From Point Particles to Gauge Field Theories: a Differential-Geometrical approach to the Structures of the Space of Solutions

by

Luca Schiavone

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in

Mathematical Engineering

Universidad Carlos III de Madrid

Advisor:

Alberto Ibort Latre

Tutor:

Alberto Ibort Latre

February 2023

PUBLISHED AND SUBMITTED CONTENT

The main results of this thesis are based on the following list of publications.

- [1].
  https://arxiv.org/abs/2208.14136.

  I am the author of this work, and it is partially included in Chapters 1, 2, 3 and 4. The material from this source included in this thesis is not singled out with typographic means and references.

- [2].
  https://arxiv.org/abs/2208.14155.

  I am the author of this work, and it is partially included in Chapters 1, 2, 3 and 4. The material from this source included in this thesis is not singled out with typographic means and references.

- [3].
  https://www.mdpi.com/2073-8994/14/1/70.

  I am the author of this work, and it is partially included in Chapters 2 and 4. The material from this source included in this thesis is not singled out with typographic means and references.

- [4].
  https://temat.es/monograficos/issue/view/vol-2

  I am the author of this work, and it is partially included in Chapters 2, 3 and 4. The material from this source included in this thesis is not singled out with typographic means and references.

CONTENTS

1. INTRODUCTION

Quantum Mechanics was born to describe phenomena happening at very small length scale, which Classical Physics was not able to predict, such as the behaviour of electrons in the atoms and the consequent atomic spectra observed. However, from the early days of its development it was clear to its founding fathers that in the energy (or length) regimes where Classical Physics successfully predicted reality, Quantum Mechanics should have produced predictions in accordance to those of Classical Physics. This belief is testified, for instance, by the following words of *N. Bohr*:

> *[...] it seems possible to throw some light on the outstanding difficulties of the quantum theory by trying to trace the analogy between the quantum theory and the classical theory of radiation as close as possible. In order to obtain the necessary connection [...] to the ordinary theory of radiation in the limit of small vibrations, we must claim that a relation, as that just proved for frequencies, will, in the limit of large n, hold also for the intensities of the different lines of the spectrum.*

contained in the treatise [5] where he posed the basis of Quantum Mechanics. With this idea in mind, thousands of scientists along the last century were motivated in studying the relation between Classical and Quantum Physics and, in particular, how to make mathematically precise Bohr's claim. This produced thousands of contributions, going from the so-called WKB APPROACH, named after *G. Wentzel*, *H. A. Kramers* and *L. Brillouin* [6]–[8], to the MOYAL BRACKET APPROACH that goes under the names of *J. E. Moyal* and *H. J. Groenewold* [9], [10], passing through the COHERENT STATES APPROACH developed by *R. J. Glauber* and *E. C. G. Sudarshan* [11], [12] and arriving to the more modern approaches in terms of WIGNER MEASURES [13], [14]. Nonetheless, we still cannot claim that we have a mathematically rigorous proof of Bohr's claim and this issue has involved (and keeps involving) some of the most brilliant minds of the last century. However, going back again to the founding fathers of Quantum Mechanics, a first milestone in this direction was already put by *P. A. M. Dirac* who, in a paper of 1925 [15] wrote:

> *We will now consider to what the expression xy-yx corresponds on the classical theory. [...] the difference of the Heisenberg product of two quantum quantities is equal to $\frac{ih}{2\pi}$ times their Poisson bracket expression. In simbols:*
>
> $$\frac{ih}{2\pi}(xy - yx) = \{\, x,\, y \,\}\,.$$

These words may be interpreted in the following way. Thanks to *W. K. Heisenberg*'s contributions it was already more or less clear that a fundamental mathematical structure appearing in Quantum Mechanics is that of Lie algebra. Indeed, nowadays we know that physical observable quantities in Quantum Mechanics can be modelled as elements of the Lie algebra of linear operators over a separable complex Hilbert space and, by looking at HEISENBERG'S EQUATION, that the dynamical content of the theory is also encoded into this structure. On the other hand, it was also clear that within Classical Mechanics the fundamental structure on the space of (classical) observables and in terms of which the dynamics is described is that of a Poisson algebra. Thus, what Dirac was claiming is that there should be some (hopefully mathematically rigorous) limiting procedure relating the "Quantum" Lie algebra structure to the "Classical" Poisson structure. This is what is often called CORRESPONDENCE PRINCIPLE or ANALOGY PRINCIPLE and is what Dirac himself in [15] referred to as QUANTUM CONDITION. The way Dirac stated the correspondence principle in 1925 is not really different to how it is formulated nowadays in most Quantum Mechanics textbooks. Indeed, even with the development of Quantum Field Theory as the theory that best describes fundamental Physics, namely, the theory of fundamental interactions, the correspondence principle stated by Dirac kept being the guiding principle in the relation between Quantum and Classical Field Theories. However, within field theories an other, fundamental, ingredient comes into play: SPECIAL RELATIVITY. Indeed, it is a well established fact that quantum field theories describing fundamental interactions should exhibit Poincaré invariance as one of the fundamental symmetries of the theory. With this in mind, and recalling the Bohr-Dirac formulation of the correspondence principle, it is clear that in classical field theories the fundamental structure in terms of which the space of observables and the dynamics must be modelled must be that of a Poisson bracket being invariant with respect to the Poincarè group (covariant Poisson bracket, for short). The first contribution in this direction was given by *R. E. Peierls* in his seminal paper [16] of 1952 where he introduced an algorithmic way to produce covariant Poisson brackets within classical field theories in terms of the causal Green's function of the linearized equations of the motion. However, as testified by the following *B. DeWitt*'s words appearing in the introduction of [17]:

> *There exists an anomaly today in the pedagogy of physics. When expounding the fundamentals of quantum field theory physicists almost universally fail to apply the lessons that relativity theory taught them early in the twentieth century. Although they usually carry out their calculations in a covariant way, in deriving their calculational rules they seem unable to wean themselves from canonical methods and Hamiltonians, which are holdovers from the nineteenth century and are tied to the cumbersome* $(3+1)$*-dimensional baggage of conjugate momenta, bigger-than-physical Hilbert spaces, and constraints. There seems to be a feeling that only canonical methods are safe; only they guarantee unitarity. This is a pity because such a belief is wrong, and it makes the foundations of field theory unnecessarily cluttered. One of the unfortunate results of this belief is that physicists, over the years, have almost totally neglected the beautiful covariant replacement for the canonical Poisson bracket that Peierls invented in 1952.*

Peierls' construction went largely unnoticed or neglected, with only a few exceptions. It was DeWitt himself who carried on Peierls' approach, applying it to the gravitational field (see [18]) and adopting Peierl's bracket in [17], [19] as the fundamental structure to construct a coherent covariant description of classical field theories.

Although providing a breakthrough in the formulation of field theories from a relativistic point of view, the above mentioned contributions lack a deep analysis of the very geometrical structures involved in the construction of the covariant brackets.

Indeed, a mathematical framework that fits the geometrical description of covariant field theories is the multi-symplectic formalism, which started to be developed few years later, with the seminal contributions given by [20]–[24]. These papers were conceived as a generalisation to field theories of the symplectic geometry approach so successful in geometrically describing Classical Mechanics. In the successive decades several contributions to the development of multi-symplectic geometry appeared. Among them, the contributions [25]–[27] (and references therein) focusing on the geometrical formulation of Lagrangian variational principles and on the generalizations of the definition of the Poincaré-Cartan form within higher order field theories should be mentioned as well as [28] (and references therein) which aims to use multi-symplectic geometry to geometrically formulate General Relativity as a field theory. Other relevant contributions to the development of multi-symplectic geometry are in the following, far from exhaustive, list [29]–[40].

After the development of multi-symplectic geometry, the construction of a covariant Poisson bracket for field theories has received a renewed interest, although under a slightly different perspective, that of studying the very differential geometrical structures involved in the construction of the brackets introduced in the literature cited above (see [41]–[49]).

The aim of this manuscript is that of offering a unified account of the description of covariant brackets covering all field theories describing fundamental interactions, i.e, all (possibly non-Abelian) gauge theories as well as General Relativity, providing a consistent analysis of the geometric and global-analytic structures involved in the construction of such a bracket. Our approach originates from an observation by *J. M. Souriau* contained in his book [50], which provides a very elegant way of preserving the covariance under the Poincaré group at each step in describing a dynamical system. The idea was that of abandoning the classical concept of phase space which do not possesses the required properties of covariance:

> *Analytical mechanics is not an outdated theory, but it appears that the categories which one classically attributes to it such as configuration space, phase space, Lagrangian formalism, Hamiltonian formalism, are, simply because they do not have the required covariance; in other words, because these categories are in contradiction with Galilean relativity. A fortiori, they are inadequate for the formulation of relativistic mechanics*

in the sense of Einstein.

and referring to the space of solutions of the equations of the motion as the carrier space on which one should settle the geometrical description of the dynamical system under investigation. Indeed, if the theory under investigation is covariant with respect to the Poincaré group, then it preserves the space of solutions, in the sense that elements of the Poincaré group map solutions into solutions. Thus, the idea underlying this manuscript is that of having in mind the Bohr-Dirac correspondence principle within the context of field theories and adopting Souriau's point of view to construct the covariant Poisson bracket. This means that we will try to exhibit a Poisson bracket structure directly defined on the space of solutions of the equations of the motion (solution space, for short) of field theories. It is interesting to recall that a closed form on the solution space were already introduced in [51] and, within the examples given by Yang-Mills theories, General Relativity and String Theory in [52] and [53], at least at a formal level from the point of view of differential geometry. Such a closed form turned out to be pre-symplectic within gauge theories, its kernel being related to the gauge invariance of the equations, and the authors do not address the problem of constructing an associated Poisson bracket. We will proceed such a step further. An analysis of this problem has been performed in [42] even if restricted only to non-gauge theories and in [44] (within the variational bi-complex approach) with the purpose of constructing a Poisson structure on the quotient of the solution space of the equations modulo gauge symmetries, while a Lie bracket on a space of gauge invariant functions is described in [54] within the so-called secondary calculus approach. With respect to these papers we adopt a slightly different point of view, namely we will provide a bracket given in terms of a suitable Poisson bivector which is *globally* defined on the solution space (prior to quotienting it modulo gauge symmetries), and this will allow us to highlight the geometrical necessity of introducing ghost fields (via a trick related to the coisotropic embedding theorem) within those gauge theories for which gauge fixing can not be global in the space of fields, i.e. for theories presenting the so called *Gribov's ambiguities* (note that, for instance, all non-Abelian gauge theories fall into this case).

Another step further we perform with respect to the existing literature is that we avoid the usual approach of dealing with the space of smooth fields as a formal differential manifold on which a formal differential calculus à la Cartan is defined and we deal, in all the examples we analyse, with well defined Banach or Hilbert manifolds on which all the geometrical objects and the differential calculus are rigorously defined.

The basic idea underlying the construction of the Poisson bracket we will perform in this manuscript is that we will try to construct a bijection

$$\Psi \ : \ \mathcal{EL}^{\Sigma} \to \mathcal{EL} \, . \tag{1.1}$$

between the solution space, that we will denote by $\mathcal{EL}$ for reasons that will be clear in the sequel, and the space of Cauchy data $\mathcal{EL}^{\Sigma}$[1].

On the other hand, we will show that within the multi-symplectic formulation of field theories a canonical 2-form on the solution space naturally emerges from the variational principle. Moreover, we will show that such a 2-form can be written as

$$\Omega \,=\, \Psi^{-1\star}\tilde{\Omega}\,, \tag{1.2}$$

where $\tilde{\Omega}$ is a suitable 2-form on $\mathcal{EL}^{\Sigma}$. Depending on the properties of $\mathcal{EL}^{\Sigma}$ and $\tilde{\Omega}$ we discuss cases (in increasing order of difficulty) in which these structures are used to define a Poisson bracket on $\mathcal{EL}$:

- For point particle mechanical systems it turns out that $\mathcal{EL}^{\Sigma}$ is a finite-dimensional manifold and that $\tilde{\Omega}$ is a symplectic form. Therefore, since Ψ^{-1} is a diffeomorphism, the form $\Psi^{-1\star}\tilde{\Omega}$ is symplectic as well and corresponds to a Poisson bracket. Note that being the spaces diffeomorphic, it is equivalent working on $(\mathcal{EL},\,\Omega)$ or on $(\mathcal{EL}^{\Sigma},\,\tilde{\Omega})$.
- Within field theories, one has that $\mathcal{EL}^{\Sigma}$ is infinite-dimensional and $\tilde{\Omega}$ is proved to be the structure emerging from the pre-symplectic constraint algorithm developed when dealing with pre-symplectic Hamiltonian systems. When the theory under investigation does not exhibit gauge symmetries, one has that $\tilde{\Omega}$ is symplectic. It then follows that $\Psi^{-1\star}\tilde{\Omega}$ is symplectic and gives rise to a Poisson bracket. Note that, again, it is equivalent working on $(\mathcal{EL},\,\Omega)$ or on $(\mathcal{EL}^{\Sigma},\,\tilde{\Omega})$.
- The last case we analyse is that of gauge theories for which the pre-symplectic constraint algorithm ends up with a degenerate 2-form $\tilde{\Omega}$. We will define a Poisson bracket by using such a pre-symplectic structure and by using a regularization technique related with the so-called coisotropic embedding theorem that allows to define a Poisson bracket on a canonical enlargement of $\mathcal{EL}^{\Sigma}$. In particular we will distinguish two sub-cases. The first one, exemplified by Abelian gauge theories, where the Poisson bracket constructed on such enlargement can be "projected" to a Poisson bracket on $\mathcal{EL}^{\Sigma}$. The second one, exemplified by non-Abelian gauge theories, in which such a projection is impossible and, thus, one is forced to work on the enlargement of $\mathcal{EL}^{\Sigma}$ obtained out of the coisotropic embedding theorem in order to have a well defined Poisson bracket. Here, we will note that the additional degrees of freedom related to the enlargement procedure can be interpreted in terms of the well known concept of GHOST FIELDS which is known to be necessary to quantize non-Abelian gauge theories within the BRST approach [55].

[1] This notation is due to the fact that along the manuscript we will denote by Σ the hypersurface of the space-time on which the field theory is settled, on which we will take Cauchy data for the equations of the motion.

The manuscript is organized as follows. Chapter 2 is a collection of mathematical preliminaries useful for the reading. In Chapter 3 we describe the multi-symplectic formulation of classical field theories giving a geometrical definition of the solution space coming from an (geometrically) intrinsic definition of the variational principle. In this chapter, as well as in the subsequent, we will consider always in parallel both the Lagrangian and the Hamiltonian formulation within particle mechanics as well as in field theories.

Chap. 4 is the core of the manuscript. In Sec. 4.1 we will see how from the intrinsic variational formulation mentioned above, a canonical 2-form emerges on the solution space. In Sec. 4.2 we will see how to use such a canonical structure in formulating the dynamical system under investigation as a pre-symplectic Hamiltonian system. It will result that this approach allows to algorithmically construct the space of Cauhcy data for our system that turns out to be equipped with a canonical 2-form as well. Finally, in Sec. 4.3 we will construct the Poisson bracket on the solution space of the theory by the aid of the canonical structure on the space of Cauchy data just mentioned.

Part of the material of the present manuscript appears in the following list of contribution of the author together with some co-authors [1]–[4], [47], [48], [56].

2. PRELIMINARIES

As it is clear from the introduction, the research project from which this manuscript originates is a very interdisciplinary one. Even if the main focus is on the study of geometric structures, it is evident that Differential Geometry is not the only branch of Mathematics involved. On the one hand, since a great part of the dissertation will take place on infinite-dimensional spaces, Functional Analysis will play a crucial role along the whole manuscript. On the other hand, since geometric structures we are interested in comes from Physical systems, Group Theory is everywhere along the discussion as well as many results from Algebra, in general, will be used here and there.

For these reasons, the program of writing a "Mathematical preliminaries" chapter turns into a very hard issue and the idea of collecting all the results needed in order to make the manuscript readable from a transversally wide audience is an hopeless one. Indeed, we abandon from the very beginning the presumption of writing a comprehensive account on all the Mathematical Methods needed to safely start the reading of this manuscript. We assume the average reader to be quite familiar with the fundamentals of Differential Geometry, Lie Group theory, Mathematical Analysis and Theoretical Physics and we intend the present chapter just as a collection of results that we consider either not standard or of crucial importance because used extensively along the text. Moreover, we take the existence of this chapter also as an opportunity to fix many notations and conventions used throughout the manuscript.

In particular, we will devote Sec. 2.1 to recall the differential calculus on, possibly infinite-dimensional, manifolds restricting ourselves to the Banach setting. Sec. 2.2 and 2.3 deal with the notions of symplectic and pre-symplectic manifold, on the issue of modelling a dynamical system as a (pre-symplectic) Hamiltonian system and how this formulation is useful in dealing with the symmetries of the dynamical system. In Sec. 2.4 we recall the main aspects of the theory of jet bundles that are the arena on which the multi-symplectic formulation of field theories addressed in the successive chapters takes place. The chapter ends with an appendix (2.A) in which the construction of the prolongation of vector fields from a fibre bundle to its first jet prolongation is recalled.

2.1. Banach differential manifolds

This section is devoted to recall the main notions about Banach differential manifolds we will use throughout the manuscript, focusing mainly on finite-dimensional differential manifolds and Banach manifolds of mappings. Again, this should not be intended as an exhaustive account on the subject, for which we refer, for instance, to [57]–[60] and references therein. We start recalling the differential calculus on Banach spaces in Sec. 2.1.1. Then we proceed with the notion of smooth Banach manifold in Sec. 2.1.2 and with the definition of a differential calculus on it in Sec. 2.1.3. We end the chapter with Sec. 2.1.4 where some notations about vector fields and differential forms on infinite-dimensional Banach manifolds, that will be useful throughout the manuscript, are fixed.

2.1.1. Differential calculus on Banach spaces

We first recall some basic concepts about Banach spaces that will be necessary for the definition of smooth Banach manifolds.

Definition 2.1.1 (Banach space)**.** *A Banach space,* $(\mathbf{E}, \| \cdot \|_{\mathbf{E}})$ *is a couple made by a (possibly infinite-dimensional) vector space together with a norm* $\| \cdot \|_{\mathbf{E}}$ *for which the vector space is complete. By norm, we mean a map:*

$$\| \cdot \|_{\mathbf{E}} \;\; : \;\; \mathbf{E} \to \mathbb{R} \;\; : \;\; b \mapsto \| \, b \, \|_{\mathbf{E}} \,, \tag{2.1}$$

such that:

- $\| \, b \, \|_{\mathbf{E}} \geq 0 \;\; \forall \, b \in \mathbf{E}$*,*
- $\| \, b \, \|_{\mathbf{E}} \, = \, 0$ *if and only if* $b \, = \, 0$*,*
- $\| \, \alpha \, b \, \|_{\mathbf{E}} \, = \, |\alpha| \| \, b \, \|_{\mathbf{E}} \;\; \forall \, \alpha \in \mathbb{R}$*,*
- $\| \, b_1 + b_2 \, \|_{\mathbf{E}} \leq \| \, b_1 \, \|_{\mathbf{E}} + \| \, b_2 \, \|_{\mathbf{E}} \;\; \forall \, b_1, \, b_2 \in \mathbf{E}$*.*

By complete we mean that any Cauchy sequence converges in $\mathbf{E}$*, that is, for any sequence* $\{ \, e_n \, \}_{n \in \mathbb{N}}$ *such that* $\| \, e_n - e_m \, \|_{\mathbf{E}} \to_{n,m \to \infty} 0$ *there exists an element* $e \in \mathbf{E}$ *such that:*

$$\lim_{n \to \infty} \| \, e_n - e \, \|_{\mathbf{E}} \, = \, 0 \, . \tag{2.2}$$

Definition 2.1.2 (Fréchet derivative)**.** *Consider two Banach spaces* $\mathbf{E}$ *and* $\mathbf{F}$ *and a map* f *from* $\mathbf{E}$ *to* $\mathbf{F}$*. We say that* f *is Frechet differentiable at* $e_0 \in \mathbf{E}$ *if there exists a bounded, linear function, say* $\delta f(e_0)$ *from* $\mathbf{E}$ *to* $\mathbf{F}$ *such that:*

$$\lim_{\| e - e_0 \|_{\mathbf{E}} \to 0} \frac{\| \, f(e) - f(e_0) - \delta f(e_0) \, [e - e_0] \, \|_{\mathbf{F}}}{\| \, e - e_0 \, \|_{\mathbf{E}}} \, = \, 0 \, . \tag{2.3}$$

$\delta f(e_0)$ *is said to be the Fréchet derivative of* f *at* e_0*. The element* e_0 *is sometimes referred to as the direction of the derivative. If* $\delta f(e)$ *exists for all* $e \in \mathbf{E}$*, then* f *is said to be Fréchet differentiable in* $\mathbf{E}$*.*

Remark 2.1.3. *It is worth recalling that a generalization of the Fréchet derivative to functions between locally convex spaces and which is not necessarily linear exists, i.e.,* *the* GATEAUX DERIVATIVE. *However, since throughout the manuscript we are going to work within the context of Banach manifolds, where it coincides with the Fréchet derivative defined above, from now on we will use simply the expression "derivative" and we will always mean "Fréchet derivative".*

Let us now define the concept of second derivative in the following way.

Definition 2.1.4 (SECOND DERIVATIVE). *Consider a differentiable function f between two Banach spaces $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ and $(\mathbf{F}, \|\cdot\|_{\mathbf{F}})$. It is said to be twice differentiable at e_0 if there exists a bounded linear function, say $\delta^2 f(e_0)$, from $\mathbf{E}$ to $\mathbf{L}(\mathbf{E}, \mathbf{F})$ such that:*

$$\lim_{\| e-e_0 \|_{\mathbf{E}} \to 0} \frac{\| \delta f(e) - \delta f(e_0) - \delta^2 f(e_0)\,[e-e_0] \,\|_{\mathbf{L}(\mathbf{E},\mathbf{F})}}{\| e - e_0 \|_{\mathbf{E}}} = 0\,, \tag{2.4}$$

where $\mathbf{L}(\mathbf{E}, \mathbf{F})$ denotes the Banach space of bounded linear functions from $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ to $(\mathbf{F}, \|\cdot\|_{\mathbf{F}})$ equipped with the norm:

$$\| A \|_{\mathbf{L}(\mathbf{E},\mathbf{F})} := \sup_{e\in\mathbf{E}} \frac{\| A[e] \|_{\mathbf{F}}}{\| e \|_{\mathbf{E}}}\,. \tag{2.5}$$

In other words, $\delta f \,:\, \mathbb{E} \to \mathbf{L}(\mathbb{E}, \mathbb{F})$ is differentiable. Moreover, f is said to be twice differentiable in $\mathbf{E}$ if it is twice differentiable at any $e \in \mathbf{E}$.

Definition 2.1.5 ($\mathcal{C}^k$ and SMOOTH). *If a function f between two Banach spaces $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$, $(\mathbf{F}, \|\cdot\|_{\mathbf{F}})$ is differentiable and $\delta f(e)$ is a continuous map with respect to the norm:*

$$\| \delta f(e) \|_{\mathbf{L}(\mathbf{E},\mathbf{F})} := \sup_{\tilde{e}\in\mathbf{E}} \frac{\| \delta f(e)\,[\,\tilde{e}\,] \,\|_{\mathbf{F}}}{\| \tilde{e} \|_{\mathbf{E}}}\,, \tag{2.6}$$

then f is said to be a $\mathcal{C}^1$ function.

