DOI: 10.1111/joes.12522

JOURNAL OF ECONOMIC SURVEYS WILEY

ARTICLE

Direct versus iterated multiperiod Value-at-Risk forecasts

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Funding information
Ministerio de Economía y Competitividad, Grant/Award Number: PDI2019-108079GB-C21/AIE/10.13039/501100011033

Abstract

Since the late nineties, the Basel Accords require financial institutions to measure their financial risk by reporting daily predictions of Value at Risk (VaR) based on 10-day returns. However, a vast part of the related literature deals with VaR predictions based on one-period returns. Given its relevance for practitioners, in this paper, we survey the literature on available procedures to estimate VaR over an h-period. First, to convert 1 day into 10-day VaR, it is popular to use the square-root-of-time (SRoT) rule, which is only satisfied under very restrictive and unrealistic properties of returns. Alternatively, direct (based on h-period returns) and iterated (based on one-period returns) two-step procedures can be implemented to obtain 10-period VaR. We also illustrate and compare the performance of these procedures in the context of popular conditionally heteroscedastic models for returns using both simulated and real data. We show that, under realistic assumptions on the distribution of returns, multiperiod VaR predictions based on iterating an asymmetric GJR model with normal or bootstrapped errors are usually preferred. We also show that, in general, direct methods could be not only biased but also inefficient.

KEYWORDS
conditional heteroscedasticity, conditional quantiles, direct forecasts, iterated forecasts, multiperiod forecasts, risk

Financial risk management has generally focused on short-term risks rather than long-term risks, and arguably this was an important component of the recent financial crisis. *Engle* (2011)

1 INTRODUCTION

Since proposed by the Basel Committee on Banking Supervision (BCBS) in 1996 and, in spite of its many limitations, Value at Risk (VaR) is still extensively used by financial institutions to measure the risk of their portfolios; see Basel Committee on Banking Supervision (1996a). Recently, in response to the subprime mortgage global crisis, the BCBS has overhauled the regulatory framework governing the minimum capital requirements for banks; see Liu and Stentoft (2021) for a recent and very detailed description of how the regulatory environment of the Basel accords has changed over time and the requirements and objectives of the current regulation. In 2019, the BCBS finalized the Fundamental Review of the Trading Book (FRTB), which is popularly known as Basel 3; see Basel Committee on Banking Supervision (2019). Although Basel 3 requires measuring risk using expected shortfall (ES) instead of VaR, the latter is still of interest for both academics and risk managers as it is used for regulatory backtesting and for setting the capital multiplier for capital requirements. Furthermore, VaR is also involved in the calculation of ES. Finally, it is important to remark that VaR has also been established in 2009 by the Solvency 2 regulation as the risk measure to be considered by insurance companies operating in the European Union; see Solvency (2009) and Frankland et al. (2019).

The VaR of a portfolio held over a period of h days is defined by the (negative) δ-quantile of the conditional distribution of h-day portfolio's returns given by

$$VaR_t^{(h)} = -\sup\left[r \mid \Pr\left(R_t^{(h)} \leq r\right) \leq \delta\right], \tag{1}$$

where $R_t^{(h)} = 100 \times (lnP_{t+h} - lnP_t)$ are h-period log-returns with P_t being the price of the portfolio at day $t = 1, \dots, T$, and $ln(P_t)$ being its natural logarithm. Note that h-day log-returns can be written as follows:

$$R_t^{(h)} = \sum_{i=1}^{h} r_{t+i}. \tag{2}$$

where $r_t \equiv R_{t-1}^{(1)} = 100 \times (lnP_t - lnP_{t-1})$ are 1-day log-returns, assumed to be a strictly stationary martingale difference. Furthermore, the multiperiod VaR can be defined as follows:

$$VaR_t^{(h)} = -\sigma_t^{(h)} q_\delta^{(h)} \tag{3}$$

where $\sigma_t^{(h)}$ is the standard deviation of $R_t^{(h)}$ conditional on the information available at time t and $q_\delta^{(h)}$ is the δ quantile of the distribution of multiperiod standardized returns, $e_t^{(h)} = \frac{R_t^{(h)}}{\sigma_t^{(h)}}$.

With respect to the related literature, we can observe that, on the one hand, there is a prolific literature devoted to VaR forecasts based on 1-day returns, r_t; see, for example, the early survey by Duffie and Pan (1997) and the more recent one by Nieto and Ruiz (2016). However, the regulatory authorities have emphasized the importance of risk horizon; see Basel Committee on Banking Supervision (2013). Furthermore, Engle (2009) points out that the key failure in forecasting VaR lies on its deterioration in forecasting multiperiod VaR. Therefore, conclusions about one-period VaRs cannot be directly extrapolated to multiperiod VaRs.

When looking at the literature on multiperiod forecasts, there are two main alternative methodologies, namely, direct and iterated forecasts with a vast literature comparing them in the context of univariate linear models. It is well known that iterated forecasts are more efficient under a correct specification of the one-step-ahead forecast model while direct forecasts are unbiased and robust against misspecification; see, for example, Pesaran et al. (2011). In general, most works within this literature favor direct forecasts; see Chevillon and Hendry (2005) and Tsay (1993), among many others. However, the advantages of the direct over iterated approaches under misspecification are not universal and depend on data characteristics and the degree of misspecification of the one-period models; see, for instance, Kang (2003), McElroy and Wildi (2013), and Proietti (2011), and the excellent overview of the literature by Chevillon (2007). Among the authors that conclude that iterated approaches outperform direct ones, Marcellino et al. (2006) study a large number of macroeconomic time series and find that iterated forecasts typically have smaller mean square forecast errors (MSFEs) than direct ones, with the relative performance of the former improving with the forecast horizon; see also Baek (2019) and Quaedvlieg (2021) for the same conclusion.[1] They attribute this better performance of iterated forecasts to the fact that the models for the higher observation frequency usually include more lags. In the context of linear multivariate VAR models, McCracken and McGillicuddy (2019) only find an empirical marked improvement of the direct approach during the Great Moderation. In general, the improvements in terms of MSFE of iterated versus direct forecasts tend to be modest.

However, in spite of the practical importance for risk managers of obtaining accurate $VaR_t^{(h)}$ forecasts, very few authors compare direct and iterated approaches in this context. As far as we know only Mancini and Trojani (2011), De Nicolo and Lucchetta (2017) and Kole et al. (2017) have aimed to this comparison. They all advocate the use of iterated forecasts when computing multiperiod VaR.

In this paper, we survey the literature on the estimation of multiperiod VaR estimation. From a methodological point of view, it is important to analyze, in the context of multiperiod VaR estimation, the trade-off between unbiasedness and efficiency of direct and iterated approaches. Moreover, the interest in a correct estimation of $VaR_t^{(h)}$ is not only academic but it also has important implications for risk managers and regulators. As mentioned above, capital requirements are tightly linked to VaR estimates with Basel 3 penalizing banks by increasing their capital requirements when their model generates too many VaR exceedances; see, for example, Jiménez-Martín et al. (2007) for capital requirements under Basel 2. Therefore, underestimating risk could lead to excessively high capital requirements with the corresponding associated opportunity costs that may affect the profitability of the institution; see, for example, McAleer (2009), Pérignon et al. (2008), and Pérignon and Smith (2010). In consequence, banks have incentives to use conservative models for VaR estimation. Although lower capital requirements may be privately optimal for banks, higher capital requirements may be socially beneficial in reducing the likelihood of system risk; see McAleer (2009) for the private and public benefits of risk management and Birn et al. (2020) for a review on the costs and benefits of bank capital requirements. However, overestimation of risk also has undesirable consequences. The effect of the exaggeration of the own level of risk is that financial institutions appear more risky than they actually are, thus generating reputation concerns about their risk management systems. This affects the perception of investors and can induce underinvestment in VaR-overstating institutions. Indeed, Jorion (2002) shows that VaR disclosures are informative about the future variability in trading revenues, thus corroborating the idea that analysts/investors may be using VaR forecasts to support investment decisions.

We not only survey the most important measures of multiperiod VaR, but also illustrate their performance by comparing them in the context of simulated data generated by realistic models for 1-day returns; we complement the simulation results in Mancini and Trojani (2011) by considering a larger set of designs to generated returns and a larger number of alternative direct and iterated procedures to estimate $VaR_t^{(h)}$. In our simulations, we take into account misspecification of the model for 1-day returns, which is central to this comparison. In particular, we generate daily returns by symmetric and asymmetric conditionally heteroscedastic models different to those fitted to estimate the conditional variance used for the iterations. Furthermore, we consider not only symmetric but also asymmetric distributions for standardized returns as those often encountered in the analysis of real data. The analysis is carried out by considering different sample sizes so that we can analyze whether the drawbacks of the direct approach could be attributable to the lack of enough observations for the estimation of the relevant parameters need to obtain the multiperiod VaRs. Our conclusions are in concordance with the related literature. We show that, in the context of volatilities with leverage effect, it is important to consider this asymmetry when modeling the one-period returns, with the advantage of iterated over direct approaches being strong. Finally, the direct and iterated procedures for forecasting $VaR_t^{(h)}$ are compared based on real returns. In particular, we analyze S&P500, Dollar/Euro exchange rates, and IBM returns. These three variables have been chosen because of their different properties in terms of the asymmetric response of volatilities to positive and negative past returns and of the asymmetry properties of the conditional distribution of 1-day returns. We show that multiperiod VaR estimates based on iterating the asymmetric GJR model for conditional volatilities with the quantile estimated by simulation are unbiased and have smaller RMSFEs than those obtained using the direct approaches. Our results are in concordance with those by Mancini and Trojani (2011), De Nicolo and Lucchetta (2017), and Kole et al. (2017) supporting the use of iterated estimates of $VaR_t^{(h)}$. Therefore, for risk managers, it is always safer to obtain multistep VaR forecasts using iterated procedures simulating the asymmetric model for volatilities with either normal or bootstrapped errors. According to our results, the direct approach should not be used in the presence of asymmetric volatilities.

The rest of the paper is organized as follows. In Section 2, we briefly describe the popular square-root-of-time (SRoT) approach and the early solutions proposed to estimate multiperiod

VaR based on the distribution of multiperiod returns. Two-step methods, either direct or iterated, are described in Section 3. In Section 4, we survey the literature comparing empirically these methods. Section 5 compares the finite sample properties of these alternative multiperiod VaR estimators using simulated data. In Section 6, multiperiod VaRs of S&P500, Dollar/Euro exchange rates, and IBM 10-day returns are obtained and compared using backtesting procedures. Finally, Section 7 concludes.

2 | SQUARE ROOT OF TIME AND THE MULTIPERIOD DISTRIBUTION OF RETURNS

The most popular procedure to compute multiperiod VaR is the SRoT, which is based on scaling the conditional variance as proposed by Basel Committee on Banking Supervision (1996b). The SRoT is not only popular between academics, who often use it as a benchmark, but it is also the conventional solution used in industry practice. According to the SRoT, $VaR_t^{(h)}$ is based on $VaR_t^{(1)}$, the one-period VaR, as follows:

$$VaR_t^{(h)} = \sqrt{h} VaR_t^{(1)}. \tag{4}$$

In spite of its popularity, the SRoT has important limitations as it is only satisfied if the conditional distribution of returns is normal and either conditional variances are constant or they follow the RiskMetrics model; see, among others, Brummelhuis and Kaufmann (2007), Danielsson and Zigrand (2006), Diebold et al. (1998), Engle (2004), Tsay (2010), and Wang et al. (2011). To see this point, consider the following model for one-period returns as assumed by Riskmetrics

$$r_t = \varepsilon_t \sigma_t \tag{5}$$

$$\sigma_t^2 = \alpha \sigma_{t-1}^2 + (1 - \alpha r_{t-1}^2), \tag{6}$$

where ε_t, the standardized one-period return at time $t = 1, \dots, T$, is assumed to be an independent and identically distributed sequence with zero mean and variance one, and α is a parameter satisfying $0 \leq \alpha \leq 1$. Note that, if $\alpha = 1$, then the conditional variance of one-period returns is constant over time. In empirical applications with daily returns, α is often estimated as being between 0.9 and 1. Given Equation (5), it is straightforward to see that $E(r_{t+h}|r_1, \dots, r_t) = 0$. Furthermore, $\sigma_{t+1}^2 = E(r_{t+1}^2|r_1, \dots, r_t)$. The conditional variance in Equation (6) can be alternatively written as

$$\sigma_t^2 = \sigma_{t-1}^2 + (1 - \alpha)(\varepsilon_{t-1}^2 - 1)\sigma_{t-1}^2, \tag{7}$$

where, taking into account that ε_t is an independent sequence, $E((1 - \alpha)(\varepsilon_{t-1}^2 - 1)\sigma_{t-1}^2|r_1, \dots, t_t) = 0$. Furthermore, the following expression of the variance of r_{t+h} conditional on $(r_1), \dots, r_t$ can be obtained from (7)

$$\begin{aligned} E(r_{t+h}^2|r_1, \dots, r_t) &= E(\sigma_{t+h}^2|r_1, \dots, r_t) = E(\sigma_{t-h-1}^2 + (1 - \alpha)(\varepsilon_{t-h-1}^2 - 1)\sigma_{t-h-1}^2|r_1, \dots, r_t) \\ &= E(\sigma_{t-h-1}^2|r_1, \dots, r_t) \end{aligned} \tag{8}$$

Therefore, when the conditional variances of one-period returns are given by the Riskmetrics model in Equations (5) and (6), then the variance of r_{t+h} conditional on the information available at time t is σ_{t+1}. Consequently

$$\sigma_t^{(h)2} = Var(R_t^{(h)}|r_1, \dots, r_t) = Var(\sum_{i=1}^{h} r_{t+i}|r_1, \dots, r_t) = \sum_{i=1}^{h} Var(r_{t+i}|r_1, \dots, r_t) = h\sigma_{t+1}^2. \quad (9)$$

Finally, putting together Equations (3) and (9), the following expression is obtained

$$VaR_t^{(h)} = -\sigma_t^{(h)} q_\delta^{(h)} = -h\sigma_{t+1}^2 q_\delta^{(h)}. \quad (10)$$

Only if $q_\delta^{(h)} = q_\delta^{(1)}$, the SRoT in Equation (4) will be satisfied. Note that this last condition requires assuming conditional normallity of one-period returns.

