Essays on Firm Growth and Financing

by

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A meus pais e a miña irmá.

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Abstract

The unifying theme of this dissertation is the behaviour of firms over their life cycle, namely regarding their financing and growth, and its macroeconomic implications. My thesis starts from a general question: how should we allocate financial resources to young firms over their first years of activity? On one side, young firms are often small, which may prevent them from accessing financial markets. On the other side, in some sectors, there is high uncertainty regarding the underlying quality of young firms. These issues might deter productive firms from growing over life. Nevertheless, those firms are worth financing from an aggregate perspective. Exploring the conditions under which productive young firms can flourish is a first-order issue to promote a well-functioning productive sector and a good economy-wide performance.

Chapter 1 of this dissertation uses micro data on Spanish companies to study the life-cycle behaviour of firm productivity and factor inputs. I find evidence that some firms are expected to grow strongly when young. Moreover, factor input data points to firm-level frictions affecting resource allocation to firms over life.

Motivated by empirical findings in Chapter 1, I build a firm-dynamics model in Chapter 2 to study whether firms with a high growth potential at birth manage to realise high growth, or whether they are deterred by the existence of firm-level frictions, namely borrowing constraints. This chapter addresses the firm-level effects of borrowing constraints, as well as their aggregate impact on the Spanish economy.

Chapter 3 studies how young, innovative firms make investment, liquidation and sale decisions in a context of high uncertainty. I show that a model where firms learn about their uncertain quality captures documented patterns in the venture capital literature, such as delayed exits and contingent stage financing.

Chapter 1. A Life-Cycle Study of Productivity and Factor Allocation of Spanish Firms. The Spanish economy has recently experienced a poor aggregate performance accompanied by the prevalence of low-growth, small firms. This chapter relates these facts to the phenomenon of high-growth-potential firms. I use a rich micro-level dataset containing financial information on companies, and I perform a study of the evolution of firm-level productivity, employment and capital over the life cycle. First, I find that Spanish firms are heterogeneous in their expected growth

rates. Second, I find that dynamic life-cycle moments in the data are informative about the allocation of factor inputs to companies, and thus about frictions affecting young firms.

Chapter 2. Factor Misallocation and High-Growth Firms in Spain. Motivated by empirical patterns on the life-cycle evolution of productivity and input allocations to firms, I study whether young firms are deterred from growing by borrowing constraints, and its macroeconomic implications. I develop a firm-dynamics general equilibrium model considering a firm-level productivity process and frictions. I calibrate two alternative models, with and without expected-growth-rate heterogeneity, to match data on input life-cycle allocations. In the model with heterogeneous expected growth, high-growth-potential firms are prevented from growing by financial frictions, and eliminating borrowing constraints generates large aggregate gains. In the model without this source of heterogeneity, there are less high-growing firms and aggregate effects from removing financial frictions are smaller.

Chapter 3. Venture Capital Investments and Learning over the Life Cycle. The life cycle of young, high-risk entrepreneurial projects and the financing of innovation has become increasingly important for economists, companies and policymakers. The objective of this chapter is to understand how high-risk firms learn over time about their unknown quality, and how this affects firm decisions and financing. I develop a model of the firm that imitates realistic features of young, high-risk companies, such as uncertainty about a firm's own quality, staged financing, exit strategies, and the realisation of period cash-flows that yield information about the firm's unobserved quality. The model captures empirical patterns of innovative firms documented in the venture capital literature – namely, delayed exit decisions and investments into companies being contingent on firm-level results over their life. I find that the ability to learn makes investment sensitive to period cash-flows in the model. A high initial quality uncertainty reflects into exit and investment strategies and may motivate firms to perform growth investments. In this context, a higher learning ability increases firm value substantially by motivating experimentation and contingent staged financing.

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Chapter 1

A Life-Cycle Study of Productivity and Factor Allocation of Spanish Firms

1.1 Introduction

It has been long argued that firm size heterogeneity is a key determinant of total factor productivity (TFP) in the overall economy (Hsieh and Klenow, 2009; Restuccia and Rogerson, 2008). Large firms are often more productive than their smaller peers and account for a large share of aggregate employment. Across countries, there exist big differences in terms of the firm size distribution even among similarly developed economies. If we look at the Spanish economy in recent years, many studies highlight a relative lack of large firms (Rubini et al., 2012) as well as a low employment growth over the firm life cycle (Hsieh and Klenow, 2014) when compared to other developed countries. On the other hand, the misallocation literature for Spain reports a bad Spanish TFP performance in recent decades, and links it to a poor allocation of resources, namely capital[1], across heterogeneous firms, which makes it difficult for most productive firms to become large (Gopinath et al., 2017; Fu and Moral-Benito, 2018; García-Santana et al., 2020).

In this chapter, motivated by the documented firm growth and aggregate TFP patterns for Spain, I study firm-life-cycle performance through the lens of the phe-

[1]The misallocation literature for Spain has often considered financial frictions as the one to blame for the poor TFP performance of the Spanish economy in recent decades. For instance, Gopinath et al. (2017) show that standard deviation of the logged marginal revenue product of capital (MRPK), a standard measure of capital misallocation, has been steadily increasing during the 2000s. They rationalise this pattern by considering a size-dependent collateral constraint at the firm level and an exogenous drop in the real interest rate. Importantly, they do not observe such a strong increase in the standard deviation of the logged marginal revenue product of labour (MRPL).

nomenon of *high-growth-potential firms* (Birch and Medoff, 1994; Haltiwanger et al., 2017; Pugsley et al., 2021). A high-growth-potential firm is a firm that is expected to exhibit very high growth rates during the first periods of its life. While reported as important for the United States (Pugsley et al., 2021), it is unclear whether the high-growth-potential phenomenon could be taking place in Spain, often depicted by the literature as a frictional environment that is particularly unsuitable for young, small firms to grow.

The objective of this chapter is to study empirically firm growth and factor hindering it in the Spanish economy. To do this, I use micro-level data of Spanish firms from *Central de Balances Integrada* (CBI) at the Bank of Spain[2]. I construct a panel of Spanish firms that contains information on financial variables and small companies. The data is particularly well-suited for studying the nature of firm growth and distortions affecting young firms in Spain. In particular, financial information allows to pin down a measure of firm-level productivity. I document firm-life-cycle facts that suggest that there is a lack of firms effectively growing in Spain. Using firm-level dynamic moments, I arrive at two empirical results. First, building on Pugsley et al. (2021)'s empirical framework, I find that there is heterogeneity in expected growth rates across Spanish firms. This margin of heterogeneity is not often considered in the firm dynamics literature. Second, dynamic moments at the firm level are informative about the existence of frictions affecting young firms in Spain. Specifically, autocorrelations of employment and capital by age suggest that firms have difficulties to adjust their inputs over the life cycle.

Literature review. This chapter relates to the existing literature in the following ways. First, it relates to the firm dynamics literature (Pugsley et al., 2021; Haltiwanger et al., 2013; Hsieh and Klenow, 2014). The main reference of this chapter is Pugsley et al. (2021). In their work, they use US administrative employment data to estimate a rich firm-level employment process with an ex-ante, life-cycle component that accounts for firm heterogeneity in expected growth rates at birth, thus giving room to the high-growth-potential firms phenomenon. This ex-ante component allows for firm-specific initial conditions at birth and firm-specific steady states after some years of life, and parameters characterising it are identified using long-run life-cycle data. They find that this component is relevant for representing firm heterogeneity in the US. In this chapter, I apply Pugsley et al. (2021)'s methodology to TFP life-cycle information for Spanish firms. Moreover, taking advantage of the CBI data, which contains not only employment data but also financial information, I assess the life-cycle allocation of employment and capital to firms, which is my main contribution. This novel view of the allocation of inputs to firms, which studies dynamic moments over the firm life cycle, is useful for identifying firm-level frictions.

[2]The source of the data is BELab. Banco de España/CORPME, Colegio de Registradores de la Propiedad y Mercantiles de España. CBI. DOI:10.48719/BELab.CBC1121_01.

Regarding documented empirical facts for Spanish firms, Rubini et al. (2012) and Hsieh and Klenow (2014) document a small share of large firms and a low life-cycle growth of manufacturing Spanish firms. I contribute to this strand of the literature by showing that these patterns also extend to all sectors in the Spanish economy, and I take advantage of the rich data to measure firm-level TFP.

Second, this chapter contributes to the literature on firm-level distortions (Cooper and Ejarque, 2003; Cooper and Haltiwanger, 2006; Cooper and Willis, 2009; Asker et al., 2014; Baley and Blanco, 2022). My contribution to this literature is the study of empirical autocorrelations of TFP, employment and capital over the firm life cycle, which can be used as a piece of evidence to identify borrowing constraints at the firm level. These moments might also be useful to identify other distortions regarded as important in the literature, e.g. capital adjustment costs, investment irreversibilities, and labour market imperfections.

Layout. This chapter is structured as follows: in section 1.2, the data for Spanish firms is presented. In section 1.3, I estimate a statistical firm-level process with ex-ante components that reflect growth-potential heterogeneity using a measure of firm-level TFP, and I discuss empirical firm-life-cycle moments. In section 1.4, I conclude.

1.2 Data

1.2.1 Description of the Dataset

I use micro-level panel data for Spanish firms. The dataset is constructed from Bank of Spain's *Central de Balances Integrada* (CBI) database, which contains yearly data at the firm level. The data broadly covers the production side of the Spanish economy, thus being representative of the population of Spanish companies. The information in CBI ultimately comes from financial statements (annual accounts) that all Spanish companies have to submit by law to the Commercial Registry (*Registro Mercantil*), as well as voluntary cooperation of a subset of these firms with the Bank of Spain. Importantly, the data does not restrict to employment data, but it also contains financial information such as value added, assets and liabilities, among others. Taking advantage of these features of the Spanish data, I pin down a measure of firm-level TFP. The data contains information on corporations, including limited liability companies (both S.A. and S.L. companies), and cooperatives. The data does not include self-employed workers. Nonetheless, the data is not restricted to listed firms. Indeed, a vast number of observations in the dataset I construct belong to unquoted firms, thus being more representative of the Spanish economy and allowing to investigate more deeply small and young firms.

The variables that are important to the analysis are firm identification number, period, year of legal creation/birth, industry identifiers, gross value added, book value of long-term assets, and yearly average number of employees, among others. My dataset covers the period 1997-2016. In my analysis, I consider all the main economic sectors (from A to S, according to the 1-digit NACE classification) in Spain, so my dataset talks to a wide range of activities including large sectors as manufacturing, construction, commerce, hotel industry, but also agriculture and financial services. I exclude government firms from the analysis. I use gross value added as the measure of nominal output at the firm level, and the book value of long-term assets as the measure of nominal capital. I deflate the book value of long-term assets using industry investment deflators, which are taken from EU-KLEMS. The measure of employment I consider throughout the chapter is the yearly average number of employees at the firm level[3]. Firm age is defined as the difference between the current observation period and the year of legal birth[4]. I consider that a firm exits the market in the last period it has reported its financial information.

Using the CBI, I adopt a life-cycle perspective and I construct a dataset considering firms that were born between 1997 and 2006, both years included, so that I have enough observations of firms during their life. Apart from selecting these specific cohorts, I select firms for which information at age 0 is available, so that it is possible to track them over their life cycle starting from the moment of (legal) birth. This dataset is named *the permanent dataset* (224,148 observations, 18,065 firms), for it only considers firms that have not exited the market before age 10, thus containing information about firms that stay operating after or at age 10. Throughout the chapter, I discuss empirical features and estimation results for this permanent dataset. These empirical patterns and results are robust to considering a *non-permanent dataset* (322,278 observations, 47,430 firms) which covers all firms meeting the cohort and age-0 information requirements regardless of the age at which they exit, thus containing information about both stayers and exiters[5]; see Appendix A.1 for the definition of the non-permanent dataset and A.2 for the robustness check. Details and discussion on data cleaning and the construction of

[3]Using the wage bill instead of the yearly average number of employees does not change substantially the empirical results of this chapter.

[4]In an unreported exercise, I have also considered an alternative definition of age, such that a firm is born (has age 0) in the first in the first firm-level observation with a yearly average number of employees greater or equal to one. This definition is closer to the age notion in Pugsley et al. (2021). Under this alternative definition of age, empirical patterns regarding low firm growth over the life cycle and life-cycle autocorrelation structures remain qualitatively similar to those presented in section 1.3.

[5]Previously to selecting specific cohorts, I have also constructed an *aggregate dataset* aiming at being representative of the universe of Spanish firms without looking at specific birth periods. In an unreported exercise, I use this dataset to analyse capital misallocation facts and total factor productivity from an economy-wide perspective over the considered period 1997-2016. The aggregate data displays similar misallocation patterns as those previously documented in the literature for Spain, which have been mentioned in footnote 1 (see Gopinath et al. (2017)).

the datasets are provided in the Appendix.

TFP measure. The data enables us to go deep into the study of the determinants of firm growth in Spain, for it includes financial information. This feature of the data allows for pinning down a measure of firm-level TFP. To do so, I impose some structure. Following Hsieh and Klenow (2009), Gopinath et al. (2017) and Restuccia and Rogerson (2017), I assume a Cobb-Douglas specification for the firm-level production function: $Y_{it} = A_{it}K_{it}^{\alpha}L_{it}^{1-\alpha}$, where i and t index a firm and an age in the data, respectively. Here, α is assumed to be homogeneous across firms and equal to 0.35. I assume that each firm produces a variety i for which it has a monopoly, and faces an isoelastic demand of the form $Y_{it} = D(P_{it}) = P_{it}^{-\sigma}$, where σ is the elasticity of substitution between different varieties. I impose $\sigma = 3$, homogeneous across firms. Given these assumptions, I use the permanent sample of Spanish firms to pin down a measure of firm-level TFP[6]:

$$TFP_{it} = \frac{(P_{it}Y_{it})^{\frac{\sigma}{\sigma-1}}}{K_{it}^{\alpha}L_{it}^{1-\alpha}} \tag{1.2.1}$$

where $P_{it}Y_{it}$ is value added, L_{it} is the yearly average number of employees, and K_{it} is the deflated book value of fixed assets.

1.3 Empirical Analysis

In order to investigate whether firm-level and aggregate patterns in the Spanish economy are linked to a low effective growth of high-growth-potential firms in the economy, I perform an empirical study of nature of firm growth and frictions in Spain. I start by estimating a rich firm-level statistical process for productivity. Then, I document evidence on firm life-cycle moments, namely autocorrelations of production factors over the life cycle. I arrive at two empirical findings. First, Spanish firms are heterogeneous in terms of their expected productivity growth rates. Second, there is evidence on firm-level frictions affecting the allocation of resources over the firm life cycle.

1.3.1 A Process for Firm-Level Productivity

Firm-level process. The process for logged firm-level productivity I consider is a simplified version of the one in Pugsley et al. (2021). I call it the GP process (for

[6]This measure includes both physical productivity in the production function and quality of the variety embedded in the demand function. In Hsieh and Klenow (2009), this measure is called "physical TFP", or TFPQ.

growth potentials). Here, i indexes firm and a indexes age. I assume the following statistical process for logged productivity[7]:

$$log(A_{ia}) = \underbrace{u_{ia}}_{\text{ex-ante}} + \underbrace{w_{ia}}_{\text{ex-post}} \tag{1.3.1}$$

where

$$\begin{aligned} u_{ia} &= \rho_u u_{i,a-1} + \theta_i, \qquad && u_{i,-1} \sim N(\mu_u, \sigma_u^2),\ \theta_i \sim N(\mu_\theta, \sigma_\theta^2) \\ w_{ia} &= \mu_w + \rho_w w_{i,a-1} + \varepsilon_{ia}, \qquad && w_{i,-1} = 0,\ \varepsilon_{ia} \sim N(0, \sigma_\varepsilon^2) \end{aligned}$$

and $|\rho_u|$, $|\rho_w| < 1$. In equation (1.3.1), $log(A_{ia})$ is assumed to be the sum of an *ex-ante* deterministic component, u_{ia}, and an *ex-post* random component resulting from shocks that realise as firms age, w_{ia}. The ex-post, shock component of the process is a standard AR(1) process. It is the ex-ante component that is the focus of our attention. The u_{ia} term represents the expected productivity growth rate of the firm at birth. According to the law of motion of u_{ia}, a firm i starts out from an initial condition $u_{i,-1}$ and it converges to a firm-specific long-run steady state $u_{i,\infty} = \frac{\theta_i}{1-\rho_u}$ as it ages. Therefore, the u component of the process captures the idea of the *growth potential* of firm i. The cross-sectional variance of long run steady states across firms is $Var\left(\frac{\theta_i}{1-\rho_u}\right) = \frac{\sigma_\theta^2}{(1-\rho_u)^2}$. In Appendix A.3, I estimate the GP process using Spanish employment data.

Although this process for $log(A_{ia})$ is relatively simple[8], it is richer than processes typically considered in the firm dynamics literature. To understand why we need a richer process to analyse the role of high-growth-potential firms, let us compare equation (1.3.1) to such often used processes. First, the standard AR(1) process $log(A_{ia}) = w_{ia} = \mu_w + \rho_w w_{i,a-1} + \varepsilon_{ia}$ does not allow for any source of heterogeneity in expected growth rates at birth and in long-run productivity steady states. Second, in a process with ex-ante heterogeneity in firm-level fixed effects, $log(A_{ia}) = \theta_i + w_{ia}$ (assuming $\rho_u = \mu_u = \sigma_u = u_{i,-1} = 0$ for all i), firms immediately converge to their long-run deterministic steady states θ_i, thus not allowing for a firm-level transition capable of generating richer growth paths that could differ across firms. Relative

[7] While the GP process has an $u + w$ structure, the process in Pugsley et al. (2021) has an $u + v + w + z$ shape. This extended process adds an extra ex-post component (an *iid* shock z) and an extra ex-ante component aiming to capture differences in the speed of convergence to firm-level steady states (a deterministic term v) to my specification. Main results from this section, namely the share of u in terms of the variance and the match of autocorrelation structures, are robust to considering that extended version of my process.

[8] The reason for considering here an $u + w$ process, simpler than the $u + v + w + z$ process in Pugsley et al. (2021) (see footnote 7), is showing that just adding the ex-ante component u to an AR(1) process is already a huge step forward in terms of matching the empirical autocorrelation structure by age of firm-level TFP, as discussed in section 1.3.1, so that it is the u component what really makes a difference. Moreover, the GP process can be plugged into a structural model without increasing the computation burden so much.

to these more traditional processes, the GP process does a better job in terms of matching autocorrelations by age of logged TFP in the data, as discussed below.

Estimation and identification. The parameter vector $\vartheta = (\rho_u, \rho_w, \sigma_\theta, \sigma_u, \sigma_\varepsilon)$ characterising the process for logged productivity is estimated from the data via a minimal distance procedure, as in Chamberlain (1984). Previous to estimation, I take out industry and cohort fixed effects of the firm-level data in logarithms. I estimate the process in order to match the empirical autocovariance of logged TFP by age. In particular, I minimize a sum of squared deviations between the theoretical $Cov(log(A_a), log(A_{a-j}))$, for any two age levels a and $a-j$, and its empirical counterpart $\widehat{Cov}(log(TFP_a), log(TFP_{a-j}))$ documented in the data. As discussed in Pugsley et al. (2021), this empirical object is used in order to identify the key parameters of the process[9], for it contains information on ex-ante and ex-post components. More specifically, the theoretical autocovariance function $Cov(log(A_a), log(A_{a-j}))$ can be written as the sum of a term that depends only on parameters characterising u (that is, ρ_u, σ_u and σ_θ) and a term that depends only on parameters characterising w (i.e. ρ_w and σ_ε):

$$Cov\big(log(A_a), log(A_{a-j})\big) = \underbrace{\Big(\sum_{k=0}^{a}\rho_u^k\Big)\Big(\sum_{k=0}^{a-j}\rho_u^k\Big)\sigma_\theta^2 + \rho_u^{2(a+1)-j}\sigma_u^2}_{\text{ex-ante term}} + \underbrace{\sigma_\varepsilon^2\rho_w^j\sum_{k=0}^{a-j}\rho_w^{2k}}_{\text{ex-post term}} \tag{1.3.2}$$

Specifically, the parameters governing the u component, and thus long-run steady states and their dispersion, ρ_u and σ_θ, are identified from the long-run autocovariance. Let us consider $h = a - j$. If we take h fixed and let a go to infinity, so that we are considering an autocovariance corresponding to a infinitely long lag j, then:

$$lim_{a\to\infty}Cov\big(log(A_a), log(A_h)\big) = \frac{1-\rho_u^{h+1}}{(1-\rho_u)^2}\sigma_\theta^2 \tag{1.3.3}$$

which equals zero if $\sigma_\theta^2 = 0$, i.e. there is no heterogeneity in long-run steady states. Put simply, the empirical long-run autocovariance by age enables us to identify persistence and dispersion parameters characterising the u component in equation (1.3.1). The idea of identifying ex-ante components using long-run ex-post data goes back to Guvenen (2007). He uses long-run life-cycle consumption data to identify ex-ante components of the income process that evolve slowly over the consumer life cycle. Here, I use long-run autocovariances to identify ex-ante components that develop over time (through parameter ρ_u).

[9]Mean parameters μ_u, μ_θ and μ_w are not identified from the autocovariance function. Identification of mean parameters requires the utilisation of additional moments. However, mean parameters are not necessary for determining the ex-ante and ex-post components of the cross-sectional variance of $log(A)$ by age, nor the autocorrelation of $log(A)$ by age, which I discuss in this section.

Table 1.1 Estimated process for logged productivity, Spain

Estimated process	ρ_u	ρ_w	σ_θ	σ_u	σ_ε	RMSE
GP process	0.208	0.782	0.406	3.864	0.411	0.038
	(0.0085)	(0.0055)	(0.0059)	(0.1583)	(0.0021)	
AR(1) process	-	0.854	-	-	0.489	0.148
		(0.0020)			(0.0021)	
AR(1) + fixed effect process	-	0.663	0.545	-	0.484	0.074
		(0.0059)	(0.0042)		(0.0019)	

Notes: the first row shows parameter estimates for the GP process for logged productivity $log(A_{ia}) = u_{ia} + w_{ia}$. The second and third rows show nested special cases of the baseline process, namely a plain AR(1) process (that is, $log(A_{ia}) = w_{ia}$), and an AR(1) process where firms draw a fixed effect θ when born (that is, $log(A_{ia}) = w_{ia} + \theta_i$). The three estimations have been performed using autocovariances of logged firm-level TFP by age calculated from the permanent sample. The standard errors of parameter estimates are in parentheses, and are small due to the very large sample size. Standard errors are calculated using a parametric bootstrap procedure with 1000 replications.

Estimated process. Estimation results are shown in the first row of Table 1.1. The estimated value for σ_θ indicates that firms are not excessively diverse in terms of their long-run steady states. This in turn relates to the empirical low dispersion of logged total factor productivity of Spanish firms (see Figure 1.4). Indeed, the variance of long-run productivity steady states is 0.263. Importantly, the share of the empirical variance of logged TFP at age 10 that is accounted by the u component is 32.14%, which is not negligible and indicates that Spanish firms display heterogeneity in their expected productivity growth rates. The estimate of ρ_u is large relative to the estimated value in Pugsley et al. (2021), and the large estimate of σ_u denotes large heterogeneity in productivity initial conditions when firms are born. Overall, the estimated GP process provides support to the idea that life-cycle, firm-specific components are important to determine firm heterogeneity in Spain.

Model fit and alternative processes. In order to reach a conclusion on the economic relevance of expected productivity growth rate heterogeneity, we need first to understand what we gain in terms of capturing micro-level moments when we depart from the standard AR(1) process in the firm dynamics literature by considering this margin of firm heterogeneity. To do so, I estimate nested productivity processes (shown in the second and third rows of Table 1.1) often seen in the literature. These are subcases of the GP process. I argue that just adding an ex-ante component u that embeds the idea that firms evolve from an initial condition at birth towards a log-run steady state (both firm-specific) improves the way we capture a key moment of the data: the autocorrelation of logged TFP by age[10], which represents the way

[10]We arrive to the same conclusions if we estimate the same processes using employment data.

this variable correlates to itself over the life cycle. This object is calculated from the standard deviation by age and the autocovariance by age, and thus contains similar information as the later.

Figure 1.1(a) shows the empirical autocorrelation of logged total factor productivity by age for Spanish firms. In the x axis, I represent age, and each of the black lines corresponds to another fixed age: the lowest line represents age 0, the next line represents age 1, and so on. Every point corresponds to an age pair. In the y axis, I represent the autocorrelation of logged TFP. The graph tells us, for every age pair, how logged TFP correlates to itself. In Figure 1.1(a), lines are decreasing as we increase age in the x axis, thus showing a life-cycle effect: if we consider longer age lags, the variable displays lower autocorrelation. However, lines eventually converge to a positive number as the age in the x axis keeps on increasing. This feature that autocorrelation does not fade away as we keep on increasing the age lag is directly related to the identification of ex-ante components and the discussion around equation (1.3.3): should there not be any ex-ante heterogeneity across firms, these long-run autocorrelations should eventually converge to zero.

Let us discuss to which extent different processes can capture these features of the empirical autocorrelation structure. In Figure 1.1(b), I show simulated autocorrelations from the estimation of a plain AR(1) process for logged productivity, $log(A_{ia}) = w_{ia}$, in the second row of Table 1.1. I compare it to the empirical autocorrelation pattern of logged TFP. The empirical feature that autocorrelations are decreasing as we increase the age lag is well represented by the AR(1) process. Nonetheless, this process is not capable of capturing the long-run convergence to a positive number that is present in the data[11]. In Figure 1.1(c), I consider an alternative setting that allows for a source of firm ex-ante heterogeneity: an AR(1) process with a heterogeneous fixed effect, $log(A_{ia}) = w_{ia} + \theta_i$, whose estimation results are shown in the third row of Table 1.1. Relative to the GP process, this process imposes $\rho_u = \mu_u = \sigma_u = 0$ and $u_{i,-1} = 0$ for every firm. As we can see, the estimation fit improves in terms of mean squared error relative to the simple AR(1). In addition, the process with a fixed effect generates positive long-run autocorrelations. However, the fixed-effect shape of the ex-ante heterogeneity generates too much persistence in life-cycle productivity, thus generating long-run autocorrelations that are too high relative to the empirical ones, when considering young ages.

In Figure 1.1(d), I show how the GP process enables us to better replicate the two features in the autocorrelation data. While it preserves the decreasing property of the curves, it allows to better replicate autocorrelations when age lags are large. The fact that we allow for firms to be heterogeneous not only in their initial conditions but also in their long-run steady states, to which they converge in time,

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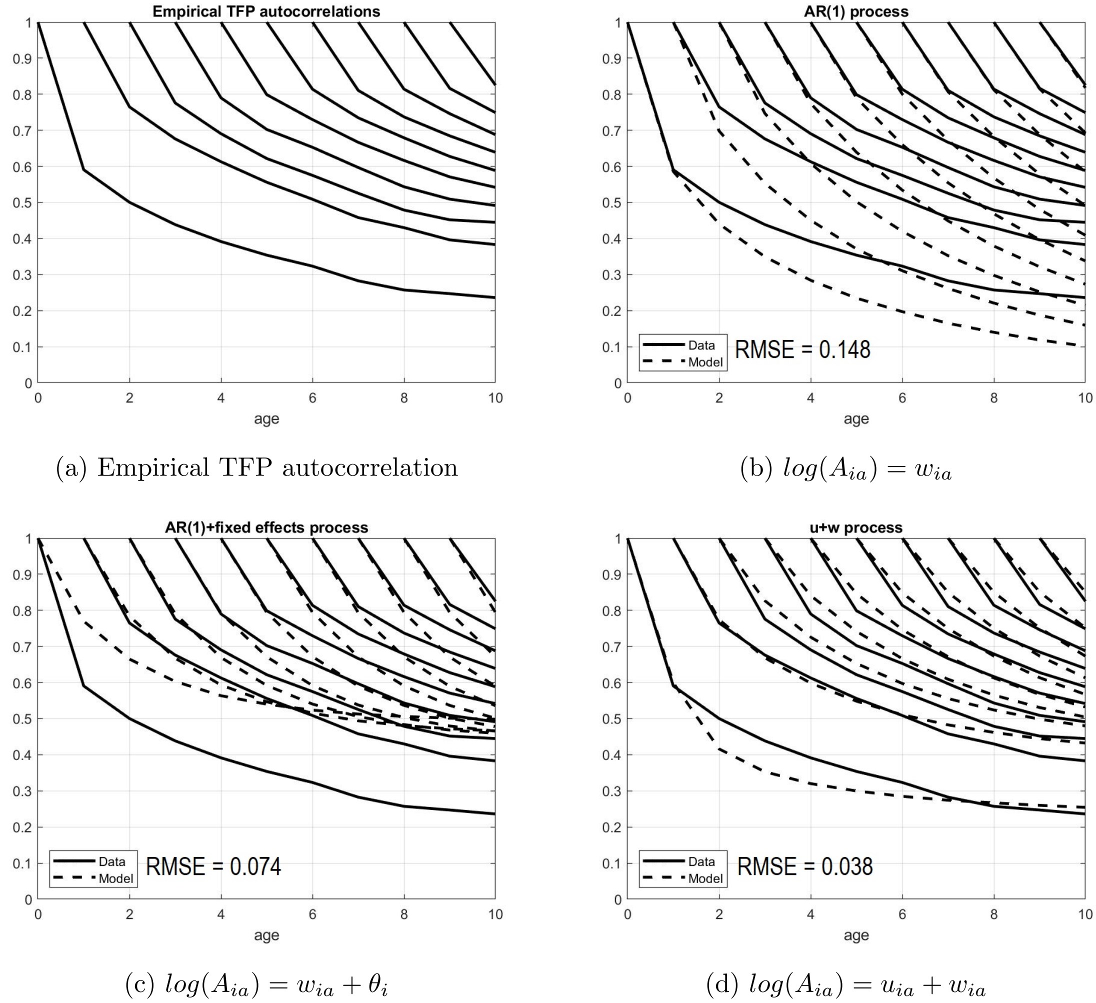
See Appendix A.3.

[11]Indeed, if we consider more than 10 periods, the autocorrelations implied by the AR(1) process would eventually converge to zero.

Figure 1.1 Productivity processes and TFP autocorrelations by age



(a) Empirical TFP autocorrelation

(b) $log(A_{ia}) = w_{ia}$

(c) $log(A_{ia}) = w_{ia} + \theta_i$

(d) $log(A_{ia}) = u_{ia} + w_{ia}$

Notes: subfigure (a) shows the empirical autocorrelation of the logarithm of TFP by age, using the permanent sample. The x axis represents autocorrelation, the y axis represents age, and each line represents a fixed age. Therefore, every point in the graph is an age pair. Subfigures (b), (c) and (d) compare the empirical autocorrelation to different models estimated in Table 1.1.

gives more flexibility and thus generates higher long-run autocorrelations than the plain AR(1), yet lower than the AR(1) process with fixed effects. In addition, the mean squared error from the estimation is notably lower than the two alternative settings. Adding an ex-ante component allowing for initial condition and long-run steady-state heterogeneity to a standard AR(1) process allows us to better capture the empirical TFP autocorrelation structure of Spanish firms, and this makes the GP process suitable for representing firm-level productivity in Spain.

1.3.2 Empirical Autocorrelation Structures

In the previous section, I have estimated the GP process in equation (1.3.1) using TFP data. If I estimate the same process using employment data (as it is done in Appendix A.3), I find that estimated parameters differ when we use Spanish employment data and Spanish TFP data. These distinct results emerge from differing autocovariance structures for these two variables; or, equivalently, differing *autocorrelation* structures for logged employment and TFP. These empirical differences indicate that Spanish firms have difficulties to adjust factors of production over their life cycle. This fact, which cannot be rationalised by a completely frictionless model of the firm, is informative of the existence of firm-level frictions affecting young Spanish firms.

Autocorrelations by age. In Figure 1.2, I compare empirical autocorrelations by age for logged TFP, employment and capital, with the same object generated by a frictionless model of the firm. Let us call s the age corresponding to each line in the graphs, and t the age that is represented in the x axis. For expositional purposes, I only show autocorrelation lines corresponding to ages $s = 0$ and $s = 4$. The complete autocorrelation maps from ages 0 to 10 can be found in Appendix A.2. Let us call $\rho_{T\hat{F}P_t, T\hat{F}P_s}$, $\rho_{\hat{L}_t, \hat{L}_s}$ and $\rho_{\hat{K}_t, \hat{K}_s}$ the autocorrelations of logged TFP, employment and capital for an age pair (t, s).

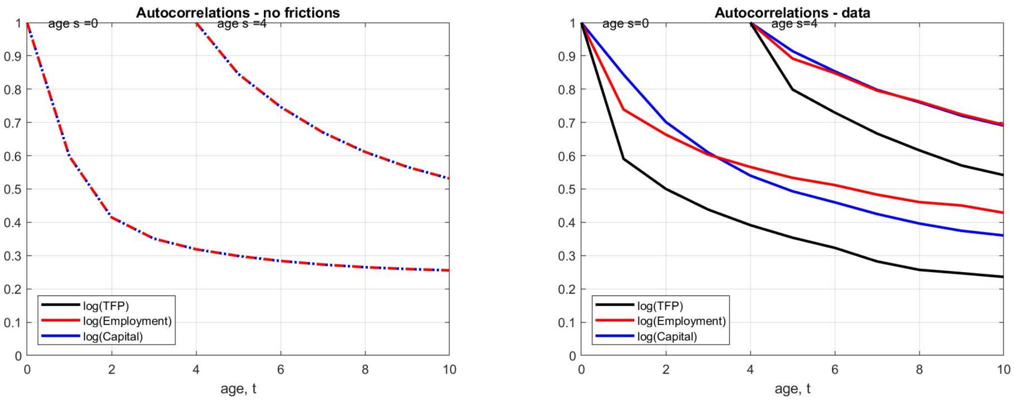
Figure 1.2(a) shows simulated autocorrelations from a frictionless model of the firm[12], considering 1,000,000 firms and taking as given the estimated GP productivity process. As we can see, a model with no firm-level frictions generates autocorrelations of logged TFP, employment and capital that coincide across the three variables for every age pair (t, s). The lines are decreasing due to the AR(1) component of the productivity process, and they converge to a strictly positive number as we increase t, keeping s fixed.

