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Complementarity, not Optimization, is the Language of Markets

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ABSTRACT Each market agent (producer or consumer) in a power market pursues its own objective, typically to maximize its own profit. As such, the specific behavior of each agent in the market is conveniently formulated as a bi-level optimization problem whose upper-level problem represents the profit seeking behavior of the agent and whose lower-level problem represents the clearing of the market. The objective function and the constraints of this bi-level problem depend on the agent's own decision variables and on those of other agents as well. Understanding the outcomes of the market requires considering and solving jointly the interrelated bi-level problems of all market agents, which is beyond the purview of optimization. Solving jointly a set of bi-level (or single-level) optimization problems that are interrelated is the purview of complementarity. In this paper and in the context of power markets, we review complementarity using a tutorial approach.

INDEX TERMS Complementarity, optimization, power markets, power systems.

I. INTRODUCTION

A. BACKGROUND

WE DESCRIBE in this paper a number of mathematical models to represent the behavior of the agents of a power market, namely, producers, consumers and market operator. These models rely on economic rationality [1].

Pursuing maximum social welfare, the market operator (or independent system operator, ISO) clears the market using an optimization problem. We assume that such optimization problem is convex or that a convex approximation of it, which is accurate enough, can be obtained.

Each producer seeks to maximize its owns profit by submitting to the market operator a strategic offer consisting of prices and quantities. The offer prices do not need to be equal to marginal cost of the producer. To identify its strategic offer, the producer solves a bi-level optimization problem whose upper-level problem represents its profit seeking behavior and whose lower-level problem represents the clearing of the market. The market clearing price and production level of the producer, derived from the lower-level problem, are used in the upper-level problem to compute the producer profit, and in turn, to derive its strategic offer.

Likewise, each consumer seeks to maximize its owns profit (or surplus) by submitting to the market operator a strategic bid consisting of prices and quantities. Bid prices do not need to be equal to marginal utility of the consumer. To identify its strategic bid, the consumer solves a bi-level optimization problem whose upper-level problem represents its profit seeking behavior and whose lower-level problem represents the clearing of the market. The market clearing price and consumption level of the consumer, derived from the lower-level problem, are used in the upper-level problem to compute the consumer profit, and in turn, to derive its strategic bid.

To analyze the market as a whole, we jointly consider and solve the interrelated bi-level problems of all producers and all consumers. Solutions that are common to all these bi-level problems are equilibrium candidates in the Nash sense [2].

The methodology that we describe allows us searching for single-strategy equilibria, but does not guarantee finding all such equilibria. Moreover, single-strategy equilibria may or may not exist, but practical experience indicates that in power markets multiple equilibria do exist. We do not search for equilibria in mixed strategies as these equilibria have a limited relevance in practice [3].

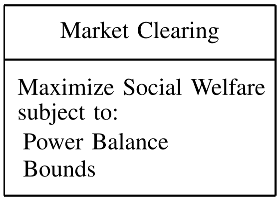

FIGURE 1. Market clearing: optimization problem.

The most common strategy to solve a bi-level problem, whose lower-level problem is convex, is to replace such lower-level problem by its Karush-Kuhn-Tucker (KKT) optimality conditions. Such conditions include *complementarity* constraints of the form:

$$0 \leq x \perp y \geq 0.$$

The above complementarity constraint is equivalent to:

$$\begin{aligned} &x \geq 0 \\ &y \geq 0 \\ &x \cdot y = 0, \end{aligned}$$

which define a complementarity condition: either $x = 0$ and $y > 0$ or the other way around $y = 0$ and $x > 0$. The case $x = 0$ and $y = 0$ is also feasible, but generally irrelevant.

The denomination *complementarity* or *complementarity theory* referring to the analysis and solution of bi-level problems and of set of interrelated bi-level problems comes from the complementarity constraints included in the KKT conditions, as described above.

B. MARKET CLEARING

The market operator typically uses an (single-level) optimization problem to clear the market. This is illustrated in Fig. 1 (lower box). The objective of such optimization problem is to maximize the social welfare, computed as the area between the demand and supply curves. Under no network congestion, the social welfare is also equal to the producers' profit (or surplus) plus the consumers' profit (or surplus). If, on the other hand, the network is congested, the network surplus needs to be added. The constraints of this problem include both power balances and production/consumption bounds.

It is most important that the optimization problem used to clear the market (lower box in Fig. 1) is convex, since convex clearing optimization problems result in meaningful clearing prices. Using a non-convex problem for market clearing is looking for trouble, as analyzed in [4].

C. STRATEGIC AGENTS

A producer or consumer generally seeks maximizing its own profit. For this, it is convenient to use a bi-level problem as illustrated in Fig. 2 (lower box). The upper-level problem of this bi-level problem represents the profit seeking behavior of the agent, while the lower-level problem (inner box of the lower box of Fig. 2) represents the clearing of the market. The upper-level problem allows the agent to determine its best

FIGURE 2. Strategic agent: MPEC.

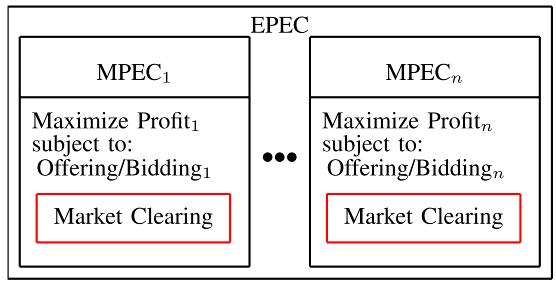

FIGURE 3. Agents' equilibria: EPEC.

offering/bidding strategy (to be transferred to the lower-level problem), while the lower-level problem provides the market clearing price (as a dual variable) needed to compute the agent's profit at the upper-level problem. Further details are provided in [3], [5], [6] and [7].

The lower-level problem (inner box of the lower box of Fig. 2) is generally replaced by its optimality (also called equilibrium) conditions, rendering a Mathematical Program with Equilibrium Constraints (MPEC). An MPEC is also referred to as a single-leader single-follower game [8], being the single-leader the agent itself and the single-follower the market.

D. EQUILIBRIUM

In a market, every agent has a say and thus, to understand the market and its outcomes, we need to consider simultaneously the behaviors of all agents. We do so by considering jointly the MPECs of all market agents and taking into account that such MPECs are related to one another. The joint considerations of these MPECs constitutes an equilibrium problem (set of interrelated optimization problems) of bi-level problems, referred to as Equilibrium Problem with Equilibrium constraints (EPEC) [9]–[11]. It is also referred to as multiple-leader common-follower game [8], being the multiple leaders the agents and the common follower the market. Fig. 3 illustrates an EPEC of n agents.

