{\mathcal{P}}) = \sum_i -m_i(e^1_{\tau_i} \times e^1_{\mathcal{P}_i}).$$

For the sake of simplicity, we introduce the following notation that we will use for some of our matrix representations. Let $\mathbf{A}$ be an $lp \times kh$ matrix where $l, p, k, h \in \mathbb{N}$. For our purposes, it is sometimes practical to study only a specific part of a matrix. Let us now consider $\mathbf{A}$ as an $l \times k$ matrix where each element (or block) of $\mathbf{A}$ is an $p \times h$ matrix. Note that from the context it should also be clear how one can obtain the rest of the matrix from an arbitrary block. Then we represent $\mathbf{A}$ as

$$\mathbf{A} = \left(l \left\{ \overbrace{\left\{ \qquad \right\}}^{k} \right. \right)_{p\times h}.$$

Example 3.19
Let $\mathbf{A}$ be an $k3 \times 3$ matrix where the i-th 3×3 block of $\mathbf{A}$ has the following form

$$\begin{pmatrix} n_i & 0 & 0\\ 0 & m_i & 0\\ 0 & 0 & l_i \end{pmatrix} \quad \text{for } i = 0, \ldots, k$$

, where n_i, m_i and l_i are non-zero and relatively prime. Thus, with the previous notation, we can write

$$\mathbf{A} = \left(k \left\{ \begin{matrix} n_i & 0 & 0\\ 0 & m_i & 0\\ 0 & 0 & l_i \end{matrix} \right\} \right).$$

Furthermore, if we consider $\mathbf{A}$ *as a map from* $\mathbb{Q}^3$ *to* $\mathbb{Q}^{3k}$ *we have*

$$rk(ker(\mathbf{A})) = 0$$
$$rk(Im(\mathbf{A})) = 3 \;\forall k \in \mathbb{N}.$$

With the above notation, the boundary operators can be written as follows

$$\partial_4 = \begin{array}{c} \\ e^1_{\mathcal{T}^2_x} \times e^2_{\mathcal{P}} \\ e^1_{\mathcal{T}^2_y} \times e^2_{\mathcal{P}} \end{array} \begin{pmatrix} e^2_{\mathcal{T}^2} \times e^2_{\mathcal{P}} \\ 0 \\ 0 \end{pmatrix}$$

$$\partial_3 = \begin{array}{c} \\ e^0_{\mathcal{T}^2} \times e^2_{\mathcal{P}} \\ e^1_{\tau_i} \times e^1_{\mathcal{P}_i} \end{array} \begin{pmatrix} & e^1_{\mathcal{T}^2_x} \times e^2_{\mathcal{P}} & e^1_{\mathcal{T}^2_y} \times e^2_{\mathcal{P}} & \\ & 0 & 0 & \\ k\{ & -m_i & n_i & \} \end{pmatrix} \text{ for } i = 1, \ldots, k,$$

where k denotes the number of 1-dimensional cones in Σ and $\begin{pmatrix} m_i \\ n_i \end{pmatrix}$ is the generator of $\tau_i \in \Sigma$ with $\dim(\tau_i) = 1$. Note that $k \geq 3$ and $\begin{pmatrix} m_i \\ n_i \end{pmatrix}$ are linearly independent. This implies

$$\text{rk}(\ker(\partial_3)) = 0$$
$$\text{rk}(\text{im}(\partial_3)) = 2.$$

Similarly, we get

$$\partial_2 = \begin{array}{c} \\ e^0_{\tau_i} \times e^1_{\mathcal{P}_i} \end{array} \begin{pmatrix} & e^0_{\mathcal{T}^2} \times e^2_{\mathcal{P}} & \overbrace{e^1_{\tau_i} \times e^1_{\mathcal{P}_i}}^{k} & \\ k\{ & 1 & 0 & \} \end{pmatrix}.$$

Accordingly, we have

$$\text{rk}(\ker(\partial_2)) = k$$
$$\text{rk}(\text{im}(\partial_2)) = 1.$$

Bear in mind that the explicit form of ∂_2 depends on the chosen orientation of $|\mathcal{P}|$. However, ker and Im of ∂_2 are independent of the chosen orientation. ∂_1 is simply $\partial_1^{\mathcal{P}}$ which is the boundary operator of the regular CW structure on $|\mathcal{P}|$. Hence, we have

$$\text{rk}(\ker(\partial_1)) = 1$$
$$\text{rk}(\text{im}(\partial_1)) = k - 1.$$

We have proven the following proposition.

Proposition 3.20
Let Σ be a complete 2-dimensional fan and f_1 denotes the number of 1-dimensional cones in Σ. Then we have

$$\begin{aligned}
&rk(\mathbf{H}_4(X_\Sigma;\mathbb{Q})) = 1,\\
&rk(\mathbf{H}_3(X_\Sigma;\mathbb{Q})) = 0,\\
&rk(\mathbf{H}_2(X_\Sigma;\mathbb{Q})) = f_1 - 2,\\
&rk(\mathbf{H}_1(X_\Sigma;\mathbb{Q})) = 0,\\
&rk(\mathbf{H}_0(X_\Sigma;\mathbb{Q})) = 1.
\end{aligned}$$

Corollary 3.21
X_Σ is a rational homology manifold, i.e. the associated Betti numbers satisfy the Poincare duality.

At this point, we can compute the homology groups of the link of the point $x \in (X_\Sigma)_0$. Let σ_x be the dual cone to x in Σ. Let τ_1 and τ_2 be the 1-dimensional cones with $\tau_1, \tau_2 \prec \sigma$. Then, we have the following chain complex for the link of x, $\mathcal{L}_x$.

$$\begin{aligned}
&\mathcal{C}_3(\mathcal{L}_x) = \mathbb{Q}\left\langle e^1_{\mathcal{I}} \times e^2_{\mathcal{T}^2}\right\rangle,\\
&\mathcal{C}_2(\mathcal{L}_x) = \mathbb{Q}\left\langle e^1_{\mathcal{I}} \times e^1_{\mathcal{T}^2_x}\right\rangle \bigoplus \mathbb{Q}\left\langle e^1_{\mathcal{I}} \times e^1_{\mathcal{T}^2_y}\right\rangle,\\
&\mathcal{C}_1(\mathcal{L}_x) = \mathbb{Q}\left\langle e^1_{\mathcal{I}} \times e^0_{\mathcal{T}^2}\right\rangle \bigoplus \mathbb{Q}\left\langle e^0_{\mathcal{I}_0} \times e^1_{\mathcal{S}^1_0}\right\rangle \bigoplus \mathbb{Q}\left\langle e^0_{\mathcal{I}_1} \times e^1_{\mathcal{S}^1_1}\right\rangle,\\
&\mathcal{C}_0(\mathcal{L}_x) = \mathbb{Q}\left\langle e^0_{\mathcal{I}_0} \times e^0_{\mathcal{S}^1_0}\right\rangle \bigoplus \mathbb{Q}\left\langle e^0_{\mathcal{I}_1} \times e^0_{\mathcal{S}^1_1}\right\rangle.
\end{aligned}$$

We get the following boundary operators

$$\partial_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \partial_2 = \begin{pmatrix} 0 & 0 \\ -m_1 & n_1 \\ -m_2 & n_2 \end{pmatrix},\ \partial_1 = \begin{pmatrix} +1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{3.9}$$

where $\begin{pmatrix} m_1 \\ n_1 \end{pmatrix}$ and $\begin{pmatrix} m_2 \\ n_2 \end{pmatrix}$ are the generators of τ_1 and τ_2, respectively. Hence, we have

$$\begin{aligned}
&\mathrm{rk}(\mathbf{H}_3(\mathcal{L}_x;\mathbb{Q})) = 1,\\
&\mathrm{rk}(\mathbf{H}_2(\mathcal{L}_x;\mathbb{Q})) = 0,\\
&\mathrm{rk}(\mathbf{H}_1(\mathcal{L}_x;\mathbb{Q})) = 0,\\
&\mathrm{rk}(\mathbf{H}_0(\mathcal{L}_x;\mathbb{Q})) = 1.
\end{aligned}$$

As expected, $\mathcal{L}_x$ is a rational homology sphere. But at this point, it is also worthwhile to study the homology groups of $\mathcal{L}_x$ with integral coefficient. It is easy to see that $\mathbf{H}_1(\mathcal{L}_x;\mathbb{Z}) = 0$ if and only if $\det \begin{pmatrix} -m_1 & n_1 \\ -m_2 & n_2 \end{pmatrix} = \pm 1$. Thus, $\mathcal{L}_x$ is a homological sphere if the later condition holds. This means that X_σ is smooth if the generators of all $\sigma \in \Sigma$ with $\dim(\sigma) = 2$ satisfy the previous condition. However, there is also an algebraic description of the singularities of toric varieties. For example in theorem (1.3.12) in [5], a toric variety X_σ is smooth if and only if the minimal generators of $\sigma \subset \mathcal{N} \bigotimes \mathbb{R}$ form a part of an $\mathbb{Z}$-basis of the lattice $\mathcal{N}$ for each $\sigma \in \Sigma$. Note that for real 4-dimensional toric varieties, this translates to the same condition that we obtained from our topological approach.

In summary, we found out that for any complete 2-dimensional fan the associated toric variety X_Σ satisfies the Poincaré duality rationally. In other words, there is no need to employ the theory of intersection spaces. For a point $x \in (X_\Sigma)_0$ to be smooth, we require the link of x to be a homological sphere. This means that the minimal generators of the dual cone to x form an $\mathbb{Z}$-basis of the lattice $\mathcal{N}$. We obtain the same result as the algebraic geometric approach to the matter.

3.2.2 *Real 6-dimensional toric varieties*

In this section, we give a CW-Cell decomposition of 6-dimensional toric varieties. As mentioned earlier we need a slight modification on the common CW structure of each $\mathcal{T}^2$ here. The goal is to ensure that each collapse $\mathcal{T}^3/\pi(\tau) \longrightarrow \mathcal{T}^3/\pi(\sigma)$, where $\tau \prec \sigma$ with $\sigma, \tau \in \Sigma$, is cellular. Here again, Σ is a complete fan.

Now let τ be a 1-dimensional cone in Σ which is generated by $\begin{pmatrix} n \\ m \\ l \end{pmatrix}$, where $\gcd(n, m, l) = 1$. Thus, we have $\tau = \operatorname{Span}\begin{pmatrix} n \\ m \\ l \end{pmatrix}$ and $\tau^\perp = \operatorname{Span}\left(\begin{pmatrix} -m \\ n \\ 0 \end{pmatrix}, \begin{pmatrix} -l \\ 0 \\ n \end{pmatrix}\right)$. This gives us the following decomposition of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (-m)\begin{pmatrix} \frac{-m}{\Delta} \\ \frac{n}{\Delta} \\ 0 \end{pmatrix} + (-l)\begin{pmatrix} \frac{-l}{\Delta} \\ 0 \\ \frac{n}{\Delta} \end{pmatrix} + (n)\begin{pmatrix} \frac{n}{\Delta} \\ \frac{m}{\Delta} \\ \frac{l}{\Delta} \end{pmatrix},$$

where $\Delta = l^2 + n^2 + m^2$. We can write the above decomposition as

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = (lm)\begin{pmatrix}\frac{m^2+l^2}{lm}\frac{1}{\Delta}\\ -\frac{mn}{lm}\frac{1}{\Delta}\\ -\frac{nl}{ml}\frac{1}{\Delta}\end{pmatrix} + (n)\begin{pmatrix}\frac{n}{\Delta}\\ \frac{m}{\Delta}\\ \frac{l}{\Delta}\end{pmatrix}.$$

Similarly, we have

$$\begin{pmatrix}0\\1\\0\end{pmatrix} = (nl)\begin{pmatrix}-\frac{nm}{nl}\frac{1}{\Delta}\\ \frac{n^2+l^2}{nl}\frac{1}{\Delta}\\ -\frac{ml}{nl}\frac{1}{\Delta}\end{pmatrix} + (m)\begin{pmatrix}\frac{n}{\Delta}\\ \frac{m}{\Delta}\\ \frac{l}{\Delta}\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = (nm)\begin{pmatrix}-\frac{nl}{nm}\frac{1}{\Delta}\\ -\frac{ml}{nm}\frac{1}{\Delta}\\ \frac{n^2+m^2}{nm}\frac{1}{\Delta}\end{pmatrix} + (l)\begin{pmatrix}\frac{n}{\Delta}\\ \frac{m}{\Delta}\\ \frac{l}{\Delta}\end{pmatrix}.$$

With these in hand, we endow $\tau^\perp$ with the following CW structure. $i\begin{pmatrix}\frac{m^2+l^2}{lm}\frac{1}{\Delta}\\ -\frac{mn}{lm}\frac{1}{\Delta}\\ -\frac{nl}{ml}\frac{1}{\Delta}\end{pmatrix}$, $i\begin{pmatrix}-\frac{nm}{nl}\frac{1}{\Delta}\\ \frac{n^2+l^2}{nl}\frac{1}{\Delta}\\ -\frac{ml}{nl}\frac{1}{\Delta}\end{pmatrix}$, and $i\begin{pmatrix}-\frac{nl}{nm}\frac{1}{\Delta}\\ -\frac{ml}{nm}\frac{1}{\Delta}\\ \frac{n^2+m^2}{nm}\frac{1}{\Delta}\end{pmatrix}$ are considered to be 0-cells where $i \in \mathbb{Z}$. A 1-cell starts at $i\begin{pmatrix}\frac{m^2+l^2}{lm}\frac{1}{\Delta}\\ -\frac{mn}{lm}\frac{1}{\Delta}\\ -\frac{nl}{ml}\frac{1}{\Delta}\end{pmatrix}$ and ends at $(i+1)\begin{pmatrix}\frac{m^2+l^2}{lm}\frac{1}{\Delta}\\ -\frac{mn}{lm}\frac{1}{\Delta}\\ -\frac{nl}{ml}\frac{1}{\Delta}\end{pmatrix}$ (similarly for $\begin{pmatrix}-\frac{nm}{nl}\frac{1}{\Delta}\\ \frac{n^2+l^2}{nl}\frac{1}{\Delta}\\ -\frac{ml}{nl}\frac{1}{\Delta}\end{pmatrix}$ and $\begin{pmatrix}-\frac{nl}{nm}\frac{1}{\Delta}\\ -\frac{ml}{nm}\frac{1}{\Delta}\\ \frac{n^2+m^2}{nm}\frac{1}{\Delta}\end{pmatrix}$). Note that

$$\begin{pmatrix}\frac{m^2+l^2}{lm}\frac{1}{\Delta}\\ -\frac{mn}{lm}\frac{1}{\Delta}\\ -\frac{nl}{ml}\frac{1}{\Delta}\end{pmatrix} + \begin{pmatrix}-\frac{nm}{nl}\frac{1}{\Delta}\\ \frac{n^2+l^2}{nl}\frac{1}{\Delta}\\ -\frac{ml}{nl}\frac{1}{\Delta}\end{pmatrix} + \begin{pmatrix}-\frac{nl}{nm}\frac{1}{\Delta}\\ -\frac{ml}{nm}\frac{1}{\Delta}\\ \frac{n^2+m^2}{nm}\frac{1}{\Delta}\end{pmatrix} = 0.$$

The above relation ensures that we can form two triangles with the above vectors. We consider these triangles as 2-cells of $\tau^\perp$. From the construction, it is clear that the previous CW structure is periodic concerning $\psi(\mathbb{Z}^3)$, where ψ denotes the natural projection from $\mathbb{R}^3$ onto $\tau^\perp$. The induced CW structure on $\mathcal{T}^3/\phi(\tau) \cong \tau^\perp/\psi(\mathbb{Z}^3) \cong \mathcal{T}^2$ can be described as follows.

$$\mathcal{T}^2 = (e^2_{\mathcal{T}^2_1} \cup e^2_{\mathcal{T}^2_2}) \bigcup (e^1_{\mathcal{T}^2_x} \cup e^1_{\mathcal{T}^2_y} \cup e^1_{\mathcal{T}^2_z}) \bigcup e^0_{\mathcal{T}^2}. \tag{3.10}$$

With an appropriate orientation we have the following boundary operators.

$$\partial_2^{\mathcal{T}^2} = \begin{pmatrix}-1 & 1\\ -1 & 1\\ -1 & 1\end{pmatrix},\ \partial_1^{\mathcal{T}^2} = \begin{pmatrix}0 & 0 & 0\end{pmatrix}.$$

Schematically we have

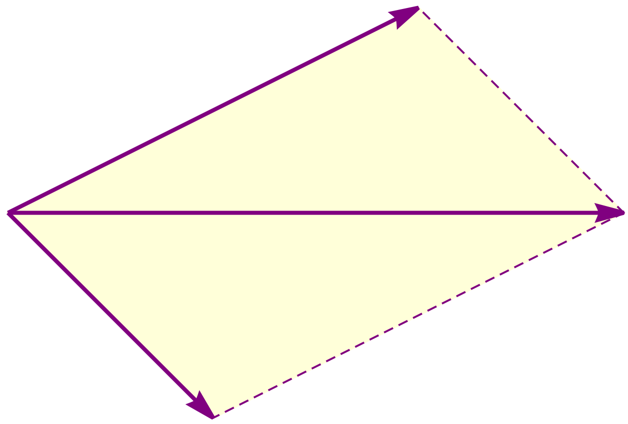

Figure 3.6: CW structure on $\mathcal{T}^2$ with two 2-Cells. Note that we identify the opposite outer sides as in the common CW structure on $\mathcal{T}^2$.

There is yet another case that we need to consider and study. Without loss of generality assume that $\tau = \text{span}\begin{pmatrix} n \\ m \\ 0 \end{pmatrix}$. This yields

$$\tau^{\perp} = \text{span}\left(\begin{pmatrix} -m \\ n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Applying the previous construction induces the following CW structure on $\mathcal{T}^3/\pi(\tau) \cong \mathcal{T}^2$.

$$\mathcal{T}^2 = e^2_{\mathcal{T}^2} \bigcup (e^1_{\mathcal{T}^2_z} \cup e^1_{\mathcal{T}^2_y}) \bigcup e^0_{\mathcal{T}^2},$$

which is the common CW structure on $\mathcal{T}^2$ with the well-known boundary operators. From the previous section, we know that other collapses are automatically cellular. Thus, it is enough to equip each $\mathcal{T}^3/\phi(\tau) \cong \mathcal{T}^2$ with the appropriate CW structure. To determine the boundary operators of X_Σ associated with this CW structure, one could also go further and investigate the collapses $\mathcal{T}^3/\phi(\tau) \longrightarrow \mathcal{T}^3/\phi(\sigma)$, such that $\tau \prec \sigma$, $\dim(\tau) = 1$, $\dim(\sigma) = 2$ and $\tau, \sigma \in \Sigma$. But as it turns out, we don't need these collapsing data explicitly for our purposes.

At last, we aim to compute the homology groups of an arbitrary 6-dimensional toric variety.

Example 3.22
Let Σ be a complete fan and $\mathcal{P}$ the associated dual polytope to Σ. Let f_2, f_1 and f_3 denote the

number of 2-dimensional, 1-dimensional and 3-dimensional cones of Σ, respectively. With the above considerations, we can endow $X_{\mathcal{P}}$, the associated toric variety to $\mathcal{P}$, with the following CW structure:

$$
\begin{aligned}
\mathcal{C}_6(X_{\mathcal{P}}) &= \mathbb{Q}\,\langle e^3_{\mathcal{T}^3} \times e^3_{\mathcal{P}} \rangle \\
\mathcal{C}_5(X_{\mathcal{P}}) &= \bigoplus_{i=1}^{3} \mathbb{Q}\,\langle e^2_{\mathcal{T}^3_i} \times e^3_{\mathcal{P}} \rangle \\
\mathcal{C}_4(X_{\mathcal{P}}) &= \bigoplus_{i=1}^{3} \mathbb{Q}\,\langle e^1_{\mathcal{T}^3_i} \times e^3_{\mathcal{P}} \rangle \bigoplus_{i=1}^{\gamma} \left(\mathbb{Q}\,\langle e^2_{(\mathcal{T}^2_{\gamma_i})_1} \times e^2_{\mathcal{P}_{\gamma_i}} \rangle \oplus \mathbb{Q}\,\langle e^2_{(\mathcal{T}^2_{\gamma_i})_2} \times e^2_{\mathcal{P}_{\gamma_i}} \rangle \right) \\
&\quad \bigoplus_{j=1}^{\omega} \mathbb{Q}\,\langle e^2_{\mathcal{T}^2_{\omega_j}} \times e^2_{\mathcal{P}_{\omega_j}} \rangle \\
\mathcal{C}_3(X_{\mathcal{P}}) &= \mathbb{Q}\,\langle e^0_{\mathcal{T}^3} \times e^3_{\mathcal{P}} \rangle \bigoplus_{i=1}^{\gamma} \bigoplus_{l=1}^{3} \mathbb{Q}\,\langle e^1_{(\mathcal{T}^2_{\gamma_i})_l} \times e^2_{\mathcal{P}_{\gamma_i}} \rangle \bigoplus_{j=1}^{\omega} \bigoplus_{l=1}^{2} \mathbb{Q}\,\langle e^1_{(\mathcal{T}^2_{\omega_j})_l} \times e^2_{\mathcal{P}_{\omega_j}} \rangle \\
\mathcal{C}_2(X_{\mathcal{P}}) &= \bigoplus_{i=1}^{\gamma} \mathbb{Q}\,\langle e^0_{(\mathcal{T}^2_{\gamma_i})} \times e^2_{\mathcal{P}_{\gamma_i}} \rangle \bigoplus_{j=1}^{\omega} \mathbb{Q}\,\langle e^0_{(\mathcal{T}^2_{\omega_j})} \times e^1_{\mathcal{P}_{\omega_j}} \rangle \bigoplus_{l=1}^{f_2} \mathbb{Q}\,\langle e^1_{\mathcal{S}^1_l} \times e^1_{\mathcal{P}_l} \rangle \\
\mathcal{C}_1(X_{\mathcal{P}}) &= \bigoplus_{i=0}^{f_1} \mathbb{Q}\,\langle e^0_{\mathcal{S}^1_i} \times e^1_{\mathcal{P}_i} \rangle \\
\mathcal{C}_0(X_{\mathcal{P}}) &= \bigoplus_{i=0}^{f_3} \mathbb{Q}\,\langle e^0_{\mathcal{P}_i} \rangle \,,
\end{aligned}
$$

, where γ and ω are the number of 2-dimensional tori with three and two 1-cells, respectively. $e^2_{\gamma_i}$ is the 2-cell of $\mathcal{P}$, endowed with the regular CW structure, which is attached to $\mathcal{T}^2_{\gamma_i}$, where $i = 1, ..., \gamma$, and similarly for $e^2_{\mathcal{P}_{\omega_j}}$. Hence, we have $\gamma + \omega = f_1$

Let $\begin{pmatrix} n_{\gamma_i} \\ m_{\gamma_i} \\ l_{\gamma_i} \end{pmatrix}$ and $\begin{pmatrix} n_{\omega_i} \\ m_{\omega_i} \\ 0 \end{pmatrix}$, with $i = 1, ..., \gamma$ and $j = 1, ..., \omega$, be the generator of 1-dimensional cones in Σ dual to $e^2_{\mathcal{P}_{\gamma_i}}$ and $e^2_{\mathcal{P}_{\omega_j}}$, respectively, where we consider the 2-cells as 2-dimensional faces of $\mathcal{P}$. Note that $\omega + \gamma \geq 4$. Hence, we get the following boundary operators.

$$
\partial_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$\partial_5 = \begin{array}{r} \\ e^1_{\mathcal{T}^3_1}\times e^3_{\mathcal{P}} \\ e^1_{\mathcal{T}^3_2}\times e^3_{\mathcal{P}} \\ e^1_{\mathcal{T}^3_3}\times e^3_{\mathcal{P}} \\ e^2_{(\mathcal{T}^2_{\gamma_i})_1}\times e^2_{\mathcal{P}_{\gamma_i}}\ \gamma\{ \\ e^2_{(\mathcal{T}^2_{\gamma_i})_2}\times e^2_{\mathcal{P}_{\gamma_i}}\ \gamma\{ \\ e^2_{\mathcal{T}^2_{\omega_j}}\times e^2_{\mathcal{P}_{\omega_j}}\ \omega\{ \end{array} \begin{array}{ccc} e^2_{\mathcal{T}^3_1}\times e^3_{\mathcal{P}} & e^2_{\mathcal{T}^3_2}\times e^3_{\mathcal{P}} & e^2_{\mathcal{T}^3_3}\times e^2_{\mathcal{P}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ m_{\gamma_i}n_{\gamma_i}l^2_{\gamma_i} & m^2_{\gamma_i}n_{\gamma_i}l_{\gamma_i} & 0 \\ 0 & m^2_{\gamma_i}n_{\gamma_i}l_{\gamma_i} & m_{\gamma_i}n^2_{\gamma_i}l_{\gamma_i} \\ 0 & -m_{\omega_j} & n_{\omega_j} \end{array}$$

$$\partial_4 = \begin{array}{r} \\ e^0_{\mathcal{T}^3}\times e^3_{\mathcal{P}} \\ e^1_{(\mathcal{T}^2_{\gamma_i})_1}\times e^2_{\mathcal{P}_{\gamma_i}}\ \gamma\{ \\ e^1_{(\mathcal{T}^2_{\gamma_i})_2}\times e^2_{\mathcal{P}_{\gamma_i}}\ \gamma\{ \\ e^1_{(\mathcal{T}^2_{\gamma_i})_3}\times e^2_{\mathcal{P}_{\gamma_i}}\ \gamma\{ \\ e^1_{(\mathcal{T}^2_{\omega_j})_1}\times e^2_{\mathcal{P}_{\omega_j}}\ \omega\{ \\ e^1_{(\mathcal{T}^2_{\omega_j})_2}\times e^2_{\mathcal{P}_{\omega_j}}\ \omega\{ \end{array} \begin{array}{cccccc} e^1_{\mathcal{T}^3_1}\times e^3_{\mathcal{P}} & e^1_{\mathcal{T}^3_2}\times e^3_{\mathcal{P}} & e^1_{\mathcal{T}^3_3}\times e^3_{\mathcal{P}} & e^2_{(\mathcal{T}^2_{\gamma_i})_1}\times e^2_{\mathcal{P}_{\gamma_i}} & e^2_{(\mathcal{T}^2_{\gamma_i})_2}\times e^2_{\mathcal{P}_{\gamma_i}} & e^2_{\mathcal{T}^2_{\omega_j}}\times e^2_{\mathcal{P}_{\omega_j}} \\ 0 & 0 & 0 & \overbrace{0}^{\gamma} & \overbrace{0}^{\gamma} & \overbrace{0}^{\omega} \\ m_{\gamma_i}l_{\gamma_i} & 0 & 0 & 1 & -1 & 0 \\ 0 & n_{\gamma_i}l_{\gamma_i} & 0 & 1 & -1 & 0 \\ 0 & 0 & n_{\gamma_i}m_{\gamma_i} & 1 & -1 & 0 \\ -m_{\omega_j} & 0 & 0 & 0 & 0 & 0 \\ 0 & n_{\omega_j} & 0 & 0 & 0 & 0 \end{array}$$

$$\partial_3 = \begin{array}{r} \\ e^0_{\mathcal{T}^2_{\gamma_i}}\times e^2_{\mathcal{P}_{\gamma_i}}\ \gamma\{ \\ e^0_{\mathcal{T}^2_{\omega_j}}\times e^2_{\mathcal{P}_{\omega_j}}\ \omega\{ \\ e^1_{\mathcal{S}^1_i}\times e^1_{\mathcal{P}_l}\ f_2\{ \end{array} \begin{array}{cccccc} e^0_{\mathcal{T}^3_1}\times e^3_{\mathcal{P}} & e^1_{(\mathcal{T}^2_{\gamma_i})_1}\times e^2_{\mathcal{P}_{\gamma_i}} & e^1_{(\mathcal{T}^2_{\gamma_i})_2}\times e^2_{\mathcal{P}_{\gamma_i}} & e^1_{(\mathcal{T}^2_{\gamma_i})_3}\times e^2_{\mathcal{P}_{\gamma_i}} & e^1_{(\mathcal{T}^2_{\omega_j})_1}\times e^2_{\mathcal{P}_{\omega_j}} & e^1_{(\mathcal{T}^2_{\omega_j})_2}\times e^2_{\mathcal{P}_{\omega_j}} \\ 1 & \overbrace{0}^{\gamma} & \overbrace{0}^{\gamma} & \overbrace{0}^{\gamma} & \overbrace{0}^{\omega} & \overbrace{0}^{\omega} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{C}_1 & \mathsf{C}_2 & \mathsf{C}_3 & \mathsf{C}_4 & \mathsf{C}_5 \end{array}$$

$$\partial_2 = \begin{array}{r} \\ e^0_{\mathcal{S}^1_l}\times e^1_{\mathcal{P}_l}\ f_2\{ \end{array} \begin{array}{ccc} e^0_{\mathcal{T}^2_{\gamma_i}}\times e^2_{\mathcal{P}_{\gamma_i}} & e^0_{\mathcal{T}^2_{\omega_j}}\times e^2_{\mathcal{P}_{\omega_j}} & e^1_{\mathcal{S}^1_l}\times e^1_{\mathcal{P}_l} \\ \overbrace{\mathsf{P}_1}^{\gamma} & \overbrace{\mathsf{P}_2}^{\omega} & \overbrace{0}^{f_2} \end{array}$$

, where C_i, $i = 1, ..., 5$ are collapsing data, that we do not need explicitly. P_i, $i = 1, 2$ are also purely determined by the regular CW structure on $\mathcal{P}$. Then, one can easily deduce that

$$rk(\mathbf{H}_6(X_{\mathcal{P}};\mathbb{Q})) = 1,$$
$$rk(\mathbf{H}_5(X_{\mathcal{P}};\mathbb{Q})) = 0.$$

Moreover, we have

$$\begin{aligned} rk(Im(\partial_5)) &= 3, \\ rk(ker(\partial_4)) &= \gamma + \omega, \\ rk(Im(\partial_4)) &= \gamma + 3, \\ rk(ker(\partial_2)) &= rk(ker(\partial_2^{\mathcal{P}})) + f_2, \\ rk(Im(\partial_2)) &= rk(Im(\partial_2^{\mathcal{P}})) \\ \partial_1 &= \partial_1^{\mathcal{P}} \end{aligned}$$

, where $\partial_i^{\mathcal{P}}$ denote the boundary operators of $\mathcal{P} \cong \mathcal{D}^3$. Thus, we have

$$rk(\mathbf{H}_4(X_{\mathcal{P}};\mathbb{Q})) = f_1 - 3$$

Taking the Poincare duality into account, one can argue that

$$rk(\mathbf{H}_2(X_{\mathcal{P}};\mathbb{Q})) = f_1 - 3 - b$$

, where b for non-singular 6-dimensional toric variety is zero. Thus, we can simply show that

$$\begin{aligned} rk(Im(\partial_3)) &= -(f_1 - f_2 - b - 3) + rk(Im(\partial_3^{\mathcal{P}})), \\ rk(ker(\partial_3)) &= (f_1 - f_2 - b - 3) + (3\gamma + 2\omega). \end{aligned}$$

Finally, we arrive at

$$rk(\mathbf{H}_3(X_{\mathcal{P}};\mathbb{Q})) = 3f_1 - f_2 - b - 6.$$

Remark 3.23
There is yet another way of the determination of the parameter b. By looking at ∂^3, we can write

$$rk(ker(\partial_3)) = 3\gamma + 2\omega - f_2 + b' + rk(ker(\partial_3^{\mathcal{P}})),$$

where b' determines the number of rows in ∂^3, which are linearly dependent. This implies

$$rk(Im(\partial_3)) = f_2 - b' + rk(Im(\partial_3^{\mathcal{P}}))$$

Hence, we get

$$rk(\mathbf{H}_2(X_{\mathcal{P}};\mathbb{Q})) = b'.$$

Using the Poincare duality for non-singular toric varieties, we get $b' = f_1 - 3$. Thus, we reparametrize b' as follows.

$$b' = f_1 - 3 - b,$$

where obviously $b = 0$ for non-singular toric varieties. This yields

$$rk(\mathbf{H}_3(X_{\mathcal{P}};\mathbb{Q})) = 3f_1 - f_2 - b - 6.$$

The previous considerations lead to the following proposition.

Proposition 3.24
Let $X_{\mathcal{P}}$ be a 6-dimensional toric variety associated to a complete rational fan Σ, which is dual to the polytope $\mathcal{P}$. Let f_2 and f_1 be the number of 2- and 1-dimensional cones in Σ, respectively. Then, we have

$$\begin{aligned}
rk(\mathbf{H}_6(X_{\mathcal{P}};\mathbb{Q})) &= 1,\\
rk(\mathbf{H}_5(X_{\mathcal{P}};\mathbb{Q})) &= 0,\\
rk(\mathbf{H}_4(X_{\mathcal{P}};\mathbb{Q})) &= f_1 - 3,\\
rk(\mathbf{H}_3(X_{\mathcal{P}};\mathbb{Q})) &= 3f_1 - f_2 - b - 6,\\
rk(\mathbf{H}_2(X_{\mathcal{P}};\mathbb{Q})) &= f_1 - 3 - b,\\
rk(\mathbf{H}_1(X_{\mathcal{P}};\mathbb{Q})) &= 0,\\
rk(\mathbf{H}_0(X_{\mathcal{P}};\mathbb{Q})) &= 1,
\end{aligned}$$

where the parameter b is determined by the exact form of Σ.

Remark 3.25
In the next section, we try to give a geometric description of b, or at least for some special cases. there is yet another approach in [13], which computes the homology groups of 6-dimensional toric varieties, using spectral sequences.

Remark 3.26
As mentioned here [13], b is not combinatorial invariant.

3.2.3 *Singularities of 6-dimensional toric varieties*

Let Σ be a complete fan in $\mathbb{R}^3$ and $\mathcal{P}$ be the associated dual polytope to it. As mentioned earlier, the link of a point $x \in X_4 - X_2$ is simply $\mathcal{S}^1$. For $x \in X_2 - X_0$, we employ the construction of links introduced earlier. Let $\tau \in \Sigma$ such that $p(x) \in \text{int}(\tau)$, where p is the natural project $p : X_\Sigma \longrightarrow |\mathcal{P}|$. Then, we have $\mathcal{S}_\tau = \{\sigma_1, \sigma_2, \text{int}(|\mathcal{P}|)\}$, where σ_1 and σ_2 are the two 2-dimensional neighboring faces of τ. Hence as in the example 3.15, we have $\mathcal{M}_\tau \cong \mathcal{I}$. However, in contrast to the example 3.15, here we have $\mathcal{T}^3/\pi(\delta(\tau)) \cong \mathcal{S}^1$. But in the end, $\mathcal{L}_\tau$, the link of x, has the same CW structure as in example 3.15.

$$\mathcal{L}_\tau \cong \left(\text{int}(\mathcal{I}) \times \mathcal{T}^3/(\mathcal{T}^3/\pi(\delta(\tau)))\right) \bigcup \left(\{0\} \times (\mathcal{T}^3/\delta(\pi(\sigma_1)))/(\mathcal{T}^3/\pi(\delta(\tau)))\right) \bigcup \left(\{1\} \times (\mathcal{T}^3/\delta(\pi(\sigma_2)))/(\mathcal{T}^3/\pi(\delta(\tau)))\right).$$

Note that with the same argument as in the remark 3.16, we conclude that $\mathcal{L} \ncong \mathcal{S}^1 \times \mathcal{S}^2$. The computation method goes along the same line as in the corollary 3.21. The homology groups of the link remain unchanged compared with the corollary 3.21.

$$\begin{aligned}
&\mathrm{rk}(\mathbf{H}_3(\mathcal{L}_\tau;\mathbb{Q})) = 1,\\
&\mathrm{rk}(\mathbf{H}_2(\mathcal{L}_\tau;\mathbb{Q})) = 0,\\
&\mathrm{rk}(\mathbf{H}_1(\mathcal{L}_\tau;\mathbb{Q})) = 0,\\
&\mathrm{rk}(\mathbf{H}_0(\mathcal{L}_\tau;\mathbb{Q})) = 1.
\end{aligned}$$

Remark 3.27
Studying the homology groups of $\mathcal{L}_\tau$ with integral coefficient yields the same result for the smoothness of a 2-dimensional stratum in X_Σ as in the previous case. This means that if the two generators of $\delta(\tau)$ in $\Sigma = \mathcal{N} \otimes \mathbb{R}$ do not form a part of a $\mathbb{Z}$-basis of the lattice $\mathcal{N}$ then the stratum is singular.