If a function f between two Banach spaces $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$, $(\mathbf{F}, \|\cdot\|_{\mathbf{F}})$ is twice differentiable and $\delta^2 f(e)$ is a continuous map with respect to the norm:

$$\| \delta^2 f(e) \|_{\mathbf{L}(\mathbf{E},\mathbf{L}(\mathbf{E},\mathbf{F}))} := \sup_{\tilde{e}\in\mathbf{E}} \frac{\| \delta^2 f(e)\,[\,\tilde{e}\,] \,\|_{\mathbf{L}(\mathbf{E},\mathbf{F})}}{\| \tilde{e} \|_{\mathbf{E}}}\,, \tag{2.7}$$

where $\mathbf{L}(\mathbf{E}, \mathbf{L}(\mathbf{E}, \mathbf{F}))$ denotes the space of bounded linear functions from $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ to $\Big(\mathbf{L}(\mathbf{E}, \mathbf{F}), \|\cdot\|_{\mathbf{L}(\mathbf{E},\mathbf{F})}\Big)$, then f is said to be a $\mathcal{C}^2$ function. Recursively, it is possible to define the concept of $\mathcal{C}^k$ function for any $k \in \mathbb{N}_+$.

In particular if a function is a $\mathcal{C}^k$ function for any $k \in \mathbb{N}_+$, it is said to be a SMOOTH *($\mathcal{C}^\infty$) function.*

Definition 2.1.6 ($\mathcal{C}^k$ and SMOOTH DIFFEOMORPHISMS). *A $\mathcal{C}^k$ (resp. smooth) diffeomorphism between two Banach spaces $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$, $(\mathbf{F}, \|\cdot\|_{\mathbf{F}})$ is a $\mathcal{C}^k$ (resp. smooth) function between $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ and $(\mathbf{F}, \|\cdot\|_{\mathbf{F}})$ admitting an inverse and such that the inverse function is a $\mathcal{C}^k$ (resp. smooth) function itself.*

Note that in the next section we will often need the concept of $\mathcal{C}^k$ or smooth functions (or diffeomorphisms) between open sets of Banach spaces, say $U \subset \mathbf{E}$, $V \subset \mathbf{F}$. The definitions in this case can be given analogously, by replacing $\mathbf{E}$ and $\mathbf{F}$ by U and V in the definitions 2.1.4, 2.1.5, 2.1.6.

2.1.2. Banach smooth manifolds

We are now ready to define the concept of SMOOTH BANACH MANIFOLD.

Definition 2.1.7 (ATLAS). *An* ATLAS $\mathscr{A}$ *over a set* $\mathcal{M}$ *is a collection* $\{U_j, \psi_j\}_{j\in\mathbb{J}}$ *(*$\mathbb{J}$ *denoting an index set) where* $\bigcup_{j\in\mathbb{J}} U_j = \mathcal{M}$ *and* ψ_j *is, for all* $j \in \mathbb{J}$*, a bijection between* U_j *and an open set of a fixed Banach space* $\mathbf{E}$*. Any pair* (U_j, ψ_j) *is called a* CHART *on* $\mathcal{M}$*. The charts of the atlas are also required to be compatible in the sense that for any pair* (U_j, U_k) *such that* $U_j \cap U_k \neq \emptyset$*, the function:*

$$\psi_k \circ \psi_j^{-1} \quad : \quad \psi_j(U_j \cap U_k) \to \psi_k(U_j \cap U_k)\,, \tag{2.8}$$

is a $\mathcal{C}^k$ *diffeomorphism between an open set of* $\mathbf{E}$ *and itself.*

Definition 2.1.8 (DIFFERENTIAL STRUCTURE). *A* $\mathcal{C}^k$ DIFFERENTIAL STRUCTURE*,* $\mathscr{D}$*, over a set* $\mathcal{M}$ *is an equivalence class of* $\mathcal{C}^k$ *atlases on* $\mathcal{M}$ *where two atlases* $\mathscr{A}_1$*,* $\mathscr{A}_2$ *are said to be equivalent if* $\mathscr{A}_1 \cup \mathscr{A}_2$ *is a* $\mathcal{C}^k$ *atlas.*

An atlas and, consequently, a differential structure are said to be SMOOTH if the maps $\psi_k \circ \psi_j^{-1}$ of Def. 2.1.7 are $\mathcal{C}^\infty$.

Definition 2.1.9 (BANACH MANIFOLD). *A* $\mathcal{C}^k$ *(resp.* SMOOTH*)* BANACH MANIFOLD *is a triple* $(\mathcal{M}, \mathcal{A}, \mathbf{E})$ *where:*

- $\mathcal{M}$ *is a set;*
- $\mathscr{D}$ *is a* $\mathcal{C}^k$ *(resp. smooth)* DIFFERENTIAL STRUCTURE *on* $\mathcal{M}$*.*
- $\mathbf{E}$ *is the Banach space where the charts of the atlases of* $\mathscr{D}$ *take values, called the* MODEL SPACE *of the maifold;*

A smooth Banach manifold is often denoted simply by $\mathcal{M}$*, omitting an explicit reference to the model space and the differential structure. When the model space* $\mathbf{E}$ *is a Hilbert space, the manifold is said to be a* HILBERT MANIFOLD*.*

From now on, we will focus on smooth Banach manifolds even if most of the concepts we will introduce can be properly discussed within $\mathcal{C}^k$ Banach manifold for a suitable k.

Along the whole manuscript we will mainly consider two kinds of smooth Banach manifolds:

- finite-dimensional smooth Banach manifolds, namely, smooth Banach manifolds modelled over $\mathbf{E} = \mathbb{R}^n$ for some finite n. In this case the charts of the atlas we will use will be denoted by $(U_j,\, \psi_j)_{j \in \mathbb{J}}$ with:

$$\psi_j \;\;:\;\; \mathcal{M} \supseteq U_j \to \tilde{U}_j \subseteq \mathbb{R}^n \;\;:\;\; m \mapsto \psi_j(m) \,=\, x \,=\, (x^1, ..., x^n)\,. \tag{2.9}$$

- suitable (Banach) completions of spaces of maps between two finite dimensional smooth manifolds, say M of dimension l and N of dimension k, namely, $\mathbf{E} = \mathcal{F}(M;\, N)$. In this case the charts of the atlas we will use will be denoted by $(U_j,\, \psi_j)_{j \in \mathbb{J}}$, with:

$$\psi_j \;\;:\;\; \mathcal{M} \supseteq U_j \to \tilde{U}_j \subseteq \mathcal{F}(M;\, N) \;\;:\;\; \phi \mapsto \psi_j(\phi) \,=\, \phi^a \tag{2.10}$$

  where a is an index running over the dimension of the manifold N.

We will often refer to the chart maps ψ_j as SYSTEM OF LOCAL COORDINATES on $\mathcal{M}$.

Definition 2.1.10 (OPEN). *Given a smooth Banach manifold $\mathcal{M}$, an OPEN SET of $\mathcal{M}$ is a subset $\mathcal{O} \subset \mathcal{M}$ such that for all $m \in \mathcal{O}$ there exists a chart $(U,\, \psi)$ such that $m \in U$ and $U \subset \mathcal{O}$.*

It is possible to prove ([57, Prop. 3.1.6]) that a smooth Banach manifold, equipped with its open sets, is a topological space.

Definition 2.1.11 ($\mathcal{C}^k$ and SMOOTH MAPS BETWEEN BANACH MANIFOLDS). *A map f from the Banach manifold $\mathcal{M}$ (modelled on the Banach space $\mathbf{E}$) to the Banach manifold $\mathcal{N}$ (modelled on the Banach space $\mathbf{F}$) is said to be a $\mathcal{C}^k$ function at $m \in \mathcal{M}$ if, given any chart $(U,\, \psi)$ around $m \in \mathcal{M}$ and any chart $(V,\, \phi)$ around $f(m) \in \mathcal{N}$, the function $\phi \circ f \circ \psi^{-1}$ is a $\mathcal{C}^k$ function from $\psi(U) \subset \mathbb{E}$ to $\phi(V) \subset \mathbf{F}$ in the sense of Banach spaces, following the definitions of the previous section.*

2.1.3. Tangent and cotangent bundles and Cartan's differential calculus

Let us now give the definition of the TANGENT BUNDLE over a smooth Banach manifold. In order to do so, we should first define the concept of TANGENCY OF CURVES over a manifold.

Definition 2.1.12 (OPERATIONAL TANGENT SPACE). *Given a smooth Banach manifold $\mathcal{M}$, the OPERATIONAL TANGENT SPACE at a point m is defined to be the space of operators:*

$$\delta \;\;:\;\; \mathcal{C}^\infty(U) \to \mathbb{R}\,, \tag{2.11}$$

with $U \subset \mathcal{M}$, which are linear and satisfy a Leibniz rule at m, i.e.:

$$\begin{aligned} \delta(f+g) \,&=\, \delta f + \delta g\,, \\ \delta(fg) \,&=\, \delta(f)g + f\delta(g) \end{aligned} \tag{2.12}$$

and, accordingly, any derivation is called an OPERATIONAL TANGENT VECTOR *at* $m \in \mathcal{M}$*. One proves that any derivation at* $m \in U \subset \mathcal{M}$ *can be written, given the chart* (U, ψ) *around* m *and an element* v_m *of the model space, in terms of the Fréchet derivative as follows:*

$$\delta_{v_m} \quad : \quad \mathcal{C}^\infty(U) \to \mathbb{R} \quad : \quad f \mapsto \delta\left(f \circ \psi^{-1}\right)(\psi(m))[v_m]\,, \tag{2.13}$$

where $\delta\left(f \circ \psi^{-1}\right)(\psi(m))[v_m]$ *is the derivative of the function* $f \circ \psi$ *on the Banach model space* $\mathbf{E}$ *at* $\psi(m)$ *along the direction* v_m*.*

Remark 2.1.13. *Another approach to define the tangent space of a smooth Banach manifold is the so called* KINEMATICAL APPROACH *(see [57]), where tangent vectors at a point are defined to be equivalence classes of curves passing through the point being tangent at that point. Such an approach is more often used within the context of finite-dimensional smooth Banach manifolds where the two approaches indeed coincide. Within the more general context of infinite-dimensional Banach manifolds the two approaches do not coincide in general, even if every kinematical tangent vector is also an operational tangent vector (see [61]). The converse is true only if the Banach model space of the Banach manifold is* REFLEXIVE *and satisfies the so called* BORNOLOGICAL APPROXIMATION PROPERTY *(see [62, Theorem 28.7]).*

From now on, we will simply use the expression "tangent vector" referring to "operational tangent vector".

In the case of finite-dimensional smooth manifolds, a tangent vector at some point m, i.e. a derivation at m, can be expressed in the chart (2.9), as:

$$V^j \frac{\partial}{\partial x^j}\left(f \circ \psi^{-1}\right)(x)\bigg|_{x=\psi(m)}\,, \tag{2.14}$$

where V^j are real numbers and Einstein convention over repeated indices is used. Essentially it was used the fact that the derivations along the coordinates x^j are a basis for the derivations at a point and any derivation can be written as a linear combination of them. Usually, a tangent vector is simply denoted by:

$$V = V^j \frac{\partial}{\partial x^j}\Big|_x\,. \tag{2.15}$$

Regarding the second kind of smooth Banach manifold we will consider, namely, those modelled over $\mathcal{F}(M;\, N)$ with chart (2.10), we will explain the notational convention on tangent vectors in Sec. 2.1.4.

Definition 2.1.14 (TANGENT BUNDLE)**.** *The* TANGENT BUNDLE *over a smooth Banach manifold is defined to be the following set:*

$$\mathbf{T}\mathcal{M} := \bigsqcup_{m \in \mathcal{M}} \mathbf{T}_m\mathcal{M}\,. \tag{2.16}$$

The natural projection $(m,\, v_m = \delta(\,\cdot\,)(m)[v_m]) \mapsto m$ *is denoted by* τ *and it is called* TANGENT BUNDLE PROJECTION*.*

To describe the differential structure of $\mathbf{T}\mathcal{M}$ as a smooth Banach manifold, it is necessary to first introduce the concept of TANGENT OF A MAP.

Definition 2.1.15 (TANGENT OF A MAP). *Consider two smooth Banach manifolds, say $\mathcal{M}$ and $\mathcal{N}$, and a $\mathcal{C}^k$ $(k \geq 1)$ function, f, from $\mathcal{M}$ to $\mathcal{N}$. The TANGENT MAP of f, say Tf is the following map:*

$$Tf \quad : \quad \mathbf{T}\mathcal{M} \to \mathbf{T}\mathcal{N} \quad : \quad (m,\, \delta(\,\cdot\,)(m)[v_m]) \mapsto (f(m),\, \delta(\,\cdot \circ f\,)(m)[v_m])\,, \tag{2.17}$$

where the dot $(\,\cdot\,)$ gives the action of the operator δ upon any smooth real valued map on $\mathcal{M}$. The corresponding map from $\mathbf{T}_m\mathcal{M}$ to $\mathbf{T}_{f(m)}\mathcal{N}$ given by the restriction of Tf to $\mathbf{T}_m\mathcal{M}$ is denoted by $T_m f$ or df_m or $f_\star$ and is called PUSH FORWARD of f.

Now, we are ready to claim that $\mathbf{T}\mathcal{M}$ is a smooth Banach manifold modelled over $\mathbf{E} \times \mathbf{E}$ and where the differential structure is made by atlases whose charts are $(\,\tau^{-1}(U_j),\, T\psi_j\,)_{j\in\mathbb{J}}$ (see [57]), where i denotes a canonical map (that exists, as it is proven in [57]) from $(U_j \times \mathbf{E})$ to an open set inside $\mathbf{T}\mathcal{M}$.

With the notion of tangent map at hand, we are able to define the concepts of IMMERSION, SUBMERSION and EMBEDDING of smooth Banach manifolds.

Definition 2.1.16 (IMMERSION). *Given two smooth Banach manifolds, $\mathcal{M}$ and $\mathcal{N}$, and a $\mathcal{C}^k$ (resp. smooth) map from $\mathcal{M}$ to $\mathcal{N}$, say $\mathfrak{i}$, we say that $\mathfrak{i}$ is a $\mathcal{C}^k$ (resp. smooth) IMMERSION if the map $f_\star$ is injective $\forall\, m \in \mathcal{M}$ and its image is closed in $\mathbf{T}_{f(m)}\mathcal{N}$ and admits a complement.*

Definition 2.1.17 (EMBEDDING). *An immersion $\mathfrak{i} \;:\; \mathcal{M} \to \mathcal{N}$ is a (smooth) EMBEDDING if it is a homeomorphism onto its image $\mathfrak{i}(\mathcal{M}) \subseteq \mathcal{N}$ with respect to the subspace topology on $\mathfrak{i}(\mathcal{M})$ induced by the topology on $\mathcal{N}$.*

Definition 2.1.18 (SUBMANIFOLD). *A smooth Banach manifold $\mathcal{N}$ is said to be a (immersed) SUBMANIFOLD of a smooth Banach manifold $\mathcal{M}$, if $\mathcal{N}$ is a subset of $\mathcal{M}$ and if the natural inclusion map $\mathfrak{i} \;:\; \mathcal{N} \to \mathcal{M}$ is a smooth immersion. Moreover, it is said to be an EMBEDDED SUBMANIFOLD if $\mathfrak{i}$ is a smooth embedding.*

The concept of tangent bundle is a particular instance of the more general concept of VECTOR BUNDLE over a smooth Banach manifold.

Definition 2.1.19 (VECTOR BUNDLE STRUCTURE). *Given a set S, a LOCAL BUNDLE CHART on it is defined as a pair $(U,\, \psi)$, with $U \subset S$ and ψ a bijection between U and $U' \times \mathbf{F}$ for some Banach space $\mathbf{F}$. A VECTOR BUNDLE ATLAS is a family $\{\, U_j,\, \psi_j\,\}_{j\in\mathbb{J}}$ of local bundle charts such that $\bigcup_j U_j = S$ and the functions ψ_j are compatible in the same sense of the chart maps of a smooth Banach manifold. Then, a VECTOR BUNDLE STRUCTURE on S is an equivalence class of vector bundle atlases, say $\mathcal{B}$, where two atlases are said to be equivalent if their union[2] is a vector bundle atlas.*

[2]The union of two atlases $\{\, U_j,\, \psi_j\,\}_{j\in\mathbb{J}}$ and $\{\, V_i,\, \phi_i\,\}_{i\in\mathbb{I}}$ reads the atlas $\{\, U_j, V_i,\, \psi_j,\, \phi_i\,\}_{j\in\mathbb{J},\, i\in\mathbb{I}}$.

Definition 2.1.20 (Vector bundle)**.** *A* vector bundle *is a set equipped with a vector bundle structure, say $\mathcal{V} = (S, \mathcal{B})$. Given a vector bundle, its* base manifold *reads:*

$$\mathcal{M} = \left\{ p \in S \ : \ \exists (U, \psi) \in \mathcal{V} \text{ and } u \in U \ : \ p = \psi^{-1}(u, 0) \right\} . \tag{2.18}$$

As it is proven in [57], $\mathcal{M}$ is a submanifold of S and there always exists a smooth surjective submersion π from S to $\mathcal{M}$. Sometimes, a vector bundle is denoted by specifying $\mathcal{V}$, the projection π and the base $\mathcal{M}$, say $(\mathcal{V}, \pi, \mathcal{M})$. Moreover, given a vector bundle, it is possible, for instance, to construct the tangent bundle over it by considering, at each point its tangent space. Furthermore, one could also consider, at each point p, only those tangent vectors V_p such that $\pi_\star V_p = 0$, which are called VERTICAL TANGENT VECTORS. The space of vertical tangent vectors is called the VERTICAL TANGENT SPACE and denoted by $\mathbf{V}_p\pi$. Now, we can define the concept of VERTICAL BUNDLE over a smooth fibre bundle, which will be useful throughout the manuscript.

Definition 2.1.21 (Vertical bundle)**.** *Given a vector bundle $(S, \mathcal{V})$, its* vertical bundle *is:*

$$\mathbf{V}\pi = \bigsqcup_{p \in \mathcal{V}} \mathbf{V}_p \pi , \tag{2.19}$$

where π is the projection of $(S, \mathcal{V})$ onto its base.

The vertical bundle over a fibre bundle is a smooth Banach manifold with respect to the same differential structure that makes the tangent bundle over a smooth Banach manifold, a smooth Banach manifold.

For smooth vector bundles the concept of SUBBUNDLE will be relevant.

Definition 2.1.22 (Subbundle)**.** *A* subbundle *$\mathcal{W}$ of a vector bundle $(S, \mathcal{V})$ is a subset $\mathcal{W} \subset \mathcal{V}$ such that for each $m \in \mathcal{M}$ there exists a chart $(\pi^{-1}(U), \psi)$ of $\mathcal{V}$ around m:*

$$\psi \ : \ \pi^{-1}(U) \to U' \times \mathbf{F} , \tag{2.20}$$

for some $U' \subset \mathbf{E}$, $\mathbf{E}$ being the Banach model space of $\mathcal{M}$, and for some Banach space $\mathbf{F}$, such that there exists a closed subspace $\mathbf{G}$ of $\mathbf{F}$ admitting a closed complement for which:

$$\psi(\pi^{-1}(U) \cap \mathcal{W}) = U' \times (\mathbf{G} \times \{ 0 \}) . \tag{2.21}$$

Definition 2.1.23 (Vector field)**.** *A $\mathcal{C}^k$ (resp. smooth)* vector field *over a smooth Banach manifold $\mathcal{M}$ is a $\mathcal{C}^k$ (smooth resp.) map $\sigma \ : \ \mathcal{M} \to \mathbf{T}\mathcal{M}$ which is the right inverse of the tangent bundle projection, that is $\tau \circ \sigma = \mathbb{1}_{\mathcal{M}}$. A* vector field *over $\mathcal{M}$ can therefore be represented locally as maps:*

$$X \ : \ U_j \ni m \to \mathbf{T}_m \mathcal{M} , \tag{2.22}$$

that on the intersections $U_j \cap U_k$ are suitably glued together and:

$$\sigma \ : \ \mathcal{M} \to \mathbf{T}\mathcal{M} \ : \ m \mapsto (m, X(m)) . \tag{2.23}$$

The space of smooth vector fields over a smooth Banach manifold $\mathcal{M}$ will be denoted by $\mathfrak{X}(\mathcal{M})$ and it can be proved that each element X of $\mathfrak{X}(\mathcal{M})$ gives a derivation of $\mathcal{C}^\infty(\mathcal{M})$ that we will denote by δ_X. The set $\mathfrak{X}(\mathcal{M})$ is a (infinite-dimensional) Lie algebra equipped with the Lie bracket:

$$[\,\cdot\,,\,\cdot\,] \;:\; (X,\,Y) \mapsto [X,\,Y]\,, \tag{2.24}$$

where $[X,\,Y]$ is the vector field on $\mathcal{M}$ that at m agree with the unique tangent vector $[X,\,Y]_m$ associated with the derivation $\delta_X\delta_Y - \delta_Y\delta_X\Big|_m$.

Having said that a vector field gives a derivation of $\mathcal{C}^\infty(\mathcal{M})$ and having in mind the notation (2.15) for tangent vectors in the finite-dimensional case, we will denote, in this case, vector fields locally as:

$$V \;=\; V^j(x)\frac{\partial}{\partial x^j}\,, \tag{2.25}$$

where $V^j(x)$ are functions on $\psi_j(U_j) \subset \mathbb{R}^n$. The corresponding notation in the infinite-dimensional case will be explained in Sec. 2.1.4.

A smooth vector field X over a smooth Banach manifold $\mathcal{M}$ defines a class of curves over $\mathcal{M}$, its INTEGRAL CURVES.

Definition 2.1.24 (INTEGRAL CURVES)**.** *Given a smooth vector field X over a smooth Banach manifold $\mathcal{M}$, an* INTEGRAL CURVE *through $m \in \mathcal{M}$ is defined to be a curve γ through $m \in \mathcal{M}$ satisfying the following ordinary differential equation (ODE, for short):*

$$\frac{d}{ds}\gamma(s) \;=\; X(\gamma(s))\,, \tag{2.26}$$

for all $s \in \mathbb{I} \subset \mathbb{R}$ where $\mathbb{I}$ is the interval of definition of γ, which gives, for each point $\gamma(s)$ in $\mathcal{M}$, a relation in $\mathbf{T}_{\gamma(s)}\mathcal{M}$ between the tangent vector to the curve $\frac{d\gamma}{ds}$ and the image $X(\gamma(s))$.

Given a chart $(U,\,\psi)$ around $m \in \mathcal{M}$, the ODE over $\mathcal{M}$ defining the integral curves of X can be rephrased as an ODE over the model Banach space of $\mathcal{M}$, say $\mathbf{E}$:

$$\frac{d}{ds}\tilde{\gamma}(s) \;=\; \tilde{X}(\tilde{\gamma}(s))\,, \tag{2.27}$$

where $\tilde{\gamma}(s) := \psi\,[\,\gamma(s)\,]$ and $\tilde{X}(e)$ is defined by the tangent map of ψ as follows:

$$T\psi \;:\; \mathbf{T}\mathcal{M} \supset \mathbf{T}U \to \mathbf{E}\times\mathbf{E} \;:\; (m,\,v_m \;=\; X(m)) \mapsto \Big(\psi(m) \;=\; e,\,[\,T_m\psi(\,v_m\,)\,]_e \;=:\; \tilde{X}(e)\Big)\,, \tag{2.28}$$

where e is a point of $\mathbf{E}$. Eq. (2.27) has a unique solution for any fixed initial condition $\tilde{\gamma}(0) \;=\; e_0$ since $\tilde{X}(e)$ is a smooth (and, thus, Lipschitz) function on $\mathbf{E}$ by virtue of the PICARD THEOREM. We will always keep this assumption for any vector field we will consider. Consequently, Eq. (2.26) has a unique solution for any fixed initial condition $\gamma(0) \;=\; m_0$. We will denote it by $\gamma(s;\,m_0)$ where $s \in \mathbb{I} \subseteq \mathbb{R}$ is the evolution parameter of the curve γ in a suitable defining domain $\mathbb{I}$.