E. CONTRIBUTION AND PAPER ORGANIZATION

The contribution of this paper is simple: providing a tutorial overview of the mathematical models needed to understand power markets, market agents' behavior and the outcomes of such markets. For additional details, we provide a number of representative references.

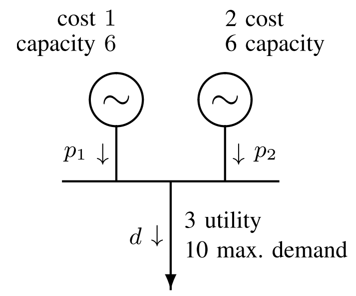

FIGURE 4. Three-agent market.

The rest of this paper is organized as follows. Section II reviews market clearing, Section III analyzes the strategic behavior of market-agents, Section IV considers market equilibria among agents, and Section V concludes listing pressing research needs. To provide adequate background, Appendix A reviews the Karush-Kuhn-Tucker optimality conditions and Appendix B the diagonalization algorithm.

II. MARKET CLEARING

This section reviews succinctly optimization models for market clearing. A simple example is discussed first, followed by a general formulation.

A. ILLUSTRATIVE EXAMPLE

The key objective of the market operator is to clear the market, which can be represented by a (single-level) optimization problem. This is illustrated below through an example.

We consider a market comprising two producers (upper part of Fig. 4) and one consumer (lower part of Fig. 4). Producer 1 has a capacity of 6 power units and a marginal cost $c_1 = \$1$ per energy unit, while producer 2 has a capacity of 6 power units as well and a marginal cost $c_2 = \$2$ per energy unit. The consumer has a maximum demand of 10 power units and a marginal utility of $u = \$3$ per energy unit.

If the productions of producers 1 and 2 are denoted by p_1 and p_2, respectively, and the demand of the consumer by d, the market operator clears the market seeking maximum social welfare, or minimum minus social welfare, z, by solving the following primal optimization problem:

$$\min_{p_1,p_2,d\geq 0} \quad z = 1\cdot p_1 + 2\cdot p_2 - 3\cdot d \tag{1a}$$
$$\text{s. t.} \quad p_1 + p_2 - d = 0 : \ \lambda \tag{1b}$$
$$p_1 \leq 6 : \ \mu_1^p \tag{1c}$$
$$p_2 \leq 6 : \ \mu_2^p \tag{1d}$$
$$d \leq 10 : \ \mu^d, \tag{1e}$$

where the objective function (1a) is the minus social welfare (cost minus utility), constraint (1b) is the power balance, and constraints (1c)-(1e) are bounds. Dual variable λ (of power balance constraint (1b)) is the market clearing price, and μ_1^p, μ_2^p and μ^d are dual variables related to bounds.

The solution of problem (1) can be easily obtained by inspection and is: $p_1^* = 6, p_2^* = 4, d^* = 10, \lambda^* = 2$ and $z^* = -16$. That is, since the marginal utility of the consumer is higher than the marginal cost of the most expensive producer and there is enough production capacity to supply the whole demand, the whole demand is served and thus $d^* = 10$, the cheapest producer produces at capacity, $p_1^* = 6$, and the most expensive producer covers the remaining demand, $p_2^* = 4$. Since the marginal producer (the producer not producing at capacity) is producer 2, the market clearing price is $\lambda^* = 2$. Finally, the minus social welfare (objective function value) is $z^* = -16$.

The profit or surplus (production × (price − marginal cost)) of producer 1 is $\pi_1^p = 6\cdot(2-1) = 6$ and that of producer 2, $\pi_2^p = 4\cdot(2-2) = 0$, while the profit or surplus (demand × (marginal utility − price)) of the consumer is $\pi^d = 10\cdot(3-2) = 10$.

The total producer surplus ($\pi_1^p + \pi_2^p = 6 + 0 = 6$) plus the total consumer surplus ($\pi^d = 10$) is the social welfare, w. We verify that in the considered market condition $w = -z^*$

The dual problem [12] of primal problem (1) is:

$$\max_{\lambda\in\mathbb{R};\mu_1^p,\mu_2^p,\mu^d\leq 0} \quad z = 6\cdot\mu_1^p + 6\cdot\mu_2^p + 10\cdot\mu^d \tag{2a}$$
$$\text{s. t.} \quad \lambda + \mu_1^p \leq 1 : \ p_1 \tag{2b}$$
$$\lambda + \mu_2^p \leq 2 : \ p_2 \tag{2c}$$
$$-\lambda + \mu^d \leq -3 : \ d. \tag{2d}$$

Considering primal and dual problems (1) and (2), respectively, generic production offers o_1 (instead of $c_1 = 1$) and o_2 (instead of $c_2 = 2$), and generic demand bid b (instead of $u = 3$), the necessary and sufficient optimality conditions of either of these problems are [12]:

$$\begin{cases} \text{(a)} & p_1 + p_2 - d = 0 \\ \text{(b)} & p_1 \leq 6, p_2 \leq 6, d \leq 10 \\ \text{(c)} & p_1 \geq 0, p_2 \geq 0, d \geq 0 \\ \text{(d)} & \lambda + \mu_1^p \leq o_1 \\ \text{(e)} & \lambda + \mu_2^p \leq o_2 \\ \text{(f)} & -\lambda + \mu^d \leq -b \\ \text{(g)} & \lambda \in \mathbb{R}; \mu_1^p \leq 0, \mu_2^p \leq 0, \mu^d \leq 0 \\ \text{(h)} & o_1\cdot p_1 + o_2\cdot p_2 - b\cdot d = 6\cdot\mu_1^p + 6\cdot\mu_2^p + 10\cdot\mu^d \end{cases} \tag{3}$$

The system of equalities and inequalities (3) includes primal constraints: conditions (a) and (b), dual constraints: conditions (d) to (f), primal and dual variable bounds: conditions (c) and (g), respectively, and condition (h) that states that the primal objective function value equals the dual objective function value at the optimizer (strong duality theorem [12]).

The solutions of problems (1) or (2) are obtained solving (3) with values $o_1 = c_1 = 1, o_2 = c_2 = 2$ and $b = u = 3$.

The system of equalities and inequalities (3) is a particular instance of the more general KKT conditions for convex optimization problems. We remind the reader that KKT conditions are necessary and sufficient optimality conditions for convex optimization problems [12].

Finally, we note that instead of (1) or (2), if convenient, we can use (3).

B. MATHEMATICAL FORMULATION

The general formulation of problem (1) above is:

$$\min_{\boldsymbol{x}} \quad f(\boldsymbol{x}) \tag{4a}$$
$$\text{s. t.} \quad \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0} : \boldsymbol{\lambda} \tag{4b}$$
$$\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0} : \boldsymbol{\mu}, \tag{4c}$$

where $\boldsymbol{x}$ is the decision variable vector, $f(\boldsymbol{x})$ the minus social welfare, and $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}$ and $\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}$ equality and inequity constraints, respectively.