Figure 1.2(b) shows the empirical autocorrelation structures for ages $s = 0$ and $s = 4$. They do not coincide with simulated autocorrelations for a frictionless model, and are different across the three logged variables if we consider the same age pair. Indeed, autocorrelations are notably higher for logged employment and logged capital than for logged TFP, so that employment and capital levels across different ages (t, s) are more positively correlated than the corresponding TFP levels.

As discussed in section 1.3.1, the empirical pattern of autocorrelation of logged TFP is convex as we keep s fixed and increase t, and it converges to a value that is above zero when t is sufficiently large, thus suggesting the presence of firm-specific long-term steady states. The autocorrelation of logged employment displays an

[12] In Appendix A.3, I discuss the frictionless model in detail and show that it must be the case that no difference across the three variables should appear in terms of autocorrelations if there are no firm-level frictions.

Figure 1.2 Autocorrelations by age: frictionless vs. empirical



(a) Simulated, frictionless (b) Data

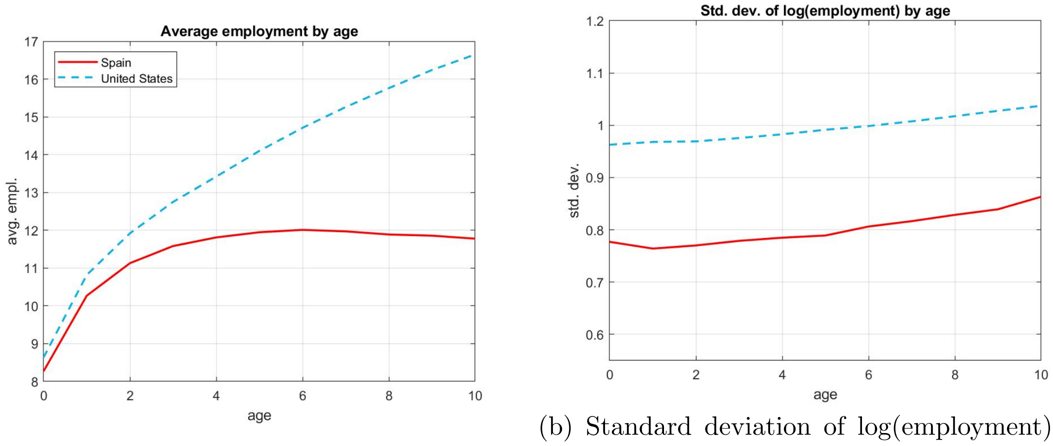
Notes: subfigure (a) shows simulated autocorrelations for logged TFP, employment and capital for different pairs of ages (t, s), for a simulated permanent sample of 1,000,000 firms, using the estimated productivity process for Spain in the first row of Table (1.1) and a frictionless setting that is described in Appendix A.3. Subfigure (b) shows the same object from the data, using the permanent sample of Spanish firms. The x-axis represents age t, and each line in the graph corresponds to another age s, either age 0 or age 4.

almost linear pattern for lags $t - s > 1$. Regarding the autocorrelation of logged capital, given an age s, it is close to the autocorrelation of logged employment in the permanent sample, and it is by all means larger than the autocorrelation of logged TFP. Importantly, these differences in autocorrelations exist over the life cycle, even when firms get older. The age $s = 4$ lines indicate that factors of production seem to be relatively difficult to adjust even if we consider firms after some periods of activity. For both $s = 0$ and $s = 4$ and for small age lags $t - s$, the autocorrelation of logged capital is slightly larger than the autocorrelation of logged employment, suggesting that, in the short term, capital adjusts less than employment. This is expected if firms face financial frictions and/or capital adjustment costs that prevent them from optimally adapting their capital stock to different short-term contingencies[13]. For $s = 0$, this pattern of $\rho_{\hat{L}_t,\hat{L}_s} < \rho_{\hat{K}_t,\hat{K}_s}$ reverts for larger time lags $t - s$, so that employment is adjusting less in the medium term.

Overall, the observed differences in empirical autocorrelation structures of logged firm-level TFP, employment and capital for young firms point to the existence of underlying frictions that may be interacting with the firm-level productivity process and may be preventing high-growth-potential firms from achieving high levels of growth. Given the standard view in the misallocation literature for Spain, the

[13]Indeed, capital distortions alone may be inducing not only a greater persistence of capital but also, to some extent, an autocorrelation of logged employment above that of TFP and below that of capital. If firms are constrained in the amount of capital they can choose, they will not optimally adjust capital and will thus be choosing to adjust labour, but not as much as they would have desired, due to the complementarity of factors in the production function.

Figure 1.3 Life-cycle moments, Spain and the United States



(a) Average employment by age

(b) Standard deviation of log(employment) by age

Notes: the Spanish moments (solid, red line) over the life cycle are calculated using the permanent sample. The moment in (a) is computed using yearly average number of employees in levels. The moment in (b) is computed using the logarithm of the yearly average number of employees. In both cases, I take out industry and cohort fixed effects in order to examine an average industry and cohort in the economy. The US moments (dashed, blue line) come from the balanced sample in Pugsley et al. (2021).

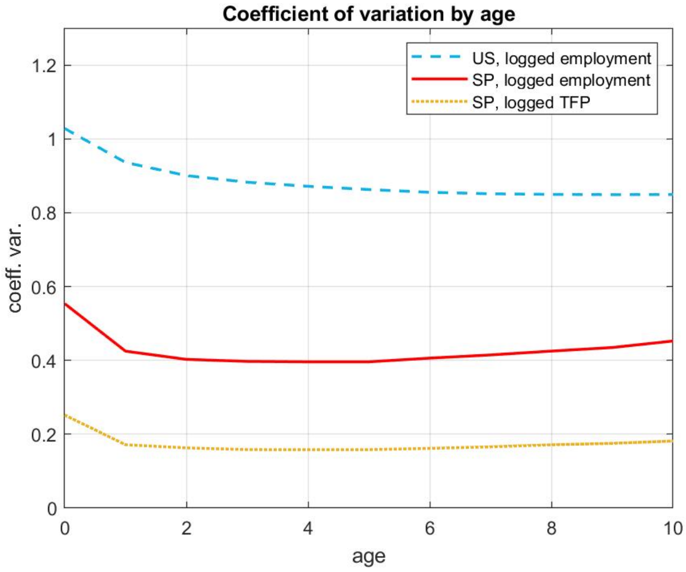
main culprit of the slow adjustment of capital is a firm-level borrowing constraint (García-Posada and Mora-Sanguinetti, 2014; Gopinath et al., 2017; Ruiz-García, 2021). Nevertheless, it may also be the case that adjustment costs in capital (Cooper and Haltiwanger, 2006), as well as labour market imperfections (Cooper and Willis, 2009), affect autocorrelation structures. The statistical process in section 1.3.1 alone is unable to replicate empirical autocorrelation patterns, since this setting considers no friction whatsoever. Hence, any such differences across $\rho_{T\hat{F}P_t,T\hat{F}P_s}$, $\rho_{\hat{L}_t,\hat{L}_s}$ and $\rho_{\hat{K}_t,\hat{K}_s}$ must come from departures from the frictionless setting. Given an appropriate structural model that explicitly accounts for frictions at the firm level, we can estimate the parameters characterising these distortions using empirical autocorrelations by age in Figure 1.2, which seem a plausible source of information to identify firm-level frictions.

1.3.3 Other Evidence on the Lack of High-Growing Firms

Using the Spanish data, I have found that Spanish firms display heterogeneity in expected growth rates, and that there is evidence on the existence of frictions preventing firms from easily adjusting their factors of production. In this section, I show more life-cycle evidence supporting the idea that Spain lacks high-growth-potential firms that are effectively realising their growth.

Previous studies have shown that manufacturing Spanish firms display low av-

Figure 1.4 Coefficient of variation by age, logged employment and TFP



Notes: the Spanish moments over the life cycle are calculated using the permanent sample, considering the logarithm of employment and TFP respectively. In both cases, I take out industry and cohort fixed effects in order to examine an average industry and cohort in the economy. The coefficient of variation for US employment is calculated using moments from the balanced sample in Pugsley et al. (2021).

erage growth over their life cycle (Hsieh and Klenow, 2014). The permanent sample I use in this chapter confirms this pattern for the whole productive sector in Spain. Figure 1.3(a) shows the average employment by age for Spanish firms over their first ten years of life. In order to benchmark the discussion, I plot the same object for the United States; the US moments are taken from a permanent sample of firms in Pugsley et al. (2021). On average, Spanish firms grow until age 6 but, differently from US firms, they stop growing from then on. The line displays a concave pattern so that Spanish average firm growth slows down after some years, even becoming negative. In the United States, the curve is strictly increasing even after ten years, so that the average firm still grows beyond age 10. The existence of frictions deterring Spanish firms (and, in particular, a high-potential subset of firms) from growing might be contributing to this empirical life-cycle pattern.

The lack of high-growing firms in Spain might also be reflecting in firm heterogeneity over the firm life cycle in Spain. Figure 1.3(b) shows the cross-sectional standard deviation of logged employment by age for young Spanish firms, in the permanent sample. Again, I plot the same object for the United States in Pugsley et al. (2021)'s balanced sample, to benchmark the numbers for Spain. As we can see, Spanish firms are not very diverse in their employment levels over different age groups, and this lack of diversity prevails over the life cycle[14]. Figure 1.3(b) sug-

[14]The documented low heterogeneity in terms of logged employment is also present if we

gests that Spanish firms are on average closer to the average employment by age than firms in the United States. Figure 1.4 confirms this idea, for it considers the coefficient of variation by age (that is, the ratio of the standard deviation by age to the mean by age) for logged employment and TFP. The coefficient of variation of logged TFP is lower than that of logged employment. This relates to the low heterogeneity in ex-ante components estimated in section 1.3.1, which contributes to a low firm heterogeneity overall.

Apart from these life-cycle facts and the well-known misallocation facts (see footnotes 1 and 5), other moments may point in the direction that highly productive firms may be deterred from growing in Spain, so that large firms end up being of less importance to the aggregate economy. Using the permanent sample, I compute the quantiles of the employment distribution by age to look at the top 5% of Spanish firms and their share of aggregate employment by age. I find that, after 15 years of firm life, top-5% firms in the employment distribution at that age account for 28.42% of total employment at age 15[15]. The corresponding quantile is $25.18 \approx 25$ employees. I interpret this 95th quantile as being low, thus denoting a lack of aggregate importance of top firms in the Spanish economy. Other empirical patterns that reinforce the idea of the absence of high-growing firms are discussed in Appendix A.2.

1.4 Conclusion

In this chapter, I perform a life-cycle study of firm-level productivity and factors of production for Spanish firms. I take advantage of a rich panel data for Spanish companies, containing information about financial variables at the firm level and about the life cycle of small and young firms. First, I look into the possibility that some firms in Spain have a high expected growth at birth. I use the methodology proposed by Pugsley et al. (2021) to study the evolution of TFP over firm life, which considers a productivity process with a life-cycle component embedding heterogeneity in expected growth rates at birth. I find that explicitly accounting for the idea that some firms may be determined to grow strongly at birth helps to better capture the micro evidence on firm-level TFP. Second, I look into the evolution of TFP, employment and capital to reach a conclusion on how well factor inputs are allocated throughout the firm life cycle. By looking at the dynamic moments of

consider the non-permanent sample (containing stayers and exiters; see Appendix A.2), as well as if we consider a permanent sample using an alternative definition of age based on employment (see footnote 4).

[15]To the best of my knowledge, there is no study that allows me to benchmark this number. Indeed, Pugsley et al. (2021) do not report this moment for their US sample. In addition, they claim that their structural model is unable to match well the right tail of the employment distribution by age.

input data (namely, empirical autocorrelations by age), I find supportive evidence that the allocation of inputs to Spanish firms is affected by firm-level frictions.

The documented patterns on autocorrelations of inputs by age suggest that there are frictions affecting firm growth in Spain. Yet, it remains to study how these distortions affect firms with a high growth potential, and how this translates into aggregate outcomes. Chapter 2 addresses these issues by proposing a structural model that explicitly accounts for firm-level borrowing constraints and considers a productivity process that embeds the idea of high-growth-potential firms.

A Appendix to Chapter 1

A.1 Dataset Construction and Data Cleaning

I build three datasets combining firm-level data from *Central de Balances Integrada* (CBI) with investment deflators at the 2-digit industry level (NACE) for Spain from EU-KLEMS. All these data correspond to the period 1997-2016. As mentioned in the main text, I consider sectors from A to S, according to the 1-digit NACE classification, thus including range of activities including large sectors as manufacturing, construction, commerce, hotel industry, but also agriculture and financial services. I do not consider government firms. Before merging the data from CBI with capital deflators, it is necessary to realise that some firms may change their industry indicators over time, which is problematic when it comes to controlling for industry fixed effects. I assume that firms belong to the industry for which they have a larger number of observations, according to their industry indicators from CBI. I also define an *age* variable as the difference between the current year and the year of legal creation of the firm, which may be different from the first year the firm reports its information to the Commercial Registry, and also from the first year in which the firm has a positive value for logged employment.

The datasets I construct are an *aggregate dataset* (considers all cohorts), a *non-permanent dataset* (firms born in period 1997-2006 with information at age 0), and a *permanent dataset* (firms born in period 1997-2006 with information at age 0 that have not exited before or at age 10). The main text utilises the permanent sample, thus focusing on stayers, and I leave the non-permanent dataset for robustness checks. I require that firms have reported their information at the period of *legal* birth (which I define as age 0), so that we are able to track Spanish firms from their first years of activity. In an unreported exercise, I consider an alternative definition of age which is in line with that in Pugsley et al. (2021), as mentioned in footnote 4. In that check, I define firm birth as the first period reported by a firm in which a firm's yearly average number of employees is greater than one. Under this alternative age definition, my permanent sample contains a greater number of firms of observations, and the empirical life-cycle patterns discussed in the main text remain qualitatively similar.

The data cleaning process departs from an initial dataset that has 16,124,897 firm-year observations. In order to get the permanent sample in the main text, as well as the non-permanent sample used for robustness checks in section A.2, I follow these steps:

1. I drop firm-year observations that belong to government firms, in order to focus on the private sector. I drop 19,312 observations in this step.
2. I drop firms that have had an invalid zip code at some point in time. I drop

firms that have had an invalid age at some point in time. I drop 185,008 observations in this step.

3. I drop firm-year observations with a missing or zero value for value added (revenue minus the cost of materials), long-term assets, wage bill and average number of employees. I drop 6,302,875 observations in this step.

4. I drop firms that have had a value lower than 1 for gross value added (revenue minus the cost of materials), capital, wage bill, or average number of employees at some point in time. I drop 4,379,898 observations in this step.

5. I drop firm-year observations with a value added below 1 (1,000 euros) or above 10,000,000 (10 billion euros), or with a wage bill below 1 (1,000 euros), or with an average number of employees above 1 million. Besides, for observations with a value added above 25,000 (25 million euros), I also require that the consistency requirements in accounting and units of measure established by *Central de Balances* are satisfied[16], since observations with an extreme value of value added are suspicious of being misreported. I drop 8,176 observations in this step.

6. I drop firm-year observations that have an average number of employees larger or equal than 3000 and do not satisfy the staff coherence requirement of *Central de Balances*. I drop 693 observations in this step.

7. I drop firm-year observations that have a zero or a missing value for depreciation or long-term liabilities; and missing values of taxes over profits, short-term assets, and liquid assets. Then, I drop firm-year observations that are below the 1 percentile or above the 99 percentile of the distribution of the following variables: ratio of wage bill to value added, depreciation, taxes over profits, long-term liabilities, and liquid assets. I drop observations with a ratio of liquid assets over total assets greater than 1.1. I drop 544,137 observations in this step.

8. I drop firms which have some observations with very low values for certain variables while having very high values for different, yet related variables. Specifically, firms that have observations below the 1 percentile of the distribution of value added, long-term assets, wage bill or the average number of employees, and at the same time, above the 99 percentile of the distribution of any other of these variables. I drop 1,939 observations in this step.

[16]In a robustness exercise, I build the aggregate dataset by just considering firm-year observations that satisfy the *Central de Balances*' consistency and staff coherence criteria, without further cleaning apart from the non-negativity and non-missing-values steps. Results are similar to those in the main text.

9. I drop firm-year observations whose level of total nominal assets is above 12,500 (12,500,000 euros) and that do not satisfy the *Central de Balances* consistency criterion. I drop 4,695 observations in this step.

10. Drop outliers for financial variables: firm-year observations below the 1 percentile or above the 99 percentile of the distribution of: debt over assets, liquid assets over assets, cash-flow over assets, and real investment. Drop firm-year observations with an investment rate above 10. I drop 126,172 observations in this step.

11. Once productivity and marginal revenue products have been calculated, I drop firms that have been at some point in time above the 99.5 percentile or below the 0.5 percentile of the distribution of physical productivity (A), revenue productivity ($TFPR$), marginal revenue product of capital ($MRPK$), or marginal revenue product of labour ($MRPL$). I also drop observations corresponding to a negative value of logged physical productivity A. Finally, to clean some weird observations in the initial reports, I eliminate observations at age 0 for which physical productivity is below (above) the 97.5 (2.5) percentile *within that age group*. I drop 471,865 observations in these steps. After these previous cleaning steps, we obtain the *aggregate dataset* (4,080,065 observations).

12. Departing from the aggregate dataset, I drop firms that were not born in between 1997 and 2006, both years included in this period (2,512,650 observations), and those firms that do not have financial information at period 0 (1,245,137 observations). I drop 3,757,787 observations in this step. I eventually obtain the *non-permanent dataset* (322,278 observations).

13. Departing from the non-permanent dataset, I drop firms whose last financial report belongs to an age smaller than 10. I drop 98,130 observations in this step. I obtain the *permanent dataset* (224,148 observations).

A.2 Alternative Samples

Throughout the main text, the permanent sample is used to discuss empirical patterns and to get estimation results. This Appendix has two objectives. First, I show that the main empirical patterns and estimation results of the chapter remain similar when we consider the non-permanent sample instead of the permanent one. Second, I discuss differing empirical patterns between the permanent and the non-permanent samples that reinforce the idea that Spanish young firms, particularly most productive ones, face difficulties to grow.

Robustness Checks Using the Non-Permanent Sample

It may well be argued that our permanent sample is not representative of the population of Spanish firms, for it does not consider exiting firms during the first ten years of firm life. Indeed, non-exiters may be firms with somewhat special characteristics relative to the whole population - particularly, a larger size in terms of employment. I consider the non-permanent dataset as described in section 1.2. This dataset includes not only firms that have been documented to still exist after ten periods of life, but also those firms that have exited before age 10. Thus, the number of observations increases from 224,148 in the permanent sample to 322,278 in the non-permanent sample (the number of firms increases from 18,065 to 47,430). The object of this section is to analyse the non-permanent sample to address whether this is a concern for the empirical life-cycle patterns discussed in the main text, namely mean employment by age, standard deviations of logged employment and TFP by age, and the estimated process for firm-level productivity.

Mean employment and standard deviations by age. As suspected, it can be shown that firms in the permanent sample are larger over the life cycle than firms in the non-permanent sample in terms of the average employment by age. Figure 1.5 compares the average employment by age of the permanent and the non-permanent samples for Spain. Even though firms in the non-permanent sample are smaller, on average, than those from the sample in the main text, the key motivating pattern of the chapter remains: Spanish firms typically display low growth over their life cycle in terms of employment. This fact holds regardless of whether we consider exiters or stayers in our sample, as the slopes of the average employment profiles of the two samples show.

As for the pattern of the standard deviation of logged employment by age, Figure 1.6(a) shows that the empirical pattern in Figure 1.3(b) is still present if I consider a larger sample of firms that are smaller in size, on average. In particular, the cross-sectional deviation of logged employment is slightly increasing in age. Figure 1.6(b) represents the coefficient of variation for logged employment and TFP by age for both the permanent and the non-permanent sample. Thus, even if we consider the non-permanent sample, which aims to be more representative of the population of Spanish firms by including exiters and a larger number of small firms, firm heterogeneity in logged employment and TFP over the life cycle does not differ much from that discussed in the main text.

Figure 1.5 Life-cycle average employment: permanent vs. non-permanent sample

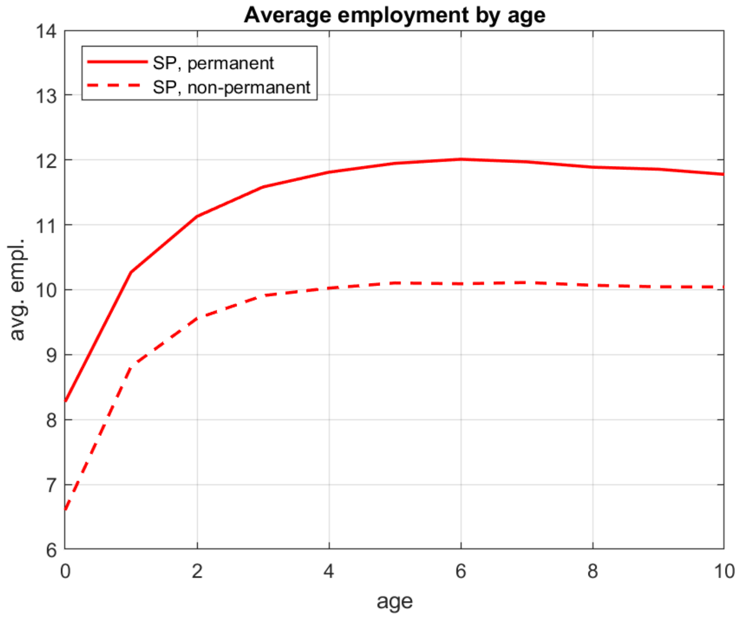
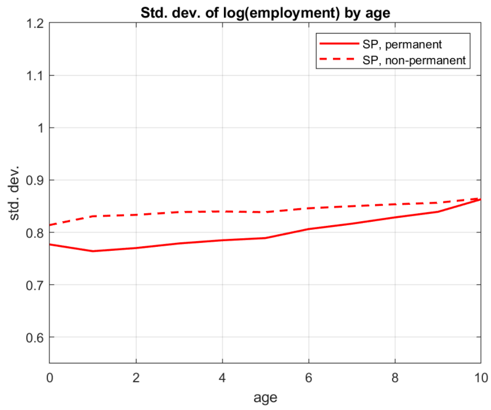
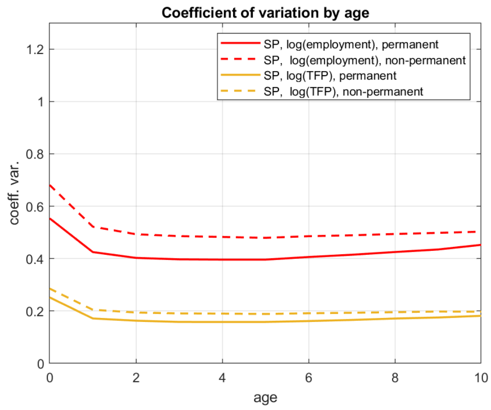


Figure 1.6 Life-cycle firm heterogeneity: permanent vs. non-permanent sample



(a) Standard deviation by age



(b) Coefficient of variation by age

Table 1.2 Estimated process for logged productivity, Spain (non-permanent sample)

Estimated process	ρ_u	ρ_w	σ_θ	σ_u	σ_ε	RMSE
GP process	0.482	0.577	0.326	1.636	0.508	0.032
	(0.0076)	(0.0044)	(0.0044)	(0.0297)	(0.0009)	
AR(1) process	-	0.832	-	-	0.537	0.003
		(0.0014)			(0.0015)	
AR(1) + fixed effect process	-	0.606	0.566	-	0.553	0.067
		(0.0037)	(0.0029)		(0.0012)	

Notes: the first row shows parameter estimates for the GP process for logged productivity $log(A_{ia}) = u_{ia} + w_{ia}$. The second and third rows show nested special cases of the baseline process, namely a plain AR(1) process (that is, $log(A_{ia}) = w_{ia}$), and an AR(1) process where firms draw a fixed effect θ when born (that is, $log(A_{ia}) = w_{ia} + \theta_i$). The three estimations have been performed using autocovariances of logged firm-level TFP by age calculated from the non-permanent sample. The standard errors of parameter estimates are in parentheses, and are small due to the very large sample size. Standard errors are calculated using a parametric bootstrap procedure with 1000 replications.

Process for firm-level productivity. In the main text, I have estimated the statistical process for firm-level productivity in equation (1.3.1) by using firm-level TFP data that was pinned down using the permanent sample. In this section, I use the non-permanent sample to estimate the GP process, as well as the simple AR(1) process and a process containing an AR(1) component and a firm-level fixed effect. Table 1.2 shows estimated parameters and standard errors for these processes using the non-permanent sample.

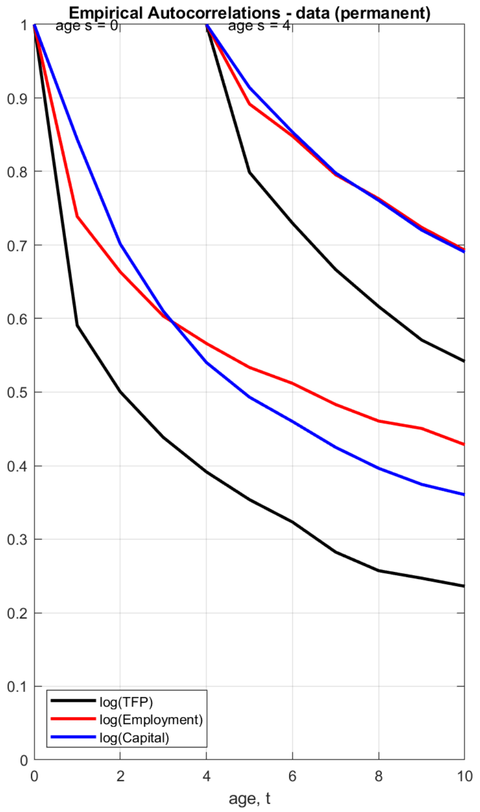
As we can see, when compared to results in Table 1.1, estimated parameters for the GP process only change slightly with respect to those in the main text. In particular, parameters governing the u component are all statistically significant, which indicates the presence of ex-ante firm heterogeneity as the one described by the initial-condition-plus-steady-state structure of the GP process. In addition, while the estimated cross-sectional variance of long-run steady states – that is, $\sigma_\theta^2/(1-\rho_u)^2$, was equal to 0.263 for the permanent sample, it is equal to 0.396 for the non-permanent sample. This reinforces the main empirical finding regarding firm heterogeneity in the main text: the non-permanent sample confirms that Spanish firms are heterogeneous in their expected productivity growth rates. Regarding other estimated statistical processes for firm-level productivity, namely the plain AR(1) and the AR(1)-plus-fixed-effect processes, estimation results are robust to using the permanent and the non-permanent sample.

Empirical autocorrelations by age. A key object of the discussion in the main text is the empirical autocorrelation by age of several variables – namely, logged employment, capital, and TFP. It is of particular interest to check that the distinctive features of the permanent sample that inspired the calibration of the firm-dynamics model in section 2.2 in Chapter 2 are also present if we consider exiters – that is, the non-permanent sample.

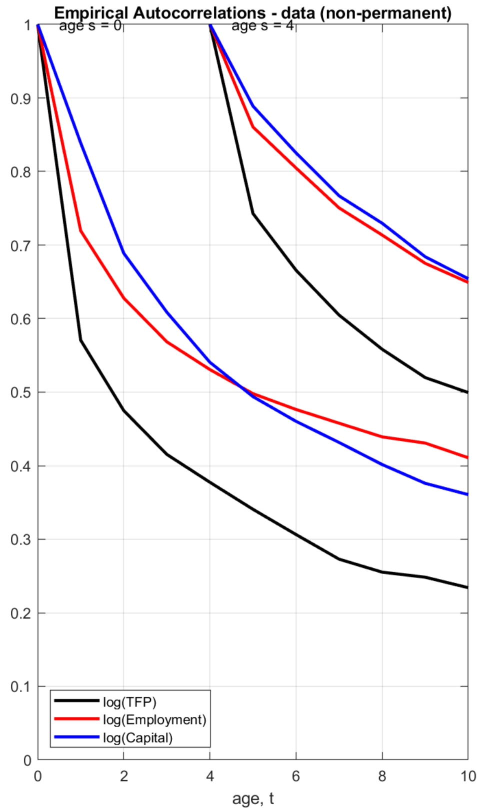
Figure 1.7 compares the empirical autocorrelations by age of the variables of interest across the permanent and the non-permanent sample. In particular, for clearness, I show autocorrelation curves for age 0 and age 4 (at is was done in Figure 1.2(b) in the main text). I find that, for these two ages, the autocorrelation curves of the three logged variables (TFP, employment, and capital) calculated using the non-permanent sample look very similar to those in the main text, which correspond to the permanent sample. Thus, the fact that we are considering a sample with exiters does not make a great difference in empirical autocorrelation structures – particularly in empirical autocorrelations of factors of production at age 0, which are our targets in the model calibration in section 2.2.3 in Chapter 2.

Figure 1.8 shows the complete map of autocorrelations (that is, for several age pairs (t, s) corresponding to young firms) for logged employment, logged capital and logged TFP. It compares empirical autocorrelations from the permanent sample with those from the non-permanent sample. As we can see, there is not a very large difference between empirical autocorrelations by age between the two samples. Interestingly, autocorrelations by age of the three logged variables are slightly smaller for the non-permanent sample. This is due to the fact that, although empirical autocovariances by age (the numerator of the autocorrelation expression) are larger for the sample of exiters, the fact that the non-permanent sample contains more firms increases its cross-sectional standard deviation by age (the denominator of the autocorrelation expression) in the three variables considered (see, for instance, Figure 1.6(a) for the case of logged employment). As a result, there is a small divergence between the two samples in terms of empirical autocorrelations.

Figure 1.7 Autocorrelations by age: permanent vs. non-permanent sample



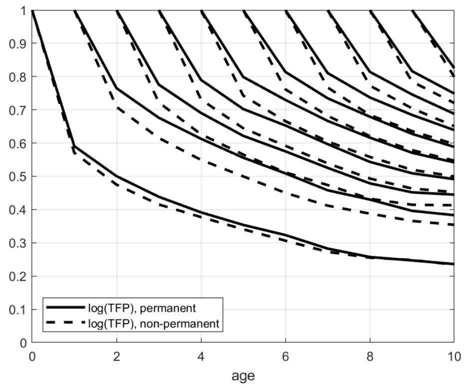
(a) Permanent sample



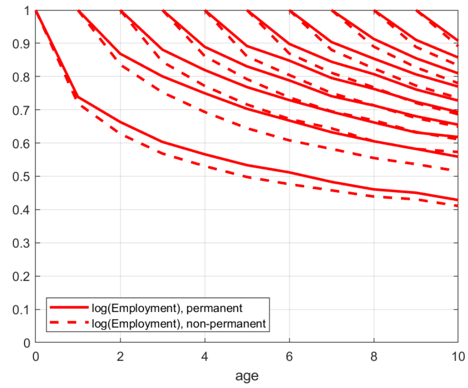
(b) Non-permanent sample

Notes: subfigure (a) shows the autocorrelations for logged TFP, employment and capital for different pairs of ages (t, s) from the permanent sample of Spanish firms. Subfigure (b) shows the same object from the non-permanent sample of Spanish firms. In both subfigures, the x-axis represents age t, and each line in the graph corresponds to another age s, either age 0 or age 4.