In example 3.18, we constructed the link of a connected component of X_0 (a point) in a real 6-dimensional toric variety. Here, we generalize the example 3.18 for an arbitrary real 6-dimensional compact toric variety.
Let $x \in X_0$. We consider x as a 0-dimensional face of $\mathcal{P}$. $\mathcal{S}_x$ consists of $\mathrm{int}(\mathcal{P})$ and all 1- and 2-dimensional faces of $\mathcal{P}$, τ, with $x \prec \tau$. Following the introduced construction of links, $|\mathcal{M}_x|$ is simply a 2-dimensional convex polygon with f_{x_1} vertices and f_{x_2} 1-dimensional faces, where f_{x_1} and f_{x_2} denote the number of 1- and 2-dimensional neighboring faces of x in Σ, respectively. The convexity of $|\mathcal{P}|$ yields $f_{x_1} = f_{x_2}$. Hence, we have the following relations.

$$\begin{aligned}
p_{\mathcal{L}_x}^{-1}(\mathrm{int}(\mathcal{M}_x)) &= \mathcal{T}^3 \times \mathrm{int}(\mathcal{M}_x) \\
p_{\mathcal{L}_x}^{-1}(\mathrm{int}(\tau)) &= \mathcal{T}^2 \times \mathrm{int}(\tau) \qquad \text{for } \tau \in \mathcal{M}_x \text{ with } \dim(\tau) = 1 \\
p_{\mathcal{L}_x}^{-1}(\nu) &= \mathcal{S}^1 \times \nu \qquad \text{for } \nu \in \mathcal{M}_x \text{ with } \dim(\nu) = 0.
\end{aligned}$$

At this point, we can endow $\mathcal{L}_v$ with a CW structure and compute the corresponding homology groups. As before, we equip $\mathcal{T}^3$ with the common CW structure and each $\mathcal{T}^2$ with an appropriate CW structure such that each collapse map becomes cellular as we discussed earlier. Each 1-dimensional face of $\mathcal{M}_x$, τ_i, is associated to a 2-dimensional face of $\mathcal{P}$, which is dual to a 1-dimensional cone in Σ. Let a and b be the numbers of such 1-dimensional cones in Σ, whose generators have no non-zero entry and at least one zero-entry, respectively. Take into consideration that $a + b \geq 3$. Let $\begin{pmatrix} n_{a_i} \\ m_{a_i} \\ l_{a_i} \end{pmatrix}$ and $\begin{pmatrix} n_{b_i} \\ m_{b_i} \\ 0 \end{pmatrix}$ with $i = 1, \dots, a$ and $j = 1, \dots, b$ be the generators of these 1-dimensional cones in Σ.

Remark 3.28

Without loss of generality and for the sake of simplicity, we only consider $\begin{pmatrix} n_{b_i} \\ m_{b_i} \\ 0 \end{pmatrix}$ *as generators of 1-dimensional faces in* σ *with at least one zero-entry. However, one can easily deduce that the arguments used here can be generalized for the more general case, and our results remain intact.*

Accordingly, we get the following chain complex for $\mathcal{L}_x$, where we use the regular CW structure on $\mathcal{M}_x$.

$$\begin{aligned}
\mathcal{C}_5(\mathcal{L}_x) &= \mathbb{Q}\langle e^3_{\mathcal{T}^3} \times e^2_{\mathcal{M}_x}\rangle \\
\mathcal{C}_4(\mathcal{L}_x) &= \bigoplus_{i=1}^{3} \mathbb{Q}\langle e^2_{\mathcal{T}^3_i} \times e^2_{\mathcal{M}_x}\rangle \\
\mathcal{C}_3(\mathcal{L}_x) &= \bigoplus_{i=1}^{3} \mathbb{Q}\langle e^1_{\mathcal{T}^3_i} \times e^2_{\mathcal{M}_x}\rangle \bigoplus_{i=1}^{a} \left(\mathbb{Q}\langle e^2_{(\mathcal{T}^2_{a_i})_1} \times e^1_{(\mathcal{M}_{x_a})_i}\rangle \oplus \mathbb{Q}\langle e^2_{(\mathcal{T}^2_{a_i})_2} \times e^1_{(\mathcal{M}_{x_a})_i}\rangle\right) \\
&\quad \bigoplus_{j=1}^{b} \mathbb{Q}\langle e^2_{(\mathcal{T}^2_{b_j})} \times e^1_{(\mathcal{M}_{x_b})_i}\rangle \\
\mathcal{C}_2(\mathcal{L}_x) &= \mathbb{Q}\langle e^0_{\mathcal{T}^3} \times e^2_{\mathcal{M}_x}\rangle \bigoplus_{i=1}^{a}\bigoplus_{l=1}^{3} \mathbb{Q}\langle e^1_{(\mathcal{T}^2_{a_i})_l} \times e^1_{(\mathcal{M}_{x_a})_i}\rangle \bigoplus_{j=1}^{b}\bigoplus_{l=1}^{2} \mathbb{Q}\langle e^1_{(\mathcal{T}^2_{b_j})_l} \times e^1_{(\mathcal{M}_{x_b})_j}\rangle \\
\mathcal{C}_1(\mathcal{L}_x) &= \bigoplus_{i=1}^{a} \mathbb{Q}\langle e^0_{(\mathcal{T}^2_{a_i})} \times e^1_{(\mathcal{M}_{x_a})_i}\rangle \bigoplus_{j=1}^{b} \mathbb{Q}\langle e^0_{(\mathcal{T}^2_{b_j})} \times e^1_{(\mathcal{M}_{x_b})_j}\rangle \\
\mathcal{C}_0(\mathcal{L}_x) &= \bigoplus_{l=0}^{f_{x_1}} \mathbb{Q}\langle e^0_{\mathcal{M}_{x_l}}\rangle .
\end{aligned}$$

Remark 3.29
In the above chain complex, $e^1_{(\mathcal{M}_{x_a})_i}$ *and* $e^1_{(\mathcal{M}_{x_b})_j}$ *are those 1-cells of* $\mathcal{M}_x$*, whose associated* $\mathcal{T}^2$ *in* $\mathcal{L}_x$ *has three and two 1-cells ,respectively. Similarly, we have labeled each* $\mathcal{T}^2$ *with either* a_i *or* b_j *according to the generator of the collapsed 1-dimensional cone in* Σ*.*

Consequently, we obtain the following boundary operators for the above CW structure on $\mathcal{L}_x$.

$$\partial_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\partial_4 = \begin{array}{r} \\ e^1_{\mathcal{T}^3_1}\times e^2_{\mathcal{M}x} \\ e^1_{\mathcal{T}^3_2}\times e^2_{\mathcal{M}x} \\ e^1_{\mathcal{T}^3_3}\times e^2_{\mathcal{M}x} \\ e^2_{(\mathcal{T}^2_{a_i})_1}\times e^1_{(\mathcal{M}x_a)_i}\, a\{ \\ e^2_{(\mathcal{T}^2_{a_i})_2}\times e^1_{(\mathcal{M}x_a)_i}\, a\{ \\ e^2_{(\mathcal{T}^2_{b_j})}\times e^1_{(\mathcal{M}x_b)_j}\, b\{ \end{array} \begin{array}{ccc} e^2_{\mathcal{T}^3_1}\times e^2_{\mathcal{M}x} & e^2_{\mathcal{T}^3_2}\times e^2_{\mathcal{M}x} & e^2_{\mathcal{T}^3_3}\times e^2_{\mathcal{M}x} \\ \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ m_{a_i}n_{a_i}l^2_{a_i} \\ 0 \\ 0 \end{array} \right. & \begin{array}{c} 0 \\ 0 \\ 0 \\ m^2_{a_i}n_{a_i}l_{a_i} \\ m^2_{a_i}n_{a_i}l_{a_i} \\ -m_{b_j} \end{array} & \left. \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ m_{a_i}n^2_{a_i}l_{a_i} \\ n_{b_j} \end{array} \right) \end{array} \tag{3.11}$$

$$\partial_3 = \begin{array}{r} \\ e^0_{\mathcal{T}^3}\times e^2_{\mathcal{M}x} \\ e^1_{(\mathcal{T}^2_{a_i})_1}\times e^1_{(\mathcal{M}x_a)_i}\, a\{ \\ e^1_{(\mathcal{T}^2_{a_i})_2}\times e^1_{(\mathcal{M}x_a)_i}\, a\{ \\ e^1_{(\mathcal{T}^2_{a_i})_3}\times e^1_{(\mathcal{M}x_a)_i}\, a\{ \\ e^1_{(\mathcal{T}^2_{b_j})_1}\times e^1_{(\mathcal{M}x_b)_j}\, b\{ \\ e^1_{(\mathcal{T}^2_{b_j})_2}\times e^1_{(\mathcal{M}x_b)_j}\, b\{ \end{array} \left(\begin{array}{cccccc} e^1_{\mathcal{T}^3_1}\times e^2_{\mathcal{M}x} & e^1_{\mathcal{T}^3_2}\times e^2_{\mathcal{M}x} & e^1_{\mathcal{T}^3_3}\times e^2_{\mathcal{M}x} & e^2_{(\mathcal{T}^2_{a_i})_1}\times e^1_{(\mathcal{M}x_a)_i} & e^2_{(\mathcal{T}^2_{a_i})_2}\times e^1_{(\mathcal{M}x_a)_i} & e^2_{(\mathcal{T}^2_{b_j})}\times e^1_{(\mathcal{M}x_b)_j} \\ & & & \overbrace{\quad}^{a} & \overbrace{\quad}^{a} & \overbrace{\quad}^{b} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ m_{a_i}l_{a_i} & 0 & 0 & 1 & -1 & 0 \\ 0 & n_{a_i}l_{a_i} & 0 & 1 & -1 & 0 \\ 0 & 0 & n_{a_i}m_{a_i} & 1 & -1 & 0 \\ -m_{b_j} & 0 & 0 & 0 & 0 & 0 \\ 0 & n_{b_j} & 0 & 0 & 0 & 0 \end{array} \right) \tag{3.12}$$

In order to avoid any ambiguity on ∂_3, we describe the explicit form of ∂_3 in more details. Let $l \in \{1,2,3\}$. The rows associated to $e^1_{(\mathcal{T}^2_{a_i})_l} \times e^1_{(\mathcal{M}_{x_a})_i}$ have only non-zero entries on the columns labeled by $e^1_{\mathcal{T}^3_l} \times e^2_{\mathcal{M}_x}$, $e^2_{(\mathcal{T}^2_{a_i})_1} \times e^1_{(\mathcal{M}_{x_a})_i}$ and $e^2_{(\mathcal{T}^2_{a_i})_2} \times e^1_{(\mathcal{M}_{x_a})_i}$. Similarly, the columns labeled by $e^1_{\mathcal{T}^3_l} \times e^2_{\mathcal{M}_x}$ have only non-zero entires on the rows labeled by $e^1_{(\mathcal{T}^2_{a_i})_l} \times e^1_{(\mathcal{M}_{x_a})_i}$ and possibly on $e^1_{(\mathcal{T}^2_{b_j})_1} \times e^1_{(\mathcal{M}_{x_b})_j}$ or $e^1_{(\mathcal{T}^2_{b_j})_2} \times e^1_{(\mathcal{M}_{x_b})_j}$. Lastly, the columns associated to $e^2_{(\mathcal{T}^2_{a_i})_1} \times e^1_{(\mathcal{M}_{x_a})_i}$ and $e^2_{(\mathcal{T}^2_{a_i})_1} \times e^1_{(\mathcal{M}_{x_a})_i}$ have only non-zero entries on the rows labeled by $e^1_{(\mathcal{T}^2_{a_i})_1} \times e^1_{(\mathcal{M}_{x_a})_i}$, $e^1_{(\mathcal{T}^2_{a_i})_2} \times e^1_{(\mathcal{M}_{x_a})_i}$ and $e^1_{(\mathcal{T}^2_{a_i})_3} \times e^1_{(\mathcal{M}_{x_a})_i}$. For instance, let us start with $i = 1$. Consider the following part of ∂_3.

$$\begin{pmatrix} m_{a_1}l_{a_1} & 0 & 0 & 1 & -1 & 0 \\ 0 & n_{a_1}l_{a_1} & 0 & 1 & -1 & 0 \\ 0 & 0 & n_{a_1}m_{a_1} & 1 & -1 & 0 \end{pmatrix}$$

Adding $e^2_{(\mathcal{T}^2_{a_i})_1} \times e^1_{(\mathcal{M}_{xa})_i}$ and $e^2_{(\mathcal{T}^2_{a_i})_2} \times e^1_{(\mathcal{M}_{xa})_i}$ to columns, and $e^1_{(\mathcal{T}^2_{a_i})_1} \times e^1_{(\mathcal{M}_{xa})_i}$, $e^1_{(\mathcal{T}^2_{a_i})_2} \times e^1_{(\mathcal{M}_{xa})_i}$ and $e^1_{(\mathcal{T}^2_{a_i})_3} \times e^1_{(\mathcal{M}_{xa})_i}$ to rows gives us

$$\begin{pmatrix} m_{a_1}l_{a_1} & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & n_{a_1}l_{a_1} & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & n_{a_1}m_{a_1} & 1 & -1 & 0 & 0 & 0 \\ m_{a_1}l_{a_1} & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & n_{a_1}l_{a_1} & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & n_{a_1}m_{a_1} & 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

Adding more columns and rows goes along the same lines.

Remark 3.30
Bear in mind that the explicit form of ∂_4 depends on the relative positioning of the images of $e^1_{\mathcal{T}^3_1}$, $e^1_{\mathcal{T}^3_2}$ and $e^1_{\mathcal{T}^3_3}$ under the collapse maps in $e^2_{(\mathcal{T}^2_{a_i})_1}$ and $e^2_{(\mathcal{T}^2_{a_i})_2}$. For instance we have chosen the CW structures on $\mathcal{T}^2$ such that the image of $e^1_{\mathcal{T}^3_2}$ lies between the images of $e^1_{\mathcal{T}^3_1}$ and $e^1_{\mathcal{T}^3_3}$. Obviously, ker and Im of ∂_4 remain intact.

Computing the ker and Im of the above boundary operators is straightforward.

$$\begin{aligned} &\mathrm{rk}(\ker(\partial_5)\,) = 1,\ \mathrm{rk}(\mathrm{Im}(\partial_5)\,) = 0, \\ &\mathrm{rk}(\ker(\partial_4)\,) = 0,\ \mathrm{rk}(\mathrm{Im}(\partial_4)\,) = 3, \\ &\mathrm{rk}(\ker(\partial_3)\,) = a + b = f_{x_1}, \end{aligned}$$

which yield

$$\begin{aligned} &\mathrm{rk}(\mathbf{H}_5(\mathcal{L}_x;\mathbb{Q})) = 1 \\ &\mathrm{rk}(\mathbf{H}_4(\mathcal{L}_x;\mathbb{Q})) = 0 \\ &\mathrm{rk}(\mathbf{H}_3(\mathcal{L}_x;\mathbb{Q})) = f_{x_1} - 3. \end{aligned}$$

The homology groups below the middle degree can be obtained using the rational Poincaré duality. We assume that $\mathcal{L}_x$ would not satisfy the rational Poincaré duality. This would result in an existence of a singularity in $\mathcal{L}_x$, which would imply that there is a rationally singular i-dimensional stratum in X_Σ, where $i \geq 2$. This is in contradiction with our previous observations.

Remark 3.31
There is yet another way for computing the homology groups below the middle degree. We can use our previous method of orthogonal decomposition and compute ∂_2. ∂_2 will have the following form.

$$\partial_2 = \begin{array}{r} \\ e^0_{\mathcal{T}^2_{a_i}} \times e^1_{(\mathcal{M}_{xa})_i}\, a\{ \\ e^0_{\mathcal{T}^2_{b_j}} \times e^1_{(\mathcal{M}_{x_b})_j}\, b\{ \\ e^1_{\mathcal{S}^1_i} \times e^0_{(\mathcal{M}_x)_i}\, a+b\{ \end{array} \begin{array}{cccccc} e^0_{\mathcal{T}^3_1} \times e^2_{\mathcal{M}_x} & e^1_{(\mathcal{T}^2_{a_i})_1} \times e^1_{(\mathcal{M}_{xa})_i} & e^1_{(\mathcal{T}^2_{a_i})_2} \times e^1_{(\mathcal{M}_{xa})_i} & e^1_{(\mathcal{T}^2_{a_i})_3} \times e^1_{(\mathcal{M}_{xa})_i} & e^1_{(\mathcal{T}^2_{b_j})_1} \times e^1_{(\mathcal{M}_{x_b})_j} & e^1_{(\mathcal{T}^2_{b_j})_2} \times e^1_{(\mathcal{M}_{x_b})_j} \\ 1 & \overbrace{0}^{a} & \overbrace{0}^{a} & \overbrace{0}^{a} & \overbrace{0}^{b} & \overbrace{0}^{b} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & [& & & &] \end{array}$$

It can be shown that rows associated to the cells of the form $e^1_{\mathcal{S}^1_i} \times e^0_{(\mathcal{M}_x)_i}$ are linearly independent. This implies

$$\begin{aligned} rk(ker(\partial_2)\,) &= (3a+2b)-(a+b) = 2a+b \\ rk(im(\partial_2)\,) &= a+b+1. \end{aligned}$$

Bear in mind that the upper left part of ∂_2 is merely determined by the orientation of $|\mathcal{M}_x|$. Similarly, for ∂_1 we have

$$\partial_1 = \begin{array}{c} \\ e^0_{\mathcal{S}^1_i}\times e^0_{(\mathcal{M}_x)_i}\, a+b\Big\{ \end{array} \begin{pmatrix} \overset{e^0_{\mathcal{T}^2_{a_i}}\times e^1_{(\mathcal{M}_{x_a})_i}}{\overbrace{z_i}^{a}} & \overset{e^0_{\mathcal{T}^2_{b_j}}\times e^1_{(\mathcal{M}_{x_b})_j}}{\overbrace{y_j}^{b}} & \overset{e^1_{\mathcal{S}^1_i}\times e^0_{(\mathcal{M}_x)_i}}{\overbrace{0}^{a+b}} \end{pmatrix},$$

where again y_j's and z_i's are solely determined by the orientation of $|\mathcal{M}_x|$. Using $|\mathcal{M}_x| \cong \mathcal{D}^2$ yields

$$\begin{aligned} rk(ker(\partial_1)\,) &= a+b+1, \\ rk(im(\partial_1)\,) &= f_{x_1}-1, \\ rk(ker(\partial_0)\,) &= f_{x_1}. \end{aligned}$$

In any case, we have shown the following proposition.

Proposition 3.32
Let Σ be a complete 3-dimensional fan and X_Σ the associated toric variety and $\mathcal{L}_x$ be the link of $x \in (X_\Sigma)_0$. Let f_{x_1} be the number of 1-dimensional faces of $\mathcal{P}$, the dual polyhedron to Σ, with x as a proper face. Then we have

$$\begin{aligned} rk(\mathbf{H}_5(\mathcal{L}_x;\mathbb{Q})\,) &= 1, \\ rk(\mathbf{H}_4(\mathcal{L}_x;\mathbb{Q})\,) &= 0, \\ rk(\mathbf{H}_3(\mathcal{L}_x;\mathbb{Q})\,) &= f_{x_1}-3, \\ rk(\mathbf{H}_2(\mathcal{L}_x;\mathbb{Q})\,) &= f_{x_1}-3, \\ rk(\mathbf{H}_1(\mathcal{L}_x;\mathbb{Q})\,) &= 0, \\ rk(\mathbf{H}_0(\mathcal{L}_x;\mathbb{Q})\,) &= 1. \end{aligned}$$

Corollary 3.33
A point $x \in (X_\Sigma)_0$ is rationally singular if x in $\mathcal{P}$ has more than three 1-dimensional neighboring faces.

3.2.4 *Link of an isolated singularity in an 8-dimensional toric variety*

In this section, we study the link of an isolated singularity in an 8-dimensional toric variety. First, we look at the homology groups of such a link. After some remarks on homology groups, we briefly look at the natural stratification of the link.

Example 3.34
In this example, we employ our previous methods to compute the homology groups of the link of an isolated singularity in an 8-dimensional toric variety. However, there are some crucial remarks that we need to make before we start. Let $\mathcal{L}$ be the link of a point $x \in X_0$ in an 8-dimensional toric variety X, with common stratification. The underlying polytope, based on 3.9, is a 3-dimensional convex polytope. Please note that here, for the sake of simplicity, we endow each torus with the common CW structure, despite the previous example where we considered different types of CW structures on $\mathcal{T}^2$. However, our treatment can be easily generalized. Let $\mathcal{M}$ be the underlying polytope of $\mathcal{L}$ .i.e. there is natural projection $p : \mathcal{L} \longrightarrow \mathcal{M}$, in the sense of 3.9. f_1 is defined to be the number of 1-dimensional cones in Σ, the dual fan to $\mathcal{M}$. Similarly, f_2 denotes the number of 2-dimensional cones in Σ. Thus, $\mathcal{L}$ can be provided with the following CW structure.

$$
\begin{aligned}
\mathcal{C}_7(\mathcal{L}) &= \mathbb{Q}\,\langle e^4_{\mathcal{T}^4} \times e^4_{\mathcal{M}} \rangle \\
\mathcal{C}_6(\mathcal{L}) &= \bigoplus_{i=1}^{4} \mathbb{Q}\,\langle e^3_{\mathcal{T}^4_i} \times e^3_{\mathcal{M}} \rangle \\
\mathcal{C}_5(\mathcal{L}) &= \bigoplus_{i=1}^{6} \mathbb{Q}\,\langle e^2_{\mathcal{T}^4_i} \times e^2_{\mathcal{M}} \rangle \bigoplus_{j=1}^{f_1} \mathbb{Q}\,\langle e^3_{\mathcal{T}^3} \times e^2_{\mathcal{M}_j} \rangle \\
\mathcal{C}_4(\mathcal{L}) &= \bigoplus_{i=1}^{4} \mathbb{Q}\,\langle e^1_{\mathcal{T}^4} \times e^3_{\mathcal{M}} \rangle \bigoplus_{j=1}^{f_1} \bigoplus_{l=1}^{3} \mathbb{Q}\,\langle e^2_{\mathcal{T}^3_l} \times e^2_{\mathcal{M}_j} \rangle \\
\mathcal{C}_3(\mathcal{L}) &= \mathbb{Q}\,\langle e^0_{\mathcal{T}^4} \times e^3_{\mathcal{M}} \rangle \bigoplus_{j=1}^{f_1} \bigoplus_{l=1}^{3} \mathbb{Q}\,\langle e^1_{(\mathcal{T}^3_l)} \times e^2_{\mathcal{M}_j} \rangle \bigoplus_{k=1}^{f_2} \mathbb{Q}\,\langle e^2_{\mathcal{T}^2} \times e^1_{\mathcal{M}_k} \rangle \\
\mathcal{C}_2(\mathcal{L}) &= \bigoplus_{i=0}^{f_1} \mathbb{Q}\,\langle e^0_{\mathcal{T}^3} \times e^2_{\mathcal{M}_i} \rangle \bigoplus_{j=1}^{f_2} \bigoplus_{l=1}^{2} \mathbb{Q}\,\langle e^2_{\mathcal{T}^2_l} \times e^1_{\mathcal{M}_j} \rangle . \\
\mathcal{C}_1(\mathcal{L}) &= \bigoplus_{i=1}^{f_2} \mathbb{Q}\,\langle e^0_{\mathcal{T}^2} \times e^1_{\mathcal{M}_i} \rangle \bigoplus_{j=1}^{f_2-f_1+2} \mathbb{Q}\,\langle e^1_{\mathcal{S}^1} \times e^0_{\mathcal{M}_j} \rangle \\
\mathcal{C}_0(\mathcal{L}) &= \bigoplus_{i=1}^{f_2-f_1+2} \mathbb{Q}\,\langle e^0_{\mathcal{M}_i} \rangle
\end{aligned}
$$

Similar to 3.22, we get the following results.

$$
\begin{aligned}
&rk(\mathbf{H}_7(\mathcal{L};\mathbb{Q})\,) = 1, \\
&rk(\mathbf{H}_6(\mathcal{L};\mathbb{Q})\,) = 0, \\
&rk(\mathbf{H}_5(\mathcal{L};\mathbb{Q})\,) = f_1 - 4,
\end{aligned}
$$

$$\partial_4 = \begin{array}{r} \\ e^0_{\mathcal{T}^4}\times e^3_{\mathcal{M}} \\ e^1_{\mathcal{T}^3_1}\times e^2_{\mathcal{M}_i}\ \ f_1\{ \\ e^1_{\mathcal{T}^3_2}\times e^2_{\mathcal{M}_i}\ \ f_1\{ \\ e^1_{\mathcal{T}^3_3}\times e^2_{\mathcal{M}_i}\ \ f_1\{ \\ e^2_{\mathcal{T}^2}\times e^1_{\mathcal{M}_j}\ \ f_2\{ \end{array} \begin{pmatrix} e^1_{\mathcal{T}^4_1}\times e^3_{\mathcal{M}} & e^1_{\mathcal{T}^4_2}\times e^3_{\mathcal{M}} & e^1_{\mathcal{T}^4_3}\times e^3_{\mathcal{M}} & e^1_{\mathcal{T}^4_4}\times e^3_{\mathcal{M}} & e^2_{\mathcal{T}^3_1}\times e^2_{\mathcal{M}} & e^2_{\mathcal{T}^3_2}\times e^2_{\mathcal{M}} & e^2_{\mathcal{T}^3_3}\times e^2_{\mathcal{M}} \\ 0 & 0 & 0 & 0 & \overbrace{0}^{f_1} & \overbrace{0}^{f_1} & \overbrace{0}^{f_1} \\ \mathsf{C}_1 & \mathsf{C}_2 & \mathsf{C}_3 & \mathsf{C}_4 & 0 & 0 & 0 \\ \mathsf{C}_5 & \mathsf{C}_6 & \mathsf{C}_7 & \mathsf{C}_8 & 0 & 0 & 0 \\ \mathsf{C}_9 & \mathsf{C}_{10} & \mathsf{C}_{11} & \mathsf{C}_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathsf{D}_1 & \mathsf{D}_2 & \mathsf{D}_3 \end{pmatrix}$$

$$rk(ker(\partial_4)) = 3f_1 - f_2 - b_1$$
$$rk(Im(\partial_4)) = f_2 + b_1 + 4.$$

Note that, similar to the 6-dimensional toric varieties, $b_1 = 0$ for non-singular links.

$$\partial_3 = \begin{array}{r} \\ e^0_{\mathcal{T}^3}\times e^2_{\mathcal{M}_k}\ \ f_1\{ \\ e^1_{\mathcal{T}^2_1}\times e^1_{\mathcal{M}_l}\ \ f_2\{ \\ e^1_{\mathcal{T}^2_2}\times e^1_{\mathcal{M}_l}\ \ f_2\{ \end{array} \begin{pmatrix} e^0_{\mathcal{T}^4}\times e^3_{\mathcal{M}} & e^1_{\mathcal{T}^3_1}\times e^2_{\mathcal{M}_i} & e^1_{\mathcal{T}^3_2}\times e^2_{\mathcal{M}_i} & e^1_{\mathcal{T}^3_3}\times e^2_{\mathcal{M}_i} & e^2_{\mathcal{T}^2}\times e^1_{\mathcal{M}_j} \\ \partial_3^{\mathcal{M}} & \overbrace{0}^{f_1} & \overbrace{0}^{f_1} & \overbrace{0}^{f_1} & \overbrace{0}^{f_2} \\ 0 & \mathsf{E}_1 & \mathsf{E}_2 & \mathsf{E}_3 & 0 \\ 0 & \mathsf{E}_4 & \mathsf{E}_5 & \mathsf{E}_6 & 0 \end{pmatrix}$$

$$rk(ker(\partial_3)) = 3f_1 - f_2 - b'$$
$$rk(Im(\partial_3)) = 2f_2 + b'$$

$$\partial_2 = \begin{array}{r} \\ e^0_{\mathcal{T}^2}\times e^1_{\mathcal{M}_j}\qquad f_2\{ \\ e^1_{\mathcal{S}^1}\times e^0_{\mathcal{M}_k}\ \ f_2-f_1+2\{ \end{array} \begin{pmatrix} e^0_{\mathcal{T}^3}\times e^2_{\mathcal{M}_l} & e^1_{\mathcal{T}^2_1}\times e^1_{\mathcal{M}_i} & e^1_{\mathcal{T}^2_2}\times e^1_{\mathcal{M}_i} \\ \overbrace{\partial_2^{\mathcal{M}}}^{f_1} & \overbrace{0}^{f_2} & \overbrace{0}^{f_2} \\ 0 & \mathsf{F}_1 & \mathsf{F}_2 \end{pmatrix}$$

$$rk(ker(\partial_2)) = f_2 + f_1 - 2$$

Here again, we use Poincaré duality and argue that

$$rk(\mathbf{H}_2(\mathcal{L};\mathbb{Q})\ ) = f_1 - 4 - b_2,$$

where $b_2 = 0$ *for non-singular links. This yields*

$$-b' = f_2 - 2 - b_2.$$

Finally, we get

$$\begin{aligned} rk(\mathbf{H}_7(\mathcal{L};\mathbb{Q})\ ) &= 1, \\ rk(\mathbf{H}_6(\mathcal{L};\mathbb{Q})\ ) &= 0, \\ rk(\mathbf{H}_5(\mathcal{L};\mathbb{Q})\ ) &= f_1 - 4, \\ rk(\mathbf{H}_4(\mathcal{L};\mathbb{Q})\ ) &= 3f_1 - f_2 - 6 - b_1, \\ rk(\mathbf{H}_3(\mathcal{L};\mathbb{Q})\ ) &= 3f_1 - f_2 - 6 - (b_1 + b_2), \\ rk(\mathbf{H}_2(\mathcal{L};\mathbb{Q})\ ) &= f_1 - 4 - b_2, \\ rk(\mathbf{H}_1(\mathcal{L};\mathbb{Q})\ ) &= 0, \\ rk(\mathbf{H}_0(\mathcal{L};\mathbb{Q})\ ) &= 1. \end{aligned}$$

Remark 3.35
Here, we generalize the above consideration. Assume that for each $\mathcal{T}^2$ *we have an arbitrary number of 1-cells such that all collapses are cellular. Thus, we get*

$$\begin{aligned} rk(Im(\partial_3\ ) &= nf_2 + b', \\ rk(Ker(\partial_2\ ) &= nf_2 - (f_2 - f_1 + 2), \end{aligned}$$

where n *denotes the number of 1-cells. It yields*

$$rk(\mathbf{H}_2(\mathcal{L};\mathbb{Q})\ ) = f_1 - f_2 - 2 - b'.$$

With the same argument as above, we get

$$-b' = f_2 - 2 - b_2,$$

which is the same result, as before.
It is easy to show

$$rk(Im(\partial_5) = 6.$$

Thus, $rk(Im(\partial_5)$ *does not depend on the exact form of the chosen CW structure of* $\mathcal{T}^3$*. Because* $rk(\mathbf{H}_4(\mathcal{L};\mathbb{Q}))$ *is also independent of the chosen CW structure, it follows that* $rk(ker(\partial_4)$ *is independent of the chosen CW structure. Finally, by cutting out the rational singularities of* $\mathcal{L}$*, using the Lefschetz duality on the resulting space, and using the long exact sequence of relative homology we can show that* $rk(\mathbf{H}_3(\mathcal{L};\mathbb{Q}))$ *is the same as claimed.*

Remark 3.36
Later on, by using the theory of intersection spaces, we will show $b_1 = 0$.

Using 3.9, we get the following stratification on $\mathcal{L}$.

$$\mathcal{L} = \mathcal{L}_7 \supset \mathcal{L}_3 \supset \mathcal{L}_1,$$

where

$$\mathcal{L}_1 \cong \bigsqcup_{i=1}^{f_2 - f_1 + 2} \mathcal{S}^1.$$

Showing that the link bundle of $\mathcal{L}$, considered as a pseudomanifold, is trivial goes along the same line as in 3.1. Accordingly, one can easily show that the link of a connected component of $\mathcal{L}_0$ has the same CW structure as a link of an isolated singularity in a 6-dimensional toric variety. This means that we can easily determine whether a connected component of $\mathcal{L}_0$ is, at least rationally, singular or regular. Consequently, we arrive at the following corollary.

Corollary 3.37
$\mathcal{L}$ *in the above example is a rational homology sphere, if* $f_1 = 4$ *and* $f_2 = 6$.

4 PSEUDO-TORIC VARIETIES

The main goal of this chapter is to leave the world of toric varieties and introduce the notion of pseudo-toric varieties. As we have seen in the earlier section, the link of a stratum in a toric variety with co-dimension 4 is a 3-homological sphere. For instance, consider a point x in the bottom stratum of a 4-dimensional toric variety. As we have seen, the link of x is always a 3-homological sphere. However, as we will see, we can slightly modify the definition of toric varieties, such that the link of x is not a 3-homological sphere anymore but rather homeomorphic to $\mathcal{S}^1 \times \mathcal{S}^2$. But note that we can equip the new pseudo-toric variety with the same stratification as a 4-dimensional toric variety. However, in a pseudo-toric variety, there are certain conditions that need to be fulfilled. The most important one is that the combination of collapsing maps, which we will define in details later, should be transitive.

4.1 DEFINITIONS AND EXAMPLES

Definition 4.1
A pseudo-toric variety is a triple $(\mathcal{P}, \Gamma, X)$ such that $\mathcal{P}$ is a rational convex polytope, in the sense of 2.1, and Γ is a set of collapsing maps together with a topological space X. Each $\tau, \sigma, \gamma \in \mathcal{P}$ with $\dim(\tau) < \dim(\sigma) < \dim(\gamma)$ should fulfill the following conditions.

- *Let $\dim(\mathcal{P}) = n$ and $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n \cong \mathcal{T}^n$ the projection map then*

$$\exists p_\tau \in \Gamma \text{ such that } p_\tau : \mathcal{T}^n \longrightarrow \mathcal{T}^{n-\dim(\tau)} \text{ where } \pi^{-1}(\mathcal{T}^{n-\dim(\tau)})$$
$$\text{is a rational } (n\text{-}\dim(\tau))\text{-plane with } 0 \in \pi^{-1}(\mathcal{T}^{n-\dim(\tau)}) \subset \mathbb{R}^n.$$

- *We require also that $\exists p_{\sigma\tau} \in \Gamma$ such that $p_{\sigma\tau} : \mathcal{T}^{n-\dim(\sigma)} \longrightarrow \mathcal{T}^{n-\dim(\tau)}$, where $\mathcal{T}^{n-\dim(\tau)}$ and $\mathcal{T}^{n-\dim(\sigma)}$ are both rational subtori of $\mathcal{T}^n$ in the above sense.*

- *We require transitivity of collapsing maps which means*

$$p_{\gamma\sigma} p_{\sigma\tau} = p_{\gamma\tau}.$$

- *There is also a projection map $p : \mathcal{P} \times \mathcal{T}^n \longrightarrow X$ constructed as follows.*
  For $x \in \mathcal{P}$, $\{x\} \times \mathcal{T}^n$ under the projection map p_σ is collapsed to $\{x\} \times \mathcal{T}^{n-\dim(\sigma)}$, where σ is the unique open cell of p containing x.

Remark 4.2
With the same argument as in 3.11, we can show that a pseudo-toric variety is a topological pseudomanifold with a trivial link bundle.

Remark 4.3
Consider the pseudo-toric variety $(\mathcal{P}, \Gamma, X)$ and $\Sigma_{\mathcal{P}}$, which is the dual fan of the polyhedron $\mathcal{P}$. If $\Sigma_{\mathcal{P}}$ and Γ both yield the same collapsing data, then X is a toric variety associated with $\Sigma_{\mathcal{P}}$.