If problem in (4) is convex or has been convexified [13], the optimality conditions below (KKT conditions) identify optimal solutions:

$$\begin{cases} \nabla_{\boldsymbol{x}} f(\boldsymbol{x}) + \boldsymbol{\lambda}^\top \nabla_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{\mu}^\top \nabla_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0} \\ \boldsymbol{h}(\boldsymbol{x}) = 0, \quad \boldsymbol{0} \leq \boldsymbol{\mu} \perp \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0} \end{cases} \tag{5}$$

where $\boldsymbol{0} \leq \boldsymbol{x} \perp \boldsymbol{y} \leq \boldsymbol{0}$ is equivalent to $\boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{y} \leq \boldsymbol{0}$ and $\boldsymbol{x} \cdot \boldsymbol{y} = 0$.

Note that deriving KKT conditions is a relatively simple exercise. For example, the solver EMP,[1] which is available in GAMS, derives KKT conditions automatically.

KKT conditions can be directly solved by using appropriate solvers, such as PATH[2] or MILES.[3] Alternatively, they can be solved using an auxiliary problem with an arbitrary objective function and the KKT conditions (linearized or not) as constraints [5].

III. STRATEGIC AGENTS

This section reviews bi-level problems for producers and consumers as strategic market agents. A *strategic* market agent actively pursues altering the market clearing price to its own benefit.

We discuss first simple examples and provide then a general formulation.

A. STRATEGIC AGENT EXAMPLES

We consider first that producer 1 is strategic, but producer 2 and the consumer are not. The bi-level problem to be solved by producer 1 is:

$$\max_{o_1} \pi_1^p = (\lambda - 1) \cdot p_1 \tag{6a}$$
$$\text{s. t.} \quad o_1 \geq 1 \tag{6b}$$
$$\min_{p_1, p_2, d \geq 0} \quad z = o_1 \cdot p_1 + 2 \cdot p_2 - 3 \cdot d \tag{6c}$$
$$\text{s. t.} \quad d - p_1 - p_2 = 0 : \ \lambda \tag{6d}$$
$$p_1 \leq 6, p_6 \leq 6, d \leq 10, \tag{6e}$$

where o_1 is the strategic offer price of producer 1, p_1 its production, π_1^p its profit, λ the market clearing price, p_2 and d the production of producer 2 and the consumption of the demand, respectively, and z the declared social welfare for the price offer o_1 submitted by producer 1.

[1]https://www.gams.com/latest/docs/UG_EMP.html
[2]http://pages.cs.wisc.edu/ ferris/path.html
[3]https://www.gams.com/latest/docs/S_MILES.html

TABLE 1. **Producer 1 strategies.**

o_1	p_1	p_2	d	λ	$-z$	π_1^p	π_2^p	π^d	w
$3-\varepsilon$	4	6	10	3	6	8	6	0	14
$2-\varepsilon$	6	4	10	2	10	6	0	10	16
1	6	4	10	2	16	6	0	10	16

Upper-level problem (6a)-(6b) seeks to identify the best choice for o_1 so that producer 1's profit ($\pi_1^p(\cdot)$) is maximized. Lower-level problem (6c)-(6e) clears the market and provides the market clearing price λ (dual variable of constraint (6d)), used to calculate the profit of producer 1 in the objective function (6a) of the upper-level problem.

Regarding the optimal selection of o_1 (solution of the upper-level problem) in this simple example, two strategies are worthy exploring, offering at $o_1 = 3 - \varepsilon$ (ε is a small enough positive constant) to raise the clearing price as much as possible without losing demand ($o_1 < u = 3$), and offering at $o_1 = 2 - \varepsilon$ to undercut producer 2 ($o_1 < c_2 = 2$). Outcomes for these two cases are reported in Table 1. For the sake of comparison, the last row of this table provides the competitive equilibrium.

The first solution maximizes the profit of producer 1, π_1^p, but the second one drives the profit of producer 2, π_2^p to 0. Both outcomes might be relevant for producer 1. We note that the second solution corresponds to the competitive one.

Observe that $-z$ is the declared social welfare, computed using the offer/bid prices declared by producers/consumers, while w is the true social welfare, computed using the true marginal costs/utilities of producers/consumers. In the competitive case, $-z = w$, but in strategic cases, this is not generally so.

We consider second that producer 2 is strategic, but that producer 1 and the consumer are not. The bi-level problem to be solved by producer 2 is:

$$\max_{o_2} \pi_2^p = (\lambda - 2) \cdot p_2 \tag{7a}$$
$$\text{s. t.} \quad o_2 \geq 2 \tag{7b}$$
$$\min_{p_1, p_2, d \geq 0} \quad z = 1 \cdot p_1 + o_2 \cdot p_2 - 3 \cdot d \tag{7c}$$
$$\text{s. t.} \quad d - p_1 - p_2 = 0 : \ \lambda \tag{7d}$$
$$p_1 \leq 6, p_6 \leq 6, d \leq 10. \tag{7e}$$

Regarding the optimal selection of o_2 in this simple example, a single strategy is worthy to explore, offering at $o_2 = 3 - \varepsilon$ to raise the clearing price as much as possible without losing demand ($o_2 < u = 3$). The outcome is provided in Table 2. For the sake of comparison, the last row of this table provides the competitive equilibrium.

We consider third that the consumer is strategic, but that producers 1 and 2 are not. The bi-level problem to be solved

TABLE 2. **Producer 2 strategy.**

o_2	p_1	p_2	d	λ	$-z$	π_1^p	π_2^p	π^d	w
$3-\varepsilon$	6	4	10	3	12	12	4	0	16
2	6	4	10	2	16	6	0	10	16

TABLE 3. **Consumer strategies.**

b	p_1	p_2	d	λ	$-z$	π_1^p	π_2^p	π^d	w
$2+\varepsilon$	6	4	10	2	6	6	0	10	16
$1+\varepsilon$	6	0	6	1	0	0	0	12	12
3	6	4	10	2	16	6	0	10	16

by the consumer is:

$$\max_{b\geq 0} \pi^d = (3-\lambda)\cdot d \tag{8a}$$
$$\text{s. t. } b \leq 3 \tag{8b}$$
$$\min_{p_1,p_2,d\geq 0} z = 1\cdot p_1 + 2\cdot p_2 - b\cdot d \tag{8c}$$
$$\text{s. t. } d - p_1 - p_2 = 0 : \lambda \tag{8d}$$
$$p_1 \leq 6, p_6 \leq 6, d \leq 10. \tag{8e}$$

Regarding the optimal selection of b in this simple example, two strategies are worthy to explore, bidding at $b = 2 + \varepsilon$ to lower the clearing price as much as possible without losing demand ($b > c_2 = 2$), and bidding at $b = 1 + \varepsilon$ to lower the clearing price ever further at the cost of losing some demand ($b > c_1 = 1$). Outcomes for these two cases are reported in Table 3. For the sake of comparison, the last row of this table provides the competitive equilibrium.