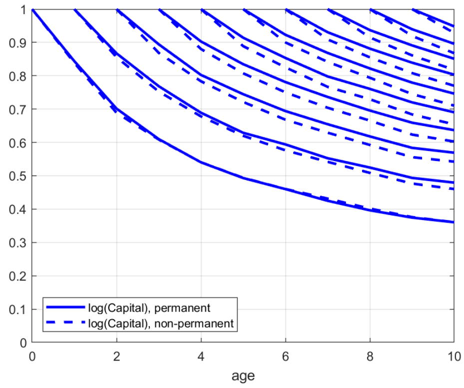
Figure 1.8 Autocorrelation structures by age: permanent vs. non-permanent



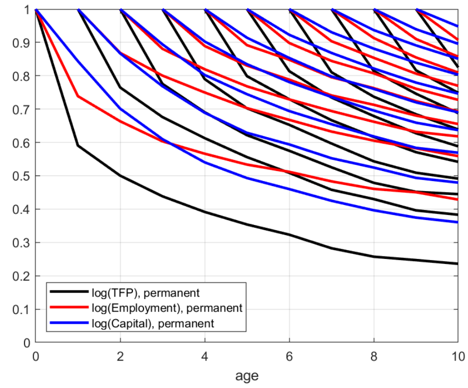
(a) Logged TFP



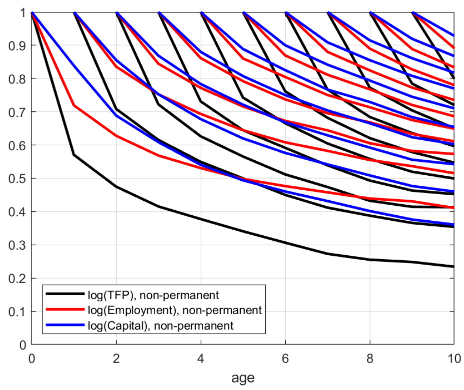
(b) Logged employment



(c) Logged capital



(d) All, permanent



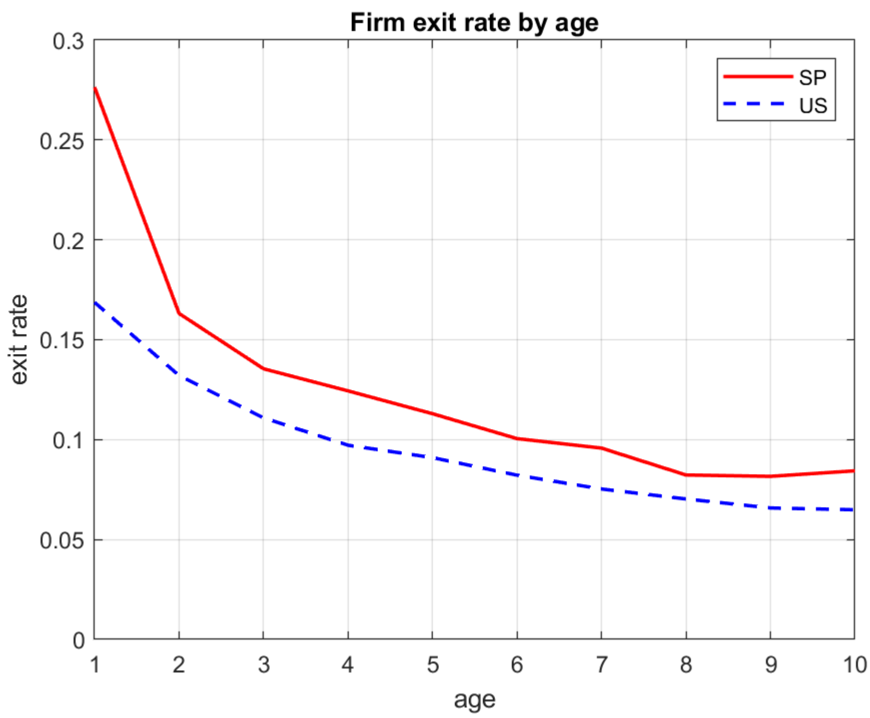
(e) All, non-permanent

Notes: subfigures represent autocorrelations by age for logged TFP, employment and capital, for different age pairs (t, s). The x-axis represents age t, and each line in the graphs corresponds to another age s such that $0 \leq s \leq t$. Moments are calculated using the permanent sample of Spanish firms.

Empirical Patterns Differing Across Permanent and Non-Permanent Samples

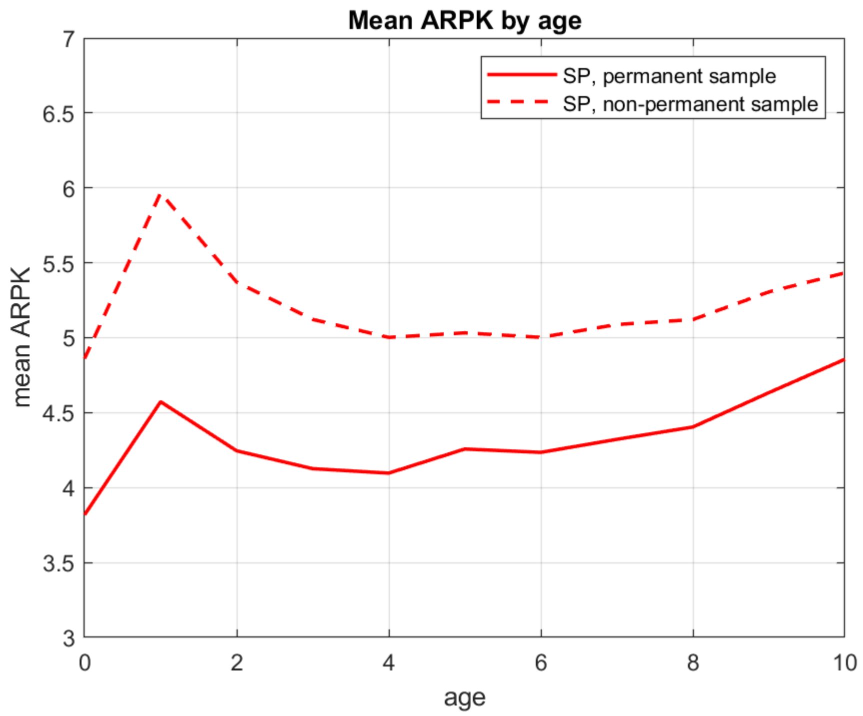
As shown in Appendix A.2, although the main empirical patterns of interest are similar if we consider the permanent and the non-permanent samples, there are some differences across the two datasets, which come from the fact that some firms are exiting the non-permanent sample at some point between age 0 and age 10. However, as I briefly argue here, these differences reinforce the main hypothesis of this chapter: young Spanish firms seem to face difficulties to grow and get financing. Indeed, these difficulties mitigate the aggregate role of high-growth-potential firms. In particular, the non-permanent sample provides suggestive evidence for the average revenue product of capital (ARPK) by age, and the exit rate by age.

Figure 1.9 Firm exit rate by age: Spain vs. US



Exit rates by age. First of all, the permanent sample remains silent about the firm exit rate by age, by construction. From the non-permanent sample, we are able to pin down the number of firms that exit every period. I compute the exit rate of and age s as the ratio of firms that exit at age s, (i.e. firms whose last observation takes place at age $s-1$) over the total number of firms alive at age s. In Figure 1.9, I compare the exit rate by age of Spanish firms in the non-permanent sample with the exit rate of US firms, taken from the unbalanced sample in Pugsley et al. (2021). As we can see from the graph, the exit rate by age of Spanish firms is substantially high over the first 5 years of activity. For age 1, the exit rate is about 28%. For ages 2 to 10, the exit rate ranks between about 16% and 8% and shows a decreasing pattern with firm age. The exit rate profile is steeper than that of US firms, particularly at young ages. This is indicative of a lot of firm exit taking place in the youngest ages in Spain. Although not discussed in the main text, this might relate to firm-level frictions preventing firms not only from growing, but also from surviving in the market.

Figure 1.10 Mean ARPK by age: permanent vs. non-permanent sample



Mean ARPK by age. The permanent and the non-permanent samples are different in the mean average revenue product of capital (ARPK) by age of firms therein. Figure 1.10 represents that object over the life cycle, for the permanent and the non-permanent samples. On average, firms in the non-permanent sample display a noticeably greater ARPK than their permanent-sample counterparts over the first 5 years of activity. After that age, mean average revenue products of capital tend to slightly equalise across the two samples, but mean ARPK remains higher for the sample that considers exiters. The average revenue product of capital, which is defined as the nominal product of the firm divided by its stock of real capital, is often considered as an indirect measure of the intensity of firm-level financial frictions that prevent firms from achieving their optimal level of capital. Thus, a higher mean ARPK of exiters could potentially be interpreted as high-productivity firms facing problems in order to fulfill their desired investments and exiting the market when they are young, thus reflecting difficulties of some productive firms to grow and remain active.

Table 1.3 Estimated process for logged employment, Spain and United States

Estimated process	ρ_u	ρ_w	σ_θ	σ_u	σ_ε	RMSE
SP, GP process	0.165	0.846	0.491	3.139	0.318	0.019
	(0.0124)	(0.0050)	(0.0063)	(0.2316)	(0.0017)	
US, GP process	0.119	0.934	0.646	5.604	0.300	0.038
	(0.0015)	(0.0003)	(0.0011)	(0.0701)	(0.0003)	

Notes: the graph shows parameters estimates for the GP process for logged employment $\ell_{ia} = u_{ia} + w_{ia}$ for Spain and the United States. The first row corresponds to estimates for Spain (SP), using autocovariances of logged employment by age calculated from the permanent sample for Spain. The second row corresponds to estimates for the United States (US), using the same empirical moments from the balanced sample in Pugsley et al. (2021). The standard errors of parameter estimates are in parentheses, and are small due to the very large sample size. Standard errors are calculated using a parametric bootstrap procedure with 1000 replications.

A.3 Statistical Process and Autocorrelations

A Process for Firm-Level Employment

Estimated process. I consider the GP process in equation (1.3.1) for ℓ_{ia}, where ℓ_{ia} is the logarithm of employment, instead of logged productivity. I estimate the process using employment data. Results from estimation are shown in the first row of Table 1.3. The second row shows estimated parameters from estimation using the corresponding moments for the balanced sample of US firms in Pugsley et al. (2021), in order to benchmark the results for Spain. Regarding the ex-post component, the estimated ρ_w is smaller in Spain than in the United States, which translates into a lower variance of the w component, according to equation (1.3.2). As for the u component, Spanish firms display a lower dispersion in long-run steady states than US firms. Although the estimated ρ_u is slightly higher for Spain than for the United States (which indicates slower convergence to firm-level steady states in terms of employment), a lower estimate of σ_θ results in the dispersion of $u_{i,\infty}$ being noticeable smaller in the Spanish case. Specifically, while the cross-sectional dispersion of steady states $\frac{\sigma_\theta^2}{(1-\rho_u)^2}$ equals 0.538 for the United States, it is only 0.346 for Spanish firms, indicating that there are less Spanish firms with extreme values for their long-run steady states. This is evocative of the existence of distinct patterns regarding heterogeneity in expected growth rates between Spain and the United States, and may indicate a lower presence of high-growth-potential firms in Spain.

I want to study whether lower heterogeneity in ex-ante components translates into a lower employment dispersion over the life cycle for Spain. To do this, I calculate the fraction of the variance of logged employment over the life cycle that is accounted by the ex-ante term u. I find that ex-ante components account for a considerable amount of the life-cycle employment dispersion, and thus contribute to explaining the low employment heterogeneity over the life cycle in Spain. If we consider US moments, the variance of the u component can explain 49.85% of the empirical variance for US firms at age 10. In Spain, it explains 46.39% of the dispersion of logged employment ten years after birth[17]. Apart from the variance of long-run steady-states being lower in Spain, a lower estimate of ρ_w decreases the variance of ex-post shocks, which is also bringing down life-cycle dispersion.

Ex-ante heterogeneity and firm size. Long-run employment steady states display a noticeably low dispersion in Spain. We may wonder to which extent this is reflecting in life-cycle employment patterns as we let firms age and grow, with an eye on firms that are on the top of the employment distribution by age. To study this, I simulate 1,000,000 paths for ℓ_{ia} given the estimated parameters for Spain and for the US, in Table 1.3. I use simulated data to compute (i) the fraction of employment by age accounted for firms that are above the 95th quantile of the employment distribution at age 15; and (ii) the slope of the average employment by age curve[18]. The first of these moments gives us an idea on the existence of firms that have a large size after several periods of activity and account for a considerable fraction of employment within their age group once they are older. The second is motivated by the different slopes for Spain and the United States in Figure 1.3(a). I want to study whether these two objects are sensitive to changes in parameters determining long-run steady-state heterogeneity, namely ρ_u and σ_θ.

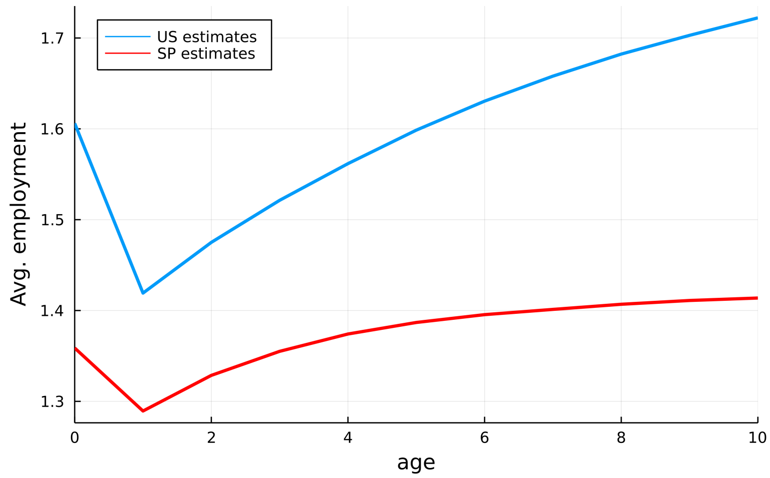
The simulated moments I obtain do not match those in the data, for they were not targeted in the estimation. However, they are indicative of the effects of long-run steady state heterogeneity, which is notably different under the SP and the US parameterisations. First, regarding the simulated employment share of top 5% age-15 firms, I find that these firms account for 20.99% of total employment at age 15 for the SP parameterisation, and for a 28.46% percent for the US estimates. These differences suggest that the relevance of top firms for aggregate employment by age is sensitive to parameters characterising the u component in (1.3.1). More specifically, the lower variance of long-run steady states (represented by a lower σ_θ) is driving this result[19], indicating that a lower heterogeneity in long-term ex-ante components has a negative effect on the size of top firms. Second, the slope of the simulated average employment by age from ages 5 to 10 is lower if we consider the SP estimates. As we see in Figure 1.11, although the empirical mean employment by age was not

[17] The numbers for the cross-sectional variance of long-run steady states and the importance of the u component for explaining the empirical cross-sectional deviation of employment by age are robust to the estimation of the $u + v + w + z$ process in Pugsley et al. (2021); see footnote 7.

[18] As mentioned in footnote 9, the average employment by age is not targeted in estimations, so the focus is on how parameters, specifically σ_θ, affect the slope of this moment over the life cycle.

[19] If we perform comparative statics exercise with other parameters, such as σ_u and ρ_w, changes in these parameters do not affect this simulated moment significantly. The same happens if we consider the slope of the average employment by age.

Figure 1.11 Simulated average employment by age, different parameterisations

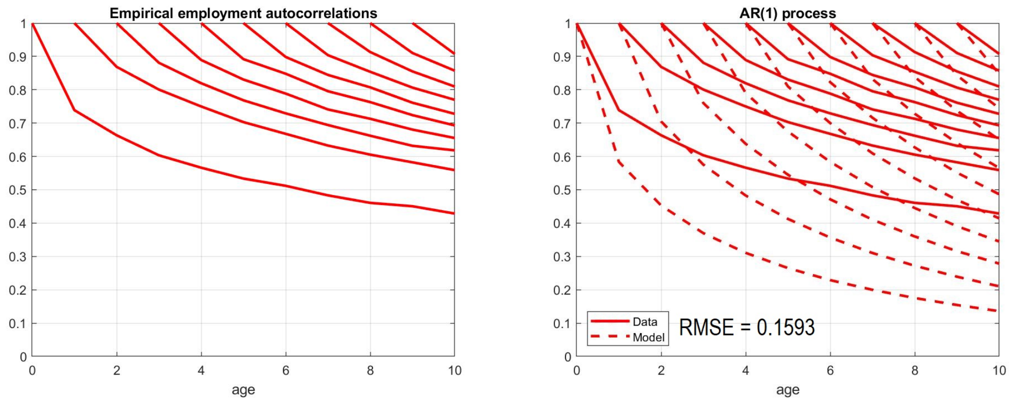


targeted by the estimation in Table 1.3, the different estimations capture relatively well the empirical fact in Figure 1.3(a) that the slope of the average employment by age is steeper in the United States than in Spain from age 5 to age 10. Differences in steady-state dispersions are reflected in the slope of the curve. This suggests that a lower diversity in terms of expected growth rates translates into Spanish firms displaying a less steep average size profile over their life cycle, all else equal.

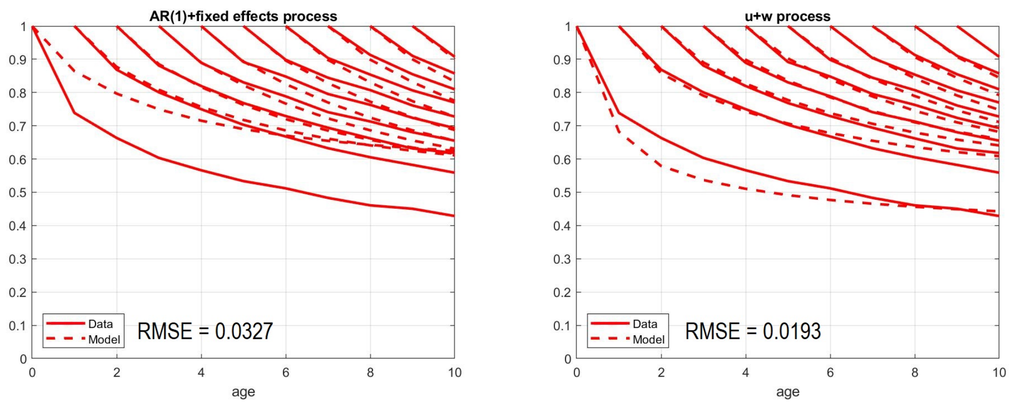
To conclude, the estimation of the GP process using employment data from Spain and the United States indicates that Spanish firms display relative low dispersion in terms of their employment long-run steady states. I find that a substantial percentage of the variance of employment by age in Spain is explained by ex-ante components, which contributes to a lower size diversity. Finally, I show that low steady-state heterogeneity reduces the aggregate prominence of top firms, and renders the age profile of average employment less steep.

Autocorrelation of Logged Employment and Alternative Processes

Figure 1.12 Productivity processes and employment autocorrelations by age



(a) Empirical employment autocorrelation (b) $log(L_{ia}) = w_{ia}$



(c) $log(L_{ia}) = w_{ia} + \theta_i$ (d) $log(L_{ia}) = u_{ia} + w_{ia}$

Notes: subfigure (a) shows the empirical autocorrelation of the logarithm of employment by age, using the permanent sample. The x axis represents autocorrelation, the y axis represents age, and each line represents a fixed age. Therefore, every point in the graph is an age pair. Subfigures (b), (c) and (d) compare the empirical autocorrelation to different models: an AR(1) process, an AR(1) process with fixed effects, and the GP process in the first row of Table 1.3, respectively.

Frictionless Setting and Autocorrelation Structures

In section 1.3.2, I use the fact that a frictionless model of the firm generates autocorrelation structures by age for logged productivity, employment and capital that are *equal for the three variables*, for every age pair. That is, $\rho_{\hat{A}_t,\hat{A}_s} = \rho_{\hat{L}_t,\hat{K}_s} = \rho_{\hat{K}_t,\hat{K}_s}$ for every pair of ages (t, s)[20]. This appendix proves this sentence. I explicitly de-

[20] Although I use letter A, I am referring to firm-level TFP when I talk about the data or simulations using the frictionless model. However, it turns out that if I simulate frictionless L_t and K_t series given a process for productivity A_t and I use equation (1.2.1) and the simulated

scribe a frictionless and stationary model of the firm and argue that, under such a setting, no differences could arise in autocorrelation structures across the three variables of interest. Within this framework, the problem of the firm is dynamic, as it will become apparent, although it can be shown to be equivalent (in terms of first-order conditions) to a simple static model. This frictionless setting also serves as an undistorted reference for the more elaborated theoretical framework in section 2.2 in Chapter 2.

The environment is as follows. There is a discrete-time economy inhabited by just one firm (for illustration purposes) that lives for infinite periods. In every period t, the firm owns a stock of capital K_{t-1} brought from previous period, it takes its idiosyncratic productivity A_t as given and, after observing it, chooses how much labour L_t to hire and how much investment good I_t to buy in order to produce output Y_t. I assume a Cobb-Douglas production function of the form $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$, where $\alpha \in (0,1)$ and A_t is firm-level productivity, and a demand function for the good produced by the firm $Y_t = P_t^{-\sigma}$, where $\sigma > 0$ and P_t is the price of the final good. These two assumptions on production and demand yield the revenue function $P_t Y_t = A_t^{\gamma} K_t^{\eta} L_t^{\xi}$, where $\gamma = 1 - \frac{1}{\sigma}, \eta = \alpha(1 - \frac{1}{\sigma})$ and $\xi = (1-\alpha)(1-\frac{1}{\sigma})$, also present in the structural model of section 2.2 in Chapter 2. The problem of the firm is dynamic, in that capital K_t is owned, and not rented, by the firm. The law of motion of capital is standard: $K_t = I_t + (1-\delta)K_{t-1}$, where δ is the depreciation rate. Notice the timing assumption that the firm enters period t with stock K_{t-1} and chooses I_t and K_t once the value for A_t is known, and K_t becomes productive immediately[21]. Input prices are exogenous to the firm and are constant over time. In particular, the price of hiring labour at every period t is ω, and the price of the investment good is p_I (which I set to one in the main text, thus considering a one-sector model). I also assume that the firm discounts period payoffs at an exogenous rate r. Productivity A_t evolves according to an exogenous idiosyncratic process. In this section, I consider a productivity process of the same form in (1.3.1), with $a = t$. This implies, similarly to the statistical model, that the firm draws θ and u_{-1} at birth from independent normal distributions, and has a long-run steady-state productivity level $\frac{\theta}{1-\rho_u}$. Importantly, in this simple model, the firm faces no friction whatsoever. Let us consider $\hat{A}_t = log(A_t), \hat{L}_t = log(L_t)$ and $\hat{K}_t = log(K_t)$. I will show that, in this setting, it must be the case that the autocorrelation between

data, the simulated measure of TFP coincides with firm-level productivity A_t.

[21] A different assumption would be to consider a one-period lag for capital to become productive (time-to-build in capital). I do not make such an assumption for simplicity, in part. But, more importantly, it has been argued that this time-to-build feature is a source of marginal revenue product of capital dispersion, since the firm is making a decision under uncertainty that may not be ex-post optimal. Thus, time-to-build in capital can be seen as a form of capital adjustment friction (see Gopinath et al. (2017) and Asker et al. (2014)). As a matter of fact, the time-to-build assumption affects the expressions of firm first-order conditions, thus rendering autocorrelations of logged productivity, labour and capital different across the three variables, even in the absence of distortions other than the productive lag of capital.

logged employment, logged productivity and logged capital for any age pair (t, s) is the same for the three variables, $\rho_{\hat{A}_t,\hat{A}_s} = \rho_{\hat{L}_t,\hat{K}_s} = \rho_{\hat{K}_t,\hat{K}_s}$ for every (t, s). The problem of the firm, expressed in standard dynamic programming notation[22], is:

$$V(K_{-1}, \theta, u, w) = \max_{L,K} \quad \left\{ A^{\gamma} K^{\eta} L^{\xi} - \omega L - p_I\big(K - (1-\delta)K_{-1}\big) + \frac{1}{1+r} E_t[V(K, \theta, u', w')] \right\}$$

$$\text{s.t.} \qquad log(A) = u + w$$

Developing first-order conditions and defining $r_K := p_I\big(1 - \frac{1}{1+r}(1-\delta)\big)$, I find the following relationships between productivity, employment and capital:

$$L_t = \frac{r_K}{\omega}\frac{\xi}{\eta}K_t \implies \hat{L}_t = log\left(\frac{r_K}{\omega}\frac{\xi}{\eta}\right) + \hat{K}_t$$

and

$$K_t = \left[(\frac{\xi}{\omega})^{\xi} (\frac{\eta}{r_K})^{1-\xi} \right]^{\frac{1}{1-\xi-\eta}} A_t^{\frac{\gamma}{1-\xi-\eta}} \implies \hat{K}_t = \frac{1}{1-\xi-\eta} log\left((\frac{\xi}{\omega})^{\xi} (\frac{\eta}{r_K})^{1-\xi} \right) + \frac{\gamma}{1-\xi-\eta}\hat{A}_t$$

It can be easily shown that the first-order conditions of this dynamic problem are equivalent to the first-order conditions of a static problem where the firm rents capital, instead of owning it, thus yielding the same result in terms of allocations and autocorrelation structures. Therefore, instead of solving a dynamic problem numerically, I can solve a simple static problem, so that the size of the state space is not an issue, as far as no firm-level frictions are considered.

What we learn from first-order conditions is that this frictionless model of the firm yields linear relationships between the logarithm of A_t, L_t and K_t, so that $\hat{A}_t$, $\hat{L}_t$ and $\hat{K}_t$ are perfectly (contemporaneously) correlated. Since this is the case, then it is easy to check that their autocorrelations by age, $\rho_{\hat{A}_t,\hat{A}_s}$, $\rho_{\hat{L}_t,\hat{L}_s}$ and $\rho_{\hat{K}_t,\hat{K}_s}$ are all equal, given age pair (t, s).

[22]i.e. K_{-1} stands for K_{t-1}, K stands for K_t, K' stands for K_{t+1}, etc.

Chapter 2

Factor Misallocation and High-Growth Firms in Spain

2.1 Introduction

From the empirical analysis in Chapter 1, we arrive at two results. First, there are some firms in Spain that are expected to display high growth at the moment of birth. Second, the allocation of factor inputs to young Spanish firms is affected by firm-level frictions. The objective of this chapter is to study how high-growth firms and distortions interact – i.e. whether Spanish high-growth firms are prevented from growing due to the existence of borrowing constraints, and how this translates into aggregate total factor productivity and output.

Motivated by firm-level findings in Chapter 1, a structural model of the firm is developed. I model a stationary economy in which firms are born, produce, and exit over the life cycle. Importantly, there exist firm-level frictions affecting labour and capital choices of firms. Specifically, I consider convex adjustment costs in capital and a collateral constraint limiting capital and labour (thus serving as a working-capital constraint) allocations. Within this framework, I consider two alternative firm-level productivity processes. The first process explicitly accounts for heterogeneity in expected productivity growth rates across firms (thus allowing for *high-growth-potential firms*); the second process is a standard process from the firm dynamics literature without this margin of heterogeneity. I calibrate the structural model for each of these alternative processes, to match empirical firm-life-cycle moments documented in Chapter 1. The model embedding the first productivity process matches satisfactorily the empirical autocorrelations of logged TFP, employment and capital, and captures the life-cycle profile of average employment by age. The model with the more standard process is worse at capturing these empirical dynamic moments.

I use the two calibrated models to perform quantitative experiments. I find that, in the economy with heterogeneity in expected growth rates, a 1.11% of firms are classified as high-growth-potential firms. Although this is a small fraction of firms, it accounts for a 3.30% of aggregate TFP and a 7.15% of aggregate output. Nevertheless, only 36.63% of these firms are effectively growing, which indicates that they are affected by frictions. If I eliminate borrowing constraints, there are substantial increases in aggregate TFP (a 20.52% increase), output (22.96%), and the number of effectively high-growing firms. In addition, a 76.20% of high-growth-potential firms realise enough growth, and this group of firms increases its aggregate relevance to 4.30% of TFP and 10.59% of output. The working-capital constraint in the model is an important feature that helps to generate large aggregate results. In the economy with a more standard productivity process, however, there are no high-growth-potential firms, and large and high-growing firms are less in number and aggregate importance. Eliminating the friction in this economy generates lower gains in aggregate TFP (14.07%) and output (13.59%) than in the economy with growth-potential heterogeneity. Therefore, using a model that omits this source of heterogeneity would lead us to underestimate the aggregate productivity and output gains from removing financial frictions.

Quantitative exercises show that explicitly accounting for life-cycle features in the productivity process (namely, differences in expected growth rates) altogether with frictions is important not only for capturing empirical firm-level moments, but it also has sizable aggregate implications. If some firms slowly transition to a high productivity level as they age, they start their life earning low profits and performing low investments. After some years, these firms attain their long-run potential, but they are still harmed by borrowing constraints and are inefficiently small. The elimination of financial frictions strongly benefits this group of firms and generates substantial aggregate gains.

Literature review. This chapter relates to the economics literature in several ways. First, it is related to the literature on firm dynamics and individual heterogeneity (Pugsley et al., 2021; Haltiwanger et al., 2013; Hsieh and Klenow, 2014; Moll, 2014; Buera and Shin, 2011; Guvenen, 2007). The main reference is Pugsley et al. (2021). Similarly to Chapter 1, I consider here a firm-level productivity process in the spirit of the employment process in the empirical part of Pugsley et al. (2021). However, our structural models differ. The authors rely in a frictionless theoretical framework, and find that a small fraction of firms that have a high expected growth rate represents an important share of average employment by age and aggregate output[1]. My contribution to their work is to study how life-cycle components in the firm productivity process interact with explicitly modelled frictions

[1]More specifically, they find that a 5.4% of US firms, which they call *gazelles*, account for about 25% of average employment by age 19 years after birth. Moreover, they show that a small decline in the fraction of gazelles that happened in time (from 6.4% in the 1979-1985 period to 5.3% in the 1986-1993. period) has translated into a 4.85% fall in aggregate output.

to capture firm-level moments in the data and generate aggregate effects. Buera and Shin (2011) and Moll (2014) develop theoretical models in which heterogeneous firms face collateral constraints. In these works, a standard firm-level productivity process with persistent shocks is considered. They show that if productivity shocks are highly persistent, as it is often the case in the literature, aggregate productivity losses from financial frictions in steady state are small. I contribute to these works by showing that explicitly accounting for life-cycle components in the productivity process (which are in turn informed from the firm life-cycle evidence), in addition to ex-post shocks, allows to generate large steady-state aggregate effects of financial frictions.

Second, this chapter is tightly related to the capital misallocation literature in Spain using firm micro-level data. Following the tradition initiated by Restuccia and Rogerson (2008) and Hsieh and Klenow (2009), several authors have studied the role of a bad allocation of resources, particularly capital, and financial frictions in Spain in recent decades (see, for example, Gopinath et al. (2017), Almunia et al. (2018), Fu and Moral-Benito (2018), García-Santana et al. (2020) and Ruiz-García (2021)). None of these papers consider the idea that firms may be heterogeneous in their expected growth rates, and they do not study the aggregate importance of high-growth-potential firms. I contribute to this literature by considering life-cycle components in the productivity processes that explicitly allow for such phenomena, and their aggregate effects when interacted with financial frictions.

Layout. This chapter is structured as follows: in section 2.2, I develop a structural model of the firm with frictions and a firm-level productivity process, and I calibrate two alternative models to the Spanish data. In section 2.3, I use calibrated models to perform quantitative experiments and I discuss results. In section 2.4, I conclude.

2.2 Structural Model

2.2.1 Firms

I consider a stationary small open economy with heterogeneous firms, exogenous firm entry, and both exogenous and endogenous firm exit. Time is discrete and infinite. Every period, a continuum of firms with mass 1 enters the economy and starts producing. I call age $t = 0$ this initial productive stage in a firm's life, and entrants are called age-0 firms. After age 0, a firm may exit the economy or continue its activity. I study firm decisions over the life cycle, and assume that labour supply is fixed over time, $\overline{L}$. At every period, firms are heterogeneous in their productivity levels and in their capital stocks.

Technology. At every period, a firm i of age t that has not exited the economy produces a variety of the final good of which it is a monopolist, following Melitz (2003). Production of firm i at age t is called Y_{it}, and it requires using hired labour L_{it} and capital K_{it} that is owned by the firm. Period payoffs are discounted at an exogenously given discount rate r. Each firm i faces a downward-sloping demand curve $Y_{it} = P_{it}^{-\sigma}$, where P_{it} is the price of the variety, and $\sigma > 0$ is the elasticity of substitution between varieties. All varieties are produced according to a Cobb-Douglas technology $Y_{it} = A_{it}K_{it}^{\alpha}L_{it}^{1-\alpha}$, where the capital share α is assumed to be equal for all firms, and A_{it} is the productivity of firm i at age t. Consistent with the TFP measure used in Chapter 1, I impose $\sigma = 3.0$ and $\alpha = 0.35$. Given these assumptions, we can express the revenue of firm i at age t as $P_{it}Y_{it} = \left(A_{it}L_{it}^{1-\alpha}K_{it}^{\alpha}\right)^{1-\frac{1}{\sigma}}$. I express the logarithm of A_{it} as a function $log(A_{it}) = h(\varphi_{it})$, where φ_{it} is a vector of firm-level characteristics that are taken as given by the firm and vary over its life cycle. Specifically, φ_{it} may include ex-post shocks to productivity and/or ex-ante components, depending on the life-cycle productivity process assumed.

Capital accumulation. The capital stock of a firm evolves as it ages, starting from an exogenous initial draw that varies across firms. When a firm is born, it draws an initial capital stock $K_{i,-1}$ from a time-invariant distribution. Then, when a firm starts producing at age 0, it uses the initial capital draw to produce, so that $K_{i0} = K_{i,-1}$ is exogenous. More specifically, I assume that $K_{i,-1}$ is drawn from a bivariate log-normal distribution of initial capital and productivity. The parameters characterising this exogenous distribution are chosen to match cross-sectional variances and covariances of logged capital and logged TFP in the data for age 0 – that is, $\sigma^2_{\hat{K}_0} = 2.30$, $\sigma^2_{T\hat{F}P_0} = 1.01$ and $\sigma^2_{\hat{K}_0,T\hat{F}P_0} = -0.07$[2]. This allows initial capital draws to replicate the data on capital at age zero well.