Example 4.4
Let $\mathcal{P}$ be the dual polytope associated with Σ ,an arbitrary complete rational fan. Let $\mathcal{T}^2 \cong \mathcal{S}^1_x \times \mathcal{S}^1_y$. We define

$$\begin{aligned}\Gamma := \{ & \forall \tau \in \mathcal{P} \textit{ with } \dim(\tau) = 1 \textit{ let } \; p_\tau : \mathcal{T}^2 \longrightarrow \mathcal{S}^1_x \\ & \textit{and} \\ & \forall o \in \mathcal{P} \textit{ with } \dim(o) = 0 \textit{ let } \; p_o : \mathcal{T}^2 \longrightarrow * \}.\end{aligned}$$

We claim that $(\mathcal{P}, \Gamma, X)$ is a pseudo-toric variety. First of all, it is straightforward to verify that Γ fulfills the required conditions in 4.1. As mentioned earlier, we can stratify pseudo-toric varieties similar to toric varieties. Thus, one would expect the following canonical stratification.

$$X_4 \supset X_2 \supset X_0,$$

where we X_{2i} define as in 3.1. However, the link of each connected component of the top stratum is homeomorphic to a $\mathcal{S}^1$. Note that the proof of the latter statement goes along the same line as in the case of toric varieties. Thus, the following filtration yields a stratification of the pseudo toric variety.

$$X_4 \supset X_0.$$

In the next step, we want to describe links of the isolated singularities in the above pseudo-toric variety. Construction of the links goes along the same line as in 3.15. However, due to the particular form of collapsing maps the link of an isolated singularity in the above pseudo-toric variety is not a 3-homological sphere. It is rather homeomorphic to $\mathcal{S}^1 \times \mathcal{S}^2$. At this point, we can compute homology groups of the above pseudo-toric variety. Again, we can endow a pseudo-toric variety with CW structure in the same way as we did for toric varieties. However, for a pseudo-toric variety, we obtain the boundary maps from the set of collapsing maps. Now let us compute the boundary operators. As in 3.19, let f_1 be the number of 1-dimensional faces of $\mathcal{P}$, then we have $f_1 \geq 3$.

$$\partial_4 = \begin{array}{c} \\ e^1_{\mathcal{T}^2_x} \times e^2_{\mathcal{P}} \\ e^1_{\mathcal{T}^2_y} \times e^2_{\mathcal{P}} \end{array} \begin{array}{c} e^2_{\mathcal{T}^2} \times e^2_{\mathcal{P}} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{array}$$

$$\partial_3 = \begin{array}{c} \\ e^0_{\mathcal{T}^2} \times e^2_{\mathcal{P}} \\ e^1_{\tau_i} \times e^1_{\mathcal{P}_i} \end{array} \begin{pmatrix} & e^1_{\mathcal{T}^2_x} \times e^2_{\mathcal{P}} & e^1_{\mathcal{T}^2_y} \times e^2_{\mathcal{P}} & \\ & 0 & 0 & \\ f_1 \{ & 1 & 0 & \} \end{pmatrix} \textit{ for } i = 1, \ldots, f_1,$$

$$\partial_2 = \begin{array}{c} \\ e^0_{\tau_i} \times e^1_{\mathcal{P}_i} \end{array} \left(\begin{array}{cc} e^0_{\mathcal{T}^2} \times e^2_{\mathcal{P}} & e^1_{\tau_i} \times e^1_{\mathcal{P}_i} \\ & \overbrace{}^{f_1} \\ f_1\{ \; 1 & 0 \; \} \end{array} \right).$$

Thus, we have

$$\begin{aligned} rk(ker(\partial_4)) &= 1 \\ rk(im(\partial_4)) &= 0 \\ rk(ker(\partial_3)) &= 1 \\ rk(im(\partial_3)) &= 1 \\ rk(ker(\partial_2)) &= f_1 \\ rk(im(\partial_2)) &= 1, \end{aligned}$$

and recall ∂_1 is simply $\partial_1^{\mathcal{P}}$, which is the boundary operator of the regular CW structure on $|\mathcal{P}|$. Hence, we have

$$\begin{aligned} rk(\mathbf{H}_4(X_\Sigma; \mathbb{Q})) &= 1, \\ rk(\mathbf{H}_3(X_\Sigma; \mathbb{Q})) &= 1, \\ rk(\mathbf{H}_2(X_\Sigma; \mathbb{Q})) &= f_1 - 1, \\ rk(\mathbf{H}_1(X_\Sigma; \mathbb{Q})) &= 0, \\ rk(\mathbf{H}_0(X_\Sigma; \mathbb{Q})) &= 1. \end{aligned}$$

We call the above pseudo-toric variety **the totally singular 4-dimensional pseudo-toric variety** with f_1 faces. In contrast to a 4-dimensional toric variety, a totally singular pseudo-toric variety does not satisfy the Poincaré duality rationally.

Example 4.5
Let $(\mathcal{P}, \Gamma, X)$ be a pseudo-toric variety, such that $\mathcal{P} \cong \mathcal{D}^3$. X can be given the same CW structure as the toric variety in 3.22. Here, we use the same labeling as in 3.22.

$$\partial_5 = \begin{array}{r} \\ e^1_{\mathcal{T}^3_1} \times e^3_{\mathcal{P}} \\ e^1_{\mathcal{T}^3_2} \times e^3_{\mathcal{P}} \\ e^1_{\mathcal{T}^3_3} \times e^3_{\mathcal{P}} \\ e^2_{(\mathcal{T}^2_{\gamma_i})_1} \times e^2_{\mathcal{P}_{\gamma_i}} \; \gamma\{ \\ e^2_{(\mathcal{T}^2_{\gamma_i})_2} \times e^2_{\mathcal{P}_{\gamma_i}} \; \gamma\{ \\ e^2_{\mathcal{T}^2_{\omega_j}} \times e^2_{\mathcal{P}_{\omega_j}} \; \omega\{ \end{array} \left(\begin{array}{ccc} e^2_{\mathcal{T}^3_1} \times e^3_{\mathcal{P}} & e^2_{\mathcal{T}^3_2} \times e^3_{\mathcal{P}} & e^2_{\mathcal{T}^3_3} \times e^2_{\mathcal{P}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ m_{\gamma_i} n_{\gamma_i} l^2_{\gamma_i} & m^2_{\gamma_i} n_{\gamma_i} l_{\gamma_i} & 0 \\ 0 & m^2_{\gamma_i} n_{\gamma_i} l_{\gamma_i} & m_{\gamma_i} n^2_{\gamma_i} l_{\gamma_i} \\ 0 & -m_{\omega_j} & n_{\omega_j} \end{array} \right)$$

$$\partial_4 = \begin{array}{r} \\ e^0_{\mathcal{T}^3} \times e^3_{\mathcal{P}} \\ e^1_{(\mathcal{T}^2_{\gamma_i})_1} \times e^2_{\mathcal{P}_{\gamma_i}} \gamma\{ \\ e^1_{(\mathcal{T}^2_{\gamma_i})_2} \times e^2_{\mathcal{P}_{\gamma_i}} \gamma\{ \\ e^1_{(\mathcal{T}^2_{\gamma_i})_3} \times e^2_{\mathcal{P}_{\gamma_i}} \gamma\{ \\ e^1_{(\mathcal{T}^2_{\omega_j})_1} \times e^2_{\mathcal{P}_{\omega_j}} \omega\{ \\ e^1_{(\mathcal{T}^2_{\omega_j})_2} \times e^2_{\mathcal{P}_{\omega_j}} \omega\{ \end{array} \begin{pmatrix} e^1_{\mathcal{T}^3_1} \times e^3_{\mathcal{P}} & e^1_{\mathcal{T}^3_2} \times e^3_{\mathcal{P}} & e^1_{\mathcal{T}^3_3} \times e^3_{\mathcal{P}} & e^2_{(\mathcal{T}^2_{\gamma_i})_1} \times e^2_{\mathcal{P}_{\gamma_i}} & e^2_{(\mathcal{T}^2_{\gamma_i})_2} \times e^2_{\mathcal{P}_{\gamma_i}} & e^2_{\mathcal{T}^2_{\omega_j}} \times e^2_{\mathcal{P}_{\omega_j}} \\ & & & \overbrace{0}^{\gamma} & \overbrace{0}^{\gamma} & \overbrace{0}^{\omega} \\ 0 & 0 & 0 & & & \\ m_{\gamma_i} l_{\gamma_i} & 0 & 0 & 1 & -1 & 0 \\ 0 & n_{\gamma_i} l_{\gamma_i} & 0 & 1 & -1 & 0 \\ 0 & 0 & n_{\gamma_i} m_{\gamma_i} & 1 & -1 & 0 \\ -m_{\omega_j} & 0 & 0 & 0 & 0 & 0 \\ 0 & n_{\omega_j} & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that ∂_6 remains unchanged. Thus, we have

$$rk(ker(\partial_6)) = 1,\ rk(Im(\partial_6)) = 0$$

However, in contrast to toric varieties, we have

$$rk(ker(\partial_5)) = n,\ rk(Im(\partial_5)) = 3 - n$$
$$rk(ker(\partial_4)) = \gamma + \omega + \left[\frac{n}{2}\right],\ rk(Im(\partial_4)) = \gamma + 3 - \left[\frac{n}{2}\right]$$

, where $n = 0, 1, 2$. With the same argument, as in 3.22, we can show that

$$\mathbf{H}_1(X) = 0.$$

This implies that

$$rk(ker(\partial_2)) = f_2 - 1.$$

Here again, we use the same argument, as in 3.22, and conclude that

$$\mathbf{H}_2(X) = f_1 - 3 - b'.$$

, where $b' = 0$, for non singular toric varieties. Thus, we have

$$
\begin{aligned}
&rk(\mathbf{H}_6(X_{\mathcal{P}};\mathbb{Q})) = 1,\\
&rk(\mathbf{H}_5(X_{\mathcal{P}};\mathbb{Q})) = n,\\
&rk(\mathbf{H}_4(X_{\mathcal{P}};\mathbb{Q})) = f_1 - 3 + n + \left[\frac{n}{2}\right],\\
&rk(\mathbf{H}_3(X_{\mathcal{P}};\mathbb{Q})) = 3f_1 - f_2 - b' - 6 + \left[\frac{n}{2}\right],\\
&rk(\mathbf{H}_2(X_{\mathcal{P}};\mathbb{Q})) = f_1 - 3 - b',\\
&rk(\mathbf{H}_1(X_{\mathcal{P}};\mathbb{Q})) = 0,\\
&rk(\mathbf{H}_0(X_{\mathcal{P}};\mathbb{Q})) = 1.
\end{aligned}
$$

Remark 4.6
As an example for $n = 2$, one could consider the following pseudo-toric variety. Let $\mathcal{T}^3 = \mathcal{S}^1_x \times \mathcal{S}^1_y \times \mathcal{S}^1_z$, then we define

$$
\begin{aligned}
\Gamma := \{ &\forall \sigma \in \mathcal{P} \textit{ with } \dim(\tau) = 2 \textit{ let } \; p_\sigma : \mathcal{T}^3 \longrightarrow \mathcal{S}^1_x \times \mathcal{S}^1_y,\\
&\forall \tau \in \mathcal{P} \textit{ with } \dim(\tau) = 1 \textit{ let } \; p_\tau : \mathcal{T}^3 \longrightarrow \mathcal{S}^1_x\\
&\textit{and}\\
&\forall o \in \mathcal{P} \textit{ with } \dim(o) = 0 \textit{ let } \; p_o : \mathcal{T}^3 \longrightarrow * \}.
\end{aligned}
$$

Now let $(\mathcal{P}, \Gamma, X)$ be the associated pseudo-toric variety. One can easily check that

$$
\partial_5 = \begin{array}{r} \\ e^1_{\mathcal{T}^3_x}\times e^3_{\mathcal{P}} \\ e^1_{\mathcal{T}^3_y}\times e^3_{\mathcal{P}} \\ e^1_{\mathcal{T}^3_z}\times e^3_{\mathcal{P}} \\ e^2_{\mathcal{T}^2_{\omega_j}}\times e^2_{\mathcal{P}_{\omega_j}} \;\omega\{ \end{array}
\begin{pmatrix} e^2_{\mathcal{T}^3_{xy}}\times e^3_{\mathcal{P}} & e^2_{\mathcal{T}^3_{yz}}\times e^3_{\mathcal{P}} & e^2_{\mathcal{T}^3_{xz}}\times e^2_{\mathcal{P}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}
$$

$$
\partial_4 = \begin{array}{r} \\ e^0_{\mathcal{T}^3}\times e^3_{\mathcal{P}} \\ e^1_{(\mathcal{T}^2_{\omega_j})_x}\times e^2_{\mathcal{P}_{\omega_j}} \;\omega\{ \\ e^1_{(\mathcal{T}^2_{\omega_j})_y}\times e^2_{\mathcal{P}_{\omega_j}} \;\omega\{ \end{array}
\begin{pmatrix} e^1_{\mathcal{T}^3_x}\times e^3_{\mathcal{P}} & e^1_{\mathcal{T}^3_y}\times e^3_{\mathcal{P}} & e^1_{\mathcal{T}^3_z}\times e^3_{\mathcal{P}} & e^2_{\mathcal{T}^2_{\omega_j}}\times e^2_{\mathcal{P}_{\omega_j}} \\ 0 & 0 & 0 & \overbrace{0}^{\omega} \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
$$

Hence, we have

$$
\begin{aligned}
&rk(\mathbf{H}_6(X_{\mathcal{P}};\mathbb{Q})) = 1,\\
&rk(\mathbf{H}_5(X_{\mathcal{P}};\mathbb{Q})) = 2,\\
&rk(\mathbf{H}_4(X_{\mathcal{P}};\mathbb{Q})) = f_1.\\
&rk(\mathbf{H}_3(X_{\mathcal{P}};\mathbb{Q})) = 3f_1 - f_2 - b' - 5,\\
&rk(\mathbf{H}_2(X_{\mathcal{P}};\mathbb{Q})) = f_1 - 3 - b',\\
&rk(\mathbf{H}_1(X_{\mathcal{P}};\mathbb{Q})) = 0,\\
&rk(\mathbf{H}_0(X_{\mathcal{P}};\mathbb{Q})) = 1.
\end{aligned}
$$

Note that to calculate the parameter b', we need to know the exact form of the inclusion of faces, which is beyond the scope of this example. We call the above pseudo-toric variety a ***6-dimensional totally singular pseudo-toric variety****.*

5 Q-PSEUDOMANIFOLDS AND LEFSCHTZ DUALITY

As we have seen in previous chapters, the link of a connected component of a 4-co-dimensional stratum in a toric variety is a rational homological 3-sphere. For instance, consider a 4-dimensional toric variety. As described above, the link of each isolated singularity is a rational homological 3-sphere. However, such a toric variety satisfies the Poincaré duality rationally. The question that one may ask is, to what extent can we consider strata with rational homological spherical links as regular and still hope to obtain the Poincaré duality? To answer this question, we need to introduce the notion of $\mathbb{Q}$-pseudomanifolds.

Definition 5.1 ($\mathbb{Q}$-pseudomanifold)
We call the topological space X with the filtration

$$X \supset X_i \supset X_{i-1} \supset \cdots \supset X_0 \tag{5.1}$$

a $\mathbb{Q}$-pseudomanifolds if there exists a filtration of X,

$$X \supset X_{i+k} \supset \cdots \supset X_{i+1} \supset X_i \supset \cdots \supset X_0,$$

such that X is a pseudomanifold concerning it and the link of each connected component of $X_{i+(j+1)} - X_{i+j}$ for $j = 1, \ldots k-1$ in X is a rational homological sphere.
We call the filtration in 5.1 a $\mathbb{Q}$-stratification of X.

Definition 3.3 covers pseudomanifolds without boundary. However, as it will turn out, we need to extend the previous definition to the case, wherein the chosen filtration consists of a closed subset of X with co-dimension 1, which we consider as the boundary of X. The following definition is due to [7].

Definition 5.2 (∂-pseudomanifold)
An n-dimensional ∂-stratified pseudomanifold is a pair $(X, \mathcal{B})$ together with filtration on X such that:

1. *$X - \mathcal{B}$ with the induced filtration $(X - \mathcal{B})^i = (X - \mathcal{B}) \cap X^i$ is an n-dimensional stratified pseudo manifold.*
2. *$\mathcal{B}$ with the induced filtration $\mathcal{B}^{i-1} = \mathcal{B} \cap X^i$ is an $n-1$ dimensional stratified pseudo manifold.*
3. *$\mathcal{B}$ has an open filtered collar neighborhood in X, i.e. there exists a neighborhood $\mathcal{N}$ of $\mathcal{B}$ and a filtered homeomorphism $\mathcal{N} \longrightarrow [0,1) \times \mathcal{B}$ (where $[0,1)$ is given the trivial filtration) that takes $\mathcal{B}$ to $\{0\} \times \mathcal{B}$.*

$\mathcal{B}$ is called the boundary of X and is also denoted with ∂X.

However, the question that may arise is to what extent the boundary, $\mathcal{B}$, is intrinsic to the topology of the space. The following consideration is again due to [7].
Let $(\mathcal{M}, \partial\mathcal{M})$ be a paracompact n-dimensional manifold with boundary. First, we consider the trivial filtration on $\mathcal{M}$ so that $\mathcal{M}$ itself is the only non-empty stratum. On the flip side, suppose X is the filtered space $\partial\mathcal{M} \subset \mathcal{M}$. In the first case, $(\mathcal{M}, \partial\mathcal{M})$ is a stratified pseudomanifold, although, in the second instance, the third condition in the definition 5.2 would not be satisfied. Despite the seeming contradiction, the following proposition from [8] ensures that when there are no one-co-dimensional strata, the boundary does depend only on the underlying space X and not the choice of a specific filtration (without one co-dimensional stratum.).

Proposition 5.3
Let $(X, \partial X)$ and $(X', \partial X')$ be ∂ stratified pseudomanifolds of dimension n with no one-co-dimensional strata and let $h : X \longrightarrow X'$ be homeomorphism (which is not required to be filtration preserving). Then h takes ∂X onto $\partial X'$.

Example 5.4
A 6-dimensional toric variety X with

$$X = X_6 \supset X_0$$

is a $\mathbb{Q}$-pseudomanifold.

Example 5.5
The link of an isolated singularity in an 8-dimensional toric variety, $\mathcal{L}$, is a $\mathbb{Q}$-pseudomanifold with the following $\mathbb{Q}$-stratification.

$$\mathcal{L} = \mathcal{L}_7 \supset \mathcal{L}_1.$$

Now, in a 6-dimensional toric variety, X, let $\mathcal{C}_x$ be a conical neighborhood of $x \in X_0$, such that $\mathcal{C}_x \cap \mathcal{C}_{x'} = \varnothing$, $\forall x, x' \in X_0$. Removing all $\mathcal{C}_x$ from X gives us a pseudomanifold with boundary, $(\mathcal{M}, \partial\mathcal{M})$, where now all links are rational homological spheres. The question that arises here is, does $(\mathcal{M}, \partial\mathcal{M})$ satisfies the Lefschetz duality?

Definition 5.6
A perversity

$$\bar{p} : \mathbb{Z}_{2\geq} \longrightarrow \mathbb{Z}$$

is a function such that

$$\bar{p}(2) = 0$$
$$\bar{p}(k+1) - \bar{p} \in \{1, 0\}.$$

The complementary perversity $\bar{q}$ of $\bar{p}$ is the one with

$$\bar{p}(k) + \bar{q}(k) = k - 2.$$

Definition 5.7
We denote the i-th intersection homology group of X with the coefficient A and the perversity $\bar{p}$ with

$$\mathbf{I}^{\bar{p}}\mathbf{H}_i(X;A).$$

Similarly, for the i-th intersection cohomology group, we have

$$\mathbf{I}_{\bar{p}}\mathbf{H}^i(X;A).$$

For an introduction to the theory of intersection homology, the reader may consult [7] or [12].

Proposition 5.8
Let X be an n-dimensional compact oriented topological pseudomanifold with the following stratification

$$X = X_n \supset X_i \supset X_0$$

and trivial link bundles. We also require that the link of $x \in X_i - X_0$ is an $(n-i-1)$-dimensional rational homological sphere. For each $x_i \in X_0$, let $\mathcal{C}_i$ be a cone-like open neighbourhood of x_i, such that $\mathcal{C}_i \cap \mathcal{C}_j = \varnothing$ for $i \neq j$. Let $\mathcal{M} = X \setminus \bigsqcup \mathcal{C}_i$. Then

$$(\mathcal{M}, \partial\mathcal{M} = \bigsqcup_{x_i \in X_0} \mathcal{L}_i),$$

where $\mathcal{L}_i$ is the link of $x_i \in X_0$, is pseudomanifold with boundary in the sense of 5.2. We claim that $(\mathcal{M}, \partial\mathcal{M})$ satisfies the Lefschetz duality for ordinary homology, rationally.

Proof. We start with the theorem (8.3.9) from [7].

Theorem 5.9
Suppose $\mathcal{R}$ is a Dedekind domain, and let X be a compact n-dimensional $\mathcal{R}$-oriented locally $(\bar{p};\mathcal{R})$-torsion-free stratified pseudomanifold with boundary. Then, we have isomorphism

$$\mathcal{D} : \mathbf{I}_{\bar{p}}\mathbf{H}^i(X;\mathcal{R}) \longrightarrow \mathbf{I}^{\bar{q}}\mathbf{H}_{n-i}(X,\partial X;\mathcal{R})$$
$$\mathcal{D} : \mathbf{I}_{\bar{p}}\mathbf{H}^i(X,\partial X;\mathcal{R}) \longrightarrow \mathbf{I}^{\bar{q}}\mathbf{H}_{n-i}(X;\mathcal{R}),$$

induced by the cap product with the fundamental class.

Remark 5.10
We work over $\mathbb{Q}$, and hence $(\bar{p}, \partial)$ is torsion-free, automatically. For further information, the reader may consult [7]. The orientability of $(\mathcal{M}, \partial\mathcal{M})$ follows from the orientability of X.

Hence, it remains to show

$$\mathbf{I}^{\bar{p}}\mathbf{H}_*(\mathcal{M}, \partial\mathcal{M}) \cong \mathbf{H}_*(\mathcal{M}, \partial\mathcal{M})$$
$$\mathbf{I}^{\bar{q}}\mathbf{H}_*(\mathcal{M}) \cong \mathbf{H}_*(\mathcal{M}),$$

concerning Q-coefficient, which we will omit for the rest of our proof. At this point, we use the induced stratification on $(\mathcal{M}, \partial\mathcal{M})$,

$$\mathcal{M} = \mathcal{M}_n \supset \mathcal{M}_i.$$

For each connected component of $\mathcal{M}_i$, $\mathcal{A}_i$, we choose an open cone-like neighborhood, $\mathcal{U}_i$, such that

$$\mathcal{U}_i \cong \mathcal{A}_i \times \mathcal{C}(\mathcal{L}_{\mathcal{A}_i})$$

, where $\mathcal{L}_{\mathcal{A}_i}$ is the link of $\mathcal{A}_i$. We also require

$$\mathcal{U}_i \cup \mathcal{U}_j = \varnothing \text{ for } i \neq j.$$

Now, let $\mathcal{M}' = \mathcal{M} \setminus \bigsqcup \mathcal{U}_i$. $(\mathcal{M}', \partial\mathcal{M}')$ is a manifold with boundary. Consequently, we arrive at

$$\begin{aligned} \mathbf{I}^{\bar{p}}\mathbf{H}_*(\mathcal{M}', \partial\mathcal{M}') &\cong \mathbf{H}_*(\mathcal{M}', \partial\mathcal{M}') \\ \mathbf{I}^{\bar{p}}\mathbf{H}_*(\mathcal{M}') &\cong \mathbf{H}_*(\mathcal{M}'). \end{aligned}$$

Let $\mathcal{V} = \mathcal{M}'$ and $\mathcal{U} = \bigsqcup \mathcal{U}_i$, which implies $\mathcal{V} \bigcap \mathcal{U} = \bigsqcup(\mathcal{A}_i \times \mathcal{L}_{\mathcal{A}_i})$. Note that $\mathcal{V}$ and $\mathcal{V} \bigcap \mathcal{U}$ are manifolds, and hence we get

$$\begin{aligned} \mathbf{I}^{\bar{p}}\mathbf{H}_*(\mathcal{V}) &\cong \mathbf{H}_*(\mathcal{V}) \\ \mathbf{I}^{\bar{p}}\mathbf{H}_*(\mathcal{V} \bigcap \mathcal{U}) &\cong \mathbf{H}_*(\mathcal{V} \bigcap \mathcal{U}). \end{aligned}$$

We use theorem 5.2.25 from [7].

Theorem 5.11
Suppose X is a filtered space and $\mathcal{M}$ is an n-dimensional manifold. Then we have the following isomorphism

$$\mathbf{H}_*(\mathcal{C}_*(\mathcal{M})) \otimes \mathcal{I}^{\bar{p}}\mathcal{C}_*(X)) \xrightarrow{\cong} \mathbf{I}^{\bar{p}}\mathbf{H}_*(\mathcal{M} \times X),$$

where $\mathcal{I}^{\bar{p}}\mathcal{C}_*$ is the singular intersection chain complex with the perversity $\bar{p}$.

We also employ the following elementary proposition from [12].

Proposition 5.12
Suppose X is a compact topological pseudomanifold of dimension $m \geq 1$. Then, for a perversity $\bar{p}$, we have

$$\mathbf{I}^{\bar{p}}\mathbf{H}_j(\mathcal{C}(X)) \cong \begin{cases} \mathbf{I}^{\bar{p}}\mathbf{H}_j(X), & \text{if } j < m - 1 - \bar{p}(m) \\ 0, & \text{otherwise.} \end{cases}$$

This yields

$$\mathrm{rk}(\mathbf{I}^{\bar{p}}\mathbf{H}_j(\mathcal{C}(\mathcal{L}_{\mathcal{A}_i}))) = \begin{cases} 1, & \text{if } j = 0 \\ 0, & \text{otherwise,} \end{cases}$$

where we used that $\mathcal{L}_{\mathcal{A}_i}$ is a homological sphere. Hence, we get $\mathbf{I}^{\bar{p}}\mathbf{H}_*(\mathcal{A}_i \times \mathcal{C}(\mathcal{L}_{\mathcal{A}_i})) \cong \mathbf{H}_*(\mathcal{A}_i)$. And finally, we obtain

$$\mathbf{I}^{\bar{p}}\mathbf{H}_*(\mathcal{U}) \cong \mathbf{H}_*(\mathcal{U}).$$

Using the Mayer-Vietoris sequence for ordinary and intersection homology and employing the 5-lemma gives us the desired isomorphism

$$\mathbf{I}^{\bar{p}}\mathbf{H}_*(\mathcal{M}) \cong \mathbf{H}_*(\mathcal{M}).$$

Similarly, we show

$$\mathbf{I}^{\bar{p}}\mathbf{H}_*(\partial\mathcal{M}) \cong \mathbf{H}_*(\partial\mathcal{M}).$$

In the end, we use the long exact sequence for both relative intersection and ordinary homology. Employing the 5-lemma once again gives us

$$\mathbf{I}^{\bar{p}}\mathbf{H}_*(\mathcal{M}, \partial\mathcal{M}) \cong \mathbf{H}_*(\mathcal{M}, \partial\mathcal{M}).$$

Using 5.9 concludes the proof. □

Definition 5.13
We call the process of removing the disjoint union of the cone-like neighborhood of the i-th stratum of X, as in the above proof, ***cutting out*** *the i-th stratum of X.*

Corollary 5.14
Let X be an n-dimensional compact oriented $\mathbb{Q}$-pseudomanifold with the following $\mathbb{Q}$-stratification.

$$X = X_n \supset X_0.$$

Let $(\mathcal{M}, \partial\mathcal{M})$ be the topological pseudomanifold obtained by cutting out the bottom stratum. Then $(\mathcal{M}, \partial\mathcal{M})$ satisfies the Lefschetz duality for ordinary homology rationally.

Proof. The required triviality condition in 5.8 can be simply dropped off as follows. Consider a system of trivialization of link bundles, with finitely many open sets. We show the duality on each open set and construct $(\mathcal{M}, \partial\mathcal{M})$ inductively.
Assume now that X has the following stratification

$$X = X_n \supset X_{n-2} \supset \cdots \supset X_1 \supset X_0.$$

Cutting out all singular strata of X gives us a manifold with boundary. We use the method of the previous proof and glue back each rationally regular stratum inductively, starting with the top stratum. □

Taking all the above considerations into account, we proved the following statement which is the main theorem of this section.

Theorem 5.15
Let X be an n-dimensional compact oriented $\mathbb{Q}$-pseudomanifold with the following $\mathbb{Q}$-stratification.

$$X = X_n \supset X_i \supset \cdots \supset X_0.$$

Let $(\mathcal{M}, \partial\mathcal{M})$ be the pseudomanifold obtained by cutting out all strata in the $\mathbb{Q}$-stratification. Then $(\mathcal{M}, \partial\mathcal{M})$ satisfies the Lefschetz duality rationally, concerning the ordinary homology.

Corollary 5.16
Let X be an n-dimensional closed compact oriented $\mathbb{Q}$-pseudomanifold with the following $\mathbb{Q}$-stratification

$$X = X_n.$$

*In other words, let X be a $\mathbb{Q}$-**manifold**. Then X satisfies the Poincaré Duality rationally.*

Example 5.17
Let X be a 4-dimensional toric variety. Then X satisfies the Poincaré Duality rationally.

Example 5.18
Let X be a 6-dimensional toric variety. The pseudomanifold obtained by cutting out X_0 satisfies the Lefschetz duality, rationally.

Part III

INTERSECTION SPACES OF TORIC VARIETIES WITH ISOLATED SINGULARITIES

The second part of this work aims to study the intersection spaces of toric varieties. Readers may consult [3] for an introduction to the theory of intersection spaces. The theory of intersection spaces, which has been introduced in [3], can be applied to the pseudomanifolds with isolated singularities. However, here [2] Banagl's theory of intersection spaces has been improved for more general classes of pseudomanifolds with non-isolated singularities. In this part, we first apply Banagl's theory of intersection spaces on the 4-dimensional toric varieties. After that, we generalize the theory of intersection spaces for $\mathbb{Q}$-pseudomanifolds with isolated singularities. Finally, we apply the generalized theory to 6-dimensional toric varieties.

6 INTERSECTION SPACES AND ISOLATED SINGULARITIES

In this chapter, first, we briefly introduce the theory of intersection spaces. In this work, we only consider the duality of Betti numbers for middle perversity. Throughout this chapter, we work with rational coefficients unless otherwise specified. Having the main theorem introduced, we look at 4-dimensional toric varieties as an example, although they satisfy the Poincaré duality rationally. As the first solid instance, we apply the theory to the 4-dimensional totally singular pseudo toric varieties. For a detailed introduction the reader may consult [3]. All credits of introductory section goes to Prof. Dr. Markus Banagl.

Definition 6.1
A CW-complex $\mathcal{K}$ is called n-segmented if it contains a sub-complex $\mathcal{K}_{<n} \subset \mathcal{K}$ such that

$$\mathbf{H}_r(\mathcal{K}_{<n}) = 0 \ \text{for } r \geq n$$
and
$$i_* : \mathbf{H}_r(\mathcal{K}_{<n}) \xrightarrow{\cong} \mathbf{H}_r(\mathcal{K}) \ \text{for } r < n,$$

where i is the inclusion of $\mathcal{K}_{<n}$ into $\mathcal{K}$.

The following lemma gives a condition for which a CW complex is n-segmented.

Lemma 6.2
Let $\mathcal{K}$ be an n-dimensional CW complex. If its group of n-cycles has a basis of cells, then $\mathcal{K}$ is n-segmented.

Proposition 6.3
Let $\mathcal{K}$ be an n-dimensional, n-segmented CW complex and suppose $\mathcal{K}_{<n} \subset \mathcal{K}$ is a subcomplex with the properties in 6.2 and such that $(\mathcal{K}_{<n})^{n-1} = \mathcal{K}^{n-1}$. If the group of n-cycle of $\mathcal{K}$ has a basis of cells, then $\mathcal{K}_{<n}$ is unique, namely

$$\mathcal{K}_{<n} = \mathcal{K}^{n-1} \cup \bigcup_{\alpha} \{y_\alpha\},$$

where $\{y_\alpha\}$ is the set of n-cells of $\mathcal{K}$ that are not cycles.

Definition 6.4
A (homological) n-truncation structure is a quadruple $(\mathcal{K}, {}^{\mathcal{K}}/_{n}, h, \mathcal{K}_{<n})$, where

1. *${}^{\mathcal{K}}/_{n}$ is an n-dimensional CW complex with $({}^{\mathcal{K}}/_{n})^{n-1} = \mathcal{K}^{n-1}$ and such that the group of n-cycle of ${}^{\mathcal{K}}/_{n}$ has a basis of cells.*
2. *$h : {}^{\mathcal{K}}/_{n} \longrightarrow \mathcal{K}^n$ is the identity on $\mathcal{K}^{n-1}$ and a cellular homotopy equivalence rel. $\mathcal{K}^{n-1}$.*

3. *$\mathcal{K}_{<n} \subset \mathcal{K}/n$ is a subcomplex with the properties in 6.2 with respect to $\mathcal{K}/n$ and such that $(\mathcal{K}_{<n})^{n-1} = \mathcal{K}^{n-1}$.*

As a consequence of the above definition, we have the following lemma.

Lemma 6.5
Assume that the CW complex $\mathcal{K}$ can be endowed with a homological n-truncation $(\mathcal{K}, \mathcal{K}/n, h, \mathcal{K}_{<n})$. Let

$$f : \mathcal{K}_{<n} \xhookrightarrow{\text{incl.}} \mathcal{K}/n \xrightarrow{h} \mathcal{K}.$$

Then

$$f_* : \mathbf{H}_i(\mathcal{K}_{<n}) \longrightarrow \mathbf{H}_i(\mathcal{K})$$

is an isomorphism for $i < n$, and

$$\mathbf{H}_i(\mathcal{K}_{<n}) = 0$$

for $i \geq n$.

Let X be an n-dimensional compact oriented topological pseudomanifold with isolated singularities. For $x_i \in X_0$, let $\mathcal{L}_i$ be the associated link. Assume that all links can be given a homological k-truncation, where $k = n - 1 - \bar{p}(n)$, for the perversity $\bar{p}$. Let $\mathcal{M}$ be the manifold obtained by cutting out all isolated singularities of X. Then, we have

$$\partial \mathcal{M} = \bigsqcup_i \mathcal{L}_i.$$

Let

$$\mathcal{L}_{<k} = \bigsqcup_i (\mathcal{L}_i)_{<k}$$

and define a homotopy class

$$g : \mathcal{L}_{<k} \xrightarrow{f} \partial \mathcal{M} \xhookrightarrow{\text{incl.}} \mathcal{M},$$

where $f = \bigsqcup_i f_i$.

Definition 6.6
The perversity $\bar{p}$ intersection space $I^{\bar{p}}X$ of X is defined to be

$$I^{\bar{p}}X = cone(g) = \mathcal{M} \bigcup_g \mathcal{C}(\mathcal{L}_{<k}).$$

Theorem 6.7
Let X be an n-dimensional compact oriented topological pseudomanifold with only isolated singularities. Let $\bar{p}$ and $\bar{q}$ be complementary perversities. Then, we have the duality isomorphism.

$$d : \widetilde{\mathbf{H}}_r(I^{\bar{p}}X)^* \xrightarrow{\cong} \widetilde{\mathbf{H}}_{n-r}(I^{\bar{q}}X),$$

where

$$\widetilde{\mathbf{H}}_r(I^{\bar{p}}X)^* = Hom(\widetilde{\mathbf{H}}_r(I^{\bar{p}}X), \mathbb{Q}).$$

Proof. The detailed construction of the above duality isomorphism can be found in the proof of the theorem 2.12 in [3]. □

Remark 6.8
Let $k = n - 1 - \bar{p}(n)$. Then, we have the following isomorphisms.
For $r > k$

$$\mathbf{H}_r(\mathcal{M}) \xrightarrow{\cong} \widetilde{\mathbf{H}}_r(I^{\bar{p}}X)$$
$$\widetilde{\mathbf{H}}_{n-r}(I^{\bar{q}}X) \xrightarrow{\cong} \mathbf{H}_{n-r}(\mathcal{M}, \partial\mathcal{M}).$$

For $r < k$

$$\widetilde{\mathbf{H}}_r(I^{\bar{p}}) \xrightarrow{\cong} \mathbf{H}_r(\mathcal{M}, \partial\mathcal{M})$$
$$\mathbf{H}_{n-r}(\mathcal{M}) \xrightarrow{\cong} \widetilde{\mathbf{H}}_{n-r}(I^{\bar{q}}X).$$

Corollary 6.9
If $n = \dim(X)$ is even, then the difference between the Euler characteristic of $\widetilde{\mathbf{H}}_*(I^{\bar{p}}X)$ and $\mathbf{I}^{\bar{p}}\mathbf{H}_*(X)$ is given by

$$\chi(\widetilde{\mathbf{H}}_*(I^{\bar{p}}X)) - \chi(\mathbf{I}^{\bar{p}}\mathbf{H}_*(X)) = -2\chi_{<n-1-\bar{p}}(\mathcal{L}),$$

where $\mathcal{L}$ is the disjoint union of the links of the isolated singularities of X.
If $n = \dim(X)$ is odd, then

$$\chi(\widetilde{\mathbf{H}}_*(I^{\bar{n}}X)) - \chi(\mathbf{I}^{\bar{n}}\mathbf{H}_*(X)) = (-1)^{\frac{n-1}{2}} b_{\frac{n-1}{2}}(\mathcal{L}),$$

where $b_{\frac{n-1}{2}}(\mathcal{L})$ is the middle dimensional Betti number of $\mathcal{L}$ and $\bar{n}$ is the upper middle perversity.
Regardless of the parity of n, the identity

$$rk(\widetilde{\mathbf{H}}_k(I^{\bar{p}}X)) + rk(\mathbf{I}^{\bar{p}}\mathbf{H}_k(X)) = rk(\mathbf{H}_k(\mathcal{M})) + rk(\mathbf{H}_k(\mathcal{M}, \mathcal{L}))$$

always holds in degree $k = n - 1 - \bar{p}$, where $\mathcal{M}$ is the exterior of the singular set of X.