Obviously, the empirical success of the SRoT depends on both the properties of the returns and the methodology used to estimate their conditional variances and obtain the corresponding one-period VaRs. As far as we know, very few works support using the SRoT. Among them, one can find Beltratti and Morana (1999), who analyze Deutsche Mark-US Dollar exchange rates and fit GARCH and FIGARCH models. They show that the exceptions of 10-day VaR estimates are approximately correct. Singleton (2006) concludes that scaling volatilities in the context of a GARCH model leads to errors, which are small relative to the error tolerance of most risk managers. Kole et al. (2017) also conclude empirically that the RiskMetrics approach based on scaling forecasts of the volatilities constructed from daily returns in combination with the empirical distribution of standardized returns is not bad. However, there is ample evidence about the use of SRoT generating $VaR_t^{(10)}$ forecasts with large negative biases mainly for $\delta = 0.01$; see Colacito and Engle (2010), Danielsson and Zigrand (2006), Lönnbark (2016), McNeil and Frey (2000), Saadi and Rahman (2008), Wang et al. (2011), Wang et al. (2018), Wong and So (2003), and Zhou et al. (2016) for negative biases. On the contrary, few authors have also found positive biases of SRoT generating over-conservative multiperiod VaR estimates; see, for example, Müller et al. (1990), Jorion (2001), and Wong (2020). Wang et al. (2011) carries out an analysis to disentangle the sensitivity of SRoT to different characteristics of financial returns in the context of obtaining one-period VaR using historical simulation (HS). Recently, Wang et al. (2018) propose a modified SRoT that reduces bias.

It is important to note that negative biases of VaR estimates have important practical implications for financial institutions. Basel 3 penalizes banks by increasing their capital requirements when their models generate too many VaR exceedances, incentivating them to use more conservative models for VaR estimation. As mentioned above, although higher capital requirements may be socially beneficial in reducing the likelihood of system risk, lower capital requirements may be privately optimal for banks. Consequently, using the SRoT does not seems to be a good strategy for financial institutions to calculate multiperiod VaR and, as a result, Basel 3 explicitly disallows scaling procedures to estimate multiperiod risk measures. Furthermore, given the negative effects of VaR underestimation, the US Federal Reserve suggested using alternative procedures to estimate the multiperiod VaR, with several of them been consequently proposed in the related literature; see Laubsch and Ulmer (1999).

Instead of using the SRoT, one can directly obtain the distribution of multiperiod returns with the multiperiod VaR defined as the desired quantile of this distribution. Consequently, some authors propose computing $VaR_t^{(h)}$ by approximating the density of $R_t^{(h)}$ using analytical methods based on higher moments. Among these analytical approximations, one can use, for instance, the Gram–Charlier or the Cornish–Fisher expansions; see, for example, Alexander et al. (2013) and Boudt et al. (2009). However, Lönnbark (2016) shows that the Gram–Charlier approximation can have large positive biases when forecasting $VaR_t^{(10)}$ for $\delta = 0.01$ while Simonato (2011) argues that, for some skewness and kurtosis combinations, these approaches may generate densities with negative values or nonmonotone quantile functions and, consequently, they may fail to generate valid VaR estimates. Furthermore, Simonato (2011) carries out Monte Carlo experiments to show that the Johnson approximation has better properties estimating the unconditional VaR; see also Alexander et al. (2013) for the same conclusion in an empirical application.

Finally, instead of obtaining the distribution of multiperiod returns, Liu and Stentoft (2021) propose estimating $VaR_t^{(h)}$ by HS, computing the desired empirical quantile of overlapping returns. They show that, although estimates of multiperiod VaR's obtained by HS are rejected using traditional backtesting measures, they require lower capital requirements than alternative direct and iterated two-step procedures. The HS estimates, which minimize average capital requirements, appear to be misspecified and produce inferior forecasts of the regulatory risk measures.

3 | TWO-STEP PROCEDURES

Triggered by the good performance of two-step methods in the context of estimating one-period VaR, many authors propose using these methods for estimating $VaR_t^{(h)}$. Two-step methods are based on, first, modeling the conditional variance of returns using a GARCH-type model and, second, computing the quantile of standardized returns. There are two main approaches to estimate the multiperiod $VaR_t^{(h)}$ defined as in Equation (3). On the one hand, one can use the direct approach, which specifies directly a model for $\sigma_t^{(h)}$ with the information available at time t being $R_{t-h}^{(h)}, R_{t-2h}^{(h)}, \ldots$ On the other hand, $VaR_t^{(h)}$ can be estimated using the more popular iterated approaches, which are based on estimating $\sigma_t^{(h)}$ by iterating the model specified for one-period returns, r_t. Iterated approaches are efficient if this latter model is correctly specified while the direct approach has the advantage of being robust against its misspecification. Furthermore, it is important to remark that the temporal-aggregation formulae needed for some iterated approaches are only available for some restrictive classes of models; see Christoffersen and Diebold (2000). Finally, note that, regardless of whether a direct or iterated approach is used, $\sigma_t^{2(h)}$ can be obtained by modeling the portfolio at the asset level (based on a multivariate conditionally heteroscedastic model) or by a complete portfolio aggregation (using a univariate model for the portfolio returns). Santos et al. (2013) suggest that one-period VaR forecasts based on modeling the portfolio returns using multivariate models have better properties that those based on univariate models while Kole et al. (2017) support this conclusion in the context of multiperiod VaRs. However, as far as we are concerned, there are not further studies on cross-sectional aggregation when estimating multiperiod VaR and, consequently, the rest of this survey deals with univariate models.

3.1 | Direct procedures

The direct approach to estimate multiperiod VaR has been advocated by, for example, Diebold et al. (1998) and Hoga (2019), with the former authors focusing on conditional variances and not on VaR. A direct approach to forecast multiperiod VaR, defined as in Equation (3), specifies a model for returns at the relevant horizon, $R_t^{(h)}$, as follows:

$$R_t^{(h)} = \varepsilon_t^{(h)} \sigma_t^{(h)}, \tag{11}$$

where $\varepsilon_t^{(h)}$ is assumed to be an independent and identically distributed sequence with zero mean and variance one. To avoid mechanical correlations between multiperiod returns, the conditional variance, $\sigma_t^{(h)2}$, is often specified for nonoverlapping returns given by $R_{h+1}^{(h)}, R_{2h+1}^{(h)}, R_{3h+1}^{(h)}, \ldots$. Note that, by using nonoverlapping returns, the sample size available for estimation is reduced to $[\frac{T}{h}]$ observations. Alternatively, using the full sample of overlapping returns, $R_2^{(h)}, R_3^{(h)}, R_4^{(h)}, \ldots$ is prone to inference problems; see, for example, the results in Valkanov (2003) in the context of regression models based on overlapping returns.

Given that conditional variances usually respond asymmetrically to positive and negative past returns, one popular asymmetric specification of conditional variances in the context of VaR estimation is the GJR specification of Glosten et al. (1993) according to which, for $t = h + 1, 2h + 1, 3h + 1, \ldots$,

$$\sigma_t^{(h)2} = \omega_h + \alpha_h R_{t-h}^{(h)2} + \beta_h \sigma_{t-h}^{(h)2} + \gamma_h R_{t-h}^{(h)2} I(R_{t-h}^{(h)} < 0), \tag{12}$$

where $I(\cdot)$ is the indicator function, which takes value one if the argument is true and zero otherwise, and $\sigma_1^{(h)2} = \frac{\omega_h}{1-\alpha_h - F_0\gamma_h - \beta_h}$, the marginal variance of multiperiod returns, with F_0 being the distribution function of $\varepsilon_t^{(h)}$ evaluated at zero; see, among others, Bams et al. (2017), Colacito and Engle (2010), Kole et al. (2017), Kuester et al. (2006), and Mancini and Trojani (2011), who advocate fitting GJR models to estimate the one-period VaR and Alexander et al. (2013, 2021) for the moments of multiperiod returns in the context of GJR models. The parameters of model (12) satisfy the usual positivity and stationarity conditions. Obviously, the symmetric GARCH(1,1) model proposed by Bollerslev (1986) and Engle (1982) is obtained from Equation (12) when $\gamma_h = 0$.[2] Drost and Nijman (1993) derive the relation between the parameters of the GARCH model for different values of h and those for $h = 1$ while Sbrana and Silvestrini (2013) derive this relationship for the IGARCH model of RiskMetrics and Meddahi and Renault (2004) extend the results to the asymmetric GJR model. It is important to note that $\sigma_t^{(h)2}$ defined as in model (12) is the conditional variance of $R_t^{(h)}$ given $R_{t-h}^{(h)}, R_{t-2h}^{(h)}, \ldots$ instead of the conditional variance given $R_{t-1}^{(h)}, R_{t-2}^{(h)}, \ldots$

In practice, the parameters required to obtain $\sigma_t^{(h)2}$ are unknown and need to be estimated and substituted in Equation (12). The most popular estimator of these parameters is Gaussian Quasi-maximum likelihood (G-QML) based on nonoverlapping multiperiod returns, with the number of returns reduced by a factor of h with respect to the available daily returns; see Hoga (2019). It is important to point out that Basel 3 allows risk to be calculated with overlapping returns. Very recently, Patton et al. (2019) also propose estimating the parameters in Equation (12) by minimizing a loss function especially designed for quantiles. This alternative estimator of the parameters is implemented by Liu and Stentoft (2021). The asymptotic properties of the G-QML estimator are revised by Francq and Zakoïan (2009) and Hamadeh and Zakoïan (2011) for GARCH and GJR models, respectively.

Alternatively, when $\varepsilon_t^{(h)}$ has a non-normal distribution, the parameters can be estimated by maximum likelihood (ML), maximizing the corresponding likelihood. In particular, in this survey, we consider two popular further distributions for standardized returns, namely, the Student with ν degrees of freedom and the Skewed-generalized-t (SGT) distribution defined by Theodosiou (1998). Note that the SGT distribution with parameter $k = 2$ is the same as the distribution proposed by Hansen (1994) and implemented by Anatolyev and Petukhov (2016), Bali and Theodossiou (2008), Fenunou et al. (2016), Le (2020), Theodossiou (2015), Wong and So (2003), and Zhou et al. (2016), among others; see Aas and Haff (2006), Li and Nadarajah (2020), and Nadarajah *et al* (2017) for surveys of asymmetric Student distributions often assumed in the context of financial econometrics. Fan et al. (2014) deal with the distribution of the QML estimator when the maximized likelihood is non-Gaussian. Finally, note that the marginal variances required to compute the starting values of the conditional volatilities, $\sigma_t^{(h)2}$, can be estimated using the corresponding sample variances of multiperiod returns.

The VaR in Equation (3) also depends on $q_\delta^{(h)}$, the δ% quantile of the distribution of $e_T^{(h)}$, which can be obtained by assuming a particular distribution as, for example, in Wong and So (2003), who consider that $e_T^{(h)}$ has either a normal or a SGT distribution. If the assumed distribution is non-Gaussian, its parameters can be estimated by ML together with all other parameters of the conditional variance. Alternatively, one can estimate $q_\delta^{(h)}$ by simulation. One of the most popular simulation methods to estimate $q_\delta^{(h)}$, which does not assume a particular distribution for multistep standardized returns, is conditional historical simulation (CHS), as proposed by Hull and White (1998). CHS estimates $q_\delta^{(h)}$ as the empirical quantile of standardized residuals, $\hat{e}_T^{(h)}$. It is well known that estimating a quantile by inverting the empirical distribution may not be efficient; see, for example, Modarres et al. (2002). Alternatively, these latter authors propose a weighted estimator using information on the mean being zero and the variance being one. As proposed by Barone-Adesi et al. (2008), it is also possible to bootstrap from the empirical distribution of standardized residuals. This procedure is known as filtered historical simulation (FHS). Finally, some authors estimate $q_\delta^{(h)}$ using extreme value theory (EVT) as independently proposed by Danielsson and de Vreis (2000) and McNeil and Frey (2000). In this case, the center of the distribution is estimated by bootstrapping while the extremes are obtained by using the Hill (1975) estimator . This alternative is particularly tailored for models with heavy-tailed errors. However, while intuitive, it lacks a solid theoretical foundation.