Definition 2.1.25 (Flow of a vector field). *Given a smooth vector field X over a smooth Banach manifold $\mathcal{M}$, its* flow *is the map:*

$$F^X \; : \; \mathcal{M} \times \mathbb{R} \supset \mathcal{O} \to \mathcal{M} \; : \; (m_0,\, s) \mapsto F^X_s(m_0) \,:=\, \gamma(s;\, m_0)\,, \tag{2.29}$$

where $\mathcal{O}$ is the so-called flow domain *of X.*

Definition 2.1.26 (Distribution). *A* distribution *over a smooth Banach manifold $\mathcal{M}$ is a collection of subspaces of $\mathbf{T}_m\mathcal{M}$ for all $m \in \mathcal{M}$, say $\mathcal{D}_m \subset \mathbf{T}_m\mathcal{M}$.*

Definition 2.1.27 (Regular distribution). *A* regular distribution *over a smooth Banach manifold $\mathcal{M}$ is a subbundle of $\mathbf{T}\mathcal{M}$.*

The concepts of involutive and integrable distributions, related by Frobenius' theorem, are relevant.

Definition 2.1.28 (Involutive distribution). *A regular distribution $\mathcal{D}$ over a smooth Banach manifold $\mathcal{M}$ is said to be* involutive *if for any couple of vector fields X, Y taking value, at each point, in $\mathcal{D}$, then $[X,\, Y]$ takes value, at each point, in $\mathcal{D}$.*

Definition 2.1.29 (Integrable distribution). *A regular distribution $\mathcal{D}$ over a smooth Banach manifold $\mathcal{M}$ is said to be* integrable *if at each $m \in \mathcal{M}$ there exists a submanifold $\mathcal{N} \subset \mathcal{M}$ with $m \in \mathcal{N}$ such that $\mathbf{T}\mathcal{N}$ coincides with $\mathcal{D}$ restricted to $\mathcal{N}$.*

Theorem 2.1.30 (Frobenius' theorem). *A distribution $\mathcal{D}$ over a smooth Banach manifold $\mathcal{M}$ is integrable iff it is involutive.*

Let us now pass to the definition of the cotangent bundle over a smooth Banach manifold and to the definition of a Cartan's differential calculus.

Definition 2.1.31 (Cotangent bundle). *The* cotangent bundle *over a smooth Banach manifold is defined to be the following set:*

$$\mathbf{T}^\star\mathcal{M} \,:=\, \bigsqcup_{m\in\mathcal{M}} \mathbf{T}^\star_m\mathcal{M}\,, \tag{2.30}$$

where $\mathbf{T}^\star_m\mathcal{M}$ is the dual to the vector space $\mathbf{T}_m\mathcal{M}$, i.e. the space of continuous linear functions on $\mathbf{T}_m\mathcal{M}$, which is $\mathbf{L}(\mathbf{E},\, \mathbb{R})$. Here, we will denote by $\rho \; : \; \mathbf{T}^\star\mathcal{M} \to \mathcal{M}$ the projection onto the first factor.

To describe the differential structure of the cotangent bundle, let us stress that, in general, it is possible to define more general tensor bundles by considering the set made by disjoint unions of the space of (n,k)-tensors on $\mathcal{M}$, say the space ${\mathbf{T}^{(n,k)}}_m\mathcal{M}$ of smooth multilinear maps from the cartesian product of n copies of $\mathbf{E}^\star$ and k copies of $\mathbf{E}$ to $\mathbb{R}$, and equipping it with a suitable differential structure being also a vector bundle structure. The bundles obtained in this way are denoted by $\mathbf{T}^{(n,k)}\mathcal{M}$. Note that $\mathbf{T}^{(1,0)}\mathcal{M} = \mathbf{T}\mathcal{M}$ and $\mathbf{T}^{(0,1)}\mathcal{M} = \mathbf{T}^\star\mathcal{M}$.

Definition 2.1.32 (PULL-BACK OF FUNCTIONS). *Given a real valued $\mathcal{C}^k$ $(k \geq 1)$ function $\mathscr{F}$ on a smooth Banach manifold $\mathcal{N}$ and a $\mathcal{C}^k$ $(k \geq 1)$ map from another Banach manifold $\mathcal{M}$ to $\mathcal{N}$, say f, the* PULL-BACK *of $\mathscr{F}$ via f is the real valued $\mathcal{C}^k$ $(k \geq 1)$ function on $\mathcal{N}$ defined by:*

$$f^\star \mathscr{F} \,=\, \mathscr{F} \circ f\,. \tag{2.31}$$

The notion of pull-back can be extended to elements of ${\mathbf{T}^{(n,k)}}_m\mathcal{M}$.

Definition 2.1.33 (PULL-BACK OF TENSORS). *Given an element of ${\mathbf{T}^{(n,k)}}_n\mathcal{N}$ for some smooth Banach manifold $\mathcal{N}$, say T, and a $\mathcal{C}^l$ $(l \geq 1)$ map f from another smooth Banach manifold $\mathcal{M}$ to $\mathcal{N}$, the* PULL-BACK *of T via f is the element of ${\mathbf{T}^{(n,k)}}_m\mathcal{M}$ (with $f(m) = n$) given by:*

$$f^\star T_n(\alpha_1,\, ...,\, \alpha_n,\, v_1,\, ...,\, v_k) \,=\, T_m(f^{-1\star}\alpha_1,\, ...,\, f^{-1\star}\alpha_n,\, f_\star v_1,\, ...,\, f_\star v_k)\,. \tag{2.32}$$

Now, we are ready to claim that $\mathbf{T}^\star\mathcal{M}$ is a smooth Banach manifold modelled over $\mathbf{E} \times \mathbf{E}^\star$ and where the differential structure is made by atlases whose charts are $(\rho^{-1}(U_j),\, \psi_j \times \psi_j^{-1\star})_{j\in\mathbb{J},\, m\in U_j}$ where $\psi^{-1\star}$ is the pull-back map on tensors defined above (see [57]). It is also a vector bundle according to Def. 2.1.20. In this case the base is $\mathcal{M}$ and the projection is the projection onto the first factor, that we denote by ρ.

Definition 2.1.34 (DIFFERENTIAL 1-FORMS). *A $\mathcal{C}^k$ (resp. smooth) 1-*FORM *over a smooth Banach manifold $\mathcal{M}$ is a $\mathcal{C}^k$ (resp. smooth) section of $\mathbf{T}^\star\mathcal{M}$, i.e. a point by point $\mathcal{C}^k$ (resp. smooth) inverse of the cotangent bundle projection ρ, i.e. map σ such that $\rho \circ \sigma \,=\, \mathbb{1}_\mathcal{M}$. A 1-*FORM *over $\mathcal{M}$ can be locally written in terms of maps:*

$$\alpha \;\;:\;\; m \in U_j \to \mathbf{T}^\star_m\mathcal{M}\,, \tag{2.33}$$

such that on the intersections $U_j \cap U_k$ are suitably glued together and:

$$\sigma \;\;:\;\; \mathcal{M} \to \mathbf{T}\mathcal{M} \;\;:\;\; m \mapsto (m,\, \alpha(m))\,. \tag{2.34}$$

Having in mind the expression (2.25) for a vector field over $\mathcal{M}$ in the finite-dimensional case, a 1-form, i.e. a linear functional over $\mathfrak{X}(\mathcal{M})$ will be locally denoted (when acting on vector fields), in the finite-dimensional case, as:

$$\alpha(V) \,=\, \alpha_j(x)V^j(x)\,, \tag{2.35}$$

or simply as:

$$\alpha \,=\, \alpha_j(x)\mathrm{d}x^j\,, \tag{2.36}$$

where $\mathrm{d}x^j$ is the dual basis[3] to the basis of derivations $\left\{ \frac{\partial}{\partial x^j} \right\}_{j=1,\dots,n}$. The corresponding notation in the infinite-dimensional case will be explained in Sec. 2.1.4.

[3]In the sense of vector spaces.

As for $\mathbf{T}^{\star}\mathcal{M}$, it is possible to show that $\mathbf{T}^{(0,n)}\mathcal{M}$ is a smooth Banach manifold modelled over $\mathbf{E} \times \mathbf{E}^{\star} \times ... \times \mathbf{E}^{\star}$ (n copies of $\mathbf{E}^{\star}$) and where the differential structure is made by atlases whose charts are $(\rho^{n-1}(U_j),\, \psi_j \times \psi_j^{-1\star})_{j\in\mathbb{J},\, m\in U_j}$ where $\psi_j^{-1\star}$ is the pull-back map on tensors defined above and $\rho^n \;:\; \mathbf{T}^{(0,n)}\mathcal{M} \to \mathcal{M}$ is the projection onto the first factor. It is also a vector bundle according to Def. 2.1.20. In this case the base is $\mathcal{M}$ and the projection is ρ^n.

Definition 2.1.35 ($(0,n)$-TENSOR FIELDS)**.** *A $\mathcal{C}^k$ (resp. smooth) $(0,n)$-TENSOR FIELD (or a n-COVECTOR FIELD) over a smooth Banach manifold $\mathcal{M}$ is a $\mathcal{C}^k$ (resp. smooth) section of $\mathbf{T}^{(0,n)}\mathcal{M}$, i.e. $\mathcal{C}^k$ (resp. smooth) right inverse of the bundle projection ρ^n. Locally a n-COVECTOR FIELD over $\mathcal{M}$ can be written as a $\mathcal{C}^k$ (resp. smooth) association of an element of $\mathbf{T}^{(0,n)}{}_m\mathcal{M}$ to each point of $\mathcal{M}$:*

$$\alpha \;:\; \mathcal{M} \to \mathbf{T}^{(0,n)}\mathcal{M} \;:\; m \mapsto (m,\, \alpha(m))\,. \tag{2.37}$$

n-covector fields being skew-symmetric, that is, satisfying:

$$\alpha(X_1, ..., X_n) \;=\; -\alpha(X_{\sigma(1)}, ..., X_{\sigma(n)})\,, \tag{2.38}$$

for any odd permutation σ of the indices $(1, ..., n)$ and where $X_j \in \mathfrak{X}(\mathcal{M}) \;\; \forall\, j$, is called a n-FORM. The space of n-forms over a smooth Banach manifold $\mathcal{M}$ is denoted by $\Omega^n(\mathcal{M})$.

As for $\mathbf{T}^{(0,n)}\mathcal{M}$, it is possible to show that $\mathbf{T}^{(n,0)}\mathcal{M}$ is a smooth Banach manifold modelled over $\mathbf{E} \times \mathbf{E} \times ... \times \mathbf{E}$ (n copies of $\mathbf{E}$) and where the differential structure is made by atlases whose charts are $(U_j \times \mathbf{T}^{(n,0)}{}_m\mathcal{M},\, \psi_j \times \prod_n T\psi_j)_{j\in\mathbb{J},\, m\in U_j}$. It is also a vector bundle according to Def. 2.1.20. In this case the base is $\mathcal{M}$ and the projection is the projection onto the first factor that we denote by τ^n.

Definition 2.1.36 ($(n,0)$-TENSOR FIELDS)**.** *A $\mathcal{C}^k$ (resp. smooth) $(n,0)$-TENSOR FIELD over a smooth Banach manifold $\mathcal{M}$ is a $\mathcal{C}^k$ (resp. smooth) section of $\mathbf{T}^{(n,0)}\mathcal{M}$, i.e. a $\mathcal{C}^k$ (resp. smooth) right inverse of the bundle projection τ^n. A $(n,0)$-VECTOR FIELD over $\mathcal{M}$ can be locally written as a $\mathcal{C}^k$ (resp. smooth) association of an element of $\mathbf{T}^{(n,0)}{}_m\mathcal{M}$ to each point of $\mathcal{M}$:*

$$W \;:\; \mathcal{M} \to \mathbf{T}^{(n,0)}\mathcal{M} \;:\; m \mapsto (m,\, W(m))\,. \tag{2.39}$$

$(n,0)$-tensor fields being skew-symmetric, that is, satisfying:

$$\alpha_1 \otimes ... \otimes \alpha_n(W) \;=\; -\alpha_{\sigma(1)} \otimes ... \otimes \alpha_{\sigma(n)}(W)\,, \tag{2.40}$$

for any odd permutation σ of the indices $(1, ..., n)$ and where $\alpha_j \in \Omega^1(\mathcal{M}) \;\; \forall\, j$, are called a n-VECTOR FIELDS. The space of n-vector fields over a smooth Banach manifold $\mathcal{M}$ is denoted by $\Lambda^n(\mathcal{M})$.

It is possible to show that $\mathbf{T}^{(n,k)}\mathcal{M}$ is a smooth Banach manifold modelled over $\mathbf{E}\times\mathbf{E}\times...\times\mathbf{E}\times\mathbf{E}^{\star}\times...\times\mathbf{E}^{\star}$ (n copies of $\mathbf{E}$ and k copies of $\mathbf{E}^{\star}$) and where the differential structure is made by atlases whose charts are $(U_j \times \mathbf{T}^{(n,k)}{}_m\mathcal{M},\, \psi_j \times \psi^{-1\star})_{j\in\mathbb{J},\, m\in U_j}$ where $\psi^{-1\star}$ is the pull-back of tensors defined in Def. 2.1.33. It is also a vector bundle according to Def. 2.1.20. In this case the base is $\mathcal{M}$ and the projection is, again, the projection onto the first factor.

Given a n-form and a l-form, there is a way to canonically construct a $(n+k)$-form, the so called WEDGE PRODUCT, $\wedge$.

Definition 2.1.37 (WEDGE PRODUCT). *Consider a n-form α and a l-form β over a smooth Banach manifold $\mathcal{M}$. Their* WEDGE PRODUCT *is the $(n+l)$-form defined by:*

$$(\alpha \wedge \beta)(X_1, ..., X_{n+l}) \;=\; \frac{1}{n!l!}\sum_{\sigma} \mathrm{sign}(\sigma)\alpha(X_{\sigma(1)}, ..., X_{\sigma(n)})\, \beta(X_{\sigma(n+1)}, ..., X_{\sigma(n+l)})\,, \tag{2.41}$$

where σ is any permutation of the indices $(1, ..., n+l)$.

Definition 2.1.38 (EXTERIOR DERIVATIVE). *The* EXTERIOR DERIVATIVE, *say* d, *of a n-form α, is a map from $\Omega^n(\mathcal{M})$ to $\Omega^{n+1}(\mathcal{M})$ such that:*

$$\mathrm{d}\alpha(X_0,\, ...,\, X_n) \;=\; \sum_{j=0}^{n}(-1)^j\alpha(X_0,\, ...,\, \hat{X}_j,\, ...,\, X_n) + \sum_{j<k}\alpha([X_j,\, X_k],\, X_1,\, ...,\, \hat{X}_j,\, ...,\, \hat{X}_k,\, ...,\, X_n)\,. \tag{2.42}$$

Definition 2.1.39 (CONTRACTION). *For a fixed $X \in \mathfrak{X}(\mathcal{M})$ and a n-form α, the* CONTRACTION *of α along X, say $i_X\alpha$, is a map from $\Omega^n(\mathcal{M})$ to $\Omega^{n-1}(\mathcal{M})$ such that:*

$$i_X\alpha(X_1, ..., X_n) \;=\; \alpha(X,\, X_1,\, ...,\, X_n)\,. \tag{2.43}$$

Definition 2.1.40 (LIE DERIVATIVE OF n-FORMS). *For a fixed $X \in \mathfrak{X}(\mathcal{M})$ and a n-form α over a smooth Banach manifold $\mathcal{M}$, the* LIE DERIVATIVE, *say $\mathcal{L}$, of α along X is the n-form:*

$$\mathcal{L}_X\alpha \;=\; \frac{d}{ds}\left(F_s^{X\star}\alpha\right)_{t=0}\,. \tag{2.44}$$

It is possible to prove (see [58, Sec. 7.6]) the following useful properties of the pull-back, the exterior derivative, the interior product and the Lie derivative:

- $\mathcal{L}_X \;=\; i_X \circ \mathrm{d} + \mathrm{d} \circ i_X \;\; \forall\, X \in \mathfrak{X}(\mathcal{M})$,
- $\mathrm{d} \circ \mathrm{d} \;=\; 0$,
- $f^\star \circ \mathrm{d} \;=\; \mathrm{d} \circ f^\star$ for all smooth functions f from a smooth Banach manifold $\mathcal{M}$ to a smooth Banach manifold $\mathcal{N}$,
- $\mathcal{L}_X \circ \mathrm{d} \;=\; \mathrm{d} \circ \mathcal{L}_X \;\; \forall\, X \in \mathfrak{X}(\mathcal{M})$.

The condition $\mathrm{d} \circ \mathrm{d} \;=\; 0$ gives the DE RHAM COMPLEX:

$$\mathcal{C}^\infty(\mathcal{M}) \;=:\; \Omega^0(\mathcal{M}) \xrightarrow{\;\mathrm{d}\;} \Omega^1(\mathcal{M}) \xrightarrow{\;\mathrm{d}\;} ... \tag{2.45}$$

Elements α for which $\mathrm{d}\alpha \;=\; 0$ are called CLOSED, elements of the type $\alpha \;=\; \mathrm{d}\beta$ are called EXACT.

2.1.4. Notational conventions for the differential calculus on Banach manifolds modelled over spaces of maps

As we said in the previous section, along the manuscript we will only consider finite-dimensional smooth manifolds or spaces of maps between two finite-dimensional smooth manifolds. In particular we will always consider cases in which the "range manifold" N is a fibre bundle over the "domain manifold" M, i.e., our smooth Banach manifold $\mathcal{M}$ will be a suitable completion (as we will see in more detail) of the space of sections of $\pi \;:\; N \to M$. Consider a system of local coordinates denoted by $\{\, x^\mu \,\}_{\mu=1,\dots,\mathrm{dim}M}$ on M and a system of fibered local coordinates denoted by $\{\, x^\mu,\, y^A \,\}_{\mu=1,\dots,\mathrm{dim}M;\, A=1,\dots,\mathrm{dim}N}$ on N. By using the characterization of the tangent space to a space of sections given in [62, Theorem 42.20], it is possible to see that the tangent space to $\mathcal{M}$ at ϕ is isomorphic to the space of π-vertical vector fields on the finite-dimensional manifold N defined along the image of ϕ in N. Denote a tangent vector to $\mathcal{M}$ at ϕ as $\mathbb{X}_\phi$ and denote by X an extension of $\mathbb{X}_\phi$ to a vector field on the finite-dimensional manifold N defined on a neighborhood of the image of ϕ in N. The flow of X, say F_s^X defines a curve on $\mathcal{M}$, say $\phi_s \,=\, F_s^X \circ \phi$. This is a consequence of the fact that, since X is π-vertical, then:

$$\pi \circ \phi_s \,=\, \pi \circ \underbrace{F_s^X \circ \phi}_{\mathbb{1}_M \circ \phi} \,=\, \pi \circ \phi \,=\, \mathbb{1}_M\,. \tag{2.46}$$

We will always assume M to be equipped with a volume form, vol_M. Consider on $\mathcal{M}$ a function of the type:

$$\mathscr{F} \;:\; \mathcal{M} \supset \psi(U) \to \mathbb{R} \;:\; \left(\mathscr{F} \circ \psi^{-1}\right)(\phi) \,=\, \int_M \phi^\star \left[\, F(x,\, y) vol_M \,\right] \,=\, \int_M F(x,\, \phi(x)) vol_M\,, \tag{2.47}$$

where ψ is the chart map (2.10) and $F \;:\; N \to \mathbb{R}$ is an integrable function with respect to vol_M. With a slight abuse of notation we will denote $\mathscr{F} \circ \psi^{-1}$ simply by $\mathscr{F}$. The differential of $\mathscr{F}$ along the direction of $\mathbb{X}_\phi$ is defined to be:

$$\delta_{\mathbb{X}_\phi} \mathscr{F}_\phi \,=\, \frac{d}{ds} \int_M {\phi_s}^\star \left[\, F(x,\, y) vol_M \,\right]\Big|_{s=0}\,, \tag{2.48}$$

which, as it will be proved in Sec. 3.1.1, amounts to:

$$\delta_{\mathbb{X}_\phi} \mathscr{F}_\phi \,=\, \int_M \phi^\star \left[\, \mathcal{L}_X\, F(x,\, y) vol_M \,\right] \,=\, \int_M \phi^\star \left[\, X^A \frac{\partial F}{\partial y^A} vol_M \,\right], \tag{2.49}$$

where the last equality follows from the fact that X is π-vertical and where following the notation (2.25):

$$X \,=\, X^A \frac{\partial}{\partial y^A}\,, \tag{2.50}$$

X_A being a function on N. Computing the pull-back via ϕ, one gets:

$$\delta_{\mathbb{X}_\phi} \mathscr{F}_\phi \,=\, \int_M X^A(x,\, \phi(x)) \frac{\partial F}{\partial y^A}\Big|_{\phi(x)} vol_M \,=:\, \int_M \mathbb{X}_\phi^A \frac{\delta F}{\delta \phi^A}\, vol_M\,. \tag{2.51}$$

Therefore, by looking at the latter equation, and in analogy with the notation (2.15), we will denote the derivation $\mathbb{X}_\phi$ (acting on the algebra $\mathcal{C}^\infty(\mathcal{M})$) as:

$$\mathbb{X}_\phi \,=\, \mathbb{X}^A_\phi \frac{\delta}{\delta \phi^A}\,, \tag{2.52}$$

where an integration over M is implicitly assumed. Consequently, in analogy to the notation (2.25) we will denote vector fields on $\mathcal{M}$ by:

$$\mathbb{X} \,=\, \mathbb{X}^A \frac{\delta}{\delta \phi^A}\,, \tag{2.53}$$

where $\mathbb{X}^A$ is a function on $\mathcal{M}$.

On the other hand, we will always consider covectors on $\mathcal{M}$ at ϕ, of the type:

$$\alpha_\phi(\mathbb{X}_\phi) \,=\, \int_M \phi^\star \left[\, i_X \bar{\alpha} \,\right] \tag{2.54}$$

where X is an extension of $\mathbb{X}_\phi$ of the type considered above and $\bar{\alpha}$ is what some authors call a semibasic $(l+1)$-form on N (l being the dimension of M), i.e. a differential form on N which vanishes when contracted along two π-vertical vector fields on N. It has the form:

$$\bar{\alpha} \,=\, \bar{\alpha}_A \mathrm{d}y^A \wedge vol_M\,. \tag{2.55}$$

Consequently α_ϕ reads:

$$\alpha_\phi(\mathbb{X}_\phi) \,=\, \int_M \phi^\star \left[\, \bar{\alpha}_A X^A vol_M \,\right] \,=\, \int_M \bar{\alpha}_A(\phi(x)) \mathbb{X}^A \, vol_M\,. \tag{2.56}$$

Therefore, by looking at the latter equation and in analogy with the notation (2.52) we will denote the covector α_ϕ simply by:

$$\alpha_\phi \,=\, \alpha_{A\phi} \delta\phi^A\,, \tag{2.57}$$

where $\left\{\, \delta\phi^A \,\right\}_{A=1,\dots,\mathrm{dim}N}$ denotes a basis of the dual space of $\mathbf{T}_\phi\mathcal{M}$ and a differential 1-form $\alpha \in \Omega^1(\mathcal{M})$ by:

$$\alpha \,=\, \alpha_A \delta\phi^A\,, \tag{2.58}$$

where α_A is a function on $\mathcal{M}$.