Proof. A proof of the above corollary can be found in [3]. □

6.1 INTERSECTION SPACES OF REAL 4-DIMENSIONAL TORIC VARIETIES

Initially, we look at the homological truncation of links in 4-dimensional toric varieties. Obviously, by links, we mean the links of isolated singularities.
But first, we start with rather an easy example.

Example 6.10
Consider $\mathcal{T}^2$ endowed with the common CW structure on

$$\mathcal{T}^2 = e^2_{\mathcal{T}^2} \bigcup (e^1_{\mathcal{T}^2_z} \cup e^1_{\mathcal{T}^2_y}) \bigcup e^0_{\mathcal{T}^2}.$$

Then, $\mathcal{T}^2_{<2}$ is simply $\mathcal{S}^1 \vee \mathcal{S}^1$ or in other words

$$\mathcal{T}^2_{<2} = (e^1_{\mathcal{T}^2_z} \cup e^1_{\mathcal{T}^2_y}) \bigcup e^0_{\mathcal{T}^2}.$$

Now, let $\mathcal{T}'^2$ be a torus with the following CW structure introduced here 3.10,

$$\mathcal{T}'^2 = (e^2_{\mathcal{T}'^2_1} \cup e^2_{\mathcal{T}'^2_2}) \bigcup (e^1_{\mathcal{T}'^2_x} \cup e^1_{\mathcal{T}'^2_y} \cup e^1_{\mathcal{T}'^2_z}) \bigcup (e^0_{\mathcal{T}'^2}).$$

$\mathcal{T}'^2$ is 2-segmented because

$$\mathcal{T}'^2_{<2} = e^2_{\mathcal{T}'^2_2} \bigcup (e^1_{\mathcal{T}'^2_x} \cup e^1_{\mathcal{T}'^2_y} \cup e^1_{\mathcal{T}'^2_z}) \bigcup e^0_{\mathcal{T}'^2}.$$

with the boundary operators

$$\partial_2^{\mathcal{T}'^2_{<2}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and} \partial_1^{\mathcal{T}'^2_{<2}} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

satisfy the required conditions in 6.2.

Remark 6.11
In the above example, the group of 2-cycles of $\mathcal{T}^2$ has a basis of cells. Thus, based on 6.2, $\mathcal{T}^2$ is 2-segmented. However, although the group of 2-cycles of $\mathcal{T}'^2$ does not possess a basis of cells, $\mathcal{T}'^2$ is still 2-segmented and $\mathcal{T}'^2_{<2}$ is not unique. Furthermore, $(\mathcal{T}^2, id, \mathcal{T}^2, \mathcal{T}^2_{<2})$ is a homological n-truncation structure in the sense of 6.4.

As we mentioned earlier, a 4-dimensional toric variety, X_Σ, cannot possess a rationally singular stratum. Accordingly, although we can construct IX_Σ, the rational Poincaré duality is already satisfied. Nevertheless, as our first example, we will exhibit the construction of the intersection spaces of 4-dimensional toric varieties.

Example 6.12 (4-dimensional toric variety with a singularity)
For $n = 4$, we have $k = 4 - 1 - \bar{m}(4) = 2$. Let Σ be a 2-dimensional complete fan and X_Σ the associated toric variety stratified as follows.

$$X_\Sigma = (X_\Sigma)_4 \supset (X_\Sigma)_0 = \{v\}.$$

Let $\mathcal{L}_v$ be the link of v. From 3.9 we know that $\ker(\partial_2^{\mathcal{L}_v}) = 0$. This implies

$$\begin{aligned} &\mathcal{C}_2((\mathcal{L}_v)_{<2}) = \mathcal{C}_2(\mathcal{L}_v), \\ &\mathcal{C}_1((\mathcal{L}_v)_{<2}) = \mathcal{C}_1(\mathcal{L}_v), \\ &\mathcal{C}_0((\mathcal{L}_v)_{<2}) = \mathcal{C}_0(\mathcal{L}_v), \end{aligned}$$

with $\partial_i^{(\mathcal{L}_v)_{<2}} = \partial_i^{(\mathcal{L}_v)}$, for $i = 0, 1, 2$.
As noted earlier, $g : (\mathcal{L}_v)_{<2} \hookrightarrow \mathcal{M}$ is an inclusion, where $\mathcal{M}$ is the manifold obtained by cutting out X_0. Remember that in 3.15 we constructed $\mathcal{M}_v$, such that $|\mathcal{M}_v| \subset |\mathcal{P}_\Sigma|$, where $\mathcal{P}_\Sigma$ is the dual polygon to Σ. Let $|\widetilde{\mathcal{P}}|$ be the polygon obtained by removing the open cone $\mathring{\mathcal{C}}(|\mathcal{M}_v|)$ from $|\mathcal{P}_\Sigma|$. Thus, we can write

$$|\mathcal{P}_\Sigma| = |\widetilde{\mathcal{P}}| \bigcup_{|\mathcal{M}_v|} \overline{\mathring{\mathcal{C}}(|\mathcal{M}_v|)}.$$

At this point, we slightly modify the already introduced regular cw structure on $|\mathcal{P}_\Sigma|$. First, we endow $|\widetilde{\mathcal{P}}|$ with the regular CW structure. $|\mathcal{M}_v|$ is a sub-CW complex of $|\widetilde{\mathcal{P}}|$. We label cells in $|\mathcal{M}_v|$ explicitly with $\mathcal{M}_v$. Cells in $|\widetilde{\mathcal{P}}| \backslash |\mathcal{M}_v|$ are labeled with $\widetilde{\mathcal{P}}$. We glue $|\widetilde{\mathcal{P}}|$ and $\overline{\mathring{\mathcal{C}}(|\mathcal{M}_v|)}$ along $|\mathcal{M}_v|$ to obtain $|\mathcal{P}_\Sigma|$. Finally, we label cells in $\left(|\widetilde{\mathcal{P}}| \bigcup_{|\mathcal{M}_v|} \overline{\mathring{\mathcal{C}}(|\mathcal{M}_v|)}\right) \backslash |\widetilde{\mathcal{P}}|$ with $\mathcal{C}$. This gives us the following chain complex.

$$\begin{aligned} \mathcal{C}_4(IX_\Sigma) =& \mathbb{Q}\left\langle e^2_{\mathcal{T}^2} \times e^2_{\widetilde{\mathcal{P}}} \right\rangle \\ \mathcal{C}_3(IX_\Sigma) =& \mathbb{Q}\left\langle e^1_{\mathcal{T}^2_x} \times e^2_{\widetilde{\mathcal{P}}} \right\rangle \oplus \mathbb{Q}\left\langle e^1_{\mathcal{T}^2_y} \times e^2_{\widetilde{\mathcal{P}}} \right\rangle \oplus \mathbb{Q}\left\langle e^2_{\mathcal{T}^2} \times e^2_{\mathcal{M}_v} \right\rangle \\ & \oplus \mathbb{Q}\left\langle e^1_{\mathcal{T}^2_x} \times e^2_{\mathcal{C}} \right\rangle \oplus \mathbb{Q}\left\langle e^1_{\mathcal{T}^2_y} \times e^2_{\mathcal{C}} \right\rangle \\ \mathcal{C}_2(IX_\Sigma) =& \mathbb{Q}\left\langle e^0_{\mathcal{T}^2} \times e^2_{\widetilde{\mathcal{P}}} \right\rangle \bigoplus_{\substack{\tau_i \in \Sigma \\ \dim(\tau_i)=1}} \mathbb{Q}\left\langle e^1_{\tau_i} \times e^1_{\widetilde{\mathcal{P}}_i} \right\rangle \\ & \oplus \mathbb{Q}\left\langle e^1_{\mathcal{T}^2_x} \times e^1_{\mathcal{M}_v} \right\rangle \oplus \mathbb{Q}\left\langle e^1_{\mathcal{T}^2_y} \times e^1_{\mathcal{M}_v} \right\rangle \\ & \oplus \mathbb{Q}\left\langle e^0_{\mathcal{T}^2} \times e^2_{\mathcal{C}} \right\rangle \oplus \mathbb{Q}\left\langle e^1_{\tau_1} \times e^1_{\mathcal{C}_1} \right\rangle \oplus \mathbb{Q}\left\langle e^1_{\tau_2} \times e^1_{\mathcal{C}_2} \right\rangle \\ \mathcal{C}_1(IX_\Sigma) =& \bigoplus_{\substack{\tau_i \in \Sigma \\ \dim(\tau_i)=1}} \mathbb{Q}\left\langle e^0_{\tau_i} \times e^1_{\widetilde{\mathcal{P}}_i} \right\rangle \\ & \oplus \mathbb{Q}\left\langle e^0_{\mathcal{T}^2} \times e^1_{\mathcal{M}_v} \right\rangle \oplus \mathbb{Q}\left\langle e^1_{\tau_1} \times e^0_{\mathcal{M}_{v_1}} \right\rangle \oplus \mathbb{Q}\left\langle e^1_{\tau_2} \times e^0_{\mathcal{M}_{v_2}} \right\rangle \\ & \oplus \mathbb{Q}\left\langle e^1_{\tau_1} \times e^1_{\mathcal{C}_1} \right\rangle \oplus \mathbb{Q}\left\langle e^1_{\tau_2} \times e^1_{\mathcal{C}_2} \right\rangle, \end{aligned}$$

where, without loss of generality, we assume that ν *is dual to* $\sigma_{12} \in \Sigma$ *generated by* τ_1 *and* τ_2*. Then, we obtain the following boundary operators.*

$$\partial_4 = \begin{array}{c} \\ e^1_{\mathcal{T}^2_x} \times e^2_{\widetilde{\mathcal{P}}} \\ e^1_{\mathcal{T}^2_y} \times e^2_{\widetilde{\mathcal{P}}} \\ e^1_{\mathcal{T}^2_x} \times e^2_{\mathcal{C}} \\ e^1_{\mathcal{T}^2_y} \times e^2_{\mathcal{C}} \\ e^2_{\mathcal{T}^2} \times e^2_{\mathcal{M}_\nu} \end{array} \begin{array}{c} e^2_{\mathcal{T}^2} \times e^2_{\widetilde{\mathcal{P}}} \\ \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right) \end{array},$$

$$\partial_3 = \begin{array}{c} e^0_{\mathcal{T}^2} \times e^2_{\widetilde{\mathcal{P}}} \\ e^0_{\mathcal{T}^2} \times e^2_{\mathcal{C}} \\ e^1_{\tau_i} \times e^1_{\widetilde{\mathcal{P}_i}} \; f_1\{ \\ e^1_{\tau_1} \times e^1_{\mathcal{C}_1} \\ e^1_{\tau_2} \times e^1_{\mathcal{C}_2} \\ e^1_{\mathcal{T}^2_x} \times e^1_{\mathcal{M}_\nu} \\ e^1_{\mathcal{T}^2_x} \times e^1_{\mathcal{M}_\nu} \end{array} \overset{\begin{array}{ccccc} e^1_{\mathcal{T}^2_x} \times e^2_{\widetilde{\mathcal{P}}} & e^1_{\mathcal{T}^2_y} \times e^2_{\widetilde{\mathcal{P}}} & e^1_{\mathcal{T}^2_x} \times e^2_{\mathcal{C}} & e^1_{\mathcal{T}^2_y} \times e^2_{\mathcal{C}} & e^2_{\mathcal{T}^2} \times e^1_{\mathcal{M}_\nu} \end{array}}{\left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -m_i & n_i & 0 & 0 & 0 \\ 0 & 0 & -m_1 & n_1 & 0 \\ 0 & 0 & -m_2 & n_2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array} \right)},$$

$$\partial_2 = \begin{array}{c} e^0_{\tau_i} \times e^1_{\widetilde{\mathcal{P}_i}} \; f_1\{ \\ e^0_{\mathcal{T}^2} \times e^1_{\mathcal{M}_\nu} \\ e^0_{\tau_1} \times e^1_{\mathcal{C}_1} \\ e^0_{\tau_2} \times e^1_{\mathcal{C}_2} \\ e^1_{\tau_1} \times e^0_{\mathcal{M}_{\nu_1}} \\ e^1_{\tau_2} \times e^0_{\mathcal{M}_{\nu_2}} \end{array} \overset{\begin{array}{ccccccccc} e^0_{\mathcal{T}^2} \times e^2_{\widetilde{\mathcal{P}}} & e^0_{\mathcal{T}^2} \times e^2_{\mathcal{C}} & e^1_{\tau_1} \times e^1_{\widetilde{\mathcal{P}_1}} & e^1_{\tau_2} \times e^1_{\widetilde{\mathcal{P}_2}} & e^1_{\tau_i} \times e^1_{\widetilde{\mathcal{P}_i}} \, (i \neq 1,2) & e^1_{\tau_1} \times e^1_{\mathcal{C}_1} & e^1_{\tau_2} \times e^1_{\mathcal{C}_2} & e^1_{\mathcal{T}^2_x} \times e^1_{\mathcal{M}_\nu} & e^1_{\mathcal{T}^2_y} \times e^1_{\mathcal{M}_\nu} \end{array}}{\left(\begin{array}{ccccccccc} (\partial^{\mathcal{P}}_2)_{1\,i} & (\partial^{\mathcal{P}}_2)_{2\,i} & 0 & 0 & \overbrace{0}^{f_1-2} & 0 & 0 & 0 & 0 \\ (\partial^{\mathcal{P}}_2)_{1\,f_1+1} & (\partial^{\mathcal{P}}_2)_{2\,f_1+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\partial^{\mathcal{P}}_2)_{1\,f_1+2} & (\partial^{\mathcal{P}}_2)_{2\,f_1+2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\partial^{\mathcal{P}}_2)_{1\,f_1+3} & (\partial^{\mathcal{P}}_2)_{2\,f_1+3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -m_1 & n_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -m_1 & n_1 \end{array} \right)},$$

$$\partial_1 = \begin{array}{c} e^0_{\mathcal{P}_j} \; f_1+2\{ \end{array} \overset{\begin{array}{cccccc} e^0_{\tau_i} \times e^1_{\widetilde{\mathcal{P}_i}} & e^0_{\mathcal{T}^2} \times e^1_{\mathcal{M}_\nu} & e^0_{\tau_1} \times e^1_{\mathcal{C}_1} & e^0_{\tau_2} \times e^1_{\mathcal{C}_2} & e^1_{\tau_1} \times e^0_{\mathcal{M}_{\nu_1}} & e^1_{\tau_2} \times e^0_{\mathcal{M}_{\nu_2}} \end{array}}{\left(\begin{array}{cccccc} \overbrace{(\partial^{\mathcal{P}}_1)_{ij}}^{f_1} & (\partial^{\mathcal{P}}_1)_{f_1+1\,j} & (\partial^{\mathcal{P}}_1)_{f_1+2\,j} & (\partial^{\mathcal{P}}_1)_{f_1+3\,j} & 0 & 0 \end{array} \right)},$$

where by $\partial^{\mathcal{P}}_*$ *we denote the boundary operators of* $\mathcal{P}$*. The lower indices of* $(\partial^{\mathcal{P}}_*)$ *determine to which element of* $\partial^{\mathcal{P}}_*$ *we refer. Computing the homology groups of* $I^{\bar{n}}X$ *is an easy calculation and we get*

$$rk(\widetilde{\mathbf{H}}_4(I^{\bar{n}}X)) = 0,$$
$$rk(\widetilde{\mathbf{H}}_3(I^{\bar{n}}X)) = 0,$$
$$rk(\widetilde{\mathbf{H}}_2(I^{\bar{n}}X)) = f_1 - 2$$
$$rk(\widetilde{\mathbf{H}}_1(I^{\bar{n}}X)) = 0,$$
$$rk(\widetilde{\mathbf{H}}_4(I^{\bar{n}}X)) = 0.$$

In the above instance, we considered a 4-dimensional toric variety with only a single singularity. However, we can easily generalize the previous example to an arbitrary number of singularities. Consequently, we proved the following theorem.

Theorem 6.13
Let X be a 4-dimensional toric variety with the following stratification

$$X = X_4 \supset X_0 = \{\bigsqcup_{i=1}^{n} v_i\}.$$

Then, we have

$$rk(\widetilde{\mathbf{H}}_4(I^{\bar{n}}X)) = 0,$$
$$rk(\widetilde{\mathbf{H}}_3(I^{\bar{n}}X)) = n-1,$$
$$rk(\widetilde{\mathbf{H}}_2(I^{\bar{n}}X)) = f_1 - 2$$
$$rk(\widetilde{\mathbf{H}}_1(I^{\bar{n}}X)) = n-1,$$
$$rk(\widetilde{\mathbf{H}}_4(I^{\bar{n}}X)) = 0.$$

Remark 6.14
Note that $\mathbf{I}^{\bar{p}}\mathbf{H}_*(X) \cong \mathbf{H}_*(X)$, for a 4-dimensional toric variety. However, in this case, applying the identities in 6.9 will result in trivialities.

As we have seen earlier, a totally singular 2-dimensional pseudo-toric variety does not satisfy the Poincare duality, even rationally. In the following example, our goal is to construct the associated intersection space.

Example 6.15
Let $(\mathcal{P}, \Gamma, X)$ be a totally singular 4-dimensional pseudo-toric variety as in 4.4. In this example, we want to set up the corresponding intersection space to X and compute the homology groups of IX. As before, we use the canonical stratification on X. Let $x_i \in X_0$. But recall, $\mathcal{L}_{x_i}$, the link of x_i in X, is not a homological 3-sphere anymore but rather homeomorphic to $\mathcal{S}^1 \times \mathcal{S}^2$.Thus, an easy consideration yields

$$(\mathcal{L}_{x_i})_{<2} = \mathcal{S}^1, \quad \forall x_i \in X_0.$$

Here, instead of using the former CW structure, we employ the Mayer–Vietoris sequence to obtain the associated homology groups of IX. Let $p : X \longrightarrow \mathcal{P}$ be the projection from the pseudo-toric variety *to the underlying 2-dimensional polygon. As in the last example, let* $\widetilde{\mathcal{P}}$ *be the polygon obtained by removing the open cones* $\mathcal{U} = \bigsqcup_i \mathring{\mathcal{C}}((\mathcal{L}_{x_i})_{<2})$. *Then, we set*

$$\mathcal{U} = \bigsqcup_i \mathring{\mathcal{C}}((\mathcal{L}_{x_i})_{<2})$$

$$\mathcal{V} = p^{-1}(\widetilde{\mathcal{P}}).$$

Hence, we get

$$\cap := \mathcal{U} \cap \mathcal{V} = \bigsqcup_i \mathcal{S}^1.$$

The above construction yields

$$\mathcal{U} \simeq *.$$

Computing the homology groups of $\mathcal{V}$ *is a bit more complicated. We pursue our computation with an example. Assume that* $\mathcal{P}$ *is a 3-simplex. Figure 6.1 shows the associated intersection space, where we have truncated the links in the red regions. In the above notation,* $\mathcal{U}$ *corresponds to the disjoint union of the red areas.*

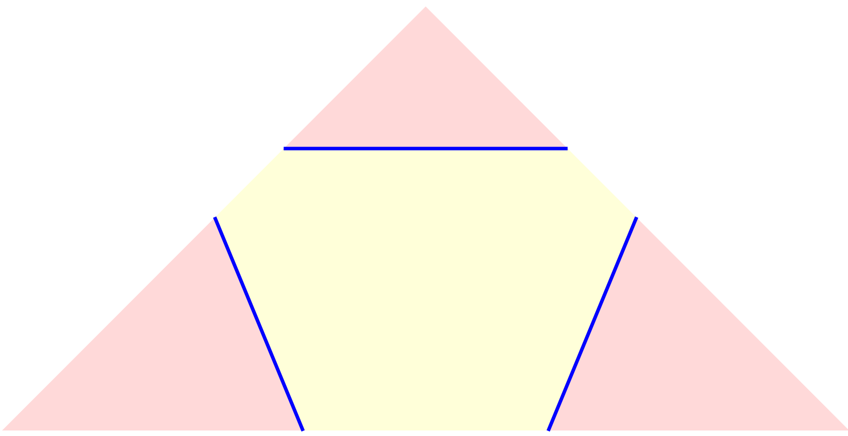

Figure 6.1: IX of the totally singular pseudo-toric variety associated with a 3-simplex.

In the figure 6.2, we have shown $\mathcal{U}$ *and* $\mathcal{V}$*, separately.*

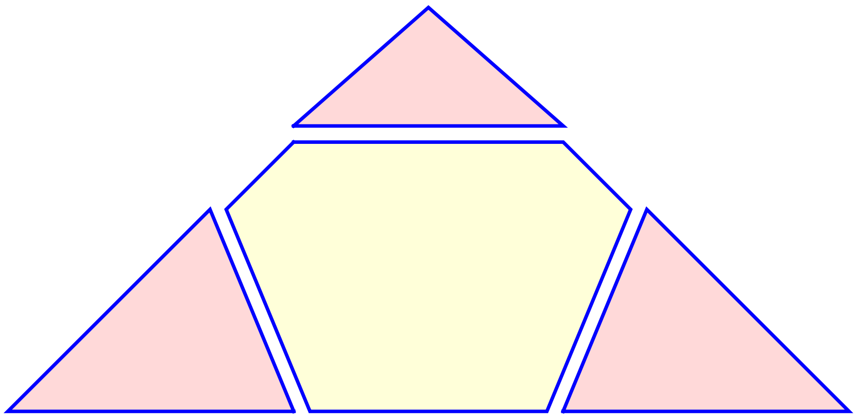

Figure 6.2: In our notation, the disjoint union of red areas corresponds to $\mathcal{U}$, where the yellow region denotes $\mathcal{V}$.

In figure 6.3, we take a closer look at $\mathcal{V}$. Each point on red lines is attached to a $\mathcal{S}^1$, and similarly, each point on blue lines and each green dot are attached to a $\mathcal{T}^2$. At this point, one could endow $\mathcal{V}$, similarly to toric varieties, with CW structure and compute the homology groups. Although this method is still doable in 4-dimensional cases, we use a homotopy model of $\mathcal{U}$, which is more computational-friendly. As we will see in 6-dimensional cases for computational purposes, we should use an adequate homotopy model of $\mathcal{U}$.

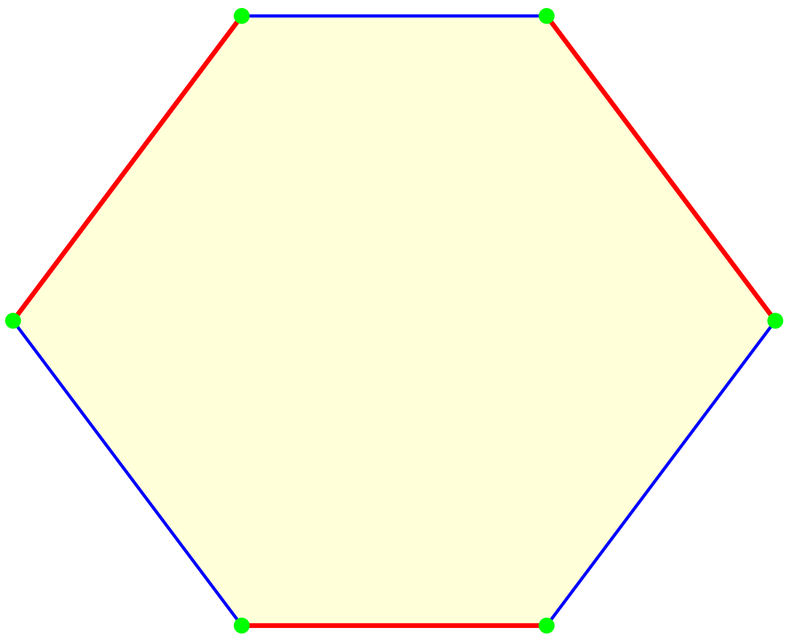

Figure 6.3: Each point on red lines is attached to a $\mathcal{S}^1$ and similarly each point on blue lines and each green dot are attached to a $\mathcal{T}^2$.

Each red line is homeomorphic to a $\mathcal{S}^1 \times [0,1]$, *which we can obviously shrink homotopically to* $\mathcal{S}^1$. *Thus, we define the following map on* $\mathcal{V}$. *We map each red line via the projection* $h : \mathcal{S}^1 \times [0,1] \longrightarrow \mathcal{S}^1$ *to* $\mathcal{S}^1$ *and use the identity map on the rest of* $\mathcal{V}$. *One can easily check that the above map is continuous, and thus, define a homotopy equivalence.*
As shown in figure 6.4, the above consideration results in a 3-simplex, where to each green dot we attach a $\mathcal{S}^1$, *and to each other point a* $\mathcal{T}^2$ *is attached. Note that the collapsing data can be extracted from* Γ.

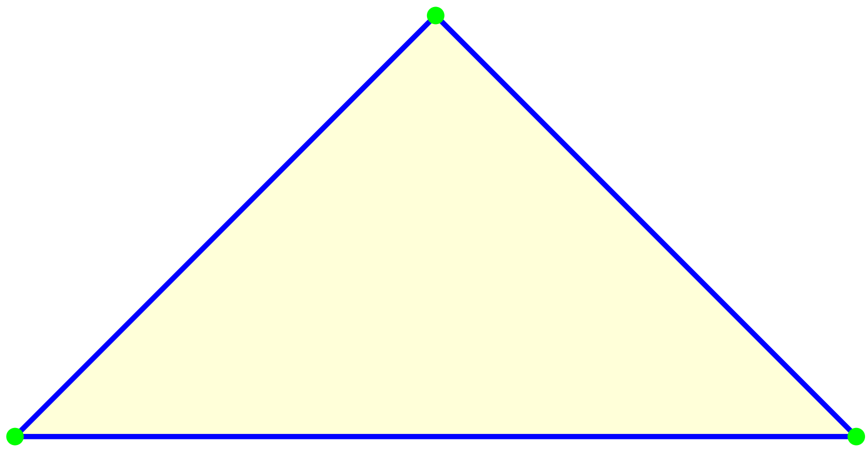

Figure 6.4: A homotopy model of $\mathcal{V}$, where we shrank each red line in 6.3 to a green dot.

Hence, for the general case, we can show that $\mathcal{V}$ *is homotopically equivalent to* $\mathcal{V}'$ *constructed as follows. Consider* $\mathcal{P}$, *where we attach to each vertex of it an* $\mathcal{S}^1$, *and to each point of open 1- and 2-cells we attach an* $\mathcal{T}^2$. *Now, we can compute the homology groups of* $\mathcal{V}'$ *easily and we get*

$$\begin{aligned}
rk(\mathbf{H}_4(\mathcal{V})) &= 0\\
rk(\mathbf{H}_3(\mathcal{V})) &= f_1 - 1\\
rk(\mathbf{H}_2(\mathcal{V})) &= f_1 - 1\\
rk(\mathbf{H}_1(\mathcal{V})) &= 1\\
rk(\mathbf{H}_0(\mathcal{V})) &= 1
\end{aligned}$$

On the other hand, we have the following Mayer Vietoris sequence.

$$\begin{aligned}
0 &\longrightarrow \mathbf{H}_4(\cap_{<2}) \longrightarrow \mathbf{H}_4(\mathcal{U}) \oplus \mathbf{H}_4(\mathcal{V}) \longrightarrow \mathbf{H}_4(IX)\\
&\longrightarrow \mathbf{H}_3(\cap_{<2}) \longrightarrow \mathbf{H}_3(\mathcal{U}) \oplus \mathbf{H}_3(\mathcal{V}) \longrightarrow \mathbf{H}_3(IX)\\
&\longrightarrow \mathbf{H}_2(\cap_{<2}) \longrightarrow \mathbf{H}_2(\mathcal{U}) \oplus \mathbf{H}_2(\mathcal{V}) \longrightarrow \mathbf{H}_2(IX)\\
&\longrightarrow \mathbf{H}_1(\cap_{<2}) \longrightarrow \mathbf{H}_1(\mathcal{U}) \oplus \mathbf{H}_1(\mathcal{V}) \longrightarrow \mathbf{H}_1(IX)\\
&\longrightarrow \mathbf{H}_0(\cap_{<2}) \longrightarrow \mathbf{H}_0(\mathcal{U}) \oplus \mathbf{H}_0(\mathcal{V}) \longrightarrow \mathbf{H}_0(IX) \longrightarrow 0.
\end{aligned}$$

First, we can conclude $rk(\mathbf{H}_3(IX)) = f_1 - 1$*. Using the duality of intersection spaces, we immediately get* $rk(\mathbf{H}_1(IX)) = f_1 - 1$*. Finally, from the exactness of the above sequence, we can deduce* $rk(\mathbf{H}_4(IX)) = 2f_1$*. Hence, we have*

$$
\begin{aligned}
rk(\widetilde{\mathbf{H}_4}(IX)) &= 0\\
rk(\widetilde{\mathbf{H}_3}(IX)) &= f_1 - 1\\
rk(\widetilde{\mathbf{H}_2}(IX)) &= 2f_1\\
rk(\widetilde{\mathbf{H}_1}(IX)) &= f_1 - 1\\
rk(\widetilde{\mathbf{H}_0}(IX)) &= 0.
\end{aligned}
$$

As the last remark, note that using the direct computation via CW structure yields the same results as expected.

7 GENERALIZATION OF BANAGL'S CONSTRUCTION FOR Q-PSEUDOMANIFOLDS

In this chapter, we generalize the theory of intersection spaces for isolated singularities. Our main result is that we can use the theory of intersection spaces for Q-pseudomanifolds with isolated singularities. To show this, we should revisit the proof of 6.7 in [3]. The central idea is to indicate that the main ingredients of the previous proof remain intact, even for Q-pseudomanifold with isolated singularities. Before proving our main theorem, we want to compute the intersection homology of Q-pseudomanifolds with isolated singularities. But first of all, we need to generalize the previous definition of intersection spaces that we made.

Let X be an n-dimensional compact oriented Q-pseudomanifold with isolated singularities, which means X has the following Q-stratification

$$X = X_n \supset X_0.$$

For $x_i \in X_0$, let $\mathcal{L}_i$ be the associated link. Assume that all links can be endowed with a homological k-truncation, where $k = n - 1 - \bar{p}(n)$, for the perversity $\bar{p}$. Let $\mathcal{M}$ be the Q-manifold obtained by cutting out all isolated singularities of X. Then, we have

$$\partial\mathcal{M} = \bigsqcup_i \mathcal{L}_i.$$

Let

$$\mathcal{L}_{<k} = \bigsqcup_i (\mathcal{L}_i)_{<k},$$

and define a homotopy class

$$g : \mathcal{L}_{<k} \xrightarrow{f} \partial\mathcal{M} \xhookrightarrow{\text{incl.}} \mathcal{M},$$

where $f = \bigsqcup_i f_i$.

Definition 7.1
The perversity $\bar{p}$ generalized intersection space $I^{\bar{p}}X$ of X is defined to be

$$I^{\bar{p}}X = cone(g) = \mathcal{M} \bigcup_g \mathcal{C}(\mathcal{L}_{<k}).$$

Proposition 7.2
Let X be an n-dimensional $\mathbb{Q}$-pseudomanifold with isolated singularities, where $n \geq 1$. Then for perversity $\bar{p}$, we have

$$\mathbf{I}^{\bar{p}}\mathbf{H}_r(X) = \begin{cases} \mathbf{H}_r(\mathcal{M}, \mathcal{L}), & r > k \\ \mathbf{H}(\mathcal{M}), & r < k, \end{cases}$$

where $\mathcal{M}$ is again the $\mathbb{Q}$-manifold obtained by cutting out the isolated singularities and $\mathcal{L} = \bigsqcup \mathcal{L}_{x_i}$, where for each $x_i \in X_0$, $\mathcal{L}_{x_i}$ is the associated link to x_i, and $k = n - 1 - \bar{p}(n)$.

Proof. The proof goes along the same line as in 5.15. We also make use of the Mayer–Vietoris sequence. As in the previous example 6.15, we cut out cone-like neighborhoods of isolated singularities. The obtained topological space is a pseudomanifold with boundary, $(\mathcal{M}, \partial\mathcal{M})$. It is easy to show that $(\mathcal{M}, \partial\mathcal{M})$ is a $\mathbb{Q}$-manifold with boundary. Let $\mathcal{C}(\mathcal{L}_{x_i})$ be the removed cone-like neighborhood of x_i. We set $\mathcal{V} = \mathcal{M}$ and $\mathcal{U} = \bigsqcup_i \mathcal{C}(\mathcal{L}_{x_i})$. Thus, we have $\cap := \mathcal{U} \cap \mathcal{V} = \bigsqcup_i \mathcal{L}_{x_i}$. At this point, we employ proposition 4.7.2 from [12].

Proposition 7.3
Suppose X is a compact topological pseudomanifold of dimension $m \geq 1$. Then for a perversity $\bar{p}$,

$$\mathbf{I}^{\bar{p}}\mathbf{H}_r(\mathcal{C}(X)) = \begin{cases} \mathbf{I}^{\bar{p}}\mathbf{H}_r(X), & r < m - 1 - \bar{p}(m) \\ 0, & \textit{otherwise.} \end{cases}$$

The Mayer–Vietoris sequence for intersection homology yields

$$\cdots \longrightarrow \mathbf{I}^{\bar{p}}\mathbf{H}_r(\cap) \longrightarrow \mathbf{I}^{\bar{p}}\mathbf{H}_r(\mathcal{U}) \oplus \mathbf{I}^{\bar{p}}\mathbf{H}_r(\mathcal{V}) \longrightarrow \mathbf{I}^{\bar{p}}\mathbf{H}_r(X) \longrightarrow \dots.$$

For $r > k$, we have $\mathbf{I}^{\bar{p}}\mathbf{H}_r(\mathcal{U}) = 0$. First, we consider the long exact sequence for relative ordinary homology

$$\cdots \longrightarrow \mathbf{H}_r(\partial\mathcal{M}) \longrightarrow \mathbf{H}_r(\mathcal{M}) \longrightarrow \mathbf{H}_r(\mathcal{M}, \partial\mathcal{M}) \longrightarrow \dots.$$

Using 5-lemma and the same arguments as in 5.8 yields

$$\mathbf{I}^{\bar{p}}\mathbf{H}_r(X) \cong \mathbf{H}_r(\mathcal{M}, \mathcal{L}), \text{for } r > k.$$

For $r < k$, we use the Mayer–Vietoris sequence for ordinary homology and 5-lemma again. Here, one should note that $\mathcal{L}$ is a $\mathbb{Q}$-manifold and thus, as we have shown earlier in the proof of 5.8, the ordinary and intersection homology agree. Thus, we have

$$\mathbf{I}^{\bar{p}}\mathbf{H}_r(X) \cong \mathbf{H}_r(\mathcal{M}), \text{for } r < k.$$

This concludes the proof. □

We are now at the point where we can start with the proof of the main theorem of this section. In [3], Banagl shows more than just the duality of Betti numbers of intersection spaces. Although we are only interested in the duality of the Betti numbers of intersection spaces, we can generalize theorem 2.12 in [3] in its general form for Q-pseudomanifolds. For a brief introduction to reflective algebra, the reader may consult [3].

Theorem 7.4
Let X be an n-dimensional compact oriented Q-pseudomanifold with only isolated singularities. Let $\bar{p}$ and $\bar{q}$ be complementary perversities. Assume that link of each isolated singularity admits an homological $(n-1-\bar{p}(n))$-truncation.
Then

1. *The pair $(\widetilde{\mathbf{H}}_*(I^{\bar{p}}X), \mathbf{I}^{\bar{p}}\mathbf{H}_*(X))$ is $(n-1-\bar{p}(n))$-reflective across the homology of the links and*
2. *$(\widetilde{\mathbf{H}}_*(I^{\bar{p}}X), \mathbf{I}^{\bar{p}}\mathbf{H}_*(X))$ and $(\widetilde{\mathbf{H}}_*(I^{\bar{q}}X), \mathbf{I}^{\bar{q}}\mathbf{H}_*)$ are n-dual reflective pairs.*

Remark 7.5
As mentioned earlier, we only consider homology with rational coefficients.
Note also that because the underlying space, X, is a Q-pseudomanifold with isolated singularities, from the context it should be clear that by $I^{\bar{p}}X$, we refer to the generalized intersection space of X.