3.2 | Iterated procedures

Assuming that one-period standardized returns, $\varepsilon_t^{(1)}$, are an independent sequence, the conditional variance needed to compute $VaR_t^{(h)}$ in Equation (3) is given by

$$\sigma_t^{(h)2} = \sum_{i=1}^{h} \sigma_{t+i|t}^2, \tag{13}$$

where $\sigma_{t+i|t}^2$ is the variance of one-period returns, r_{t+i}, conditional on the information available at time t, which can be obtained recursively for the particular model fitted to r_t. If the conditional variance of one-period returns is specified as a GJR model, the i-step-ahead conditional variance is given by

$$\sigma_{t+i|t}^2 = \begin{cases} \sigma_{t+1}^2, & i = 1 \\ \sigma_r^2 + (\alpha + \beta + F_0\gamma)^{i-1}\left(\sigma_{t+1}^2 - \sigma_r^2\right), & i \geq 2, \end{cases} \tag{14}$$

where $\sigma_{t+1}^2 = \omega + \alpha r_t^2 + \beta\sigma_t^2 + \gamma r_t^2 I(r_t < 0)$ and $\sigma_r^2 = \frac{\omega}{1-\alpha-\beta-F_0\gamma}$. Consequently, the conditional variance of future multiperiod returns is given by

$$\sigma_t^{(h)2} = h\sigma_r^2 + \frac{1-(\alpha+\beta+F_0\gamma)^h}{1-(\alpha+\beta+F_0\gamma)}\left(\sigma_{t+1}^2 - \sigma_r^2\right). \tag{15}$$

Once $\sigma_t^{(h)2}$ is estimated by the iterated approach, estimates of $q_\delta^{(h)}$ can be obtained by assuming a particular distribution of $e_t^{(h)}$. Normality is assumed by Tsay (2010), Liu and Stentoft (2021), and Wong and So (2003), among others while Barendsen et al. (in press) assume a Student-t distribution with 5 and 30 degrees of freedom, and Skewed-Student distributions have been assumed by Liu and Stentoft (2021) and Wong and So (2003). If this distribution is not Gaussian, then its parameters can be estimated by matching the moments of multiperiod returns to the theoretical ones implied by the model specified to one-period returns; see Lönnbark (2016), Wong and So (2003), So and Wong (2012), and Zhou et al. (2016) for the good behavior of this estimator.

Alternatively, one can assume a particular distribution of the standardized one-period returns, $\varepsilon_t = \frac{r_t}{\sigma_{t|t-1}}$. In this case, the corresponding distribution of multiperiod returns is not generally available in closed-form. Brummelhuis and Guégan (2005) study the tail behavior of the conditional probability of $R_T^{(h)}$ when σ_t^2 is specified as a GARCH(1,1) process and ε_t is an independent sequence of standard normal variables while Wong and So (2003) carry out Monte Carlo experiments to analyze the distribution of $e_T^{(h)}$ when ε_t is either normal or Student-5. After assuming a particular distribution of ε_t, several authors propose simulating paths of $r_{t+h|t}$ using the estimated conditional variance and the assumed distribution in order to compute $VaR_t^{(h)}$; see, for example, Bams et al. (2005), Christoffersen (2003), Degiannakis, Floros and Dent (2013), Degiannakis et al. (2014), Degiannakis and Potamia (2017), Engle (2004), and Wong and So (2003). Even if the distribution of standardized returns is known and $h = 1$, in order to compute $q_\delta^{(h)}$ one should take into account that one is using returns standardized using estimated volatilities; see Hartz et al. (2006) and Taniai and Taniguchi (2008) for point corrections of the VaR estimates that take into account this estimation error. Consider, for example, that the model assumed for one-period returns is a GJR model with normal errors. In this case, at each moment of time t, N paths of future returns are generated recursively, for $j = 1, \ldots, N$ and $i = 1, \ldots, h$, as follows:

$$r_{j,T+i} = \hat{\sigma}_{j,T+i}\varepsilon_{j,T+i} \tag{16}$$

$$\hat{\sigma}_{j,T+i+1}^2 = \hat{\omega} + \hat{\alpha} r_{j,T+i}^2 + \hat{\beta}\hat{\sigma}_{j,T+i}^2 + \hat{\gamma} r_{j,T+i}^2 I(r_{j,T+i} < 0), \tag{17}$$

where $j = 1, \ldots, N$ represents each of the paths, $\hat{\omega}, \hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$ are either ML or QML estimates of the corresponding parameters, and $\varepsilon_{j,t+i}$ are random draws from the normal distribution. The $VaR_t^{(h)}$ is then computed as the corresponding quantile of the empirical distribution of $R_{j,t}^{(h)} = \sum_{i=1}^{h} r_{j,t+i}$, $j = 1, \ldots, N$. We denote this procedure as SIM.

If no assumptions are made neither on the distribution of $e_t^{(h)}$ nor on that of ε_t, $\varepsilon_{j,t+i}$ in Equation (16) can be obtained by simulation, using bootstrap methods. The most popular method is FHS, which consists on simulating paths of $r_{j,t+i}$ as in Equations b(16) and (17) but with $\varepsilon_{j,t+i}$ being random draws from the empirical distribution of the standardized returns; see Hsieh (1993) for a very early proposal of using simulated paths and Giannopoulos (2003), Engle (2004), Le (2020), Mancini and Trojani (2011), Liu and Stentoft (2021), Pritsker (2006), and Wong (2020) for the implementation of FHS with multiperiod returns. An important issue is the nonrobustness of several resampling procedures used to compute multiperiod VaR estimates. It is known that a few large observations are sufficient to cause the break down of quantile estimates based on nonparametric residual bootstrap and, consequently, multiperiod VaR forecasts can be strongly affected by a few large observations; see Mancini and Trojani (2011) and Trucíos et al. (2017), who propose robustified procedures to estimate GARCH models in the presence of outliers.

Finally, the quantile, $q_{\delta}^{(h)}$, can also be estimated by EVT; see Mancini and Trojani (2011) for the good performance of this approach.

Table 1 summarizes the main point estimators of $VaR_t^{(h)}$ available in the literature and some selected references implementing each of them.

3.3 | Some further two-step procedures

Several authors propose alternative procedures to the direct and iterated procedures described above. Among them, Taylor (1999, 2000) proposes estimating multiperiod VaR using quantile regression. Recently, Le (2020) and Chen et al. (2021) propose procedures combining the quantile regression and the MIDAS approach of Ghysels (2014). MIDAS uses daily data to produce directly multiperiod volatility forecasts and can thus be viewed as a middle ground between the direct and the iterated approaches.

Alternatively, some authors propose estimating multiperiod VaR using intra-daily data and realized volatilities; see, for example, Beltratti and Morana (2005) and Louzis et al. (2013). It is important to note that the conclusions about using ultra-high-frequency returns to compute multiperiod VaR are mixed. While Beltratti and Morana (2005) conclude that using ARFIMA models in the context of realized volatilty provide a superior performance, Degiannakis and Potamia (2017) show that the accuracy of multiperiod VaR forecasts does not improve with respect to estimating them using daily data.

Finally, Hoogerheide and van Dijk (2010) also propose a two-step approach based on Bayesian procedures.

3.4 | A quick look at multiperiod expected shortfall

As mentioned above, ES is gaining popularity as the risk measure that should be reported by financial institutions to assess their risk level. However, the research about multiperiod is still extremely scarce with just a bunch of works available in this topic. In this subsection, we describe those that we are aware about.

The multiperiod ES is given by the expected loss when multiperiod returns are below the multiperiod VaR, $VaR_t^{(h)}$, as follows:

$$ES_t^{(h)} = -E\left[R_t^{(h)} | R_t^{(h)} \leq -VaR_t^{(h)}\right] = -\frac{1}{\delta}\int_0^{\delta} VaR_t^{(h)}(u)\,du, \tag{18}$$

where $VaR_t^{(h)}(u)$ stands for the multiperiod VaR for a probability level of u.

TABLE 1 Summary of procedures to estimate multiperiod VaR. A few representative papers are reported for each procedure.

One-step procedures	SRoT	Distribution of multiperiod returns	Historical simulation
	Singleton (2006)	Simonato (2011)	Liu and Stentoft (2021)
	Wang et al. (2018)	Alexander et al. (2013)	
	So and Wong (2012)	Boudt et al. (2009)	
	Kole et al. (2017)		
	Tsay (2010)		
Two-step procedures			
	Conditional variance	**Conditional quantile**	
Direct			
	GARCH	Hill	Hoga (2019)
	Asymmetric-GARCH	Student	Kole et al. (2017)
		HS	Kole et al. (2017)
Iterated			
	GARCH	Normal	Tsay (2010); Liu and Stentoft (2021)
		Skew-Stud	Liu and Stentoft (2021); Degiannakis and Potamia (2017)
		Simulation based on normality	Bams et al. (2005); Liu and Stentoft (2021)
		Simulation based on Skew-Stu	Degiannakis et al. (2017); Liu and Stentoft (2021)
		Simulation based on bootstrap	Le (2020); Liu and Stentoft (2021); Pristker (2006)
		Simulation based on EVT	Le (2020); Liu and Stentoft (2021)
	Asymmetric GARCH	Normal	Wong and So (2003)
		Student	Kole et al. (2017)
		Skew-Stud	Wong and So (2003); Lönnbark (2016)
		Simulation based on normality	Engle (2004); Wong and So (2003)
		Simulation based on Skew-Stu	Wong and So (2003)
		Simulation based on bootstrap	Engle (2004); Le (2020); Mancini and Trojani (2011)
			Lönnbark (2016); Kole et al. (2017)
		Simulation based on EVT	Le (2020); Mancini and Trojani (2011)

When the multistep VaR is given by Equation (3), the ES is given by

$$ES_t^{(h)} = -\sigma_t^{(h)} E\left[e_t^{(h)} | e_t^{(h)} \le q_\delta^{(h)}\right] \tag{19}$$

It is important to remark that, when the VaR is underestimate (overestimated), the ES is also underestimated (overestimated). There are several proposals in the literature to estimate the ES based on two-step procedures. So and Wong (2012) propose three different estimators of the multiperiod ES. First, assuming that $e_t^{(h)}$ is normal, they propose estimating the multiperiod VaR as follows:

$$ES_t^{(h)} = -\sigma_t^{(h)} \phi(q_\delta^{(h)}), \tag{20}$$

where $\sigma_t^{(h)}$ is obtained by the exact conditional variance of $R_t^{(h)}$ under a QGARCH model for one-period returns as described by Wong and So (2003), and $\phi(\bullet)$ is the standard normal probability density function.

Second, assuming that $e_t^{(h)}$ has a Skewed-Student distribution, So and Wong (2012) propose using the exact conditional kurtosis of $R_t^{(h)}$ as proposed by Wong and So (2003) to estimate its parameters. Then, the multiperiod ES is obtained by using the expression of the expectation of the estimated distribution.

Finally, So and Wong (2012) propose using the simulation procedures described above in Equations (16) and (17), to obtain replicates of multiperiod returns, $R_{j,t}^{(h)}$, $j = 1, \dots, N$. Then, the multiperiod ES is given by

$$ES_t^{(h)} = -\frac{1}{N\delta} \sum_{j=1}^{N} R_{j,t}^{(h)} I\left(R_{j,t}^{(h)} < VaR_t^{(h)}\right), \tag{21}$$

where $I(\bullet)$ takes value one when the argument is true and zero otherwise. In an empirical application to seven worldwide financial indices, they conclude that the last two estimators are unbiased.

Alternatively, Degiannakis et al. (2013) propose an alternative estimator of multiperiod ES, which is also based on assuming normality, although they obtain the multiperiod VaR for different probability levels. Denoting the multiperiod VaR at time t for a probability level of α by $VaR_t^{(h)}(\alpha)$, their proposed estimator of the multiperiod ES is given by

$$ES_t^{(h)} = \frac{1}{R} \sum_{i=1}^{R} VaR_t^{(h)}(1 - 0.05 + i0.05(R+1)^{-1}) \tag{22}$$

where R is the number of slices in which the interval $[0, \delta]$ is divided.

Very recently, Barendsen et al. (in press) propose estimating multiperiod ES using Equation (19) with $\sigma_t^{(h)2}$ given as in Equation (15) and assuming a particular distribution of the multistep quantile, in particular, $q_\delta^{(h)}$ being the δ-quantile of a Student-t distribution with either $\nu = 5$ or 30 degrees of freedom. In this case

$$E\left[\varepsilon_t^{(h)} | \varepsilon_t^{(h)} \le q_\delta^{(h)}\right] = \sqrt{\frac{\nu - 2}{\nu}} \frac{\nu + (g_\delta)^2}{\nu - 1} \frac{G_\nu(g_\delta)}{\delta} \tag{23}$$

with g_δ being the δ quantile of the standard Student-t distribution with ν degress of freedom and

$$G_\nu(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}\left(1 - \frac{x^2}{\nu}\right)^{-\frac{\nu-1}{2}}, \tag{24}$$

where $\Gamma(\bullet)$ is the Gamma function. In an application to compute 10-period VaR and ES of FTSE-100 returns, Barendsen et al. (in press) conclude that $\text{VaR}^{(10)}$ is underestimated while $\text{ES}^{(10)}$ forecasts are too pessimistic.