At ages greater than zero, capital stock follows a standard law of motion of the form $K_{it} = I_{it} - (1-\delta)K_{i,t-1}$, where I_{it} is the investment exerted at age t and δ is the depreciation rate. Importantly, when a firm chooses to invest, it pays a convex capital adjustment cost, which is assumed to take a standard quadratic specification in investment:

$$C(K_{i,t-1}, K_{it}) = \frac{\psi}{2}\frac{\left(K_{it} - (1-\delta)K_{i,t-1}\right)^2}{K_{i,t-1}}$$

where $\psi > 0$ parameterises the intensity of the convex adjustment cost. The cost is paid in the period when the investment effort is exerted and when the new capital becomes productive.

[2] I also discipline initial capital draws to capture the average level of capital at age 0 in the data, which equals 115.6. Notice that the covariance between logged initial capital and logged initial TFP in the data is very close to zero. Indeed, the correlation coefficient of these two logged variables is −0.047 in the data, and 0.035 in levels. This justifies the adoption of the exogenous initial capital draws assumption: should capital be optimally chosen at age $t = 0$, the correlation between initial capital and initial TFP would be larger, contrary to the data.

Markets of factors. At every period in time, capital is supplied inelastically in the global capital market, and firms take the interest rate r as given. Regarding labour, firms take the stationary price of labour ω as given, which is determined in equilibrium to clear the labour market, given the fixed aggregate supply of labour. Importantly, I assume that firms are subject to a borrowing constraint at the firm level that works as a upper bound on investment and labour expenditures according to:

$$\lambda^L \omega L_{it} + I_{it} \leq \lambda K_{i,t-1} \tag{2.2.1}$$

This borrowing constraint limiting investment follows closely Cooper and Ejarque (2003) and it enables me to keep the state space of the firm problem small – in particular, by not including debt. The upper bound $\lambda K_{i,t-1}$ is linear in the capital owned by the firm at the beginning of age t, and it is parameterised by λ. If $\lambda = \infty$, neither investment nor labour expenditures are restricted by the borrowing constraint. On one side, a positive λ imposes a bound from above to investment, and therefore functions as a restriction on (fixed) capital. On the other side, it limits labour expenditures ωL_{it} up to a fraction λ^L, and thus represents a working-capital constraint, which is frequent in the business cycle literature (Jermann and Quadrini, 2008; Quadrini, 2011). It embeds the idea that firm access to current assets, or working capital, is also limited to some extent, and that labour expenditures are correlated to current assets. This positive correlation between labour expenditures and current assets is indeed a feature of the Spanish sample: the unconditional correlation between these two variables in the data is 0.57, and the correlation by age between current assets and labour expenditures fluctuates between 0.50 and 0.59 over the first ten periods after firm birth. A positive λ^L generates a wedge in labour that allows the model to better replicate empirical autocorrelations of employment by age, while not increasing the computational burden. If we instead set $\lambda^L = 0$, employment autocorrelations generated by the model are too low relative to those in the data. In section 2.3.3, I study an alternative economy without working-capital constraints.

Firm entry and exit. Entrants and incumbents coexist in the economy. Firm entry is exogenous. There is a mass 1 of age-0 entrants that start producing every period, given K_{i0} and φ_{i0}. After age 0, firms may exit either exogenously or endogenously. At the beginning of age $t > 0$, exogenous exit may happen with probability $\nu \in (0, 1)$. If the firm does not exit exogenously at age $t > 0$, it may exit endogenously and obtain a value of zero. Otherwise, if the firm decides to remain in the market, it must pay a fixed operation cost $f_{OP} > 0$. Then, the firm chooses $P_{it}, Y_{it}, I_{it}, K_{it}$ and L_{it} to produce.

Firm problem. Let us adopt a dynamic programming notation and omit subscript

i for exposition purposes. The problem faced by an entrant (i.e. an age-0 firm) is that of choosing how much labour to hire in order to produce. Both K_{-1} and φ are taken as given by the firm at entry. I assume that labour at age 0 is chosen optimally. For simplicity, I assume that labour expenditures at age 0 are not restricted by a borrowing constraint. This results in firm labour demand at age 0 being a function of the level of the initial capital draw and productivity components at age 0, $g_0^L(K_{-1}, \varphi)$.

Incumbent firms that have not exited exogenously (i.e. age-t firms, with $t > 0$) enter age t with a firm-level state (K_{-1}, φ). The age-t incumbent decides whether to exit endogenously, thus having the following value:

$$V(K_{-1}, \varphi) = \max\{0, V^{NE}(K_{-1}, \varphi)\}$$

Upon not exiting, the optimisation problem of an incumbent with state (K_{-1}, φ) is:

$$\begin{aligned} V^{NE}(K_{-1}, \varphi) = \max_{L,\, K} \quad & \Big\{ A^{\gamma} L^{\xi} K^{\eta} - \big(K - (1-\delta)K_{-1}\big) - \omega L - f_{OP} \\ & - \frac{\psi}{2} \frac{\big(K - (1-\delta)K_{-1}\big)^2}{K_{-1}} + \frac{1}{1+r}(1-\nu)E[V(K, \varphi') \mid \varphi] \Big\} \\ \text{s.t.} \quad & log(A) = h(\varphi), \\ & \lambda^L \omega L + K \leq \big(\lambda + (1-\delta)\big) K_{-1} \end{aligned}$$

From the problem of the incumbent, we get an exit policy $x(K_{-1}, \varphi) \in \{0, 1\}$ (where $x = 1$ if the firm decides to exit), and labour and capital policies, $g^L(K_{-1}, \varphi)$ and $g^K(K_{-1}, \varphi)$.

Aggregation and market clearing. Let $s = (K_{-1}, \varphi)$ be the state of a firm at a point of its life, and S be a set of possible values for s. Let $\Gamma_t(S)$ be the measure of age-t firms in S. At age 0, there is a mass 1 of firms that are distributed across capital and productivity according to a distribution $\Gamma_0(S)$. This distribution is exogenous, provided that firms draw initial capital and productivity components from exogenous distributions. Total labour demand at age 0 is thus:

$$L_0^D(\omega) = \int_s g_0^L(s; \omega) d\Gamma_0(s)$$

where I make explicit the dependence of individual labour demands on the equilibrium price of labour, ω. At age 1, endogenous capital accumulation and the exogenous evolution of the productivity vector (which depends on the specific process we assume) give rise to a transition law for s. Thus, the measure of age-1 firms in S satisfies:

$$\Gamma_1(S') = \int_s F(S' \mid s, g_K, g_x)[1 - x(s)](1 - \delta)d\Gamma_0(s)$$

where $F(S' \mid s, g_K, g_x)$ is the transition law for the state vector; I make it explicitly dependent on capital and exit policies. As a result, the measure of age-1 firms in the economy is $M_1 = \int_s d\Gamma_1(s)$, and their total labour supply is $L_1^D(\omega) = \int_s g^L(s;\omega)d\Gamma_1(s)$. We can pin down the distribution of age-t firms for any t, $\Gamma_t(S)$, the mass of age-t firms in the economy M_t, and the total labour demand of age-t firms $L_t^D(\omega)$. Summing age-t masses for all t gives us the (endogenous) mass of firms in the stationary economy.

Labour market clearing implies that aggregate labour supply equals aggregate labour demand, which is the sum of total labour demands of all cohorts of firms:

$$\overline{L} = \sum_{t=0}^{\infty} M_t L_t^D(\omega)$$

The stationary-equilibrium wage ω is then determined from this equation.

2.2.2 Productivity Processes

I consider two alternative productivity processes at the firm-level. These two processes differ in the existence of ex-ante heterogeneity in expected growth rates. The first process is the GP process in equation (1.3.1) in Chapter 1 (with $a = t$). The GP process explicitly considers both ex-ante and ex-post heterogeneity in productivity growth rates. The life-cycle evolution of the ex-ante component u is known at birth, while the ex-post w component is stochastic. Before production starts at age 0, each firm draws and observes a tuple $(\theta_i, u_{i,-1})$ that determines the evolution of the ex-ante component towards a firm-specific long-run steady state. Under the GP process, the vector of productivity components of firm i at age t is $\varphi_{it} = (\theta_i, u_{it}, w_{it})$. In section 1.3.1 of Chapter 1, I have shown that the GP process captures empirical micro-level patterns of Spanish firms, namely life-cycle autocorrelations of logged TFP, better than a more standard process that ignores firm heterogeneity in expected growth rates.

The second process for logged productivity does not consider ex-ante heterogeneity in expected growth rates, as it is standard in the firm dynamics literature. I call it the NGP process (for *no growth potentials*). It reads:

$$log(A_{it}) = \underbrace{\theta_i}_{\text{ex-ante}} + \underbrace{w_{it} + z_{it}}_{\text{ex-post}} \tag{2.2.2}$$

where

Table 2.1 GP and NGP processes for logged productivity, Spain

Estimated process	ρ_u	ρ_w	σ_θ	σ_u	σ_ε	σ_z	RMSE
GP process	0.208	0.782	0.406	3.864	0.411	-	0.038
	(0.0085)	(0.0055)	(0.0059)	(0.1583)	(0.0021)		
NGP process	-	0.820	0.522	-	0.339	0.382	0.070
		(0.0059)	(0.0041)		(0.0040)	(0.0035)	

Notes: the first row replicates parameter estimates for the GP specification of the process for logged productivity $log(A_{it}) = u_{it} + w_{it}$, which was first shown in the first row of Table 1.1 in Chapter 1. The second row shows parameter estimates for the NGP specification in equation (2.2.2). The standard errors of parameter estimates are in parentheses, and are small due to the very large sample size. Standard errors are calculated using a parametric bootstrap procedure with 1000 replications.

$$
\begin{aligned}
&\theta_i \sim N(\mu_\theta, \sigma_\theta^2) \\
&w_{it} = \rho_w w_{i,t-1} + \varepsilon_{it}, && w_{i,-1} = 0,\ \varepsilon_{it} \sim N(0, \sigma_\varepsilon^2) \\
&z_{it} \sim N(0, \sigma_z^2)
\end{aligned}
$$

The NGP process has an ex-ante component that consists of a firm-specific fixed effect θ_i drawn from a normal distribution previous to age 0. This fixed effect does not display any transition towards a long-run steady state, since $\rho_u = \mu_u = \sigma_u = 0$ and $u_{i,-1} = 0$, and remains constant over the firm life cycle[3]. Regarding the ex-post component, the firm receives persistent AR(1) shocks and, additionally, an iid shock z_{it} every period. The presence of z is the only difference with respect to the "AR(1)+fixed effect" process in the third row of Table 1.1 in Chapter 1, and it is added here to compensate for excessive autocorrelations of logged TFP generated by the "AR(1)+fixed effect" process (see Figure 1.1(c) in Chapter 1) and to improve the match of empirical autocovariances. Under this process, the vector of productivity components of firm i at age t is $\varphi_{it} = (\theta_i, w_{it}, z_{it})$.

I estimate each of these two processes autonomously using empirical autocovariances of logged TFP by age, as in section 1.3.1 of Chapter 1. The estimates for the GP and the NGP processes are shown in Table 2.1. The first row of the table replicates the "GP process" row in Table 1.1 in Chapter 1. The second row shows estimates of $(\rho_w, \sigma_\theta, \sigma_\varepsilon, \sigma_z)$ for the NGP process. As we can see, the match of empirical autocorrelations is worse in terms of RMSE when we consider the NGP process relative to the GP process[4]. Once productivity processes are estimated, I plug them

[3] In this setting, we can see θ_i as a firm-specific long-run steady state to which the firm converges immediately at birth.

[4] Indeed, when we consider an AR+fixed-effects process without the iid shock z, the RMSE equals 0.076, so this alternative process is similar to the AR+fixed-effects in the third row of Table 1.1 in Chapter 1 in terms of goodness of fit. The most important difference with respect to this process is that the estimate of ρ_w increases when we add an iid shock.

Table 2.2 Calibration of structural model, different productivity processes

Parameter	Definition	GP economy	NGP economy
Set a priori			
r	Discount rate	0.087	0.087
δ	Depreciation rate	0.05	0.05
ν	Probability of exogenous exit	0.10	0.10
ω	Price of labour	1.0	1.0
σ	Elasticity of substitution	3.0	3.0
α	Production function	0.35	0.35
μ_w	Ex-post component w, average	0.0	0.0
Set to target moments			
ψ	Convex capital adjustment cost	0.372	0.441
λ	Upper bound in borrowing constraint	0.306	0.220
λ^L	Working-capital constraint parameter	0.415	0.495
f_{OP}	Period fixed operation cost	0.820	0.939
μ_θ	Ex-ante component θ, average	1.551	1.167
μ_u	Ex-ante component u, average	-1.522	–
ρ_u	Ex-ante component u, persistence	0.208	–
ρ_w	Ex-post component w, persistence	0.782	0.820
σ_θ	Ex-ante component θ, variance	0.406	0.522
σ_u	Ex-ante component u, variance	3.864	–
σ_ε	Ex-post component ε, variance	0.411	0.339
σ_z	iid shock z, variance	–	0.382

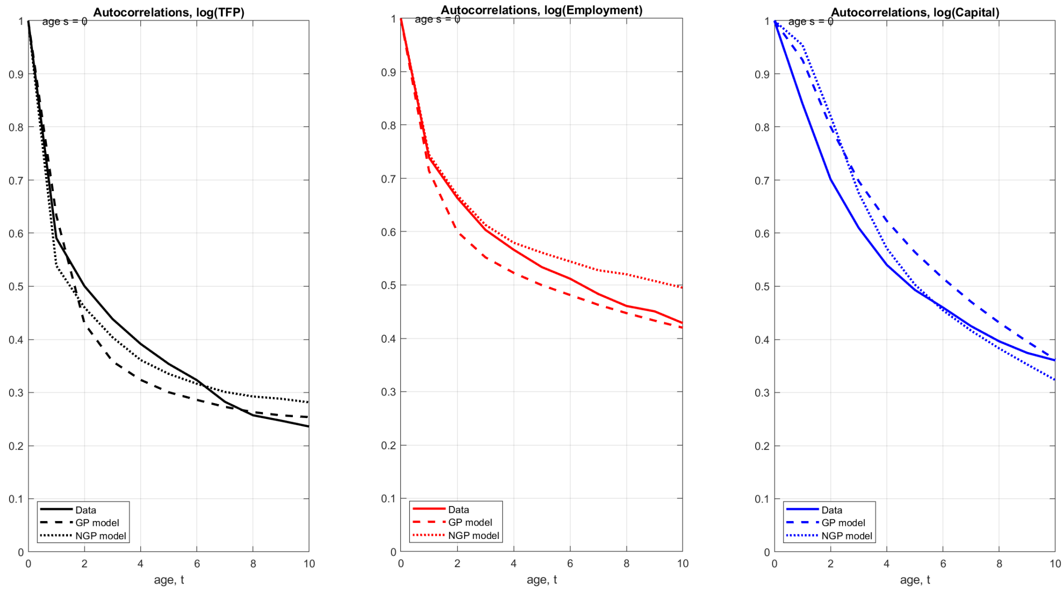
into the structural model, each in turn, and I calibrate the rest of parameters.

2.2.3 Calibration

This section discusses the calibration of the structural model considering each of the two productivity processes in turn. Parameters for the GP and the NGP cases are shown in Table 2.2. I discuss how each of the two calibrated models is capable of replicating a set of relevant targeted and untargeted moments.

After having estimated the corresponding productivity process autonomously, as described above, I calibrate parameters $(\psi, \lambda, \lambda^L, \mu_u, \mu_\theta, f_{OP})$ to match the empirical autocorrelations by age of logged employment and capital at age 0, as well as the empirical average employment by age at age 0 (leaving the rest of the average employment profile over the life cycle untargeted). All targets come from the permanent sample. Specifically, my targeted autocorrelations are the three lines in Figure 1.2(b) in Chapter 1, corresponding to age $s = 0$, for they contain enough information about how labour and capital behave over the firm life cycle. I assume that the mean parameter of the AR(1) component of the productivity process, μ_w, is equal to zero in both processes, following Pugsley et al. (2021). Regarding parameters set a priori, they coincide in both models. I set the elasticity of substitution equal to 3.0

FIGURE 2.1 Targeted moments: GP and NGP models



(a) Logged TFP (b) Logged employment (c) Logged capital

Notes: subfigures (a), (b) and (c) respectively show autocorrelations for logged TFP, employment and capital for age 0. The x-axis represents age t, and lines in the graph correspond to another age $s = 0$. Solid lines correspond to autocorrelations from the data. Dashed lines correspond to autocorrelations for a simulated permanent sample using the GP model and parameter values in Table 2.2, column "GP economy". Dotted lines correspond to autocorrelations for a simulated permanent sample using the NGP model and parameter values in Table 2.2, column "NGP economy".

and the capital share in the production function equal to 0.35, following Gopinath et al. (2017). The depreciation rate $\delta = 0.05$ and the probability of exogenous exit $\nu = 0.1$ are standard in the firm dynamics literature. I assume an interest rate of 0.087. Finally, I normalise the aggregate supply of labour $\overline{L}$ in order to have a price $\omega = 1$ in equilibrium. Using parameters in Table 2.2, I solve the two models and then I simulate 500,000 firms over 20 years in each setting.

Targeted moments. Let us first show how the GP model and the NGP model behave in terms of replicating empirical autocorrelations of logged labour and capital at age zero[5]. In Figure 2.1, I show autocorrelations of logged TFP, employment and capital for age zero. Solid lines correspond to empirical autocorrelations from the Spanish sample at age $s = 0$. Dashed and dotted lines correspond to simulated

[5]The discussion in section 1.3.1 and Figure 1.1 in Chapter 1 regarding the autocorrelation of logged TFP generated also applies here, since parameters in the GP process were estimated previously. Indeed, the estimated NGP process is inferior than the GP process in terms of replicating the entire TFP autocorrelation map.

autocorrelations using permanent samples of firms from the GP and the NGP model, respectively. The properties imposed on the structural model, namely the value of parameters $(\psi, \lambda, \lambda^L, \mu_\theta, f_{OP})$, affect the relative position and curvature of the autocorrelation curves at age zero in both models. In particular, a higher λ, i.e a less tight borrowing constraint, translates all else equal into a lower autocorrelation of logged capital, but also reflects in a more attenuated manner (due to $\lambda^L < 1$) in the autocorrelation of labour, which also becomes lower. A higher λ^L increases all else equal the labour wedge, thus shifting the autocorrelation of logged employment up[6].

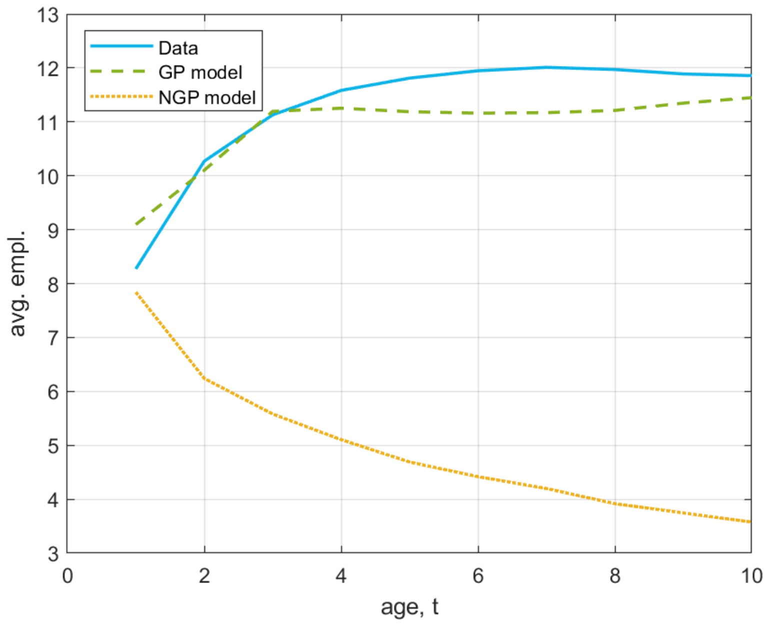
Both the GP and the NGP model are able to replicate reasonably well autocorrelations for logged employment and capital in the data. The properties I impose on the GP model (dashed lines) generate long-run autocorrelations (i.e. autocorrelations between age 0 and age 10) in line with those in the data, while the NGP model (dotted lines) misses these long-run autocorrelations. The GP model captures well the slope of the solid blue line, corresponding to the autocorrelation of capital, thus suggesting the appropriateness of considering both a financial friction (parameterised by λ) an a convex capital adjustment cost (parameterised by ψ). Indeed, a sufficiently large $\psi = 0.372$ and a sufficiently low $\lambda = 0.306$ are able to shift the blue curve up, further enough from the TFP, black curve. However, in the short term, it generates a too high autocorrelation of capital. Autocorrelation of labour is well captured by the GP model, specially in the long run. The convex, decreasing shape in the empirical curve is present in the simulated curve as well. This is owing to the presence of a working-capital constraint: the positive value for $\lambda^L = 0.415$ generates a sufficiently large autocorrelation of labour over the firm life cycle, thus allowing for a good match of the data. Interestingly, in the long run, the dashed red and blue lines are close to each other, in line with the data.

Regarding the NGP model, it also matches reasonably well targeted autocorrelations. Differently from the GP model, it fails at capturing long-run autocorrelations of logged TFP, employment and capital, but it does a better job in capturing short-run autocorrelations. The worse match of long-run moments is due to the fixed-effect nature of θ and the lack of flexibility of the NGP process. This idea also extends to long-run autocorrelations of inputs, and to higher-ages autocorrelations[7]. In particu-

[6] These effects are similar if we consider higher-ages autocorrelations (i.e for ages $s > 0$), which are untargeted in my calibrations. In Appendix A.1, I discuss how the GP and NGP model capture higher-ages autocorrelations.

[7] Higher-ages autocorrelations (for ages $s > 0$) are untargeted in my calibrations. In Appendix A.1, I discuss how the GP and the NGP models replicate them. As shown there, for ages higher than age 0, the GP model replicates well the autocorrelations of logged TFP (as it was also shown in Figure 1.1(d) in Chapter 1) and employment, but generates autocorrelations of capital that are too high in comparison to the data. In spite of this, relative positions of the three curves for these upper ages are preserved by the model. In the NGP model, higher-ages autocorrelations are too low relative to the TFP and employment data, both for short and long lags, indicating the lower ability of the NGP model to replicate these life-cycle objects.

Figure 2.2 Average employment by age: GP and NGP models



Notes: the blue, solid line shows the empirical average employment by age for Spanish firms, calculated using the permanent sample and yearly average number of employees in levels. I take out industry and cohort fixed effects in order to examine an average industry and cohort in the economy. The green, dashed line shows the average employment by age in a simulated permanent sample using the GP model, given the baseline parameterisation in Table 2.2, column "GP economy". The yellow, dotted line shows the same simulated object using the NGP model with parameters in Table 2.2, column "NGP economy" .

lar, it generates a long-run autocorrelation of logged employment that is higher than that in the data, and a long-run autocorrelation of logged capital that is lower than the empirical one. It is worth noting that, due to this, the NGP model generates a sizable long-run difference between the red and blue dotted lines. This difference between long-run autocorrelations of capital and employment is much smaller in the data, as shown in Figure 1.2(b) in Chapter 1. Overall, both the GP and the NGP model provide a good match of the empirical autocorrelation patterns, but the GP model replicates better the long term and the NGP model is better at capturing short-term trends.

Notice that, relative to the GP model, calibrated parameters $(\psi, \lambda, \lambda^L, f_{OP})$ in Table 2.2 are larger in the NGP model. Given equation (2.2.2), a firm reaches its steady-state level of productivity immediately at birth. Therefore, if these parameters had the same (lower) values as in the GP model, this would allow firms to adjust production factors more strongly over their first years of life. As a result, simulated autocorrelations of employment and capital would be lower than in the data. Matching this piece of evidence using the NGP model thus requires that friction parameters are larger relative to the GP model.

Untargeted moments. In Figure 2.2, I show how the GP and the NGP models behave in terms of untargeted moments – specifically, the average employment by age over the life cycle for ages greater than zero, for a permanent sample of firms, which is of particular interest to us. The GP model replicates well the empirical concave shape of the curve for the Spanish sample, which is notably different from the US pattern in Figure 1.3(a) in Chapter 1. Nevertheless, the NGP model is not capable of capturing such an empirical feature. The average Spanish firm is shrinking as it ages: average employment declines from 7.84 at age 0 to 3.75 at age 10, at odds with the data.

This declining average size profile is partially related to the high calibrated values of parameters representing frictions, but it is the lack of expected growth that is key in shaping the curve. Provided that the expected productivity growth rate at birth is equal to zero for all firms, the average Spanish firm starts its activity with an initial capital endowment that is too large. As a consequence, it is optimal for the average firm to disinvest over time, so that the simulated average capital decreases as firms age, and low capital accumulation renders Spanish firms small after some periods[8]. To sum up, the GP model is superior to the NGP model in terms of capturing relevant untargeted empirical moments of the firm life cycle.

The concave age profile for average employment in the data denotes that Spanish firms have on average difficulties to grow strongly during their first years of life. Indeed, the two calibrated models suggest that frictions faced by firms are playing a role in determining average firm growth. In the next section, I perform quantitative experiments to study how borrowing constraints affect the allocation of production factors, and how they relate to this pattern of slow growth over the life cycle and to the aggregate performance of the Spanish economy.

2.3 Quantitative Experiments

I use the calibrated GP and NGP models to assess how financial frictions interact with the firm-level productivity process, how borrowing constraints deter Spanish firms from growing over their life cycle, and the resulting effects on aggregate output and total factor productivity. I pay particular attention to two subgroups of firms, those that have a high expected productivity growth at birth and those that effectively realise a high level of employment growth. As we shall see, the aggregate importance of these firms is different in the GP and in the NGP model.

I start out by introducing two definitions of high-growth firms, an *ex-ante* def-

[8]Larger adjustment costs of factors also play a role here. However, quantitative analysis shows that relaxing different firm-level frictions shifts the average employment by age curve up, but does not revert its decreasing pattern as firms age.

inition and an *ex-post* one[9], that are then applied to simulated firms. First, let us define *ex-ante high-growth potential firms* (labeled HGP). A firm is an HGP firm if the average yearly growth rate of its deterministic component in the productivity process in levels (that is, e^u) is greater or equal than 20% over the first 5 years of life, and if e^u is greater or equal than a given level at some point in the first 5 periods of life. Specifically, I require that u is greater or equal than two standard deviations above the mean of firm-level long-run steady states in the cross-section of simulated firms, so that $e^u \geq e^{\mu_\theta/(1-\rho_u)+2\sigma_\theta/(1-\rho_u)}$, at some point during the first five years. I thus restrict to firms whose expected productivity after some years of activity not only has grown a lot, but also it ends up being very high. This definition follows closely the ex-ante definition in Pugsley et al. (2021). Second, let us define *ex-post, effective high-growth firms* (labeled EHG). A firm is an EHG firm if the average yearly growth rate of firm-level employment L is greater or equal than 20% over the first 5 years of life, and if L is greater than a given level (I consider $L \geq 25$) at some point in the first 5 periods of life. EHG firms are those who have effectively realised high growth rates *in terms of employment* and ended up being sufficiently large.

It may be the case that firms labeled as HGP do not fit the definition of EHG, and vice versa. For example, we can imagine a firm that is not an HGP firm but received good realisations of the ex-post productivity shock w over its first years of life, so it ended up displaying effective growth and fitting the EHG definition. Similarly, a firm that is ex-ante categorised as HGP may be prevented from realising its growth due to a very stringent borrowing constraint. The share of EHG firms among the group of HGP firms provides information on the extent to which high-growth-potential firms are deterred from effectively growing in the Spanish economy.

[9] I also consider, as robustness checks, definitions such as "a firm is an EHG (or an HGP) firm if it multiplies its size (or ex-ante component) by X during the first 5 years of live. Qualitative results from these alternative definitions are similar to the ones presented here.

Table 2.3 Baseline economies: GP and NGP model

	GP economy	NGP economy
Panel A: Firm life-cycle characteristics		
% ex-ante HGP firms	1.11	0.0
% ex-post EHG firms	6.53	0.36
Average employment, age 19	12.66	2.74
% ex-ante HGP that grew ex-post	36.63	–
Panel B: Aggregate variables in steady state		
Equilibrium price ω	1	1
% of firms above 250 employees	0.04	0.002
% of aggregate employment, firms above 250 employees	1.15	0.13
Panel C: Aggregate relevance of EHG firms		
% aggregate TFP	11.10	1.20
% aggregate output	28.22	2.68
% aggregate employment	20.33	1.80
% aggregate capital	17.23	0.94
Panel D: Aggregate relevance of HGP firms		
% aggregate TFP	3.30	0.0
% aggregate output	7.15	0.0
% aggregate employment	4.30	0.0
% aggregate capital	3.50	0.0

Notes: the first column shows simulated moments for the GP model (with heterogeneity in expected productivity growth rates) in Table 2.2, column "GP economy". The second column shows these simulated moments for the NGP model (without heterogeneity in expected productivity growth rates) in Table 2.2, column "NGP economy". Results in Panel A refer to characteristics of simulated firms in each of the two settings. Panel B shows results concerning aggregate outcomes. Panels C and D focus on the aggregate importance of the subgroups of effective high-growth firms (EHG) and high-growth-potential firms (HGP).

2.3.1 Baseline Economies

I study the differences between the GP model and the NGP model in terms of the number of EHG firms, the fraction of HGP firms that end up displaying sufficient growth, and the aggregate role of these groups of firms. To do this, I consider the GP and the NGP models and simulate 500,000 firms over 80 years, in each of them. Table 2.3 shows simulated moments for the GP and the NGP economies.

Firm-life-cycle and aggregate moments. Panel A in Table 2.3 shows simulated firm life-cycle moments of interest – namely, the fraction of HGP and EHG firms over all firms, the average employment by age at age 19 of all operating simulated firms[10], and the fraction of HGP firms that are also EHG firms.

Let us first discuss results for the GP economy. In this economy, only 1.11% of all firms fit the definition of HGP firms. Nevertheless, 6.53% of firms display high effective employment growth and become sufficiently large to fit the EHG definition. EHG firms are distinct to the average firm in some margins: EHG firms have, on average, a higher initial capital endowment (185 vs. 116 in the overall simulated GP sample); they have higher realised ex-post productivity shocks w_t averaged over the first 5 years of life (0.31 vs. 0 in the overall sample); and HGP firms are disproportionately represented in the EHG group of firms (6.21% of EHG firms are also HGP, vs. 1.11% in the overall economy). On one side, EHG firms are on average more productive than the average firm in the GP economy, due to higher ex-ante and/or ex-post productivity components. On the other side, EHG firms hold, on average, larger endowments of capital when they are born, which allows them to overcome more easily the financial friction in (2.2.1) when young and realise their growth. This suggests that initial capital is linked to firm growth in Spain[11]. Average employment by age at age 19 of all operating firms is 12.66 employees, in line with the pattern shown in Figure 2.2. This denotes difficulties for the average firm to grow over its life cycle. Importantly, only 36.63% of firms fitting the HGP definition end up fitting the EHG definition. This indicates that high-growth-potential firms are facing difficulties that prevent them from effectively realising their growth.

In the NGP economy, however, there are no HGP firms. This is due to the fact that the ex-ante component of the productivity process does not evolve over time, and hence it is expected to grow at a rate of zero for all firms in the economy. Importantly, the number of EHG firms in the NGP economy is much lower than that in the GP economy: 0.36% of firms fit the definition. In addition, average employment at age 19 of firms in the NGP economy is 2.74. The model with no heterogeneity in expected productivity growth rates is generating smaller firms on average and less high-growing firms than the GP model.

I calculate the share of simulated firms that have more than 250 employees, as well as the fraction of total employment they account for. These numbers are shown in Panel B of Table 2.3. In the GP economy, firms above 250 employees represent a share of 0.04% of total firms. Even if very small, the GP model reflects the reduced presence of very large firms in Spain. This small fraction of firms accounts for 1.15% of aggregate employment in the GP model. The numbers generated by the NGP model are smaller: firms above 250 employees account for a 0.002% of firms and

[10] Here, I adopt a long-term perspective by discussing this moment for age 19. Considering lower age horizons, as in Figure 2.2, yields similar intuitions.

[11] Similar conclusions about the characteristics of EHG firms are obtained under the NGP model, both in the baseline and in the counterfactual economy. I only discuss these characteristics in the GP economy, for briefness.

a 0.13% of total employment in the economy. Thus, under the NGP model, there are less large firms, and they are less relevant to the economy than those in the GP economy.

Aggregate relevance of high-growth firms. I compute the fraction of aggregate TFP, output, capital and employment that is accounted by EHG firms (Panel C in Table 2.3) and HGP firms (Panel D in Table 2.3) in the GP and NGP economies. To do so, I first calculate aggregate variables in an economy considering all simulated firms. Second, I recompute aggregate variables in the same economy, but this time excluding simulated EHG (or HGP) firms from the calculation. Differences in aggregate variables between these two calculations give us the fraction of each aggregate variable accounted by EHG (or HGP) firms. I find that, while EHG firms account for a very substantial fraction of aggregate variables in the GP economy, they represent a notably lower fraction of the aggregate economy in the NGP economy. The higher frictions faced by firms in the NGP economy (see Table 2.2) and the absence of a transition towards high firm-level productivity steady states gives room to this muted role of EHG firms in this economy.