Proof. For the proof, we mainly mimic the proof of theorem 2.12 in [3]. However, through our former considerations, we show that even if we consider Q-Pseudomanifolds with isolated singularities, the main ingredients of the proof remain intact.
We start with the braid of the triple

$$\mathcal{L}_{<k} \xrightarrow{f} \mathcal{L} \xrightarrow{j} \mathcal{M}.$$

Here, we use proposition 7.2, and hence, the same argument as in the proof of theorem 2.12 in [3] can be applied. The rest of the proof relies on the Lefschetz duality of $(\mathcal{M}, \partial\mathcal{M})$ and the Poincaré duality of $\partial\mathcal{M}$. For the Lefschetz duality, we employ 5.15. $\partial\mathcal{M}$ is a Q-manifold, and therefore it satisfies the Poincaré duality rationally, using 5.16. This concludes our proof. □

Corollary 7.6
Let X be an n-dimensional, compact, oriented, topological Q-pseudomanifold with only isolated singularities, which means we have the following Q-stratification

$$X_n \supset X_0.$$

Let $\bar{p}$ and $\bar{q}$ be complementary perversities. Then, we have the duality isomorphism.

$$d : \widetilde{\mathbf{H}}_r(I^{\bar{p}}X)^* \xrightarrow{\cong} \widetilde{\mathbf{H}}_{n-r}(I^{\bar{q}}X),$$

where

$$\widetilde{\mathbf{H}}_r(I^{\bar{p}}X)^* = Hom(\widetilde{\mathbf{H}}_r(I^{\bar{p}}X), \mathbb{Q}).$$

Proof. Construction of the duality isomorphism goes along the same line as in the proof of theorem 2.12 in [3] and using 5.16 and 5.15. □

Corollary 7.7
Let X be an n-dimensional, compact, oriented $\mathbb{Q}$-pseudomanifold with only isolated singularities. Let $I^{\bar{p}}X$ be the associated generalized intersection spaces. If $n = \dim(X)$ is even, then the difference between the Euler characteristic of $\widetilde{\mathbf{H}}_*(I^{\bar{p}}X)$ and $\mathbf{I}^{\bar{p}}\mathbf{H}_*(X)$ is given by

$$\chi(\widetilde{\mathbf{H}}_*(I^{\bar{p}}X)) - \chi(\mathbf{I}^{\bar{p}}\mathbf{H}_*(X)) = -2\chi_{<n-1-\bar{p}}(\mathcal{L}),$$

where $\mathcal{L}$ is the disjoint union of the links of the isolated singularities of X.
If $n = \dim(X)$ is odd, then

$$\chi(\widetilde{\mathbf{H}}_*(I^{\bar{n}}X)) - \chi(\mathbf{I}^{\bar{n}}\mathbf{H}_*(X)) = (-1)^{\frac{n-1}{2}} b_{\frac{n-1}{2}}(\mathcal{L}),$$

where $b_{\frac{n-1}{2}}(\mathcal{L})$ is the middle dimensional Betti number of $\mathcal{L}$ and $\bar{n}$ is the upper middle perversity. Regardless of the parity of n, the identity

$$rk(\widetilde{\mathbf{H}}_k(I^{\bar{p}}X)) + rk(\mathbf{I}^{\bar{p}}\mathbf{H}_k(X)) = rk(\mathbf{H}_k(\mathcal{M})) + rk(\mathbf{H}_k(\mathcal{M}, \mathcal{L}))$$

always holds in degree $k = n - 1 - \bar{p}$, where $\mathcal{M}$ is the $\mathbb{Q}$-manifold obtained by cutting out the isolated singularities of X.

8 INTERSECTION SPACES OF REAL 6-DIMENSIONAL TORIC VARIETIES

As mentioned in 5.18, a 6-dimensional toric variety is a $\mathbb{Q}$-pseudomanifold with isolated singularities. Based on our generalization of the theory of intersection spaces, a 6-dimensional toric variety can be a subject of the generalized theory. Although we won't compute the homology groups of the intersection spaces directly via CW structures, we will show that considering the middle perversity, links of isolated singularities are 3-segmented. Using Lefschetz duality, we compute homology groups of generalized intersection spaces of 6-dimensional toric varieties. As a side remark, we will also show that 7.7 yields the well-known Euler formula for 3-dimensional convex polytypes.
Let $X_{\mathcal{P}}$ be a 6-dimensional toric variety associated with $\mathcal{P}$, the underlying polytope. We can endow $X_{\mathcal{P}}$ with the following $\mathbb{Q}$-stratification.

$$X_{\mathcal{P}} = X_6 \supset X_0.$$

We have indicated here 3.32, and proven separately here 5.8, that links of isolated singularities are $\mathbb{Q}$-manifolds and satisfy the Poincaré duality rationally.

Proposition 8.1
Let X be a 6-dimensional toric variety with the following $\mathbb{Q}$-stratification

$$X_6 \supset X_0.$$

For each $x_i \in X_0$, let $\mathcal{L}_i$ be the associated link. We can 3-truncate $\mathcal{L}_i$ homologically.

Proof. Recall that we have computed ∂_3 of an arbitrary $\mathcal{L}_i$ in 3.12. From each $\mathcal{T}^2$ with a 2-cell, we remove the 2-cell. At last, from each $\mathcal{T}^2$ with two 2-cells, we omit the 2-cell with the negative sign in the boundary operator. We denote the obtained CW structure by $\mathcal{L}_i'$. Hence, we get

$$\partial_3^{\mathcal{L}_i'} = \begin{array}{r} \\ e^0_{\mathcal{T}^3}\times e^2_{\mathcal{M}_x} \\ e^1_{(\mathcal{T}^2_{\gamma_i})_1}\times e^1_{(\mathcal{M}_{x_a})_i}\gamma\{ \\ e^1_{(\mathcal{T}^2_{\gamma_i})_2}\times e^1_{(\mathcal{M}_{x_\gamma})_i}\gamma\{ \\ e^1_{(\mathcal{T}^2_{\gamma_i})_3}\times e^1_{(\mathcal{M}_{x_\gamma})_i}\gamma\{ \\ e^1_{(\mathcal{T}^2_{\omega_j})_1}\times e^1_{(\mathcal{M}_{x_\omega})_j}\omega\{ \\ e^1_{(\mathcal{T}^2_{\omega_j})_2}\times e^1_{(\mathcal{M}_{x_\omega})_j}\omega\{ \end{array} \begin{array}{cccc} e^1_{\mathcal{T}^3_1}\times e^2_{\mathcal{M}_x} & e^1_{\mathcal{T}^3_2}\times e^2_{\mathcal{M}_x} & e^1_{\mathcal{T}^3_3}\times e^2_{\mathcal{M}_x} & \overbrace{e^2_{(\mathcal{T}^2_{\gamma_i})_1}\times e^1_{(\mathcal{M}_{x_\gamma})_i}}^{\gamma} \\ 0 & 0 & 0 & 0 \\ m_{\gamma_i} l_{a_i} & 0 & 0 & 1 \\ 0 & n_{a_i} l_{a_i} & 0 & 1 \\ 0 & 0 & n_{\gamma_i} m_{\gamma_i} & 1 \\ -m_{\omega_j} & 0 & 0 & 0 \\ 0 & n_{b_j} & 0 & 0 \end{array}. \tag{8.1}$$

Clearly, we have

$$\mathrm{im}(\partial_3^{\mathcal{L}'_i}) = \mathrm{im}(\partial_3^{\mathcal{L}_i}) \text{ and } \ker(\partial_3^{\mathcal{L}'_i}) = 0,$$

and the inclusion $incl : \mathcal{L}'_i \hookrightarrow \mathcal{L}_i$ induces

$$incl_j : \mathbf{H}_j(\mathcal{L}'_i) \xrightarrow{\cong} \mathbf{H}_j(\mathcal{L}_i) \text{ for } j \leq 2.$$

Obviously,

$$\mathbf{H}_j(\mathcal{L}'_i) = 0 \text{ for } j > 2.$$

□

Theorem 8.2
Let $X_{\mathcal{P}}$ be an arbitrary 6-dimensional toric variety, where $\mathcal{P}$ is the underlying polytope. Let Σ be the dual fan to $\mathcal{P}$. We denote the 1-dimensional, 2-dimensional, and 3-dimensional cones of Σ with f_1, f_2, and f_3, respectively. Then, we have

$$\begin{aligned}
rk(\widetilde{\mathbf{H}}_6(I^{\bar{n}}X)) &= 0\\
rk(\tilde{\mathbf{H}}_5(I^{\bar{n}}X)) &= f_3 - 1\\
rk(\tilde{\mathbf{H}}_4(I^{\bar{n}}X)) &= f_1 - 3 - b\\
rk(\tilde{\mathbf{H}}_3(I^{\bar{n}}X)) &= 2(3f_1 - f_2 - b - 6)\\
rk(\tilde{\mathbf{H}}_2(I^{\bar{n}}X)) &= f_1 - 3 - b\\
rk(\tilde{\mathbf{H}}_1(I^{\bar{n}}X)) &= f_3 - 1\\
rk(\tilde{\mathbf{H}}_0(I^{\bar{n}}X)) &= 0,
\end{aligned}$$

where $I^{\bar{n}}X$ is obviously the generalized intersection space associated to the $\mathbb{Q}$-pseudomanifold $X_{\mathcal{P}}$, and the parameter b is introduced and has been studied here 3.22.

Proof. We start by cutting out the isolated singularities of $X_{\mathcal{P}}$, in the sense of 5.13. We denote the resulting $\mathbb{Q}$-manifold by $(\mathcal{M}, \partial\mathcal{M})$. $(\mathcal{M}, \partial\mathcal{M})$ satisfies the Lefschetz duality rationally. We can endow $\mathcal{M}$ with CW-structure similarly to the toric variety $X_{\mathcal{P}}$. Now, let $\mathcal{M}/\partial\mathcal{M}$ be the topological pseudomanifold obtained by coning-off the boundary of $(\mathcal{M}, \partial\mathcal{M})$. From the construction, we can deduce

$$\mathbf{H}_i(X) \cong \mathbf{H}_i(\mathcal{M}/\partial\mathcal{M}) \cong \mathbf{H}_i(\mathcal{M}, \partial\mathcal{M}) \text{ for } i \geq 2.$$

Hence, we get

$$\begin{aligned}
\mathrm{rk}(\mathbf{H}_6(\mathcal{M}, \partial\mathcal{M})) &= \mathrm{rk}(\mathbf{H}_0(\mathcal{M})) = 1\\
\mathrm{rk}(\mathbf{H}_5(\mathcal{M}, \partial\mathcal{M})) &= \mathrm{rk}(\mathbf{H}_1(\mathcal{M})) = 0\\
\mathrm{rk}(\mathbf{H}_4(\mathcal{M}, \partial\mathcal{M})) &= \mathrm{rk}(\mathbf{H}_2(\mathcal{M})) = f_1 - 3\\
\mathrm{rk}(\mathbf{H}_3(\mathcal{M}, \partial\mathcal{M})) &= \mathrm{rk}(\mathbf{H}_3(\mathcal{M})) = 3f_1 - f_2 - b - 6\\
\mathrm{rk}(\mathbf{H}_2(\mathcal{M}, \partial\mathcal{M})) &= \mathrm{rk}(\mathbf{H}_4(\mathcal{M})) = f_1 - 3 - b.
\end{aligned}$$

At last, we can easily via the CW structure show that $\mathrm{rk}(\mathbf{H}_6(\mathcal{M})) = 0$. Consequently, by considering the long exact sequence of relative homology, we arrive at

$$0 \longrightarrow \mathbf{H}_6(\mathcal{M}, \partial\mathcal{M}) \longrightarrow \mathbf{H}_5(\partial\mathcal{M}) \longrightarrow \mathbf{H}_5(\mathcal{M}) \longrightarrow 0.$$

This implies

$$\mathrm{rk}(\mathbf{H}_1(\mathcal{M}, \partial\mathcal{M})) = \mathrm{rk}(\mathbf{H}_5(\mathcal{M})) = f_3 - 1.$$

Before dealing with the homology groups of the generalized intersection space, we need to compute the intersection homology groups of $X_{\mathcal{P}}$. Here [14], it has been shown that the odd degree of intersection homology groups of toric varieties vanish. However, in our example, we merely need the above result for the middle degree. Using 7.2, we finally get

$$\begin{aligned}
\mathrm{rk}(\mathbf{I}^{\bar{n}}\mathbf{H}_6(X)) &= 1\\
\mathrm{rk}(\mathbf{I}^{\bar{n}}\mathbf{H}_5(X)) &= 0\\
\mathrm{rk}(\mathbf{I}^{\bar{n}}\mathbf{H}_4(X)) &= f_1 - 3\\
\mathrm{rk}(\mathbf{I}^{\bar{n}}\mathbf{H}_3(X)) &= 0\\
\mathrm{rk}(\mathbf{I}^{\bar{n}}\mathbf{H}_2(X)) &= f_1 - 3\\
\mathrm{rk}(\mathbf{I}^{\bar{n}}\mathbf{H}_1(X)) &= 0\\
\mathrm{rk}(\mathbf{I}^{\bar{n}}\mathbf{H}_0(X)) &= 1.
\end{aligned}$$

8.1 ensures the existence of the associated generalized intersection space of $X_{\mathcal{P}}$. By using the constructed duality isomorphisms in the proof of 7.4, we can read off the homology groups of $I^{\bar{n}}X$, above and below the middle degree.

$$\begin{aligned}
&\mathbf{H}_r(\mathcal{M}) \xrightarrow{\cong} \widetilde{\mathbf{H}}_r(I^{\bar{n}}X) \text{ for } r > k\\
&\mathbf{H}_r(\mathcal{M}, \partial\mathcal{M}) \xrightarrow{\cong} \widetilde{\mathbf{H}}_r(I^{\bar{n}}X) \text{ for } r < k.
\end{aligned}$$

Finally, we use 7.7 and compute $\mathrm{rk}(\tilde{\mathbf{H}}_3(I^{\bar{n}}X))$. □

Remark 8.3
We employ the first identity on Euler characteristic on 7.7. Hence, we have

$$\begin{aligned}
&\big(0 - (f_3 - 1) + (f_1 - 3 - b) - 2(3f_1 - f_2 - b - 6) + (f_1 - 3 - b)\\
&- (f_3 - 1) + 0\big) - \big(1 - (f_1 - 3) - (f_1 - 3) + 1\big)\\
&= -2\big(2 + 0 + \sum_{i=1}^{f_3}(f_1^i - 3)\big),
\end{aligned}$$

where f_1^i denotes the number of 1-dimensional neighboring faces of the i-th vertex of $\mathcal{P}$. We can also interpret f_1^i as the number of 1-dimensional faces of $\mathcal{M}_i$, where $\mathcal{M}_i$ is the underlying 2-dimensional polygon of the link of the i-th vertex. This implies

$$\sum_{i=1}^{f_3}(f_1^i - 3) = 2f_2 - 3f_3$$

Then, we have

$$\begin{aligned}&\big(-2f_3 - 2f_1 + 2f_2 + 8\big)\\ &-\big(2f_1 - 4\big)\\ &= \big(4f_2 + 4f_3\big).\end{aligned}$$

Finally, we get

$$f_2 - f_3 - f_1 = 2,$$

which is the well-known Euler formula for convex 3-dimensional polytope.

Corollary 8.4
Considering the above remark, we arrive at

$$\begin{aligned}&rk(\tilde{\mathbf{H}}_6(I^{\bar{n}}X)) = 0\\ &rk(\tilde{\mathbf{H}}_5(I^{\bar{n}}X)) = f_2 - f_1 - 3\\ &rk(\tilde{\mathbf{H}}_4(I^{\bar{n}}X)) = f_1 - 3 - b\\ &rk(\tilde{\mathbf{H}}_3(I^{\bar{n}}X)) = 2(3f_1 - f_2 - b - 6)\\ &rk(\tilde{\mathbf{H}}_2(I^{\bar{n}}X)) = f_1 - 3 - b\\ &rk(\tilde{\mathbf{H}}_1(I^{\bar{n}}X)) = f_2 - f_1 - 3\\ &rk(\tilde{\mathbf{H}}_0(I^{\bar{n}}X)) = 0\end{aligned}$$

Remark 8.5
In the above corollary, we have

$$\chi(\tilde{\mathbf{H}}_*(I^{\bar{n}}X)) = 2(6 - f_1),$$

which is even. Compare this result with the fact that the Euler characteristic of a 6-dimensional manifold which is also always even.

Remark 8.6
In the above example, by knowing the homology groups of $I^{\bar{n}}X$, we can determine the intersection homology of $X_{\mathcal{P}}$. However, on the contrary, computing the intersection homology groups do not even determine the combinatorial data of the fan. Roughly speaking, the homology groups of intersection spaces, in the above example, see more structure than the intersection homology groups.

Remark 8.7
We can fit all the above data into the Mayer-Vietoris sequence, nicely.
Let $I^{\bar{n}}X$ be the associated generalized intersection space to a 6-dimensional toric variety, X. Then we set $\mathcal{U} \simeq \mathcal{M}$, where $\mathcal{M}$ is the $\mathcal{Q}$-manifold obtained by cutting out the isolated singularities of X. We also set $\mathcal{V} \cong \mathcal{C}(\sqcup \mathcal{L}_{<3})$. Hence, we get $\cap := \mathcal{U} \cap \mathcal{V} \simeq \mathcal{L}_{<3}$. Thus, we have

$$
\begin{aligned}
0 &\longrightarrow \overbrace{\mathbf{H}_6(\cap)}^{=0} \longrightarrow \overbrace{\mathbf{H}_6(\mathcal{U})}^{=0} \xrightarrow{\cong} \overbrace{\mathbf{H}_6(I^{\bar{n}}X)}^{=0} \longrightarrow \overbrace{\mathbf{H}_5(\cap)}^{=0} \longrightarrow \overbrace{\mathbf{H}_5(\mathcal{U})}^{\cong \mathbb{Q}^{f_3-1}} \xrightarrow{\cong} \overbrace{\mathbf{H}_5(I^{\bar{n}}X)}^{\cong \mathbb{Q}^{f_3-1}} \\
&\longrightarrow \overbrace{\mathbf{H}_4(\cap)}^{=0} \longrightarrow \overbrace{\mathbf{H}_4(\mathcal{U})}^{\cong \mathbb{Q}^{f_1-3-b}} \xrightarrow{\cong} \overbrace{\mathbf{H}_4(I^{\bar{n}}X)}^{\cong \mathbb{Q}^{f_1-3-b}} \longrightarrow \overbrace{\mathbf{H}_3(\cap)}^{=0} \longrightarrow \overbrace{\mathbf{H}_3(\mathcal{U})}^{\cong \mathbb{Q}^{3f_1-f_2-b-6}} \longrightarrow \overbrace{\mathbf{H}_3(I^{\bar{n}}X)}^{\cong \mathbb{Q}^{2(3f_1-f_2-b-6)}} \\
&\longrightarrow \overbrace{\mathbf{H}_2(\cap)}^{\cong \mathbb{Q}^{\Sigma}} \longrightarrow \overbrace{\mathbf{H}_2(\mathcal{U})}^{\cong \mathbb{Q}^{f_1-3}} \longrightarrow \overbrace{\mathbf{H}_2(I^{\bar{n}}X)}^{\cong \mathbb{Q}^{f_1-3-b}} \longrightarrow \overbrace{\mathbf{H}_1(\cap)}^{=0} \longrightarrow \overbrace{\mathbf{H}_1(\mathcal{U})}^{=0} \longrightarrow \overbrace{\mathbf{H}_1(I^{\bar{n}}X)}^{\cong \mathbb{Q}^{f_3-1}} \\
&\xrightarrow{\cong} \overbrace{\widetilde{\mathbf{H}}_0(\cap)}^{\cong \mathbb{Q}^{f_3-1}} \longrightarrow \overbrace{\widetilde{\mathbf{H}}_0(\mathcal{U})}^{=0} \oplus \overbrace{\widetilde{\mathbf{H}}_0(\mathcal{V})}^{=0} \longrightarrow \overbrace{\widetilde{\mathbf{H}}_0(I^{\bar{n}}X)}^{=0} \longrightarrow 0,
\end{aligned}
$$

where $\Sigma := \sum_{i=1}^{f_3}(f_1^i - 3) = 2f_2 - 3f_3$.

Corollary 8.8
For each 6-dimensional toric variety, X, with $\mathbf{H}_3(X) = 0$, b is combinatorial invariant, and

$$b = \Sigma.$$

Remark 8.9
For the proof of the above corollary, we simply set $\mathbf{H}_3(\mathcal{U}) = 0$ and hence $\mathbf{H}_3(I^{\bar{n}}X) = 0$. By using the splitting lemma, we get

$$b = \Sigma = 2f_2 - 3f_3.$$

However, reformulating the assumption yields

$$b = 3f_1 - f_2 - 6.$$

Setting both expressions equal to each other will result in the Euler formula.

Part IV

INTERSECTION SPACE PAIRS

9 A BRIEF INTRODUCTION TO INTERSECTION SPACE PAIRS

In this chapter, we briefly study the topological construction of intersection space pairs introduced by M. AGUSTIN and J. FERNANDEZ DE BOBADILLA. Readers may consult [2] for a detailed study of the matter. Note that generally, we use the same notation as in [2], and all credit of this section goes to M. AGUSTIN and J. FERNANDEZ DE BOBADILLA. Before introducing the mentioned topological construction, we start with some basic definitions.

Definition 9.1

1. *A t-uple of spaces is a set of topological spaces* $(\mathcal{Z}_1, ..., \mathcal{Z}_t)$.
2. *A morphism from a t-uple of spaces into a space* $(\mathcal{Z}_1, ..., \mathcal{Z}_t) \longrightarrow \mathcal{Z}$ *is a set of morphisms* $\varphi_i : \mathcal{Z}_i \longrightarrow \mathcal{Z}$.
3. *A morphism between t-uples of spaces* $(\mathcal{Z}_1, ..., \mathcal{Z}_t) \longrightarrow (\mathcal{Z}'_1, ..., \mathcal{Z}'_t)$ *is a set of morphisms* $\varphi_i : \mathcal{Z}_i \longrightarrow \mathcal{Z}'_i$.
4. *The mapping cylinder of morphism* $\varphi = (\varphi_1, ..., \varphi_t) : (\mathcal{Z}_1, ..., \mathcal{Z}_t) \longrightarrow \mathcal{Z}$, $cyl(\varphi)$, *is the union of the t-uple* $(\mathcal{Z}_1, ..., \mathcal{Z}_t) \times [0,1]$ *with* $(\mathcal{Z}, Im(\varphi_1), ..., Im(\varphi_t))$ *with the equivalence relation* $\sim$ *such that for* $i = 0, ..., t$ *and for every* $x \in \mathcal{Z}_i$, *we have* $(x, 1) \sim \varphi_i(x)$.

Definition 9.2
Let $\sigma : (\mathcal{Z}_1, \ldots, \mathcal{Z}_t) \longrightarrow \mathcal{B}$ *be a locally trivial fibration of t-uples of spaces. The cone of* σ *over the base* $\mathcal{B}$ *is the locally trivial fibiration*

$$\pi : cyl(\sigma) \longrightarrow \mathcal{B},$$

where $cyl(\sigma)$ *is the mapping cylinder of* σ, $\pi(x,t) := \sigma(x)$ *for* $(x,t) \in (\mathcal{Z}_1, \ldots, \mathcal{Z}_t) \times [0,1]$ *and* $\pi(b) := b$ *for* $b \in \mathcal{B}$ *(The definition of* π *is compatible with the identifications made to construct* $cyl(\sigma)$*). The cone over a fibration has a canonical vertex section*

$$s : \mathcal{B} \longrightarrow cyl(\sigma)$$

sending any $b \in \mathcal{B}$ *to the vertex of the cone* $(cyl(\sigma))_b$.

Definition 9.3
Let (X, Y) *be a pair of topological spaces and let*

$$X_{d-k} \supset X_{d-k-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

be a topological pseudomanifold such that X_{d-k} *is a subspace of* Y. *We say that the pair* (X, Y) *has a conical structure with respect to the stratified subspace if for every* $r \geq k$ *there exists an open neighbourhood* TX_{d-r} *of* $X_{d-r} \backslash X_{d-r-1}$ *in* $X \backslash X_{d-r-1}$ *with the following properties:*

1. *Let $\overline{TX_{d-r}}$ be the closure of TX_{d-r} in X. There is a locally trivial fibration of $2(r-k+1)$-uples of spaces*

$$\begin{array}{c}(\overline{TX_{d-r}}\backslash X_{d-r-1})\cap(X,Y,\overline{TX_{d-k}},X_{d-k},\overline{TX_{d-k-1}},X_{d-k-1},\dots,\overline{TX_{d-r+1}},X_{d-r+1})\\ \downarrow{\scriptstyle\sigma_{d-r}}\\ X_{d-r}\backslash X_{d-r-1}\end{array}$$

such that its restriction to the boundary

$$\begin{array}{c}(\partial\overline{TX_{d-r}}\backslash X_{d-r-1})\cap(X,Y,\overline{TX_{d-k}},X_{d-k},\overline{TX_{d-k-1}},X_{d-k-1},\dots,\overline{TX_{d-r+1}},X_{d-r+1})\\ \downarrow{\scriptstyle\sigma^{\partial}_{d-r}}\\ X_{d-r}\backslash X_{d-r-1}\end{array}$$

is a locally trivial fibration.

2. *The fibration σ_{d-r} is the cone of σ^{∂}_{d-r} over the base $X_{d-r}\backslash X_{d-r-1}$.*

3. *Let $k\le r_1<r_2\le d$ and consider the isomorphism induced by property (2).*

$$\overline{TX_{d-r_1}}\cap(\overline{TX_{d-r_2}}\backslash X_{d-r_2-1})\cong(\overline{TX_{d-r_1}}\cap(\partial\overline{TX_{d-r_2}}\backslash X_{d-r_2-1}))\times[0,1]/\sim,$$

where $\sim$ is the equivalence relation of the mapping cylinder.
If we remove X_{d-r_1-1} in both part of this isomorphism, we obtain an isomorphism

$$\phi_{r_1,r_2}:(\overline{TX_{d-r_1}}\backslash X_{d-r_1-1})\cap\overline{TX_{d-r_2}}\cong((\overline{TX_{d-r_1}}\backslash X_{d-r_1-1})\cap\partial\overline{TX_{d-r_2}})\times[0,1).$$

Note that since X_{d-r_2} is contained in X_{d-r_1-1}, the vertex section of the cone is not included in the previous spaces.
With this notation, we have the equality

$$(\sigma_{d-r_1})_{|(\overline{TX_{d-r_1}}\backslash X_{d-r_1-1})\cap\overline{TX_{d-r_2}}}=\phi^{-1}_{r_1,r_2}\circ((\sigma_{d-r_1})_{|(\overline{TX_{d-r_1}\backslash X_{d-r_1-1}})\cap\partial\overline{TX_{d-r_2}}},Id_{[0,1)}\circ\phi_{r_1,r_2},$$

that is, the fibration σ_{d-r_1} in the intersection $(\overline{TX_{d-r_1}}\backslash X_{d-r_1-1})\cap\overline{TX_{d-r_2}}$ is determined by its resteriction to $(\overline{TX_{d-r_1}}\backslash X_{d-r_1-1})\cap\partial\overline{TX_{d-r_2}}$.

4. *Let $k\le r_1<r_2\le d$. If $\partial\overline{TX_{d-r_2}}\cap(X_{d-r_1}\backslash X_{d-r_1-1})\neq\emptyset$, then we have the following equality of $2(r_1-k+1)$-uples*

$$\begin{aligned}&(\overline{TX_{d-r_1}}\backslash X_{d-r_1-1})\cap(X,Y,\overline{TX_{d-k}},X_{d-k},\overline{TX_{d-k-1}},X_{d-k-1},\dots,\overline{TX_{d-r_1+1}},X_{d-r_1+1})\\ &\cap\partial\overline{TX_{d-r_2}}=\sigma^{-1}_{d-r_1}(\partial\overline{TX_{d-r_2}}\cap X_{d-r_1}\backslash X_{d-r_1-1}))\end{aligned}$$

and, in this space, we have

$$\sigma^{\partial}_{d-r_2}\circ\sigma_{d-r_1}=\sigma^{\partial}_{d-r_2}.$$

Definition 9.4
We say that a conical structure verifies the r-th triviality property (T_r) *if the locally trivial fibration* σ^{∂}_{d-r} *is trivial, that is, the following two properties hold for every connected component* $\mathcal{S}_{d-r}$ *of* $X_{d-r} \backslash X_{d-r-1}$.

1. *There exist as isomorphism*

$$(\sigma^{\partial}_{d-r})^{-1}(\mathcal{S}_{d-r}) \cong \mathcal{L} \times \mathcal{S}_{d-r},$$

   where $\mathcal{L}$ *denotes the* $2(r-k+1)$*-uple of the links of* $\mathcal{S}_{d-r}$ *in*

$$(X, Y, \overline{TX_{d-k}}, X_{d-k}, \overline{TX_{d-k-1}}, X_{d-k-1}, \dots, \overline{TX_{d-r+1}}, X_{d-r+1}).$$

2. *Under the identification given by property (1),* σ^{∂}_{d-r} *restricted to* $\mathcal{L} \times \mathcal{S}_{d-r}$ *is the canonical projection over* $\mathcal{S}_{d-r}$.

Definition 9.5
Let (X, Y) *be a pair of spaces with a conical structure as in definition 9.3 which verifies the r-th triviality property* (T_r) *for any r. Fix a trivialization*

$$(\sigma^{\partial}_{d-r})^{-1}(\mathcal{S}_{d-r}) \cong \mathcal{L} \times \mathcal{S}_{d-r}$$

over each connected component of each stratum. The set of all trivialization is called a ***system of trivializations for the conical structure****.*

Remark 9.6
By definition $(\sigma^{\partial}_{d-r_2})^{-1}(\mathcal{S}_{d-r_2}) \cap \sigma^{-1}_{d-r_1}(\mathcal{S}_{d-r_1})$ *and* $\sigma^{-1}_{d-r_1}((\sigma^{\partial}_{d-r_2})^{-1}(\mathcal{S}_{d-r_2}) \cap \mathcal{S}_{d-r_1})$ *are* $2(k-r+1)-uplas$*. To Simplify the notation, in the rest of this work we denote by* $(\sigma^{\partial}_{d-r_2})^{-1}(\mathcal{S}_{d-r_2}) \cap \sigma^{-1}_{d-r_1}(\mathcal{S}_{d-r_1})$ *and* $\sigma^{-1}_{d-r_1}((\sigma^{\partial}_{d-r_2})^{-1}(\mathcal{S}_{d-r_2}) \cap \mathcal{S}_{d-r_1})$ *the first components of these* $2(k-r+1)$*-uplas.*

Consider also the space $(\sigma^{\partial}_{d-r_2})^{-1}(\mathcal{S}_{d-r_2}) \cap \mathcal{S}_{d-r_1}$.
Since we have fixed a system of trivializations, we have isomorphisms

$$\mathcal{L}^{\mathcal{S}_{d-r_1}}_{d-r_2} \times \mathcal{S}_{d-r_2} \cong (\sigma^{\partial}_{d-r_2})^{-1}(\mathcal{S}_{d-r_2}) \cap \mathcal{S}_{d-r_1}$$

and

$$\mathcal{L}^{\sigma^{-1}_{d-r_1}(\mathcal{S}_{d-r_1})}_{d-r_2} \times \mathcal{S}_{d-r_2} \cong (\sigma^{\partial}_{d-r_2})^{-1}(\mathcal{S}_{d-r_2}) \cap \sigma^{-1}_{d-r_1}(\mathcal{S}_{d-r_1})$$

where $\mathcal{L}^{\mathcal{S}_{d-r_1}}_{d-r_2}$ denotes the fiber link bundle of $\mathcal{S}_{d-r_1}$ over $\mathcal{S}_{d-r_2}$ and $\mathcal{L}^{\sigma^{-1}_{d-r_1}(\mathcal{S}_{d-r_1})}_{d-r_2}$ denotes the fiber of the link bundle of $\sigma^{-1}_{d-r_1}(\mathcal{S}_{d-r_1})$ over $\mathcal{S}_{d-r_2}$. Moreover, by the property (4) of definition 9.3, we have an equality

$$\sigma^{-1}_{d-r_1}(\mathcal{S}_{d-r_1}) \cap (\sigma^{\partial}_{d-r_2})^{-1}(\mathcal{S}_{d-r_2}) = \sigma^{-1}_{d-r_1}((\sigma^{\partial}_{d-r_2})^{-1}(\mathcal{S}_{d-r_2}) \cap \mathcal{S}_{d-r_1})$$

and, again using the fixed system of trivializations, we obtain an isomorphism

$$\sigma_{d-r_1}^{-1}((\sigma_{d-r_2}^{\partial})^{-1}(\mathcal{S}_{d-r_2}) \cap \mathcal{S}_{d-r_1}) \cong c(\mathcal{L}_{d-r_1}^{X}) \times ((\sigma_{d-r_2}^{\partial})^{-1}(\mathcal{S}_{d-r_2}) \cap \mathcal{S}_{d-r_1}),$$

where $\mathcal{L}_{d-r_1}^{X}$ is the fiber of the link bundle of X over $\mathcal{S}_{d-r_1}$. Combining the previous isomorphisms, we obtain γ.

$$\begin{aligned} \gamma : c(\mathcal{L}_{d-r_1}^{X}) \times \mathcal{L}_{d-r_2}^{\mathcal{S}_{d-r_1}} \times \mathcal{S}_{d-r_2} &\xrightarrow{\cong} \mathcal{L}_{d-r_2}^{\sigma_{d-r_1}^{-1}(\mathcal{S}_{d-r_1})} \times \mathcal{S}_{d-r_2} \\ (x, y, z) &\longrightarrow (\gamma_1(x, y, z), z). \end{aligned}$$

Definition 9.7
*We say that the system of trivializations is **compatible** if for any two connected components of $\mathcal{S}_{d-r_1}$ and $\mathcal{S}_{d-r_2}$, as above, the map γ_1 does not depend on z, that is, if there exist an isomorphism*

$$\beta : c(\mathcal{L}_{d-r_1}^{X}) \times \mathcal{L}_{d-r_2}^{\mathcal{S}_{d-r_1}} \xrightarrow{\cong} \mathcal{L}_{d-r_2}^{\sigma_{d-r_1}^{-1}(\mathcal{S}_{d-r_1})}$$

such that $\gamma = (\beta, Id_{\mathcal{S}_{d-r_2}})$.

Definition 9.8
*We say that the conical structure is **trivial** if it verifies the **r-th triviality property** (T_r) for any r and there exists a compatible system of trivializations.*

Remark 9.9
We will show that for an arbitrary 6-dimensional toric variety or pseudo-toric variety

$$X = X_6 \supset X_2 \supset X_0,$$

the pair (X_6, X_2) has a conical structure concerning the stratification

$$X_2 \supset X_0.$$

However, this is generally true for an arbitrary toric variety or pseudo-toric variety. But showing this is beyond the scope of this work.

Following the same line as in [2], we introduce an inductive construction of intersection spaces. However, the procedure depends on choices made at each inductive step and may be obstructed for a given set of choices or carried until the deepest stratum. We will show that for the canonical stratification of toric varieties, we will encounter no obstruction, and we can carry out the procedure until the deepest stratum.

Definition 9.10
*Let $\sigma : (X, Y) \longrightarrow B$ be a locally trivial fibration. We say that σ admits a **fiberwise rational q-homology truncation** if there exits a morphism of pairs of spaces*

$$\phi : (X_{\leq q}, Y_{\leq q}) \longrightarrow (X, Y)$$

such that $\sigma \circ \phi$ is a locally trivial fibration and, for any $b \in B$

1. *the homomorphism in homology of fibers*

$$\mathbf{H}_i((X_{\leq q})_b, (Y_{\leq q})_b; \mathbb{Q}) \longrightarrow \mathbf{H}_i(X_b, Y_b; \mathbb{Q})$$

is an isomorphism if $i \leq q$.

2. *the homology group*

$$\mathbf{H}_i((X_{\leq q})_b, (Y_{\leq})_b; \mathbb{Q}) \cong 0 \text{ if } i > q.$$

Remark 9.11
Given a pair of spaces (X, Y) we denote by $(X, Y)_{\leq q}$ the pair $(X_{\leq q}, Y_{\leq q})$.

Definition 9.12
Given a pair of spaces (X, Y) with a conical structure as in 9.3, a fiberwise rational q-homology truncation of σ^{∂}_{d-r}

$$\begin{array}{ccc} (\partial\overline{TX_{d-r}} \backslash X_{d-r-1} \cap (X,Y))_{\leq q} & & \\ \downarrow \phi^{\partial}_{d-r} & \overset{(\sigma^{\partial}_{d-r})_{\leq q}}{\searrow} & \\ & & X_{d-r} \backslash X_{d-r-1} \\ & \overset{\sigma^{\partial}_{d-r}}{\nearrow} & \\ \partial\overline{TX_{d-r}} \backslash X_{d-r-1} \cap (X,Y) & & \end{array}$$

*is **compatible with the conical structure** if, for every $r' > r$,*

$$\sigma_{d-r'} \circ (\sigma^{\partial}_{d-r})_{\leq q} : (\sigma^{\partial}_{d-r})^{-1}_{\leq q}(\overline{TX_{d-r'}} \cap (X_{d-r} \backslash X_{d-r-1})) \longrightarrow X_{d-r'} \backslash X_{d-r'-1}$$

is the cone of

$$\sigma^{\partial}_{d-r'} \circ (\sigma^{\partial}_{d-r'})_{\leq q} : (\sigma^{\partial}_{d-r})^{-1}_{\leq q}(\partial\overline{TX_{d-r'}} \cap (X_{d-r} \backslash X_{d-r-1})) \longrightarrow X_{d-r'} \backslash X_{d-r'-1}$$

Proposition 9.13
Given a pair of spaces (X, Y) with conical structure as in 9.3, if there exist a fiberwise rational q-homology truncation of $\sigma^{\partial}_{d-r'}$, then there exist a fiberwise rational q-homology truncation of σ^{∂}_{d-r} compatible with the conical structure.