4 | COMPARING DIRECT AND ITERATED MULTIPERIOD VAR ESTIMATES

There are few works comparing direct and iterated approaches in the context of estimating multiperiod volatilities. Among them, in a recent paper, Ghysels et al. (2019) undertake a comprehensive empirical examination of multiperiod volatility forecasting approaches. In addition to the simple SRoT and the direct and iterated approaches, they consider a MIDAS regression to estimate multiperiod volatilities. The results of their study suggest that long-horizon volatility is much more predictable than previously suggested at horizons as long as 60 trading days (about three months) with MIDAS being convenient for $h > 10$; see also Christoffersen and Diebold (2000), who show that, depending on the particular model for one-period returns, the volatility forecastability ends around $h = 10$. Furthermore, Ghysels et al. (2019) also conclude that, when volatilities are represented by GARCH models, the direct approach provides worse (in the MSFE sense) forecasts than the iterated approach.

The literature on the comparison between direct and iterated approaches to estimate multiperiod VaR is even more scarce. As far as we know, Mancini and Trojani (2011) were the first aiming to this comparison. They consider simulated and real returns concluding that the direct approach does not work well. They advocate the use of iterated forecasts based on GARCH models for 1-day returns with the quantile computed using a robust bootstrap procedure.

More recently, Kole et al. (2017) also carry out an empirical comparison about the effects of temporal aggregation on the quality of multistep VaR forecasts based on a portfolio of eight indexes observed daily, weekly and biweekly from 1994 to 2014. Their conclusions also favor the iterated approach based on daily returns. They attribute this better performance to the more precise identification and propagation of shocks in the daily models.

Le (2020) and Chen et al. (2021) compare empirically multiperiod VaR estimates based on some direct, iterated and MIDAS approaches and also conclude that the direct approaches have the worst performance. The results in the latter paper favor the iterated approaches when estimating the 1% VaR while approaches based on MIDAS are favored when estimating the 2.5% and 5% VaRs.

Therefore, in the context of multiperiod VaR forecasting, most studies support the best performance of the iterated approaches when compared with the direct ones. Although this result is different from that of most works dealing with linear models, it is in concordance with the conclusions of Baek (2019), Marcellino et al. (2006), McCracken and McGillicuddy (2019), and Quaedvlieg (2021). This apparent contradiction between the performance of direct and iterated approaches in the context of linear and conditionally heteroscedastic models could be due to the different nature of misspecifications faced in both types of models. Also, it is possible that the large sample sizes required for the correct estimation of the parameters of GARCH-type models are not available when the direct approach is implemented.

5 | THE DISTRIBUTION OF MULTIPERIOD VAR ESTIMATES: A MONTE CARLO EXERCISE

In this section, we carry out Monte Carlo experiments to compare the finite sample properties of the direct and iterated procedures to forecast multiperiod VaR in the context of possibly misspecified conditionally heteroscedastic models for one-period returns. In particular, we generate $R = 1000$ series of sizes $T = 1000, 5000, 10000$ from seven different data generating processes (DGPs) for one-period returns, which are chosen to replicate the empirical characteristics often observed in real financial returns. The first three DGPs are based on the following symmetric and highly persistent GARCH(1,1) model

$$r_t = \varepsilon_t \sigma_t \tag{25}$$

$$\sigma_t^2 = 0.02 + 0.1 r_{t-1}^2 + 0.88 \sigma_{t-1}^2, \tag{26}$$

where ε_t is an independent and identically distributed white noise with variance one. We consider three (adequately standardized) alternative distributions for ε_t, namely, normal, Student with $\nu = 7$ degrees of freedom, and SGT with $\nu = 7$ degrees of freedom and asymmetry parameter $\lambda = -0.8$.

To analyze the effect of asymmetric volatilities, the second three DGPs are given by Equation (25) with the conditional variances given by the following asymmetric GJR model

$$\sigma_t^2 = (0.065 - 0.09 F_0) + 0.055 r_{t-1}^2 + 0.88 \sigma_{t-1}^2 + 0.09 r_{t-1}^2 I(r_{t-1} < 0), \tag{27}$$

and ε_t having the same three distributions mentioned above. Note that, for the two symmetric distributions, $F_0 = 0.5$ while, in the SGT distribution, $F_0 = 0.36$. Therefore, the constant in Equation (27) is 0.02 when the errors are normal or Student and it is 0.033 when they are SGT. In any case, regardless of the particular distribution of standardized returns, the marginal variance of returns in the GARCH and GJR models in Equations (26) and (27), respectively, is 1.

Finally, we deal with misspecification by considering a last DGP for one-period returns. In particular, we generate returns by Equation (25) with ε_t having a standardized GED distribution with parameter $\nu = 1.7$ and the following asymmetric stochastic volatility (ASV)

$$\begin{aligned} log\sigma_t^2 = &-0.00001 + 0.98 log\sigma_{t-1}^2 + 0.02(I(\varepsilon_{t-1} < 0) - 0.5) - 0.15\varepsilon_{t-1} \\ &-0.04[|\varepsilon_{t-1}| - E(|\varepsilon_{t-1}|)] + \eta_t \end{aligned} \tag{28}$$

where η_t is $N(0, \sigma_\eta^2)$ with $\sigma_\eta^2 = 0.02$, distributed independently from ε_{t+s} for all leads and lags; see Mao et al. (2020) for a description of the properties of this model that nets several popular stochastic volatility models designed to capture asymmetric volatility. The main objective when considering the DGP in Equation (28) is to generate returns by a model different from any of the models used in the iterated procedures. In this case, all models are misspecified.

For each DGP and Monte Carlo replicate, we compute the true $VaR_T^{(10)}$ for $\delta = 0.01$, which is the VaR level originally required by the Basel accords. Note that, in the context of insurance, the required VaR level is even smaller, $\delta = 0.005$ (Fröhlich and Weng, 2018) while Hsieh (1993) also considers $\delta = 0.005$ when calculating daily minimum capital requirements of a futures position. Furthermore, Basel Committee on Banking Supervision (2019) also recommends banks employ the 2.5% risk level to estimate financial risk. The true $VaR_T^{(10)}$ is obtained by generating paths of independent future noises $\varepsilon_{T+1}^{(i)}, \dots, \varepsilon_{T+10}^{(i)}$ from the assumed distribution for $i = 1, \dots, 200,000$. It is important to mention that, in order to generate a reliable value of the "true" $VaR_T^{(10)}$, the number of replicated paths has to be very large for small values of δ. Furthermore, the values in the extremes generate volatility paths, which can diverge as h increases. Consequently, Wong and So (2003) choose $N = 200,000$ paths for $\delta = 0.01$ and 0.05 while Hsieh (1993), Engle (2004), and Hoogerheide and van Dijk (2010) choose $N = 10,000$ when $\delta = 0.05$. Finally, Degiannakis et al. (2014) generate $N = 5000$ paths when $\delta = 0.01$ and 0.05. According to our experience, values of N below 100,000 are not enough when $\delta = 0.01$. The generated paths are inserted in the corresponding true specification of the volatility to generate paths of future returns, $r_{T+1}^{(i)}, \dots, r_{T+10}^{(i)}$, which are used to compute 10-period returns, $R_T^{(10)(i)} = \sum_{j=1}^{10} r_{T+j}^{(i)}$. Finally, the "true" $VaR_T^{(10)}$ is obtained as the empirical quantile of $R_T^{(10)(i)}$. Then, we estimate $VaR_T^{(10)}$ using each of the direct and iterated procedures described above. Finally, we compute the corresponding relative errors as $\frac{\widehat{VaR}_T^{(10)(M)} - VaR_T^{(10)}}{VaR_T^{(10)}}$, where $\widehat{VaR}_t^{(10)(M)}$ is the estimated VaR at time T obtained using procedure M.

Tables 2–4 report the averages and root MSFEs (RMSFEs) through Monte Carlo replicates of the relative errors incurred when the $VaR_T^{(10)}$ is estimated using the direct approaches and one-period returns are generated by the GARCH, GJR, and SV models, respectively. The direct approaches are based on first fitting to nonoverlapping ten-period returns either a GARCH or a GJR model. It is important to note that the effective sample size is reduced by $\frac{1}{10}$. Consequently, when the sample sizes are $T = 1000, 5000, 10000$, the effective sample sizes of nonoverlapping returns used in the estimation of the model parameters are 100, 500, and 1000 observations, respectively. After fitting the GARCH or GJR models to 10-period returns, the quantile is estimated by: (i) assuming a particular distribution for the multiperiod errors (normal, Student, or SGT); (ii) using CHS; (iii) using FHS; and (iv) using the EVT procedure. Note that, when the errors are assumed to be Student or SGT, the model parameters are estimated by ML. In all other cases, the parameters are estimated by G-QML.

Table 2 shows that, when one-period returns are generated by the GARCH model, the results are very similar regardless of whether the GARCH or GJR models are fitted to multiperiod returns. This result could be expected as the asymmetry parameter of the GJR model would be nonsignificant in the latter case. It seems that there is not a price to pay for fitting the GJR model to multiperiod returns even when one-period returns are symmetric with the RMSFEs being only slightly larger when the GJR model is fitted. Somehow more unexpected is the fact that, regardless of the particular error distribution of standardized 1-day returns, the biases and RMSFEs of multiperiod VaR estimates are minimized when $e_T^{(10)}$ is assumed to be normal. The temporal aggregation of returns generates conditional normality even if 1-day returns have distributions with excess kurtosis and/or asymmetry. If the multiperiod quantile is estimated by CHS, FHS, or EVT, the estimates of $VaR_T^{(10)}$ are negatively biased with the biases being largest when the

TABLE 2 Monte Carlo averages and RMSFEs (in brackets) of $VaR_T^{(10)}$ relative errors based on the direct approach when one-period returns are generated by a GARCH(1,1) model with Normal errors (top panel), Student errors (middle panel), and Skewed-Student errors (bottom panel). The effective size used for estimating the VaR is $\frac{T}{10}$. Figures in bold denote that the bias is not significantly different from zero at 1% significance level.

		GARCH						GJR					
		N	S	Sk	CHS	FHS	EVT	N	S	Sk	CHS	FHS	EVT
N	$T = 1000$	**.01**	−.07	−.07	−.23	−.09	**−.03**	**.02**	−.05	−.05	−.20	−.07	**.01**
o		[0.24]	[0.28]	[0.33]	[0.50]	[0.33]	[0.30]	[0.26]	[0.30]	[0.35]	[0.48]	[0.39]	[0.32]
r	$T = 5000$	**−.02**	−.10	−.10	−.14	−.11	−.10	**−.01**	−.10	−.10	−.14	−.10	−.10
m		[0.20]	[0.22]	[0.24]	[0.32]	[0.27]	[0.24]	[0.22]	[0.24]	[0.24]	[0.32]	[0.28]	[0.24]
a	$T = 10000$	**−.02**	−.11	−.11	−.13	−.12	−.12	**−.02**	−.11	−.11	−.13	−.11	−.12
l		[0.20]	[0.24]	[0.26]	[0.28]	[0.27]	[0.26]	[0.20]	[026]	[0.26]	[0.30]	[0.26]	[0.26]
S	$T = 1000$	**.03**	−.07	−.05	−.26	−.11	**−.04**	.04	−.05	−.05	−.23	−.09	**−.02**
t		[0.24]	[0.30]	[0.34]	[0.60]	[0.45]	[0.32]	[0.27]	[0.30]	[0.35]	[0.57]	[0.48]	[0.35]
u	$T = 5000$	**.00**	−.11	−.11	−.15	−.12	−.11	**.01**	−.11	−.11	−.15	−.11	−.11
d		[0.23]	[0.27]	[0.29]	[0.35]	[0.30]	[0.29]	[0.23]	[0.27]	[0.28]	[0.36]	[0.31]	[0.29]
	$T = 10000$	**−.01**	−.12	−.11	−.15	−.12	−.13	**−.00**	−.12	−.12	−.15	−.12	−.12
		[0.21]	[0.26]	[0.27]	[0.33]	[0.27]	[0.27]	[0.22]	[0.27]	[0.27]	[0.34]	[0.28]	[0.28]
S	$T = 1000$	**.02**	−.09	−.10	−.31	−.17	−.09	**.03**	−.08	−.10	−.26	−.15	−.06
G		[0.26]	[0.32]	[0.37]	[0.69]	[0.55]	[0.39]	[0.27]	[0.34]	[0.36]	[0.62]	[0.53]	[0.37]
T	$T = 5000$	**.01**	−.11	−.13	−.21	−.16	−.15	**.01**	−.10	−.14	−.20	−.16	−.15
		[0.21]	[0.25]	[0.28]	[0.40]	[0.33]	[0.31]	[0.22]	[0.25]	[0.29]	[0.39]	[0.33]	[0.32]
	$T = 10000$	**.02**	−.09	−.11	−.16	−.14	−.14	**.02**	−.09	−.13	−.16	−.14	−.16
		[0.20]	[0.24]	[0.27]	[0.33]	[0.29]	[0.29]	[0.21]	[0.24]	[0.27]	[0.34]	[0.29]	[0.34]

CHS procedure is implemented to compute the quantile. In this later case, even if $T = 10,000$, and the effective sample size used to estimate the multiperiod model is 1000, the bias estimating $VaR_T^{(10)}$ can be as large as 16%. Therefore, the direct approach would underestimate risk with dangerous potential damages for financial companies. Underestimation of the own level of risk may lead to insufficient amount of capital reserves to cover potential losses, thus increasing the risk of bankruptcy.