Regarding the 2-forms, let us consider a 2-form being the exterior derivative of 1-forms of the type just described. Using the definition of exterior derivative 2.1.38, it is a matter of direct computation to prove that:

$$\omega_\phi(\mathbb{X}_\phi,\, \mathbb{Y}_\phi) \,=\, \mathrm{d}\alpha_\phi(\mathbb{X}_\phi,\, \mathbb{Y}_\phi) \,=\, \int_M \phi^\star \left[\, i_X i_Y \mathrm{d}\bar{\alpha} \,\right]\,, \tag{2.59}$$

where, again, X and Y are extensions of $\mathbb{X}_\phi$ and $\mathbb{Y}_\phi$ to vertical vector fields on N defined in a neighborhood of the image of ϕ. Being X and Y π-vertical, the following equalities can be proved:

$$\mathrm{d}\alpha_\phi(\mathbb{X}_\phi,\, \mathbb{Y}_\phi) \,=\, \int_M \phi^\star \left[\frac{\partial \bar{\alpha}_A}{\partial y^B} X^{[B} X^{A]} \, vol_M \right] \,=\, \int_M \left.\frac{\partial \bar{\alpha}_A}{\partial y^B}\right|_{\phi(x)} \mathbb{X}^{[B}_\phi \mathbb{Y}^{A]}_\phi \, vol_M \,=:\, \int_M \frac{\delta \bar{\alpha}_A}{\delta \phi^B} \mathbb{X}^{[B}_\phi \mathbb{Y}^{A]}_\phi \, vol_M\,. \tag{2.60}$$

Thus, the notation analogous to (2.57) we will use for skew-symmetric $(0,2)$-tensors at ϕ will be:

$$\omega_\phi \,=\, \omega_{AB}[\phi]\delta\phi^A \wedge \delta\phi^B \,, \tag{2.61}$$

whereas, for 2-forms we will use the following:

$$\omega \,=\, \omega_{AB}\delta\phi^A \wedge \delta\phi^B \,. \tag{2.62}$$

2.2. Symplectic Banach manifolds and dynamical systems

This section is devoted to recall the notion of symplectic manifold within the, possibly infinite-dimensional, Banach setting. After giving, in Sec. 2.2.1, the basic definitions about symplectic manifolds, we proceed by dealing with the notion of Hamiltonian system in Sec. 2.2.2 and, in Sec. 2.2.3, with the relation between symmetries and conserved quantities within this setting.

Again, we refer to the huge existing literature on the subject for a more exhaustive account (see, for instance [63]–[66] and references therein).

2.2.1. Symplectic Banach manifolds

Definition 2.2.1 (Weakly symplectic Banach manifold)**.** *A* weakly symplectic *Banach manifold is a couple* $(\mathcal{M},\, \omega)$ *where* $\mathcal{M}$ *is a smooth Banach manifold and* ω *is a closed* 2*-form on* $\mathcal{M}$ *such that the set:*

$$K_m \,=\, \{\, V_m \in \mathbf{T}_m\mathcal{M} \;\;:\;\; \omega(V_m,\, W_m) \,=\, 0 \;\; \forall\, W_m \in \mathbf{T}_m\mathcal{M} \,\} \,, \tag{2.63}$$

i.e. the kernel *of* ω *at the point* m*, is trivial* $\forall\; m \in \mathcal{M}$*.*

Definition 2.2.2 (Strongly symplectic Banach manifold)**.** *A* strongly symplectic *Banach manifold is a couple* $(\mathcal{M},\, \omega)$ *where* $\mathcal{M}$ *is a smooth Banach manifold and* ω *is a closed* 2*-form along* $\mathcal{M}$ *such that the so called* musical map*:*

$$\flat \;\;:\;\; \mathbf{T}_m\mathcal{M} \to \mathbf{T}^\star_m\mathcal{M} \;\;:\;\; V_m \mapsto \omega_m(V_m,\, \cdot\,) \,=\, i_{V_m}\omega_m \,, \tag{2.64}$$

is an isomorphism $\forall\; m \in \mathcal{M}$*.*

Remark 2.2.3. *Note that when* $\mathcal{M}$ *is a Hilbert manifold (or, a fortiori, a smooth finite-dimensional manifold) the two notions of weakly and strongly symplectic manifold coincide since in that case the tangent space at a point is a Hilbert space which is canonically isomorphic to its dual. Therefore, any bilinear map with empty kernel induces an isomorphism between* $\mathbf{T}_m\mathcal{M}$ *and* $\mathbf{T}^\star_m\mathcal{M}$*.*

Example 2.2.4 (Weakly but not strongly symplectic manifold)**.** *Consider the space* $\ell_1 \times \ell_\infty$*. It is a Banach manifold which is actually a linear Banach space equipped with the norm:*

$$\|\,(q,\, p)\,\| \,=\, \|\,q\,\|_1 + \|\,p\,\|_\infty \tag{2.65}$$

where $q = (q_1, q_2, ..., q_k, ...)$ *denotes an element of* ℓ_1*, i.e. the space of absolutely convergent sequences,* $p = (p_1, p_2, ..., p_k, ...)$ *denotes an element of* ℓ_∞*, i.e. the space of bounded sequences,* (q, p) *denotes an element of* $\ell_1 \times \ell_\infty$ *and:*

$$\| q \|_1 = \sum_{k=1}^{\infty} | q_k | , \tag{2.66}$$

$$\| p \|_\infty = \sup_{k=1,\ldots,\infty} |p_k| . \tag{2.67}$$

We consider the system of (global) coordinates $\{ q_1, q_2, ..., p_1, p_2, ... \}$ *associated to the standard Schauder basis of the vector space. Being a vector space, the tangent space of* $\ell_1 \times \ell_\infty$ *is* $\ell_1 \times \ell_\infty$ *itself. We will denote tangent vectors to* $\ell_1 \times \ell_\infty$ *at* (q, p) *as* $V = (V_q, V_p)$ *where* $V_q \in \ell_1$ *and* $V_p \in \ell_\infty$*. In the chosen coordinate system,* $V = (V_q, V_p)$ *explicitly reads:*

$$V = (V_q, V_p) = V_{q\,k} \frac{\partial}{\partial q_k} + V_{p\,k} \frac{\partial}{\partial p_k} . \tag{2.68}$$

The set $\ell_1 \times \ell_\infty$ *is equipped with the following closed* 2*-form:*

$$\omega(V, W) = \langle W_p, V_q \rangle - \langle W_q, V_p \rangle , \tag{2.69}$$

where $\langle \cdot , \cdot \rangle$ *denotes the action of an element of* $\ell_\infty = \ell_1{}^\star$ *over* ℓ_1 *as a linear functional. It is clear that* ω *is weakly symplectic since* $\omega(V, W) = 0 \;\forall\, W$ *implies* $V = 0$*. However it is not strongly symplectic since the image of* $\flat$ *is* $\ell_\infty \times \ell_1$ *while* $\mathbf{T}^\star_{(q,p)}(\ell_1 \times \ell_\infty)$ *is* $\ell_1{}^\star \times \ell_\infty{}^\star = \ell_\infty \times \ell_\infty{}^\star \supset \ell_\infty \times \ell_1$*.*

Remark 2.2.5. *Note that analogously one can construct examples of weakly symplectic manifolds by considering* $\mathcal{M} = \mathrm{B} \times \mathrm{B}^\star$ *with* B *a non-reflexive Banach space.*

Remark 2.2.6. *Note that in the same way one can construct examples of strongly symplectic manifolds by considering* $\mathcal{M} = \mathrm{B} \times \mathrm{B}^\star$ *with* B *a reflexive Banach space.*

2.2.2. Hamiltonian systems

Definition 2.2.7 (Hamiltonian system)**.** *A* Hamiltonian system *is a triple* $(\mathcal{M}, \omega, H)$ *made by a strongly symplectic smooth manifold* $(\mathcal{M}, \omega)$ *and a smooth function* H *on* $\mathcal{M}$ *called the* Hamiltonian *of the system.*

Being $(\mathcal{M}, \omega)$ strongly symplectic, the Hamiltonian H uniquely defines a vector field, the Hamiltonian vector field, say X_H, associated to H with respect to ω, satisfying:

$$i_{X_H}\omega = \mathrm{d}H . \tag{2.70}$$

When one describes a dynamical system as a Hamiltonian system, the Hamiltonian vector field models the dynamics of the system in the sense that its integral curves represent the trajectories of the dynamical system over the manifold $\mathcal{M}$, which is often called the PHASE SPACE of the system. Note that such curves do not necessarily represent the physical trajectories of the system. Indeed, for instance, often $\mathcal{M}$ is the cotangent bundle of the space of configurations of the dynamical system, i.e. $\mathbf{T}^\star\mathcal{Q}$ and, therefore, physical trajectories are represented by the projection to $\mathcal{Q}$ of the trajectories on the phase space. Examples of physical systems modelled as Hamiltonian system are addressed along the manuscript (see, examples 3.1.7, 3.1.14).

The problem in defining a Hamiltonian system on a weakly symplectic manifold is that in that case Hamiltonian vector fields are not uniquely defined in the sense that equation (2.70) may be not well posed since, being $\flat$ not an isomorphism, $\mathrm{d}H$ may lie outside its image. Therefore, an obvious generalization of the notion of Hamiltonian system to the setting of weakly symplectic manifolds requires one to restrict to those Hamiltonian functions such that $\mathrm{d}H$ lies in the image of $\flat$.

Definition 2.2.8 ($\mathcal{C}^\infty_\omega(\mathcal{M})$)**.** *Given a weakly symplectic smooth manifold* $(\mathcal{M},\,\omega)$*, we denote by* $\mathcal{C}^\infty_\omega(\mathcal{M})$ *the algebra of smooth functions on* $\mathcal{M}$ *such that their differential lies in* $\flat(\mathbf{T}_m\mathcal{M})\ \forall\ m\in\mathcal{M}$*.*

Definition 2.2.9 (WEAK HAMILTONIAN SYSTEM)**.** *A* WEAK HAMILTONIAN SYSTEM *is a triple* $(\mathcal{M},\,\omega,\,H)$ *where* $(\mathcal{M},\,\omega)$ *is a weakly symplectic smooth manifold and* $H\in\mathcal{C}^\infty_\omega(\mathcal{M})$*.*

2.2.3. Symmetries and momentum maps

Given a Hamiltonian system $(\mathcal{M},\,\omega,\,H)$ modelling a dynamical system, if an action of a Lie group $\mathcal{G}$ is defined over $\mathcal{M}$ one may ask whether it represents a symmetry for the dynamical system under investigation, i.e. whether it leaves invariant the space of solutions of Eq. (2.70). In the sequel, we will give a more precise notion of SYMMETRY within the context of Hamiltonian systems after recalling how a Lie group can act upon a symplectic manifold. Moreover, we will recall the relation between symmetries of an Hamiltonian system with conserved quantities along the solutions of Eq. (2.70), which is the content of the celebrated NOETHER'S THEOREMS.

Definition 2.2.10 (ACTION OF $\mathcal{G}$ UPON $\mathcal{M}$)**.** *A smooth* ACTION *of a Lie group* $\mathcal{G}$ *upon the smooth manifold* $\mathcal{M}$ *is a map:*

$$\Phi\ \ :\ \ \mathcal{G}\to\mathrm{Diff}(\mathcal{M})\ \ :\ \ g\mapsto\Phi_g\,, \tag{2.71}$$

where $\mathrm{Diff}(\mathcal{M})$ *represents the group of diffeomorphisms of* $\mathcal{M}$*, such that:*

$$\Phi_{g\cdot h}\;=\;\Phi_g\circ\Phi_h\,, \tag{2.72}$$

(i.e. it is a group homomorphism between $\mathcal{G}$ *and the group of diffeomorphisms* $\mathrm{Diff}(\mathcal{M})$*) and such that the* ACTION MAP*:*

$$\Phi\ \ :\ \ \mathcal{G}\times\mathcal{M}\to\mathcal{M}\ \ :\ \ (g,\,m)\mapsto\Phi_g(m)\,, \tag{2.73}$$

is smooth.

Definition 2.2.11 (Canonical action). *Given a (weakly or strongly) symplectic manifold* $(\mathcal{M},\,\omega)$ *and a Lie group acting upon* $\mathcal{M}$*, the action is said to be* CANONICAL *if:*

$$\Phi_g^\star \omega \,=\, \omega\,, \quad \forall\, g \in \mathcal{G}\,. \tag{2.74}$$

Equivalently, provided $\mathcal{G}$ *is connected, the action is canonical if:*

$$\mathcal{L}_{X_\xi}\omega \,=\, 0\,, \tag{2.75}$$

where X_ξ *is the infinitesimal generator of (each one-parameter subgroups of)* Φ_g *with* ξ *the element in the Lie algebra* $\mathfrak{g}$ *associated with* $g \in \mathcal{G}$.

Definition 2.2.12 (Weakly Hamiltonian action). *Given a strongly symplectic manifold* $(\mathcal{M},\,\omega)$ *and a Lie group acting upon* $\mathcal{M}$*, the action is said to be* WEAKLY HAMILTONIAN *if for any* $\xi \in \mathfrak{g}$ *there exists a function, say* J_ξ *such that:*

$$i_{X_\xi}\omega \,=\, \mathrm{d}J_\xi\,. \tag{2.76}$$

Note that if an action is Hamiltonian, it is also canonical, whereas the converse is not true. Indeed, being ω closed, the fact that $i_{X_\xi}\omega$ is exact implies that:

$$\mathcal{L}_{X_\xi}\omega \,=\, i_{X_\xi}\underbrace{\mathrm{d}\omega}_{=0} + \mathrm{d}\underbrace{i_{X_\xi}\omega}_{\mathrm{d}J_\xi} \,=\, \mathrm{d}^2 J_\xi \,=\, 0\,. \tag{2.77}$$

Viceversa, the fact that $\mathrm{d}\, i_{X_\xi}\omega$ is zero does not implies that $i_{X_\xi}\omega$ is globally exact along $\mathcal{M}$.

Definition 2.2.13 (Momentum map). *Given a strongly symplectic manifold* $(\mathcal{M},\,\omega)$*, and a weakly Hamiltonian action of a Lie group* $\mathcal{G}$ *upon* $(\mathcal{M},\,\omega)$*, a smooth map:*

$$\mathbb{J} \;:\; \mathcal{M} \to \mathfrak{g}^\star \;:\; m \mapsto \mathbb{J}(m)\,, \tag{2.78}$$

satisfying:

$$\langle \mathbb{J}(m),\, \xi \rangle \,=\, J_\xi\,, \tag{2.79}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ *is the pairing between* $\mathfrak{g}$ *and* $\mathfrak{g}^\star$*, is called a* MOMENTUM MAP *for the action of* $\mathcal{G}$.

As the following proposition proves, the momentum map is equivariant with respect to the action of $\mathcal{G}$ up to a $\mathfrak{g}^\star$-valued cocycle on $\mathcal{G}$.

Proposition 2.2.14. *Given a strongly symplectic connected manifold* $(\mathcal{M},\,\omega)$ *together with a weakly Hamiltonian action of a Lie group* $\mathcal{G}$ *upon* $(\mathcal{M},\,\omega)$*, the function:*

$$\psi_{g,\xi} \;:\; \mathcal{M} \to \mathbb{R} \;:\; m \mapsto \psi_{g,\xi}(m) \,=\, J_\xi(\Phi_g \cdot m) - J_{\mathrm{Ad}_{g^{-1}}\xi}(m)\,, \tag{2.80}$$

where $\mathrm{Ad}_{\mathrm{g}^{-1}}\xi$ *is the adjoint action of* $\mathcal{G}$ *upon* $\mathfrak{g}$*, is constant along* $\mathcal{M}$ *for all* g *and* ξ*. Moreover, it satisfies:*

$$\psi_{g,\xi}(m) \,=\, \langle \sigma_g,\, \xi \rangle \;\; \forall\, m \in \mathcal{M}\,, \tag{2.81}$$

where σ is a $\mathfrak{g}^\star$-valued cocycle on $\mathcal{G}$, i.e. it is a map:

$$\sigma \;:\; \mathcal{G} \to \mathfrak{g}^\star \;:\; g \mapsto \sigma_g \,, \tag{2.82}$$

such that:

$$\sigma_{gh} \,=\, \sigma_g + \mathrm{Ad}^\star_{g^{-1}}\sigma_h \,. \tag{2.83}$$

Proof. See [57, Prop. 4.2.3]. □

Note that the function $\psi_{g,\xi}$ measures how much $\mathbb{J}$ is not equivariant with respect to the action of $\mathcal{G}$. Indeed, when $\psi_{g,\xi}$ is zero, that is, when:

$$J_\xi(\Phi_g \cdot m) \,=\, J_{\mathrm{Ad}_{g^{-1}}\xi}(m) \,, \tag{2.84}$$

or, equivalently, when:

$$\mathbb{J}(\Phi_g \cdot m) \,=\, \mathrm{Ad}^\star_{g^{-1}}(\mathbb{J}(m)) \tag{2.85}$$

$\mathrm{Ad}^\star_{g^{-1}}$ being the dual, w.r.t. the pairing $\langle \cdot \,, \cdot \rangle$, of the adjoint action $\mathrm{Ad}_{g^{-1}}$, then the momentum map is equivariant with respect to the action of $\mathcal{G}$.

Definition 2.2.15 (Strongly Hamiltonian action)**.** *Given a strongly symplectic manifold $(\mathcal{M},\, \omega)$ and a weakly Hamiltonian action of a Lie-group $\mathcal{G}$ upon $(\mathcal{M},\, \omega)$, the action is said to be strongly Hamiltonian if the momentum map $\mathbb{J}$ is equivariant with respect to the action of $\mathcal{G}$, i.e. if:*

$$\mathbb{J}(\Phi_g \cdot m) \,=\, \mathrm{Ad}^\star_{g^{-1}}(\mathbb{J}(m)) \,. \tag{2.86}$$

Before stating two relevant results about Hamiltonian actions upon a strongly symplectic manifold, recall that, given a strongly symplectic manifold $(\mathcal{M},\, \omega)$, then, a Poisson bracket structure on $\mathcal{C}^\infty(\mathcal{M})$ is canonically defined and is given by:

$$\begin{aligned} \{\cdot \,, \cdot\} \;:\; \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M}) \;:\; (f,\, g) \mapsto \{f,\, g\}_\omega &= \omega(X_f,\, X_g) \\ &= \Lambda(\mathrm{d}f,\, \mathrm{d}g) \,, \end{aligned} \tag{2.87}$$

where X_f (resp. X_g) is the Hamiltonian vector field associated with f (resp. g) via ω and Λ is a bi-vector field on $\mathcal{M}$ defined by the latter equation. The bracket defined in this way has the defining properties of a Poisson bracket, namely, it is skew-symmetric and obeys Leibniz's rule and Jacobi identity.

Recalling the definition of Schouten bracket between multivector fields on $\mathcal{M}$:

Definition 2.2.16 (Schouten bracket)**.** *Given a n-vector field and a m-vector field over a smooth Banach manifold of the type $X \,=\, X_1 \wedge \ldots \wedge X_n$ and $Y \,=$*

$Y_1 \wedge ... \wedge Y_m$, *their* SCHOUTEN BRACKET *is defined to be the* $(n+m-1)$*-vector field given by:*

$$[X,\, Y]_S \;=\; \sum_{j,k}(-1)^{j+k}[X_j,\, Y_k] \wedge X_1 \wedge ... \wedge \hat{X}_j \wedge ... \wedge \hat{Y}_k \wedge ... \wedge Y_m \,, \tag{2.88}$$

where $\hat{X}_j$ *means that the term* X_j *is missing. Such bracket is extended to generic multi-vector fields by the additional condition:*

$$[X,\, f]_S \;=\; \sum_{j}(-1)^{j-1} X_j(f) X_1 \wedge ... \wedge \hat{X}_j \wedge ... \wedge X_n \,, \tag{2.89}$$

for $f \in \mathcal{C}^\infty(\mathcal{M})$*. Notice that such a condition is well-posed only if* $\mathcal{M}$ *is modelled onto a reflexive Banach space. The case of a strongly symplectic manifold falls into this condition.*

it can be seen that [67] Λ satisfies:

$$[\Lambda,\, \Lambda]_S \;=\; 0 \,, \tag{2.90}$$

which is a consequence of the fact that ω is closed and strongly non-degenerate and which is the geometrical property which encodes the Jacobi identity satisfied by the Poisson bracket.

The following two results hold.

Proposition 2.2.17. *Given a strongly symplectic manifold* $(\mathcal{M},\, \omega)$ *and a (at least) weakly Hamiltonian action of a Lie group* $\mathcal{G}$ *upon* $\mathcal{M}$*, the mapping:*

$$J \;\;:\;\; \mathfrak{g} \to \mathcal{C}^\infty(\mathcal{M}) \;\;:\;\; \xi \mapsto J_\xi \,, \tag{2.91}$$

is a homomorphism from the Lie algebra $(\mathfrak{g},\, [\,\cdot\,,\,\cdot\,])$ *to the Poisson algebra* $(\mathcal{C}^\infty(\mathcal{M}),\, \{\,\cdot\,,\,\cdot\,\}_\omega)$*, up to the function:*

$$\Xi \;\;:\;\; \mathfrak{g}\times\mathfrak{g} \to \mathbb{R} \;\;:\;\; (\xi,\, \rho) \mapsto \Xi(\xi,\, \rho) \;=\; \mathrm{d}\langle \sigma_e,\, \rho\rangle(\xi) \,, \tag{2.92}$$

i.e.

$$\{J_\xi,\, J_\rho\} \;=\; J_{[\xi,\, \rho]} + \Xi(\xi,\, \rho) \,. \tag{2.93}$$

What is more, if the action of $\mathcal{G}$ *upon* $(\mathcal{M},\, \omega)$ *is strongly Hamiltonian, then* J *is a homomorphism, i.e.:*

$$\{J_\xi,\, J_\rho\} \;=\; J_{[\xi,\, \rho]} \,. \tag{2.94}$$

Proof. See [63] Theorem 4.2.8 and Corollary 4.2.9. □

Being the function $\Xi(\xi,\, \rho)$ a constant on $\mathcal{M}$, a consequence of the previous proposition is that the left and right hand side of Eq. (2.93) define the same Hamiltonian vector field.

Proposition 2.2.18. *Given a strongly symplectic manifold $(\mathcal{M},\,\omega)$ and a weakly Hamiltonian action of a Lie group $\mathcal{G}$ upon $(\mathcal{M},\,\omega)$, the following holds:*

$$X_{\{J_\xi, J_\rho\}} \,=\, X_{J_{[\xi,\rho]}}\,. \tag{2.95}$$

We end this section by giving the definition of SYMMETRY GROUP for an Hamiltonian system and stating NOETHER'S THEOREM.

Definition 2.2.19 (SYMMETRY GROUP)**.** *Given a Hamiltonian system $(\mathcal{M},\,\omega,\,H)$ and a canonical action of a Lie group $\mathcal{G}$ on the strongly symplectic manifold $(\mathcal{M},\,\omega)$, then $\mathcal{G}$ is said to be a SYMMETRY GROUP for $(\mathcal{M},\,\omega,\,H)$ if the Hamiltonian H is invariant along $\mathcal{G}$, that is, if:*

$$\Phi_g^\star H \,=\, H \;\;\forall\, g \in \mathcal{G}\,, \tag{2.96}$$

or, what is the same:

$$\mathcal{L}_{X_\xi} H \,=\, 0 \;\;\forall\, \xi \in \mathfrak{g}\,, \tag{2.97}$$

provided $\mathcal{G}$ is connected. Since weakly/strongly Hamiltonian actions are canonical this result applies to them as a particular case.

If the action of $\mathcal{G}$ is strongly Hamiltonian, the following celebrated NOETHER'S THEOREM holds.

Theorem 2.2.20 (NOETHER'S THEOREM)**.** *Given a Hamiltonian system $(\mathcal{M},\,\omega,\,H)$ and a (at least weakly) Hamiltonian action of a Lie group upon the strongly symplectic manifold $(\mathcal{M},\,\omega)$, then, if $\mathcal{G}$ is a connected symmetry group for the Hamiltonian system, i.e., if:*

$$\mathcal{L}_{X_\xi} H \,=\, 0 \;\;\forall\, \xi \in \mathfrak{g}\,, \tag{2.98}$$

the following holds:

$$\mathcal{L}_\Gamma J_\xi \,=\, 0 \;\;\forall\, \xi \in \mathfrak{g}\,, \tag{2.99}$$

where Γ is the dynamics of the Hamiltonian system, i.e., it is the solution of:

$$i_\Gamma \omega \,=\, \mathrm{d}H\,. \tag{2.100}$$

This means that all the functions J_ξ associated with the strongly Hamiltonian action of the symmetry group are conserved along the solutions of the Hamiltonian system.