Proof. For proof, the reader may consult [2]. □

Definition 9.14
Given a pair of spaces (X, Y) with a conical structure verifying the triviality property (T_r) for any r, choose a system of trivializations as in Definition 9.5. A fiberwise rational q-homology truncation of σ^{∂}_{d-r}

$$\begin{array}{ccc} (\partial\overline{TX_{d-r}}\backslash X_{d-r-1}\cap(X,Y))_{\leq q} & & \\ \downarrow \phi^{\partial}_{d-r} & \searrow^{(\sigma^{\partial}_{d-r})_{\leq q}} & \\ & & X_{d-r}\backslash X_{d-r-1} \\ & \nearrow_{\sigma^{\partial}_{d-r}} & \\ \partial\overline{TX_{d-r}}\backslash X_{d-r-1}\cap(X,Y) & & \end{array}$$

is ***compatible with the trivialization*** *if the following conditions hold.*

1. *Given a connected component $\mathcal{S}_{d-r}$ of $X_{d-r}\backslash X_{d-r-1}$, if $\mathcal{L}_{d-r} = (\mathcal{L}^X_{d-r}, \mathcal{L}^Y_{d-r})$ denotes the fiber of the link bundle of (X,Y) over $\mathcal{S}_{d-r}$ and*

$$(\sigma^{\partial}_{d-r})^{-1}(\mathcal{S}_{d-r}) \cong \mathcal{L}_{d-r} \times \mathcal{S}_{d-r}$$

is the isomorphism induced by the system of trivializations, then there exist a pair of spaces $(\mathcal{L}_{d-r})_{\leq q} := ((\mathcal{L}^X_{d-r})_{\leq q}, (\mathcal{L}^Y_{d-r})_{\leq q})$ such that the group $\mathbf{H}_i((\mathcal{L}^X_{d-r})_{\leq q}, (\mathcal{L}^Y_{d-r})_{\leq q}; \mathbb{Q}) \cong 0$ if $i > q$, we have an isomorphism

$$(\sigma^{\partial}_{d-r})^{-1}_{\leq q}(\mathcal{S}_{d-r}) \cong (\mathcal{L}_{\leq q} \times \mathcal{S}_{d-r})$$

and, under these identifications, $((\sigma^{\partial}_{d-r})_{\leq q})_{|(\mathcal{L}_{d-r})_{\leq q}\times\mathcal{S}_{d-r}}$ is the canonical projection and $((\phi^{\partial}_{d-r})_{\leq q})_{|(\mathcal{L}_{d-r})_{\leq q}\times\mathcal{S}_{d-r}} = (\phi_1, Id_{\mathcal{S}_{d-r}})$ where $\phi_1 : (\mathcal{L}_{d-r})_{\leq q} \longrightarrow \mathcal{L}_{d-r}$ is a morphism such that

$$\mathbf{H}_i(\phi) : \mathbf{H}_i((\mathcal{L}^X_{d-r})_{\leq q}, (\mathcal{L}^Y_{d-r})_{\leq q}; \mathbb{Q}) \longrightarrow \mathbf{H}_i(\mathcal{L}^X_{d-r}, \mathcal{L}^Y_{d-r}; \mathbb{Q})$$

is an isomorphism if $i \leq q$.

2. *Given $r' > r$ and a connected component $\mathcal{S}_{d-r'}$ of $X_{d-r'}\backslash X_{d-r'-1}$ such that*

$$(\sigma^{\partial}_{d-r'})^{-1}(\mathcal{S}_{d-r'}) \cap \mathcal{S}_{d-r} \neq \emptyset,$$

let $\mathcal{L}^{\mathcal{S}_{d-r}}_{d-r'}$ and $\mathcal{L}^{\sigma^{-1}_{d-r}(\mathcal{S}_{d-r})}_{d-r'}$ denote the fibers of the link bundles of $\mathcal{S}_{d-r}$ and $\sigma^{-1}_{d-r}(\mathcal{S}_{d-r})$ over $\mathcal{S}_{d-r'}$ respectively. Moreover, let

$$\gamma : c(\mathcal{L}^X_{d-r_1}) \times \mathcal{L}^{\mathcal{S}_{d-r_1}}_{d-r_2} \times \mathcal{S}_{d-r_2} \cong \mathcal{L}^{\sigma^{-1}_{d-r_1}(\mathcal{S}_{d-r_1})}_{d-r_2} \times \mathcal{S}_{d-r_2}$$

be the isomorphism defined previously in 9.6. Then, the image of the composition

$$\begin{array}{ccc} c((\mathcal{L}^X_{d-r})_{\leq q} \times \mathcal{L}^{\mathcal{S}_{d-r}}_{d-r'} \times \mathcal{S}_{d-r'} & \xrightarrow{(c(\phi_1), Id)} & c(\mathcal{L}^X_{d-r}) \times \mathcal{L}^{\mathcal{S}_{d-r}}_{d-r'} \times \mathcal{S}_{d-r'} \\ & & \downarrow \gamma \\ & & \mathcal{L}^{\sigma^{-1}_{d-r}(\mathcal{S}_{d-r})}_{d-r'} \times \mathcal{S}_{d-r'} \end{array}$$

is equal to $\mathcal{A} \times \mathcal{S}_{d-r'}$ for some subset $\mathcal{A} \subset \mathcal{L}^{\sigma^{-1}_{d-r}}(\mathcal{S}_{d-r})_{d-r'}$.

Remark 9.15
If the conical structure is trivial (see Definition 9.8), the condition (1) of the previous definition implies the condition (2).

Remark 9.16
As we have seen, the conical structure of toric varieties is trivial, and hence due to the previous remark, condition (1) of the previous definition implies condition (2). Furthermore, we will see that ϕ_1, defined previously, is an inclusion map, making the whole construction easier.

9.1 A TOPOLOGICAL CONSTRUCTION OF INTERSECTION SPACES

In this section, we introduce a topological construction of intersection space pairs. The method of construction goes along the same line as [2].

9.1.1 *The initial step of the induction*

Let X be a topological pseudomanifold such that the pair (X, X_{d-2}) has a conical structure concerning the stratification

$$X = X_d \supset X_{d-2} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset.$$

We consider the fixed open neighborhoods TX_{d-r} with respect to the stratification

$$X_{d-2} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset.$$

Let m be the minimum such that $X_{d-m} \backslash X_{d-m-1} \neq \emptyset$. We assume that the fibration $\sigma^{\partial}_{d-m} : \partial\overline{TX_{d-m}} \backslash X_{d-m-1} \longrightarrow X_{d-m} \backslash X_{d-m-1}$ admits a fiberwise rational $\bar{q}(m)$-homology truncation compatible with the conical structure.

Remark 9.17
If such truncation isn't possible, then the intersection space does not exist.

We consider

$$\begin{array}{ccc} (\partial\overline{TX_{d-m}} \backslash X_{d-m-1})_{\leq \bar{q}(m)} & & \\ \Big\downarrow \phi^{\partial}_{d-m} & \searrow^{(\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}} & \\ & & X_{d-m} \backslash X_{d-m-1}. \\ & \nearrow_{\sigma^{\partial}_{d-m}} & \\ \partial\overline{TX_{d-m}} \backslash X_{d-m-1} & & \end{array}$$

We construct a new space X'_m, a homotopy equivalence $\pi : X'_m \longrightarrow X$ with contractible fibers and a subspace $I^{\bar{p}}_m X \hookrightarrow X'_m$ as follows. We define the map

$$(\sigma_{d-m})_{\leq \bar{q}(m)} : cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}) \longrightarrow X_{d-m} \backslash X_{d-m-1}$$

to be the cone of the fibration $(\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}$ over $X_{d-m} \backslash X_{d-m-1}$. Property (2) in 9.3 implies that there exists a fiber bundle morphism

$$\phi_{d-m} : cyl(\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}) \longrightarrow \overline{TX_{d-m}} \backslash X_{d-m-1}$$

over the base $X_{d-m} \backslash X_{d-m-1}$ which preserves the vertex sections. Let

$$\theta_{d-m} : cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}) \longrightarrow X \backslash X_{d-m-1}$$

be the composition of the fiber bundle morphism ϕ_{d-m} with the natural inclusion of the closed subset $\overline{TX_{d-m}} \backslash X_{d-m-1}$ into $X \backslash X_{d-m-1}$. Let $cyl(\theta_{d-m})$ be the mapping cylinder of θ_{d-m}. Denote by

$$s_{d-m} : X_{d-m} \backslash X_{d-m-1} \longrightarrow cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)})$$

the vertex section. Let Z_m be the result of quotienting $cyl(\theta_{d-m})$ by the equivalence relation which identifies, for any $x \in X_{d-m} \backslash X_{d-m-1}$, the subspace $s_{d-m}(x) \times [0,1]$ to a point. Note that $cyl(\phi^{\partial}_{d-m})$ of ϕ^{∂}_{d-m} in the above diagram is a subspace both of $cyl(\theta_{d-m})$ and of Z_{d-m}. We have a natural projection map $\pi : Z_m \longrightarrow X \backslash X_{d-m-1}$ which is a homotopy equivalence whose fibers are contractible and has a natural section denoted by α_m. α_m is a closed inclusion of $X \backslash X_{d-m-1}$ into Z_m. Define X'_m as the set $Z_m \cup X_{d-m-1}$. The projection map extends to a projection

$$\pi_m : X'_m \longrightarrow X.$$

Consider in X'_m the topology spanned by all the open subsets of Z_m and the collection of subsets of the form $\pi_m^{-1}(U)$ for any open subset U of X.
Define the step m intersection space $I^{\bar{p}}_m X$ to be the subspace of X'_m given by

$$I^{\bar{p}}_m X := cyl((\sigma^{\partial}_{d-m})_{\leq \bar{q}(m)}) \times \{0\} \cup cyl(\phi^{\partial}_{d-m}) \cup (X \backslash TX_{d-m})$$

with the restricted topology.

Remark 9.18
Note that we have the equality $cyl(\phi^{\partial}_{d-m}) = \pi_m^{-1}(\partial\overline{TX_{d-m}} \backslash X_{d-m-1})$. *Then, we have the following chains of inclusions*

$$X'_m \supset X \supset X_{d-m} \supset \cdots \supset X_0,$$
$$X'_m \supset I^{\bar{p}}_m X \supset X_{d-m} \supset \cdots \supset X_0,$$

where X *is embedded in* X'_m *via section* α_m.

The following lemma follows from the above construction.

Lemma 9.19
The pairs (X'_m, X_{d-m}) *and* $(I^{\bar{p}}_m X, X_{d-m})$ *have a conical structure with respect to the stratified subspace* $X_{d-m-1} \supset \cdots \supset X_0$*, given by the following open neighbourhoods of* $X_{d-r} \backslash X_{d-r-1}$ *:* $\pi_m^{-1}(TX_{d-r})$ *is a neighbourhood in* $X' \backslash X_{d-r-1}$ *and* $\pi_m^{-1}(TX_{d-r}) \cap I^{\bar{p}}_m X$ *is a neighbourhood in* $I^{\bar{p}}_m X \backslash X_{d-r-1}$*.*

9.1.2 *The inductive step*

We first indicate that in the inductive step a fibration of link pairs must admit a fiberwise rational q-homology truncation, or we can't carry out the inductive step. The smaller space in the pair is constructed by iterated modifications of $X_{d-2} = X_{d-m}$. Define

$$I^{\bar{p}}_m(X_{d-2}) := X_{d-2}.$$

We assume by induction that, for $k \geq m$, we have constructed

1. A space X'_k and a projection

   $$\pi_k : X'_k \longrightarrow X$$

   which is a homotopy equivalence with contractible fibers, together with a section α_k providing a closed inclusion of X into X'_k.

2. Subspaces $I^{\bar{p}}_k(X_{d-2}) \subset I^{\bar{p}}_k X \subset X'_k$ such that, embedding X into X'_k via α_k, we have the topological pseudomanifold

   $$X_0 \subset X_1 \subset \cdots \subset X_{d-k-1}$$

   embedded into $I^{\bar{p}}_k(X_{d-2})$,

3. The pairs $(X'_k, I^{\bar{p}}_k X)$, $(I^{\bar{p}}_k X, I^{\bar{p}}_k(X_{d-2}))$ have respective conical structures with respect to the stratified subspace described in the previous point. The open neighbourhoods of $X_{d-r} \backslash X_{d-r-1}$ appearing in these structures are $\pi_k^{-1}(TX_{d-r})$ in $X'_k \backslash X_{d-k-1}$ and $\pi_k^{-1}(TX_{d-r}) \cap I^{\bar{p}}_k X$ in $I^{\bar{p}_k} X \backslash X_{d-k-1}$, respectively.

If $X_{d-k-1} \backslash X_{d-k-2}$ is empty we define $X'_{k+1} := X'_k$, $\pi_{k+1} := \pi'_k$, $\alpha_{k+1} := \alpha_k$, $I^{\bar{p}}_{k+1} X := I^{\bar{p}}_k X$, and $I^{\bar{p}}_{k+1}(X_{d-2}) := I^{\bar{p}}_k(X_{d-2})$.
If $X_{d-k-1} \backslash X_{d-k-2} \neq \emptyset$, since the pair $(I^{\bar{p}}_k X, I^{\bar{p}}_k(X_{d-2}))$ has a conical structure concerning the stratified subspace

$$X_0 \subset X_1 \subset \cdots \subset X_{d-k-1},$$

we have a locally trivial fibration of pairs

$$\sigma^{\partial}_{d-k-1} : (\overline{\partial \pi_k^{-1}(TX_{d-k-1})}) \backslash X_{d-k-2}) \cap (I^{\bar{p}}_k X, I^{\bar{p}}_k(X_{d-2})) \longrightarrow X_{d-k-1} \backslash X_{d-k-2}.$$

Remark 9.20
As indicated above, if the fibration does not admit a fiberwise rational $\overline{q}(k+1)$-homology truncation, then the intersection space construction cannot be completed with previous choices.

In the other case, we choose a fiberwise rational $\overline{q}(k+1)-$homology truncation compatible with the conical structure.

$$
\begin{array}{ccc}
((\partial\overline{\pi_k^{-1}(TX_{d-k-1})}\backslash X_{d-k-2} \cap (I_k^{\overline{p}}X, I_k^{\overline{p}}(X_{d-2}))_{\leq \overline{q}(k+1)} & \xrightarrow{(\sigma^{\partial}_{d-k-1})_{\leq \overline{q}(k+1)}} & \\
\downarrow \phi^{\partial}_{d-k-1} & & X_{d-k-1}\backslash X_{d-k-2}. \\
(\partial\overline{\pi_k^{-1}(TX_{d-k-1})}\backslash X_{d-k-2}) \cap (I_k^{\overline{p}}X, I_k^{\overline{p}}(X_{d-2})) & \xrightarrow{\sigma^{\partial}_{d-k-1}} &
\end{array}
$$

We construct now a homotopy equivalence $\pi_{k+1} : X'_{k+1} \longrightarrow X$ with contractible fibres and a pair of subspaces $(I^{\overline{p}}_{k+1}X, I^{\overline{p}}_{k+1}(X_{d-2})) \hookrightarrow X'_{k+1}$ as follows.
Let

$$(\sigma_{d-k-2})_{\leq \overline{q}(k+1)} : cyl((\sigma^{\partial}_{d-k-1})_{\leq \overline{q}(k+1)}) \longrightarrow X_{d-k-1}\backslash X_{d-k-2}$$

be the cone of the fibration of pairs $(\sigma^{\partial}_{d-k-1})_{\leq \overline{q}(k+1)}$ over $X_{d-k-1}\backslash X_{d-k-2}$. By property (2) of 9.3 there exists a morphism of fiber bundles of pairs

$$\phi_{d-k-1} : cyl((\sigma^{\partial}_{d-k-1})_{\leq \overline{q}(k+1)}) \longrightarrow (\overline{\pi_k^{-1}(TX_{d-k-1})}\backslash X_{d-k-2}) \cap (I_k^{\overline{p}}X, I_k^{\overline{p}}(X_{d-2}))$$

over the base $X_{d-k-1}\backslash X_{d-k-2}$ which preserves the vertex sections.
Let

$$\theta_{d-k-1} : cyl((\sigma^{\partial}_{d-k-1})_{\leq \overline{q}(k+1)}) \longrightarrow X'_k\backslash X_{d-k-2}.$$

be the composition of the fiber bundle morphism ϕ_{d-k-1} with the natural inclusion

$$(\overline{\pi_k^{-1}(TX_{d-k-1})})\backslash X_{d-k-2}) \cap (I_k^{\overline{p}}X, I_k^{\overline{p}}(X_{d-2})) \hookrightarrow X'_k\backslash X_{d-k-2}$$

$cyl(\theta_{d-k-1})$, the mapping cylinder of θ_{d-k-1}, is by definition the union of the pair $cyl((\sigma^{\partial}_{d-k-1})_{\leq \overline{q}(k+1)}) \times [0,1]$ with the pair $(X'_k\backslash X_{d-k-2})$ with the usual equivalence relation (where $(\phi_{d-k-1})_2$ denotes the second component of the fiber bundle of pairs ϕ_{d-k-1}).
Let

$$s_{d-k-1} : X_{d-k-1}\backslash X_{d-k-2} \longrightarrow cyl((\sigma^{\partial}_{d-k-1})_{\leq \overline{q}(k+1)})$$

be the vertex section. We define Z_{k+1} to be the pair of spaces which is the quotient of $cyl(\theta_{d-k-1})$ by the equivalence relation which identifies, for any $x \in X_{d-k-1} \backslash X_{d-k-2}$, the subspace $s_{d-k-1}(x) \times [0,1]$ to a point.
We denote the spaces forming the pair $Z_{k+1} = (Z^1_{k+1}, Z^2_{k+1})$ by Z_{k+1}. We have a natural projection map $\rho_{k+1} : Z^1_{k+1} \longrightarrow X'_k \backslash X_{d-k-2}$ which is a homotopy equivalence whose fibers are contractible, and has a natural section denoted by β_{k+1}. The composition

$$\pi_{k+1} := \pi_k|_{X'_k \backslash X_{d-k-2}} \circ \rho_{k+1} : Z^1_{k+1} \longrightarrow X \backslash X_{d-k-2}$$

is a homotopy equivalence with contractible fibers, and has a section $\alpha_{k+1} := \beta_{k+1} \circ \alpha_k|_{X \backslash X_{d-k-2}}$ providing

$$X \backslash X_{d-k-2} \hookrightarrow Z^1_{k+1},$$

which is a closed inclusion.
Let

$$X'_{k+1} := Z^1_{k+1} \cup X_{d-k-2}.$$

Then we have the extension of the projection maps ρ_{k+1} and π_{k+1} to the projection maps

$$\begin{aligned} \rho_{k+1} : X'_{k+1} &\longrightarrow X'_k \\ \pi_{k+1} : X'_{k+1} : X'_{k+1} &\longrightarrow X. \end{aligned}$$

Remark 9.21
Consider the topology in X'_{k+1} spanned by the all open subsets of Z_{k+1} and the collection of subsets of the form $\pi^{-1}_{k+1}(U)$ for any open subset U of X. With this topology the projections are also homotopy equivalences whose fibers are contractible, and such that the natural sections β_{k+1} and α_{k+1} extend to them as closed inclusions.

Define the step $(k+1)$ intersection space pair to be the pair of subspaces of X'_{k+1} given by

$$(I^{\overline{p}}_{k+1}X, I^{\overline{p}}_{k+1}(X_{d-2})) := cyl((\sigma^{\partial}_{d-k-1})_{\leq \overline{q}(k)}) \times \{0\} \cup cyl(\phi^{\partial}_{d-k-1}) \cup (I^{\overline{p}}_k X, I^{\overline{p}}_k X_{d-2})) \backslash \pi^{-1}_k (TX_{d-k-1}),$$

with the restricted topology.

Definition 9.22
Given a topological pseudomanifold $X = X_d \supset \cdots \supset X_0$ such that the pair (X_d, X_{d-2}) has a conical structure with respect to the stratification, we say that it has an intersection space pair if there exist successive choices of suitable fiberwise homology truncations so that the construction above can be carried up to $k = d$. In that case, the pair

$$(I^{\overline{p}}X, I^{\overline{p}}X_{d-2}) = (I^{\overline{p}}_d X, I^{\overline{p}}_d X_{d-2})$$

*is called **an intersection space pair associated with the stratification**.*

9.1.3 *Intersection space pairs for pseudomanifolds having trivial conical structures*

Let X be a topological pseudomanifold with a trivial conical structure. Fix a compatible system of trivializations. We carry the inductive construction of the intersection space pair as above, but we add the following property to properties (1)-(3) which are checked along the induction:

4. *the conical structures of the pairs* $(X'_k, I^{\overline{p}}_k X)$, $(I^{\overline{p}}_k X, I^{\overline{p}}_k(X_{d-2}))$ *with respect to*

$$X_0 \subset X_1 \subset \cdots \subset X_{d-k-1}$$

are trivial, and a compatible system of trivializations is inherited from the inductive construction.

At the initial step of the construction we have a topological pseudomanifold

$$X \supset X_{d-m} \supset \cdots \supset X_0$$

with a trivial conical structure and a compatible set of trivializations (as before the co-dimension of the first non-open stratum is m).

The compatible system of trivializations gives us a fixed trivialization of the fibration

$$\sigma^{\partial}_{d-m} : \partial\overline{TX_{d-m}} \backslash X_{d-m-1} \longrightarrow X_{d-m} \backslash X_{d-m-1}.$$

Choose a rational $\overline{q}(m)$-homology truncation of the fiber. This is always possible and elementary. Now, using the trivialization, the rational $\overline{q}(m)$-homology truncation of the fiber propagates to a fiberwise $\overline{q}(m)$-homology truncation of the fibration above. This is the truncation chosen at the initial step.

Now, using the compatibility of our system of trivializations, it is easy to show that the pairs $(X'_k, I^{\overline{p}}_m X)$, $(I^{\overline{p}}_m X, I^{\overline{p}}_m(X_{d-2}))$ satisfy the required properties (1)-(4). The compatible system of trivializations required in property (4) are inherited, by construction, by the compatible system of trivializations used at the beginning.

The inductive step of the construction is carried in the same way: the fixed trivializations propagate rational homology truncations of the corresponding fibrations of pairs of links. As stated in [2], the above considerations prove:

Theorem 9.23
If X is a topological pseudomanifold with a trivial conical structure, then there exist an intersection space pair associated with it for every perversity.

Corollary 9.24
Let X be a toric variety. Then, X has an intersection space pair for every perversity.

Finally, we want to present the main theorem of [1].

Theorem 9.25
Let X be a topological stratified pseudomanifold of dimension d with a compatible system of

trivializations. Let $(I^{\overline{p}}X, I^{\overline{p}}X_{d-2})$ *be the intersection space pair associated with the system of trivializations. There is Poincaré duality isomorphism*

$$\mathbf{H}^k(I^{\overline{p}}X, I^{\overline{p}}X_{d-2}; \mathbb{Q}) \cong Hom(\mathbf{H}^{d-k}(I^{\overline{q}}X, I^{\overline{q}}X_{d-2}; \mathbb{Q})),$$

where $\overline{q}$ *is the complementary perversity of* $\overline{p}$*.*

10 INTERSECTION SPACE PAIRS OF 6-DIMENSIONAL TORIC VARIETIES

In the previous part, we considered a 6-dimensional toric variety as a $\mathbb{Q}$-pseudomanifold and showed that the introduced generalized version of Banagl's theorem can still handle this case. However, we can still consider a 6-dimensional toric variety as a pseudomanifold with the common stratification and apply the theory of intersection space pairs. Note that throughout this section, we take only the middle perversity $\bar{n}$ into account. Thus, for the sake of simplicity, we omit it from our notation.
Let X be a 6-dimensional toric variety. We consider the following stratification of X

$$X = X_6 \supset X_2 \supset X_0.$$

First, we need to introduce our system of trivialization. We set

$$TX_0 = \bigsqcup_{x_i \in X_0} \mathring{\mathcal{C}}(\mathcal{L}_{x_i}),$$

where $\mathcal{L}_{x_i}$ is the link of x_i and $\mathring{\mathcal{C}}(\mathcal{L}_{x_i})$ denotes the open cone of $\mathcal{L}_{x_i}$ in X. Let $\bigsqcup_{i \in I} \mathcal{M}_i = X_2 - X_0$, where each $\mathcal{M}_i$ is a connected component of $X_2 - X_0$. Let $\mathcal{L}_{\mathcal{M}_i}$ be the associated link to $\mathcal{M}_i$. Let also $\mathring{\mathcal{I}}$ be the open unit interval. Then, we set

$$TX_2 \cong \bigsqcup_{i \in I} \mathring{\mathcal{C}}(\mathcal{L}_{\mathcal{M}_i}) \times \mathring{\mathcal{I}},$$

such that

$$\begin{aligned} \partial \overline{TX}_2 \cap \partial \overline{TX}_0 &= \Big(\bigsqcup_{x_i \in X_0} \mathcal{L}_{x_i} \Big) \bigcap \Big(\bigsqcup_{i \in I} \mathcal{L}_{\mathcal{M}_i} \Big) \\ &= \bigsqcup_{i \in I} \big(\mathcal{L}_{\mathcal{M}_i} \sqcup \mathcal{L}_{\mathcal{M}_i} \big). \end{aligned}$$

Remark 10.1
Let us elaborate on the above situation and our system of trivialization with an example. Let $X_{\mathcal{P}}$ be the associated toric variety to a pyramid $\mathcal{P}$.
Let us first consider $(TX_{\mathcal{P}})_0$. In our example, TX_0 is simply the disjoint union of cones over the link of isolated singularities. We assume that each vertex is a singular point. Thus, as shown in the figure 10.1, we have the disjoint union of four cones. Note that each point of the blue 2-dimensional faces of the cones is associated with a $\mathcal{T}^2$. Notice also that the 2-dimensional green faces are the intersection of the cones with the preimage of the interior of $\mathcal{P}$ under the projection of

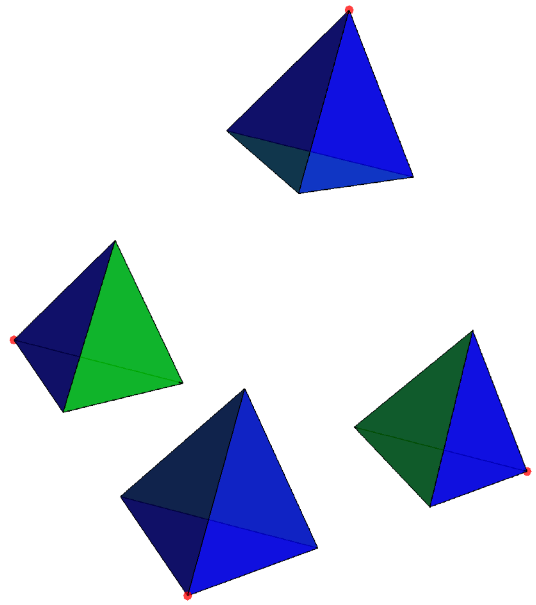

Figure 10.1: $(TX_{\mathcal{P}})_0$ of the toric variety associated with a pyramid.

the toric variety to $\mathcal{P}$. Hence, each point of them is associated with a $\mathcal{T}^3$. Red points are vertices of $\mathcal{P}$.

Let us now consider one of these cones, As shown in the figure 10.2. The yellow lines exhibit the intersection of the base of the cone with the 2-dimensional faces of $\mathcal{P}$. Consider the projection map from the toric variety to $\mathcal{P}$. Hence, the preimage of each point of the yellow lines under the projection map is homeomorphic to $\mathcal{T}^2$. With a similar argument, each brown point is associated with $\mathcal{S}^1$. Finally, from the construction, it is clear that the preimage of each point of the cyanic lines under the projection is homeomorphic to a $\mathcal{S}^1$.

Subsequently, we investigate $(TX_{\mathcal{P}})_2$. We consider the figure 10.3 and use a similar argument as before. Hence, each point of the 2-dimensional blue and green faces is associated with a $\mathcal{T}^2$ and $\mathcal{T}^3$, respectively. Additionally, we have 2-dimensional yellow faces representing the intersection of the boundary of $(TX_{\mathcal{P}})_2$ with the preimage of the interior of the $\mathcal{P}$ under the projection. Now, we consider a connected component of $(TX_{\mathcal{P}})_2$.

As shown in the figure 10.4, in addition to our prior consideration, we have purple points and orange and pink lines. Taking the intersection of the boundary of $(TX_{\mathcal{P}})_2$ and the preimage of the interior of $\mathcal{P}$ into account, we get the following. Each purple point and any point of the orange lines is associated with a $\mathcal{T}^2$. However, any point of the orange face is attached to a $\mathcal{T}^3$.

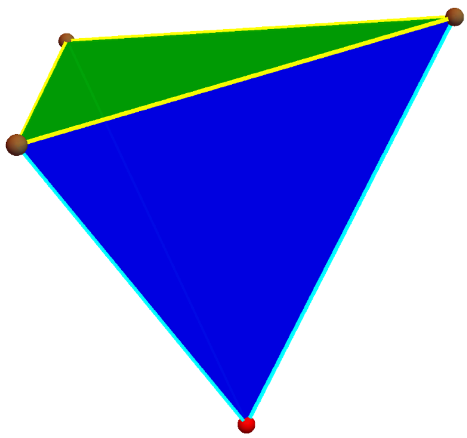

Figure 10.2: One of the cones of $(TX_{\mathcal{P}})_0$.

We can use the triviality of the link bundles of toric varieties to give a homotopy model of $(TX_{\mathcal{P}})_2$*. Consider the figure 10.5. Using the triviality of the line bundle, one can show that schematically, the pink line with both attached purple points is homeomorphic to* $\mathcal{S}^1 \times \mathcal{L}_{\mathcal{M}_i}$*. Thus, figure 10.5 is homeomorphic to* $\mathcal{S}^1 \times \mathcal{C}(\mathcal{L}_{\mathcal{M}_i})$*. Again the triviality yields that the connected component of* $(TX_{\mathcal{P}})_2$*, shown in the figure 10.4, is homeomorphic to* $\mathcal{I} \times \mathcal{S}^1 \times \mathcal{C}(\mathcal{L}_{\mathcal{M}_i})$*, where* $\mathcal{I}$ *is the unit interval. Consequently, figure 10.4 is homotopy equivalent to figure 10.5.*

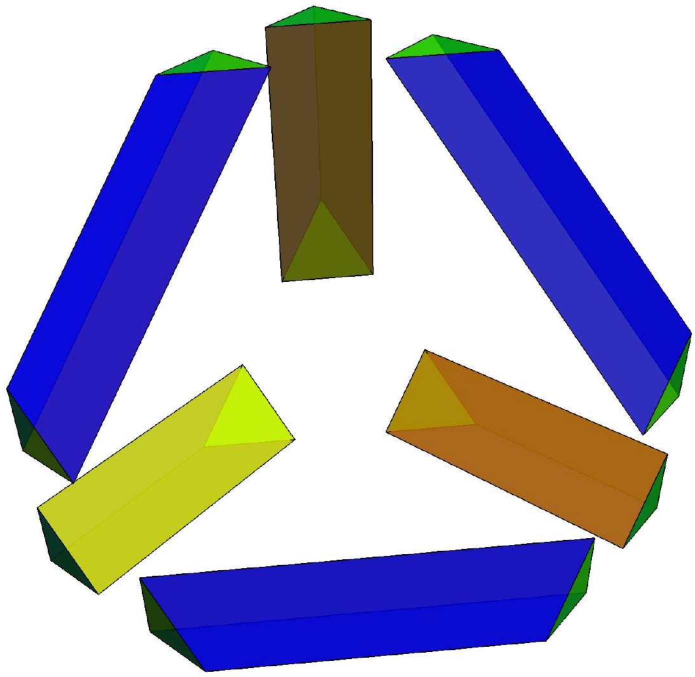

Figure 10.3: $(TX_{\mathcal{P}})_2$ of the toric variety associated to a pyramid.

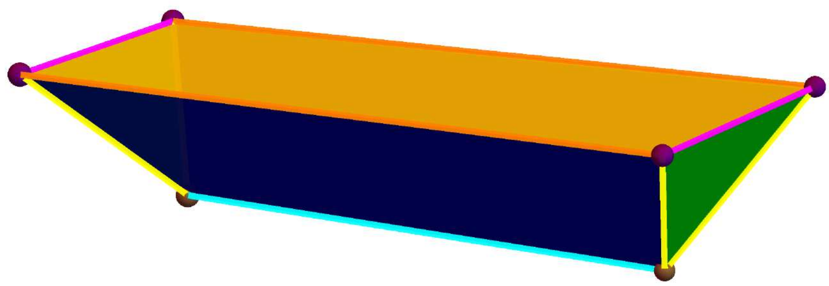

Figure 10.4: A connected component of $(TX_{\mathcal{P}})_2$.

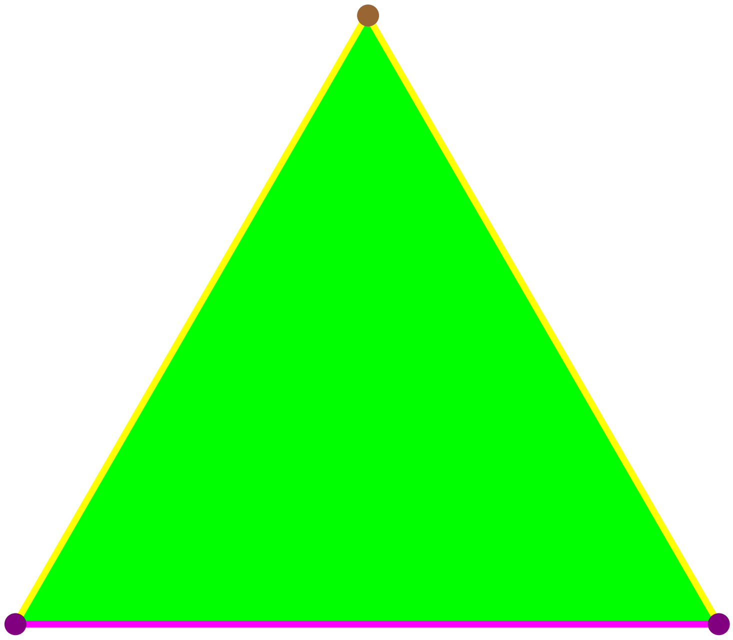

Figure 10.5: $\mathcal{S}^1 \times \mathcal{C}(\mathcal{L}_{\mathcal{M}_i})$, where the brown point is attached to a $\mathcal{S}^1$.

We can generalize our above consideration to any 6-dimensional toric variety, with a slight modification. With the trivialized neighborhood in hand, we can start with the initial step of the construction. However, geometrically for the sake of simplicity, we pursue a slightly different path from [2]. We start with cutting out the isolated singularities of X. As in the last section, the resulting topological space is a $\mathbb{Q}$-manifold-with-boundary. We have shown the situation in figure 10.6 for a toric variety associated with a pyramid. Note that the green areas are the boundary of the $\mathbb{Q}$-manifold.

To obtain a manifold, we still need to cut out the remaining singularities. For our example, we have illustrated the situation in figure 10.7. For our study, we denote the manifold obtained by cutting out the singular strata of X by $\mathcal{M}$. Before starting with the homological truncation of the links, we need to calculate the homology groups of $\mathcal{M}$.

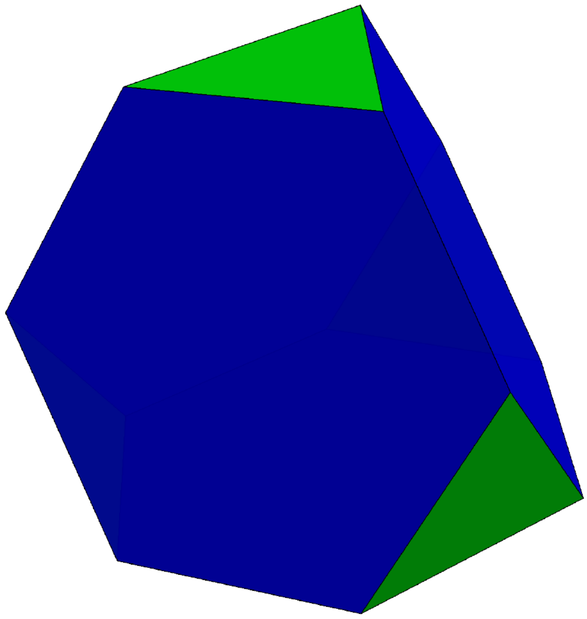

Figure 10.6: The Q-manifold obtained by cutting out the isolated singularities.