The second solution maximizes the profit of the consumer, π^d, but at the cost of consuming solely 6 units. On the other hand, the first solution allows supplying the whole load, but with a comparatively lower profit for the consumer. We note that the first solution corresponds to the competitive one.

B. BI-LEVEL MODEL OF A STRATEGIC AGENT

In general, the bi-level problem of each agent (producer or consumer) in the market can be formulated as:

$$\max_{\Xi^{(i)}} \pi^{(i)}\left(\boldsymbol{x}^{(i)}, \boldsymbol{\lambda}\right) \tag{9a}$$
$$\text{s. t. } \boldsymbol{o}^{(i)} \in \mathcal{O}^{(i)} \tag{9b}$$
$$\min_{\boldsymbol{x}} f(\boldsymbol{x}, \boldsymbol{o}) \tag{9c}$$
$$\text{s. t. } \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0} : \boldsymbol{\lambda} \tag{9d}$$
$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{o}) \leq \boldsymbol{0} : \boldsymbol{\mu}, \tag{9e}$$

where $\Xi^{(i)} = \{\boldsymbol{o}^{(i)}\} \cup \{\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}\}$.

The notation used is as follows: $\pi^{(i)}(\cdot)$ is the profit of agent (i), $\boldsymbol{x}$ the vector of optimization variables, $\boldsymbol{x}^{(i)}$ the sub-vector (of vector $\boldsymbol{x}$) of optimization variables that pertains to agent (i), $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ vectors of dual variables, $\boldsymbol{o}$ the offer/bid vector, $\boldsymbol{o}^{(i)}$ the offer/bid sub-vector (of vector $\boldsymbol{o}$) pertaining to agent (i), and $\mathcal{O}^{(i)}$ the feasible set of offers/bids of agent (i).

Upper-level problem (9a)-(9b) represents the profit of the agent (producer or consumer), while lower-level problem (9c)-(9e) represents the clearing of the market.

C. MPEC OF A STRATEGIC AGENT

Considering bi-level problem (9), and assuming that lower-level power problem (9c)-(9e) is convex or have been convexified [13], we can replace it with its corresponding KKT optimality conditions (or alternative optimality conditions) rendering the Mathematical Program with Equilibrium Constraints (MPEC) below [3], [5], [14]:

$$\max_{\Xi^{(i)}} \pi^{(i)}\left(\boldsymbol{x}^{(i)}, \boldsymbol{\lambda}\right) \tag{10a}$$
$$\text{s. t. } \boldsymbol{o}^{(i)} \in \mathcal{O}^{(i)} \tag{10b}$$
$$\nabla_{\boldsymbol{x}} f(\cdot) + \boldsymbol{\lambda}^\top \nabla_{\boldsymbol{x}} \boldsymbol{h}(\cdot) + \boldsymbol{\mu}^\top \nabla_{\boldsymbol{x}} \boldsymbol{g}(\cdot) = \boldsymbol{0} \tag{10c}$$
$$\boldsymbol{h}(\cdot) = \boldsymbol{0} \tag{10d}$$
$$\boldsymbol{0} \leq \boldsymbol{\mu} \perp \boldsymbol{g}(\cdot) \leq \boldsymbol{0}. \tag{10e}$$

MPEC (10) above is a single-level optimization problem that can be solved using conventional optimization techniques [12].

Regarding the simple examples previously considered in this section, bi-level problem (6) can be solved by solving the single-level problem:

$$\max_{o_1} \pi_1^p = (\lambda - 1)\cdot p_1 \tag{11a}$$
$$\text{s. t. } o_1 \geq 1 \tag{11b}$$
$$(3), \tag{11c}$$

where (3) is stated with $o_2 = c_2 = 2$ and $b = u = 3$.

Bi-level problem (7) can be solved by solving the single-level problem:

$$\max_{o_2} \pi_2^p = (\lambda - 2)\cdot p_2 \tag{12a}$$
$$\text{s. t. } o_2 \geq 2 \tag{12b}$$
$$(3), \tag{12c}$$

where (3) is stated with $o_1 = c_1 = 1$ and $b = u = 3$.

Finally, bi-level problem (8) can be solved by solving the single-level problem:

$$\max_{b\geq 0} \pi_d = (3-\lambda)\cdot d \tag{13a}$$
$$\text{s. t. } b \leq 3 \tag{13b}$$
$$(3), \tag{13c}$$

where (3) is stated with $o_1 = c_1 = 1$ and $o_2 = c_2 = 2$.

IV. AGENTS' EQUILIBRIA

This section reviews equilibrium models in markets with strategic agents. An example is considered first, followed by a general formulation.

A. ALL AGENTS' VIEW

We consider again the market example involving two producers and one consumer (Fig. 4). We assume that every agent is strategic. Thus, we need to simultaneously solve the three bi-level problems below, which constitutes an equilibrium problem (set of interrelated optimization problems) of bi-level problems:

$$\max_{o_1} \pi_1^p = (\lambda - 1)\cdot p_1 \tag{14a}$$
$$\text{s. t. } o_1 \geq 1 \tag{14b}$$

$$\min_{p_1,p_2,d\geq 0} \quad z = o_1 \cdot p_1 + o_2 \cdot p_2 - b \cdot d \tag{14c}$$
$$\text{s. t.} \quad d - p_1 - p_2 = 0 : \lambda \tag{14d}$$
$$p_1 \leq 6, p_6 \leq 6, d \leq 10 \tag{14e}$$
$$\max_{o_2} \pi_2^p = (\lambda - 2) \cdot p_2 \tag{14f}$$
$$\text{s. t.} \quad o_2 \geq 2 \tag{14g}$$
$$\min_{p_1,p_2,d\geq 0} \quad z = o_1 \cdot p_1 + o_2 \cdot p_2 - b \cdot d \tag{14h}$$
$$\text{s. t.} \quad d - p_1 - p_2 = 0 : \lambda \tag{14i}$$
$$p_1 \leq 6, p_6 \leq 6, d \leq 10 \tag{14j}$$
$$\max_{b\geq 0} \pi^d = (3 - \lambda) \cdot d \tag{14k}$$
$$\text{s. t.} \quad b \leq 3 \tag{14l}$$
$$\min_{p_1,p_2,d\geq 0} \quad z = o_1 \cdot p_1 + o_2 \cdot p_2 - b \cdot d \tag{14m}$$
$$\text{s. t.} \quad d - p_1 - p_2 = 0 : \lambda \tag{14n}$$
$$p_1 \leq 6, p_6 \leq 6, d \leq 10. \tag{14o}$$

Alternatively, we can solve the three interrelated MPECs below:

$$\max_{o_1} \pi_1^p = (\lambda - 1) \cdot p_1 \tag{15a}$$
$$\text{s. t.} \quad o_1 \geq 1 \tag{15b}$$
$$(3) \tag{15c}$$
$$\max_{o_2} \pi_2^p = (\lambda - 2) \cdot p_2 \tag{15d}$$
$$\text{s. t.} \quad o_2 \geq 2 \tag{15e}$$
$$(3) \tag{15f}$$
$$\max_{u} \pi^d = (3 - \lambda) \cdot d \tag{15g}$$
$$\text{s. t.} \quad u \leq 3 \tag{15h}$$
$$(3) \tag{15i}$$

We note that the set of interrelated bi-level problems (14) and the set of MPECs (15) are equivalent.