Proof. The claim comes from the following chain of equalities:

$$0 \,=\, \mathcal{L}_{X_\xi} H \,=\, i_{X_\xi}\,\mathrm{d}H \,=\, \omega(\Gamma,\,X_\xi) \,=\, -\omega(X_\xi,\,\Gamma) \,=\, -\mathcal{L}_\Gamma J_\xi\,. \tag{2.101}$$

□

2.3. Pre-symplectic Banach manifolds and dynamical systems

In this section we pass to the more general setting of pre-symplectic manifolds, defined in Sec. 2.3.1, and pre-symplectic Hamiltonian systems, described in Sec. 2.3.2. In this case the definition of a Poisson bracket as well as the correspondence between symmetries and constants of the motion turns to be more involved than within the symplectic case and they are discussed, in Sec. 2.3.4 and Sec. 2.3.5, after describing, in Sec. 2.3.3, a regularization technique related to the so called COISOTROPIC EMBEDDING THEOREM.

As for the symplectic case we refer to the literature for more details (see, for instance, [63], [66] and references therein for more details about pre-symplectic manifolds and pre-symplectic Hamiltonian systems and [65], [68], [69] for the coisotropic embedding theorem and some its applications).

2.3.1. Pre-symplectic Banach manifolds

Definition 2.3.1 (PRE-SYMPLECTIC BANACH MANIFOLD)**.** *A PRE-SYMPLECTIC Banach manifold is a couple* $(\mathcal{M},\, \omega)$ *where* $\mathcal{M}$ *is a smooth Banach manifold and* ω *is a closed* 2*-form on* $\mathcal{M}$*.*

In this section we will focus on the case in which the kernel defined in (2.63) is non trivial. Sometimes we will refer to this case as GENUINELY PRE-SYMPLECTIC case or as FAIRLY PRE-SYMPLECTIC case.

2.3.2. Pre-symplectic Hamiltonian systems and the pre-symplectic constraint algorithm

Definition 2.3.2 (PRE-SYMPLECTIC HAMILTONIAN SYSTEM)**.** *A PRE-SYMPLECTIC HAMILTONIAN SYSTEM is a triple* $(\mathcal{M},\, \omega,\, H)$ *made by a pre-symplectic manifold* $(\mathcal{M},\, \omega)$ *and a smooth function* H *on* $\mathcal{M}$ *called the* HAMILTONIAN *of the system.*

When $(\mathcal{M},\, \omega)$ is genuinely pre-symplectic, a Hamiltonian vector field associated to H may not even be defined and, in general, when defined, it is not unique. We will always assume the kernel of ω at $m \in \mathcal{M}$, say K_m, to define a regular distribution, K, on $\mathcal{M}$, in order to represent it locally around each point in terms of vector fields on $\mathcal{M}$. However, even in this pre-symplectic case, one can still speak about SOLUTIONS of an equation of the type (2.70) and these are found via the so called PRE-SYMPLECTIC CONSTRAINT ALGORITHM (PCA) that we are going to described in the next lines.

Remark 2.3.3. *Before proceeding with the description of the PCA, let us stress that, insisting in defining the concept of solution of Eq.* (2.70) *even in the genuinely* *pre-symplectic case is not just an abstract exercise. Indeed, as we will see, solutions of this type of system will obey a combination of differential equations together with a set of constraint relations and, therefore, pre-symplectic Hamiltonian systems are well suited to model those theories, such as* GAUGE THEORIES, *for which the equations of motion split into a set of evolutionary equations and a set of constraint relations. Indeed, many examples of this instance will appear in the present manuscript.*

Let us now proceed with the description of the PCA. Let us consider a pre-symplectic Hamiltonian system $(\mathcal{M},\, \omega,\, H)$ and let us consider the equation:

$$i_{X_H}\omega \;=\; \mathrm{d}H\,. \tag{2.102}$$

It is clear that, in order for this equation to be well posed, a first, obvious, condition that must hold is that the elements of the kernel of ω at the point m, K_m, should lie also in the kernel of the 1-form $\mathrm{d}H$ at m. Indeed, by contracting both sides of the latter equation along an element of K_m, the left hand side vanishes and, thus, the same must happen with the right hand side. In general, such a condition will be satisfied only at some points of $\mathcal{M}$. We denote such a set of points by $\mathcal{M}_1$:

$$\mathcal{M}_1 \;:=\; \{\, m \in \mathcal{M} \;\;:\;\; \mathrm{d}H_m(V_m) \;=\; 0 \;\; \forall\; V_m \in K_m \,\}\;, \tag{2.103}$$

where:

$$K_m \;=\; \{\, V_m \in \mathbf{T}_m\mathcal{M} \;\;:\;\; \omega(V_m,\, W_m) \;=\; 0 \;\; \forall\; W_m \in \mathbf{T}_m\mathcal{M} \,\}\;. \tag{2.104}$$

We assume $\mathcal{M}_1$ to be a smooth embedded submanifold of $\mathcal{M}$ with smooth embedding map denoted by $\mathfrak{i}_1$. The following equation is now well posed:

$$\mathfrak{i}_1^\star\left(i_{X_H}\omega - \mathrm{d}H\right) \;=\; 0\,, \tag{2.105}$$

even if nothing is said about the fact that X_H is actually tangent to $\mathcal{M}_1$, namely, that it is $\mathfrak{i}_1$-related to a vector field in $\mathfrak{X}(\mathcal{M}_1)$. However, we want that this would be actually the case since we want that along the flow of the dynamics X_H the constraints imposed by (2.103) were preserved. Imposing this last condition, that is, that the X_H satisfying (2.105) is $\mathfrak{i}_1$-related with a vector field on $\mathcal{M}_1$, it is straightforward to prove that all the $V \in \mathfrak{X}(\mathcal{M})$ such that:

$$\mathfrak{i}_1^\star\left(i_V\omega\right) \;=\; 0\,, \tag{2.106}$$

must satisfy:

$$\mathfrak{i}_1^\star\left(i_V\mathrm{d}H\right) \;=\; 0\,. \tag{2.107}$$

The set of points for which this condition is satisfied will be denoted by $\mathcal{M}_2$ and, again, it is assumed to be a smooth embedded submanifold of $\mathcal{M}_1$ with the smooth embedding map denoted by $\mathfrak{i}_2$. Now, on such $\mathcal{M}_2$ we aim to find a solution of:

$$\mathfrak{i}_2^\star\left(i_{X_H}\omega - \mathrm{d}H\right) \;=\; 0\,, \tag{2.108}$$

but, again, nothing is said about the fact that such X_H is actually tangent to $\mathcal{M}_2$. Therefore, again, we select a smooth embedded submanifold of $\mathcal{M}_2$, say $\mathcal{M}_3$, such that all the $V \in \mathfrak{X}(\mathcal{M})$ for which:

$$\mathfrak{i}_2^\star\left(i_V \omega\right) \,=\, 0\,, \tag{2.109}$$

satisfy:

$$\mathfrak{i}_2^\star\left(i_V \mathrm{d}H\right) \,=\, 0\,. \tag{2.110}$$

It is clear that we turned into an algorithmic procedure that select, at the k-th step, the following smooth manifold:

$$\mathcal{M}_k \,=\, \left\{ m \in \mathcal{M}_{k-1} \;\;:\;\; i_V \mathrm{d}H \,=\, 0 \;\;\forall\; V \in \mathbf{T}\mathcal{M}_{k-1}^{\perp} \right\}\,, \tag{2.111}$$

where:

$$\mathbf{T}\mathcal{M}_k^{\perp} \,=\, \left\{\, V \in \mathfrak{X}(\mathcal{M}) \;\;:\;\; \mathfrak{i}_k^\star\left(i_V \omega\right) \,=\, 0 \,\right\}\,. \tag{2.112}$$

Such algorithmic procedure is called the Pre-symplectic Constraint Algorithm, PCA from now on.

If there exists a finite k for which $\mathcal{M}_k \,=\, \mathcal{M}_{k+1}$ we say that the PCA converges, we call the manifold $\mathcal{M}_k$ the stable or final manifold of the algorithm and we denote it by $\mathcal{M}_k \,=\, \mathcal{M}_\infty$. Coherently, we denote the embedding $\mathfrak{i}_k$ by $\mathfrak{i}_\infty$. On the stable manifold we are left with the equation:

$$\mathfrak{i}_\infty^\star\left(i_{X_H} \omega - \mathrm{d}H\right) \,=\, 0\,, \tag{2.113}$$

which, being X_H $\mathfrak{i}_\infty$-related to a vector field, say X_H^∞ on $\mathcal{M}_\infty$, is equivalent to:

$$i_{X_H^\infty} \omega_\infty \,=\, \mathrm{d}H_\infty\,, \tag{2.114}$$

where $\omega_\infty \,=\, \mathfrak{i}_\infty^\star \omega$ and $H_\infty \,=\, \mathfrak{i}_\infty^\star H$.

At this point two situations may occur: ω_∞ may be symplectic or again pre-symplectic.

In the first case, Eq. (2.114) has a unique solution X_H^∞. Consequently, we say that the solutions of the original pre-symplectic Hamiltonian system $(\mathcal{M},\, \omega,\, H)$ are the integral curves of X_H^∞ embedded into $\mathcal{M}$ via $\mathfrak{i}_\infty$. That is, given a Cauchy datum $\gamma_0(m_\infty)$, at some point $m_\infty \in \mathcal{M}_\infty$, the unique integral curve of X_H^∞ passing through $\gamma_0(m_\infty)$ reads:

$$\gamma_\infty(s) \,=\, F_s^{X_H}\left[\,\gamma_0(m_\infty)\,\right]\,, \tag{2.115}$$

where $F_s^{X_H}$ is the flow of X_H and the unique solution of the pre-symplectic Hamiltonian system associated with the Cauchy datum $\gamma_0(m_\infty)$ reads:

$$\gamma(s) \,=\, \mathfrak{i}_\infty\left[\,\gamma_\infty(s)\,\right]\,. \tag{2.116}$$

The composition of $F_s^{X_H}$ and $\mathfrak{i}_\infty$ is evidently a one-to-one map between $\mathcal{M}_\infty$ (representing the space of Cauchy data) and its image (representing the space of solutions of the pre-symplectic Hamiltonian system). Such a bijection allows for defining a differential structure on the image manifold, namely the space of solutions and, consequently, the space of Cauchy data and the space of solutions result to be diffeomorphic. It should be stressed that the solutions constructed in this way result as solutions of a set of ODEs (the ones defining the vector field X_H) and a set of constraints equations (the ones defining the embedding of $\mathcal{M}_\infty$ into $\mathcal{M}$) as we anticipated in Remark 2.3.3. Examples of physical systems modelled as pre-symplectic Hamiltonian systems can be found along the manuscript (see examples 4.2.5, 4.2.9).

In the second case, the solution X_H^∞ is only determined up to the kernel of ω_∞. Consequently, given the integral curves of any such X_H^∞, an equivalence class of curves obtained by acting with the flow of any vector field in the kernel of ω_∞ is defined. In this case, we say that the solutions of the original pre-symplectic Hamiltonian system, are all the elements of such equivalence class of curves on $\mathcal{M}_\infty$ embedded into $\mathcal{M}$ via $\mathfrak{i}_\infty$. Therefore, such solutions result as solutions of a set of differential equations (the ones defining one of the X_H^∞) up to transformations generated by the kernel of ω_∞ (representing gauge transformations) and a set of constraint equations (the ones defining the embedding of $\mathcal{M}_\infty$ into $\mathcal{M}$). Examples of this instance are examples 4.2.6, 4.2.11, 4.2.12, 4.2.13. Here the final manifold $\mathcal{M}_\infty$ is, again, diffeomorphic (in the same sense of the previous case) to the space of solutions of the pre-symplectic Hamiltonian system, even if some of them are equivalent from the physical point of view since they are just related by gauge transformations.

2.3.3. The coisotropic embedding theorem

In the previous section we saw that the space of solutions of the equations of motion of a theory modelled as a pre-symplectic Hamiltonian system is in general a pre-symplectic manifold. However, there are various reasons for which it is more useful having a symplectic structure at hand. Among them, the fact that via a symplectic structure one may always construct a Poisson bracket and, provided with a Hamiltonian action of a Lie group, a set of conserved quantities (see Sec. 2.2.3). In this section we recall the so called COISOTROPIC EMBEDDING THEOREM, as a tool to construct a symplectic manifold starting from a pre-symplectic one. We will see in Sec. 2.3.4 how to use such a theorem to construct a Poisson bracket structure on a class of pre-symplectic manifolds. In particular Sec. 4.3.5 will be an application of such general theory where the pre-symplectic manifold is represented by the solution space of gauge theories. On the other hand, in Sec. 2.3.5, we will see how to use it to construct conserved quantities also in the pre-symplectic case.

Let us consider a pre-symplectic manifold $(\mathcal{M},\, \omega)$. The coisotropic embedding theorem states that there is a canonical (up to local symplectomorphisms) way of embedding $(\mathcal{M},\, \omega)$ into a symplectic manifold, that we denote by $(\tilde{\mathcal{M}},\, \tilde{\omega})$, of which $(\mathcal{M},\, \omega)$ is a coisotropic submanifold, namely, $\mathfrak{i}^\star \tilde{\omega} = \omega$, where $\mathfrak{i}$ is the embedding map.

First we recall the standard proof of the theorem for which we refer to *M. Gotay* [69] (see also [65] for the proof of the theorem). Then, we also provide an alternative proof of the theorem (see [70]) which is more canonical and more suited for our concrete use of the theorem in the examples considered in Sec. 4.3.

Classical coisotropic embedding theorem. Let us denote by K_m the kernel of ω at $m \in \mathcal{M}$. Assume that the distribution given by K_m for $m \in \mathcal{M}$, say K, defines a subbundle of $\mathbf{T}\mathcal{M}$, denote it by $\mathbf{K}$ and let us call it the CHARACTERISTIC BUNDLE over $\mathcal{M}$. Its dual bundle, is again a vector bundle over $\mathcal{M}$ denoted by $\mathbf{K}^\star$, where we will denote by $K^\star$ the distribution generated by $K^\star_m$. $\mathcal{M}$ can be immersed into $\mathbf{K}^\star$ as the range of the zero section σ_0 w.r.t. the bundle projection $\tau \;:\; \mathbf{K}^\star \to \mathcal{M}$. Along σ_0 the tangent space to $\mathbf{K}^\star$ splits as $\mathbf{T}_{\sigma_0(m)}\mathbf{K}^\star = \mathbf{T}_m\mathcal{M} \oplus K^\star_m$. Now, let us assume that a (closed) complement, W_m, of K_m into $\mathbf{T}_m\mathcal{M}$ exists[4]. Notice that:

$$\mathbf{T}_{\sigma_0(m)}\mathbf{K}^\star \,=\, W_m \oplus K_m \oplus K^\star_m \,. \tag{2.117}$$

Therefore, since $K_m \oplus K^\star_m$ is a symplectic vector space[5] and the original pre-symplectic structure ω is non-degenerate when contracted along elements of W_m, a symplectic structure can be constructed on $\sigma_0(\mathcal{M})$ in the following way:

$$\tilde{\omega}_0 \,=\, \tau^\star\omega + \omega_{\mathbf{K}\oplus\mathbf{K}^\star} \circ \mathrm{pr}\,, \tag{2.118}$$

where pr is the projection $\mathbf{T}_{\sigma_0(m)}\mathbf{K}^\star \to K_m \oplus K^\star_m$. Extending $\tilde{\omega}_0$ to $\mathbf{K}^\star$ we get a differential form $\tilde{\omega}_0^{\mathrm{ext}}$ such that:

$$\sigma_0^\star \tilde{\omega}_0^{\mathrm{ext}} \,=\, \omega\,. \tag{2.119}$$

However, $(\mathbf{K}^\star,\, \tilde{\omega}_0^{\mathrm{ext}})$ is not yet the desired symplectic manifold since $\tilde{\omega}_0^{\mathrm{ext}}$ is not closed, in general, outside $\sigma_0(\mathcal{M})$. However a differential form, say α, defined in a tubular neighborhood of $\sigma_0(\mathcal{M})$, $\tilde{\mathcal{M}}$, such that:

$$\mathrm{d}\alpha \,=\, -\mathrm{d}\tilde{\omega}_0^{\mathrm{ext}} \qquad \text{and} \qquad \alpha\Big|_{\sigma_0(\mathcal{M})} \,=\, 0\,, \tag{2.120}$$

can always be added to ω_0^{ext} (see [65, lemma 39.1 at page 318]). The form α is constructed out of a retraction ϕ_t of $\mathbf{K}^\star$ onto $\mathcal{M}$, given by the multiplication by a real parameter t. Given the one-parameter family of generators of ϕ_t, say X_t, the form α reads:

$$\alpha \,=\, \int_0^1 \phi_t^\star\left[\, i_{X_t}\omega \,\right] \mathrm{d}t\,. \tag{2.121}$$

Given such an α, the manifold $(\tilde{\mathcal{M}},\, \tilde{\omega})$, with $\tilde{\omega} \,=\, \tilde{\omega}_0^{\mathrm{ext}}\Big|_{\tilde{\mathcal{M}}} + \alpha$, is the desired symplectic manifold. Whether it is weakly or strongly symplectic depends on the model Banach space of our starting smooth Banach manifold. From now on, we will simply refer to it as a symplectic manifold always assuming it to be strongly symplectic and we will verify this assumption case by case in the examples we consider.

[4]This requirement is fulfilled in al the examples considered along the manuscript.

[5]The symplectic structure is the one we already used in example 2.2.4.

Coisotropic embedding theorem via a connection. The construction of the symplectic form given above is the original one of *M. Gotay* and is based on some choices, such as that of the retraction ϕ, that make it not properly canonical. Here, we propose a canonical construction, due to *Y. G. Oh* and *J. S. Park* [70], that only makes use of the splitting $\mathbf{T}\mathcal{M} = W \oplus K$ and that is more suited to our concrete use of the coisotropic embedding theorem in Sec. 4.3.

Given the splitting above, denote by P the projection:

$$P \;:\; \mathbf{T}\mathcal{M} \to K \,, \tag{2.122}$$

and consider the map:

$$T\tau \;:\; \mathbf{T}\mathbf{K}^\star \to \mathbf{T}\mathcal{M} \,. \tag{2.123}$$

The composition $T\tau \circ P$ gives a canonical map from $\mathbf{TK}^\star$ to $\mathcal{M}$ which, given any point β of $\mathbf{K}^\star$, allows for defining the following 1-form on $\mathbf{K}^\star$:

$$\vartheta^P_\beta(X) \,=\, \langle\, \alpha,\, P \circ T\tau(X)\,\rangle \,, \tag{2.124}$$

where $X \in \mathbf{T}_\mu\mathbf{K}^\star$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the pairing between K_m and $K^\star_m$. Then, the 2-form:

$$\tilde{\omega} \,=\, \tau^\star\omega - \mathrm{d}\vartheta^P \,, \tag{2.125}$$

is closed (by construction) and non-degenerate (as it can be easily seen) in a neighborhood of the zero-section σ_0 of τ. Note that now, as it is stressed by the notation ϑ^P, the construction of $\tilde{\omega}$ only makes use of the projector P. A local expression for such a 2-form can be given by writing the projection P in local coordinates in terms of the following (1, 1) (idempotent) tensor field:

$$P \,=\, P^j \otimes V_j \,, \tag{2.126}$$

where $\{\, V_j \,\}_{j=1,\ldots,\mathrm{dim}K}$ is a subset of a basis of $\mathfrak{X}(\mathcal{M})$ that generates K at each point and where $\{\, P^j \,\}_{j=1,\ldots,\mathrm{dim}K}$ are 1-forms on $\mathcal{M}$ that, acting on a vector field X, give its components along K in the basis $\{\, V_j \,\}_{j=1,\ldots,\mathrm{dim}K}$:

$$P(X) \,=\, P^j(X)V_j \,=:\, X^{vj}V_j \,\in\, \mathfrak{X}^v(\mathcal{M}) \,. \tag{2.127}$$

Denote by $\{\, \mu_j \,\}_{j=1,\ldots,\mathrm{dim}K^\star}$ the system of coordinates on $K^\star_m$ such that the 1-forms $\mathrm{d}\mu_j$ are "dual" to the 1-forms P^j in the sense that $\omega_{\mathbf{K}\oplus\mathbf{K}^\star} \circ \mathrm{pr}$ reads:

$$\omega_{\mathbf{K}\oplus\mathbf{K}^\star} \circ \mathrm{pr}(X,\, Y) \,=\, \mathrm{d}\mu_j \wedge P^j(X,\, Y) \,=\, \langle X_{\mu_j},\, Y^{vj}\,\rangle - \langle Y_{\mu_j},\, X^{vj}\,\rangle \,, \tag{2.128}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes, at each point, the pairing between K_m and its dual and where X_{μ_j} denotes the components of X along $K^\star_m$. Then, the 2-form $\tilde{\omega}$ reads:

$$\tilde{\omega} \,=\, \tau^\star\omega + \mathrm{d}\mu_j \wedge P^j + \mu_j \mathrm{d}P^j \,, \tag{2.129}$$

and its pull-back to $\mathcal{M}$ via σ_0 is:

$$\tilde{\omega}_0 \,=\, \omega + \mathrm{d}\mu_j \wedge P^j \,, \tag{2.130}$$

from which we see that the form α constructed in the previous section is:

$$\alpha \,=\, \mu_j \mathrm{d}P^j \,. \tag{2.131}$$

Remark 2.3.4. *From now on, we assume $\tilde{\omega}$ to be actually strongly symplectic and we postpone the check of this instance to a case by case analysis within the examples considered along the manuscript.*

2.3.4. Poisson brackets on pre-symplectic manifolds via coisotropic embeddings

In this section we will present the coisotropic embedding theorem as a tool to construct Poisson brackets on a class of pre-symplectic manifolds selected by geometrical properties of the connection chosen. In particular we will see that a Poisson bracket can be defined if the connection is closed or has zero curvature. In the other cases to define a Poisson bracket one is forced to work on the enlarged manifold $\tilde{\mathcal{M}}$.