Regarding HGP firms, I find that, even though HGP firms account for only a 1.11% of firms in the GP economy, they represent a 3.30% of aggregate TFP and a 7.15% of aggregate output, thus being disproportionately important for the macroeconomy. This is the case in spite of only about one third of these firms being able to effectively realise their growth. In other words, high-growth-potential firms are playing a non-negligible role in the GP economy – while being absent in the NGP economy. Still, we might wonder to which extent the aggregate role of HGP is being attenuated by the existence of firm-level frictions, particularly borrowing constraints faced by Spanish firms. To address this issue, I perform a counterfactual exercise where I shut down the borrowing constraint.

Table 2.4 No financial friction in alternative economies

	GP economy ($\lambda = \infty$)	NGP economy ($\lambda = \infty$)	NWK economy ($\lambda = \infty$)
Panel A: Firm life-cycle characteristics			
% ex-ante HGP firms	1.11	0.0	1.11
% ex-post EHG firms	12.10	1.39	18.89
Average employment, age 19	13.68	3.78	22.42
% ex-ante HGP that grew ex-post	76.20	–	83.84
Panel B: Aggregate variables in steady state			
% of firms above 250 employees	0.25	0.01	0.63
% of aggregate employment, firms above 250 employees	9.39	1.08	15.94
Equilibrium price ω	1.37	1.25	1.09
% change in aggregate TFP with respect to baseline	20.52	14.07	8.49
% change in aggregate output with respect to baseline	22.96	13.59	12.10
Panel C: Aggregate relevance of EHG firms			
% aggregate TFP	25.45	6.41	31.44
% aggregate output	52.02	13.01	62.12
% aggregate employment	38.71	8.88	47.65
% aggregate capital	29.53	3.59	38.92
Panel D: Aggregate relevance of HGP firms			
% aggregate TFP	4.30	0.0	4.26
% aggregate output	10.59	0.0	10.71
% aggregate employment	7.24	0.0	7.27
% aggregate capital	5.39	0.0	5.72

Notes: the first column shows simulated moments for the GP economy (with heterogeneity in expected productivity growth rates) in Table 2.2, column "GP economy". The second column shows these simulated moments for the NGP economy (with heterogeneity in expected productivity growth rates) in Table 2.2, column "NGP economy". The third column shows these simulated moments for an economy without working-capital constraint (labeled as the NWK economy) discussed in section 2.3.3. Results in Panel A refer to characteristics of simulated firms in each of the two settings. Panel B shows results concerning aggregate outcomes. Panels C and D focus on the aggregate importance of the subgroups of effective high-growth firms (EHG) and high-growth-potential firms (HGP).

2.3.2 Quantitative Experiment: Eliminate Borrowing Constraints

Departing from the GP and NGP economies, I eliminate the borrowing constraint faced by firms by setting $\lambda = \infty$ in the two economies[12]. Due to $\lambda = \infty$, aggregate demand of labour is larger and equilibrium wage increases to $\omega = 1.37$ in the GP economy and to $\omega = 1.25$ in the NGP economy. Table 2.4 shows results from counterfactual exercises.

[12]Changes caused by a partial relaxation of the financial friction are less intense, but go on the same direction as the total elimination of the constraint discussed here.

Firm life-cycle moments. When we eliminate financial frictions, average employment at age 19 increases in both the GP and NGP economies, in spite of the increase in the equilibrium price of labour. Besides, this change in λ increases the fraction of effective high-growth firms from 6.53% in the baseline GP economy to 12.10% (first column in Table 2.4), and from 0.36% in the baseline NGP economy to 1.39% (second column in Table 2.4). These increases indicate that borrowing constraints are actually harming a non-negligible share of young firms and deterring them from growing.

Interestingly, characteristics of effective high-growth firms change when we remove financial frictions. In the counterfactual GP economy, the average initial capital of EHG firms is smaller than that of the average firm (84 vs. 116). This indicates that, without financial frictions, having a high initial level of capital is less linked to firm growth, since more firms with smaller capital endowments at birth are managing to display high growth. It is productivity that takes on a more important role for determining which firms are EHG. If λ goes to infinity, realised ex-post productivity shocks w averaged over the first 5 periods of firm life for EHG firms increase from 0.31 in the baseline GP economy to 0.39 in the counterfactual GP economy. In addition, the fraction of EHG firms that are HGP firms is 7.08, larger than in the baseline GP calibration.

Since parameters characterising productivity processes have not changed, the percentage of HGP firms remains the same in both economies. Importantly, a substantial fraction of HGP firms would be capable of growing if financial frictions were eliminated. If λ goes to infinity, the fraction of HGP firms that are also EHG firms increases from 36.63% to 76.20%. These numbers indicate that explicitly accounting for ex-ante, life-cycle components in the productivity process is important from an economic perspective: a small subgroup of firms with high growth potentials is particularly damaged by financial frictions. This phenomenon cannot be observed through the lens of a more standard productivity process, as it can be seen when studying the NGP model.

Aggregate relevance of high-growth firms. Panels C and D in Table 2.4 show how the aggregate importance of EHG and HGP changes in the alternative economies when we eliminate the borrowing constraint. I find that EHG firms become more prominent in the GP economy without borrowing constraints, in aggregate terms. Their aggregate importance also increases in the NGP economy, but less. Regarding HGP firms, the fraction of aggregate TFP accounted by high-growth-potential firms in the GP setting increases from 3.30% to 4.30%, and their fraction of aggregate output increases from 7.15% to 10.59% if the distortion is completely alleviated.

The fact that a significant fraction of HGP firms is deterred from growing in the GP economy has aggregate implications. Eliminating financial frictions facilitates firm growth of this subset of firms: a higher percentage of HGP firms effectively realise high growth rates and they increase their aggregate relevance, even if they account for a small fraction of all firms in the GP economy. Finally, it is worth noting that HGP firms do not play any role in the NGP economy. Thus, neglecting firm heterogeneity in expected growth rates impedes us from studying a subgroup of firms that, although small in number, is capable of playing a substantial aggregate role in the economy.

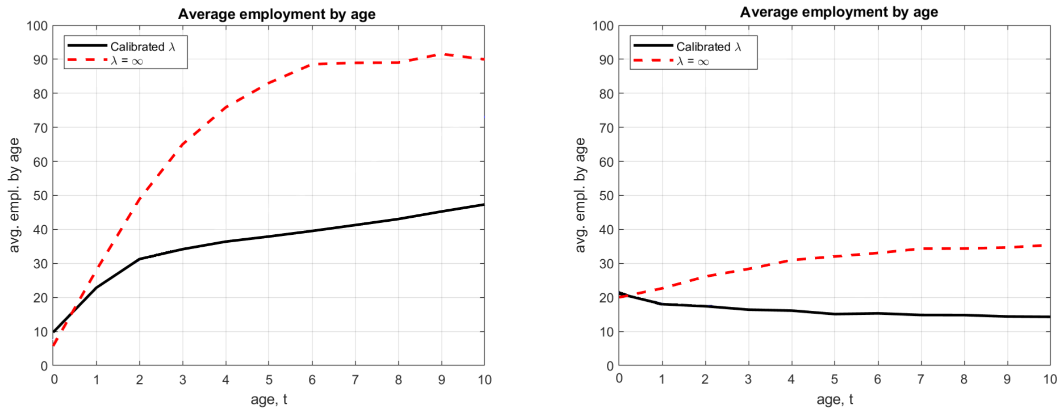
The mechanism that explains that HGP firms increase their macroeconomic relevance in the GP economy when financial frictions are removed is the following. Given the rich GP productivity process, some firms expect to become highly productive only after some years of life, due to the existence of a time-consuming transition to firm-specific steady states. As a consequence of this time delay, these firms will be earning low profits and performing low investments when they are very young. As they age, they end up attaining their long-run potential. However, in the baseline GP economy, a large fraction of these firms are still subject to borrowing constraints after some years because of the slow capital accumulation over their first years of life. Removing these constraints thus strongly benefits high-growth-potential firms, allowing them to grow and become more relevant in the economy.

Aggregate effects of eliminating borrowing constraints. In Panel B of Table 2.4, I discuss the aggregate effects of the elimination of the financial friction in the GP and NGP economies. First, let us consider the GP economy. If we set $\lambda = \infty$ keeping the rest of parameters equal, aggregate TFP increases by 20.52% and aggregate output increases by 22.96% relative to the baseline GP setting, with calibrated $\lambda = 0.306$. Besides, the number of firms that have at least 250 employees becomes 6.25 times larger and their aggregate employment share increases from 1.15% to 9.39%. In the NGP economy, I find that completely eliminating the borrowing constraint increases aggregate TFP (output) by only 14.07% (13.59%). Thus, if we had ignored the life-cycle evidence on firm heterogeneity in expected growth rates when specifying our macroeconomic model, we would have concluded that financial frictions are less important for the aggregate economy.

Firms with high ex-ante components in their productivity processes may contribute significantly to the aggregate economy. Let us analyse how these firms are affected by borrowing constraints in each of the two alternative economies. In the GP economy, I focus on the subgroup of HGP firms, which account for a 1.11% of all firms. In the NGP economy, I look at high-θ firms; in particular, at firms whose fixed effect lies above the 99 quantile of the θ distribution. By definition, these firms account for 1% of all firms in the NGP economy[13]. I observe the simulated average employment by age for these firms. I find that, although the elimination of

[13] Results are robust to considering firms above the 99 quantile of the distribution of θ also in the GP economy, instead of HGP firms.

Figure 2.3 Average employment by age of firms with a high ex-ante component



(a) GP economy, high-growth-potential firms

(b) NGP economy, high-θ firms

Notes: subfigure (a) shows simulated average employment by age of the group of high-growth-potential firms (HGP) in the GP economy, for different values of the borrowing constraint parameter λ. Subfigure (b) shows simulated average employment by age of the group of high-θ firms in the NGP economy, again for different values of λ. The solid, black line corresponds to the calibrated-λ scenario. The dashed, red line corresponds to the $\lambda = \infty$. counterfactual.

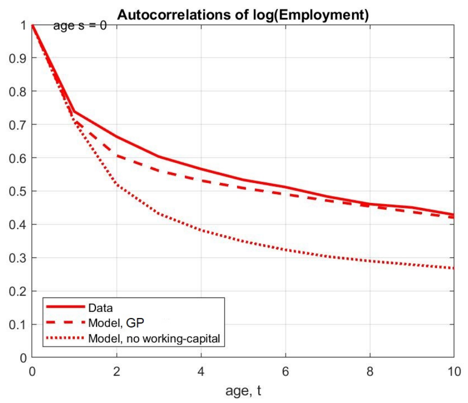
borrowing constraints is affecting the overall economy, its effects are stronger if we focus on firms with high ex-ante components.

In the baseline GP economy, I find that HGP firms display a steep pattern in terms of average employment by age, as shown in Figure 2.3(a). This curve is steeper than that of the average firm in the economy (see Figure 2.2), reaching an average size of around 37 employees after 10 years. Importantly, eliminating borrowing constraints causes HGP firms to grow strongly on average. As a consequence, HGP firms increase their macroeconomic relevance and affect positively the average employment by age for all firms.

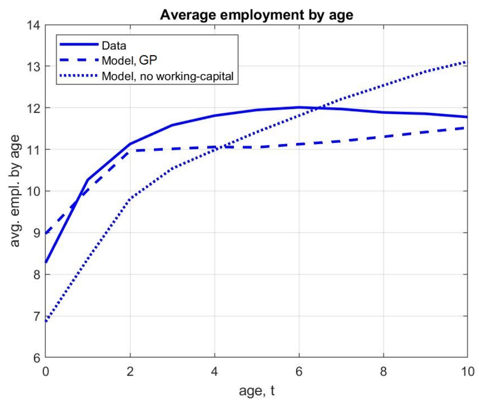
In Figure 2.3(b) for the NGP economy, we observe that, although high-θ firms start large on average, they shrink over time. Althought these firms have received high realisations of their fixed effects, the absence of a life-cycle productivity component generating expected growth makes these firms disinvest as they age, since their initial capital endowments were large at birth. This disaccumulation of capital in time is reverted when we eliminate borrowing constraints. In this case, high-θ grow more on average, but not strongly. This lack of strong growth contributes to low aggregate effects from removing financial frictions in the NGP economy. As a conclusion, accounting for firm heterogeneity in expected growth rates in our structural model, and thus allowing for a rich transition towards firm-specific steady states, matters for generating larger effects of borrowing constraints in the overall economy.

Yet, if we observe the average employment at age 19, in Tables 2.3 and 2.4 for the GP economy, we see that relaxing borrowing constraints increases this simulated

Figure 2.4 NWK economy and empirical moments



(a) Autocorrelation of logged employment



(b) Average employment by age

Notes: subfigure (a) shows autocorrelations for logged employment for age 0, from the empirical permanent sample, and for two simulated samples of 500,000 firms using the structural model: one sample for the GP model in Table 2.2, and another sample for the NWK economy discussed in this section. Subfigure (b) shows the average employment by age from the empirical permanent sample and from two simulated samples corresponding to the same models.

moment from 12.66 to 13.68 – a mild increase. That is, in spite of the sizable effects on aggregate output and TFP, the removal of the friction is incapable of generating a steep average employment by age profile like the one documented for the United States in Figure 1.3(a) in Chapter 1. This suggests that, although important for the Spanish economy, high-growth-potential firms are less productive and converge to lower steady states than their peers in the US.

2.3.3 The Role of Working Capital Constraints

In this section, I consider an environment in which firms do not face working-capital constraints – that is, $\lambda^L = 0$. I keep the rest of the calibration as in the GP model[14] in Table 2.2. I label this economy NWK, for *no-working-capital*. In this setting, equation (2.2.1) becomes simply $I_{it} \leq \lambda K_{i,t-1}$. I argue that having a working-capital constraint is important not only to generate high employment autocorrelations as those in the data, but also to generate a wedge in labour that prevents firms from achieving higher levels of growth. Importantly, I show that aggregate effects from eliminating firm-level borrowing constraints are smaller in the NWK economy than in the GP economy.

Replication of empirical moments. I simulate the NWK model in order to understand what we gain by considering a working-capital constraint in the structural model in terms of replicating moments from the Spanish data. Figure 2.4(a)

[14] A full recalibration of the $\lambda^L = 0$ model is still pending work.

shows the autocorrelation of logged employment in the Spanish data and in the GP and NWK economies. The empirical and the GP curve are close to each other. In the NWK economy, this autocorrelation is too low relative to the data – thus the appropriateness of explicitly considering working-capital constraints to better match the empirical autocorrelation pattern. The fact that there is no labour wedge in the NWK economy facilitates substitution of inputs and makes labour easier to adjust over the life cycle, contrary to data. This, in turn, has consequences on the simulated average employment by age of firms. As shown in Figure 2.4(b), if firms were not subject to a working-capital constraint, their average size over the life cycle would have a steeper profile – again, at odds with the data.

Macroeconomic effects of working-capital constraints. I simulate the NWK economy, as it was done previously for the GP and the NGP economies in section 2.3.1. If $\lambda^L = 0$, I find that 17.34% of firms in the economy are EHG firms. This is a notably larger fraction than in the baseline GP economy (with a working-capital constraint) in Table 2.3, where this share is only 6.53%. The employment share of firms above 250 employees is large (10.12%, vs. 1.15% in the baseline GP economy), and 77.70% of HGP firms are able to effectively grow strongly, while only 36.63% manage to do so in the economy with working-capital constraints. If we eliminate the working-capital constraint, firms find it easier to adjust its labour factor, and more firms manage to grow strongly.

Let us take the baseline NWK economy and perform the same quantitative experiment in section 2.3.2. Results are shown in the third column of Table 2.4. If I eliminate the financial friction in the NWK economy, aggregate TFP increases by 8.49% and aggregate output increases by 12.10%. These numbers are lower than in the GP economy, where removing the friction increases aggregate TFP by 20.52% and aggregate output by 22.96%. The reason for these lower aggregate gains is that, in the baseline GP economy, the borrowing constraint is also restricting firm labour expenditures. In this setting, removing financial frictions directly facilitates labour adjustment, since the wedge on labour is eliminated. In the no-working-capital-constraint economy, however, labour expenditures are not constrained and the labour wedge is equal to zero. Therefore, there are less gains in terms of firm growth and aggregates from eliminating financial frictions, which are only restricting the accumulation of fixed capital and not that of current assets. As a conclusion, including a working-capital constraint in the structural model has important aggregate implications, because it increases the effect of the borrowing constraint on the allocation of labour and prevents firms from growing strongly.

2.4 Conclusion

This chapter explores the role of high-growth potential firms in contributing to explain relevant facts about firm growth and the aggregate performance of the Spanish economy. In particular, the key issue that I analyse is whether firms that are expected to grow strongly when young are prevented from realising their optimal sizes due to the existence of borrowing constraints, and the effects this carries to the rest of the economy.

I propose a structural model consisting on two main elements. On one side, I consider frictions at the firm level. Specifically, firms are affected by a convex capital adjustment cost and a borrowing constraint that limits the amount of investment in fixed capital and in labour expenditures, acting as a working capital constraint. These features allow the model to capture Spanish firm-level dynamic moments. The borrowing constraint hurts firms who are highly productive, yet small in their capital endowments. On the other side, I consider two alternative productivity processes: one that accounts for ex-ante heterogeneity in expected productivity growth rates (embedding the idea of "high-growth-potential firms"), and one that does not consider this margin of heterogeneity.

I calibrate two models, one for each productivity process, and I use them to conduct quantitative experiments. If the "growth-potentials" process is considered, I find that high-growth-potential firms have difficulties to effectively realise their life-cycle growth: only about one third of these firms is capable of attaining sufficiently high sizes over their first periods of life. This phenomenon does not occur in a more standard "no-growth-potential" process. If I eliminate borrowing constraints, I find that the aggregate importance of high-growth-potential firms increases notably in the first economy, but is zero in the economy with a more standard process. Indeed, not considering heterogeneity in ex-ante growth rates would lead our model to underestimate the aggregate gains from eliminating financial frictions.

It is worth noting that several margins of firm heterogeneity have been simplified within the theory developed here. Namely, I have not considered non-convex adjustment costs in capital, which are important to replicate investment dynamics at the micro level and, through this channel, to affect firm dynamics and growth. Additionally, the theory does not consider heterogeneity in access to financial resources, debt issuance and learning about one's own growth potential. Although exciting for explaining life-cycle allocation of resources across Spanish firms, these more complicated features are beyond the scope of this chapter. Nevertheless, I believe that the model with expected growth rate heterogeneity presented here is an appropriate starting point to study these and other firm dynamics features, which are left for future research.

A Appendix to Chapter 2

A.1 Structural Model

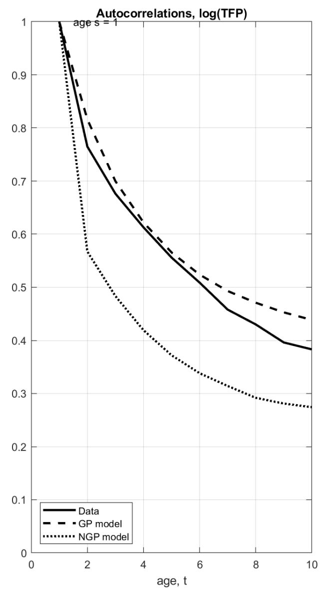
Higher-Ages Autocorrelations

In section 2.2.3, I have calibrated the GP and the NGP model in order to match empirical autocorrelations by age of logged capital and logged employment *at age 0*. The manner these age-0 empirical autocorrelation were captured by the two models was shown in Figure 2.1. Arguably, empirical autocorrelations of higher ages also provide information about the allocation of resources to firms over the life cycle. While these higher-order autocorrelations are not used as targets in my calibration, I can still simulate those untargeted higher-ages autocorrelations and see how the GP and the NGP models capture them.

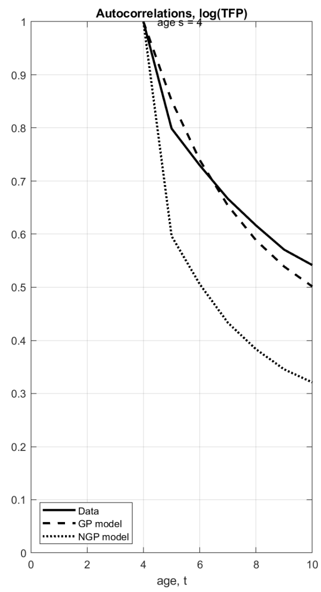
Figures 2.5, 2.6 and 2.7 show some empirical autocorrelations (bold lines) lines for different ages, respectively for logged TFP, employment and capital. In particular, I am considering ages 1, 4 and 7, which are all higher ages than age 0. I compare these higher-ages empirical autocorrelation lines with the same objects obtained from simulating the GP economy (dotted lines) and the NGP economy (dashed lines) in turn. As we can see from Figures 2.5 and 2.6, the GP model clearly outperforms the NGP model in capturing untargeted empirical autocorrelations of logged TFP and employment at ages 1, 4 and 7. The reason for that is the additional flexibility that the initial-condition-plus-steady-state structure of the firm-level productivity process in equation (1.3.1) from Chapter 1 provides, relative to the somewhat simpler process in equation (2.2.2). In particular, the model that explicitly considers firm heterogeneity in expected productivity growth rates is capable of providing a noticeably good match of the employment and TFP data over different stages of the firm life cycle, which increases its credibility as a contender to models where firm heterogeneity is less sophisticated. Regarding the empirical higher-ages autocorrelations of logged capital in Figure 2.7, the NGP model is closer to capture the curve corresponding to age 1, although both the GP and the NGP model generate allocations of capital that are too persistent relative to the data. However, when considering ages 4 and 7, the two models behave similarly.

Overall, the state-of-the art process in equation (1.3.1) from Chapter 1 makes a difference with respect to the more standard AR(1)-plus-fixed-effect process in terms of replicating empirical dynamic moments, particularly higher-ages autocorrelations of logged TFP and employment in the data.

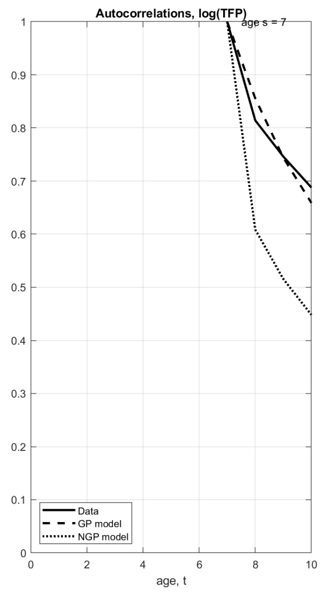
Figure 2.5 Untargeted higher-ages autocorrelations: logged TFP



(a) Age 1



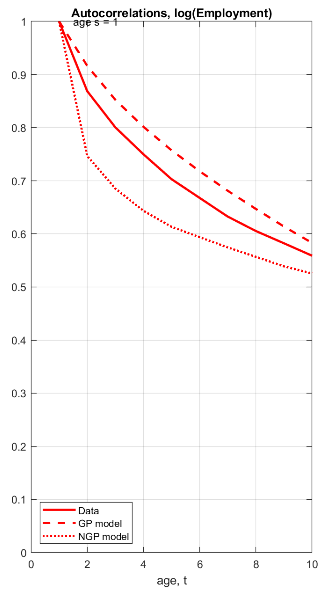
(b) Age 4



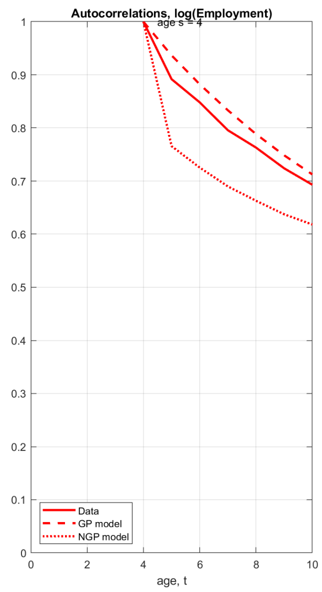
(c) Age 7

Notes: subfigures (a), (b) and (c) show autocorrelations for logged TFP, for ages $s = 1$, 4 and 7 respectively. The x-axis represents age t, and lines in the graph correspond to another age s. Solid lines correspond to autocorrelations from the data. Dashed lines correspond to autocorrelations for a simulated permanent sample using the GP model and parameter values in Table 2.2, column "GP economy". Dotted lines correspond to autocorrelations for a simulated permanent sample using the NGP model and parameter values in Table 2.2, column "NGP economy".

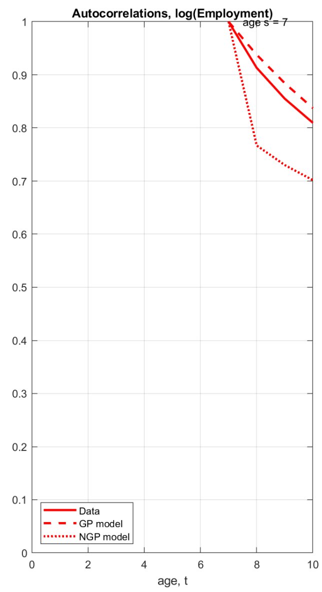
FIGURE 2.6 Untargeted higher-ages autocorrelations: logged employment



(a) Age 1



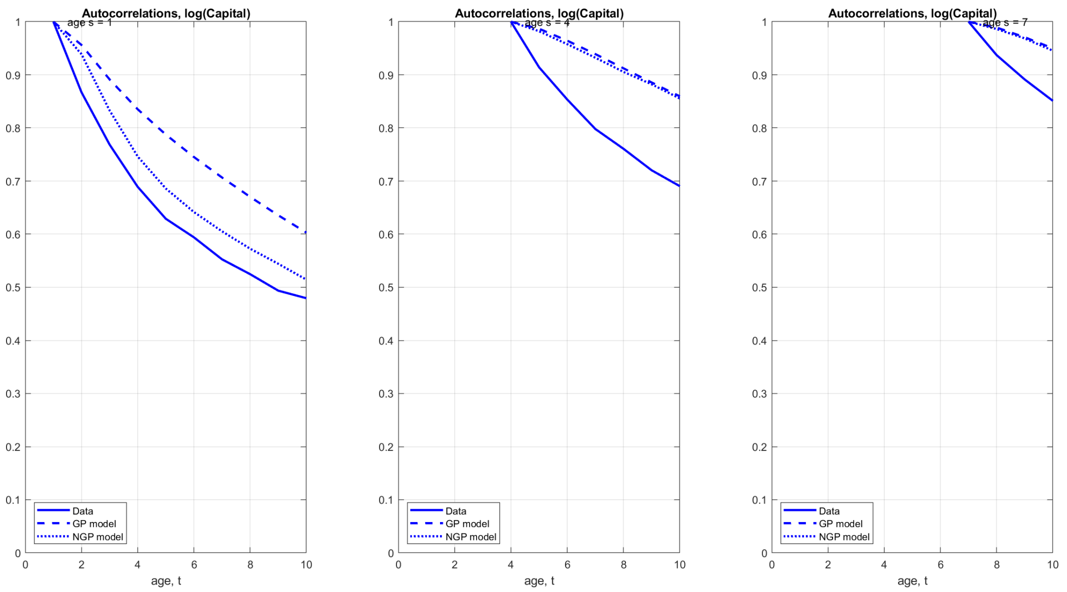
(b) Age 4



(c) Age 7

Notes: subfigures (a), (b) and (c) show autocorrelations for logged employment, for ages $s = 1$, 4 and 7 respectively. The x-axis represents age t, and lines in the graph correspond to another age s. Solid lines correspond to autocorrelations from the data. Dashed lines correspond to autocorrelations for a simulated permanent sample using the GP model and parameter values in Table 2.2, column "GP economy". Dotted lines correspond to autocorrelations for a simulated permanent sample using the NGP model and parameter values in Table 2.2, column "NGP economy".

FIGURE 2.7 Untargeted higher-ages autocorrelations: logged capital



(a) Age 1 (b) Age 4 (c) Age 7

Notes: subfigures (a), (b) and (c) show autocorrelations for logged capital, for ages $s = 1$, 4 and 7 respectively. The x-axis represents age t, and lines in the graph correspond to another age s. Solid lines correspond to autocorrelations from the data. Dashed lines correspond to autocorrelations for a simulated permanent sample using the GP model and parameter values in Table 2.2, column “GP economy”. Dotted lines correspond to autocorrelations for a simulated permanent sample using the NGP model and parameter values in Table 2.2, column “NGP economy”.

Chapter 3

Venture Capital Investments and Learning over the Life Cycle

3.1 Introduction

The life cycle of firms and the dynamics of firm growth and financing have recently become of increasing interest to economists (Luttmer, 2011; Pugsley et al., 2021). In particular, a key issue is how financial resources should be allocated to young firms over their first years of activity, so that they grow and make sound contributions to the economy. This issue is crucial when we talk about innovative, high-risk entrepreneurial projects, whose underlying quality and future prospects are unknown at birth. The objective of this chapter is to shed light on a feature of the firm life cycle that has not been studied in much detail in the context of high-risk firms and their financing: their ability to learn about the company's uncertain quality, as firm-level results are observed over time. This chapter studies how learning affects life-cycle investment and exit decisions, as well as firm value, in contexts of high uncertainty.

I present a model of the firm that imitates realistic features of high-risk, innovative companies – namely, uncertainty about a firm's own quality, staged investments in time, exit decisions and firm-level results at different ages. In this one-agent model, the owner of an entrepreneurial project carries out investments over the firm life cycle until it decides to terminate the project or to sell it to the market. While injecting funds over time, the owner learns about the uncertain quality of her firm in a Bayesian manner. The learning process is possible because the agent receives period cash-flows that convey information about the true quality of the firm. Given the high degree of uncertainty about the project, this information serves to update beliefs about the firm's unobserved quality and thus affects investment and exit decisions of the agent.

The model is capable of capturing some firm-level regularities of innovative companies – namely, it replicates empirical patterns that have been documented by the venture capital literature. By *venture capital*, I refer to a type of financial intermediation consisting of a financier, the *venture capitalist*, that buys shares of a private company. In exchange, the company receives not only funding, but also, and importantly, monitoring, networking and expertise from the venture capitalist. Venture capital is focused on the growth of young firms, having as its final goal the exit of the venture capitalist by means of the sale of the shares previously acquired[1]. The venture capital industry in the United States, often considered the paradigm for a developed venture capital industry[2], has become an important vehicle for the financing of young, innovative firms[3].

Some of the practices and conventions within venture-backed companies have been documented by Kaplan and Strömberg (2003), among others. They report that investments made by venture capitalists are contingent on firm-level results or cash-flows, and thus made in a staged manner over the firm life cycle (staged financing). Additionally, exit strategies are carefully chosen by venture capitalists, and contracts used in deals are often sophisticated convertible securities. Some explanations for these practices have been proposed in the literature using principal-agent models to study moral hazard problems (Bergemann and Hege, 1998; Schmidt, 2003; Repullo and Suarez, 2004), control rights (Marx, 1998) and tax motives (Gilson and Schizer, 2003; Ollivierre, 2010). However, little is known about how the learning process inherent to innovative, high-risk projects affects the life cycle of venture-backed companies. In this chapter, I abstract from the contracting problem between a financier and an entrepreneur and I rationalise firm-level patterns of venture capital investments, namely staged financing and exit strategies, by modelling a single agent that learns about her firm's uncertain returns over time.

I study the properties of the model and I arrive at two theoretical results. First, the possibility of learning from period cash-flows provides value to high-risk projects. In particular, I find that the ability to learn, jointly with the capability of terminating the project at every period, gives the agent an option value of waiting and updating her beliefs. This is possible if cash-flows are informative, to some extent, about the true quality of the project. Second, the ability to learn from these signals renders optimal investment decisions contingent on period cash-flow realisations, which is consistent with documented patterns at Kaplan and Strömberg (2003). Should we turn signals into completely uninformative ones (so that there is no learning), optimal investment would be unaffected by realised cash-flows.

I numerically solve the model and simulate it to better understand the implica-

[1]Usually via an initial public offering (IPO) or mergers and acquisitions (M&A).

[2]See Gompers and Lerner (2001) for a historical overview.

[3]For the sake of example, successful firms that received venture capital financing at different points of their life cycle, such as Cisco Systems, Apple, Google, Starbucks or Yahoo, are well-known.

tions of its theoretical properties. I find that investment and exit policies change over the life cycle of the firm, as more information is revealed and uncertainty decreases. Importantly, if the noise of the signal is sufficiently low, a higher degree of uncertainty results into a higher value from experimentation and a higher sensitivity of investment to cash-flow realisations. This in turn translates into a high positive contemporaneous correlation between simulated investments and cash-flows, in line with the empirical finding that fund injections and firm-level results go hand in hand. Next, I use the model to perform quantitative experiments. If we consider an environment in which learning is impossible, then cash-flow realisations do not provide information to the agent and there is no valuable belief updating. As a consequence, investment is completely insensitive to cash-flow realisations. Instead, if cash-flows are highly informative about project quality, this motivates the owner of a risky firm to continue running it and to carry out contingent investments. This translates into a greater value for firms, particularly when little is known about firm quality at birth. My findings are supportive of the idea that, when the degree of uncertainty about a firm's own quality is intense, a superior ability to learn about firm-level results over the life cycle renders the capacity to perform contingent growth investments valuable – thus being particularly interesting for innovative entrepreneurial projects.