It is particularly relevant to note that in case that no agent acts strategically, i.e., producer 1 offers at $o_1 = c_1 = 1$, producer 2 offers at $o_2 = c_2 = 2$, and the consumer bids at $b = u = 3$, EPEC (15) reduces to the simple optimization problem (1).

The general form of the equilibrium problem of bi-level problems (14) is:

$$\left.\begin{array}{l} \max_{\Xi^{(i)}} \ \pi^{(i)}\left(\boldsymbol{x}^{(i)}, \boldsymbol{\lambda}\right) \\ \text{s. t.} \ \ \boldsymbol{o}^{(i)} \in \mathcal{O}^{(i)} \\ \qquad \min_{\boldsymbol{x}} \ f(\boldsymbol{x}, \boldsymbol{o}) \\ \qquad \text{s. t.} \ \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0} : \boldsymbol{\lambda} \\ \qquad\qquad \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{o}) \leq \boldsymbol{0} : \boldsymbol{\mu} \end{array}\right\} \forall i, \tag{16}$$

and that of EPEC (15):

$$\left.\begin{array}{l} \max_{\Xi^{(i)}} \ \pi^{(i)}\left(\boldsymbol{x}^{(i)}, \boldsymbol{\lambda}\right) \\ \text{s. t.} \ \ \boldsymbol{o}^{(i)} \in \mathcal{O}^{(i)} \\ \qquad \nabla_{\boldsymbol{x}} f(\cdot) + \boldsymbol{\lambda}^\top \nabla_{\boldsymbol{x}} \boldsymbol{h}(\cdot) + \boldsymbol{\mu}^\top \nabla_{\boldsymbol{x}} \boldsymbol{g}(\cdot) = \boldsymbol{0} \\ \qquad \boldsymbol{h}(\cdot) = \boldsymbol{0} \\ \qquad \boldsymbol{0} \leq \boldsymbol{\mu} \perp \boldsymbol{g}(\cdot) \leq \boldsymbol{0} \end{array}\right\} \forall i. \tag{17}$$

In case that no agent offers/bids strategically, EPEC (17) reduces to optimization problem (4) [3].

Solving (17) is generally complex. We describe in Subsection IV-B below a solution strategy.

We note that in case that no agent acts strategically, i.e., each producer offers at its marginal cost, and each consumer bids at its marginal utility, EPEC (17) reduces to an optimization problem. Further details are provided in [3].

B. EPEC SOLUTION

The KKT optimality conditions of any of the MPECs constituting EPEC (17) (single-agent optimality conditions) can be easily obtained [3] and represented as:

$$\text{KKT}^{(i)}. \tag{18}$$

However, since any of the MPECs in EPEC (17) is generally non-convex and its constraints are generally non-regular [12], its optimality conditions as given by (18) identify points that might or might not be maxima. Thus, solution points need to be checked for optimality.

We then consider all set of conditions (18) (one per agent) and solve them jointly to search for equilibria. This constitutes a solution strategy for EPEC (17). It can be expressed as:

$$\left\{ \text{KKT}^{(i)} \quad \forall i \right., \tag{19}$$

and constitutes a system of nonlinear equalities and inequalities, generally difficult to solve.

The auxiliary problem below can be used to solve (19), i.e., to search for equilibria:

$$\max \quad \mathit{OF}(\cdot) \tag{20a}$$
$$\text{s. t.} \quad \text{KKT}^{(i)} \quad \forall i, \tag{20b}$$

where $\mathit{OF}(\cdot)$ is a suitable objective function.

Problem (20) is generally nonlinear and non-convex. Its solution can be attempted via linearization or using nonlinear solvers, such as IPOPT [15], BARON [16], or KNITRO [17].

It is important to note that since the constraints of any of the MPECs of EPEC (17) are generally non-regular, (19) identifies equilibria and other points [18]. Therefore, once potential equilibrium points (solutions of (20)) are found, a diagonalization algorithm [3] should be used to verify whether or not these points are indeed equilibria. Such algorithm can be seen as a modified version of the Gauss-Seidel method for solving systems of equations. It works as follows. Given a candidate solution for the EPEC, the algorithm verifies that no single player (producer or consumer) benefits by individually deviating from this solution. If this is the case, the resulting strategies (offers and bids) lead to a Nash equilibrium [2].

The EPECs formulated in this paper allow searching for pure-strategy equilibria. Such equilibria may or may not exist; however, our experience analyzing both operation and investment EPECs for electricity markets indicates that multiple pure-strategy equilibria do exist. On the other hand, we do not search for mixed-strategy equilibria because such equilibria have a very limited practical relevance in power markets [3].

TABLE 4. EPEC solution: offers, bid and price.

$OF(\cdot)$	o_1^*	o_2^*	b^*	λ^*	Nash?
π_1^p	1	3	3	3	NO
π_2^p	3	$3-\varepsilon$	$3-\varepsilon$	3	NO
π^d	1	2	1	1	YES
w	$2-\varepsilon$	2	2	2	YES

TABLE 5. EPEC solution: productions, demand, profits and social welfare.

$OF(\cdot)$	p_1^*	p_2^*	d^*	π_1^{p*}	π_2^{p*}	π^{d*}	w^*
π_1^p	6	4	10	12	4	0	16
π_2^p	0	6	6	0	6	0	6
π^d	6	0	6	0	0	12	12
w	6	4	10	6	0	10	16

C. EQUILIBRIUM EXAMPLE

Regarding the simple example considered throughout this paper, to solve (14), we solve the nonlinear system of equalities and inequalities [5]:

$$\begin{cases} \text{KKTs of problem (15a)} - \text{(15c)} \\ \text{KKTs of problem (15d)} - \text{(15f)} \\ \text{KKTs of problem (15g)} - \text{(15i)}, \end{cases} \tag{21}$$

by solving the auxiliary problem:

$$\max \quad OF(\cdot) \tag{22a}$$
$$\text{s. t.} \quad (21) \tag{22b}$$

KKTs (21) are provided in Appendix A.