The closed case

Consider the case where $\mathrm{d}P^j = 0$. The symplectic form $\tilde{\omega}$, at each $\tilde{m} \in \tilde{\mathcal{M}}$, is the sum of a form having components only on W_m and a form having components only on $K_m \oplus K^\star_m$. These are well defined along the whole $\tilde{\mathcal{M}}$ provided a connection on $\mathbf{K}^\star$ is fixed. We will fix it to be the pull-back of P via the projection $\tau \,:\, \mathbf{K}^\star \to \mathcal{M}$. The two forms mentioned above read:

$$\begin{aligned} \tilde{\omega}_{\tilde{m}}(\tilde{X}_W + \tilde{X}_K + \tilde{X}_{K^\star},\, \tilde{Y}_W + \tilde{Y}_K + \tilde{Y}_{K^\star}) \,&=\, \tau^\star \omega_{\tilde{m}}(\tilde{X}_W,\, \tilde{Y}_W) + \mathrm{d}\mu_j \wedge P^j_{\tilde{m}}(\tilde{X}_K + \tilde{X}_{K^\star},\, \tilde{Y}_K + \tilde{Y}_{K^\star}) \\ &=: \, \tilde{\omega}_{W_m}(\tilde{X}_W,\, \tilde{Y}_W) + \tilde{\omega}_{K_m \oplus K^\star_m}(\tilde{X}_K + \tilde{X}_{K^\star},\, \tilde{Y}_K + \tilde{Y}_{K^\star}) \end{aligned} \tag{2.132}$$

where $\tilde{X}_W \in W_m$, $\tilde{X}_K \in K_m \;\forall\, \tilde{m} \in \tilde{\mathcal{M}}$ and $\tilde{X}_{K^\star} \in K^\star_m \;\forall\, \tilde{m} \in \tilde{\mathcal{M}}$. This means that $\tilde{\omega}$ results as the direct sum of two closed forms:

$$\tilde{\omega} \,=\, \tilde{\omega}_W \oplus \tilde{\omega}_{K \oplus K^\star} \tag{2.133}$$

which, restricted respectively to the distributions W and $K \oplus K^\star$, are non-degenerate. Consequently, the inverse of $\tilde{\omega}$ is a Poisson bi-vector field on $\tilde{\mathcal{M}}$, say $\tilde{\lambda}$, which reads:

$$\tilde{\lambda} \,=\, \tilde{\lambda}_W \oplus \tilde{\lambda}_{K \oplus K^\star} \tag{2.134}$$

where $\tilde{\lambda}_W$ is a Poisson bi-vector field belonging, at each $\tilde{m} \in \tilde{\mathcal{M}}$, to $W_m \wedge W_m$ whereas $\tilde{\lambda}_{K \oplus K^\star}$ is a Poisson bi-vector field belonging, at each $\tilde{m} \in \tilde{\mathcal{M}}$ to $(\, K_m \oplus K^\star_m \,) \wedge (\, K_m \oplus K^\star_m \,)$. Since $\tilde{\lambda}$ is the Poisson bi-vector field associated to a symplectic structure, it satisfies the following:

$$[\tilde{\lambda},\, \tilde{\lambda}]_S \,=\, 0 \tag{2.135}$$

where $[\,\cdot\,,\,\cdot\,]_S$ denote the Schouten-Njienhuis brackets, which is equivalent to the fact that the bracket associated to $\tilde{\lambda}$, $\{f,\,g\} \,=\, \tilde{\lambda}(\mathrm{d}f,\,\mathrm{d}g)$, satisfies the Jacobi identity. Now, since $\tilde{\lambda}_W$ is the inverse (restricted to W) of the closed (and non-degenerate when restricted to W) form $\tilde{\omega}_W$, it satisfies:

$$\left[\tilde{\lambda}_W,\,\tilde{\lambda}_W\right]_S \,=\, 0 \tag{2.136}$$

itself. With this in mind (2.136) is a straightforward consequence. This says that on the subalgebra of functions on $\tilde{\mathcal{M}}$ being the pull-back via τ of functions on $\mathcal{M}$ (namely, functions that do not depend on μ_j and that are invariant with respect to V_j), the Poisson bracket given by $\tilde{\lambda}$ restricts to a Poisson bracket given by $\tilde{\lambda}_W$. What is more, the bi-vector field $\tilde{\lambda}_W$ satisfying (2.136) can be used to define a Poisson bracket on the pre-symplectic manifold $\mathcal{M}$. Indeed, since by construction $\tilde{\lambda}_W$ does not depend on μ_j, it is easy to see that:

$$\left[\tilde{\lambda}_W,\,\frac{\partial}{\partial\mu_j}\right]_S \,=\, 0 \quad \forall\; j = 1, ..., \mathrm{dim}K^\star\,. \tag{2.137}$$

Consequently, the bi-vector field $\tilde{\lambda}_W$ is projectable onto $\mathcal{M}$ via $\tau \;:\; \tilde{\mathcal{M}} \to \mathcal{M}$ to the following bi-vector field:

$$\lambda_W \,=\, \tau_\star\tilde{\lambda}_W \;\in \bigwedge^2(\mathcal{M})\,. \tag{2.138}$$

The latter also has a vanishing Schouten bracket with itself because of the following equalities:

$$[\lambda_W,\,\lambda_W]_S \,=\, [\tau_\star\tilde{\lambda}_W,\,\tau_\star\tilde{\lambda}_W]_S \,=\, \tau_\star\underbrace{[\tilde{\lambda}_W,\,\tilde{\lambda}_W]}_{=0} \,=\, 0 \tag{2.139}$$

and, thus, it defines a Poisson bracket on $\mathcal{M}$ in the following way:

$$\{\,f,\,g\,\} \,=\, \lambda_W(\mathrm{d}f,\,\mathrm{d}g) \tag{2.140}$$

for $f,\,g \in \mathcal{F}(\mathcal{M})$.

An example of this type is given in Sec. 4.3.5 when dealing with free Electrodyanamics.

Zero-curvature case

The previous construction can be extended to the case where P^j is not closed but only horizontally-closed, i. e., to the case where:

$$\mathrm{d}_H P^j \,=\, \mathrm{d}P^j\,(\,(\mathbb{1} - P)(\,\cdot\,),\,(\mathbb{1} - P)(\,\cdot\,)\,) \,=\, 0\,. \tag{2.141}$$

Indeed, when P^j is not closed, the structure $\tilde{\omega}$ reads:

$$\tilde{\omega} \,=\, \tilde{\omega}_W \oplus \tilde{\omega}_{K\oplus K^\star} \oplus \alpha \tag{2.142}$$

where $\tilde{\omega}_W$ is closed and, in general, $\alpha = \mu_j \mathrm{d}P^j$ has components both on W and on $K \oplus K^\star$, i.e.:

$$\begin{aligned}
\alpha_{\tilde{m}}(\tilde{X}_W + \tilde{X}_K + \tilde{X}_{K^\star},\, \tilde{Y}_W + \tilde{Y}_K + \tilde{Y}_{K^\star}) &= \alpha_{\tilde{m}}(\tilde{X}_W,\, \tilde{Y}_W) + \alpha_{\tilde{m}}(\tilde{X}_K + \tilde{X}_{K^\star},\, \tilde{Y}_K + \tilde{Y}_{K^\star}) \\
&= \alpha_{W\,\tilde{m}}(\tilde{X}_W,\, \tilde{Y}_W) + \alpha_{K\oplus K^\star\,\tilde{m}}(\tilde{X}_K + \tilde{X}_{K^\star},\, \tilde{Y}_K + \tilde{Y}_{K^\star}) \\
&= \mu_j \mathrm{d}_H P^j_{\tilde{m}}(\tilde{X}_W,\, \tilde{Y}_W) + \mu_j \mathrm{d}_V P^j(\tilde{X}_K + \tilde{X}_{K^\star},\, \tilde{Y}_K + \tilde{Y}_{K^\star}) ,
\end{aligned} \tag{2.143}$$

where $\mathrm{d}_V P^j(X,\, Y) = \mathrm{d}P^j(P(X),\, P(Y))$ and $\mathrm{d}_H P^j(X,\, Y) = \mathrm{d}P^j((\mathbb{1} - P)(X),\, (\mathbb{1} - P)(Y))$.
Now, if $\mathrm{d}_H P^j = 0$, i. e., if P has zero curvature, $\tilde{\omega}$ reads:

$$\tilde{\omega} = \tilde{\omega}_W \oplus \tilde{\omega}^\alpha_{K\oplus K^\star} \tag{2.144}$$

where $\tilde{\omega}^\alpha_{K\oplus K^\star} = \tilde{\omega}_{K\oplus K^\star} + \alpha_{K\oplus K^\star}$. Therefore, again $\tilde{\lambda}$ reads:

$$\tilde{\lambda} = \tilde{\lambda}_W \oplus \tilde{\lambda}^\alpha_{K\oplus K^\star} \tag{2.145}$$

with $\tilde{\lambda}_W$ satisfying:

$$\left[\tilde{\lambda}_W,\, \tilde{\lambda}_W\right] = 0 \tag{2.146}$$

since it comes from a closed 2-form $\tilde{\omega}_W$. Therefore, also in this case the previous construction can be performed, giving rise to the Poisson bracket (2.140) on $\mathcal{M}$.

The non-closed case

The general case where neither $\mathrm{d}_H P^j = 0$ nor $\mathrm{d}_V P^j = 0$ does not allow to directly define a Poisson bracket on $\mathcal{M}$ using the Poisson bracket defined on $\tilde{\mathcal{M}}$.

Indeed, in that case the structure $\tilde{\omega}$ reads:

$$\tilde{\omega} = \tilde{\omega}^\alpha_W \oplus \tilde{\omega}^\alpha_{K\oplus K^\star} , \tag{2.147}$$

where:

$$\tilde{\omega}^\alpha_W = \tilde{\omega}_W \oplus \alpha_W , \qquad \tilde{\omega}^\alpha_{K\oplus K^\star} = \tilde{\omega}_{K\oplus K^\star} \oplus \alpha_{K\oplus K^\star} , \tag{2.148}$$

with:

$$\alpha_W = \mu_j \mathrm{d}_H P^j , \qquad \alpha_{K\oplus K^\star} = \mu_j \mathrm{d}_V P^j . \tag{2.149}$$

However, in this case even if $\tilde{\omega}^\alpha_W$ and $\tilde{\omega}^\alpha_{K\oplus K^\star}$ are non-degenerate when restricted to W and $K \oplus K^\star$ respectively, they are not closed. Therefore, the corresponding bi-vector fields $\tilde{\lambda}^\alpha_W$ and $\tilde{\lambda}^\alpha_{K\oplus K^\star}$ such that:

$$\tilde{\lambda} = \tilde{\lambda}^\alpha_W \oplus \tilde{\lambda}^\alpha_{K\oplus K^\star} , \tag{2.150}$$

do not satisfy:

$$\left[\tilde{\lambda}^\alpha_W,\, \tilde{\lambda}^\alpha_W\right]_S = 0 , \qquad \left[\tilde{\lambda}^\alpha_{K\oplus K^\star},\, \tilde{\lambda}^\alpha_{K\oplus K^\star}\right]_S = 0 , \tag{2.151}$$

and, thus, $\tilde{\lambda}^{\alpha}_{W}$ can not define a Poisson bracket on $\mathcal{M}$.

The only natural construction in this case seems to be to consider the Poisson bracket associated with $\tilde{\omega}$ on the whole $\tilde{\mathcal{M}}$ restricted to the subalgebra of functions on $\tilde{\mathcal{M}}$ being pull-back (via τ) of functions on $\mathcal{M}$. That is, one can consider two functions $\tilde{f}$, $\tilde{g} \in \mathcal{F}(\tilde{\mathcal{M}})$ such that $\tilde{f} = \tau^{\star} f$ and $\tilde{g} = \tau^{\star} g$ with $f, g \in \mathcal{F}(\mathcal{M})$ and to consider their bracket computed with respect to $\tilde{\lambda}$

$$\{\tilde{f}, \tilde{g}\} = \tilde{\lambda}(\mathrm{d}\tilde{f}, \mathrm{d}\tilde{g}) = \tilde{\lambda}^{\alpha}_{W}(\mathrm{d}\tilde{f}, \mathrm{d}\tilde{g}) . \tag{2.152}$$

Even if, due to the fact that $\tilde{\lambda}$ is a bi-vector field coming from a symplectic structure, this bracket satisfies the Jacobi identity, it can not be used to induce a bracket on $\mathcal{M}$ because, in general, since the term α_W added to $\tilde{\omega}_W$ contains a dependence on the variables μ_j, the function $\{\tilde{f}, \tilde{g}\}$ is not the pull-back of a function on $\mathcal{M}$ as well. Indeed, in general it will be of the form

$$\{\tilde{f}, \tilde{g}\} = \tilde{h} + H \tag{2.153}$$

where $\tilde{h} = \tau^{\star} h$ with $h \in \mathcal{F}(\mathcal{M})$ and $H \in \mathcal{F}(\tilde{\mathcal{M}})$.

An example of this type is given in Sec. 4.3.5 when dealing with Yang-Mills theories.

To conclude the section, it is worth stressing that, in the first two cases considered, a way to define a Poisson structure on $\mathcal{M}$ already exists and does not make use of the coisotropic embedding theorem. It is the construction defined in [71] that the author used in [1] which gives rise to the same bracket obtained above. With this in mind, the coisotropic embedding theorem was presented in this section as a universal tool to define a Poisson bracket on a suitable enlarged space starting from a pre-symplectic manifold. Then, in some cases we saw that the Poisson bracket can be "projected" from the enlarged space to the original pre-symplectic manifold obtaining the same result obtained by applying the procedure of [71], whereas in other situations, working with the enlarged space seems to be necessary in order to have a Poisson structure.

2.3.5. Symmetries and momentum maps via coisotropic embeddings

For pre-symplectic Hamiltonian systems $(\mathcal{M}, \omega, H)$, given the action of a Lie group $\mathcal{G}$ on $\mathcal{M}$, the concepts of canonical, weakly and strongly Hamiltonian action as well as that of momentum map and symmetry group can be defined as we did in Sec. 2.2.3. However, since in this case ω does not induce an isomorphism between $\mathbf{T}\mathcal{M}$ and $\mathbf{T}^{\star}\mathcal{M}$, it is not possible to construct a one-to-one correspondence between symmetry group actions and conserved quantities as we did in Theorem 2.2.20.

In this section, we will see that the coisotropic embedding theorem can be used as a regularization technique also in this case. Indeed, we will see that the action of a Lie group upon $(\mathcal{M},\, \omega)$ can be lifted in a canonical way to the enlarged manifold $(\tilde{\mathcal{M}},\, \tilde{\omega})$ constructed out of the coisotropic embedding theorem. What is more, we will see that if the action of $\mathcal{G}$ upon $(\mathcal{M},\, \omega)$ is strongly Hamiltonian, then its lift is strongly Hamiltonian upon $(\tilde{\mathcal{M}},\, \tilde{\omega})$ and on this enlarged manifold the one-to-one correspondence between generators of such lifted action, say $\tilde{X}_\xi$, and momentum maps $\tilde{J}_\xi$ is recovered. Finally, the functions $\tilde{J}_\xi$ can be pulled-back to the original manifold $\mathcal{M}$ and it can be proved that they are conserved quantities in the sense of Theorem 2.2.20.

Consider the action of a Lie group $\mathcal{G}$ upon a pre-symplectic manifold $(\mathcal{M},\, \omega)$, say Φ. Assume it is strongly Hamiltonian. Assume also that the splitting given by the connection used to coisotropically embed $\mathcal{M}$ into the symplectic manifold $(\tilde{\mathcal{M}},\, \tilde{\omega})$ is equivariant with respect to such action. This results in the fact that P is an equivariant connection. The action Φ_g can be lifted to $\tilde{\mathcal{M}}$ to the action $\tilde{\Phi}_g$ which, in local coordinates, takes the following form:

$$\tilde{\Phi}_g(\tilde{m}) \,=\, \Big(\, \Phi_g(m),\, {\Phi_{g_\star}}^\star(\mu) \Big) \,, \tag{2.154}$$

where ${\Phi_{g_\star}}^\star$ is the dual of the map $\Phi_{g_\star}$ in the sense of maps between vector spaces, that is, it is the map defined by:

$$\langle\, {\Phi_{g_\star}}^\star(\mu),\, V_m \,\rangle \,=\, \langle\, \mu,\, \Phi_{g_\star} V_m \,\rangle \,, \tag{2.155}$$

for $\mu \in K_m^\star$. Now, as the following proposition shows, under some assumptions, such action is strongly Hamiltonian on $(\tilde{\mathcal{M}},\, \tilde{\omega})$, i.e., it is possible to define an equivariant (w.r.t. $\mathcal{G}$) momentum map associated to it.

Proposition 2.3.5. *Assume the action of $\mathcal{G}$ upon $\mathcal{M}$ is quasi-free, that is, the map:*

$$\mathfrak{i}_\mathfrak{g} \;:\; \mathfrak{g} \to \mathbf{T}_m\mathcal{M} \;:\; \xi \mapsto X_\xi \,, \tag{2.156}$$

is injective. Consider the subbundle of $\mathbf{T}\mathcal{M}$ whose fibres are $\mathrm{Im}\,\mathfrak{i}_\mathfrak{g} \cap K_m$, call it $\mathbf{K}_{GAUGE} \to \mathcal{M}$. Consider a connection P on such bundle. Consider, as above, the pull-back of P to $\tilde{\mathcal{M}}$, via $\tau \,:\, \tilde{\mathcal{M}} \to \mathcal{M}$, say $\tilde{P}$. The operator $\tilde{P}$ is, at each point, a projector onto K_m, since it is the pull-back of a projector onto K_m. Therefore, its action upon a vector field on $\tilde{\mathcal{M}}$ gives its component along K. Then, the momentum map $\tilde{\mathbb{J}}$ defined by:

$$\langle\, \tilde{\mathbb{J}}(\tilde{m}),\, \xi \,\rangle \,=\, \langle\, \mathbb{J}(m),\, \xi \,\rangle + \langle\, \mu,\, \tilde{P}(\tilde{X}_\xi) \,\rangle \,, \tag{2.157}$$

for any $\mathbb{J}$ being an equivariant momentum map for $\mathcal{G}$ acting on $\mathcal{M}$, is an equivariant momentum map for the action of $\mathcal{G}$ lifted to $\tilde{\mathcal{M}}$.

Proof. Denote by $\tilde{X}_\xi$ the lift of X_ξ to $\tilde{\mathcal{M}}$, i.e., the generator of (each one-parameters group of) the action of $\mathcal{G}$ lifted to $\tilde{\mathcal{M}}$ and by $\tilde{X}_{\xi_\mu}$ and X_ξ^v the components of $\tilde{X}_\xi$

along $K^\star$ and K respectively. That (2.157) is a momentum map for $\tilde{X}_\xi$ w.r.t. $\tilde{\omega}$ is due to the following equalities:

$$i_{\tilde{X}_\xi}\tilde{\omega} \,=\, \underbrace{i_{X_\xi}\omega}_{=\mathrm{d}\langle\,\mathbb{J}(m),\,\xi\,\rangle}\,(\,\cdot\,) + \underbrace{\langle\tilde{X}_{\xi_\mu},\,\cdot\,\rangle - \langle\,\cdot\,,\,X^v_\xi\,\rangle}_{=\mathrm{d}\langle\,\mu,\,X^v_\xi\,\rangle}\,, \tag{2.158}$$

where:

$$\mathrm{d}\langle\,\mu,\,X^v\,\rangle \,=\, \mathrm{d}\langle\,\mu,\,\tilde{P}(\tilde{X}_\xi)\,\rangle\,. \tag{2.159}$$

That $\tilde{\mathbb{J}}$ is equivariant is a consequence of the fact that $\mathbb{J}$ and P are equivariant by assumption. □

Note that, differently from the correspondence between X_ξ and $\mathbb{J}$, the correspondence between $\tilde{X}_\xi$ and $\tilde{\mathbb{J}}$ is one-to-one since $\tilde{\omega}$ is strongly symplectic. Now, as we said above, the momentum map (2.157) can be pulled-back to the original pre-symplectic manifold $(\mathcal{M},\,,\omega)$ by setting $\mu = 0$ obtaining $J_\xi \,=\, \langle\,\mathbb{J}(m),\,\xi\,\rangle$, which is a constant function along the flow of X_H since $\mathbb{J}$ is a momentum map for the strongly Hamiltonian action of $\mathcal{G}$ upon $(\mathcal{M},\,\omega,\,H)$.

2.4. Jet bundle formalism

This section is devoted to discuss first order prolongations of fibre bundles and how to construct their (affine) dual bundles.

We refer to [27], [72]–[74] and references therein, for more details and for some proofs.

2.4.1. Jet prolongations of fibre bundles

Consider a fibre bundle $\pi\ :\ \mathbb{E} \to \mathscr{M}$. Here, $\mathscr{M}$ is an n-dimensional smooth differential manifold where we denote by $\{\,x^\mu\,\}_{\mu=0,\ldots,n-1}$ a system of local coordinates on $\mathscr{M}$. We denote by $\mathcal{E}$ the typical fibre of π and by $\{\,x^\mu,\,u^a\,\}_{\mu=0,\ldots,n-1,\,a=1,\ldots,r}$ a system of fibered coordinates on $\mathbb{E}$. Sections of π are denoted by ϕ. The space of (local, in general) sections of π is denoted by $\Gamma(\pi)$. At each point m of $\mathscr{M}$ it can be defined an equivalence class of local sections, denoted by $j^1_m\phi$ made by all local sections of π whose Taylor expansions at m coincide up to first order:

$$\phi_1,\,\phi_2 \in j^1_m\phi \quad\Longleftrightarrow\quad \phi_1(m) \,=\, \phi_2(m) \;\text{ and }\; \left.\frac{\partial\,(u^a\circ\phi_1)}{\partial x^\mu}\right|_m \,=\, \left.\frac{\partial\,(u^a\circ\phi_2)}{\partial x^\mu}\right|_m\,. \tag{2.160}$$

The following facts hold[72, Sect. 4.1].

Proposition 2.4.1. *The set:*

$$\mathbf{J}^1\pi \,=\, \left\{\,j^1_m\phi\ :\ m\in\mathcal{M}\,,\ \phi\in\Gamma(\pi)\,\right\}\,, \tag{2.161}$$

is a $n + r + nr$ smooth differential manifold provided with the induced coordinate system

$$\left\{ x^\mu ,\, u^a ,\, z^a_\mu \right\}_{\mu=0,\dots,n-1,\, a=1,\dots,r} \tag{2.162}$$

where z^a_μ are defined by:

$$z^a_\mu(j^1_m\phi) \;=\; \left.\frac{\partial\left(u^a\circ\phi\right)}{\partial x^\mu}\right|_m . \tag{2.163}$$

Definition 2.4.2 (First order jet manifold)**.** *The manifold $\mathbf{J}^1\pi$ is called* FIRST JET MANIFOLD OF *π.*

Proposition 2.4.3. *$\pi_1 \;:\; \mathbf{J}^1\pi \to \mathscr{M}$ is a fibre bundle, where the projection π_1 reads:*

$$\pi_1 \;:\; \mathbf{J}^1\pi \to \mathscr{M} \;:\; j^1_m\phi \mapsto m\,. \tag{2.164}$$

Proposition 2.4.4. *$\pi^1_0 \;:\; \mathbf{J}^1\pi \to \mathbb{E}$ is an affine fibre bundle modelled over the vector bundle $\tau \;:\; \pi^\star\left(\mathbf{T}^\star\mathscr{M}\right)\otimes\mathbf{V}\pi \to \mathbb{E}$ where $\tau = \left.\tau^\star_{\mathbb{E}}\right|_{\pi^\star(\mathbf{T}^\star\mathscr{M})}\otimes\left.\tau_{\mathbb{E}}\right|_{\mathbf{V}\pi}$, with $\tau_{\mathbb{E}} \;:\; \mathbf{T}\mathbb{E}\to\mathbb{E}$ and $\tau^\star_{\mathbb{E}} \;:\; \mathbf{T}^\star\mathbb{E}\to\mathbb{E}$, and where:*

$$\pi^1_0 \;:\; \mathbf{J}^1\pi \to \mathbb{E} \;:\; j^1_m\phi \mapsto \phi(m)\,. \tag{2.165}$$

Definition 2.4.5 (First order jet bundle)**.** *The bundle $\pi_1 \;:\; \mathbf{J}^1\pi \to \mathscr{M}$ is called* FIRST ORDER JET BUNDLE *of the bundle $\pi \;:\; \mathbb{E}\to\mathscr{M}$.*

Given a section ϕ of π, it is canonically defined a particular section of π_1, called the FIRST ORDER JET PROLONGATION of ϕ, say $j^1\phi$.

Definition 2.4.6 (First order jet prolongation of a section)**.** *Given a local section ϕ of π, its first order jet prolongation is the unique local section of π_1, say $j^1\phi$, such that:*

$$j^1\phi \;:\; \mathcal{M} \to \mathbf{J}^1\pi \;:\; m \mapsto j^1_m\phi\,, \tag{2.166}$$

which, in local coordinates, reads the local section of π_1 satisfying:

$$\pi^1_0\left(j^1\phi(m)\right) \;=\; \phi(m)\;\;\forall\, m\in U_{\mathscr{M}} \;\; and \;\; \left(z^a_\mu\circ j^1\phi(m)\right) \;=\; \left.\frac{\partial\left(u^a\circ j^1\phi\right)}{\partial x^\mu}\right|_m . \tag{2.167}$$

The space of first order jet prolongations of sections of π is denoted by $\mathbf{J}^1\Gamma(\pi)$.