Lemma 10.2

Let X be a 6-dimensional toric variety associated with the 3-dimensional polytype $\mathcal{P}$. Let $\mathcal{M}$ be the manifold obtained by cutting out all the singular strata of X. Then we have

$$\begin{aligned} rk\big(\mathbf{H}_6(\mathcal{M})\big) &= 0 \\ rk\big(\mathbf{H}_5(\mathcal{M})\big) &= 0 \\ rk\big(\mathbf{H}_4(\mathcal{M})\big) &= f_1 - 1 \\ rk\big(\mathbf{H}_3(\mathcal{M})\big) &= 2f_1 - 3 \\ rk\big(\mathbf{H}_2(\mathcal{M})\big) &= f_1 - 3 \\ rk\big(\mathbf{H}_1(\mathcal{M})\big) &= 0 \\ rk\big(\mathbf{H}_0(\mathcal{M})\big) &= 1 \end{aligned}$$

, where f_1 and f_2 denote the number of 2-dimensional and 1-dimensional faces of $\mathcal{P}$, respectively.

Proof. The proof goes along a similar line as in 6.15. For computing the homology group of $\mathcal{M}$, our goal is to find an adequate homotopy model of $\mathcal{M}$. With the same argument as in 6.15, in figure 10.6, each red rectangle is homotopy equivalent to a line connecting

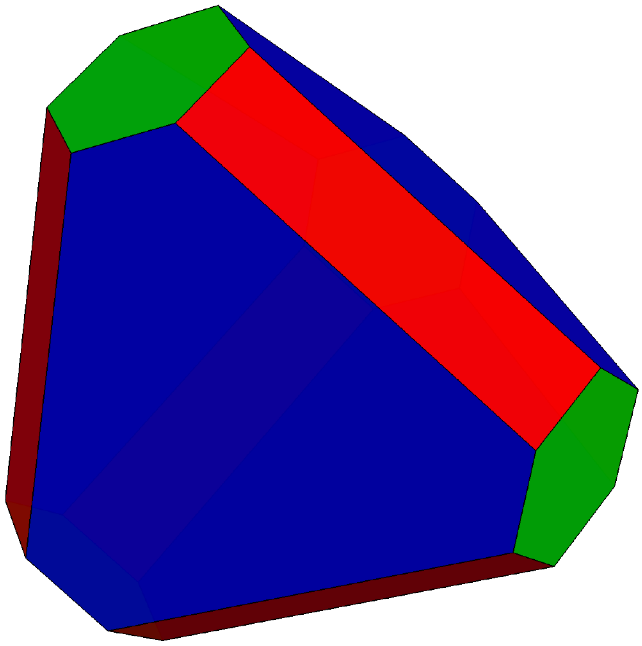

Figure 10.7: The manifold obtained by cutting out the singular strata.

the two neighboring regions. For each green area, we use the homotopy equivalence presented in 6.15. Finally, we collapse each blue 2-dimensional face to a point. We denote the number of vertices of $\mathcal{P}$ with f_3. Hence, for an arbitrary $\mathcal{P}$, the resulting space is a convex polytope with f_3 2-dimensional faces, f_2 1-dimensional faces, and f_1 vertices, where to each of vertices a $\mathcal{T}^2$ is attached and to each other points a $\mathcal{T}^3$. The collapsing data are coming from the toric variety. We denote the resulting space with $\mathcal{B}$. Calculating the homology groups of $\mathcal{B}$ is rather an easy task. We can either endow $\mathcal{B}$ with a CW structure and compute the homology groups directly or use the Mayer-Vietoris sequence. □

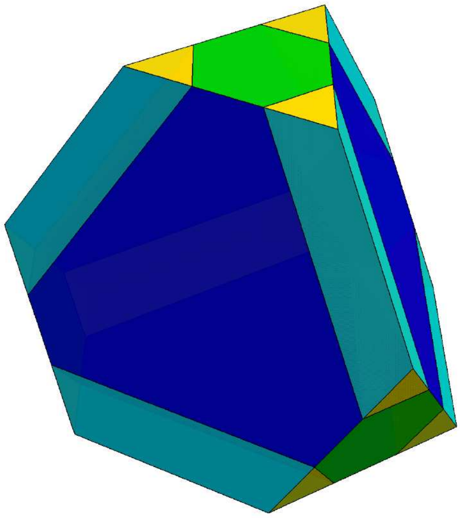

Figure 10.8: First step of induction in the construction of intersection space pairs.

In the first step of our construction, we truncate the links of the 2-dimensional stratum. As mentioned earlier, the links are homological 3-spheres, which means that any homological truncation below the top degree leaves us with a homological point. After applying the homological truncation, we attach the modified open neighborhood, TX_2, to the manifold, $\mathcal{M}$. Putting the above steps into use in our example, we have shown the situation in figure 10.8. Here one can see that the previous spatial modification altered the links of isolated singularities. Regardless, we need to compute the homology groups of the space obtained by attaching the modified TX_2 to the manifold, $\mathcal{M}$. Once again, we use the Mayer-Vietoris sequence. We choose $\mathcal{U}$ and $\mathcal{V}$, such that

$$\begin{aligned} &\mathcal{U} \simeq \mathcal{M}, \\ &\mathcal{V} \cong \bigsqcup_i \mathcal{S}^1 \times \mathcal{C}\big((\mathcal{L}_{\mathcal{M}_i})_{\leq \bar{n}(4)}\big), \\ &\cap := \mathcal{U} \cap \mathcal{V} \simeq \bigsqcup_i \mathcal{S}^1 \times (\mathcal{L}_{\mathcal{M}_i})_{\leq \bar{n}(4)}. \end{aligned}$$

Let

$$\mathcal{M}_1 := \mathcal{U} \cup \mathcal{V}. \tag{10.1}$$

Hence, we have

$$
\begin{aligned}
0 &\longrightarrow \overbrace{\mathbf{H}_6(\cap)}^{=0} \longrightarrow \overbrace{\mathbf{H}_6(\mathcal{U})}^{=0} \xrightarrow{\cong} \overbrace{\mathbf{H}_6(\mathcal{M}_1)}^{=0} \\
&\longrightarrow \overbrace{\mathbf{H}_5(\cap)}^{=0} \longrightarrow \overbrace{\mathbf{H}_5(\mathcal{U})}^{=0} \xrightarrow{\cong} \overbrace{\mathbf{H}_5(\mathcal{M}_1)}^{=0} \\
&\longrightarrow \overbrace{\mathbf{H}_4(\cap)}^{=0} \longrightarrow \overbrace{\mathbf{H}_4(\mathcal{U})}^{\cong \mathbb{Q}^{f_1-1}} \xrightarrow{\cong} \overbrace{\mathbf{H}_4(\mathcal{M}_1)}^{\cong \mathbb{Q}^{f_1-1}} \\
&\longrightarrow \overbrace{\mathbf{H}_3(\cap)}^{=0} \longrightarrow \overbrace{\mathbf{H}_3(\mathcal{U})}^{\cong \mathbb{Q}^{2f_1-3}} \xrightarrow{\cong} \overbrace{\mathbf{H}_3(\mathcal{M}_1)}^{\cong \mathbb{Q}^{2f_1-3}} \\
&\longrightarrow \overbrace{\mathbf{H}_2(\cap)}^{=0} \longrightarrow \overbrace{\mathbf{H}_2(\mathcal{U})}^{\cong \mathbb{Q}^{f_1-3}} \xrightarrow{\cong} \overbrace{\mathbf{H}_2(\mathcal{M}_1)}^{\cong \mathbb{Q}^{f_1-3}} \\
&\longrightarrow \overbrace{\mathbf{H}_1(\cap)}^{\cong \mathbb{Q}^{f_2}} \longrightarrow \overbrace{\mathbf{H}_1(\mathcal{U})}^{=0} \oplus \overbrace{\mathbf{H}_1(\mathcal{V})}^{\cong \mathbb{Q}^{f_2}} \longrightarrow \overbrace{\mathbf{H}_1(\mathcal{M}_1)}^{=0} \\
&\longrightarrow \overbrace{\widetilde{\mathbf{H}}_0(\cap)}^{\cong \mathbb{Q}^{f_2-1}} \xrightarrow{\cong} \overbrace{\widetilde{\mathbf{H}}_0(\mathcal{U})}^{=0} \oplus \overbrace{\widetilde{\mathbf{H}}_0(\mathcal{V})}^{\cong \mathbb{Q}^{f_2-1}} \longrightarrow \overbrace{\widetilde{\mathbf{H}}_0(\mathcal{M}_1)}^{=0} \longrightarrow 0.
\end{aligned}
$$

From the above sequence, one can deduce that

$$\mathbf{H}_*(\mathcal{M}) \cong \mathbf{H}_*(\mathcal{M}_1).$$

Remark 10.3
Due to $\mathbf{H}_*(\mathcal{M}) \cong \mathbf{H}_*(\mathcal{M}_1)$, one could argue that homologically the preceding spacial modification does not contribute to the construction of the intersection space pair. Naively speaking, this could indicate that we need to introduce a generalized theory of intersection space pairs for $\mathbb{Q}$-pseudomanifolds.

Before executing the next step, we need to calculate the homology groups of the altered links of isolated singularities. Following the path of our example, we can illustrate such links as in the figure 10.9. Let us consider the link of an isolated singularity, $\mathcal{L}$, as a pseudomanifold with the following stratification.

$$\mathcal{L}_5 \supset \mathcal{L}_1.$$

The homological truncation in the initial step implies a truncation on the links of each connected component of $\mathcal{L}_1$. In figure 10.9, a connected component of $\mathcal{L}_1$, namely $\mathcal{S}^1$, is attached to each vertex . In a green cone, the corresponding link is truncated. We denote the altered link by $\mathcal{L}'$.

Lemma 10.4
Let $\mathcal{L}$ be the link of an isolated singularity in a 6-dimensional toric variety. Let $f_1^{\mathcal{L}}$ be the number

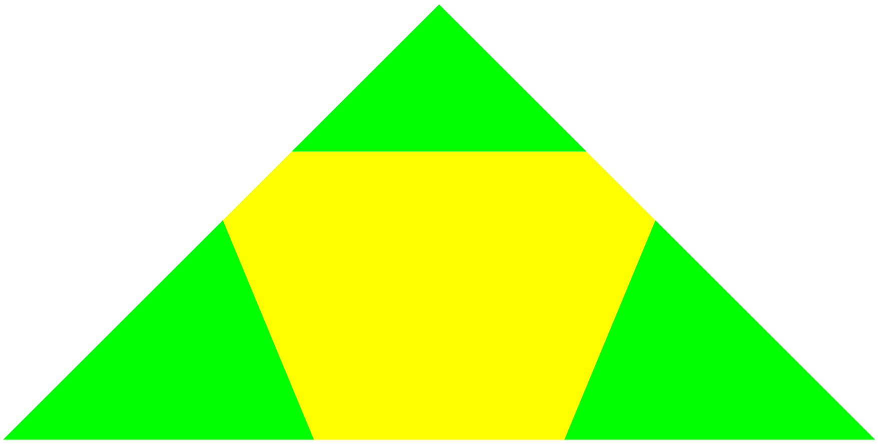

Figure 10.9: Modified link of an isolated singularity.

1-dimensional faces of the underlying 2-dimensional polygon of $\mathcal{L}$. Applying the above homological truncation, let $\mathcal{L}'$ be the associated altered link to $\mathcal{L}$. Then we have

$$
\begin{aligned}
rk\big(\mathbf{H}_5(\mathcal{L}')\big) &= 0\\
rk\big(\mathbf{H}_4(\mathcal{L}')\big) &= f_1^{\mathcal{L}} - 1\\
rk\big(\mathbf{H}_3(\mathcal{L}')\big) &= 2f_1^{\mathcal{L}} - 3\\
rk\big(\mathbf{H}_2(\mathcal{L}')\big) &= f_1^{\mathcal{L}} - 3\\
rk\big(\mathbf{H}_1(\mathcal{L}')\big) &= 0\\
rk\big(\mathbf{H}_0(\mathcal{L}')\big) &= 1
\end{aligned}
$$

Proof. For computing the homology groups of $\mathcal{L}'$, we use the Mayer-Vietoris sequence. For each connected component of $\mathcal{L}_1$, $\mathcal{S}_i^1$, let $(\mathcal{L}_i)_{\leq \bar{n}(4)}$ be the associated truncated link. Set $V \cong \bigsqcup_i \mathcal{S}_i^1 \times \mathcal{C}\big((\mathcal{L}_i)_{\leq \bar{n}(4)}\big)$. Thus, we can set $\mathcal{V} \simeq \mathcal{M}$, where $\mathcal{M}$ is the manifold obtained by cutting out the singularities of $\mathcal{L}$. Hence, we can set $\cap := \mathcal{U} \cap \mathcal{V} \simeq \bigsqcup_i \mathcal{S}_i^1 \times (\mathcal{L}_i)_{\leq \bar{n}(4)}$. First, we need to calculate the homology groups of $\mathcal{M}$. We can do this by using an adequate homotopy model of $\mathcal{M}$, similar to 6.15. Alternatively, we can equip $\mathcal{M}$ with a CW structure. In any case, we have

$$
\begin{aligned}
\mathrm{rk}\big(\mathbf{H}_5(\mathcal{M})\big) &= 0\\
\mathrm{rk}\big(\mathbf{H}_4(\mathcal{M})\big) &= f_1^{\mathcal{L}} - 1\\
\mathrm{rk}\big(\mathbf{H}_3(\mathcal{M})\big) &= 2f_1^{\mathcal{L}} - 3\\
\mathrm{rk}\big(\mathbf{H}_2(\mathcal{M})\big) &= f_1^{\mathcal{L}} - 3\\
\mathrm{rk}\big(\mathbf{H}_1(\mathcal{M})\big) &= 0\\
\mathrm{rk}\big(\mathbf{H}_0(\mathcal{M})\big) &= 1.
\end{aligned}
$$

Using the Mayer-Vietoris sequence yields the claimed result. □

Finally, we need to apply the homology truncation machinery to the modified links of isolated singularities and compute the homology groups of $I^{\bar{n}}X$. Here again, we use the Mayer-Vietoris sequence and, hence, the explicit method of truncation does not play any role for us, where it means either directly removing some cells or using a homotopy model of the modified link and then removing cells. However, one could easily show that we don't need a homotopy model. Consequently, we can perform the truncation by removing some cells. Thus, we state the following lemma with only a sketch of proof.

Lemma 10.5
Let $\mathcal{L}'$ be the modified link, in the above sense, of an isolated singularity of a 6-dimensional toric variety. Then

$$(\mathcal{L}')_{\leq 2} \hookrightarrow (\mathcal{L}')$$

is an inclusion.

Proof. Let $\mathcal{P}$ be the underlying 2-dimensional polygon of $\mathcal{L}'$. Notice that due to the modification, we will get 1-dimensional and 0-dimensional cells of $\mathcal{P}$ attached to $\mathcal{T}^3$ and $\mathcal{T}^2$, respectively. To be more specific, the 1-dimensional and 0-dimensional cells that bind green cones with the yellow region in figure 10.9 are attached to $\mathcal{T}^3$ and $\mathcal{T}^2$, respectively. Using the above CW structure, from $\partial_3^{\mathcal{L}'}$ one can easily deduce that the above claim stands by.

□

After performing the homology truncation, we can conclude with the following lemma.

Lemma 10.6
Let X be a 6-dimensional toric variety. Let $I^{\bar{n}}X$ be the associated intersection complex in middle perversity, $\bar{n}$. Then we have

$$\begin{aligned}
rk\big(\mathbf{H}_6(I^{\bar{n}}X)\big) &= 0 \\
rk\big(\mathbf{H}_5(I^{\bar{n}}X)\big) &= 0 \\
rk\big(\mathbf{H}_4(I^{\bar{n}}X)\big) &= f_1 - 1 \\
rk\big(\mathbf{H}_3(I^{\bar{n}}X)\big) &= 6f_1 - f_2 - 9 - b \\
rk\big(\mathbf{H}_2(I^{\bar{n}}X)\big) &= f_1 - 3 - b \\
rk\big(\mathbf{H}_1(I^{\bar{n}}X)\big) &= 0 \\
rk\big(\mathbf{H}_0(I^{\bar{n}}X)\big) &= 1,
\end{aligned}$$

where f_1 and f_2 denote the number of 1- and 2-dimensional cones in the associated fan, respectively.

Proof. We begin with the computation of homology groups in lower degrees. From the truncation degrees and CW structure, one can deduce that for $i \leq 2$, we have $\mathbf{H}_i(I^{\bar{n}}X) = \mathbf{H}_i(X)$. Let $\mathcal{U} \simeq \mathcal{M}_1$, where $\mathcal{M}_1$ is defined in 10. Also, we set

$$\mathcal{V} \cong \bigsqcup_{x_i \in X_0} \mathcal{C}\big((\mathcal{L}'_{x_i})_{\leq 2}\big),$$

where $(\mathcal{L}'_{x_i})_{\leq 2}$ is the modified link of the singular point x_i. Then we have

$$\cap := \mathcal{U} \cap \mathcal{V} \simeq (\mathcal{L}'_{x_i})_{\leq 2}.$$

Employing the Mayer-Vietoris sequence once more, we arrive at

$$\begin{aligned}
0 &\longrightarrow \overbrace{\mathbf{H}_6(\cap)}^{=0} \longrightarrow \overbrace{\mathbf{H}_6(\mathcal{U})}^{=0} \xrightarrow{\cong} \overbrace{\mathbf{H}_6(I^{\bar{n}}X)}^{=0} \\
&\longrightarrow \overbrace{\mathbf{H}_5(\cap)}^{=0} \longrightarrow \overbrace{\mathbf{H}_5(\mathcal{U})}^{=0} \xrightarrow{\cong} \overbrace{\mathbf{H}_5(I^{\bar{n}}X)}^{=0} \\
&\longrightarrow \overbrace{\mathbf{H}_4(\cap)}^{=0} \longrightarrow \overbrace{\mathbf{H}_4(\mathcal{U})}^{\cong \mathbb{Q}^{f_1-1}} \xrightarrow{\cong} \overbrace{\mathbf{H}_4(I^{\bar{n}}X)}^{\cong \mathbb{Q}^{f_1-1}} \\
&\longrightarrow \overbrace{\mathbf{H}_3(\cap)}^{=0} \longrightarrow \overbrace{\mathbf{H}_3(\mathcal{U})}^{\cong \mathbb{Q}^{2f_1-3}} \longrightarrow \mathbf{H}_3(I^{\bar{n}}X) \\
&\longrightarrow \overbrace{\mathbf{H}_2(\cap)}^{=\Sigma=2f_2-3f_3} \longrightarrow \overbrace{\mathbf{H}_2(\mathcal{U})}^{\cong \mathbb{Q}^{f_1-3}} \longrightarrow \overbrace{\mathbf{H}_2(I^{\bar{n}}X)}^{\cong \mathbb{Q}^{f_1-3-b}} \\
&\longrightarrow \overbrace{\mathbf{H}_1(\cap)}^{=0} \longrightarrow \overbrace{\mathbf{H}_1(\mathcal{U})}^{=0} \longrightarrow \overbrace{\mathbf{H}_1(I^{\bar{n}}X)}^{=0} \\
&\longrightarrow \overbrace{\widetilde{\mathbf{H}}_0(\cap)}^{\cong \mathbb{Q}^{f_3-1}} \xrightarrow{\cong} \overbrace{\widetilde{\mathbf{H}}_0(\mathcal{U})}^{=0} \oplus \overbrace{\widetilde{\mathbf{H}}_0(\mathcal{V})}^{\cong \mathbb{Q}^{f_3-1}} \longrightarrow \overbrace{\widetilde{\mathbf{H}}_0(I^{\bar{n}}X)}^{=0} \longrightarrow 0,
\end{aligned}$$

where f_3 denote the number of 3-dimensional cones in the associated fan. Finally, from the exactness, we can deduce that $\operatorname{rk}\big(\mathbf{H}_3(I^{\bar{n}}X)\big) = 5f_1 - f_2 - 9 - b$. □

Let us go back to our example. After performing both homological truncations, we have shown the situation in figure 10.10. The purple cones are the cones of the truncated modified links of isolated singularities. The light blue areas denote the trivialized neighborhoods of X_{d-4} with truncated links. Finally, to each point of the dark blue 2-dimensional face, a $\mathcal{T}^2$ is attached. As we have seen in lemma 10.6, $I^{\bar{n}}X$ does not satisfy the rational Poincaré duality. Here the second term in the intersection complex comes into play.
Thus, it remains to compute the homology groups of $I^{\bar{n}}X_{d-4}$. One can easily verify that here we have $X_{d-4} \cong I^{\bar{n}}X$. In other words, the homological truncations do not modify X_{d-4}.

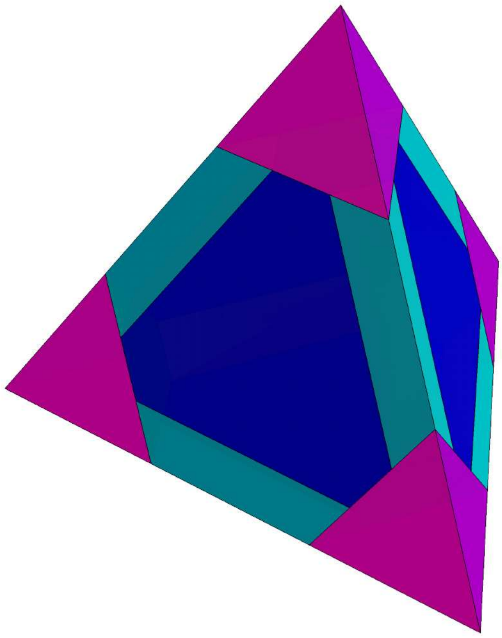

Figure 10.10: Intersection space complex, IX.

But first, let us consider a 1-dimensional face of an arbitrary 3-dimensional convex polytope, $\mathcal{P}$, associated with a complete fan, Σ. We have shown the situation in figure 10.11. The result is simply the middle black circle, which collapses to a point on both ends of the 1-dimensional face. In other words, as expected, this is the compact 2-dimensional toric variety, which is homeomorphic to S^2. Considering all 1-dimensional faces of $\mathcal{P}$ in our example, we arrive at figure 10.12. Thus, for an arbitrary $\mathcal{P}$, for each 1-dimensional face of $\mathcal{P}$, we get a $\mathcal{S}^2$, which is unified with two other spheres at its north and south poles. With the above considerations, computing the homology of X_{d-4} is an easy task. Hence, we arrive at the following lemma.

Lemma 10.7
Let X be the toric variety associated with a convex 3-dimensional polytope, $\mathcal{P}$, which is dual to a complete rational fan, Σ. We consider the common stratification of toric varieties, namely

$$X_6 \supset X_2 \supset X_0.$$

Then, we have

$$\begin{aligned} rk\big(\mathbf{H}_3(X_2)\big) &= 0 \\ rk\big(\mathbf{H}_2(X_2)\big) &= f_2 \\ rk\big(\mathbf{H}_1(X_2)\big) &= f_2 - f_1 - 1 \\ rk\big(\mathbf{H}_0(X_2)\big) &= 1, \end{aligned}$$

where f_2 and f_1 denote the number of 2-dimensional and 1-dimensional cones of Σ, respectively.

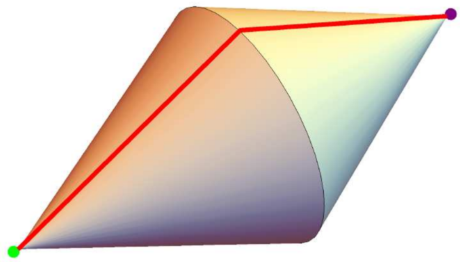

Figure 10.11: The associated toric variety to a 1-dimensional face of $\mathcal{P}$.

Having all the previous considerations in hand, we can finally state the main theorem of this chapter.

Theorem 10.8
Let X be the toric variety associated with a convex 3-dimensional polytope, $\mathcal{P}$, which is dual to a complete rational fan, Σ. We consider the common stratification of toric varieties, namely

$$X_6 \supset X_2 \supset X_0.$$

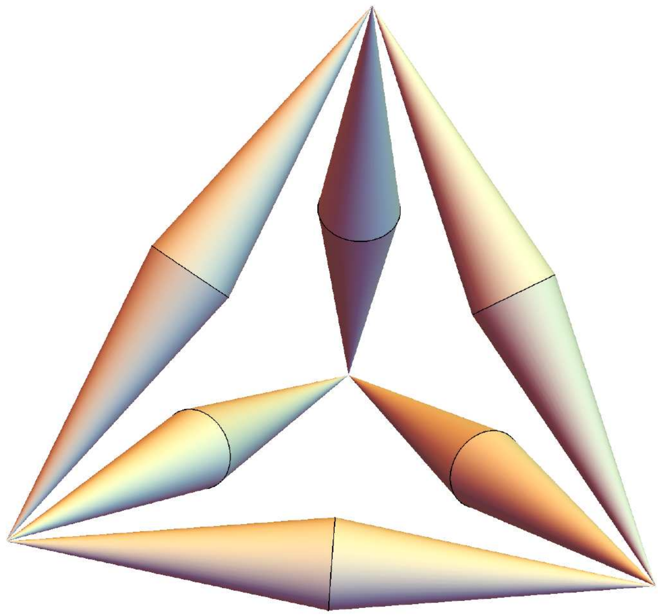

Figure 10.12: IX_{d-4} in intersection space pair.

Then, for the relative homology of intersection complex, (IX, IX_{d-4}) *in the middle perversity we have*

$$\begin{aligned}
rk\big(\mathbf{H}_6(IX, IX_{d-4})\big) &= 0\\
rk\big(\mathbf{H}_5(IX, IX_{d-4})\big) &= 0\\
rk\big(\mathbf{H}_4(IX, IX_{d-4})\big) &= f_1 - 1\\
rk\big(\mathbf{H}_3(IX, IX_{d-4})\big) &= 6f_1 - f_2 + 6\\
rk\big(\mathbf{H}_2(IX, IX_{d-4})\big) &= f_1 - 1\\
rk\big(\mathbf{H}_1(IX, IX_{d-4})\big) &= 0\\
rk\big(\mathbf{H}_0(IX, IX_{d-4})\big) &= 0,
\end{aligned}$$

where f_2 *and* f_1 *denote the number of 2-dimensional and 1-dimensional cones of* Σ*, respectively.*

Proof. The parameter b has been introduced in example 3.22. Having all the previous considerations in hand, we can prove the theorem by using the long exact sequence of relative homology and the duality of the relative Betti numbers of intersection complexes from [2].

$$
\begin{array}{l}
0 \longrightarrow \overbrace{\mathbf{H}_6(IX_{d-4})}^{=0} \longrightarrow \overbrace{\mathbf{H}_6(IX)}^{=0} \xrightarrow{\cong} \overbrace{\mathbf{H}_6(IX, IX_{d-4})}^{=0} \\
\longrightarrow \overbrace{\mathbf{H}_5(IX_{d-4})}^{=0} \longrightarrow \overbrace{\mathbf{H}_5(IX)}^{=0} \xrightarrow{\cong} \overbrace{\mathbf{H}_5(IX, IX_{d-4})}^{=0} \\
\longrightarrow \overbrace{\mathbf{H}_4(IX_{d-4})}^{=0} \longrightarrow \overbrace{\mathbf{H}_4(IX)}^{\cong \mathbb{Q}^{f_1-1}} \xrightarrow{\cong} \overbrace{\mathbf{H}_4(IX, IX_{d-4})}^{\cong \mathbb{Q}^{f_1-1}} \\
\longrightarrow \overbrace{\mathbf{H}_3(IX_{d-4})}^{=0} \longrightarrow \overbrace{\mathbf{H}_3(IX)}^{\cong \mathbb{Q}^{5f_1-f_2-9-b}} \longrightarrow \mathbf{H}_3(IX, IX_{d-4}) \\
\longrightarrow \overbrace{\mathbf{H}_2(IX_{d-4})}^{\cong \mathbb{Q}^{f_2}} \longrightarrow \overbrace{\mathbf{H}_2(IX)}^{\cong \mathbb{Q}^{f_1-3-b}} \longrightarrow \mathbf{H}_2(IX, IX_{d-4}) \\
\longrightarrow \overbrace{\mathbf{H}_1(IX_{d-4})}^{\cong \mathbb{Q}^{f_2-f_1-1}} \longrightarrow \overbrace{\mathbf{H}_1(IX)}^{=0} \longrightarrow \overbrace{\mathbf{H}_1(IX, IX_{d-4})}^{=0} \\
\longrightarrow \overbrace{\widetilde{\mathbf{H}}_0(IX_{d-4})}^{=0} \xrightarrow{\cong} \overbrace{\widetilde{\mathbf{H}}_0(IX)}^{=0} \longrightarrow \overbrace{\widetilde{\mathbf{H}}_0(IX, IX_{d-4})}^{=0} \longrightarrow 0.
\end{array}
$$

From the duality of intersection space pairs, we can deduce that

$$\operatorname{rk}\big(\mathbf{H}_4(IX, IX_{d-4})\big) = \operatorname{rk}\big(\mathbf{H}_2(IX, IX_{d-4})\big) = f_1 - 1.$$

Finally, we can obtain $\operatorname{rk}\big(\mathbf{H}_3(IX, IX_{d-4})\big) = 6f_1 - f_2 + 6$ from the exactness.

□

Remark 10.9

Consider the projection $p : X_{\mathcal{P}} \longrightarrow \mathcal{P}$. Note that, in our computation, we assumed that the preimage of each vertex and 1-dimensional face of $\mathcal{P}$ under p is singular in X. Of course, this is not generally true. However, we can carry out a similar calculation for special cases where not the preimage of each 1-dimensional face of $\mathcal{P}$ is singular in $X_{\mathcal{P}}$. The question that arises here is Should we also treat special cases in the generalization of Banagl's construction separately? Well, the answer is fortunately no! It is an advantage of considering $\mathbb{Q}$-pseudomanifolds. More precisely, in a 6-dimensional toric variety, if an isolated point is not rationally singular, then it does not contribute to the term Σ in the Mayer-Vitories in remark 8.7. Hence, it does not make any difference if we truncate the associated link to it or not, homologically.

11 INTERSECTION SPACE PAIRS OF 6-DIMENSIONAL PSEUDO-TORIC VARIETIES

In this section, we want to study the intersection space pairs of 6-dimensional pseudo-toric varieties. However, we focus on examples where the generalized theory of intersection spaces with isolated singularities is not applicable. To do so, first, we introduce some pseudo-toric varieties where the links of isolated singularities do not satisfy the Poincaré duality, even rationally. We will also show that, in this case, links of the middle stratum are not necessarily 3-rational homological spheres. Finally, we construct the intersection space pair and compute the associated relative homology groups concerning the common stratification. As noted, we use the same stratification with the same strata similar to the previous section. Consequently, each step of the construction is similar to the previous section. However, we need to replace the new inks with the old ones. To be more precise, as we have seen, the link of the middle stratum can be homeomorphic to $\mathcal{S}^1 \times \mathcal{S}^2$ for some connected components in pseudo-toric varieties. Performing the homological truncation in the initial step concerning the middle perversity function yields a circle, in contrast to the previous case where the truncated links in the initial step were merely homological points. Naively speaking, this indicates that we can't weaken the definition of pseudomanifold and manifold such that truncation of the links of the isolated singularities suffices to achieve the Poincaré duality, at least rationally, as we did earlier.

Let us consider pseudo-toric varieties introduced in 4.6. Recall that we defined such pseudo-toric varieties as **totally singular pseudo-toric varieties**. Let $\mathcal{P}$ be a convex polytope. Let $X_{\mathcal{P}}$ be the associated totally singular pseudo-toric variety to it. Let f_1 and f_2 be the number of 1-dimensional and 2-dimensional cones in the dual fan, Σ, to $\mathcal{P}$, respectively. As mentioned, we consider the following stratification of $X_{\mathcal{P}}$

$$X_{\mathcal{P}} = X_6 \supset X_2 \supset X_0.$$

Similar to our study in the last chapter, first, we introduce our system of trivialization. We set

$$TX_0 = \bigsqcup_{x_i \in X_0} \mathring{\mathcal{C}}(\mathcal{L}_{x_i}),$$

where $\mathcal{L}_{x_i}$ is the link of $x_i \in X_0$ and $\mathring{\mathcal{C}}(\mathcal{L}_{x_i})$ denotes the open cone of $\mathcal{L}_{x_i}$ in $X_{\mathcal{P}}$. Let

$$\bigsqcup_{i \in I} \mathcal{M}_i = X_2 - X_0,$$

where each $\mathcal{M}_i$ is a connected component of $X_2 - X_0$. Let $\mathcal{L}_{\mathcal{M}_i}$ be the associated link to $\mathcal{M}_i$. Let also $\mathring{\mathcal{I}}$ be the open unit interval. Then, we set

$$TX_2 \cong \bigsqcup_{i \in I} \mathring{\mathcal{C}}(\mathcal{L}_{\mathcal{M}_i}) \times \mathring{\mathcal{I}},$$

such that

$$\begin{aligned}\partial\overline{TX_2} \bigcap \partial\overline{TX_0} &= (\bigsqcup_{x_i \in X_0} \mathcal{L}_{x_i}) \bigcap (\bigsqcup_{i \in I} \mathcal{L}_{\mathcal{M}_i}) \\ &= \bigsqcup_{i \in I} (\mathcal{L}_{\mathcal{M}_i} \sqcup \mathcal{L}_{\mathcal{M}_i}).\end{aligned}$$

For more elaboration, the reader may go through remark 10.1. Note that, although the link of each $\mathcal{M}_i$ is now homeomorphic to $\mathcal{S}^1 \times \mathcal{S}^2$, we can use the same system of trivialization as before.
Our starting point, as before, is to cut out all the singular strata of $X_{\mathcal{P}}$ and compute the homology groups of the obtained manifold.

Lemma 11.1
Let $Z_{\mathcal{P}}$, be the totally singular pseudo-toric variety associated with the convex polytope $\mathcal{P}$. Let f_1 and f_2 denotes the number of 1-dimensional and 2-dimensional cones of Σ, the dual fan to $\mathcal{P}$. Let M be the manifold obtained by cutting out all singular strata of $X_{\mathcal{P}}$. Then, we have

$$\begin{aligned} rk\big(\mathbf{H}_6(M)\big) &= 0 \\ rk\big(\mathbf{H}_5(M)\big) &= 0 \\ rk\big(\mathbf{H}_4(M)\big) &= f_1 - 1 \\ rk\big(\mathbf{H}_3(M)\big) &= 2(f_1 - 1) \\ rk\big(\mathbf{H}_2(M)\big) &= f_1 \\ rk\big(\mathbf{H}_1(M)\big) &= 2 \\ rk\big(\mathbf{H}_0(M)\big) &= 1 \end{aligned}$$

Proof. The proof goes along the same line as in 10.2. Thus, here again, we find an adequate homotopy model of M. Recall that $X_{\mathcal{P}}$ is a totally singular 6-dimensional pseudo-toric variety. Hence, we have

$$M \cong \mathcal{T}^2 \times Y.$$

Y can be described as follows. Let $\mathcal{P}'$ be the polytope, which underlies the adequate homotopy model of M (see 10.2). $\mathcal{P}'$ has f_3 2-dimensional, f_2 1-dimensional and f_1 0-dimensional faces, where f_3 denotes the number of 3-dimensional cones in Σ. To each point of $\mathcal{P}'$, we attach an $\mathcal{S}^1$, and at each vertex, we collapse the attached $\mathcal{S}^1$ to a point.