Solving (22) results in the outcomes reported in Tables 4 and 5. The first column of these tables indicates the alternative objective functions $OF(\cdot)$ considered to identify candidate solutions. Then, the diagonalization algorithm described in Appendix B is used to verify whether or not these candidate solutions are Nash equilibria. This is reported in the last column of Table 4.

Four relevant candidate solutions are discussed below. We note that the fourth candidate solution (fifth row of Tables 4 and 5) corresponds to the competitive equilibrium.

1) FIRST CANDIDATE SOLUTION

The EPEC solution that entails maximum profit for producer 1 (max π_1^p), reported in the second row of Tables 4 and 5, requires that the offer price of producer 2 is $o_2 = 3$ and that the price bid of the consumer bid is $b = 3$ as well, which leads to a market clearing price of $\lambda = 3$. This renders a profit for producer 1 of $\pi_1^p = 12$, which is the maximum profit this producer can achieve. We note that the same market outcome is obtained if producer's 1 price offer is within the interval $1 \leq o_1 \leq 3-\varepsilon$. We note as well that this candidate solution is not a Nash equilibrium since the consumer has an incentive to lower its bid price b below 3. For instance, if the consumer bid price is $b = 1$, the market price decreases and the consumer's profit increases.

2) SECOND CANDIDATE SOLUTION

Analogously, the EPEC solution that achieves maximum profit for producer 2 (max π_2^p), reported in the third row of Tables 4 and 5, is not a Nash equilibrium. In this case, the offer price of producer 1 is $o_1 = 3$, while that of producer 2 is slightly below 3 ($o_2 = 3-\varepsilon$) to ensure being fully dispatched at the clearing price $\lambda = 3$. This solution is not a Nash equilibrium since producer 1, which is more competitive than producer 2 (its production cost is lower), is willing to reduce its offer price o_1 to force being dispatched.

3) THIRD CANDIDATE SOLUTION

On the other hand, the EPEC solution that achieves maximum profit for the consumer (max π^d), reported in the fourth row of Tables 4 and 5, is a Nash equilibrium. In this case, the offer price of producer 1 is $o_1 = 1$ and the bid price of the consumer $b = 1$, leading to a market clearing price $\lambda = 1$, and a consumer's profit $\pi^d = 12$. Producer 1 is fully dispatched at this clearing price, but since this price is equal to its marginal cost, its profit is 0. Producer 2, whose production cost is $c_2 = 2$, cannot lower its offer price o_2 below this value and thus, it is not dispatched. We note that for these strategies, there is no incentive for any of the two producers to raise its offer, as such action will not increase the corresponding profit. Similarly, the consumer is not willing to further decrease its bid b below producer's 1 marginal cost $c_1 = 1$, as such action would entail no energy being supplied ($d = p_1 = p_2 = 0$).

This equilibrium is illustrated in Fig. 5 (top graphic).

4) FOURTH CANDIDATE SOLUTION

A second Nash equilibrium occurs for the EPEC solution that maximizes social welfare $OF(\cdot) = w$. This equilibrium is reported in the fifth row of Tables 4 and 5. In this case, the market clearing price is $\lambda = 2$, since the price offer of producer 2 is $o_2 = 2$ and the price bid of the consumer is $b = 2$. Both producers are dispatched to satisfy the maximum demand of the consumer $d = 10$. We note that producer 1 slightly lowers its price offer below producer's 2 marginal cost, i.e., $o_1 = 2-\varepsilon$, to ensure being fully dispatched. The above strategies result in a social welfare $w = 16$. We note that neither of the two producers nor the consumer has an incentive to modify its offer or bid. Producers 1 and 2 are not willing to further raise their offer prices as such move would entail not being dispatched. Similarly, the consumer is not willing to decrease its bid b below 2 as this would entail no energy being supplied ($d = p_1 = p_2 = 0$).

This equilibrium is illustrated in Fig. 5 (center graphic). We note that in terms of clearing price, production levels and demand served, this equilibrium is unique. However, in terms of offer/bid prices, there are infinitely many *identical* equilibria, since for $o_1 \in [1, 2)$ identical values are obtained for clearing price, production levels and demand served.

For the sake of comparison, the competitive equilibrium (producers/consumers offer/bid their respective marginal costs/utilities) is also illustrated in Fig. 5 (bottom graphic).

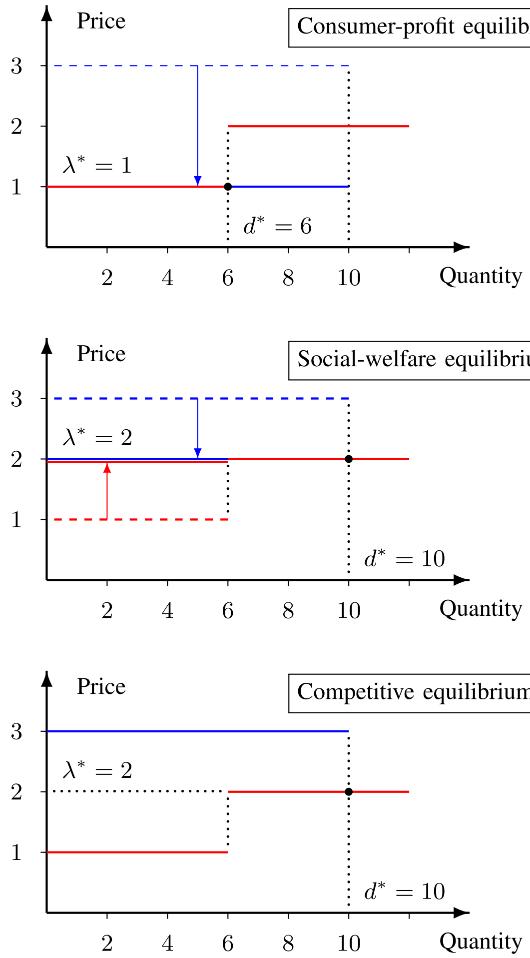

FIGURE 5. Example equilibria.

Using a game-theoretical perspective (not a complementarity one), we may recast the search for equilibria in this example as follows. Each of the two producers should select its offer price between its marginal cost (assuming economic rationality, offering below marginal cost is not considered) and the marginal utility of the consumer. The consumer should select its bid price between its marginal utility and the lowest marginal cost of the producers (to ensure a non-zero non-trivial solution). Note that offer/bid prices can be changed, but not offer/bid quantities. The search for equilibria seeks to answer the following question: Which are the sets of offer/bid prices that lead to Nash equilibria?

Finally, we note that a game-theoretical approach (not a complementarity one) is not generally insightful in power markets due to the curse of dimensionality.