Related to $\mathbf{J}^1\Gamma(\pi)$, there exists a subset of $\Omega^1(\mathbf{J}^1\pi)$ made by differential 1-forms, say η, for which elements in $\mathbf{J}^1\Gamma(\pi)$ are "critical", that is:

$$\left(j^1\phi\right)^\star\eta \;=\; 0 \;\;\forall\,\phi\in\Gamma(\pi)\,. \tag{2.168}$$

We will denote by $\Omega^1_1(\mathbf{J}^1\pi)$ such a subset of $\Omega^1(\mathbf{J}^1\pi)$ and we call them 1-contact 1-forms. In general, n-forms satisfying the property above, are called 1-contact n-forms and lie into a subset of the exterior algebra of $\mathbf{J}^1\pi$ denoted by $\Omega^n_1(\mathbf{J}^1\pi)$. The set $\Omega^n_1(\mathbf{J}^1\pi)$ defines a bilateral ideal of the exterior algebra with respect to the wedge product, since $\rho \wedge \eta \in \Omega^{n+k}_1(\mathbf{J}^1\pi)$ if $\rho \in \Omega^n_1(\mathbf{J}^1\pi)$ and $\eta \in \Omega^k_1(\mathbf{J}^1\pi)$. Such ideal is usually called the CONTACT IDEAL of $\mathbf{J}^1\pi$ [75]. Annihilators of the contact ideal generate a distribution (a non involutive one) called CARTAN DISTRIBUTION, generated by the vector fields:

$$\frac{\partial}{\partial x^\mu} + z^a_\mu \frac{\partial}{\partial u^a}\,, \quad \text{and} \quad \frac{\partial}{\partial z^a_\mu}\,. \tag{2.169}$$

Given a π-projectable vector field X along $\mathbb{E}$, there is a canonical way of defining a prolongation to a vector field along $\mathbf{J}^1\pi$, called the FIRST ORDER JET PROLONGATION of X and denoted by X^1.

Definition 2.4.7 (FIRST ORDER JET PROLONGATION). *Given a π-projectable vector field X along $\mathbb{E}$, i.e. a vector field X such that $\pi_\star X$ is a vector field along $\mathscr{M}$, say:*

$$X \;=\; X^\mu_x(x)\frac{\partial}{\partial x^\mu} + X^a_u(x,\,u)\frac{\partial}{\partial u^a}\,, \tag{2.170}$$

then its FIRST ORDER JET PROLONGATION X^1 is:

$$X^1 \;=\; X^\mu_x(x)\frac{\partial}{\partial x^\mu} + X^a_u(x,\,u)\frac{\partial}{\partial u^a} + (\,d_\mu - z^a_\nu d_\mu X^\nu_x\,)\,\frac{\partial}{\partial z^a_\mu}\,, \tag{2.171}$$

where:

$$d_\mu \;:=\; \frac{\partial}{\partial x^\mu} + z^a_\mu \frac{\partial}{\partial u^a}\,. \tag{2.172}$$

A detailed exposition about the construction of X^1 can be found in [72], [74] and we recall it in App. 2.A.

2.4.2. Affine dual of jet bundles and the Covariant Phase Space

Now, we recall the construction of the extended and reduced duals of the first order jet bundle we mentioned. We refer to [37], [72], [73] for a more extensive discussion and for some proofs.

Being $\pi^1_0 \;:\; \mathbf{J}^1\pi \to \mathbb{E}$ an affine bundle, its dual bundle can be defined by considering the affine dual of the typical fibre of π^1_0. Indeed, the typical fibre of π^1_0 over the point $e \in \mathbb{E}$ is an affine space modelled over the vector space $\mathcal{V} = \mathbf{T}^\star_m\mathscr{M} \otimes \mathbf{V}_e\pi$ whose elements can be written in terms of the basis $\{\,\mathrm{d}x^\mu\,\}_{\mu=0,\dots,n-1}$ and $\left\{\,\frac{\partial}{\partial u^a}\,\right\}_{a=1,\dots,r}$ of $\mathbf{T}^\star_m\mathscr{M}$ and $\mathbf{V}_e\pi$ as:

$$z^a_\mu \mathrm{d}x^\mu \otimes \frac{\partial}{\partial u^a}\,. \tag{2.173}$$

The affine dual of $\mathcal{V}$, say $\mathcal{V}^\dagger$, is the space of affine maps over $\mathcal{V}$, say:

$$\mathcal{V}^\dagger \;=\; \left\{\,\varrho \;:\; \mathcal{V} \to \mathbb{R} \;:\; v^a_\mu \mapsto \rho^\mu_a z^a_\mu + \rho_0\,\right\}\,. \tag{2.174}$$

It is isomorphic to the space of 1-semibasic[6] (skew-symmetric) n-covectors at $e \in \mathbb{E}$, say $\Omega^n_{1\,e}(\mathbb{E}) \simeq \mathcal{V}^\dagger$ whose generic element can, indeed, be written as:

$$\boldsymbol{w} \,=\, \rho^\mu_a \mathrm{d}u^a \wedge i_{\partial_\mu} \mathrm{d}^n x + \rho_0 \mathrm{d}^n x\,, \tag{2.175}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\mathrm{d}^n x \,=\, \mathrm{d}x^0 \wedge ... \wedge \mathrm{d}x^{n-1}$.

Definition 2.4.8 (Extended dual)**.** *The bundle over $\mathbb{E}$ obtained by replacing the typical fibre $\mathcal{V}$ of π^1_0 with its affine dual $\mathcal{V}^\dagger$ is called* EXTENDED DUAL BUNDLE *of $\pi^1_0 \;:\; \mathbf{J}^1\pi \to \mathbb{E}$ and is denoted by $\tau^1_0 \;:\; \mathbf{J}^\dagger\pi \to \mathbb{E}$.*

Proposition 2.4.9. *$\tau^1_0 \;:\; \mathbf{J}^\dagger\pi \to \mathbb{E}$ is a vector bundle, where the projection map reads:*

$$\tau^1_0 \;:\; \mathbf{J}^\dagger\pi \to \mathbb{E} \;:\; (x^\mu,\, u^a,\, \rho^\mu_a,\, \rho_0) \mapsto (x^\mu,\, u^a)\,. \tag{2.176}$$

Proposition 2.4.10. *$\tau_1 \;:\; \mathbf{J}^\dagger\pi \to \mathscr{M}$ is a fibre bundle where the projection map reads $\tau_1 := \pi \circ \tau^1_0$.*

We denote by $\{\, x^\mu,\, u^a,\, \rho^\mu_a,\, \rho_0 \,\}_{\mu=0,...,n-1,\, a=1,...,r}$ a system of coordinates on the total space $\mathbf{J}^\dagger\pi$ of the bundles just defined.

Note that the following short exact sequence of vector bundles over $\mathbb{E}$ exists:

$$0 \longrightarrow \langle\, \mathrm{d}^n x \,\rangle \longrightarrow \mathbf{J}^\dagger\pi \longrightarrow \mathcal{V}^\star \longrightarrow 0 \tag{2.177}$$

where $\mathcal{V}^\star$ is the vector bundle whose fibres are parametrized by ρ^μ_a.

Proposition 2.4.11 (Reduced dual)**.** *The quotient of $\mathbf{J}^\dagger\pi$ with respect to $\langle\, \mathrm{d}^n x \,\rangle$ is a fibre bundle over $\mathbb{E}$ which is called* REDUCED DUAL BUNDLE *of $\pi^1_0 \;:\; \mathbf{J}^1\pi \to \mathbb{E}$ and is denoted by $\delta^1_0 \;:\; \mathbf{J}^\star\pi \to \mathbb{E}$, where:*

$$\delta^1_0 \;:\; \mathbf{J}^\star\pi \to \mathbb{E} \;:\; (x^\mu,\, u^a,\, \rho^\mu_a) \mapsto (x^\mu,\, u^a)\,. \tag{2.178}$$

We will often denote it by $\mathcal{P}(\mathbb{E})$ and refer to it as the COVARIANT PHASE SPACE[7]*.*

We denote by $\{\, x^\mu,\, u^a,\, \rho^\mu_a \,\}_{\mu=0,...,n-1,\, a=1,...,r}$ a system of coordinates on the total space, $\mathbf{J}^\star\pi$, of the bundles just defined.

The fibre bundles constructed so far fit in the following diagram:

$$\begin{tikzcd} \mathbf{J}^\dagger\pi \arrow[r, "\tau^1_0"] \arrow[dr, "\tau_1"'] \arrow[rr, bend left, "\kappa"] & \mathbb{E} \arrow[d, "\pi"] & \mathbf{J}^\star\pi \arrow[l, "\delta^1_0"'] \arrow[dl, "\delta_1"] \\ & \mathscr{M} & \end{tikzcd} \tag{2.179}$$

[6]It means that they vanish when contracted along two π-vertical tangent vectors.

[7]Note that this terminology is not standard, since often people refer to the covariant phase space as the space of solutions of the equations of the motion whereas we are referring to it as the carrier space on which we will describe the evolution of our dynamical system.

κ being the projection associated with the quotient of $\mathbf{J}^{\dagger}\pi$ with respect to $\mathbb{R}$.

Given a section ϕ of the bundle $\pi\ :\ \mathbb{E} \to \mathscr{M}$ the analogue for δ_1 of the prolongation $j^1\phi$ (Def. 2.4.6) is not canonically defined. However, sections of δ_1 split into a section of π (that we will denote by ϕ) and a section of δ_0^1 (that we will denote by P), in the sense of the following proposition.

Proposition 2.4.12. *Given a section of δ_1, say χ, and given the section $\phi := \delta_0^1 \circ \chi$ of π, there exists a (not unique) section P of δ_0^1 such that $\chi = P \circ \phi$.*

The space we will interested in throughout the manuscript and where we will settle the variational calculus is the space of pairs $(\phi,\, P)$ of sections of π and δ_0^1. As we will see in Chap. 3 we will get on such space the same set of equations of motion obtained in the literature settling the variational calculus on $\Gamma(\delta_1)$ [73]. However, we choose to work with the space of pairs $(\phi,\, P)$ since it will allow, for instance, to consider constraints involving the space of momenta P (as we will do in Sec. 3.2.3) and this could not be done by working with the space $\Gamma(\delta_1)$ since, having in mind the latter proposition, elements of $\Gamma(\delta_1)$ do not uniquely define momenta P. We will denote such a space of pairs by $\Gamma^{\text{SPLIT}}(\delta_1)$ and, with a slight abuse of notation, we will still denote them by $\chi := (\phi,\, P)$, where, in the system of local coordinates adopted $u^a \circ \phi =: \phi^a$ and $\rho_a^\mu \circ P =: P_a^\mu$.

APPENDIX

2.A. First order jet prolongation of vector fields

This appendix is organized as follows. First, we recall how to prolong vertical vector fields over a fiber bundle[8] to its first order jet bundle. Then, we recall how to prolong generic vector fields from a fiber bundle to its first order jet bundle. Both procedures will be presented in two alternative ways. The first one is the most intuitive and deals with prolongations of fibered morphisms. The second one, dealing with the construction of a particular canonical bijection (surjection) of fiber bundles, even if less intuitive and more involved, is necessary since it can be applied to generic vector fields over a fiber bundle differently from the first one which only works with vertical or projectable vector fields. For the sake of simplicity of notations, we will restrict ourselves to a fibre bundle $\pi \ : \ \mathbb{E} \to \mathscr{M}$ in which $\mathscr{M}$ is an interval of the real line that we will denote by $\mathbb{I}$, even if all the constructions we make can be analogously reproduced in the case where $\mathscr{M}$ is any finite dimensional smooth manifold.

2.A.1. Prolongation of vertical vector fields

Prolongation of fibered morphisms

Definition 2.A.1 (Fibered morphism). *Consider two fibre bundles $\pi_1 \ : \ \mathcal{V}_1 \to \mathcal{M}_1$ and $\pi_2 \ : \ \mathcal{V}_2 \to \mathcal{M}_2$. A* fibered morphism *of the bundles π_1 and π_2 is a smooth map Φ from $\mathcal{V}_1$ to $\mathcal{V}_2$, such that there exists a smooth map $\bar{\Phi}$ from $\mathcal{M}_1$ to $\mathcal{M}_2$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{V}_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"] & \mathcal{V}_2 \arrow[d, "\pi_2"] \\ \mathcal{M}_1 \arrow[r, "\overline{\Phi}"] & \mathcal{M}_2 \end{tikzcd} \tag{2.180}$$

When the fibre bundles π_1 and π_2 coincide the morphisms is said to be a fibered automorphism*.*

Consider a fibered morphism of π over a diffeomorphism, i.e., such that $\overline{\Phi}$ is a diffeomorphism. Consider a section of π_1, say ϕ. Then, a section of π_2, say ϕ_Φ, is defined as the one making the following diagram commutative:

$$\begin{tikzcd} \mathcal{V}_1 \arrow[r, "\Phi"] \arrow[d, "\pi"] & \mathcal{V}_2 \arrow[d, "\pi"'] \\ \mathcal{M}_1 \arrow[u, dashed, bend left, "\phi"] & \mathcal{M}_2 \arrow[l, "\overline{\Phi}^{-1}"] \arrow[u, dashed, bend right, "\phi_\Phi"'] \end{tikzcd} \tag{2.181}$$

[8] Actually the construction is valid for generic fibered manifolds.

that is, $\Phi \circ \phi \circ \overline{\Phi}^{-1} = \phi_\Phi$. Note that $\overline{\Phi}^{-1}$ exists by the assumption that $\overline{\Phi}$ is a diffeomorphism. The first order jet prolongations of ϕ and ϕ_Φ can be considered:

$$
\begin{tikzcd}
\mathbf{J}^1\pi_1 \arrow[d, "{\pi_1}^1_0"] & \mathbf{J}^1\pi_2 \arrow[d, "{\pi_2}^1_0"] \\
\mathcal{V}_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"] & \mathcal{V}_2 \arrow[d, "\pi_2"] \\
\mathcal{M}_1 \arrow[u, dashed, bend left, "\phi"] \arrow[uu, dashed, bend left=60, "j^1\phi"] \arrow[r, shift right, "\overline{\Phi}"'] & \mathcal{M}_2 \arrow[u, dashed, bend right, "\phi_\Phi"'] \arrow[uu, dashed, bend right=60, "j^1\phi_\Phi"'] \arrow[l, shift right, "\overline{\Phi}^{-1}"']
\end{tikzcd}
\tag{2.182}
$$

Definition 2.A.2 (First order jet prolongation of an automorphism)**.** *The* FIRST ORDER JET PROLONGATION OF THE MORPHISM Φ *is defined to be the map $j^1\Phi$ that makes the following diagram commutative:*

$$
\begin{tikzcd}
\mathbf{J}^1\pi_1 \arrow[r, "j^1\Phi"] \arrow[d, "{\pi_1}^1_0"] \arrow[dd, bend right=50, "\pi_{11}"'] & \mathbf{J}^1\pi_2 \arrow[d, "{\pi_2}^1_0"] \arrow[dd, bend left=50, "\pi_{21}"] \\
\mathcal{V}_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"] & \mathcal{V}_2 \arrow[d, "\pi_2"] \\
\mathcal{M}_1 \arrow[u, dashed, bend left, "\phi"] \arrow[uu, dashed, bend left=80, "j^1\phi"] \arrow[r, shift right, "\overline{\Phi}"'] & \mathcal{M}_2 \arrow[u, dashed, bend right, "\phi_\Phi"'] \arrow[uu, dashed, bend right=80, "j^1\phi_\Phi"'] \arrow[l, shift right, "\overline{\Phi}^{-1}"']
\end{tikzcd}
\tag{2.183}
$$

that is, the one satisfying:

$$
\Phi \circ {\pi_1}^1_0 = {\pi_2}^1_0 \circ j^1\Phi \,, \qquad \overline{\Phi} \circ \pi_{11} = \pi_{21} \circ j^1\Phi \,, \tag{2.184}
$$

together with:

$$
j^1\phi_\Phi = j^1\Phi \circ j^1\phi \circ \overline{\Phi}^{-1} \,. \tag{2.185}
$$

Prolongation of vertical vector fields via prolongation of morphisms

Consider a fibre bundle $\pi : \mathbb{E} \to \mathbb{I}$, where $\mathbb{I}$ is an interval of the real line[9].

Let us consider a π-vertical vector field X along $\mathbb{E}$, that is, a vector field X along $\mathbb{E}$ such that $\pi_\star X = 0$. Denote by F_s^X its local flow. Since X projects via π onto the null vector field, F_s^X is a particular fibered automorphism of π over a diffeomorphism, namely, an automorphism over the identity:

$$
\begin{tikzcd}
\mathbb{E} \arrow[rr, "F_s^X"] \arrow[dr, "\pi"'] & & \mathbb{E} \arrow[dl, "\pi"] \\
& \mathbb{I} &
\end{tikzcd}
\tag{2.186}
$$

[9]As we already said in Sect. 2.4.1, this requirement is not necessary for most of the forthcoming definitions and constructions and we will keep it only for the sake of notational convenience.

Such automorphism can be prolonged to an automorphism $j^1F_s^X$ of $\mathbf{J}^1\pi$ as explained in the previous paragraph:

$$
\begin{tikzcd}
\mathbf{J}^1\pi \arrow[rr, "j^1F_s^X"] \arrow[d, "\pi_0^1"] & & \mathbf{J}^1\pi \arrow[d, "\pi_0^1"] \\
\mathbb{E} \arrow[rr, "F_s^X"] \arrow[dr, "\pi"'] & & \mathbb{E} \arrow[dl, "\pi"] \\
& \mathbb{I} &
\end{tikzcd}
\tag{2.187}
$$

Then, the prolongation of X is defined to be the (unique) vector field X^1 on $\mathbf{J}^1\pi$ whose local flow is $j^1F_s^X$. Consider local coordinates of the type $\{\, t,\, q^j \,\}_{j=1,\ldots,n}$ on $\mathbb{E}$ where $\{\, t \,\}$ represents local coordinates on $\mathbb{I}$ and the induced local coordinate on $\mathbf{J}^1\pi$ $\left\{\, t,\, q^j,\, q^j_t \,\right\}_{j=1,\ldots,n}$. A direct calculation shows that, given a π-vertical vector field X, whose coordinate expression is:

$$
X \;=\; X^j_q(t,q)\frac{\partial}{\partial q^j}\,,
\tag{2.188}
$$

its prolongation reads:

$$
X^1 \;=\; X^j_q(t,q)\frac{\partial}{\partial q^j} + d_t X^j_q(t,q)\frac{\partial}{\partial q^j_t}\,,
\tag{2.189}
$$

where

$$
d_t \;:=\; \frac{\partial}{\partial t} + v^j\frac{\partial}{\partial q^j}\,.
\tag{2.190}
$$

Prolongation of vertical vector fields via canonical isomorphism

Consider the bundles:

$$
\begin{tikzcd}
\mathbf{J}^1\pi \arrow[d, "\pi_1"] & \mathbf{V}\pi \arrow[d, "\nu_\pi"] \\
\mathbb{I} & \mathbb{I}
\end{tikzcd}
\tag{2.191}
$$

As a matter of fact the bundles $\mathbf{J}^1\nu_\pi$ and $\mathbf{V}\mathbf{J}^1\pi$ are isomorphic (see [72] Theorem 4.4.1 at page 125), the isomorphism denoted by i_1. An easy way to convince yourself about this fact is to use local coordinates (see [72] for the coordinate-free proof). Denote by $\{\, t, q^j \,\}_{j=1,\ldots,n}$ local coordinates on $\mathbb{E}$. Adapted coordinates on $\mathbf{J}^1\pi$ can be denoted by $\left\{\, t, q^j, q^j_t \,\right\}_{j=1,\ldots,n}$ and on $\mathbf{V}\pi_1$ by $\left\{\, t, q^j, q^j_t, v^j_q, v^j_{(q_t)} \,\right\}_{j=1,\ldots,n}$. On the other hand, on $\mathbf{V}\pi$ local coordinates can be denoted by $\left\{\, t, q^j, v^j_q \,\right\}_{j=1,\ldots,n}$ and, consequently, on $\mathbf{J}^1\nu_\pi$ by $\left\{\, t, q^j, v^j_q, q^j_t, (v_q)^j_t \right\}_{j=1,\ldots,n}$. Therefore, the isomorphism i_1 is the linear map that projects onto the identity on $\mathbb{I}$, switches q^j_t and v^j_q and that identifies $v^j_{(q_t)}$ and $(v_q)^j_t$.