Finally, in relation with the dispersion of multiperiod VaR estimates, Table 2 shows that the RMSFEs are not only rather large but also that they decrease very slowly with the sample size. This lack of efficiency is a well-known characteristic of direct approaches. Mancini and Trojani (2011), who also conclude empirically that direct methods do not work well in the context of multiperiod VaR estimation, argue that these methods suffer the inefficient use of available information, discarding nine out of 10 observations, when computing nonoverlapping 10-day returns. However, according to our results, the bad performance of direct methods does not seem to be associated only to the inefficient use of information. Table 2 shows that, for each particular procedure, the standard deviations of the $VaR_T^{(10)}$ forecasts are of the same order of magnitude regardless of the sample size. For example, when the quantile is estimated assuming normality of multiperiod standardized returns, the RMSFE is 0.24 when $T = 1000$ and decreases to 0.20 when $T = 10,000$. Using more information, the biases do not decrease and the standard deviations are similar. It

TABLE 3 Monte Carlo averages and RMSFE (in brackets) of $VaR_T^{(10)}$ relative errors based on the direct approach when one-period returns are generated by a GJR model with Normal errors (top panel), Student errors (middle panel), and Skewed-Student errors (bottom panel). The effective size used for estimating the VaR is $\frac{T}{10}$. Figures in bold denote that the bias is not significantly different from zero at 1% significance level.

		GARCH						GJR					
		N	St	SGT	CHS	FHS	EVT	N	St	SGT	CHS	FHS	EVT
N	$T = 1000$	−.12	−.21	−.28	−.49	−.32	−.24	−.10	−.19	−.27	−.42	−.28	−.20
o		[0.30]	[0.40]	[0.49]	[0.73]	[0.64]	[0.49]	[0.30]	[0.39]	[0.48]	[0.70]	[0.56]	[0.44]
r	$T = 5000$	−.14	−.24	−.29	−.37	−.33	−.32	−.13	−.22	−.29	−.35	−.31	−.30
m		[0.28]	[0.36]	[0.41]	[0.51]	[0.46]	[0.44]	[0.26]	[0.33]	[0.40]	[0.48]	[0.43]	[0.41]
a	$T = 10000$	−.14	−.24	−.29	−.35	−.33	−.33	−.12	−.22	−.28	−.33	−.30	−.31
l		[0.28]	[0.36]	[0.41]	[0.48]	[0.44]	[0.44]	[0.26]	[0.33]	[0.38]	[0.46]	[0.41]	[0.41]
S	$T = 1000$	−.11	−.22	−.26	−.53	−.34	−.25	−.09	−.20	−.26	−.46	−.32	−.22
t		[0.32]	[0.41]	[0.50]	[0.87]	[0.67]	[0.51]	[0.31]	[0.40]	[0.50]	[0.79]	[0.66]	[0.48]
u	$T = 5000$	−.11	−.13	−.26	−.30	−.40	−.35	−.34	−.12	−.24	−.31	−.40	−.33
d		[0.29]	[0.38]	[0.45]	[0.57]	[0.49]	[0.47]	[0.29]	[0.37]	[0.42]	[0.57]	[0.47]	[0.47]
t	$T = 10000$	−.12	−.25	−.28	−.37	−.34	−.34	−.11	−.24	−.30	−.36	−.32	−.32
		[0.28]	[0.38]	[0.41]	[0.52]	[0.47]	[0.46]	[0.25]	[0.35]	[0.40]	[0.50]	[0.44]	[0.43]
S	$T = 1000$	**−.01**	−.13	−.22	−.58	−.53	−.37	**.02**	−.09	−.22	−.48	−.43	−.31
T		[0.37]	[0.40]	[0.52]	[1.16]	[1.15]	[0.75]	[0.35]	[0.37]	[0.52]	[0.88]	[0.91]	[0.68]
G	$T = 5000$	−.06	−.20	−.30	−.41	−.53	−.34	.13	−.15	−.30	−.43	−.47	−.35
		[0.31]	[0.39]	[0.52]	[0.73]	[0.73]	[0.60]	[0.37]	[0.32]	[0.52]	[0.71]	[0.64]	[0.53]
	$T = 10000$	−.06	−.20	−.29	−.45	−.52	−.36	**−.02**	−.14	−.29	−.41	−.45	−.35
		[0.32]	[0.39]	[0.52]	[0.68]	[0.70]	[0.54]	[0.27]	[0.31]	[0.52]	[0.65]	[0.60]	[0.51]

TABLE 4 Monte Carlo averages and RMSFE (in brackets) of $VaR_T^{(10)}$ relative errors based on the direct approach when one-period returns are generated by a ASV model with GED errors. The effective size used for estimating the VaR is $\frac{T}{10}$. Figures in bold denote that the bias is not significantly different from zero at 1% significance level.

	GARCH						GJR					
	N	S	Sk	CHS	FHS	EVT	N	S	Sk	CHS	FHS	EVT
$T = 1000$	**−.01**	−.12	−.17	−.45	−.18	−.26	**−.05**	**−.05**	−.21	−.47	−.12	−.17
	[0.45]	[0.54]	[0.59]	[0.54]	[0.61]	[0.65]	[0.40]	[0.46]	[0.52]	[0.54]	[0.52]	[0.58]
$T = 5000$	**−.02**	−.16	−.22	−.46	−.28	−.29	**−.03**	−.08	−.27	−.46	−.28	−.30
	[0.47]	[0.55]	[0.61]	[0.53]	[0.65]	[0.66]	[0.39]	[0.45]	[0.45]	[0.53]	[0.66]	[0.67]
$T = 10,000$	**−.01**	−.16	−.22	−.47	−.31	−.31	−.04	−.08	−.28	−.48	−.21	−.22
	[0.42]	[0.51]	[0.56]	[0.53]	[0.63]	[0.63]	[0.33]	[0.38]	[0.42]	[0.52]	[0.48]	[0.48]

does not seem that the bad performance of direct procedures is just a problem associated with the sample size. If it were, we could expect RMSFEs decreasing at a larger rate with T.

Tables 3 and 4 report the results when 1-day returns are generated by the nonsymmetric GJR and SV models in Equations (27) and (28), respectively. The conclusions are mainly the same as those obtained from Table 2. The multiperiod VaR are always negatively biased (even when normality of multiperiod standardized returns is assumed), with the biases being generally even

TABLE 5 Monte Carlo averages and RMSFEs (in brackets) of $VaR_T^{(10)}$ relative errors based on the iterated approach when one-period returns are generated by a GARCH(1,1) model with Normal errors (top panel), Student errors (middle panel) and Skewed-Student errors (bottom panel). Figures in bold denote that the bias is not significantly different from zero at 1% significance level.

		GARCH						GJR					
		N	S	Sk	SIM	FHS	EVT	N	S	Sk	SIM	FHS	EVT
N	$T = 1000$	.05	**−.00**	**−.00**	**.00**	**.00**	**.00**	.05	**−.00**	**.01**	**.00**	**.01**	**.00**
o		[0.07]	[0.07]	[0.12]	[0.06]	[0.08]	[0.10]	[0.07]	[0.07]	[0.12]	[0.08]	[0.09]	[0.11]
r	$T = 5000$	.05	−.03	−.03	**.00**	**.00**	**−.00**	.05	−.03	−.03	**.00**	**.00**	**−.00**
m		[0.05]	[0.05]	[0.06]	[0.04]	[0.03]	[0.07]	[0.05]	[0.05]	[0.06]	[0.04]	[0.04]	[0.07]
a	$T = 10000$	.05	−.03	−.03	**−.00**	**−.00**	**−.00**	.05	−.03	−.03	**.00**	**−.00**	**.00**
l		[0.05]	[0.04]	[0.05]	[0.03]	[0.02]	[0.07]	[0.05]	[0.04]	[0.05]	[0.04]	[0.03]	[0.07]
S	$T = 1000$	.08	**.01**	**.01**	.03	**.01**	**.01**	.08	**.01**	.02	.03	**.01**	**.01**
t		[0.10]	[0.09]	[0.13]	[0.08]	[0.10]	[0.12]	[0.11]	[0.09]	[0.13]	[0.10]	[0.12]	[0.13]
u	$T = 5000$	.07	−.03	−.02	.02	**.00**	**−.00**	.07	−.03	−.03	.02	**−.00**	**−.00**
d		[0.08]	[0.05]	[0.06]	[0.04]	[0.05]	[0.09]	[0.08]	[0.06]	[0.07]	[0.05]	[0.06]	[0.09]
	$T = 10000$	.07	−.03	−.02	.03	**.00**	**.00**	.07	−.03	−.03	.03	**.00**	**.00**
		[0.07]	[0.04]	[0.04]	[0.05]	[0.03]	[0.07]	[0.07]	[0.04]	[0.05]	[0.05]	[0.04]	[0.08]
S	$T = 1000$	.08	**.01**	**−.01**	.03	−.03	.05	.08	**.01**	**−.01**	.03	−.03	.05
T		[0.10]	[0.09]	[0.14]	[0.08]	[0.11]	[0.12]	[0.11]	[0.10]	[0.14]	[0.10]	[0.13]	[0.14]
G	$T = 5000$	.08	−.02	−.06	.03	−.04	.05	.08	−.02	−.05	.03	−.04	.05
		[0.09]	[0.05]	[0.08]	[0.05]	[0.06]	[0.09]	[0.09]	[0.05]	[0.08]	[0.06]	[0.07]	[0.08]
	$T = 10000$	.08	−.02	−.05	.03	−.04	.05	.08	−.02	−.06	.03	−.04	0.05
		[0.08]	[0.03]	[0.06]	[0.05]	[0.06]	[0.09]	[0.08]	[0.03]	[0.07]	[0.05]	[0.06]	[0.09]

larger than those observed in Table 2. Similarly, the RMSFEs are also larger than those observed when one-period returns are generated by the symmetric GARCH model.

We now move on to analyze the finite sample properties of $VaR_T^{(10)}$ forecasts when the variance in Equation (13) is obtained by the iterated 10-period variance and the quantile is estimated using the procedures described in Section 2, namely: (i) assuming that the 10-period errors are normal, Student and SGT; (ii) EVT; (iii) simulation assuming that one-step innovations are normal with $N = 5000$ replicates; and (iv) using the FHS procedure with $B = 500,000$. Tables 5–7 report the Monte Carlo averages and RMSFEs of the relative errors when one-period returns are generated by the GARCH(1,1), GJR, and SV models, respectively.

Consider first the results when one-period returns are generated by the symmetric GARCH(1,1) model. Several important conclusions emerge from Table 5. First, as above, when dealing with the direct approach, the results are nearly identical regardless of whether the true GARCH model or the GJR model are fitted to one-period returns. Therefore, there is not a price to pay for iterating the GJR model even when the volatility of one-period returns is symmetric. Second, contrary to what we conclude when using the direct approach, the worst results in terms of bias are obtained when the multiperiod quantile is obtained by assuming normality of multiperiod standardized returns. It seems that, when the variance is computed by iterating, the aggregation effect is not generating normality. If one assumes normal multistep standardized returns, the biases are positive and moderately large. Therefore, risk is overestimated. Although risk overestimation seems

TABLE 6 Monte Carlo averages, standard deviations (in parenthesis) and RMSFEs (in brackets) of $VaR_T^{(10)}$ relative errors based on the iterated approach when one-period returns are generated by a GJR model with Normal errors (top panel), Student errors (middle panel), and Skewed-Student errors (bottom panel). Figures in bold denote that the bias is not significantly different from zero at 1% significance level.