Literature review. This chapter relates to different areas of the economics and finance literature. First and foremost, it relates to the theoretical literature on venture capital. This strand of the economic literature aims to rationalise different practices in the venture capital industry and proposes alternative explanations on why venture capital is capable of increasing the value of firms receiving this type of financing. This literature points to agency problems (Bergemann and Hege, 1998; Schmidt, 2003; Cornelli and Yosha, 2003; Repullo and Suarez, 2004), the split of control rights between entrepreneurs and venture capitalists (e.g. the capability of terminating entrepreneurial projects) (Aghion and Bolton, 1992; Marx, 1998; Hellmann, 1998; Jovanovic and Szentes, 2013), and the expertise, the ability to screen entrepreneurial projects, and the reputation of venture capitalists (Ueda, 2004; Sørensen, 2007; Piacentino, 2019) as key determinants of contractual practices and investment and exit dynamics within venture capital projects. These mechanisms are proposed to explain the use of convertible securities, staged financing, and the prevalence of venture-backed firms in the IPO market.

Bergemann and Hege (1998) propose a model to study termination decisions in a context where entrepreneurs and venture capitalists are subject to moral hazard in the allocation of funds and learn about project returns in the event of exit. They conclude that pure equity contracts cannot maximise the net present value of projects since it generates inefficiently early termination. A convertible security structure that mixes debt and equity (both retained by the venture capitalist) and links debt to the liquidation value if the project is unsuccessful is an efficient contract in their setting. Jovanovic and Szentes (2013) also study the dynamic implementation of venture-backed projects as well as the way entrepreneurs and venture capitalists match and split rents from projects in a context in which the venture capitalist has an incentive to terminate a project if she is not sufficiently optimistic about its future prospects and move on to finance a new entrepreneur. Given competition among different venture capitalists, the optimal contract is a simple equity contract, and venture capitalists add value to projects by facilitating financing and monitoring. Although these two works are the main references of this chapter, none of them explicitly models the role of intermediate results as signals that help agents engaging in high-risk projects to learn about future returns over their implementation and the way this may shape financing and exit decisions over time altogether. I contribute to this literature by proposing an explanation of staged financing contingent on intermediate results based on learning ability, which is in turn a characteristic of venture capitalists that is thought of as capable of increasing firm value.

Second, this chapter relates to the growing macroeconomic literature on venture capital. There are many recent contributions to this literature aiming to assess the economy-wide impact of the venture capital industry. In order to generate substantial aggregate effects of venture capital, these works consider particular value-adding features of venture capital financing, namely the ability to attract superior entrepreneurial talent (Opp, 2019), the expertise in product development (Ateş, 2018), and the degree of assortative matching between entrepreneurs and capitalists (Akcigit et al., 2022). Differently from these works, this chapter develops a detailed model of the firm with learning over the life cycle, as in Jovanovic (1982) and Guvenen (2007), to rationalise a superior form of fund injection that is contingent on the realisation of intermediate results and is value-enhancing at the firm level, and thus may add to previously explored explanations of the aggregate effects of the venture capital industry.

Finally, this chapter is motivated by the empirical literature on venture capital finance. The seminal article in this field is Kaplan and Strömberg (2003), which is the first one to exhaustively document the contingent nature of venture capital investments. Following this seminal work, other papers such as Kaplan and Strömberg (2004), Lerner and Schoar (2005), Cumming (2008), and more recent surveys by Gompers et al. (2020) and Ewens et al. (2022) have nurtured this part of the finance literature by providing more empirical evidence on venture capital contracts and investments. Inspired by this evidence, I propose a learning explanation to rationalise some of these empirical patterns.

Layout. This chapter is structured as follows. In section 3.2, I briefly revise trends and practices in the US venture capital market. In section 3.3, I present an investment model of the firm and I discuss theoretical properties of the model. In section 3.4, I perform a quantitative exploration of the model and I discuss the main results.

In section 3.5, I conclude.

3.2 The US Venture Capital Industry: a Brief Overview

In order to motivate this chapter, let us briefly discuss trends and recent results of the US venture capital industry, as well as firm-level practices within venture-backed companies.

Industry trends. Over the recent decades, the venture capital industry has become increasingly important in the United States, both in terms of resources allocated to it and economic outcomes from venture-backed firms. Metrick and Yasuda (2010) and NVCA (2022) provide data on capital raised by venture capital funds on a yearly basis. In 1983, total resources allocated to venture capital funds were $2.9 billion and accounted for 0.08% of the US gross domestic product. In 2021, these funds were $131.2 billion and represented 0.57% of US GDP. Between 1983 and 2021, funds allocated to venture capital have increased steadily, even during the world pandemic in 2020, with the only exception of the 2001 burst after the dotcom boom. Regarding summary output measures from the US venture capital industry, venture-backed firms broke several all-time records in 2021, as accounted by NVCA (2022). A record number of venture-backed firms, 14,411 companies, received a record amount of funding, $322 billion. In addition, the number of venture-backed exits (181 initial public offerings and 1,357 mergers and acquisitions) reached a historical peak as well. Importantly, the National Venture Capital Association also reports the relatively high prevalence of venture-backed firms in terms of employment: between 1990 and 2020, employment growth at venture-backed companies was eight times higher than that of non-venture-backed ones, and the seven largest publicly traded companies in 2021 have received venture capital financing at some point of their life cycle. As it has also been studied by Akcigit et al. (2022), these firms are disproportionately larger than their non-venture-backed counterparts in terms of employment.

Firm-level practices. The success of venture capital as a financing device for young, innovative firms in the United States is usually linked to the sophistication of venture capitalists. Practices at the firm level that prevail in the US venture capital industry were exhaustively documented by Kaplan and Strömberg (2003) in a seminal empirical study. Some of these practices are common to other venture capital markets, but others are distinct in Europe, Canada or developing countries (Cumming, 2008; Ollivierre, 2010; Lerner and Schoar, 2005). Here, I briefly describe some of these practices, namely staged financing, exit strategies and security types.

Staged financing consists in the injection of funds into firms being split into different periods of their life cycle. Staged financing has been of special interest to the theoretical literature, e.g Neher (1999) and Cornelli and Yosha (2003). According to the empirical study in Kaplan and Strömberg (2003), injection of funds of venture capitalists into firms is found to often be contingent in the realisation of favourable intermediate results or milestones within the company. The punchline of their study is that, within a venture-backed firm, cash-flow rights and other control rights that are important to the relationship between companies and financiers are separated and made contingent on different states of the world, given the high risk of entrepreneurial projects – namely, contingent on project performance[4]. This translates into venture capitalists deciding to inject money into high-risk venture-backed firms over different stages of their life.

Exit strategies refer to the decision of a venture capitalist to sell its shares of the company to the market, typically via an initial public offering (IPO) or a merger or an acquisition (M&A); or, alternatively, to liquidate the project if it turns out to be a failure. On one side, regarding the time path of successful venture-backed companies, Metrick and Yasuda (2010) report that the average sale period in the US venture capital market ranges between 3 and 7 years since the first venture capital investment takes place. This indicates that venture capitalists spend a large amount of time injecting funds and monitoring companies over their life cycle, and choose carefully when to bring the firm to the market. On the other side, a non-negligible fraction of venture-backed firms are terminated without a successful sale. In a recent survey by Gompers et al. (2020) made to venture capitalists and venture-backed firms in the United States, it has been documented that the average venture capital intermediary reports that 32% of its exits are failures (being 15% of its exits initial public offerings and 53% mergers or acquisitions). All this indicates the importance that an expected sale after some firm life periods has on within-firm decisions, but also the high frequency of failures and terminations among venture-backed companies.

Finally, Kaplan and Strömberg (2003) report the widespread usage of convertible securities in the US industry. This type of security can be roughly seen as a combination of debt and equity. Under convertible securities (also called convertible preferred stock), entrepreneurs and venture capitalists split the returns they receive from projects according to an equity share and the venture capitalist accumulates liquidation rights as she invests into the project. In the event of an exit or a liquidation, the venture capitalist can exercise the right to convert her liquidation rights

[4]This contingency guideline also shows up in other characteristics of venture capital contracts. As other examples of contingency, and thus contract sophistication, venture capitalists may vest entrepreneurs with larger cash-flow rights as performance milestones are satisfied. The convertible security structure of contracts, in turn, allows venture capitalists to increase their cash-flow rights in case of exit or liquidation of the project as they invest funds into the firm. Also, investors have an implicit control right over the firm through their ability to stage investments in time. For more on this, see Kaplan and Strömberg (2003).

into simple equity and get paid. This class of contracts are though of in the literature as having desirable efficiency properties, as in Bergemann and Hege (1998), Marx (1998) and Schmidt (2003). Importantly, they are overwhelmingly used in the US industry, to the point of a variety of classes of convertibles accounting for 95.8% of all investment rounds in the sample of Kaplan and Strömberg (2003). Interestingly, convertible securities have not been the traditional type of security in other countries, where simpler contracts are used very often in their national venture capital markets[5]. Thus, security sophistication has been indeed a special element of the US venture capital industry over the last years.

Some explanations for these practices have been proposed in the literature, e.g. agency problems (Bergemann and Hege, 1998; Schmidt, 2003; Repullo and Suarez, 2004), control rights (Marx, 1998) and tax reasons (Gilson and Schizer, 2003; Ollivierre, 2010). This chapter relates widespread industry practices, namely staged financing and exit strategies, to a feature that is inherent to innovative, high-risk projects: the learning process over the life cycle about their true quality, which is uncertain at firm birth. Since these projects have high return uncertainty, they might require a specialist, (who may be, for instance, a venture capitalist), to learn about the unknown quality of the firm over the life cycle while injecting funds to grow the project towards a successful sale. I present a model in which a firm learns about its uncertain quality over the life cycle. My model is informative on how the ability to learn relates to contingent staged financing, exit strategies, and outcomes of high-risk firms.

3.3 A Model of the Firm

3.3.1 The Environment

In this section, I describe the problem of an agent that owns an entrepreneurial project, or a firm. The agent may live and invest in her firm during several periods. I refer to the life cycle of a firm as project implementation.

Physical environment. At a given point in time, an exogenous mass 1 of age-0 agents is born, each of them owning one firm. Let us first describe what agents know at the moment of firm birth (prior to $t = 0$). An agent, indexed by i, starts its life with a draw π_{i0} from a normal population distribution, $\pi_{i0} \sim N(\Gamma_0, \Sigma_0)$,

[5]Cumming (2008) considers a sample for the European industry and reports that only 32.3% of all investments were using some class of convertible security – indeed, it is common stock that is more generalised in European venture capital markets. A similar pattern has been observed in Canada (Ollivierre, 2010). In developed countries, less sophisticated securities such as common stock or even plain debt contracts are prevailing due to legal constraints that impede enforcement of more complicated securities (Lerner and Schoar, 2005).

where Γ_0 and Σ_0 are the exogenously given cross-sectional mean and variance of π_{i0}, respectively. Thus, firms are ex-ante heterogeneous in π_{i0}. Draw π_{i0} represents the quality of the entrepreneurial project – that is, the capacity of the firm to generate cash-flows. Importantly, I assume that agents do not observe their initial draw π_{i0} at birth, and it remains unobserved over the life cycle. Nevertheless, agents know the population distribution of initial quality. Thus, the cross-sectional distribution of π_{i0} determines initial beliefs of agents. At age 0, every firm i believes that its unobserved quality is distributed according to a normal distribution with mean $\hat{\pi}_{i0} = \Gamma_0$ and variance $\sigma_{i0}^2 = \Sigma_0$. All firms in the economy have the same initial beliefs, although they are heterogeneous in their underlying unobserved quality draw[6]. This is equivalent to assuming that an agent owning a firm i is born with unobserved initial quality π_{i0}, and a duple $(\hat{\pi}_0, \sigma_0^2)$ representing initial prior beliefs that is equal for all firms. I take this bidimensional object as a primitive of the model.

I study the firm life cycle after firm birth. Time is discrete and infinite, $t = 0, 1, ...$, agents are risk neutral and discount future payoffs at an exogenous discount rate $r > 0$. In what follows, I focus on a specific firm owner and thus I omit subscript i for expositional purposes. At an age $t \geq 0$, an agent owning a firm enters the period with a prior belief $(\hat{\pi}_t, \sigma_t^2)$ about her firm's quality. The agent may decide to continue running her project or, alternatively, liquidate it or sell it to the market. If she decides to keep her firm at t, she observes period cash-flows, which are informative about the true quality of the project and thus enable the agent to update her prior beliefs, as we shall see. Once period cash-flows realise, the agent can invest in her firm to improve its unobserved quality.

Within an age $t \geq 0$ of an agent's life, there are four stages, in chronological order:

1. **Exit decision**: the agent decides whether to keep her firm, to terminate it, or to sell it to the market in exchange of a price (*keep/termination/sale*). Upon termination or sale, the agent leaves the economy.

2. **Cash-flow realisation**: upon keeping the firm, the agent receives period cash-flows.

3. **Belief updating**: given the cash-flow realisation, prior beliefs are updated.

4. **Investment decision**: after updating beliefs, the agent chooses how much to invest in her firm. The unobserved quality of the project evolves accordingly.

At the end of age t, the agent ends up with posterior beliefs $(\hat{\pi}_{t+1}, \sigma_{t+1}^2)$.

[6]These assumptions are made to represent an innovative, cutting-edge industry composed of firms whose true quality is unknown at the moment of birth and may differ across firms. In this environment, projects are inherently risky and no firm has privileged information about its true quality.

Exit and investment decisions. Agents make exit and investment decisions over the life cycle. At the beginning of age $t \geq 0$, the agent makes a discrete decision, either to cease the activity of the firm (*termination*), to sell the firm to the market (*sale*), or to continue running the firm (*keep*). If the agent chooses *termination* at age t, the firm stops its activity, the agent leaves the economy and she gets a value of zero. If the agent chooses *sale*, the firm is sold and the agent gets the discounted expected value of future cash-flows generated by the firm in exchange, and leaves the economy. The sale process is costly, and these costs are represented by an amount $C_{IPO} > 0$ that the agent has to pay if she chooses *sale*. Importantly, in the event of a sale or a termination, no further investments are made in the project. The original firm owner is the only individual that is capable of exerting effort to make the firm grow. Upon a *keep* decision at period t, the firm observes the realisation of a random variable that I label period cash-flows, $CF_t \in \mathbb{R}$, which represents intermediate revenues or results taking place during the firm life cycle. After the realisation of CF_t, the agent makes a non-negative investment, k_t. Investment is worthy to the firm in that it allows to increase the unobserved quality of the firm at age t, π_t. More specifically, π_t increases by $B > 0$ per unit of investment. Keeping the project and exerting investments has costs. First, there is a fixed cost of operation c_{op} to be paid whenever the agent chooses *keep*, regardless of the level of investment. Second, the agent pays a price p_K per unit of investment good bought. Third, there is an increasing and convex investment cost $c(k) = \frac{\psi}{2}k^2$, whose intensity is parameterised by $\psi > 0$.

Learning process. At birth, the agent does not observe the true quality of the project. However, over the life cycle, as she chooses to keep her project, she receives period cash-flows that are informative about its unobserved quality. Let us focus on the learning process that characterises project implementation, aiming to represent the dynamic experimentation that is inherent to high-risk, innovative projects. I do this by assuming that the firm updates its prior beliefs about project quality in an optimal, Bayesian manner, expressed as a Kalman filtering problem (Kalman, 1960). At birth, the firm has an initial prior belief about the distribution of π_0 that is assumed to be normal, $N(\hat{\pi}_0, \sigma_0^2)$. Under the Kalman filter setting, if the variable whose distribution we are updating is normal at any period t, the updated distribution at period $t+1$ is also normal. Therefore, the learning process describes the evolution of the duple $(\hat{\pi}_t, \sigma_t^2)$.

Two key ideas underlie belief updating. First, period cash-flows CF_t observed upon a *keep* decision are an imperfect signal about the true quality of the project, π_t. This is reflected in the following "observation equation", which represents the observable variable CF_t as a linear expression in the unobservable quality π_t and a transitory shock ε_t, which can be understood as a measurement error. The observation equation reads:

$$CF_t = \pi_t + \varepsilon_t \tag{3.3.1}$$

where $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$. The variance of ε_t, σ_ε^2, is non-negative and represents the intensity of the measurement error. If $\sigma_\varepsilon^2 > 0$, the firm is prevented from knowing with certainty whether an observed CF_t was due to its project's quality π_t, or just luck. Second, I assume that the unknown quality evolves in time according to a linear "law of motion equation". The level π_{t+1} is affected by the investment exerted at period t, so that it helps to improve the project's quality, and by its lagged value π_t. This allows us to generate persistent effects of investment in the model. The law of motion for project quality is:

$$\pi_{t+1} = \pi_t + Bk_t \tag{3.3.2}$$

where B is strictly greater than zero. The law of motion of uncertain quality makes it explicit that firm growth is only possible if effort is put at a cost. Equations (3.3.1) and (3.3.2) determine how agent's beliefs evolve over the firm life cycle. We can use equation (3.3.1) to get the distribution of CF_t conditional on our prior quality distribution, at a period t. This conditional distribution has cdf $G_t(CF_t|\hat{\pi}_t)$ and, similarly to the prior distribution, it is normal: $CF_t \sim N(\hat{\pi}_t, \sigma_t^2 + \sigma_\varepsilon^2)$. In order to derive equations for the evolution of expected quality $\hat{\pi}_t$ and quality uncertainty σ_t^2, we apply the Kalman filter. As it is discussed in Kalman (1960) and Perla et al. (2022), the Kalman filter consists in recursively finding a predictor for the unknown variable using observed imperfect measures of it, thus yielding laws of motion for its expected value and variance. This form of representing learning is becoming increasingly used in the macroeconomics literature[7]. In our context, the Kalman filter yields a point estimate for the unknown quality, which is the conditional expectation for π_t, given past observations for cash-flows and investment injections. This estimate is the mean squared error minimiser among all Borel functions on $\mathbb{R}$ with bounded variance[8]. The derivation of the Kalman filter equations is discussed in Appendix A.1. Under imperfect observability of the true quality of the project, the evolution of beliefs $(\hat{\pi}_t, \sigma_t^2)$ is characterised by the following Kalman filter equations:

$$\hat{\pi}_{t+1} = \hat{\pi}_t + Bk_t + \kappa_t(CF_t - \hat{\pi}_t) \tag{3.3.3}$$

$$\sigma_{t+1}^2 = (1 - \kappa_t)\sigma_t^2 \tag{3.3.4}$$

[7]See Guvenen (2007), Baley and Veldkamp (2021) and Farboodi and Veldkamp (2021) for applications of the Kalman filter in macroeconomics.

[8]In fact, it can be shown that predictions using the Kalman filter when the true state π_t is unobservable perform relatively well if we compare them with those from the optimal predictor (in terms of minimising the squared error) $\mathbb{E}[\pi_{t+1}|\pi_t]$ under perfect observability of π_t, which is the predictor a rational agent would use. See Perla et al. (2022) for a simple application of this idea.

and

$$\kappa_t = \frac{\sigma_t^2}{\sigma_t^2 + \sigma_\varepsilon^2} \tag{3.3.5}$$

where κ_t is the so-called Kalman gain. The Kalman gain provides an idea on how much we update the distribution mean from realisations of period cash-flows, or the sensitivity of $\hat{\pi}_{t+1}$ to different magnitudes of the observed CF_t, as it is reflected in equation (3.3.3). Importantly, it depends positively on quality uncertainty σ_t^2 and negatively on the intensity of the measurement error σ_ε^2: the updating of $\hat{\pi}_t$ is more intense when quality is very uncertain and when the imperfect measure is more accurate. At any period t, the three Kalman filter equations (3.3.3)-(3.3.5) give us the duple $(\hat{\pi}_{t+1}, \sigma_{t+1}^2)$ that characterises the updated quality distribution. Notice that the dynamics of the Kalman gain κ_t are fully determined by the dynamics of quality uncertainty σ_{t+1}^2. The Kalman gain, in turn, affects the dynamics of both the distribution mean and variance.

Value functions. Let us now discuss the problem faced by an agent that owns a firm, as well as optimal exit and investment policies. The agent chooses the investment amounts and the exit strategies to maximise the discounted expected value from her firm. There are no frictions, e.g. informational asymmetries, conflicts of interest, or hold-up problems.

First, in order to determine the value the agent gets in the event of a sale, it is necessary to make a distinction between a firm that is held by the agent and a firm that is held by the market (i.e. it has been previously sold) at age t. The key difference between these two firms is that the market is unable to perform productive investments to increase the unobserved project quality, while the agent is capable of doing so. Additionally, the agent has to pay sale costs in the event of a sale as well as period operation and investment costs, while the market is free of those. I assume, however, that both the agent and the market have the ability to terminate projects, and that, upon not terminating, projects generate period cash-flows and beliefs are updated accordingly; that is, both have the ability to learn.

Consider a firm with beliefs $(\hat{\pi}_t, \sigma_t^2)$ that has been sold to the market at some previous age. The value of an age-t firm held by the market is:

$$W_t(\hat{\pi}) = \max\{0, M_t(\hat{\pi})\} \tag{3.3.6}$$

where $M_t(\hat{\pi})$ is the expected value prior to the cash-flow realisation at t:

$$M_t(\hat{\pi}) = \int \big(CF + \frac{1}{1+r} W_{t+1}(\hat{\pi}')\big) dG_t(CF \mid \hat{\pi})$$

In this expression, $CF = \pi + \varepsilon$, and beliefs evolve according to equations (3.3.3)-

(3.3.5) – namely, $\hat{\pi}' = \hat{\pi} + \kappa_t(CF - \hat{\pi})$, since no investment can be exerted by the market-held firm. If we know market values W_t and M_t, we are able to determine the value of a firm that is kept by its original owner, who has the ability to make period investments. The value of an age-t firm held by an agent with beliefs $(\hat{\pi}_t, \sigma_t^2)$ is:

$$V_t(\hat{\pi}) = \max\{\underbrace{0}_{\text{termination}}, \underbrace{M_t(\hat{\pi}) - C_{IPO}}_{\text{sale}}, \underbrace{\int U_t(\hat{\pi}, CF) dG_t(CF|\hat{\pi})}_{\text{keep}}\} \tag{3.3.7}$$

where the value of keeping the project and receiving a cash-flow realisation CF is:

$$\begin{aligned} U_t(\hat{\pi}, CF) = \max_{k \geq 0, \hat{\pi}' \in \mathbb{R}} \quad & \left\{ CF - c(k) - p_K k - c_{op} + \frac{1}{1+r} V_{t+1}(\hat{\pi}') \right\} \\ \text{s.t.} \quad & \hat{\pi}' = \hat{\pi} + Bk + \kappa_t(CF - \hat{\pi}) \end{aligned} \tag{3.3.8}$$

and also subject to the exogenous evolution of quality uncertainty and the Kalman gain in equations (3.3.4) and (3.3.5), given initial beliefs. In words, at age t, the agent can sell the firm to the market, in which case she pays a sale cost C_{IPO} and gets the market value of the firm $M_t(\hat{\pi})$ – that is, the expected discounted value of future cash-flows[9]. By selling the project, the agent will be saving investment and operation costs. Otherwise, she can keep the project, which allows her to exert investments that increase its profitability. In the meanwhile, the agent updates her beliefs as new cash-flows arrive. Regarding belief updating, the evolution of the expected quality $\hat{\pi}$ is endogenous, for it depends on the investment decision of the agent. On the other side, notice that, given σ_0^2, equations (3.3.4) and (3.3.5) evolve exogenously and in a deterministic way. Time is a state in equations (3.3.6)-(3.3.8) (and thus the time index t) because both quality uncertainty σ_t^2 and the Kalman gain κ_t are strictly positive and vary over the firm life cycle due to belief updating, provided $0 < \sigma_\varepsilon < \infty$. Nevertheless, if we let t increase, σ_t^2 and κ_t decrease in t and infinitely old firms (either held by their initial owners or by the market) have $\sigma_\infty^2 = \kappa_\infty = 0$, thus facing a stationary problem where there is no belief updating.

We want to find the optimal investment and exit decisions made by the agent over the firm life cycle. These are investment policies $g_t^k(\hat{\pi}, CF) \in [0, \infty)$, as well as the discrete exit strategy $g_t^{exit}(\hat{\pi}) \in \{termination, sale, keep\}$. To find these policies, I solve the model numerically using value function iteration. Further details on the algorithm used to find the policy functions are presented in Appendix A.2. In the numerical solution, whose properties are discussed in the next section, the optimal

[9]We have to take into account that the market still has the option to terminate the firm, so the value of the market-held firm is bounded below.

exit policy for the agent $g_t^{exit}(\hat{\pi})$ is characterised by *expected quality thresholds* for $\hat{\pi}_t$, $\{\underline{\hat{\pi}}_t, \overline{\hat{\pi}}_t\}_{t=0}^{\infty}$ with $\underline{\hat{\pi}}_t < \overline{\hat{\pi}}_t$, such that the agent terminates the project at age t if $\hat{\pi}_t < \underline{\hat{\pi}}_t$ (when she thinks the project does not have enough quality), sells the project to the market at age t if $\hat{\pi}_t > \overline{\hat{\pi}}_t$ (when she thinks the project has enough quality), and keeps the project otherwise, for expected qualities lying in between the upper and lower thresholds[10]. By backward induction, we can eventually find the value that the agent gets from owning the firm at birth, $V_0(\hat{\pi}_0)$, taking as given the initial prior belief.

3.3.2 Learning, Investment and Exit Decisions

Previous to the quantitative exploration of the model solution, it is necessary to understand the mechanisms operating in the model presented above. In particular, it is paramount to study how *learning* affects exit choices and firm investment over the life cycle. This section illustrates the role of learning. A *learning* scenario is one where the firm receives period cash-flows that are partially informative about the true quality of projects, so that $\sigma_\varepsilon^2 > 0$, and beliefs $(\hat{\pi}_t, \sigma_t^2)$ are updated as cash-flows realisations arrive. Belief updating is possible as far as the signal is not too noisy, i.e. if $\sigma_\varepsilon^2 < \infty$. Otherwise, I will be talking about a *non-learning* scenario, where beliefs are not updated over time.

Learning Affects Exit Decisions

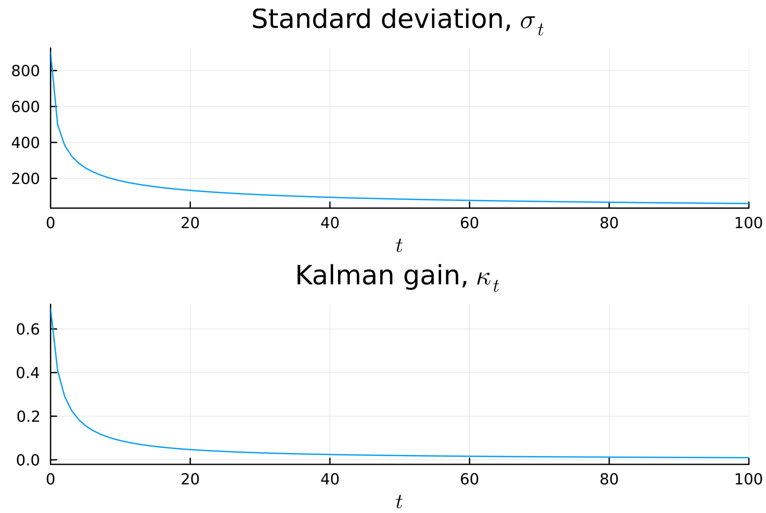
Let us first illustrate how belief updating affects discrete decisions. To do this, I abstract here from investment and I consider a firm that has been sold to the market, which does not have the ability to invest[11]. This firm is capable of updating beliefs over time and make termination decisions.

First of all, timely differences in belief updating are mapped to timely differences in σ_t^2 and κ_t, whose exogenous evolution affects the way cash-flow realisations generate surprises and change expected quality $\hat{\pi}_t$ over time. Figure 3.1 shows how the standard deviation σ_t and the Kalman gain κ_t evolve in time according to equations (3.3.4) and (3.3.5), departing from an initial belief of $\sigma_0 = 900$. As we can see, both σ_t and κ_t have a decreasing profile in time, and they converge to zero as time goes to infinity. As a consequence, belief updating will be less notorious for old firms than for young firms, since quality uncertainty fades away with age. Indeed, let us discuss first what happens to infinitely old firms in terms of value. Consider the stationary value of a firm that has been sold to the market. Given $\sigma_\varepsilon^2 > 0$ and equations (3.3.4)

[10]Similarly, market-held projects have an optimal termination policy characterised by an expected quality threshold, such that projects are terminated if $\hat{\pi}_t$ lies below a cutoff.

[11]Considering investment and the problem of the original firm owner does not change the main takeaway of this section regarding learning, value and discrete decisions.

Figure 3.1 Belief updating: standard deviation and Kalman gain



Notes: the two graphs respectively show the life-cycle evolution of the standard deviation characterising beliefs of a firm, σ_t, and the corresponding Kalman gain, κ_t, for 100 periods of life. The standard deviation and Kalman gain profiles have been calculated using equations (3.3.4) and (3.3.5), respectively. I assume an initial standard deviation of $\sigma_0 = 900$ and a measurement error of $\sigma_\varepsilon = 600$.

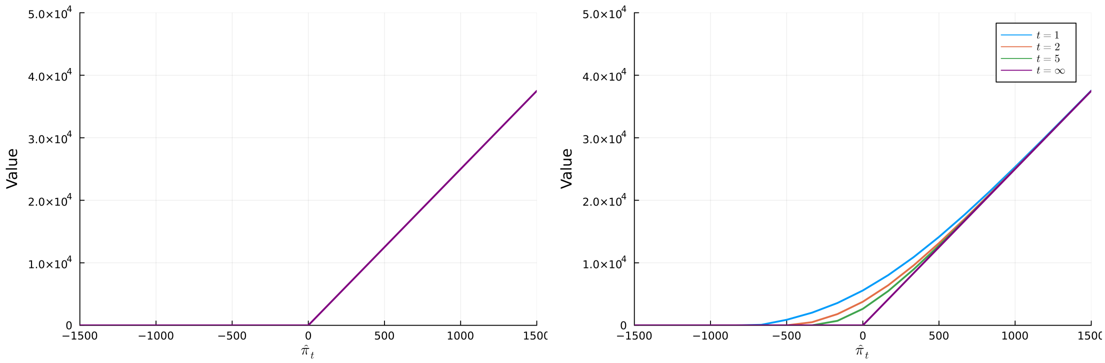
and (3.3.5), if we let time go to infinity we get $\kappa_\infty = \sigma^2_\infty = 0$. Thus, infinitely old firms know with certainty their true quality (that is, $\hat{\pi}_\infty = \pi_\infty$) and do not update their beliefs. As a consequence, the stationary value of a firm of (expected) quality $\hat{\pi}$ is:

$$W_\infty(\hat{\pi}) = \max\{0, \frac{\hat{\pi}}{1-\beta}\} \tag{3.3.9}$$

where $\beta = 1/(1+r)$. Figure 3.2(a) plots the stationary value $W_\infty(\hat{\pi})$ for different values of $\hat{\pi}$. From the graph, it is clear that the optimal exit decision of the stationary firm consists of terminating if its quality lies below zero, and keep on receiving cash-flows forever otherwise.

Let us now see what happens over the life cycle of a firm that has been sold to the market. At ages $t < \infty$, both the quality uncertainty σ^2_t and the Kalman gain κ_t are strictly positive, as far as the noise of the signal does not go to infinity. As a consequence, there is a non-trivial belief updating over time. In particular, expected quality by the firm evolves according to $\hat{\pi}_{t+1} = \hat{\pi}_t + \kappa_t(CF_t - \hat{\pi}_t)$. The value of this age-$t$ firm is given by equation (3.3.6). Figure 3.2(b) shows this value function for different firm ages (namely, ages 1, 2 and 5). Differently from the stationary function, value has a curvy, strictly convex shape. Importantly, at younger ages, value is weakly higher than the stationary value, and it decreases as the firm ages. The differences in value over the firm life cycle ultimately come from differences in belief updating over time, which translate into timely differences in the Kalman

Figure 3.2 Market value: stationary and life cycle



(a) Stationary value (b) Life-cycle value

Notes: subfigure (a) plots the stationary value of a firm that has been sold to the market in equation (3.3.9), as a function of the expected quality of the firm. Subfigure (b) compares this stationary value to different life-cycle values (for ages 1, 2 and 5) of a firm that has been sold to the market in equation (3.3.6). Life-cycle values are again plotted as a function of expected quality.

gain.

To illustrate the effects learning has on value, let us consider an age-0 firm whose expected value is slightly below zero, e.g. $\hat{\pi}_0 = -100$, and let us compare it with an age-∞ firm with $\hat{\pi}_\infty = -100$. As we see from Figure 3.2(b), the age-0 firm is getting a positive value and decides not to terminate the project at age 0. In turn, the infinitely old firm with $\hat{\pi}_\infty = -100$ decides to terminate the project and gets a value of zero. The difference between these two firms is that the young firm has room for belief updating. At age 0, project quality is yet pretty uncertain, since σ_0^2 is high. Consequently, the Kalman gain at age 0 is also large, given equation (3.3.5). Therefore, choosing to keep the project alive and to observe a new cash-flow observation CF_0 is going to yield an optimistic belief update (i.e. an increase in $\hat{\pi}_0$) with a very high probability.

Still, there is also a high probability that CF_0 is low, and that beliefs are updated pessimistically. However, the possibility to terminate generates a positive option value. Should the firm receive a very bad cash-flow realisation, it can avoid receiving a time-lasting stream of poor results by just leaving the economy. Put simply, young firms can cut negative cash-flow streams if they expect them, and so do old firms; but, differently from old firms, there is room for positive belief updating for young firms, and thus for attaining higher future cash-flows. As a result, the value of an age-0 firm can be positive even if it starts its life expecting a relative poor quality. This is entirely due to the possibility of learning under uncertainty. Indeed, if we consider a non-learning environment such that period cash-flows are completely uninformative, then $\kappa_t = 0$ for all t, no belief updating is possible, and firm value at ages $t < \infty$ coincides with the stationary value shown in Figure 3.2(a).