The resulting space is Y. Computing the homology groups of Y is rather an easy task, using CW-structure. Hence, we get

$$\begin{aligned}
\operatorname{rk}\big(\mathbf{H}_4(Y)\big) &= 0 \\
\operatorname{rk}\big(\mathbf{H}_3(Y)\big) &= 0 \\
\operatorname{rk}\big(\mathbf{H}_2(Y)\big) &= f_1 - 1 \\
\operatorname{rk}\big(\mathbf{H}_1(Y)\big) &= 0 \\
\operatorname{rk}\big(\mathbf{H}_0(Y)\big) &= 1
\end{aligned}$$

Using Künneth theorem proves the claim. □

In the first step of the construction, we truncate the links of the 2-dimensional stratum. In contrast to a toric variety, we have

$$\mathcal{L}_{\mathcal{M}_i} = \mathcal{S}^1 \times \mathcal{S}^2,$$

for each $\mathcal{M}_i$. Thus, we simply get

$$(\mathcal{L}_{\mathcal{M}_i})_{\leq \bar{n}(4)} \cong \mathcal{S}^1,$$

for each $\mathcal{M}_i$. As before, after applying the homological truncation, we attach the modified open neighborhood, TZ_2, to the manifold, M. Again, the crucial point to mention is that the previous spatial modification altered the links of isolated singularities. However, our next goal is to compute the homology groups of the space obtained by attaching the modified TZ_2 to the manifold, M. Once again, we use the Mayer-Vietoris sequence. We choose $\mathcal{U}$ and $\mathcal{V}$, such that

$$\begin{aligned}
&\mathcal{U} \simeq M, \\
&\mathcal{V} \cong \bigsqcup_i \mathcal{S}^1 \times \mathcal{C}\big((\mathcal{L}_{\mathcal{M}_i})_{\leq \bar{n}(4)}\big), \\
&\cap := \mathcal{U} \cap \mathcal{V} \simeq \bigsqcup_i \mathcal{S}^1 \times (\mathcal{L}_{\mathcal{M}_i})_{\leq \bar{n}(4)} \cong \bigsqcup_i \mathcal{S}^1 \times \mathcal{S}^1.
\end{aligned}$$

Let

$$M_1 := \mathcal{U} \cup \mathcal{V}.$$

Hence, we have

$$\begin{aligned}
0 \longrightarrow & \overbrace{\mathbf{H}_6(\cap)}^{0} \longrightarrow \overbrace{\mathbf{H}_6(\mathcal{U})}^{0} \xrightarrow{\cong} \overbrace{\mathbf{H}_6(M_1)}^{0} \\
\longrightarrow & \overbrace{\mathbf{H}_5(\cap)}^{0} \longrightarrow \overbrace{\mathbf{H}_5(\mathcal{U})}^{0} \xrightarrow{\cong} \overbrace{\mathbf{H}_5(M_1)}^{0} \\
\longrightarrow & \overbrace{\mathbf{H}_4(\cap)}^{0} \longrightarrow \overbrace{\mathbf{H}_4(\mathcal{U})}^{\cong \mathbb{Q}^{f_1-1}} \xrightarrow{\cong} \overbrace{\mathbf{H}_4(\mathcal{M}_1)}^{\cong \mathbb{Q}^{f_1-1}} \\
\longrightarrow & \overbrace{\mathbf{H}_3(\cap)}^{0} \longrightarrow \overbrace{\mathbf{H}_3(\mathcal{U})}^{\cong \mathbb{Q}^{2(f_1-1)}} \longrightarrow \overbrace{\mathbf{H}_3(M_1)}^{\cong \mathbb{Q}^{b_3^{M_1}}} \\
\longrightarrow & \overbrace{\mathbf{H}_2(\cap)}^{\cong \mathbb{Q}^{f_2}} \longrightarrow \overbrace{\mathbf{H}_2(\mathcal{U})}^{\cong \mathbb{Q}^{f_1}} \longrightarrow \overbrace{\mathbf{H}_2(M_1)}^{\cong \mathbb{Q}^{b_2^{M_1}}} \\
\longrightarrow & \overbrace{\mathbf{H}_1(\cap)}^{\cong \mathbb{Q}^{2f_2}} \longrightarrow \overbrace{\mathbf{H}_1(\mathcal{U})}^{\cong \mathbb{Q}^{2}} \oplus \overbrace{\mathbf{H}_1(\mathcal{V})}^{\cong \mathbb{Q}^{f_2}} \longrightarrow \overbrace{\mathbf{H}_1(M_1)}^{\cong \mathbb{Q}} \\
\longrightarrow & \overbrace{\widetilde{\mathbf{H}}_0(\cap)}^{\cong \mathbb{Q}^{f_2-1}} \xrightarrow{\cong} \overbrace{\widetilde{\mathbf{H}}_0(\mathcal{U})}^{0} \oplus \overbrace{\widetilde{\mathbf{H}}_0(\mathcal{V})}^{\cong \mathbb{Q}^{f_2-1}} \longrightarrow \overbrace{\widetilde{\mathbf{H}}_0(M_1)}^{0} \longrightarrow 0.
\end{aligned}$$

Finally, from the exactness, we get

$$b_3^{M_1} - b_2^{M_1} = f_1$$

Lemma 11.2

Let $\mathcal{L}$ be the link of an isolated singularity in a totally singular pseudo-toric variety. Let $\mathcal{L}'$ be the modified link of the singularity, after performing the initial step of the topological construction of the intersection space pair, as described above. Let $f_1^{\mathcal{L}}$ denotes the number of 1-dimensional faces of the underlying 2-dimensional polytope of $\mathcal{L}$. Then, we have

$$\begin{aligned}
rk\big(\mathbf{H}_5(\mathcal{L}')\big) &= 0 \\
rk\big(\mathbf{H}_4(\mathcal{L}')\big) &= f_1^{\mathcal{L}} - 1 \\
rk\big(\mathbf{H}_3(\mathcal{L}')\big) &= 3(f_1^{\mathcal{L}} - 1) \\
rk\big(\mathbf{H}_2(\mathcal{L}')\big) &= 2(f_1^{\mathcal{L}} - 1) \\
rk\big(\mathbf{H}_1(\mathcal{L}')\big) &= 1 \\
rk\big(\mathbf{H}_0(\mathcal{L}')\big) &= 1
\end{aligned}$$

Proof. For the proof, we use the Mayer-Vietoris sequence. But before specifying $\mathcal{U}$ and $\mathcal{V}$, one should note that we have $\mathcal{L}' \cong \mathcal{S}^1 \times Z$, where Z is the space obtained by collapsing the $\mathcal{S}^1$ to a point on each torus of $\mathcal{L}'$. This decomposition is allowed, because we are dealing with a totally singular pseudo-toric variety. Now, we employ the same method as in 10.4. Considering $\mathcal{U}$, we can factor out another $\mathcal{S}^1$. Hence, we have $\mathcal{U} \cong \mathcal{S}^1 \times Y$, where we obtain Y in a similar manner to Z. Using a similar adequate homotopy model of $\mathcal{U}$ as in 6.15, we can compute the homology group of $\mathcal{U}$ or respectively Y. First, we compute the homology groups of Y. Hence, we have

$$\begin{aligned}
\operatorname{rk}\big(\mathbf{H}_3(Y)\big) &= 0 \\
\operatorname{rk}\big(\mathbf{H}_2(Y)\big) &= f^{\mathcal{L}} - 1 \\
\operatorname{rk}\big(\mathbf{H}_1(Y)\big) &= 0 \\
\operatorname{rk}\big(\mathbf{H}_0(Y)\big) &= 1
\end{aligned}$$

Using Künneth theorem, we get

$$\begin{aligned}
\operatorname{rk}\big(\mathbf{H}_4(\mathcal{U})\big) &= 0 \\
\operatorname{rk}\big(\mathbf{H}_3(\mathcal{U})\big) &= f^{\mathcal{L}} - 1 \\
\operatorname{rk}\big(\mathbf{H}_2(\mathcal{U})\big) &= f^{\mathcal{L}} - 1 \\
\operatorname{rk}\big(\mathbf{H}_1(\mathcal{U})\big) &= 1 \\
\operatorname{rk}\big(\mathbf{H}_0(\mathcal{U})\big) &= 1
\end{aligned}$$

On the other hand , we have $\mathcal{V} \cong \bigsqcup_{i=1}^{f_1^{\mathcal{L}}} \mathcal{C}(\mathcal{S}^1)$ and hence, $\cap := \mathcal{U} \cap \mathcal{V} \simeq \bigsqcup_{i=1}^{f_1^{\mathcal{L}}} \mathcal{S}^1$. Using the Mayer-Vietoris sequence and Künneth theorem yields the claimed results. □

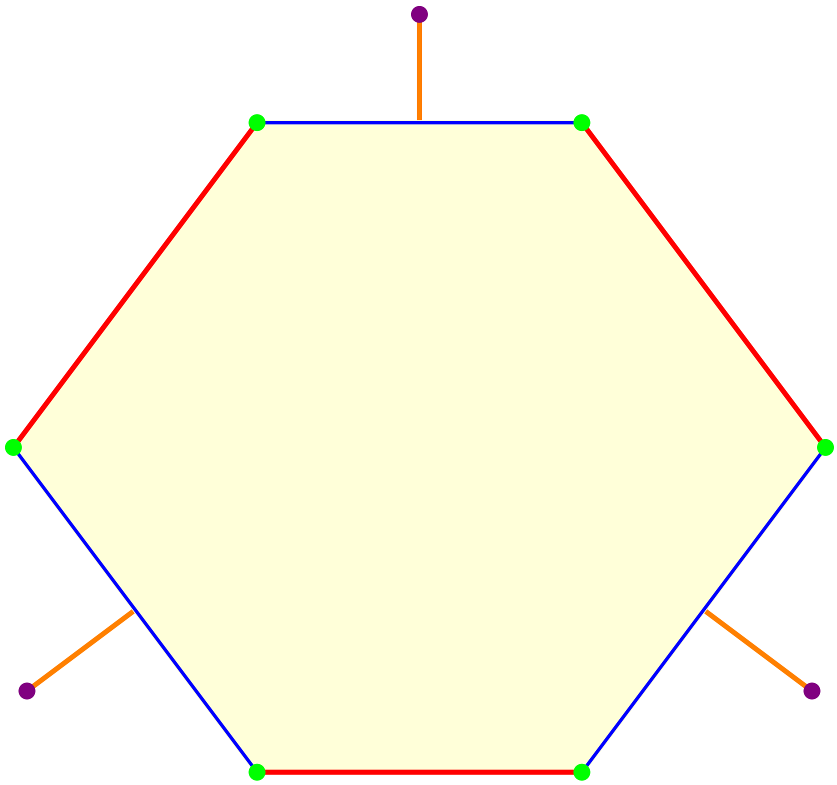

Figure 11.1: Modified link of an isolated singularity after performing the first step of the construction of intersection space pair.

Remark 11.3
Let us assume that in the lemma 11.2, the link of an isolated singularity has an underlying 2-dimensional polytope with three 1-dimensional faces. Schematically, we can describe the modified link after performing the initial step of the induction as follows. In the figure 11.1, to each purple point a $\mathcal{S}^1$ is attached. Each point of orange lines is attached to a $\mathcal{T}^2$. Each point on blue lines is associated with a $\mathcal{T}^3$ and finally, to each green point and to each point of red lines a $\mathcal{T}^2$ is attached. But one should note that we shrank the truncated areas to the orange lines and purple points, using homotopy equivalences similar to 6.4. Thus, this is a homotopy model of the truncated link.

Lemma 11.4
Let IX be the first element of the intersection space pair associated to an 6-dimensional totally singular pseudo-toric variety, $X_{\mathcal{P}}$, where $\mathcal{P}$ is the underlying convex polytope. Let f_1 and f_2 be

the number of 1-dimensional and 2 dimensional cones of Σ, the dual fan to $\mathcal{P}$, respectively. Then for the betti numbers of IX, we have

$$b_6^{IX} = 0$$
$$b_5^{IX} = 0$$
$$b_4^{IX} = f_1 - 1$$
$$b_3^{IX} - b_2^{IX} = 4f_1 + 3f_2 - 7$$
$$b_1^{IX} = 0$$
$$b_0^{IX} = 1$$

Proof. The proof goes along similar lines as in 10.6. We set $\mathcal{U} \simeq M_1$, where we defined M_1 above. Let $\mathcal{V} \cong \bigsqcup_{i=1}^{f_3} \mathcal{C}(\mathcal{L}'_{\leq n(6)})$, where f_3 is the number of the vertices of $\mathcal{P}$. Hence, we get $\cap := \mathcal{V} \cap \mathcal{U} = \bigsqcup_{i=1}^{f_3} \mathcal{L}'_{\leq n(6)}$. Using the Mayer-Vietoris sequence proves the claim. □

At this point, we can finally prove the main theorem of this chapter.

Theorem 11.5
Let $X_{\mathcal{P}}$ be the totally singular pseudo-toric variety associated with a convex polytope, $\mathcal{P}$, with the common stratification, as before. Let $(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})$ be the associated intersection space pair. Then, we have

$$b_6^{(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})} = 0$$
$$b_5^{(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})} = 0$$
$$b_4^{(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})} = f_1 - 1$$
$$b_3^{(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})} = 6f_1 + 3f_2 + 7$$
$$b_2^{(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})} = f_1 - 1$$
$$b_1^{(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})} = 0$$
$$b_0^{(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})} = 0$$

Proof. Note that even in the case of a totally singular pseudo-toric variety, we have the same $I(X_{d-2})_{\mathcal{P}}$ as in the last chapter. Hence, we use the exact sequence of relative homology.

$$\begin{aligned}
0 &\longrightarrow \overbrace{\mathbf{H}_6(I(X_{\mathcal{P}})_{d-2})}^{0} \longrightarrow \overbrace{\mathbf{H}_6(IX_{\mathcal{P}})}^{0} \xrightarrow{\cong} \overbrace{\mathbf{H}_6(IX_{\mathcal{P}}, I(X_{\mathcal{P}})_{d-2})}^{0}\\
&\longrightarrow \overbrace{\mathbf{H}_5(I(X_{\mathcal{P}})_{d-2})}^{0} \longrightarrow \overbrace{\mathbf{H}_5(IX_{\mathcal{P}})}^{0} \xrightarrow{\cong} \overbrace{\mathbf{H}_5(IX_{\mathcal{P}}, I(X_{\mathcal{P}})_{d-2})}^{0}\\
&\longrightarrow \overbrace{\mathbf{H}_4(I(X_{\mathcal{P}})_{d-2})}^{0} \longrightarrow \overbrace{\mathbf{H}_4(IX_{\mathcal{P}})}^{\cong \mathbb{Q}^{f_1-1}} \xrightarrow{\cong} \overbrace{\mathbf{H}_4(IX_{\mathcal{P}}, I(X_{\mathcal{P}})_{d-2})}^{\cong \mathbb{Q}^{f_1-1}}\\
&\longrightarrow \overbrace{\mathbf{H}_3(I(X_{\mathcal{P}})_{d-2})}^{0} \longrightarrow \overbrace{\mathbf{H}_3(IX_{\mathcal{P}})}^{\cong \mathbb{Q}^{b_3^{IX_{\mathcal{P}}}}} \xrightarrow{i} \mathbf{H}_3(IX_{\mathcal{P}}, I(X_{\mathcal{P}})_{d-2})\\
&\xrightarrow{j} \overbrace{\mathbf{H}_2(I(X_{\mathcal{P}})_{d-2})}^{\cong \mathbb{Q}^{f_2}} \xrightarrow{k} \overbrace{\mathbf{H}_2(IX_{\mathcal{P}})}^{\cong \mathbb{Q}^{b_3^{IX_{\mathcal{P}}}-4f_1-3f_2-7}} \xrightarrow{l} \mathbf{H}_2(IX_{\mathcal{P}}, I(X_{\mathcal{P}})_{d-2})\\
&\xrightarrow{m} \overbrace{\mathbf{H}_1(I(X_{\mathcal{P}})_{d-2})}^{\cong \mathbb{Q}^{f_2-f_1-1}} \longrightarrow \overbrace{\mathbf{H}_1(IX_{\mathcal{P}})}^{0} \longrightarrow \overbrace{\mathbf{H}_1(IX_{\mathcal{P}}, I(X_{\mathcal{P}})_{d-2})}^{0}\\
&\longrightarrow \overbrace{\widetilde{\mathbf{H}}_0(I(X_{\mathcal{P}})_{d-2})}^{0} \longrightarrow \overbrace{\widetilde{\mathbf{H}}_0(IX_{\mathcal{P}})}^{0} \longrightarrow \overbrace{\widetilde{\mathbf{H}}_0(IX_{\mathcal{P}}, I(X_{\mathcal{P}})_{d-2})}^{0} \longrightarrow 0.
\end{aligned}$$

Due to the importance of the theorem, we go through each step of the computation. $b_1^{(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})} = 0$ follows from the exactness. But from the duality, it follows that $b_2^{(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})} = f_1 - 1$. Using the exactness yields

$$\begin{aligned}
\mathrm{rk}(im(i)) &= b_3^{IX_{\mathcal{P}}} = \mathrm{rk}(ker(j))\\
\mathrm{rk}(im(j)) &= b_3^{(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})} - b_3^{IX} = \mathrm{rk}(ker(k))\\
\mathrm{rk}(im(k)) &= f_2 - (b_3^{(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})} - b_3^{IX_{\mathcal{P}}}) = \mathrm{rk}(ker(l))\\
\mathrm{rk}(im(l)) &= b_3^{IX_{\mathcal{P}}} - 4f_1 - 3f_2 - 7 - (f_2 - (b_3^{(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})} - b_3^{IX}))\\
&= b_3^{(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})} - 4f_1 - 4f_2 - 7\\
\mathrm{rk}(im(m)) &= f_1 - 1 - (b_3^{(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})} - 4f_1 - 4f_2 - 7) = f_2 - f_1 - 1,
\end{aligned}$$

which implies

$$b_3^{(IX_{\mathcal{P}}, I(X_{d-2})_{\mathcal{P}})} = 6f_1 + 3f_2 + 7$$

□

12 INTERSECTION SPACE OF A LINK OF AN ISOLATED SINGULARITY IN AN 8-DIMENSIONAL TORIC VARIETY

In this chapter, we want to investigate the associated intersection space to a link, $\mathcal{X}$, of an isolated singularity in an 8-dimensional toric variety. We use the theorem introduced in [3] in section 2.9 by Banagl. We generalize the above theorem for $\mathbb{Q}$-pseudomanifold, using the same technique regarding isolated singularities. We consider the middle perversity. As a $\mathbb{Q}$-pseudomanifold, we can endow $\mathcal{X}$ with the following $\mathbb{Q}$-stratification,

$$\mathcal{X} = \mathcal{X}_7 \supset \mathcal{X}_{7-6}.$$

Hence, the truncation degree happens to be

$$k = c - 1 - \bar{n}(c) = 3,$$

where $c = 6$. For now, let X be an n-dimensional, compact, stratified $\mathbb{Q}-$pseudomanifold with two strata,

$$X = X_n \supset X_{n-c}.$$

The singular set $\Sigma = X_{n-c}$ is thus an $(n-c)$-dimensional closed $\mathbb{Q}$-manifold and singularities are not isolated, unless $c = n$. Let Σ_i be a connected component of Σ. Assume that X has a trivial link bundle, that is, a neighborhood of each Σ_i in X looks like $\Sigma_i \times \mathring{\mathcal{C}}(\mathcal{L})$, where $\mathcal{L}$ is a $(c-1)$-dimensional closed $\mathbb{Q}$-manifold, the link of Σ_i. For simplicity and as shown before, we assume that $\ker(\partial_k^{\mathcal{L}})$ has a basis k-cells, or in other words, $\mathcal{L}$ is k-segmented. Hence, no completion is needed in the sense of section 2.9 in [3]. Let $\mathcal{L}_{<k}$ be the truncated link.

Remark 12.1
Note that the above assumption holds for the link of an isolated singularity in a 6-dimensional toric variety. With similar reasoning, we can show that this also stands for the link of any connected component of $\mathcal{X}$, where $\mathcal{X}$ is the link of an isolated singularity in an 8-dimensional toric variety. Thus, in the above situation we define the generalized intersection space as follows.

Definition 12.2
The perversity $\bar{p}$ intersection space $I^{\bar{p}}X$ of X is defined to be

$$I^{\bar{p}}X = \mathcal{C}(g) = \mathcal{M} \cup_g \mathcal{C}(\Sigma \times \mathcal{L}_{<k}),$$

where g is defined to be the composition of

$$\Sigma \times \mathcal{L}_{<k} \hookrightarrow \Sigma \times \mathcal{L} = \partial\mathcal{M} \xhookrightarrow{j} \mathcal{M}.$$

$\mathcal{M}$ is the $\mathbb{Q}$-manifold obtained by removing singularities of X.

Theorem 12.3
Let X be an n-dimensional, compact, oriented, stratified $\mathbb{Q}$-pseudomanifold with singular stratum Σ of dimension $(n-c)$, and trivial link bundle. We assume the link of each connected component of Σ, $\mathcal{L}$ is $k = c - 1 - \bar{p}(c)$-segmented. We assume X, Σ and $\mathcal{L}$ are oriented compatibly. Let $I^{\bar{p}}X$ and $I^{\bar{q}}X$ be generalized $\bar{p}$- and $\bar{q}$ intersection spaces of X with $\bar{p}$ and $\bar{q}$ complementary perversities. Then there exists a generalized Poincaré duality isomorphism

$$D : \widetilde{\mathbf{H}}^{n-r}(I^{\bar{p}}X) \xrightarrow{\cong} \widetilde{\mathbf{H}}_r(I^{\bar{q}}X).$$

We want to apply the above theorem on the link of an isolated singularity in an 8-dimensional toric variety, $\mathcal{X}$.

Lemma 12.4
Let $\mathcal{X}$ be the link of an isolated singularity in an 8-dimensional toric variety. We consider the following $\mathbb{Q}$-stratification on $\mathcal{X}$.

$$\mathcal{X} = \mathcal{X}_7 \supset \mathcal{X}_1.$$

Let $\mathcal{L}$ be the link of a connected component of $\mathcal{X}_1$, a $\mathcal{S}^1$. Let $f_1^{\mathcal{L}}$ be the number of 1-dimensional faces of the underlying polytope of $\mathcal{X}$, $\mathcal{P}$, with x as a proper face, where $x = \pi(\mathcal{S}^1)$ and $\pi : \mathcal{X} \longrightarrow \mathcal{P}$ is the projection map. Then for the Betti numbers, we have

$$\begin{aligned}
b_5^{\mathcal{L}} &= 1 \\
b_4^{\mathcal{L}} &= 0 \\
b_3^{\mathcal{L}} &= f_1^{\mathcal{L}} - 3 \\
b_2^{\mathcal{L}} &= f_1^{\mathcal{L}} - 3 \\
b_1^{\mathcal{L}} &= 0 \\
b_5^{\mathcal{L}} &= 1
\end{aligned}$$

Proof. The proof goes along similar lines as the construction of links of toric varieties. With the same procedure, we determine the underlying 2-dimensional polytope of $\mathcal{L}$, $\mathcal{M}$. Now, let us consider the preimage of $\mathcal{M}$ under the projection π, $\pi^{-1}(\mathcal{M})$. Because of $\mathcal{X}_1 = \sqcup \mathcal{S}^1$, we need to collapse the $\mathcal{S}^1$ associated with x, on each point of $\pi^{-1}(\mathcal{M})$. Consider that we can carry out this step due to the fan construction. In other words, local triviality allows us to carry out the above step. The associated $\mathcal{S}^1$ to x corresponds to a 1-dimensional line in Σ, where Σ is the corresponding fan of the 8-dimensional toric variety. On the other hand, the corresponding cone of the $\mathcal{S}^1$ in Σ is a proper face of all higher-dimensional neighboring faces of x in Σ. The resulting space is a topological space with the underlying polytope $\mathcal{M}$, such that to each point of an open n-dimensional face of $\mathcal{M}$, we attach a $\mathcal{T}^{n+1}$, where $n = 0, 1, 2$. The collapsing data or, in other words, the maps $p_{ij} : \mathcal{T}^i \longrightarrow \mathcal{T}^j$, where $i > j$ and $i, j \in \{0, 1, 2\}$ are extracted from Σ. Hence, we can use the method presented in 3.32 to prove the claim. □

Remark 12.5
Considering the above proof, we studied the corresponding fan to the 8-dimensional toric variety, Σ. Obviously, we can extract the collapsing data for $\mathcal{X}$ from Σ. However, we can associate $\mathcal{P}$, in the above proof, with a dual fan, $\Sigma_{\mathcal{P}}$. Then $\Sigma_{\mathcal{P}}$ will be a complete fan in $\mathbb{R}^3$, and we could be able to extract the collapsing data from $\Sigma_{\mathcal{P}}$. Studying this matter in full generality is beyond the scope of this work.

Remark 12.6
Using the same CW complex as in 3.32 justifies the assumption that the map

$$f : \mathcal{L}_{<k} \hookrightarrow \mathcal{L}$$

is an inclusion.

Lemma 12.7
Let M be the $\mathbb{Q}$-manifold obtained by removing all singularities of $\mathcal{X}$, where $\mathcal{X}$ is a link of an isolated singularity in an 8-dimensional toric variety. Then for the Bettie number of M, we have

$$b_4^M = 3f_1 - f_2 - 6 - (b_1 + b_2)$$
$$b_3^M = 3f_1 - f_2 - 6 - b_1$$
$$b_2^M = f_1 - 4$$
$$b_1^M = 0$$
$$b_0^M = 1,$$

where $f_1$and f_2 are the number of 2- , 1- and faces of $\mathcal{P}$, the underlying 3-dimensional polytope of $\mathcal{X}$. b_1 and b_2 are defined in 3.34.

Proof. M is a $\mathbb{Q}$-manifold and satisfies the Lefschetz duality, rationally. Thus, we have

$$\text{Hom}(\mathbf{H}^k(M;\mathbb{Q});\mathbb{Q}) \cong \mathbf{H}_{7-k}(M,\partial M;\mathbb{Q})$$

From the CW structure of M it is clear that

$$\mathbf{H}_i(X) \cong \mathbf{H}_i(M/\partial M)$$

for $i \geq 3$, because conning off the boundary, ∂M, only modifies the lower dimensional cells. Finally, we use $\widetilde{\mathbf{H}}_j(M/\partial M) \cong \widetilde{\mathbf{H}}_j(M,\partial M)$. □

At this point, we can prove the main theorem of this chapter.

Theorem 12.8
Let $\mathcal{X}$ be the link of an isolated singularity of an 8-dimensional toric variety. Let $I^{\bar{n}}\mathcal{X}$ be the associated

generalized intersection space to $\mathcal{X}$ concerning middle perversity $\bar{n}$, as defined above. Then for the Betti number of $I^{\bar{n}}\mathcal{X}$, we have

$$
\begin{aligned}
b_7^{I^{\bar{n}}\mathcal{X}} &= 0\\
b_6^{I^{\bar{n}}\mathcal{X}} &= f_2 - f_1 + 1\\
b_5^{I^{\bar{n}}\mathcal{X}} &= f_2 - 4 - b_2\\
b_4^{I^{\bar{n}}\mathcal{X}} &= 3f_1 - f_2 - 6 - b_2\\
b_3^{I^{\bar{n}}\mathcal{X}} &= 3f_1 - f_2 - 6 - b_2\\
b_2^{I^{\bar{n}}\mathcal{X}} &= f_2 - 4 - b_2\\
b_1^{I^{\bar{n}}\mathcal{X}} &= f_2 - f_1 + 1\\
b_0^{I^{\bar{n}}\mathcal{X}} &= 1
\end{aligned}
$$

, where f_1, f_2 and f_3 are the number of 2- , 1- and 0-dimensional faces of $\mathcal{P}$, the underlying 3-dimensional polytope of $\mathcal{X}$. b_2 is defined in 3.34.

Proof. For the proof, we us the Mayer-Vietoris sequence. Let M be the $\mathbb{Q}$-manifold obtained by removing the singularities of $\mathcal{X}$. We set $\mathcal{U} \simeq M$. Let $\mathcal{V} \cong \mathcal{C}(\sqcup_i \mathcal{S}^1 \times (\mathcal{L}_i)_{<k})$, where i goes over the connected component of $\mathcal{X}_1$. Hence, we get $\cap := \mathcal{U} \cap \mathcal{V} \simeq \sqcup_i \mathcal{S}^1 \times (\mathcal{L}_i)_{<k}$. Consequently, we have

$$
\begin{aligned}
0 &\longrightarrow \overbrace{\mathbf{H}_7(\cap)}^{\cong 0} \longrightarrow \overbrace{\mathbf{H}_7(\mathcal{U})}^{\cong 0} \xrightarrow{\cong} \overbrace{\mathbf{H}_7(I^{\bar{n}}\mathcal{X})}^{\cong 0}\\
&\longrightarrow \overbrace{\mathbf{H}_6(\cap)}^{\cong 0} \longrightarrow \mathbf{H}_6(\mathcal{U}) \xrightarrow{\cong} \mathbf{H}_6(I^{\bar{n}}\mathcal{X})\\
&\longrightarrow \overbrace{\mathbf{H}_5(\cap)}^{\cong 0} \longrightarrow \mathbf{H}_5(\mathcal{U}) \xrightarrow{\cong} \mathbf{H}_5(I^{\bar{n}}\mathcal{X})\\
&\longrightarrow \overbrace{\mathbf{H}_4(\cap)}^{\cong 0} \longrightarrow \overbrace{\mathbf{H}_4(\mathcal{U})}^{\cong \mathbb{Q}^{3f_1-f_2-6-(b_1+b_2)}} \longrightarrow \mathbf{H}_4(I^{\bar{n}}\mathcal{X})\\
&\longrightarrow \overbrace{\mathbf{H}_3(\cap)}^{\cong \mathbb{Q}^{3f_1-f_2-6}} \longrightarrow \overbrace{\mathbf{H}_3(\mathcal{U})}^{\cong \mathbb{Q}^{3f_1-f_2-6-b_1}} \longrightarrow \mathbf{H}_3(I^{\bar{n}}\mathcal{X})\\
&\longrightarrow \overbrace{\mathbf{H}_2(\cap)}^{\cong \mathbb{Q}^{3f_1-f_2-6}} \longrightarrow \overbrace{\mathbf{H}_2(\mathcal{U})}^{\cong \mathbb{Q}^{f_1-4}} \longrightarrow \mathbf{H}_2(I^{\bar{n}}\mathcal{X})\\
&\longrightarrow \overbrace{\mathbf{H}_1(\cap)}^{\cong \mathbb{Q}^{f_3}} \longrightarrow \overbrace{\mathbf{H}_1(\mathcal{U})}^{\cong 0} \longrightarrow \mathbf{H}_1(I^{\bar{n}}\mathcal{X})\\
&\longrightarrow \overbrace{\widetilde{\mathbf{H}}_0(\cap)}^{\cong \mathbb{Q}^{f_3-1}} \longrightarrow \overbrace{\widetilde{\mathbf{H}}_0(\mathcal{U})}^{\cong 0} \oplus \overbrace{\widetilde{\mathbf{H}}_0(\mathcal{V})}^{\cong 0} \longrightarrow \overbrace{\widetilde{\mathbf{H}}_0(I^{\bar{n}}\mathcal{X})}^{\cong 0} \longrightarrow 0
\end{aligned}
$$

From the exactness, it follows immediately that

$$b_1^{I\mathcal{X}} = f_3 - 1 = f_2 - f_1 + 1.$$

Thus, the duality yields

$$b_6^{I\mathcal{X}} = f_3 - 1 = f_2 - f_1 + 1.$$

Furthermore, we know that

$$\mathbf{H}_i(\partial M) \cong \mathbf{H}_i(\mathcal{S}^1 \times \mathcal{L}_{<3}) \ \text{ for } i \leq 3$$

By definition, we have $\mathbf{H}_i(M) = \mathbf{H}_i(\mathcal{U})$. We consider the above Mayer-Vietoris exact sequence and the exact sequence of the relative homology of the pair $(M, \partial M)$. By using the 5-lemma, we get

$$b_3^{I\mathcal{X}} = b_3^{(M,\partial M)} = b_4^M,$$

whereas, in the last equality, we used the Lefschetz duality. By using the duality of intersection spaces we get

$$b_3^{I\mathcal{X}} = b_4^{I\mathcal{X}} = 3f_1 - f_2 - 6 - (b_1 + b_2).$$

However, using the exactness at $\mathbf{H}_3(\cap) \longrightarrow \mathbf{H}_3(\mathcal{U})$ and considering that $0 \leq b_1$, it immediately follows that

$$b_1 = 0,$$

where b_1 is defined in 3.34. Finally, we get $b_2^{I\mathcal{X}} = f_2 - 2 - b_2$. □

Corollary 12.9
Let $\mathcal{X}$ be the link of an isolated singularity in an 8-dimensional toric variety. Then, we have

$$
\begin{aligned}
&rk(\mathbf{H}_7(\mathcal{X};\mathbb{Q})\ ) = 1,\\
&rk(\mathbf{H}_6(\mathcal{X};\mathbb{Q})\ ) = 0,\\
&rk(\mathbf{H}_5(\mathcal{X};\mathbb{Q})\ ) = f_1 - 4,\\
&rk(\mathbf{H}_4(\mathcal{X};\mathbb{Q})\ ) = 3f_1 - f_2 - 6,\\
&rk(\mathbf{H}_3(\mathcal{X};\mathbb{Q})\ ) = 3f_1 - f_2 - 6 - b_2,\\
&rk(\mathbf{H}_2(\mathcal{X};\mathbb{Q})\ ) = f_1 - 4 - b_2,\\
&rk(\mathbf{H}_1(\mathcal{X};\mathbb{Q})\ ) = 0,\\
&rk(\mathbf{H}_0(\mathcal{X};\mathbb{Q})\ ) = 1,
\end{aligned}
$$

where we defined b_2 here 3.34.

Part V

CONCLUSION

13 CONCLUSION

Finally, we are at the point where we can summarize all obtained results in this work and make outlines for future researches.
We have shown that in toric varieties, the link of each connected component of the 4-co-dimensional stratum is always a 3-homological sphere. Thus, for an arbitrary 2n-dimensional toric variety X, we get the following $\mathbb{Q}$-stratification

$$X = X_{2n} \supset X_{2(n-3)} \supset \cdots \supset X_0.$$

Based on the notion of $\mathbb{Q}$-pseudomanifolds, we generalized the theory of intersection spaces with isolated singularities.

Remark 13.1
We claim that the theory of intersection spaces with isolated singularities can not be generalized further. In other words, we can not find a weaker notion of pseudomanifolds or a coarser stratification such that the construction of intersection spaces yields any form of Poincaré duality of the ordinary homology theory.

Consider our main object of study, 6-dimensional toric varieties. We compare Betti numbers of the associated generalized intersection spaces, intersection space pairs, and intersection homology in the following table.

b_* \ Theory	$\mathbf{H}_*(IX)$	$\mathbf{H}_*(IX, IX_{d-2})$	$\mathbf{IH}_*(X)$
b_6	0	0	1
b_5	$f_2 - f_1 - 1$	0	0
b_4	$f_2 - 3 - b$	$f_1 - 1$	$f_1 - 3$
b_3	$2(3f_1 - f_2 - b - 6)$	$6f_1 - f_2 + 6$	0
b_2	$f_2 - 3 - b$	$f_1 - 1$	$f_1 - 3$
b_1	$f_2 - f_1 - 1$	0	0
b_0	1	0	1

Table 13.1: The associated Betti numbers of three different theories.

At first glimpse, it is easy to deduce that b_*^{IX}'s are not combinatorial invariant. In other words, we can determine the Betti numbers of X using b_*^{IX}. However, on the other side, the associated Betti numbers of the intersection homology do not even fix the combinatorial data of the fan. The theory of intersection space pairs removes the parameter b, but it still carries the number of cones in the associated fan.

Remark 13.2
We claim that the theory of intersection space pairs can be generalized to $\mathbb{Q}$-pseudomanifolds with trivial link bundles. We also assert that the Betti numbers of real $2n$-dimensional toric varieties include exactly $n-2$ parameters, which are not combinatorial invariant. Finally, we claim that the generalized intersection space pair of an arbitrary toric variety carries all non-combinatorial invariant parameters of the Betti numbers of the toric variety.

Next, we look at the Betti numbers of a link of an isolated singularity in an 8-dimensional toric variety, $\mathcal{X}$. We also constructed the associated intersection space, $I\mathcal{X}$. We can summarize our findings in the following table.

b_* \ Theory	$\mathbf{H}_*(\mathcal{X})$	$\mathbf{H}_*(I\mathcal{X})$
b_7	1	0
b_6	0	$f_2 - f_1 + 1$
b_5	$f_1 - 4$	$f_1 - 4 - b$
b_4	$3f_1 - f_2 - 6$	$3f_1 - f_2 - 6 - b$
b_3	$3f_1 - f_2 - 6 - b$	$3f_1 - f_2 - 6 - b$
b_2	$f_1 - 4 - b$	$f_1 - b - 4$
b_1	0	$f_2 - f_1 - 1$
b_0	1	1

Table 13.2: The Betti numbers of $\mathcal{X}$ and $I\mathcal{X}$.

As before, f_1 and f_2 denote the numbers of 1- and 2-dimensional cones in the associated fan to $\mathcal{X}$, respectively. Note that in the above table, we renamed the parameter b_2 defined here 3.34 to b to prevent any confusion.

Remark 13.3
Consider a real $2n$-dimensional toric variety, X. We claim that the link of $X_{2i} - X_{2(i-1)}$, $\mathcal{L}$, is an n-dimensional $\mathbb{Q}$-pseudomanifold with the following $\mathbb{Q}$-stratification.

$$\mathcal{L}_{2(n-i)-1} \supset \mathcal{L}_{2(n-i)-7} \supset \mathcal{L}_{2(n-i)-9} \supset \cdots \supset \mathcal{L}_1 \cong \sqcup \mathcal{S}^1.$$

We also claim that the Betti numbers of $\mathcal{L}$ have exactly $(n-i-3)$ parameters, which are not combinatorial invariants.

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DECLARATION

I declare that this thesis has been composed solely by myself and that it has not been submitted, in whole or in part, in any previous application for a degree. Except where states otherwise by reference or acknowledgment, the work presented is entirely my own.

Heidelberg, February 2023

Shahryar Ghaed Sharaf