		GARCH						GJR					
		N	S	Sk	SIM	FHS	EVT	N	S	Sk	SIM	FHS	EVT
N	$T = 1000$	−.07	−.14	−.20	−.13	−.13	−.14	−.07	−.13	−.19	**.00**	**.00**	**.00**
o		[0.11]	[0.18]	[0.26]	[0.16]	[0.18]	[0.19]	[0.09]	[0.15]	[0.24]	[0.06]	[0.08]	[0.10]
r	$T = 5000$	−.07	−.17	−.22	−.13	−.13	−.14	−.07	−.15	−.21	**.00**	**.00**	**.00**
m		[0.10]	[0.19]	[0.24]	[0.15]	[0.15]	[0.17]	[0.07]	[0.16]	[0.22]	[0.04]	[0.04]	[0.06]
a	$T = 10000$	−.07	−.17	−.22	−.13	−.13	−.13	−.07	−.16	−.21	**.00**	**.00**	**.00**)
l		[0.10]	[0.19]	[0.24]	[0.15]	[0.15]	[0.16]	[0.07]	[0.16]	[0.21]	[0.03]	[0.03]	[0.06]
S	$T = 1000$	−.05	−.13	−.17	−.10	−.13	−.14	−.04	−.12	−.17	.02	**.00**	**.01**
t		[0.11]	[0.18]	[0.24]	[0.15]	[0.19]	[0.17]	[0.08]	[0.16]	[0.23]	[0.09]	[0.11]	[0.12]
u	$T = 5000$	−.05	−.17	−.21	−.11	−.14	−.14	−.04	−.15	−.20	.03	.02	**−.01**
d		[0.09]	[0.19]	[0.23]	[0.14]	[0.17]	[0.18]	[0.05]	[0.16]	[0.21]	[0.06]	[0.05]	[0.12]
e	$T = 10000$	−.05	−.16	−.21	−.10	−.13	−.13	−.04	−.16	−.21	.02	**−.00**	**−.01**
n		[0.09]	[0.18]	[0.23]	[0.13]	[0.15]	[0.17]	[0.04]	[0.16]	[0.22]	[0.03]	[0.03]	[0.11]
S	$T = 1000$	.08	**−.01**	−.16	−.11	−.26	.28	.09	**.01**	−.17	−.06	−.08	−.11
G		[0.12]	[0.11]	[0.22]	[0.15]	[0.32]	[0.29]	[0.12]	[0.11]	[0.23]	[0.16]	[0.23]	[0.14]
T	$T = 5000$	.07	−.06	−.21	−.12	−.28	.28	.09	−.02	−.21	−.04	−.06	−.09
		[0.09]	[0.08]	[0.22]	[0.14]	[0.30]	[0.28]	[0.10]	[0.05]	[0.22]	[0.10]	[0.03]	[0.10]
	$T = 10000$	.06	−.06	−.22	−.12	−.28	.28	.09	−.03	−.22	−.03	−.05	−.09
		[0.07]	[0.07]	[0.23]	[0.14]	[0.29]	[0.28]	[0.09]	[0.05]	[0.23]	[0.09]	[0.09]	[0.10]

to be less dangerous than risk underestimation, it also has some undesirable consequences, as mentioned in the Introduction. Except when the multiperiod quantile is obtained by assuming normality, the biases estimating multiperiod VaR are nearly zero regardless of the particular procedure implemented to estimate the quantile. This conclusion is in concordance with the empirical results by Kole et al. (2017), who conclude that the distribution choice is of less importance when estimating multiperiod VaR. Third, comparing the RMSFEs reported in Table 5 with those reported for the direct procedures in Table 2, we can observe a dramatic decrease in the dispersion of multiperiod VaR forecasts that could be expected given that the effective sample size used for estimation is 10 times larger and we are assuming the true DGP for one-period returns. However, the decrease is even larger than expected, with the RMSFEs of the iterated approaches ranging from 0.02 to 0.14 while those of the direct approaches range from 0.24 to 0.69. Also note that the RMSFEs are slightly larger when the EVT procedure is implemented to estimate the quantile and are generally minimized when the quantile is estimated by simulation assuming that one-period returns are conditionally normally distributed.

To analyze the role of misspecification in the iterated multiperiod VaR estimates, we also generate one-period returns by the GJR model in Equation (27) and fit both the misspecified GARCH(1,1) and the true GJR models. The results are reported in Table 6. If one-period returns are symmetric, we can observe large negative biases incurred when the GARCH model is fitted to one-period returns before iterating. On the contrary, if the one-period returns have an asymmetric distribution, the biases can be either positive or negative; compare with results in Mancini and

TABLE 7 Monte Carlo averages and RMSFEs (in brackets) of $VaR_T^{(10)}$ relative errors based on the iterated approach when one-period returns are generated by an ASV model with GED errors. Figures in bold denote that the bias is not significantly different from zero at 1% significance level.

	GARCH						GJR					
	N	S	Sk	SIM	FHS	EVT	N	S	Sk	SIM	FHS	EVT
$T = 1000$	−.09	**−.01**	−.07	−.32	−.36	−.36	−.12	−.05	−.04	−.07	−.09	−.09
	[0.28]	[0.29]	[0.34]	[0.28]	[0.31]	[0.30]	[0.25]	[0.25]	[0.27]	[0.33]	[0.32]	[0.32]
$T = 5000$	−.08	−.03	−.12	−.34	−.37	−.38	−.11	−.08	−.09	−.10	−.11	−.15
	[0.27]	[0.29]	[0.34]	[0.28]	[0.29]	[0.30]	[0.25]	[0.25]	[0.28]	[0.32]	[0.31]	[0.32]
$T = 10000$	−.09	−.02	−.11	−.32	−.36	−.35	−.12	−.02	−.08	−.08	−.09	−.09
	[0.25]	[0.28]	[0.33]	[0.27]	[0.28]	[0.29]	[0.24]	[0.24]	[0.27]	[0.32]	[0.32]	[0.32]

Trojani (2011) who only consider the GJR model with Student-5 errors and $T = 2000$. However, it is important to point out that the biases reported in Table 6, when the multiperiod volatility is obtained by iterating the misspecified GARCH model, are of the same order of magnitude and generally smaller than those observed in Table 3 when using the direct approach. Therefore, it seems that there is not any reason for using the direct approach to estimate multiperiod VaR, not even being robust against misspecification. Also note that the RMSFEs are much smaller than those reported in Table 3. Therefore, according to our results, even in the case of misspecification of the conditional variance, using the iterated approaches seems to be less damaging for estimating the multiperiod VaR than using the direct approaches. Finally, when looking at the multiperiod VaR estimates obtained by iterating the true GJR model fitted to one-period results, we observe that, if one-period returns have a symmetric conditional distribution, the multiperiod VaR estimates are unbiased as far as the quantile is estimated by either simulation, bootstrap, or EVT. Among these alternatives, the dispersion is minimum when the quantile is obtained by simulation assuming that one-period returns are normal and maximum if it is estimated using EVT. On the other hand, if the distribution of returns is asymmetric, the best results are obtained by iterating the GJR model and estimating the quantile by assuming that multiperiod returns have a Student-*t* distribution; see Spierdijk (2016) for the problems of bootstrapping when estimating the VaR with asymmetric conditional distributions.

Finally, Table 7 reports the results when one-period returns are generated by the ASV model and the misspecified GARCH and GJR models are iterated. We can see that the best results are obtained when the GJR model is fitted to 1-day returns and the quantile is estimated by assuming a Student-*t* distribution.

Summarizing, the simulation results of this section, we observe a large decrease in dispersion when using iterated approaches as compared to direct approaches, supporting the increase in efficiency when iterated methods are used instead of direct approaches to forecast multiperiod VaRs. This conclusion is in concordance with conclusions by Marcellino et al. (2006), who show that, in the context of forecasting macroeconomic variables using linear models, iterated forecasts typically outperform the direct forecasts in terms of MSFEs. However, while improvements in Marcellino et al. (2006) are modest, we observe large improvements when forecasting $VaR_T^{(10)}$. These results also support the empirical results by Kole et al. (2017) who conclude that, for their particular, portfolio, iterating the GJR model gives the best VaR forecasts. The large variances of multiperiod VaR forecasts based on direct procedures cannot be only attributed to the inefficient use of information; compare the results for $T = 10,000$ in the direct procedures and for $T = 1000$

in the iterated procedures with the effective sample size used for estimation being the same for both methods. However, if one would like to use the direct approach, regardless of the conditional distribution of one-period returns, the results are best when fitting a GJR model estimating the quantile by assuming normality of multistep standardized returns. On the other hand, the best performance of iterated approaches, based on iterating the GJR model, depends on the distribution of one-day returns. If this distribution is symmetric, the results are best when the quantile is obtained by simulation assuming that 1-day returns are normal. However, if the distribution is asymmetric, the best performance is obtained when the quantile is estimated by assuming that multistep standardized returns are Student.

6 | EMPIRICAL ILLUSTRATION OF ALTERNATIVE MULTIPERIOD VAR ESTIMATES

In this section, we illustrate the performance of the direct and iterated procedures described above when forecasting VaR in practice. We obtain one-step-ahead 10-day VaR predictions of three series of financial returns, namely, S&P500, Dollar/Euro, and IBM returns, observed daily from January 1, 1995, to December 12, 2018, with $T = 6000$. One-day log-returns are obtained as usual as r_t for $t = 2, 3, \dots, 6000$. We also compute nonoverlapping 10-day log-returns as $R^{(10)}_{t-10}$, for $t = 11, 21, 31, \dots, 5991$. S&P500 and Dollar/Euro returns have been chosen because they are ubiquitous in the literature dealing with the empirical analysis of financial returns. Furthermore, Table 8, which reports several descriptive statistics of r_t for each of the three series of 1-day and 10-day returns, shows that they have different stylized facts. When looking at 1-day S&P500 returns, we can observe that their distribution is leptokurtic and negatively skewed and the cross-correlation between absolute and lagged returns is negative suggesting asymmetric volatility. The Dollar/Euro returns are closer to normality with symmetric volatilities. Finally, the volatility of IBM is also symmetric with the distribution of returns being leptokurtic but symmetric. As often observed in the related literature, 10-day returns have larger skewness (in absolute value) and stronger asymmetric effects while the kurtosis is smaller than those of 1-day returns; see, among others, Ghyselset al. (2016), Le (2020), Neuberger (2012), and Wong (2020) for the same empirical result and Alexander et al. (2021), Berd et al. (2007), Colacito and Engle (2010), Engle (2004, 2011), Fama and French (2018), Meddahi and Renault (2004), and Wong and So (2003), who explain why the presence of asymmetric volatilities increase the asymmetry of multiperiod returns even in the case of one-period innovations being symmetric. The three series of returns considered have been chosen to represent common stylized facts of daily returns often observed in real time series.

We fit the GJR model with normal, Student, and SGT errors to 1- and 10-day returns with the parameters estimated by ML. Note that the ML estimates of the parameters obtained by maximizing the Gaussian likelihood are G-QML estimates when the errors are not assumed to be normal. Also, given the previous Monte Carlo results, we focus on the more general GJR model and do not fit the GARCH model. The parameter estimates, reported in Table 9 for the GJR model with SGT errors, are in concordance with the sample moments reported in Table 8. When looking at the parameter estimates for 1-day returns, we can observe that Dollar/Euro returns can be represented by a symmetric and persistent GARCH model with normal errors. On the other hand, volatilities of S&P500 and IBM 1-day returns are characterized by strong leverage effects and their errors are leptokurtic with asymmetric and symmetric distributions, respectively. When looking at the parameter estimates corresponding to 10-day returns, we can see that the persistence of

TABLE 8 Sample moments of (a) 1-day log-returns and (b) nonoverlapping 10-day log-returns, for S&P500, Dollar/Euro, and IBM observed from January 1, 1995 to December 12, 2018.

	S&P500	DOLLAR/EURO	IBM
(a) Oneday log-returns			
Mean	0.000	−0.002	0.000
Median	0.100	0.000	0.000
Max	11.000	3.733	12.400
Min	−9.500	−2.781	−16.900
Stand. dev.	1.200	0.614	1.700
Skewness	−0.300	0.068	−0.100
Kurtosis	11.40	4.54	10.80
$Corr(y_t, y_{t-1})$	−0.100	−0.024	0.000
$Corr(\lvert y_t\rvert, \lvert y_{t-1}\rvert)$	0.200	0.088	0.200
$Corr(\lvert y_t\rvert, y_{t-1})$	−0.100	0.017	0.000
(b) Nonoverlapping 10-day log-returns			
Mean	0.294	−0.017	0.310
Median	0.628	0.077	0.356
Max	9.956	7.503	27.143
Min	−15.590	−8.647	−18.917
Stand. dev.	3.117	1.893	5.313
Skewness	−0.848	−0.211	0.281
Kurtosis	5.564	4.020	5.730
$Corr(y_t, y_{t-1})$	−0.026	0.054	0.061
$Corr(\lvert y_t\rvert, \lvert y_{t-1}\rvert)$	0.254	0.061	0.222
$Corr(\lvert y_t\rvert, y_{t-1})$	−0.227	−0.040	−0.069

TABLE 9 ML parameter estimates of GJR models with SGT errors based on 1-day log-returns and nonoverlapping 10-day log-returns for S&P500, Dollar/Euro, and IBM. Asymptotic standard errors in parenthesis.