Learning Affects Investment

Let us now come back to the problem of the agent who owns a firm and has the ability to perform growth investments on it. Let us illustrate how belief updating over the life cycle shapes investment decisions. For illustration purposes, I discuss a simplified version of the model. The results from this model extend to the more general setting, which is simulated in section 3.4. I find that learning makes investment by the agent contingent on intermediate cash-flows, in line with documented empirical patterns discussed in section 3.2.

Here, I present a three-period version of the model of the firm. There is an agent that owns a project and lives for three periods: $t = 0, 1, 2$. Initial beliefs at age 0 are described by a duple $(\hat{\pi}_0, \sigma_0^2)$, and may be updated over time. At period 2, the agent chooses whether to terminate the project or to sell it to the market, leaving the economy right after in either case. I assume that the discrete decision only takes place at age 2, and not before, but the mechanism discussed here is also present in a model with discrete decisions every period, as shown in section 3.4. If the agent decides to sell the project to the market, she gets the expected value of period cash-flows at age 2 – that is, $\hat{\pi}_2$. For exposition purposes, I abstract from operation and sale costs. Leaving these considerations aside, the timing of events is similar to that in section 3.3.1.

Let us first discuss the non-learning case – i.e. cash-flows are completely uninformative about the true quality of projects, and prior beliefs are never updated. An implication of this is that the Kalman gain equals zero at every period $t = 0, 1, 2$ of firm life. At period 2, the firm with expected quality $\hat{\pi}_2$ has value $V_2(\hat{\pi}_2) = \max\{0, \hat{\pi}_2\}$. Thus, in period 1 we have:

$$V_1(\hat{\pi}_1) = E_{\varepsilon_1}[U_1(\hat{\pi}_1, CF_1)]$$

where

$$U_1(\hat{\pi}_1, CF_1) = \max_{k_1 \geq 0} \left\{ CF_1 - \frac{\psi}{2} k_1^2 - p_K k_1 + \frac{1}{1+r} \max\{0, \hat{\pi}_2\} \right\}$$

subject to equation (3.3.3), which reads $\hat{\pi}_2 = \hat{\pi}_1 + Bk_1$ in this case, and where $CF_1 = \pi_1 + \varepsilon_1$. Recall that fund injection k_1 is chosen after cash-flow CF_1 is realised. I adopt a backward induction perspective. Conditional on the agent choosing to terminate at period 2, then she gets value $CF_1 - \frac{\psi}{2} k_1^2 - p_K k_1$ and she optimally chooses $k_1^T = 0$. Otherwise, conditional on choosing sale at period 2, the firm gets value $CF_1 - \frac{\psi}{2} k_1^2 - p_k k_1 + \frac{1}{1+r}(\hat{\pi}_1 + Bk_1)$ and chooses $k_1^S = \frac{1}{\psi}(\frac{1}{1+r} B - p_K)$, which is greater than zero if B is large enough. The sale choice takes place whenever $\frac{1}{1+r}(\hat{\pi}_1 + Bk_1^S) \geq 0$ or, equivalently, when $\hat{\pi}_1 + \frac{B}{\psi}(\frac{1}{1+r} B - p_K) \geq 0$. The optimal investment decision in period 1 is:

$$k_1^*(\hat{\pi}_1) = \begin{cases} \frac{1}{\psi}\big(\frac{1}{1+r}B - p_K\big) & \text{if } \hat{\pi}_1 + \frac{B}{\psi}\big(\frac{1}{1+r}B - p_K\big) \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

Notice that optimal investment is weakly increasing in $\hat{\pi}_1$: the fact that the firm enters the period with a high expected quality may alter the discrete decision towards selling instead of terminating, and thus may generate a positive jump in investment. Indeed, positive investment will take place for any expected quality such that $\hat{\pi}_1 \geq -\frac{B}{\psi}\big(\frac{1}{1+r}B - p_K\big)$. Provided that fund injections have a positive marginal productivity B that is sufficiently large relative to the price of the investment good, the firm would be willing to invest and sell the project right after even for some negative values of $\hat{\pi}_1$. The same logic discussed here can be extended to period 0 to obtain $k_0^*(\hat{\pi}_0)$, also weakly increasing in $\hat{\pi}_0$. Nevertheless, staged financing is not contingent on period cash-flows – i.e CF_t does not affect optimal investment k_t^*.

To arrive at this important result, let us now consider a bounded and sufficiently low σ_ε^2, so that there is *learning*, i.e. belief updating over time. In this case, the investment problem at period 1 is similar to that in the non-learning case, but the Kalman gain κ_1 is strictly positive and equation (3.3.3) reads $\hat{\pi}_2 = \hat{\pi}_1 + Bk_1 + \kappa_1(CF_1 - \hat{\pi}_1)$, where the extra term depends on CF_1, and on σ_1^2 and σ_ε^2 via the Kalman gain. Following the same logic as before, optimal investment at period 1 is:

$$k_1^*(\hat{\pi}_1, CF_1) = \begin{cases} \frac{1}{\psi}\big(\frac{1}{1+r}B - p_K\big) & \text{if } \hat{\pi}_1 + \frac{B}{\psi}\big(\frac{1}{1+r}B - p_K\big) + \kappa_1(CF_1 - \hat{\pi}_1) \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

Now, the fact that $\kappa_1 > 0$ makes both $\hat{\pi}_1$ and CF_1 capable of generating jumps in investment due to the discrete decision. In particular, the firm will invest and sell the project afterwards if $\hat{\pi}_1 \geq -\frac{1}{1-\kappa_1}\big(\frac{B}{\psi}(\frac{1}{1+r}B - p_K) + \kappa_1 CF_1\big)$. For example, imagine κ_1 is large. This can be caused either by a very uncertain environment (high σ_1^2) or by a very high learning ability (low σ_ε^2), or both. In this high-κ environment, a negative CF_1 realisation may induce the firm to choose to terminate the project instead of selling it at period 2, and thus may prevent it from investing, even if $\hat{\pi}_1$ is positive and large enough. We can extend this logic to period 0 (although more unpleasantly from an algebraic point of view) and get a similar intuition for $k_0^*(\hat{\pi}_0, CF_0)$. The key insight from this illustrative model is that, jointly with the possibility of terminating or selling the project, *learning is a force that makes investment contingent on period cash-flows.*

As we see next, the mechanism presented here also operates in the infinite-periods model of section 3.3.1. The ability to learn about firm-level outcomes generates interesting patterns of staged financing, a phenomenon that has been of great interest to the venture capital literature (Neher, 1999; Cornelli and Yosha, 2003). Specifically,

Table 3.1 Parameterisation of baseline model

Parameter	Definition	Value
Initial beliefs		
$\hat{\pi}_0$	Expected quality at birth	-50
σ_0	Quality uncertainty (standard deviation) at birth	900
Other parameters		
r	Discount rate	0.042
B	Marginal productivity of effort	2
σ_ε	Standard deviation of measurement error	600
ψ	Intensity of convex investment cost	4.5
p_K	Price of investment good	1
c_{op}	Fixed operation cost	400
C_{IPO}	Fixed sale cost	3500

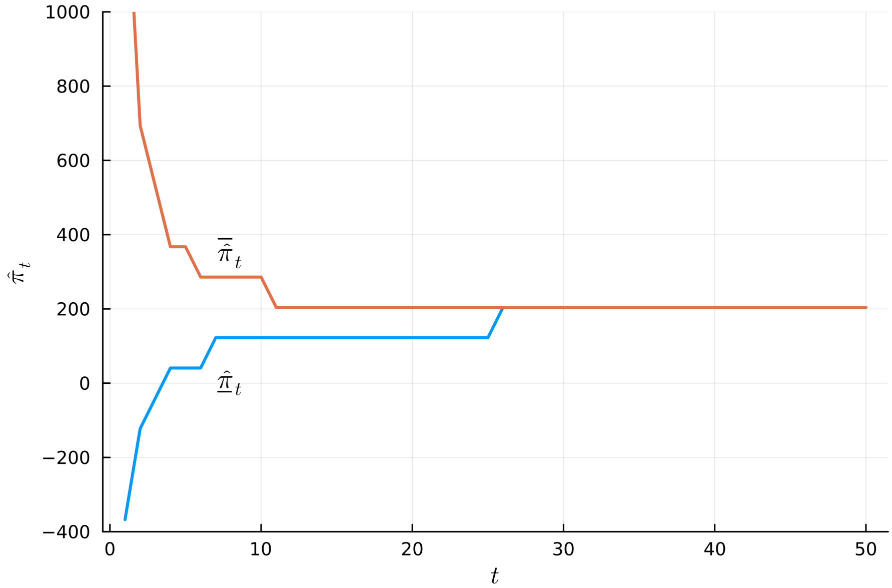
learning is shown to be a theoretical mechanism capable of rationalising contingent fund injections, an important feature of real-life US venture-backed firms, as documented by Kaplan and Strömberg (2003).

3.4 Quantitative Exploration

In this part of the chapter, I discuss the numerical solution of the complete, infinite-periods model described in section 3.3.1, given the baseline parameterisation in Table 3.1. The theoretical mechanisms described in section 3.3.2, through which learning affects exit and investment decisions, are operating in the model. In my baseline parameterisation, I consider that agents owning firms start out project implementation with initial beliefs $\hat{\pi}_0 = -50$ and $\sigma_0 = 900$. The assumption on initial expected quality $\hat{\pi}_0$ guarantees that if there were no initial quality uncertainty ($\sigma_0 = 0$), keeping the rest of parameterisation equal, the agent would immediately terminate the project in the first period of firm life. On the other hand, the assumption that $\sigma_0 = 900$ aims at representing a quite intense initial quality uncertainty. I assess whether quality uncertainty is sufficiently high to motivate the agent to keep the firm, to update her beliefs about the project's quality, and to perform growth investments. In particular, I study whether learning is beneficial for the agent in terms of the value of the firm, as well as other simulated moments of interest.

The baseline parameterisation in Table 3.1 has been chosen for the model to display two reasonable decision-making patterns of agents owning entrepreneurial projects. First, I parameterise the dynamic model such that an infinitely old agent would always decide not to keep the project. That is, the stationary value of an agent of age $t = \infty$ is such that the agent either sells the project to the market, if

Figure 3.3 Sale and termination thresholds, baseline model



Notes: the graph shows optimal sale $\overline{\hat{\pi}}_t$ and termination $\underline{\hat{\pi}}_t$ thresholds for expected quality $\hat{\pi}_t$ for different periods of an agent's life cycle. The lines are obtained from the numerical solution of the optimisation problem of the agent, given the parameterisation in Table 3.1. The orange line corresponds to the sale threshold $\overline{\hat{\pi}}_t$ and displays a decreasing pattern in time. The blue line corresponds to the termination threshold $\overline{\hat{\pi}}_t$ and is increasing in time.

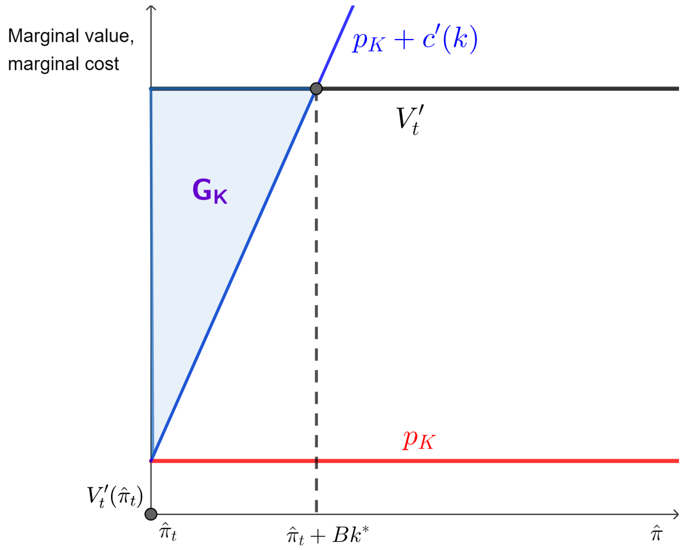
she knows that the project is profitable, or terminates it otherwise[12]. The parameterisation allows to avoid a situation in which agents keep their projects forever and growth investments are made eternally. This would be at odds with documented venture capital industry facts, where the objective of venture capitalists is precisely to exit the firm after some years, either by cashing out if the firm is profitable or by terminating it if it turns out to be a failure (Metrick and Yasuda, 2010). Second, model parameters are chosen such that, over the first years of the agent's life cycle, the firm is kept by the owner for a non-empty set of initial beliefs – i.e. there is room for the agent to keep the project, invest and learn, and she may not sell or terminate immediately. Again consistently with facts in Metrick and Yasuda (2010), this aims to capture the experimentation process venture capitalists incur when they start providing resources and guidance to high-risk firms.

3.4.1 Exit and Investment Decisions

Exit policy over the life cycle. Consider the numerical solution for the baseline model in Table 3.1. Let us first discuss exit policies of agents owning projects in this environment. In Figure 3.3, we observe the optimal termination threshold $\underline{\hat{\pi}}_t$ and

[12]Regarding implied discrete choices, this stationary value is similar to that shown in Figure 3.2(a).

FIGURE 3.4 Old agents, investment decisions



the optimal sale threshold $\overline{\hat{\pi}_t}$ for an agent over the firm life cycle. As we observe, as her firm gets old, the agent optimally keeps it for a smaller set of expected qualities $\hat{\pi}_t$. The upper threshold $\overline{\hat{\pi}_t}$ is decreasing in t, so that the agent is more eager to sell the project to the market as it gets old given a level of $\hat{\pi}_t$. The lower threshold $\underline{\hat{\pi}_t}$ is increasing in t, implying that, for the same level of expected quality, the agent is more willing to terminate the project as age increases.

Figure 3.3 shows that agents never keep their firms when they are older than a certain age, under parameter values in Table 3.1. For a very large t, the sale and the termination threshold coincide. This is due to the fact that σ_t^2 and the Kalman gain are zero for infinite ages, as implied by equations (3.3.4) and (3.3.5). When firms are very old, they already know their true quality pretty accurately, and cash-flow surprises do not lead to strong updates in their expected quality. For infinitely old agents, similarly to Figure 3.2(a), the ability to learn does not provide any option value from not terminating the project. This leads them to either terminate or sell, but they never keep and invest.

Figure 3.4 represents the investment decision of an infinitely old agent ($t = \infty$) that has a sufficiently high expected quality $\hat{\pi}_t$ and corresponding value $V(\hat{\pi}_t)$. This agent does not get any option value from keeping the project, and will only keep it if her investments are sufficiently profitable. As in Figure 3.2(a), value is linear for high expected qualities, and thus its derivative does not vary with $\hat{\pi}_t$. If the agent has already decided to keep her project, the marginal cost of investment $p_k + c'(k)$ is lower than the marginal value from investing, for small amounts of k, and optimal investment is a fixed positive amount (k^* in Figure 3.4). Thus, provided that $c(k)$ is convex, the old firm would thus find profitable to invest a positive amount[13]. The

[13]Without a convex cost, the problem of the firm would not be well-defined, and the agent would like to keep the project forever and make infinite investments to increase project's quality

benefit from investing and increasing the project's quality is represented by the blue area G_K in the graph. Nevertheless, parameters values in Table 3.1 are chosen such that $c_{op} > G_K$, so that the gains from investing when old are not large enough to cover operation costs. As a consequence, an agent decides not to keep her firm after some age, and she either sells it or terminates it depending on what she knows about her project's quality.

When agents are younger, however, projects are kept for intermediate values of $\hat{\pi}_t$, instead of being terminated or sold. First, the reason why a young agent would prefer to keep her project *instead of terminating* it is that, as far as $\hat{\pi}_t$ is not too low, the high σ_t when young translates into an option value from not terminating the project: given the ability to learn the agent has, a positive updating of $\hat{\pi}_t$ can happen with a large probability, and negative expected cash-flows can be cut down by the possibility of terminating the firm in the future. Importantly, this additional option value from not terminating may compensate the operation cost of keeping, c_{op}. In these cases, once c_{op} is sunk, an agent that keeps her firm finds it profitable to invest as far as her realised period cash-flows CF_t (or, equivalently, her future expected quality $\hat{\pi}_{t+1}$) are high enough.

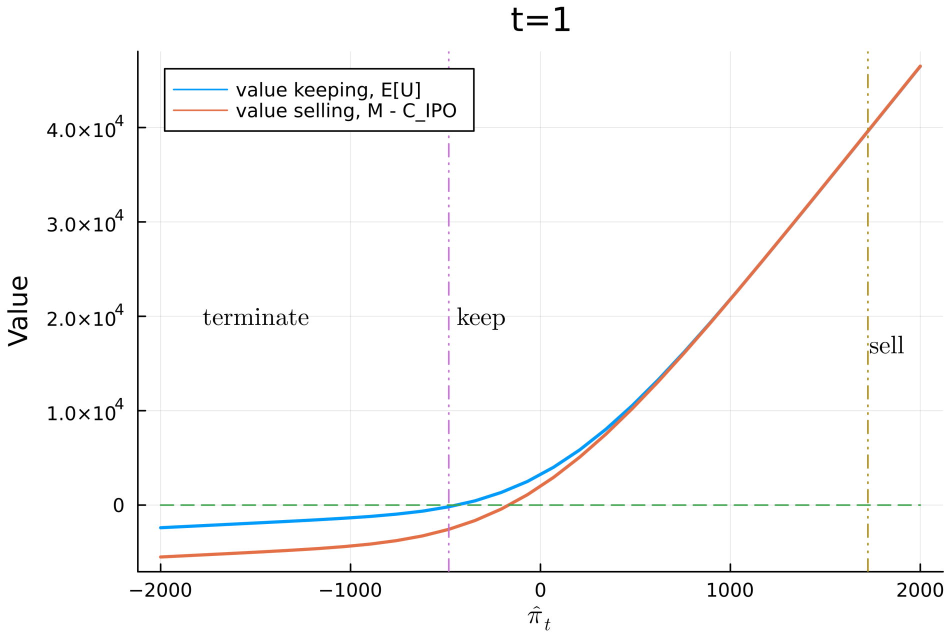
However, since the market can also learn about the project's quality, ability to learn (a low σ_ε) alone cannot explain why an agent would prefer to keep her project when young *instead of selling* it, if $\hat{\pi}_t$ is sufficiently high. The reason for this is that the sale cost C_{IPO} in Table 3.1 is large – in other words, it is costly to pass the firm to the market and to start market learning. Thus, although once sold to the market a firm is equally capable of updating beliefs, the original owner of the firm can learn from cash-flow realisations in a cheaper manner, for she does not have to pay C_{IPO} while she keeps her firm[14]. As a result, a young agent can delay paying sale costs by keeping her project and still benefit from the option value from learning, thus compensating c_{op} and investing afterwards.

Figure 3.5 shows the values of keeping and selling for different expected qualities, for an age-1 agent (thus having a high degree of quality uncertainty). For very low values of $\hat{\pi}_t$, expected cash-flows are so low that keeping or selling the project today result in a future termination with a very high probability, so the agent prefers to terminate immediately, given positive c_{op} and C_{IPO}. For very high values of $\hat{\pi}_t$, the probability of the market liquidating the project is almost zero, so the agent prefers not to delay the sale anymore. For expected qualities in between the sale and termination threshold, the agent prefers not to terminate the project and obtains an option value due to high uncertainty and the ability to learn. Still, sale is delayed in that region. On one side, given the high degree of uncertainty, upon a sale today

and value V. The difference between V'_t and p_K in Figure 3.4 makes that explicit.

[14]Indeed, if I set the sale cost to zero in a quantitative exercise, I find that no young agent would be willing to keep the project. An agent would be willing to sell the firm to the market if $\hat{\pi}_t$ is above some (negative) expected quality.

Figure 3.5 Young agents, exit decisions



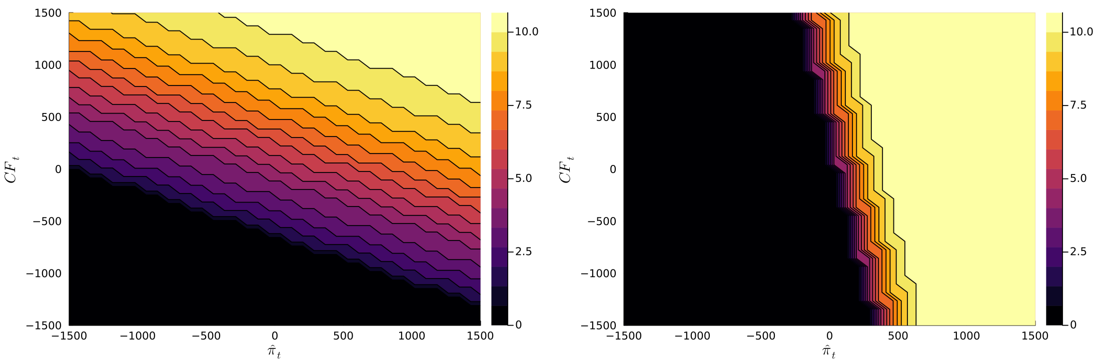
Notes: the graph shows, for an agent of age 1 owning a firm, the value she gets from terminating (dashed, green line), keeping (solid, orange line), and selling her firm to the market (solid, blue line). Sale and termination thresholds $\overline{\hat{\pi}}_t$ and $\underline{\hat{\pi}}_t$ for age 1 and *termination*, *keep* and *sale* regions are made explicit in the graph. Values are plotted against expected quality.

the market may liquidate the project afterwards with a positive probability, which brings down the market price today $M_t(\hat{\pi})$. On the other side, selling the project to the market and starting market learning is expensive, provided that C_{IPO} is large. This sale cost imposes a wedge between the value of keeping and the value of selling. Thus, the agent prefers to keep the project and get extra benefits from investing[15].

Investment policy over the life cycle. In Figure 3.6, I show the investment policy function $g_t^k(\hat{\pi}, CF)$ of an agent for two periods of time, $t = 0$ and $t = 5$. As argued in section 3.3.2 for a three-period model, learning makes optimal investment depend positively on CF_t. In the complete, numerically solved model, the investment policy is contingent on period cash-flows for both periods 0 and 5. Indeed, if I simulate the life cycle of an agent that makes decisions according to these policies, I find that the contemporaneous correlation between period cash-flows and investments is equal to 0.75, which denotes that fund injections and intermediate results are comoving in the simulated data. This is in line with documented facts in Kaplan and Strömberg (2003) regarding staged financing. It is worth noting that the frictionless model in this chapter generates contingent investment that is induced by

[15]If we shut down the ability to invest after keeping ($B = 0$), agents that keep their projects today but sell them right after in the next period with a very high probability (i.e. those with a quite high $\hat{\pi}_t$) would prefer to sell their projects immediately. This indicates that the extra benefits from investing are also a reason for keeping projects instead of selling them.

Figure 3.6 Investment policy function $g_t^k(\hat{\pi}, CF)$



(a) $t = 0$ (b) $t = 5$

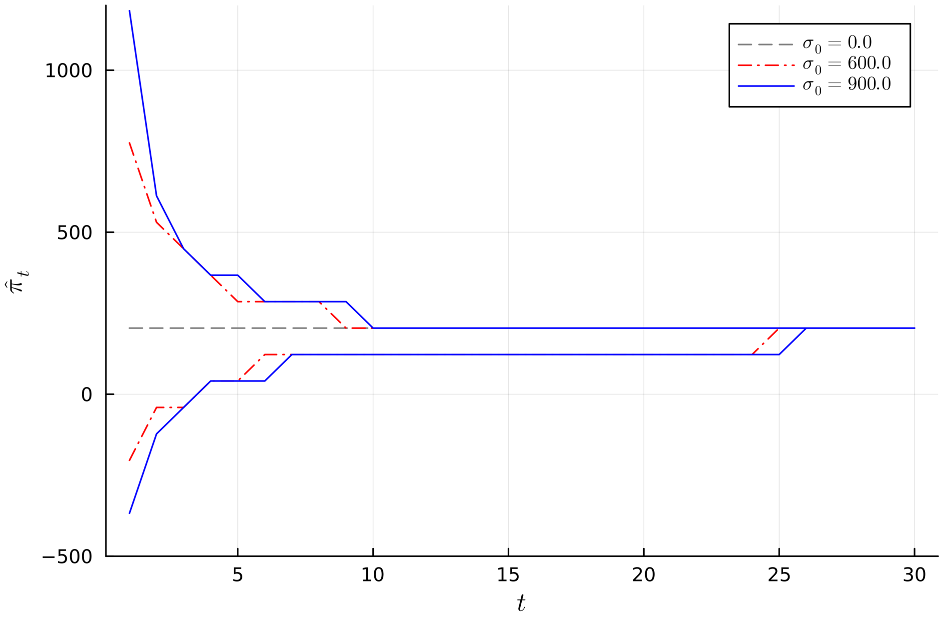
Notes: subfigure (a) represents the optimal investment policy at age 0 as a function of expected quality and period cash-flows, $g_0^k(\hat{\pi}, CF)$. Subfigure (b) represents the same object corresponding to age 5. Policy functions are obtained from the numerical solution of the optimisation problem of the agent, given the parameterisation in Table 3.1. Darker colours represent lower amounts of investment chosen by the agent, and lighter colours represent higher investment amounts.

the ability to learn and, differently from other papers in the firm dynamics literature, does not rely on financial frictions to yield a high correlation between investments and cash-flows.

As we can see from Figure 3.6, the degree of sensitivity of the optimal investment policy to period cash-flows changes over the life cycle of the firm. By comparing subfigures 3.6(a) and 3.6(b), we observe that the optimal investment policy is more sensitive to CF_t realisations when the firm is very young relative to older periods, given the same value for $\hat{\pi}_t$. For example, imagine a firm that enters period t (either 0 or 5) with $\hat{\pi}_t = 500$. If $t = 0$, a very bad cash-flow realisation (e.g. $CF_t = -1000$) would cause the firm not to invest, thus setting $g_0^k(\hat{\pi}, CF) = 0$, while a moderately good cash-flow realisation (e.g. $CF_t = 10$) would trigger a positive investment, being $g_0^k(\hat{\pi}, CF)$ slightly above 5. If $t = 5$, instead, these two alternative cash-flow realisations would yield optimal investments of around 9 and 10.5 respectively, spanning a smaller range of investment values and thus showing a less exacerbate reaction to intermediate results.

Again, the mechanism causing these different sensitivities is the fact that σ_t^2 is decreasing in time. When the agent is young, there is room for her initial expected beliefs $\hat{\pi}_0$ to vary, given the large initial uncertainty σ_0. This variation in beliefs may induce shifts in exit decisions and trigger strong investment reactions. However, after 5 periods, σ_5 is noticeably smaller than σ_0, as it has been shown in Figure 3.1. Thus, the older agent has more evidence that the true quality of the project is close to 500, i.e. that the project is sufficiently good. By that period, cash-flow realisations are less likely to cause strong changes in beliefs that induce shifts in discrete decisions, thus not giving room to sizable variations in investment.

FIGURE 3.7 Sale and termination thresholds, different σ_0



Notes: the graph shows optimal sale $\overline{\pi}_t$ and termination $\underline{\pi}_t$ thresholds for expected quality $\hat{\pi}_t$ for different periods of an agent's life cycle, considering three distinct levels of initial uncertainty σ_0. The lines are obtained from the numerical solution of the optimisation problem of the agent, given different levels of σ_0. The dashed line line represents the termination-sale threshold when there is no initial uncertainty. The dotted and the solid lines represent sale and termination thresholds for higher levels of initial uncertainty.

It is important to highlight that the shape of the exit and investment policies is possible given our assumption that $\sigma_\varepsilon = 600$. If the agent was not able to learn from cash-flows when keeping her project, equations (3.3.4) and (3.3.5) would not imply a decreasing pattern in σ_t^2, and the Kalman gain would be equal to zero. As I discuss later in this section, an agent that is completely unable to learn from period cash-flow realisations has an investment policy that is completely insensitive to period cash-flows, regardless of the level of σ_t^2.

3.4.2 The Role of Uncertainty

Here, I simulate the parameterised model and I discuss how policies and simulated outcomes change when we modify the initial quality uncertainty of the agent, represented by parameter σ_0. The role of initial uncertainty on exit and investment policies can be easily understood along the lines of the discussions of Figures 3.3 and 3.6: a higher σ_0 manifests through an increase in the Kalman gain, thus making the agent more predisposed to continue running the firm over the life cycle (via an increase in the option value of keeping). Besides, a higher Kalman gain makes investment policy $g_t^k(\hat{\pi}, CF)$ more sensitive to cash-flow realisations. The opposite occurs if the project is less uncertain at birth. Regarding the exit policy, Figure 3.7 shows how sale and termination thresholds get broader as we consider projects that are initially more uncertain. The higher the initial uncertainty, the higher the likelihood that the agent decides to keep the project over the life cycle.

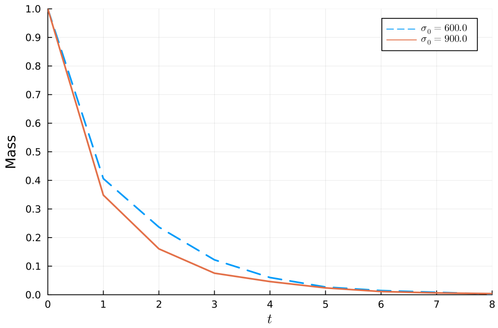
Simulated moments and value. Let us perform simulations to study how several outcomes are affected by the level of initial uncertainty. For that, I consider different levels of σ_0. For each of these levels, I solve the model (keeping the rest of the parameterisation as in Table 3.1) and I simulate 500,000 agents that are born with a firm. All of these simulated agents have homogeneous initial beliefs, but heterogeneous unobserved initial qualities, which are drawn from the exogenous population distribution of π_0. I look at several simulated moments of interest – namely, the mass of projects that are kept/terminated/have been sold by age, the total investment by age, the sale rate (i.e. the percentage of firms that are eventually sold at some period of life), and the value of the agent at firm birth.

First, in Figure 3.8(a), we see that a slightly larger mass of agents decides to keep their firms at young ages when we consider a level uncertainty $\sigma_0 = 600$ that is relatively low compared to the baseline level of 900. The reason for this is that, although the sale and termination bands are broader when we consider a high σ_0 (see Figure 3.7), the value of initial uncertainty also affects the population distribution of π_0. The higher the level of uncertainty, the larger the population variance of unobserved qualities across firms, and thus the larger the mass of firms with extreme values of π_0. This second effect prevails, and thus more firms decide either to terminate or to sell when uncertainty is high.

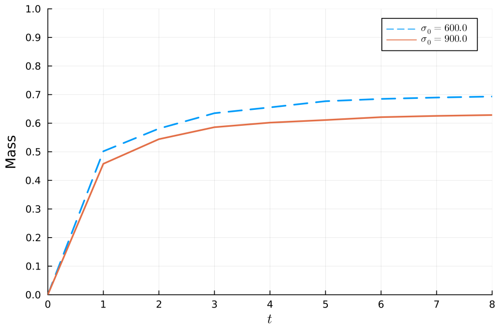
In Figures 3.8(b) and 3.8(c), we see that, when initial quality uncertainty is higher, more projects are sold and less projects are terminated at young ages. As a result of a high σ_0, agents that choose to keep their projects when young receive a high option value from continuing their firms instead of selling or terminating them immediately. These agents carry out contingent investments and eventually manage to sell their projects at high market prices. Finally, Figure 3.8(d) shows that total investment in the economy is larger for younger firms. As they age, many agents decide to sell them to the market or liquidate them, and this generates a decrease of total investment in age.