V. RESEARCH NEEDS

Pressing research needs include:

1) Since a number of real-world markets use non convex problems for market clearing, appropriate convexification techniques (such as convex relaxations [19]) are needed to represent such problems within equilibrium models.
2) Since binary variables might be needed to represent specific market constraints, developing necessary and or sufficient optimality conditions for problems with binary variables is most needed [20].
3) An equilibrium model for a realistic market typically results in a large-scale problem. Since such problem often embodies a decomposable structure, tailored decomposition techniques to address such large-scale equilibrium problems are needed [21].
4) The increasing interdependency between the power and natural-gas markets, which are independently operated, needs to be represented in equilibrium models for power markets [22].
5) Finally, we note that the proposed framework can be used to analyze a distribution market that includes a number of proactive prosumers and is managed by a distribution system operator. Research is most needed in this area as well [23], [24].

APPENDIX

A. KARUSH–KUHN–TUCKER OPTIMALITY CONDITIONS

For the sake of generality, we extend the market setting described in the body of the paper (two producers and one consumer) to consider $|I|$ producers and $|J|$ consumers, where I and J are the sets of producers and consumers, respectively. Additionally, i and k are equivalent indices for producers, i.e., $i, k \in I$, and j and l, equivalent indices for consumers, i.e., $j, l \in J$.

Producers MPECs (11) and (12) are generalized as:

$$\max_{\Xi^p_i} \; (\lambda - c_i)p_i \tag{23a}$$

s. t.

$$o_i \ge c_i : \; \eta^p_i \tag{23b}$$

$$\sum_{k\in I} p_k - \sum_{j\in J} d_j = 0 : \; \delta^p_i \tag{23c}$$

$$0 \le p_k \le \hat{p}_k : \; \check{\beta}^{pp}_{ik}, \hat{\beta}^{pp}_{ik}, \quad \forall k \in I \tag{23d}$$

$$0 \le d_j \le \hat{d}_j : \; \check{\beta}^{pd}_{ij}, \hat{\beta}^{pd}_{ij}, \quad \forall j \in J \tag{23e}$$

$$\mu^p_k + \lambda \le o_k : \; \alpha^{pp}_{ik}, \quad \forall k \in I \tag{23f}$$

$$\mu^d_j - \lambda \le -b_j : \; \alpha^{pd}_{ij}, \quad \forall j \in J \tag{23g}$$

$$\mu^p_k \le 0 : \; \theta^{pp}_{ik}, \quad \forall k \in I \tag{23h}$$

$$\mu^d_j \le 0 : \; \theta^{pd}_{ij}, \quad \forall j \in J \tag{23i}$$

$$\sum_{k\in I} o_k p_k - \sum_{j\in J} b_j d_j - \sum_{k\in I} \hat{p}_k \mu^p_k - \sum_{j\in J} \hat{d}_j \mu^d_j = 0 : \; \gamma^p_i \tag{23j}$$

where $\Xi^p_i = \{o_i, p_k, d_j, \lambda, \mu^p_k, \mu^d_j\}\ \forall k \in I$ and $\forall j \in J$ is the set of producer's i decision variables. Parameters $\hat{p}_i$ and $\hat{d}_j$ represent producer's i capacity and consumer's j maximum demand, respectively. Dual variables are indicated at their corresponding constraints following a colon.

Similarly, MPEC (13) is generalized as:

$$\max_{\Xi_j^d} (u_j - \lambda)d_j \quad (24a)$$

s. t.

$$b_j \le u_j : \ \eta_j^d \quad (24b)$$

$$\sum_{i\in I} p_i - \sum_{l\in J} d_l = 0 : \ \delta_j^d \quad (24c)$$

$$0 \le p_i \le \hat{p}_i : \ \check{\beta}_{ji}^{dp}, \hat{\beta}_{ji}^{dp}, \quad \forall i \in I \quad (24d)$$

$$0 \le d_l \le \hat{d}_l : \ \check{\beta}_{jl}^{dd}, \hat{\beta}_{jl}^{dd}, \quad \forall l \in J \quad (24e)$$

$$\mu_i^p + \lambda \le o_i : \ \alpha_{ji}^{dp}, \quad \forall i \in I \quad (24f)$$

$$\mu_l^d - \lambda \le -b_l : \ \alpha_{jl}^{dd}, \quad \forall l \in J \quad (24g)$$

$$\mu_i^p \le 0 : \ \theta_{ji}^{dp}, \ \forall i \in I \quad (24h)$$

$$\mu_l^d \le 0 : \ \theta_{jl}^{dd}, \ \forall l \in J \quad (24i)$$

$$\sum_{i\in I} o_i p_i - \sum_{l\in J} b_l d_l - \sum_{i\in I} \hat{p}_i \mu_i^p - \sum_{l\in J} \hat{d}_l \mu_l^d = 0 : \ \gamma_j^d \quad (24j)$$

where $\Xi_j^d = \{b_j, p_i, d_l, \lambda, \mu_i^p, \mu_l^d\}$ $\forall l \in J$ and $\forall i \in I$, is the set of consumer's j decision variables.

The KKT conditions of producer's i MPEC (23) are:

$$\frac{\partial \mathcal{L}_i}{\partial o_i} = -\eta_i^p - \alpha_{ii}^{pp} + p_i \gamma_i^p = 0 \quad (25a)$$

$$\frac{\partial \mathcal{L}_i}{\partial p_i} = c_i - \lambda + \delta_i^p - \check{\beta}_{ii}^{pp} + \hat{\beta}_{ii}^{pp} + \gamma_i^p o_i = 0 \quad (25b)$$

$$\frac{\partial \mathcal{L}_i}{\partial p_k} = \delta_i^p - \check{\beta}_{ik}^{pp} + \hat{\beta}_{ik}^{pp} + \gamma_i^p o_k = 0 \quad \forall (k \ne i) \in I \quad (25c)$$

$$\frac{\partial \mathcal{L}_i}{\partial d_j} = -\delta_i^p - \check{\beta}_{ij}^{pd} + \hat{\beta}_{ij}^{pd} - \gamma_i^p b_j = 0 \quad \forall j \in J \quad (25d)$$

$$\frac{\partial \mathcal{L}_i}{\partial \lambda} = -p_i + \sum_{k\in I} \alpha_{ik}^{pp} - \sum_{j\in J} \alpha_{ij}^{pd} = 0 \quad (25e)$$

$$\frac{\partial \mathcal{L}_i}{\partial \mu_k^p} = \alpha_{ik}^{pp} + \theta_{ik}^{pp} - \hat{p}_k \gamma_i^p = 0 \quad \forall k \in I \quad (25f)$$

$$\frac{\partial \mathcal{L}_i}{\partial \mu_j^d} = \alpha_{ij}^{pd} + \theta_{ij}^{pd} - \hat{d}_j \gamma_i^p = 0 \quad \forall j \in J \quad (25g)$$