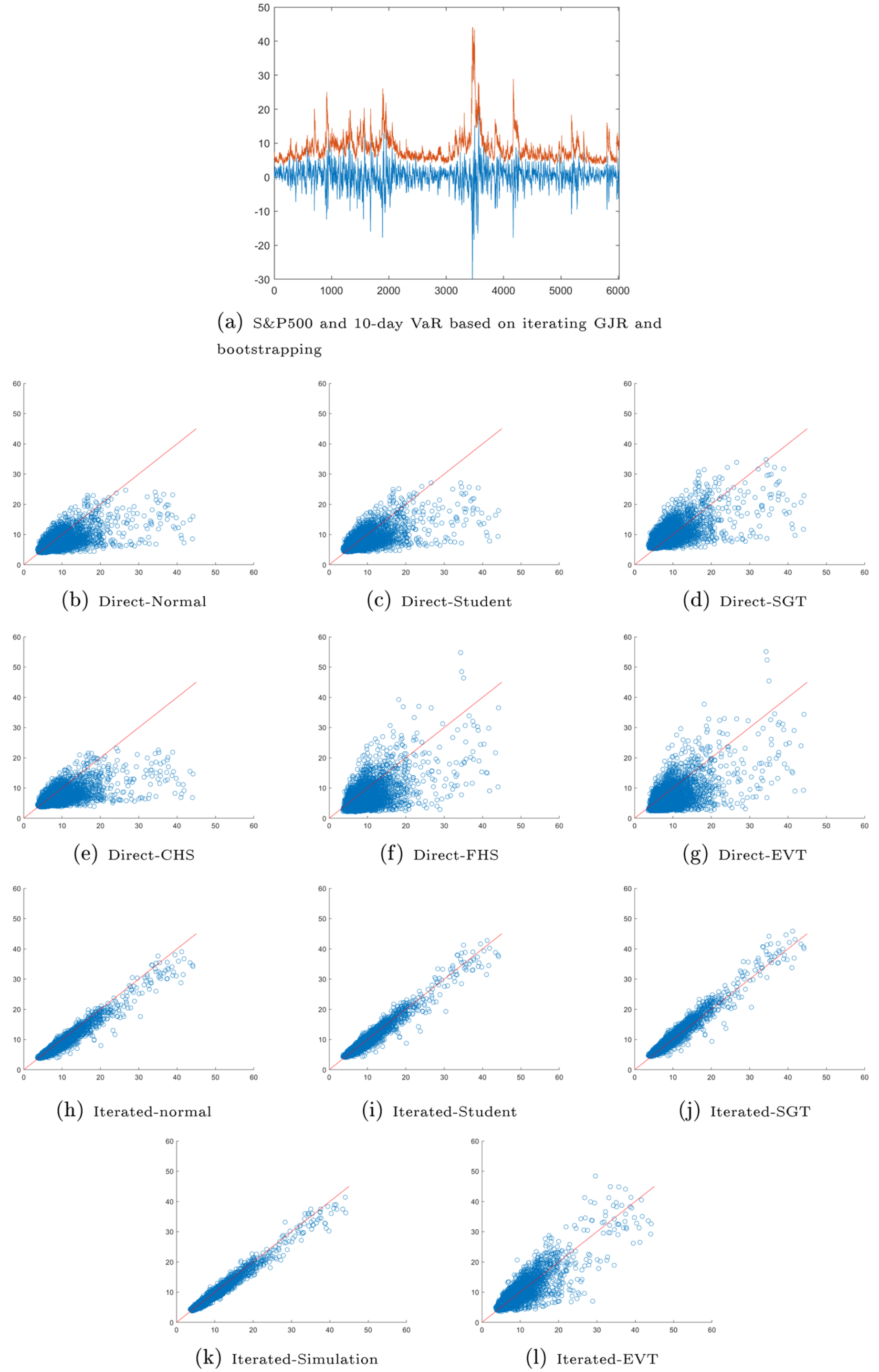
	S&P500		DOLLAR/EURO		IBM	
	1-day	10-day	1-day	10-day	1-day	10-day
ω	0.017	0.795	0.001	0.123	0.017	0.824
	(0.00)	(0.23)	(0.00)	(0.01)	(0.01)	(1.12)
α	0.000	0.000	0.024	0.069	0.036	0.058
	(0.00)	(0.02)	(0.00)	(0.00)	(0.01)	(0.00)
β	0.897	0.787	0.968	0.867	0.940	0.872
	(0.00)	(0.02)	(0.00)	(0.00)	(0.03)	(0.01)
γ	0.192	0.323	0.013	0.056	0.098	0.043
	(0.00)	(0.03)	(0.00)	(0.00)	(0.02)	(0.02)
ν	7.76	11.57	10.17	70.23	4.63	7.19
	(0.64)	(32.53)	(1.69)	(54.94)	(0.15)	(4.58)
λ	−0.138	−0.371	0.002	−0.146	−0.021	−0.058
	(0.00)	(0.00)	(0.00)	(0.01)	(0.00)	(0.00)

volatilities (measured by $\alpha + \beta + 0.5\gamma$) is smaller. However, while the ARCH parameter is larger for 10-day returns, the estimated asymmetry parameter, δ, is larger for 10-day S&P500 and Dollar/Euro returns while it is smaller for IBM returns. Finally, with respect to the parameters of the error distribution, the degrees of freedom of the Student are clearly larger when estimated with 10-day returns indicating that they are closer to normality. However, the asymmetry parameter, λ, is also larger in absolute value suggesting larger asymmetry in the conditional distribution of 10-day returns. Therefore, the conditional distribution of 10-day returns is non-normal but different than the non-normal distribution of 1-day returns.

No attempt is made to identify the best model in terms of goodness of fit or ability to pass diagnostic tests. Our goal is to compare the empirical performance of the models considered when predicting multiperiod VaRs. The estimated GJR models for 1- and 10-day returns are implemented to obtain daily in-sample estimates of $VaR_t^{(10)}$. First, we estimate $VaR_t^{(10)}$ using the direct approach based on the GJR model estimated for 10-day returns and computing the quantile using each of the six procedures described above. Similarly, daily one-step-ahead forecasts of $VaR_t^{(10)}$ are obtained by iterating the GJR model estimated for 1-day returns and computing the quantile by each of the six procedures described in the previous section. The top panels of Figures 1–3 plot the series of returns together with the estimated $VaR_t^{(10)}$ obtained iterating the GJR models estimated for 1-day returns and estimating the quantile using FHS for S&P500, Dollar/Euro, and IBM returns, respectively. These figures also represent scatter plots of these iterated-GJR-FHS estimates of $VaR_t^{(10)}$ against each of the other estimated $VaR_t^{(10)}$.

Consider first the results for S&P500 plotted in Figure 1, which shows that the $VaR_t^{(10)}$ predictions obtained by the direct approach are usually below the iterated GJR-FHS VaRs; recall that Table 3 reports negative biases for direct multiperiod VaR estimates when 1-day returns are generated by a GJR model with asymmetric and leptokurtic errors. As also shown in the Monte Carlo experiments, this is specially the case when the quantile is estimated by the empirical quantile (CHS). This result is also in concordance with the conclusions in Wong (2020), who analyze S&P500 daily returns and show that multiperiod VaRs based on GARCH models for the conditional variances underestimate risk. We can also observe the large dispersion of direct predictions, with the dispersion being larger as the VaR increases. The direct and iterated VaR predictions are similar for small values of $VaR_t^{(10)}$. However, as $VaR_t^{(10)}$ increases, the differences between direct and iterated estimates are larger. Finally, in concordance with the simulation results in Table 6, we can also observe that, when estimating $VaR_t^{(10)}$ by iterating the GJR model, the results are very similar regardless of the particular procedure implemented to estimate the quantile. The only remarkable conclusion from the iterated multiperiod VaR predictions plotted in Figure 1 is that the predictions obtained using EVT have somehow a larger dispersion. This could be due to the fact that EVT needs a large number of observations to work properly for extreme quantiles.

Consider now the estimates of the Dollar/Euro $VaR_t^{(10)}$ plotted in Figure 2. In this case, there are not strong differences in average when the direct approaches are implemented to estimate the multiperiod VaR and when it is estimated by iterating the GJR model with the quantile estimated by FHS. However, the dispersion of the direct estimates is larger and, as above, when looking at the results for S&P500 returns, the variability of direct multiperiod VaRs increases with the VaR; compare with the results reported in Table 2 when returns are generated by a GARCH model with conditionally symmetric errors and $T = 5000$. On the other hand, in concordance with the results reported in Table 5, regardless of the particular approach used to estimate the quantile, the results for the iterated multiperiod VaRs are almost identical.



(a) S&P500 and 10-day VaR based on iterating GJR and bootstrapping

(b) Direct-Normal (c) Direct-Student (d) Direct-SGT

(e) Direct-CHS (f) Direct-FHS (g) Direct-EVT

(h) Iterated-normal (i) Iterated-Student (j) Iterated-SGT

(k) Iterated-Simulation (l) Iterated-EVT

FIGURE 1 (a) S&P500 10-period returns (blue) together with $VaR_t^{(10)}$ estimates (red) obtained iterating the GJR model with bootstrap errors (FHS). Scatter plots of $VaR_t^{(10)}$ estimates obtained by (b) Direct-normal; (c) Direct-Student; (d) Direct-SGT; (e) Direct-CHS; (f) Direct-FHS; (g) Direct-EVT; (h) Iterated-Normal; (i) Iterated-Student; (j) Iterated-SGT; (k) Iterated-Simulation; and (l) Iterated-EVT, against GJR-FHS VaR estimates. The red line represents the 45ž line. [Colour figure can be viewed at wileyonlinelibrary.com]

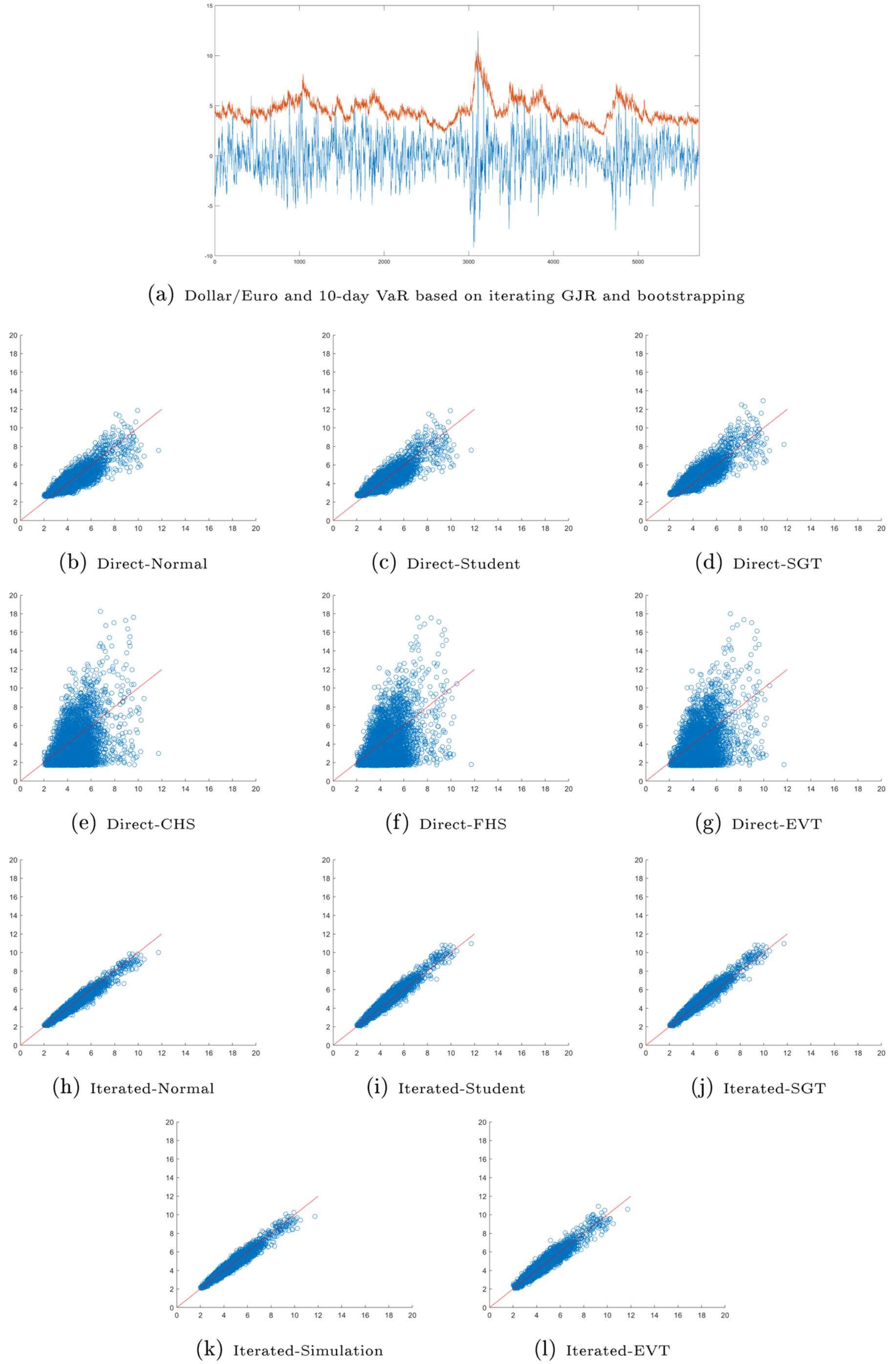


FIGURE 2 (a) Dollar-Euro exchange rate 10-period returns (blue) together with $VaR_t^{(10)}$ estimates (red) obtained iterating the GJR model with bootstrap errors (FHS). Scatter plots of $VaR_t^{(10)}$ estimates obtained by (b) Direct-Normal; (c) Direct-Student; (d) Direct-SGT; (e) Direct-CHS; (f) Direct-FHS; (g) Direct-EVT; (h) Iterated-Normal; (i) Iterated-Student; (j) Iterated-SGT; (k) Iterated-Simulation; and (l) Iterated-EVT, against GJR-FHS VaR estimates. The red line represents the 45ž line. [Colour figure can be viewed at wileyonlinelibrary.com]