In Figure 3.9, I consider a range of levels of σ_0 from 0 to 1800, and I study how initial uncertainty affects the sale rate of firms, as well as their initial value. Figure 3.9(a) shows the percentage of firms eventually sold by their owners for different σ_0. If initial uncertainty is very low, the sale rate takes on a value of zero. The low expected quality of projects at birth, $\hat{\pi}_0 = -50$, leads agents to liquidate their firms right away. However, for levels of σ_0 above 350, the probability that a project is eventually sold to the market is monotonically increasing in the

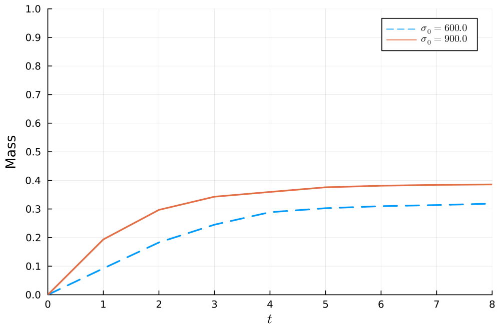
Figure 3.8 Exit and investment dynamics and initial quality uncertainty σ_0



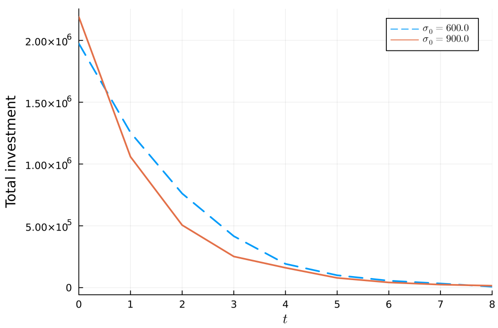
(a) Cumulative mass choosing *keep*



(b) Cumulative mass choosing *term*



(c) Cumulative mass choosing *sale*

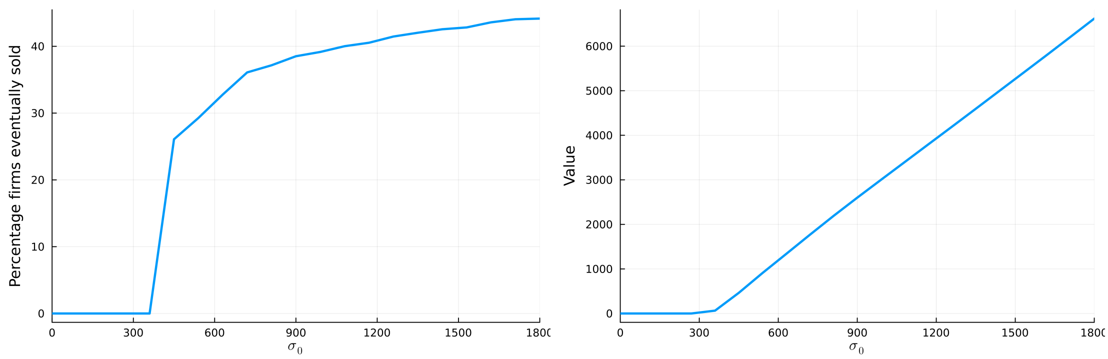


(d) Total investment by age

Notes: for every age, and a simulated sample of 500,000 agents, I show the cumulative mass of agents choosing to keep their project (subfigure (a)), to terminate the project (subfigure (b)), to sell the project (subfigure (c)), and the total investment by age in the simulated economy (subfigure (d)). The dashed, blue line corresponds to an initial level of uncertainty $\sigma_0 = 600$. The solid, orange line corresponds to a higher level of uncertainty, $\sigma_0 = 900$.

initial uncertainty, thus indicating that a sufficiently high risk may induce agents to not terminate them immediately and to carry out growth investments, given the higher likelihood of optimistic belief updatings. Indeed, as shown in Figure 3.9(b), a higher σ_0 increases the initial value of the agent. A larger σ_0 raises the Kalman gain in the learning process about the project's quality and makes keeping the project and making the appropriate contingent investment more valuable, provided that termination is always possible if belief updating turns out to be pessimistic. Agents owning more uncertain firms thus engage in more valuable experimentation, for they can learn more about its project's prospects and improve upon an immediate sale or termination.

Figure 3.9 Simulated outcomes and initial quality uncertainty σ_0



(a) Sale rate

(b) Value at period $t = 0$

Notes: subfigure (a) shows the fraction of simulated agents that eventually sell their firms at some point in their life, for simulated samples of 500,000 agents with different initial uncertainty levels. Subfigure (b) shows the initial value of an agent with initial expected quality $\hat{\pi}_0 = -50$, for different levels of initial uncertainty. I consider a range of levels for σ_0 from 0 to 1800.

3.4.3 The Role of Learning

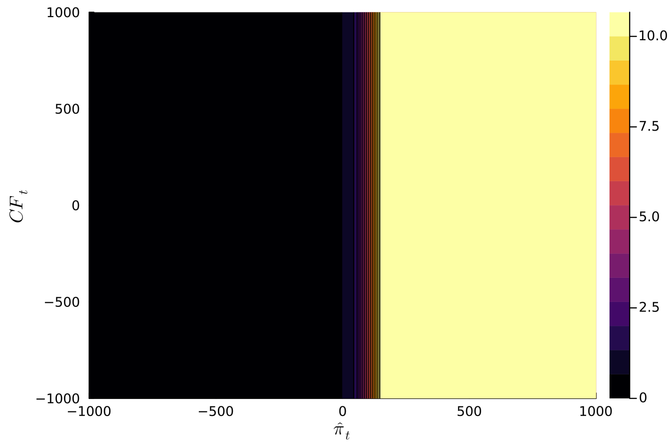
So far, we have assumed that firms can infer information about the true quality of the project from period cash-flows. We represent this ability of firms by a parameter $\sigma_\varepsilon = 600$, which shows up in the "observation equation" (3.3.1). This section studies what happens to policies, simulated moments and value when we challenge this parameter assumption.

The non-learning scenario. Let us first depart from the baseline parameterisation and consider instead an extreme non-learning case, where I impose the prior belief $(\hat{\pi}_0, \sigma_0^2)$ for all t (so that there is no belief updating), while keeping the rest of the parameterisation in Table 3.1. Under the non-learning scenario, an agent does not have access to a useful learning technology that allows her to update the quality distribution. In Figure 3.10, I show that the investment policy at period $t = 0$ of such an agent is completely insensitive to cash-flow realisations, and is positive only if the agent expects already a high quality for her firm[16]. Underlying this figure is the theoretical mechanism illustrated the three-period model from section 3.3.2: when the Kalman gain equals zero, no cash-flow realisation can induce a change in discrete decisions, and thus it does not affect investment.

Simulated moments and value. To see what happens if agents have access to better learning technologies relative to a non-learning environment, let us consider different values for σ_ε while keeping the rest of the parameterisation in Table 3.1, and

[16]This result holds for any positive σ_0. Imposing $(\hat{\pi}_0, \sigma_0^2)$ for all t implies that $\kappa_t = 0$ for all t, thus rendering g_t^k insensitive to cash-flow realisations, regardless of the value of initial uncertainty.

Figure 3.10 Investment policy $g_t^k(\hat{\pi}, CF)$ at period $t = 0$, no learning

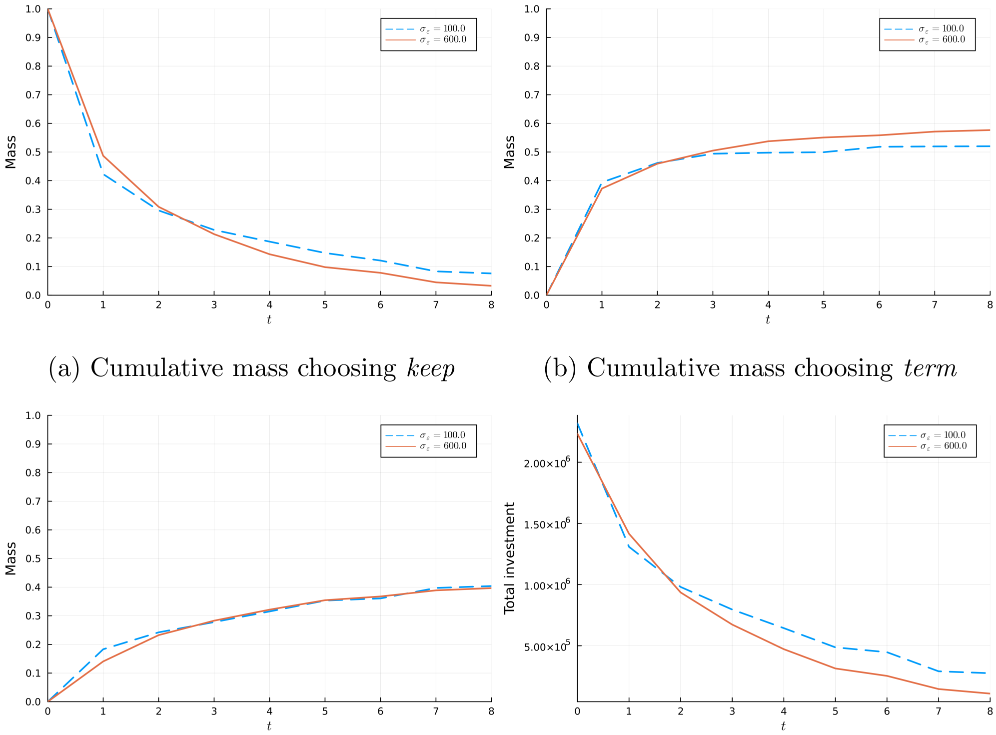


Notes: the graph represents the optimal investment policy at age 0 as a function of expected quality and period cash-flows, $g_0^k(\hat{\pi}, CF)$, in a non-learning scenario. The policy function is obtained from the numerical solution of the optimisation problem of the agent, imposing the prior belief $(\hat{\pi}_0, \sigma_0^2)$ for all t and the rest of the parameterisation in Table 3.1. Darker colours represent lower amounts of investment chosen by the agent, and lighter colours represent higher investment amounts.

let us see how simulated outcomes are affected. Figure 3.11 shows the simulated (for 500,000 firms) mass of firms that are kept, terminated and sold over the life cycle, as well as total investment by age, for two different levels of σ_ε. A better learning technology (i.e. a lower σ_ε) induces some of the simulated agents to keep the project for more periods than in the model with more noisy signals. Figure 3.11(a) shows that almost 10% of agents decide to keep the project after 9 years when $\sigma_\varepsilon = 100$, versus 3% in the baseline $\sigma_\varepsilon = 600$. As a consequence, total investment in the economy at older ages increases (Figure 3.11(d)). Figure 3.12 considers a range of values for σ_ε, from 0 to 5000, and how they affect the mass of firms eventually sold and the value at firm birth. As we can see, the lower the noise of period cash-flows, the higher the sale rate in the economy and the higher the value of owning a risky project. For very noisy signals, beliefs are barely updated and waiting to receive cash-flows is useless. Thus, agents decide to terminate the project and get zero value. As we decrease σ_ε, CF_t becomes a more significant signal of π_t, which gives incentives for agents to continue keeping their projects and inject funds in a contingent manner. This increases the chances that projects are eventually sold, and thus increases the value of owning risky firms.

Finally, Figure 3.13 shows how the initial value of risky companies changes with the noise of period cash-flows, for a high ($\sigma_0 = 900$) and a low ($\sigma_0 = 600$) level of initial uncertainty. As we can see, the increase in firm value due to having access to more informative signals is particularly notorious when the agent owns a more risky project. This gives us a powerful reason to believe that, when projects are highly uncertain, their owners' ability to learn turns out to be a particularly valuable

Figure 3.11 Exit and investment dynamics and learning σ_ε



(a) Cumulative mass choosing *keep*

(b) Cumulative mass choosing *term*

(c) Cumulative mass choosing *sale*

(d) Total investment by age

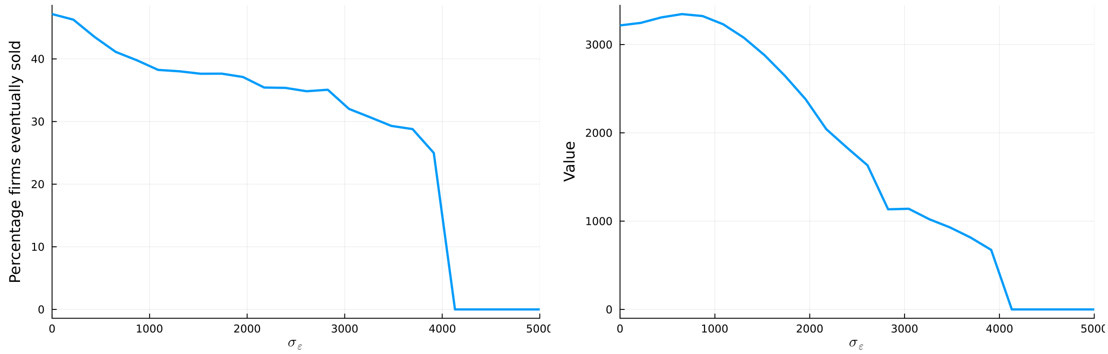
Notes: for every age, and a simulated sample of 500,000 agents, I show the cumulative mass of agents choosing to keep their project (subfigure (a)), to terminate the project (subfigure (b)), to sell the project (subfigure (c)), and the total investment by age in the simulated economy (subfigure (d)). The dashed, blue line corresponds to a high level of learning ability $\sigma_\varepsilon = 100$. The solid, orange line corresponds to a lower level of learning ability, $\sigma_\varepsilon = 600$.

skill. Given a low σ_ε, high-σ_0 firms avoid liquidations and benefit from specialised, contingent funding (which the market is unable to provide), thus increasing their value.

3.5 Conclusion

In this chapter, I present a model of the firm that mimics realistic features of young, innovative entrepreneurial projects – namely, venture-backed companies in the United States. The model explicitly considers uncertain returns, staged financing, exit decisions and intermediate results of firms over the life cycle. In this one-agent model, an agent implements a project by making investment and exit decisions over time. Importantly, the agent receives (as far as the project is not stopped) cash-flows every period that convey information about the true quality of the firm. Therefore, intermediate cash-flows allow for learning to take place over the

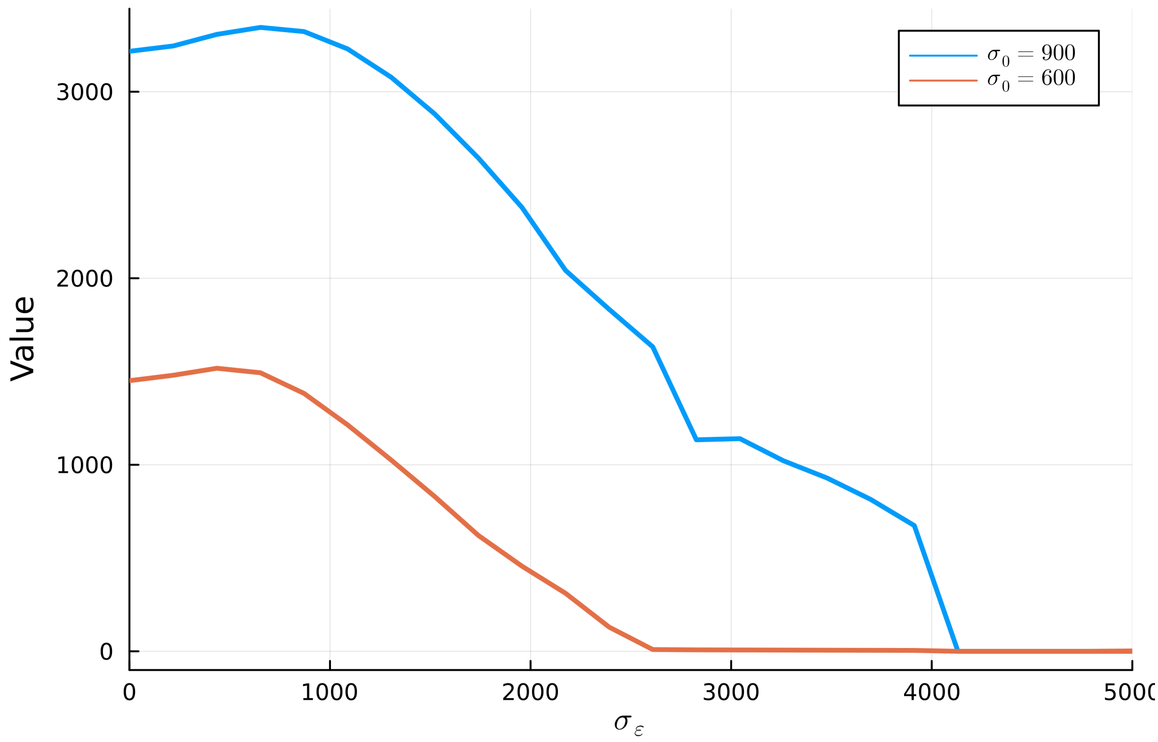
FIGURE 3.12 Simulated outcomes and learning σ_ε



(a) Sale rate (b) Value at period $t = 0$

Notes: subfigure (a) shows the fraction of simulated agents that eventually sell their firms at some point in their life, for simulated samples of 500,000 agents with different learning abilities, σ_ε. Subfigure (b) shows the initial value of an agent with initial expected quality $\hat{\pi}_0 = -50$, for different levels of learning ability. I consider a range of levels for σ_ε from 0 to 5000.

FIGURE 3.13 Value and learning, two levels of initial uncertainty



Notes: the graph shows the initial value of an agent with initial expected quality $\hat{\pi}_0 = -50$, for different levels of learning ability. I consider a range of levels for σ_ε from 0 to 5000. The blue line corresponds to a level of initial uncertainty of $\sigma_0 = 900$. The red line corresponds to a lower level of initial uncertainty, $\sigma_0 = 600$.

firm life cycle, and this affects the agent's decisions. By parameterising, solving and simulating the model, I arrive at two main quantitative findings. First, I find that optimal decisions regarding when to continue running the firm, when to sell it to the market or when to liquidate it depend crucially on the uncertainty regarding the firm's unobserved quality. In this context of uncertain returns, the agent's capability of updating her beliefs about the project (that is, the ability to learn), and of doing it in a cheaper manner than the market (given the large costs implicit in the sale process), is determinant for her to decide when to keep and when to stop or sell the project. Second, I find that, if a sufficiently good learning technology is available to the agent, she uses period cash-flows as informative signals of the true quality of the project and injects funds accordingly. As a result, it is optimal for the agent to make investments that are contingent on realised period cash-flows. Therefore, the single-agent model with learning is capable of rationalising the empirical fact that fund injections into US venture-backed firms are contingent in the realisation of intermediate results (Kaplan and Strömberg, 2003).

From the model simulation, I find that if the noise of the signal is sufficiently low, higher quality uncertainty translates into a higher value from implementing projects, a higher sensitivity of investment to cash-flow realisations, and a high positive contemporaneous correlation between simulated investments and cash-flows, in line with empirically documented patterns. If we shut down learning, keeping high-risk projects is not valuable and investment is completely insensitive to cash-flow realisations. In that situation, improving the learning technology motivates experimentation and contingent investment, thus resulting in value gains for firms. Thus, we may think of venture capital as being a means of financing that is sophisticated enough (at least in the United States) to be close to the kind of the learning agent in my model.

These findings support the idea that a superior ability to learn helps the owners of highly uncertain entrepreneurial projects to increase their value and motivates sophisticated contingent investments. The model in this chapter represents a setting where a firm owner chooses investment and exit strategies in order to maximise the net present value of her firm. The agent running the firm is not subject to any sort of contracting friction. However, in reality, we do not have just one agent implementing a high-risk project, but generally we have two parties – e.g. an entrepreneur and a venture capitalist. As it has been well acknowledged by the literature, the fact that two agents implement innovative projects may generate incentive problems (Marx, 1998; Bergemann and Hege, 1998; Cornelli and Yosha, 2003) that are though as motivating the usage of real-life, complicated securities (Kaplan and Strömberg, 2003; Cumming, 2008). Our model in section 3.3.1 does not inform about contractual choices that different parties may make in order to implement an innovative project. An alternative setting with contracting would consider an entrepreneur that owns the firm, and a venture capitalist that injects funds in it. These two agents implement the project jointly via some financial contract – either simple equity (Cumming, 2008; Ollivierre, 2010) or convertible securities (Kaplan and Strömberg, 2003; Schmidt, 2003). Still, even if this chapter studies the life-cycle behaviour of a single agent, I show that a one-agent model is sufficient to study the effects of learning over the life cycle of risky projects, and it gives an idea on what a good contracting environment should yield in terms of investment and exit practices in those firms. As a matter of fact, documented practices in the US venture capital market seem to present both the widespread usage of exit strategies in time and contingent fund injections, two features that the model replicates well. Other important possible extensions of this work, such as the competing role of traditional sources of financing, like banks, or alternative financial contracts, are left for future research.

A Appendix to Chapter 3

A.1 Derivation of the Kalman Filter Equations

We depart from a generalised form of equations (3.3.1) and (3.3.2):

$$CF_t = C\pi_t + \varepsilon_t$$
$$\pi_{t+1} = A\pi_t + Bk_t + w_t$$

where $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ and $w_t \sim N(0, \sigma_w^2)$; and from the conditional distribution of CF_t on $\hat{\pi}_t$ that we obtain by taking expectations and variance to expression (3.3.1):

$$CF_t \sim N(C\hat{\pi}_t, C^2\sigma_t^2 + \sigma_\varepsilon^2)$$

In the main text, I assume that $A = C = 1$, and that there is no shock in the law of motion equation, so that $\sigma_w^2 = 0$. The first step is to get a **filtering distribution** – that is, the distribution of quality once we have observed the realisation of the imperfect measure CF. This "filtered" quality is denoted by π_t^F, and it is our aim to get an estimate of it, $\hat{\pi}_t^F$. For that purpose, we perform the following regression of the unobserved state ($\pi_t = \pi_t^F - \hat{\pi}_t$) on the difference between the realisation of cash-flows and their prediction (a "surprise" relative to the expected value of CF_t at period t):

$$\pi_t^F - \hat{\pi}_t = \beta(CF_t - C\hat{\pi}_t) + v_t$$

The estimator that minimises the mean square error (it is in this sense that I talk about optimality here) using the information available is $\hat{\beta} = \dfrac{Cov(\pi_t^F - \hat{\pi}_t, CF_t - C\hat{\pi}_t \mid \hat{\pi}_t)}{Var(CF_t - C\hat{\pi}_t \mid \hat{\pi}_t)}$, from which we find $\hat{\beta} = C\sigma_t^2(C^2\sigma_t^2 + \sigma_\varepsilon^2)^{-1}$. This allows us to write a point estimate for the filtered quality and the corresponding variance:

$$\hat{\pi}_t^F = \hat{\pi}_t + C\sigma_t^2(C^2\sigma_t^2 + \sigma_\varepsilon^2)^{-1}(CF_t - C\hat{\pi}_t)$$
$${\sigma^F}_t^2 = \sigma_t^2 - \sigma_t^2 C^2 \sigma_t^2 (C^2\sigma_t^2 + \sigma_\varepsilon^2)^{-1}$$

which are the parameters that characterise the filtering distribution $\pi_t^F \sim N(\hat{\pi}_t^F, {\sigma^F}_t^2)$. This filtering distribution gives probabilities of different qualities *after* we filter out the prior by the new information provided by the realisation of CF_t, i.e. it is the distribution of π_t conditional on CF_t. In other words, we are applying Bayesian updating to our prior using the realised period cash-flows. The next step is to move from filtering to prediction. I use the filtering distribution and the law of motion (3.3.2) to get the **updated distribution** of quality π_{t+1}. Since new information about CF_t has arrived, we can use the filtered random variable π_t^F instead of π_t for predicting the future value of π_{t+1}. In other words, we turn equation (3.3.2) into $\pi_{t+1} = A\pi_t^F + Bk_t + w_t$. By taking expectations and the variance in this expression, we immediately get the two first Kalman filter equations:

$$\hat{\pi}_{t+1} = (A - \kappa_t C)\hat{\pi}_t + Bk_t + \kappa_t CF_t$$

$$= A\hat{\pi}_t + Bk_t + \kappa_t(CF_t - C\hat{\pi}_t)$$

and

$$\sigma_{t+1}^2 = (A^2 - AC\kappa_t)\sigma_t^2 + \sigma_w^2$$

where κ_t is the Kalman gain, whose expression is the third Kalman filter equation:

$$\kappa_t = AC\sigma_t^2(C^2\sigma_t^2 + \sigma_\varepsilon^2)^{-1}$$

A.2 Algorithm to Solve the One-Agent Model

Here, I replicate the value functions I have shown in section 3.3.1, which correspond to a firm that is held by the market (values W and M) and to a firm held by its original owner (values V and U). In either case, exit and investment decisions are made to maximise the total surplus from the firm.

The value of an age-t firm *held by the market* with period beliefs $(\hat{\pi}_t, \sigma_t^2)$ is:

$$W_t(\hat{\pi}) = \max\{0, M_t(\hat{\pi})\}$$

where $M_t(\hat{\pi})$ is given by:

$$M_t(\hat{\pi}) = \int \big(CF + \frac{1}{1+r} W_{t+1}(\hat{\pi}')\big) dG_t(CF \mid \hat{\pi})$$

where $CF = \pi + \varepsilon$ (observation equation (3.3.1)), and beliefs evolve according to equations (3.3.3)-(3.3.5) – namely, $\hat{\pi}' = \hat{\pi} + \kappa_t(CF - \hat{\pi})$. In turn, the value of an age-t firm *held by its original owner* with period beliefs $(\hat{\pi}_t, \sigma_t^2)$ is:

$$V_t(\hat{\pi}) = \max\{\underbrace{0}_{\text{termination}}, \underbrace{M_t(\hat{\pi}) - C_{IPO}}_{\text{sale}}, \underbrace{\int U_t(\hat{\pi}, CF) dG_t(CF|\hat{\pi})}_{\text{keep}}\}$$

where the value of the *keep* decision is:

$$U_t(\hat{\pi}, CF) = \max_{k \geq 0, \hat{\pi}' \in \mathbb{R}} \left\{ CF - c(k) - p_K k - c_{op} + \frac{1}{1+r} V_{t+1}(\hat{\pi}') \right\}$$
$$\text{s.t.} \qquad \hat{\pi}' = \hat{\pi} + Bk + \kappa_t (CF - \hat{\pi})$$

and, again, subject to equations (3.3.4) and (3.3.5). Since the evolution of the variance of the project, σ_t^2, and the Kalman gain, κ_t, are fully exogenous, we can use the initial condition on quality uncertainty, σ_0^2, and equations (3.3.4) and (3.3.5) to get the entire life-cycle path for the variance and the Kalman gain. An important feature of these two variables is that they converge to their respective limit values as t goes to infinity. If the Kalman filter converges at an age T_{conv}, then, for $t > T_{conv}$, the problem of the firm (either market-held or owner-held) is stationary – that is, value functions W, M, V and U do not depend on time from age T_{conv} onwards. That being said, and taking as given sequences $\{\sigma_t^2\}_{t=0}^{\infty}$ and $\{\kappa_t\}_{t=0}^{\infty}$, we can rewrite the optimisation problem faced by the owner-held firm at a generic age $t > 0$ after the period cash-flow has realised as:

$$U_t(\hat{\pi}, CF) = \max_{k \geq 0} \left\{ CF - c(k) - p_K k - c_{op} + \frac{1}{1+r} V_{t+1}(\hat{\pi} + Bk + \kappa_t (CF - \hat{\pi})) \right\}$$

which is a problem where the agent's control k and state $(\hat{\pi}, CF)$ imply a future value V_{t+1}. Importantly, the fact that the Kalman gain varies over time makes this a non-stationary problem for ages $t < T_{conv}$, i.e. until convergence of σ_t^2 and κ_t is achieved. Thus, I index values and policies by time in the exposition.

We want to find the investment policy $g_t^k(\hat{\pi}, CF) \in [0, \infty)$ and the exit policy $g_t^{exit}(\hat{\pi}) \in \{termination, sale, keep\}$. Given the max structure of $V_t(\hat{\pi})$, for convenience, let us express the exit strategy via possibly time-variant thresholds for $\hat{\pi}_t$ that make the agent indifferent between any pair of discrete choices. In the main text, I show that it is indeed the case that policies from the solved model are threshold strategies, given a wide range of alternative parameterisations. I find that these thresholds are $\underline{\hat{\pi}}_t$ and $\overline{\hat{\pi}}_t$, with $\underline{\hat{\pi}}_t < \overline{\hat{\pi}}_t$, such that the firm decides to terminate for $\hat{\pi}_t$ below $\underline{\hat{\pi}}_t$, to sell above $\overline{\hat{\pi}}_t$, and to continue running the firm herself in between these two thresholds. The algorithm I use to solve the model is value function iteration. Iterating on the value function, I get the stationary value for market-held firms and for owner-held firms. For those, I consider that the Kalman gain is equal to its limit value, which I call $\overline{\kappa}$ here. I first get the stationary values W_∞ and M_∞ (as well as the stationary termination-sale policy for a market-held firm) and then, introducing the stationary M_∞ in equation (3.3.7) with $\kappa_t = \overline{\kappa}$, I find stationary values V_∞, U_∞ and stationary policies $g_\infty^k(\hat{\pi}, CF)$ and $g_\infty^{exit}(\hat{\pi})$ for the owner-held firm. Having found the stationary functions, I use a backward-induction procedure starting from age $T = T_{conv}$ such that $\kappa_T = \overline{\kappa}$ to period 0, to get the value functions when the economy is not stationary – that is, W_t, M_t, V_t and U_t, and the corresponding non-stationary policies.

The complete value-function-iteration algorithm is:

1. Given σ_0^2, find exogenous sequences $\{\sigma_t^2\}_{t=0}^{\infty}$ and $\{\kappa_t\}_{t=0}^{\infty}$. Find the limit value for the Kalman gain, $\overline{\kappa}$.

2. **Value function iteration (stationary ages)**: for a market-held firm, find the fixed point of:

$$TW_\infty(\hat{\pi}) = \max\{0, M_\infty(\hat{\pi})\}$$

where

$$M_\infty(\hat{\pi}) = \int \big(CF + \frac{1}{1+r} W_\infty(\hat{\pi}')\big) dG_\infty(CF \mid \hat{\pi})$$

which is an operator that gets TW as a function of W. Notice that I am using the limit value for the Kalman gain, $\overline{\kappa}$, which prevents us from indexing values by time (I index here by ∞ to make explicit the stationary age). Once this is done, for an owner-held firm, and using M_∞ as an input, find the fixed point of:

$$TV_\infty(\hat{\pi}) = \max\{0, M_\infty(\hat{\pi}) - C_{IPO}, \int U_\infty(\hat{\pi}, CF) dG_\infty(CF|\hat{\pi})\}$$

where

$$U_\infty(\hat{\pi}, CF) = \max_{k \geq 0} \left\{ CF - c(k) - p_K k - c_{op} + \frac{1}{1+r} V_\infty(\hat{\pi} + Bk + \overline{\kappa}(CF - \hat{\pi})) \right\}$$

which is an operator that gets TV as a function of V. Notice the limit value for the Kalman gain, $\overline{\kappa}$. The stationary investment policy is obtained using the Nelder-Mead optimisation routine, which does not rely on derivatives[17]. The stationary policy, characterised by thresholds $\{\underline{\hat{\pi}}_\infty, \overline{\hat{\pi}}_\infty\}$, is obtained by comparing the stationary values the agent gets if she terminates the firm, if she sells it to the market (in exchange of a stationary market price M_∞), and if she keeps it, for different values of $\hat{\pi}$.

3. **Backward induction (non-stationary ages)**: consider age $T = T_{conv}$ such that $\kappa_T = \overline{\kappa}$. At that period, value functions are stationary, i.e. $M_T = M_\infty$, $V_T = V_\infty$, and so on. Then, we iterate backwards on market-held and owner-held values as follows:

[17] This routine is more stable than a derivative-based approach, such as an LBFGS routine. This alternative routine, in turn, yields the same results as those in the main text.

(i) (Period T) Consider a discrete grid for $\hat{\pi}$ and compute $W_T(\hat{\pi})$ and $V_T(\hat{\pi})$ for every value in the discrete grid. Interpolate $W_T(\hat{\pi})$ and $V_T(\hat{\pi})$, to get objects $\tilde{W}_T(\hat{\pi})$ and $\tilde{V}_T(\hat{\pi})$ defined for every $\hat{\pi}_T$ in the real line.

(ii) (Period $T-1$) In order to solve for U_{T-1}, consider discrete grids for $\hat{\pi}_{T-1}$ and CF_{T-1}. Use the interpolated $\tilde{V}_T(\hat{\pi}_T)$ to solve:

$$U_{T-1}(\hat{\pi}, CF) = \max_{k \geq 0} \left\{ CF - c(k) - p_K k - c_{op} + \frac{1}{1+r} \tilde{V}_T\big(\hat{\pi} + Bk + \kappa_{T-1}(CF - \hat{\pi})\big) \right\}$$

for every pair $(\hat{\pi}_{T-1}, CF_{T-1})$ in the discrete grid. Find the optimal investment policy $g^k_{T-1}(\hat{\pi}, CF)$ for every pair $(\hat{\pi}_{T-1}, CF_{T-1})$ in the discrete grid. Similarly, use the interpolated $\tilde{W}_T(\hat{\pi}_T)$ to find $M_{T-1}(\hat{\pi}, CF)$ for every pair $(\hat{\pi}_{T-1}, CF_{T-1})$ in the discrete grid.

(iii) For each $\hat{\pi}_{T-1}$ in the discrete grid, interpolate $M_{T-1}(\hat{\pi}, CF)$ with respect to CF. Similarly, interpolate $U_{T-1}(\hat{\pi}, CF)$ with respect to CF. For each $\hat{\pi}_{T-1}$ in the discrete grid, we get $\tilde{M}_{T-1}(\hat{\pi}, CF)$ and $\tilde{U}_{T-1}(\hat{\pi}, CF)$ that is defined for every CF in the real line. Then, take expectations (integrate) these objects with respect to CF to get the value:

$$V_{T-1}(\hat{\pi}) = \max\{0, \tilde{M}_{T-1}(\hat{\pi}) - C_{IPO}, \int \tilde{U}_{T-1}(\hat{\pi}, CF) dG_{T-1}(CF|\hat{\pi})\}$$

for each $\hat{\pi}_{T-1}$ in the discrete grid (take into account that the distribution of CF_{T-1} is known given state $\hat{\pi}_{T-1}$). By comparison of the three objects within the maximum, we can get quality thresholds $\underline{\hat{\pi}}_{T-1}$ and $\overline{\hat{\pi}}_{T-1}$, which drive the *termination/keep/sale* policy of the firm at period $T-1$. Similarly, we can get value for the market-held firm at period $T-1$, $W_{T-1}(\hat{\pi})$, and the corresponding termination-sale policy for the market.

Again, we can get interpolated objects $\tilde{W}_{T-1}(\hat{\pi})$ and $\tilde{V}_{T-1}(\hat{\pi})$, defined for every π_{T-1} in the real line.

(iv) (Periods $T-2$, ..., 0) Continue backwards until finding $V_0(\hat{\pi}_0)$ and optimal policies $g^k_t(\hat{\pi}, CF)$ and $g^{exit}_t(\hat{\pi})$ for all ages $t \geq 0$.

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