$$0 \le o_i - c_i \perp \eta_i^p \ge 0 \quad (25h)$$

$$0 \le p_k \perp \check{\beta}_{ik}^{pp} \ge 0 \quad \forall k \in I \quad (25i)$$

$$0 \le \hat{p}_k - p_k \perp \hat{\beta}_{ik}^{pp} \ge 0 \quad \forall k \in I \quad (25j)$$

$$0 \le d_j \perp \check{\beta}_{ij}^{pd} \ge 0 \quad \forall j \in J \quad (25k)$$

$$0 \le \hat{d}_j - d_j \perp \hat{\beta}_{ij}^{pd} \ge 0 \quad \forall j \in J \quad (25l)$$

$$0 \le o_k - \mu_k^p - \lambda \perp \alpha_{ik}^{pp} \ge 0, \quad \forall k \in I \quad (25m)$$

$$0 \le \lambda - \mu_j^d - b_j \perp \alpha_{ij}^{pd} \ge 0, \quad \forall j \in J \quad (25n)$$

$$0 \ge \mu_k^p \perp \theta_{ik}^{pp} \ge 0 \quad \forall k \in I \quad (25o)$$

$$0 \ge \mu_j^d \perp \theta_{ij}^{pd} \ge 0 \quad \forall j \in J \quad (25p)$$

$$\sum_{k\in I} p_k - \sum_{j\in J} d_j = 0 \quad (25q)$$

$$\sum_{k\in I} o_k p_k - \sum_{j\in J} b_j d_j - \sum_{k\in I} \hat{p}_k \mu_k^p - \sum_{j\in J} \hat{d}_j \mu_j^d = 0 \quad (25r)$$

where $\mathcal{L}_i$ is the Lagrangian function of problem (23).

The Karush–Kuhn–Tucker conditions of consumer's j MPEC (24) are:

$$\frac{\partial \mathcal{L}_j}{\partial b_j} = \eta_j^d + \alpha_{jj}^{dd} - d_j \gamma_j^d = 0 \quad (26a)$$

$$\frac{\partial \mathcal{L}_j}{\partial p_i} = \delta_j^d - \check{\beta}_{ji}^{dp} + \hat{\beta}_{ji}^{dp} + \gamma_j^d o_i = 0 \quad \forall i \in I \quad (26b)$$

$$\frac{\partial \mathcal{L}_j}{\partial d_j} = \lambda - u_j - \delta_j^d - \check{\beta}_{jj}^{dd} + \hat{\beta}_{jj}^{dd} - \gamma_j^d b_j = 0 \quad (26c)$$

$$\frac{\partial \mathcal{L}_j}{\partial d_l} = -\delta_j^d - \check{\beta}_{jl}^{dd} + \hat{\beta}_{jl}^{dd} - \gamma_j^d b_l = 0 \quad \forall (l \ne j) \in J \quad (26d)$$

$$\frac{\partial \mathcal{L}_j}{\partial \lambda} = d_j + \sum_{i\in I} \alpha_{ji}^{dp} - \sum_{l\in J} \alpha_{jl}^{dd} = 0 \quad (26e)$$

$$\frac{\partial \mathcal{L}_i}{\partial \mu_i^p} = \alpha_{ji}^{dp} + \theta_{ji}^{dp} - \hat{p}_i \gamma_j^d = 0 \quad \forall i \in I \quad (26f)$$

$$\frac{\partial \mathcal{L}_j}{\partial \mu_l^d} = \alpha_{jl}^{dd} + \theta_{jl}^{dd} - \hat{d}_l \gamma_j^d = 0 \quad \forall l \in J \quad (26g)$$

$$0 \le u_j - b_j \perp \eta_j^d \ge 0 \quad (26h)$$

$$0 \le p_i \perp \check{\beta}_{ji}^{dp} \ge 0 \quad \forall i \in I \quad (26i)$$

$$0 \le \hat{p}_i - p_i \perp \hat{\beta}_{ji}^{dp} \ge 0 \quad \forall i \in I \quad (26j)$$

$$0 \le d_l \perp \check{\beta}_{jl}^{dd} \ge 0 \quad \forall j \in J \quad (26k)$$

$$0 \le \hat{d}_l - d_l \perp \hat{\beta}_{jl}^{dd} \ge 0 \quad \forall l \in J \quad (26l)$$

$$0 \le o_i - \mu_i^p - \lambda \perp \alpha_{ji}^{dp} \ge 0, \quad \forall i \in I \quad (26m)$$

$$0 \le \lambda - \mu_l^d - b_l \perp \alpha_{jl}^{dd} \ge 0, \quad \forall l \in J \quad (26n)$$

$$0 \ge \mu_i^p \perp \theta_{ji}^{dp} \ge 0 \quad \forall i \in I \quad (26o)$$

$$0 \ge \mu_l^d \perp \theta_{ji}^{dd} \ge 0 \quad \forall l \in J \quad (26p)$$

$$\sum_{i\in I} p_i - \sum_{l\in J} d_l = 0 \quad (26q)$$

$$\sum_{i\in I} o_i p_i - \sum_{l\in J} b_l d_l - \sum_{i\in I} \hat{p}_i \mu_i^p - \sum_{l\in J} \hat{d}_l \mu_l^d = 0 \quad (26r)$$

where $\mathcal{L}_j$ represent the Lagrangian function of problem (24).

B. DIAGONALIZATION CHECKING

We consider vector $\mathbf{y} \in \mathbb{R}^{|I|+|J|}$ that includes both strategic offers and bids, i.e., o_i $\forall i \in I$ and b_j $\forall j \in J$, respectively. Specifically, y_m represents the strategic offer or bid of player $m \in M \equiv (I \cup J)$.

Then, the diagonalization algorithm to verify whether or not an optimal solution of problem (22), $\mathbf{y}^s = \{y_1^s, \ldots, y_m^s, \ldots, y_{|M|}^s\}$, is a Nash equilibrium works as follows:

1) Set initial values for the players' offers/bids as $y_m^0 = y_m^s$ $\forall m \in M$. Set $y_m^1 = y_m^0$ $\forall m \in M$.
2) Repeat for $m = 1, \ldots, |M|$ (the order is immaterial):
   a) Solve MPEC (23) if player m is a producer or MPEC (24) if it is a consumer to obtain the optimal solution y_m^*. The values of other players' offers/bids $y_{m'}^1$ $\forall (m' \ne m) \in M$ are kept fixed.
   b) Set y_m^1 equal to y_m^*.

3) If $|y_m^1 - y_m^0| \leq \varepsilon \; \forall m \in M$, the solution $\mathbf{y}^s = \{y_1^s, \ldots, y_m^s, \ldots, y_{|M|}^s\}$ is a Nash equilibrium, else, it is not.

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