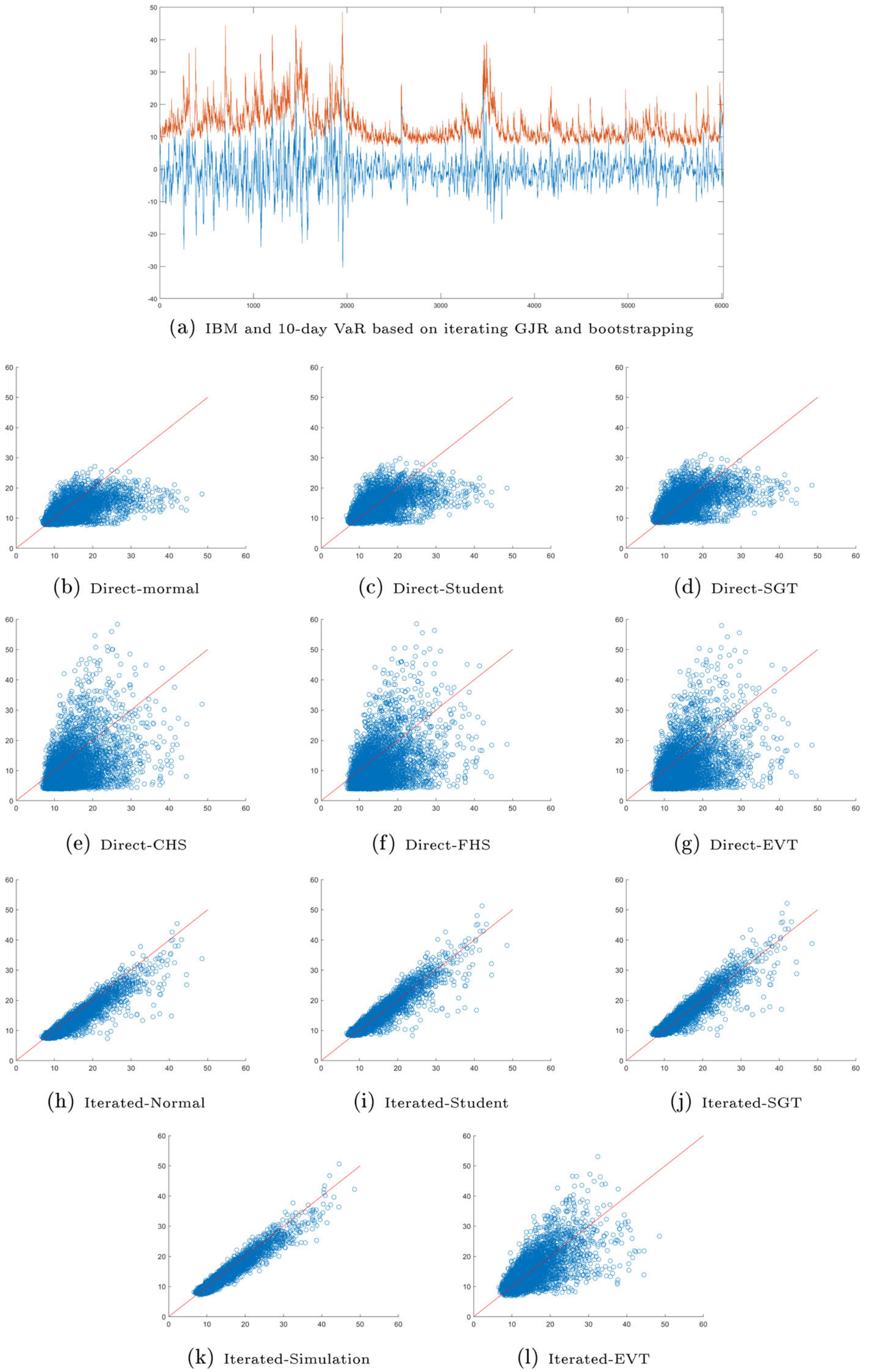


FIGURE 3 (a) IBM 10-period returns (blue) together with $VaR_t^{(10)}$ estimates (red) obtained iterating the GJR model with bootstrap errors (FHS). Scatter plots of $VaR_t^{(10)}$ estimates obtained by (b) Direct-Normal; (c) Direct-Student; (d) Direct-SGT; (e) Direct-CHS; (f) Direct-FHS; (g) Direct-EVT; (h) Iterated-Normal; (i) Iterated-Student; (j) Iterated-SGT; (k) Iterated-Simulation; and (l) Iterated-EVT, against GJR-FHS VaR estimates. The red line represents the 45ž line. [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 10 Rows K and C report p-values of Kupiec (1995) and Christoffersen (1998) backtesting tests, respectively. PCLF row reports $1000 \times C(m)$, obtained if the estimates of multiperiod VaR are not rejected by both K and C tests.

	DIRECT						ITERATED					
	N	**S**	**Sk**	**CHS**	**FHS**	**EVT**	**N**	**S**	**Sk**	**SIM**	**FHS**	**EVT**
S&P 500												
K	0.00	0.00	0.06	0.00	0.00	0.00	0.07	0.07	0.07	0.13	0.76	0.74
C	0.00	0.00	0.00	0.00	0.00	0.00	0.19	0.19	0.19	0.32	0.92	0.32
PQLF	-	-	-	-	-	-	-	-	-	0.40	0.61	0.88
Dollar/Euro												
K	0.76	0.74	0.28	0.16	0.08	0.00	0.28	0.08	0.08	0.28	0.08	0.08
C	0.06	0.61	0.55	0.00	0.22	0.00	0.55	0.22	0.22	0.55	0.22	0.22
PQLF	-	0.30	0.31	-	-	-	0.27	-	-	0.26	-	-
IBM												
K	0.00	0.00	0.00	0.00	0.00	0.00	0.74	0.08	0.08	0.28	0.28	0.76
C	0.00	0.00	0.00	0.00	0.00	0.00	0.93	0.22	0.22	0.55	0.55	0.87
PQLF	-	-	-	-	-	-	0.78	-	-	0.73	0.76	1.14

Finally, we consider the results for IBM $VaR_t^{(10)}$, which are reported in Figure 3. We can observe that the multiperiod VaR estimates obtained using the direct approaches are much smaller than those obtained by iterating the GJR model and estimating the quantile by FHS. This is the case mainly when the quantile is obtained by assuming a particular distribution for multiperiod standardized returns; see the simulation results reported in Table 3 for returns generated by a GJR model with Student errors and $T = 5000$. In Figure 3, we can also observe mild negative biases when using the iterated approaches assuming a particular distribution for 10-day standardized returns. The dispersion of the $VaR_t^{(10)}$ predictions obtained by iterating the GJR and using EVT to estimate the quantile is larger while the predictions obtained by simulation assuming normality are very similar to those obtained by bootstrapping.

To assess the adequacy of the $VaR_t^{(10)}$ estimates plotted in Figures 1–3, we backtest the in-sample daily one-step-ahead forecasts of $VaR_t^{(10)}$ using the popular tests by Kupiec (1995) and Christoffersen (1998); see Degiannakis and Potamia (2017), Kole et al. (2017), Le (2020), and Mancini and Trojani (2011), who also test for their iterated estimates of $VaR_t^{(10)}$ using these tests and Barendsen, Kole and van Dijk (in press), who show that the effect of parameter estimation on the size of these tests is somehow minor. Table 10, which reports the p-values of both tests, confirms the Monte Carlo results about the adequacy of iterated methods to compute one-step-ahead forecasts of 1% $VaR_t^{(10)}$. The one-step-ahead forecasts of $VaR_t^{(10)}$ of S&P500 and IBM returns are rejected for a 10% significance level by at least one of the Kupiec (1995) or the Christoffersen (1998) tests when they are obtained by the direct approach. In the case of S&P500 returns, characterized by asymmetric volatilities and an asymmetric conditional distribution, models based on iterating the GJR model with the quantile estimated simulating, bootstrap or EVT are adequate to forecast the $VaR_t^{(10)}$; see also the empirical results in Le (2020) about the adequacy of iterating the GJR model with either FHS or EVT when estimating the 1% 10-day VaR in the context of a very large number of series of returns. The results for IBM are similar, although iterating the GARCH model with normal 10-day standardized returns is also valid. Looking at the results for the Dollar/Euro returns, we can observe that they are also in concordance with the results obtained from the Monte Carlo experiments when both the volatilities and the conditional distribution are symmetric. In this case, the direct approach based on the GJR model with normal errors or the iterated approach based on simulating the GJR model with SIM are not rejected.

It is important to note that multiperiod VaRs forecasts obtained by iterating the GJR by simulation with the errors assumed to be normal are never rejected.

Finally, the estimates of multiperiod VaR not rejected by both of the two backtesting tests above are compared following Lopez (1999) by selecting the procedure that minimizes the loss function $C(m) = \sum_{t=1}^{T} C_t^{(m)}$, where the index m stands for each of the models considered for forecasting $VaR_t^{(10)}$. In particular, we chose the consistent predictive quantile loss function (PQLF), implemented by, for example, Bao et al. (2006), Giacomini and Komunjer (2005), Kole et al. (2017), and Le (2020). PQLF is given by

$$C_t^{(m)} = [0.01 - I(R_t < VaR_t)][R_t - VaR_t]. \tag{29}$$

Note that because of the asymmetry of the PQLF, VaR violations lead to a larger loss. Table 10, which reports the values of $C(m)$, shows that iterating the GJR model by simulation assuming normal errors (SIM) is nearly always the procedure with smallest losses.

According to the two-stage backtesting procedure described above, a risk manager can select a simple direct procedure to compute the multistep $VaR_t^{(10)}$ when the 1-day returns have symmetric volatilities or volatilities with a very mild leverage effect. In this case, the model favored by the QLF loss function is the GJR model with Student errors. However, when 1-day returns have asymmetric volatilities, the results clearly favor $VaR_t^{(10)}$ forecasts based on iterated procedures. In particular, when dealing with S&P500 and IBM returns, the iterated GJR model with SIM errors lead to $VaR_t^{(10)}$ forecasts with minimum PQLF.

7 | CONCLUSIONS

Multiperiod VaR is a key element of the Basel capital regulations. In this paper, we carry out a comprehensive survey of popular direct and iterated procedures to obtain multiperiod VaR forecasts. The scarce literature comparing both alternative methodologies clearly favors iterated procedures as compared with either SRoT or direct procedures. This is in contrast with the main results found in the context of linear time series models, which tend to favor direct approaches. This apparent contradiction could be due to the loss of information involved in the direct approaches, which is harmless in the context of linear models but could be pernicious when dealing with nonlinearities.

Based on some Monte Carlo experiments, we show that, when computing the 1% 10-day VaR, differences between direct and iterated approaches can be large when volatilities are asymmetric and/or the conditional distribution of returns is non-normal. In these cases, the direct approach generates, in general, forecasts of $VaR_t^{[10]}$ with large negative biases, with dangerous potential damages for financial companies. Underestimation of the own level of risk may lead to insufficient amount of capital reserves to cover potential losses, thus increasing the risk of bankruptcy. Multiperiod VaR estimates based on iterating the GJR model with one-period standardized returns assumed to be normal or with their quantile estimated by bootstrapping are generally unbiased and have smaller RMSFEs than those of alternative approaches.

The conclusions in this paper are important for risk managers. It is always safer to obtain multistep VaR forecasts using iterated procedures based on simulating the asymmetric model for volatilities with either normal or bootstrapped errors. Never use the direct approach if you observe asymmetric volatilities.

Finally, it is important to note that the Basel 3 regulations allow 10-day VaR forecasts to be calculated with overlapping returns. Research in this area has also been very scarce; see Frankland et al. (2019) for a discussion of the issues caused by using serially dependent overlapping returns in the context of estimation of marginal cumulants and statistical testing. However, as far as we are concerned, only Sun et al. (2009) consider the effects on VaR estimation of using overlapping returns, concluding that VaR is underestimated in this case. It could be interesting to investigate whether explicitly modeling the dependencies of overlapping returns can help recreasing these biases; see, for example, Richardson and Smith (1991), Taylor and Fang (2018), and Wong (2020) for applications to testing based on overlapping returns with corrected dependencies and Giannopoulos (2003) for an application to multistep VaR forecasting. Recently, Hedegaard and Hodrick (2016) have proposed a GMM methodology to estimate GARCH-M models using all the high-frequency data while maintaining the low frequency forecasting period. Consequently, it could be of practical interest further research on direct approaches based on overlapping observations.

ACKNOWLEDGMENTS

Financial Support from the Spanish Government contract Grant PDI2019-108079GB-C21/AIE/10.13039/501100011033 (MINECO/FEDER) is gratefully acknowledged by the first author. We are also very grateful to two anonymous referees, for very constructive comments. Any remaining errors are obviously our responsibility.

Funding for APC: Universidad Carlos III de Madrid (Read & Publish Agreement CRUE-CSIC 2022)

DATA AVAILABILITY STATEMENT

The authors confirm that the link to the data supporting the findings of this study is available within the article.

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ENDNOTES

[1] Baek (2019) proposes a modification of the direct forecast procedure consisting in imposing a smoothing parameter on the first differences of parameters across horizons. This modification outperforms the conventional iterated and direct approaches.

[2] Some authors have also analyzed the performance of multiperiod VaR estimates assuming long-memory volatility by fitting FIGARCH models. However, we do not pursue this avenue given that there is no evidence that the FIGARCH model improves VaR estimates over the short-memory GARCH model; see, for example, Beltratti and Morana (1999), Wu and Shieh (2007) and Degiannakis et al. (2013).

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How to cite this article: Ruiz, E., & Nieto, M. R. (2023). Direct versus iterated multiperiod Value-at-Risk forecasts. *Journal of Economic Surveys*, *37,*915–949. https://doi.org/10.1111/joes.12522