This isomorphism can be used to prolong a π-vertical vector field on $\mathbb{E}$ to a π_1-vertical vector field over $\mathbf{J}^1\pi$. Indeed, a π-vertical vector field on $\mathbb{E}$ is a section, say X, of the following bundle:

$$
\begin{tikzcd}
\mathbb{E} \arrow[dr, "\pi"'] \arrow[rr, dashed, bend right, "X"] & & \mathbf{V}\pi \arrow[ll, "\nu"'] \arrow[dl, "\nu_\pi"] \\
& \mathbb{I} &
\end{tikzcd}
\tag{2.192}
$$

As a bundle morphism over the identity, it can be prolonged to a bundle morphism j^1X:

$$
\begin{tikzcd}
\mathbf{J}^1\pi \arrow[d, "\pi^1_0"'] \arrow[rr, dashed, "j^1X"'] \arrow[ddr, bend right=60, "\pi_1"'] & & \mathbf{J}^1\nu_\pi \arrow[d, "\nu_\pi{}^1_0"] \arrow[ddl, bend left=60, "\nu_{\pi 1}"] \\
\mathbb{E} \arrow[dr, "\pi"'] \arrow[rr, dashed, bend right, "X"] & & \mathbf{V}\mathbb{E} \arrow[ll, "\nu"'] \arrow[dl, "\nu_\pi"] \\
& \mathbb{I} &
\end{tikzcd}
\tag{2.193}
$$

j^1X is not yet a (π_1-vertical) vector field over $\mathbf{J}^1\pi$ since it is not a section of $\mathbf{V}\mathbf{J}^1\pi \to \mathbf{J}^1\pi$. However, its image via the isomorphism i_1 is, as it is proven in [72] (section 4.4).

$$
\begin{tikzcd}
& \mathbf{V}\pi_1 \arrow[dl, "\nu_{\pi_1}"] \arrow[dr, shift left, "i_1^{-1}"] & \\
\mathbf{J}^1\pi \arrow[ur, dashed, bend left, "X^1"] \arrow[d, "\pi^1_0"'] \arrow[rr, dashed, "j^1X"'] \arrow[ddr, bend right=60, "\pi_1"'] & & \mathbf{J}^1\nu_\pi \arrow[ul, shift left, "i_1"] \arrow[d, "\nu_\pi{}^1_0"] \arrow[ddl, bend left=60, "\nu_{\pi 1}"] \\
\mathbb{E} \arrow[dr, "\pi"'] \arrow[rr, dashed, bend right, "X"] & & \mathbf{V}\mathbb{E} \arrow[ll, "\nu"'] \arrow[dl, "\nu_\pi"] \\
& \mathbb{I} &
\end{tikzcd}
\tag{2.194}
$$

Again, an easy way to convince yourself about this fact is by using local coordinates. Consider a vertical vector field over $\mathbb{E}$, say:

$$
X \;=\; X^j_q(t,q)\frac{\partial}{\partial q^j}\,. \tag{2.195}
$$

As a bundle morphism between π and ν_π it reads:

$$
X \;\;:\;\; \mathbb{E} \to \mathbf{V}\pi \;\;:\;\; (t, q^j) \mapsto (t, q^j, X^j_q(t, \mathbf{q}))\,. \tag{2.196}
$$

The first order jet prolongation of such a morphism is readily computed by using the definition (2.183):

$$
j^1X \;\;:\;\; \mathbf{J}^1\pi \to \mathbf{J}^1\nu_\pi \;\;:\;\; (t,\, q^j,\, q^j_t) \mapsto \left(t,\, q^j,\, X^j_q(t,q),\, q^j_t,\, \frac{d}{dt}X^j_q(t,q)\right)\,. \tag{2.197}
$$

Finally:

$$
X^1 := i_1 \circ j^1X \;\;:\;\; \mathbf{J}^1\pi \to \mathbf{V}\pi_1 \;\;:\;\; (t,\, q^j,\, q^j_t) \mapsto \big(t,\, q^j,\, q^j_t,\, X^j_q(t,q),\, \tfrac{d}{dt}X^j_q(t,q)\big)\,. \tag{2.198}
$$

Thus:

$$X^1 \;=\; X_q^j(t,q)\frac{\partial}{\partial q^j} + \frac{d}{dt}X_q^j(t,q)\frac{\partial}{\partial q_t^j}\,. \tag{2.199}$$

2.A.2. Prolongation of generic vector fields

Prolongation of projectable vector fields via prolongation of morphisms

If the vector field we want to prolong is a π-projectable one, the procedure described for vertical ones which makes use of the concept of prolongation of morphism must be slightly modified. Indeed, consider a π-projectable vector field on $\mathbb{E}$, say X, and denote its projection to $\mathbb{I}$ by $\overline{X}$. Denote the local flow of X by F_s^X. Provided with the assumption of π-projectability, it is actually a bundle morphism over a diffeomorphism on $\mathbb{I}$, namely, the local flow of $\overline{X}$, say $F_s^{\overline{X}}$. Then, the first order jet prolongation of the morphism F_s^X is the one closing the diagram:

$$\begin{tikzcd}
U_1 \arrow[r, "j^1F_s^X"] \arrow[d, "\pi_0^1"] \arrow[dd, bend right=60, "\pi_1"'] & \tilde{U}_1 \arrow[d, "\pi_0^1"] \arrow[dd, bend left=60, "\pi_1"] \\
U \arrow[r, "F_s^X"] \arrow[d, "\pi"] & \tilde{U} \arrow[d, "\pi"] \\
U_0 \arrow[u, dashed, bend left, "\phi"] \arrow[uu, dashed, bend left=90, "j^1\phi"] \arrow[r, shift right, "F_s^{\overline{X}}"'] & \tilde{U}_0 \arrow[l, shift right, "F_s^{\overline{X}\,-1}"'] \arrow[u, dashed, bend right, "\phi_s"'] \arrow[uu, dashed, bend right=90, "j^1\phi_s"']
\end{tikzcd} \tag{2.200}$$

where U_0 and U_1 are open sets of $\mathbb{I}$ and $\mathbb{E}$ on which $F_s^{\overline{X}}$ and F_s^X are, respectively, defined, U^1 is the image of U_0 under $j^1\phi$ and $\tilde{U}_0$, $\tilde{U}$ and $\tilde{U}_1$ are the images of U_0, U and U_1 under $F_s^{\overline{X}}$, F_s^X and $j^1F_s^X$ respectively. Consequently, the first order jet prolongation of the vector field X can be defined as the unique vector field on $\mathbf{J}^1\pi$ whose local flow is the $j^1F_s^X$ defined above. Its coordinate form is seen to be (see [27] page 32 or [74] section 1.7):

$$X^1 \;=\; X_t\frac{\partial}{\partial t} + X_q^j\frac{\partial}{\partial q^j} + \left(\frac{d}{dt}X_q^j - q_t^j\,\frac{d}{dt}X_t\right)\frac{\partial}{\partial q_t^j}\,. \tag{2.201}$$

Prolongation of vector fields via canonical surjection

For generic vector fields on $\mathbb{E}$ the lifting procedure described for vertical ones that makes use of the isomorphism i_1 does not work. Indeed, in this case the vector field X is a section of the tangent bundle $\mathbf{T}\mathbb{E}$. The jet prolongation of X seen as a fibered morphism is a morphism between the $\mathbf{J}^1\pi$ and $\mathbf{J}^1\tau_\pi$ appearing in the following diagram:

$$
\begin{tikzcd}
\mathbf{J}^1\pi \arrow[rr, dashed, "j^1X"'] \arrow[d, "\pi^1_0"] & & \mathbf{J}^1\tau_\pi \arrow[d, "\tau_{\pi}{}^1_0"] \\
\mathbb{E} \arrow[rr, dashed, bend right, "X"'] \arrow[dr, "\pi"'] & & \mathbf{T}\mathbb{E} \arrow[ll, "\tau"'] \arrow[dl, "\tau_\pi"] \\
& \mathbb{M} &
\end{tikzcd}
\tag{2.202}
$$

Therefore, again, it is not actually a vector field over $\mathbf{J}^1\pi$. However, in this case an analogue of the isomorphism i_1 does not exist, simply because $\mathbf{J}^1\tau_\pi$ is not isomorphic to $\mathbf{T}\mathbf{J}^1\pi$. This can be easily seen by noting that $dim\mathbf{J}^1\tau_\pi = 3 + 4n$ (n being the dimension of the fibre of π) while $dim\mathbf{T}\mathbf{J}^1\pi = 2(1 + 2n) = 2 + 4n$. However a canonical surjection from $\mathbf{J}^1\tau_\pi$ to $\mathbf{T}\mathbf{J}^1\pi$ can be constructed in order to obtain a vector field on $\mathbf{J}^1\pi$ from j^1X.

Canonical surjection from $\mathbf{J}^1\tau_\pi$ to $\mathbf{T}\mathbf{J}^1\pi$ As it is proven in [72] (definition 4.4.7 and proposition 4.4.8 at page 132), a map r_1 can be canonically constructed which is a surjection from $\mathbf{J}^1\tau_\pi$ to $\mathbf{T}\mathbf{J}^1\pi$. Moreover, it is a bundle morphism over the identity between $\mathbf{J}^1\tau_\pi$ and $\mathbf{T}\mathbf{J}^1\pi$ seen as bundles over $\mathbf{T}\mathbb{E}$. We recall very briefly its definition. It is defined upon specifying its action over the first order jet prolongation of a generic section ψ of τ_π, say $j^1\psi$, which is a section of $\tau_{\pi 1} = \tau_\pi \circ \tau_{\pi}{}^1_0$:

$$
\begin{tikzcd}
& & \mathbf{J}^1\tau_\pi \arrow[dl, "\tau_{\pi}{}^1_0"'] \\
\mathbb{E} \arrow[d, "\pi"'] & \mathbf{T}\mathbb{E} \arrow[l, "\tau"'] \arrow[dl, "\tau_\pi"] \arrow[d, "T\pi"] & \\
\mathbb{M} \arrow[ur, dashed, bend left, "\psi"] \arrow[uurr, dashed, bend right, "j^1\psi"'] & \mathbf{T}\mathbb{M} \arrow[l, "\tau_{\mathbb{M}}"] &
\end{tikzcd}
\tag{2.203}
$$

For any fixed ψ, the sections $\psi_v = \tau \circ \psi$ and $\psi_h = T\pi \circ \psi$ of π and $\tau_{\mathbb{M}}$ are defined:

$$
\begin{tikzcd}
& & \mathbf{J}^1\tau_\pi \arrow[dl, "\tau_{\pi}{}^1_0"'] \\
\mathbb{E} \arrow[d, "\pi"'] & \mathbf{T}\mathbb{E} \arrow[l, "\tau"'] \arrow[dl, "\tau_\pi"] \arrow[d, "T\pi"] & \\
\mathbb{M} \arrow[u, dashed, bend left, "\psi_v"] \arrow[ur, dashed, bend left, "\psi"] \arrow[uurr, dashed, bend right, "j^1\psi"'] \arrow[r, dashed, bend right, "\psi_h"'] & \mathbf{T}\mathbb{M} \arrow[l, "\tau_{\mathbb{M}}"] &
\end{tikzcd}
\tag{2.204}
$$

ψ_h can be seen as the "horizontal" part of ψ, while ψ_v can be used to extract a canonical vertical part of ψ. Indeed, it is readily proved that $T\psi_v \circ \psi_h$ is again a section of τ_π and that $\psi - T\psi_v \circ \psi_h$ is $T\pi$-vertical (see [72] proposition 4.4.8 at page 132). Now, the "horizontal" part of ψ gives rise to a map onto $\mathbf{TJ}^1\pi$ upon composing it with $Tj^1\phi$:

$$
\begin{tikzcd}
\mathbf{J}^1\pi \arrow[d, "\pi^1_0"] & \mathbf{TJ}^1\pi \arrow[l] \arrow[d, "T\pi^1_0"] & \mathbf{J}^1\tau_\pi \arrow[dl, "\tau_\pi{}^1_0"'] \\
\mathbb{E} \arrow[d, "\pi"] & \mathbf{T}\mathbb{E} \arrow[l, "\tau"'] \arrow[d, "T\pi"] \arrow[dl, "\tau_\pi"] & \\
\mathbb{I} \arrow[u, dashed, bend left, "\psi_v"] \arrow[uu, dashed, bend left=60, "j^1\psi_v"] \arrow[ur, dashed, bend left, "\psi"] \arrow[uurr, dashed, bend right=20, "j^1\psi"'] \arrow[r, dashed, bend right, "\psi_h"'] & \mathbf{T}\mathbb{I} \arrow[l, "\tau_{\mathbb{I}}"] \arrow[uu, dashed, bend right=60, "Tj^1\psi_v"'] &
\end{tikzcd}
\tag{2.205}
$$

On the other hand, the vertical part of ψ gives rise to a map onto $\mathbf{TJ}^1\pi$ upon taking the first jet and composing it with the isomorphism i_1 as we did for vertical vector fields. Thus, the canonical surjection is defined via its action on a generic $j^1\psi$ in the following way

$$
r_1(j^1_r\psi) \;=\; i_1\left[j^1_r(\psi - T\psi_v \circ \psi_h)\right] + \left(Tj^1\psi_v\right)\psi_{hr}\,. \tag{2.206}
$$

r being a point in $\mathbb{I}$. Explicitly, given the section ψ:

$$
\psi \;\;:\;\; \mathbb{I} \to \mathbf{T}\mathbb{E} \;\;:\;\; t \mapsto (t,\, \phi(t),\, X_t(t),\, X_q(t)) \tag{2.207}
$$

then:

$$
\psi_h \;\;:\;\; \mathbb{I} \to \mathbf{T}\mathbb{I} \;\;:\;\; t \mapsto (t,\, X_t(t)) \tag{2.208}
$$

$$
\psi_v \;\;:\;\; \mathbb{I} \to \mathbb{E} \;\;:\;\; t \mapsto (t, \phi(t)) \tag{2.209}
$$

Consequently:

$$
j^1\psi_v \;\;:\;\; \mathbb{I} \to \mathbf{J}^1\pi \;\;:\;\; t \mapsto (t,\, \phi(t),\, \dot{\phi}(t)) \tag{2.210}
$$

$$
Tj^1\psi_v \;\;:\;\; \mathbf{T}\mathbb{I} \to \mathbf{TJ}^1\pi \;\;:\;\; (t,\, v_t) \mapsto (t,\, \phi(t),\, \dot{\phi}(t),\, v_t,\, v_t\dot{\phi}(t),\, v_t\ddot{\phi}(t)) \tag{2.211}
$$

$$
T\psi_v \;\;:\;\; \mathbf{T}\mathbb{I} \to \mathbf{T}\mathbb{E} \;\;:\;\; (t,\, v_t) \mapsto (t,\, \phi(t),\, v_t,\, v_t\dot{\phi}(t)) \tag{2.212}
$$

and, thus:

$$
T\psi_v \circ \psi_h \;\;:\;\; \mathbb{I} \to \mathbf{T}\mathbb{E} \;\;:\;\; t \mapsto (t,\, \phi(t),\, X_t(t),\, X_t(t)\dot{\phi}(t)) \tag{2.213}
$$

$$
Tj^1\psi_v \circ \psi_h \;\;:\;\; \mathbb{I} \to \mathbf{TJ}^1\pi \;\;:\;\; t \mapsto (t,\, \phi(t),\, \dot{\phi}(t),\, X_t(t),\, X_t(t)\dot{\phi}(t),\, X_t(t)\ddot{\phi}(t)) \tag{2.214}
$$

from which it is clear that $T\psi_v \circ \psi_h$ is a section of τ_π. Finally:

$$
\psi - T\psi_v \circ \psi_h \;\;:\;\; \mathbb{I} \to \mathbf{T}\mathbb{E} \;\;:\;\; t \mapsto (t,\, 0,\, 0,\, X_q(t) - X_t(t)\dot{\phi}(t)) \tag{2.215}
$$

from which it is clear that $\psi - T\psi_v \circ \psi_h$ is $T\pi$-vertical. We have:

$$j^1(\psi - T\psi_v \circ \psi_h) \;\; : \;\; \mathbb{I} \to \mathbf{J}^1\tau_\pi \;\; : \;\; t \mapsto \Big(t, 0,\, 0,\, X_q(t) - X_t(t)\dot{\phi}(t),$$
$$0,\, \frac{d}{dt}X_q(t) - \frac{d}{dt}X_t(t)\dot{\phi}(t) - X_t(t)\ddot{\phi}(t) \Big) \tag{2.216}$$

and, thus:

$$i_1\left[j^1(\psi - T\psi_v \circ \psi_h)\right] \;\; : \;\; \mathbb{I} \to \mathbf{TJ}^1\pi \;\; : \;\; t \mapsto \Big(t,\, 0,\, 0,\, 0,\, X_q(t) - X_t(t)\dot{\phi}(t),$$
$$\frac{d}{dt}X_q(t) - \frac{d}{dt}X_t(t)\dot{\phi}(t) - X_t(t)\ddot{\phi}(t) \Big) \tag{2.217}$$

Now, taking into account (2.217) and (2.214), we obtain the action of the map r_1 over the first order jet prolongation of a section ψ:

$$r_1(j^1\psi) \;\; : \;\; \mathbb{I} \to \mathbf{TJ}^1\pi \;\; : \;\; t \mapsto \left(t,\, \phi(t),\, \dot{\phi}(t),\, X_t(t),\, X_q(t),\, \frac{d}{dt}X_q(t) - \dot{\phi}(t)\frac{d}{dt}X_t(t) \right) . \tag{2.218}$$

From the latter equation, the expression of r_1 as surjection from $\mathbf{J}^1\tau_\pi$ to $\mathbf{TJ}^1\pi$ is readily obtained:

$$r_1 \;\; : \;\; \mathbf{J}^1\tau_\pi \to \mathbf{TJ}^1\pi \;\; : \;\; \left(t,\, q^j,\, v_t,\, v^j_q,\, q^j_t,\, (v_t)_t,\, (v^j_q)_t \right) \mapsto \left(t,\, q^j,\, q^j_t,\, v_t,\, v^j_q,\, (v^j_q)_t - q_t(v_t)_t \right) \tag{2.219}$$

The map r_1 can be used to prolong a generic vector field on $\mathbb{E}$ in the following way. Let X be a vector field on $\mathbb{E}$ and let ϕ be a generic section of π. Then, the first order jet prolongation of X is defined as follows:

$$X^1_{j^1_r\phi} \;:=\; r_1\left[j^1_r(X \circ \phi)\right] . \tag{2.220}$$

Explicitly, given the vector field:

$$X \;=\; X_t(t,\, q)\frac{\partial}{\partial t} + X^j_q(t,\, q)\frac{\partial}{\partial q^j} , \tag{2.221}$$

its first jet prolongation reads:

$$X^1 \;=\; X_t(t,\, q)\frac{\partial}{\partial t} + X^j_q(t,\, q)\frac{\partial}{\partial q^j} + \left[\frac{d}{dt}X^j_q - q^j_t\frac{d}{dt}X_t \right] \frac{\partial}{\partial q^j_t} . \tag{2.222}$$

3. MULTI-SYMPLECTIC FORMULATION OF CLASSICAL FIELD THEORIES

In this chapter we pose the basis for the geometrical description of dynamical systems both finite and infinite-dimensional that we will carry on in the successive chapters. In particular we will describe the multi-symplectic formulation of Mechanical systems both in the Lagrangian, addressed in 3.1.1, and in the Hamiltonian setting, addressed in 3.1.2. We also devote a section (Sec. 3.1.3) to Hamiltonian mechanical systems with additional constraints where we see how to use the Lagrange multipliers theorem within the multi-symplectic framework. Following the same path, we proceed by addressing first order[10] Lagrangian field theories, in Sec. 3.2.1, Hamiltonian field theories, addressed in 3.2.2, and Hamiltonian field theories with additional constraints, addressed in 3.2.3. In particular both in Mechanics and within field theories, we will show how multi-symplectic geometry provide a suitable framework to develop an intrinsic (geometric) variational formulation of dynamical systems.

Again we refer to the existing literature for more details and for some proofs. In particular we refer, for instance, to [27], [76] and references therein for more details about the multi-symplectic formulation of Lagrangian and Hamiltonian mechanical systems. Regarding field theories the literature is huge and we report here a, far from exhaustive, list of references where more details can be found regarding the multi-symplectic formulation of classical field theories both in the Lagrangian and in the Hamiltonian formalism [22], [28], [35]–[38], [74], [77].

3.1. A trivial example: point particle Mechanics

As we said, we start with the multi-symplectic formulation of finite dimensional mechanical systems that, as it will become clear, can be seen as a trivial example of field theories over a $0 + 1$-dimensional space-time.

3.1.1. Lagrangian formulation

It is a well established fact (see [27] and references therein) that a natural setting for the Lagrangian description of a mechanical system of the "Newtonian type" (i. e. whose evolution is described by means of second order differential equations) is the first order jet bundle of a suitable fibration. We start by assuming that to

[10]As we will see this means we will consider only theories where the Lagrangian depends on the fields of the theory and their derivatives up to the first order.

every dynamical system it is associated an n-dimensional differential manifold, say $\mathcal{Q}$, called CONFIGURATION SPACE (see [78, Chapter 3] for a detailed discussion about the construction of the configuration space starting from experimental data and about its mathematical, physical and philosophical implications; see also [79]).

Consider the following trivial fibre bundle:

$$\begin{tikzcd} \mathbb{Q} \arrow[d, "\pi"] \\ \mathbb{I} \end{tikzcd} \tag{3.1}$$

where $\mathbb{Q} = \mathbb{I} \times \mathcal{Q}$, $\mathbb{I}$ is an interval of $\mathbb{R}$ and π is the canonical projection onto the second factor. Trajectories of the dynamical system can be modelled as particular sections of π:

$$\begin{tikzcd} \mathbb{Q} \arrow[d, "\pi"] \\ \mathbb{I} \arrow[u, bend left, dashed, "\gamma"] \end{tikzcd} \tag{3.2}$$

emerging as solutions of a variational problem that we are going to describe.

Consider the first order jet bundle of π, $\mathbf{J}^1\pi$, introduced in 2.4.1:

$$\begin{tikzcd} \mathbf{J}^1\pi \arrow[d, "\pi^1_0"] \arrow[dd, bend right=60, "\pi_1"'] \\ \mathbb{Q} \arrow[d, "\pi"] \\ \mathbb{I} \end{tikzcd} \tag{3.3}$$

equipped with the system of local coordinates $\{\, t,\, q^j,\, v^j \,\}_{j=1,\dots,n}$.

Definition 3.1.1 (LAGRANGIAN). *A LAGRANGIAN is a π_1-horizontal 1-form on $\mathbf{J}^1\pi$, i. e., a differential 1-form on $\mathbf{J}^1\pi$, say λ, such that $i_V\lambda = 0 \quad \forall\, V$ being π_1-vertical. In the system of local coordinates chosen it reads:*

$$\lambda = \mathscr{L}(t, q, v)\, \mathrm{d}t\,, \tag{3.4}$$

where $\mathscr{L}$ is a smooth function on $\mathbf{J}^1\pi$.

In terms of it, an ACTION FUNCTIONAL can be defined on the space of sections of π as follows.

Definition 3.1.2 (ACTION FUNCTIONAL). *Given a Lagrangian over $\mathbf{J}^1\pi$, an ACTION FUNCTIONAL is a real-valued function on the space of sections of π, $\Gamma(\pi)$, given by:*

$$\mathscr{S} \;:\; \Gamma(\pi) \to \mathbb{R} \;:\; \gamma \mapsto \mathscr{S}_\gamma = \int_{\mathbb{I}} \left(j^1\gamma \right)^\star \lambda \tag{3.5}$$

whose coordinate expression is:

$$\mathscr{S}_\gamma = \int_{\mathbb{I}} \mathscr{L}(t,\, \gamma(t),\, \dot{\gamma}(t))\mathrm{d}t\,, \tag{3.6}$$

and where $j^1\gamma \in \Gamma(\pi_1)$ is the first order jet prolongation of the section γ defined in 2.4.1.

Remark 3.1.3. *Let us stress that two problems arise from the latter definition. First, the integral in* (3.5) *may be not well defined for all $\gamma \in \Gamma(\pi)$. Moreover, $\Gamma(\pi)$ is the space of smooth sections of a fibre bundle and, therefore, it is not a Banach manifold, in general even if it would be very useful to deal with it as a Banach manifold in order to being allowed to use the whole machinery introduced in 2.1. In order to work on a space of γs for which the action is well defined and which has the structure of a Banach manifold, the idea we will use is the following. We will consider a subset of $\Gamma(\pi)$, say $\Gamma(\pi)_{\mathbf{E}}$, of suitably regular sections, such that* (3.5) *is well defined on it, such that it is locally isomorphic to a Banach space $(\mathbf{E},\, \|\cdot\|)$ and such that $\mathscr{S}$ is continuous with respect to $\|\cdot\|$. Then, we will locally consider the completion $\overline{\Gamma(\pi)_{\mathbf{E}}}^{\|\cdot\|} =: \mathcal{F}_{\mathbb{Q}}$. The action functional is now well defined on the Banach manifold $\mathcal{F}_{\mathbb{Q}}$ since it can be extended by continuity to the completion above.*

From now on, from the abstract point of view, we will proceed by considering $\mathscr{S}$ to be directly defined on the Banach manifold $\mathcal{F}_{\mathbb{Q}}$ but the fact that a suitable Banach norm exists on $\Gamma(\pi)$, in which $\mathscr{S}$ is continuous, should be proved case by case in the examples considered.

Clearly, (3.5) is defined up to the addition of an element in the contact ideal to the Lagrangian λ. As we will see in the sequel, such an ambiguity can be fixed by asking that the first variation of $\mathscr{S}$ split into a boundary contribution plus a "bulk" contribution.

Let us first describe how to compute the first variation of $\mathscr{S}$. Consider a tangent vector to $\mathcal{F}_{\mathbb{Q}}$ at γ, say $\mathbb{X}_\gamma$ being π-vertical. Then, consider an "extension" of it to a vector field, say X, on a neighborhood $U^{(\gamma)}$ of the image of γ in $\mathbb{Q}$, i. e., a (π-vertical) vector field on $U^{(\gamma)}$ which coincides with $\mathbb{X}_\gamma$ when restricted to the image of γ. Being X π-vertical, its flow F_s^X is, for each value of the parameter s, a bundle morphism of π over the identity. Then, $\gamma_s := F_s^X \circ \gamma$ is again a section of π as the following chain of equalities shows:

$$\pi \circ \gamma_s = \pi \circ F_s^X \circ \gamma = \mathbb{1} \circ \underbrace{\pi \circ \gamma}_{=\mathbb{1}} = \mathbb{1}\,. \tag{3.7}$$

Thus $\gamma_s := F_s^X \circ \gamma$ is a family of sections of π commonly called a "variation" of the section γ induced by the tangent vector $\mathbb{X}_\gamma$. Note that $\gamma_0 = \gamma$. The functional:

$$\mathscr{S}_{\gamma_s} = \int_{\mathbb{I}} \left(j^1\gamma_s\right)^\star \lambda \tag{3.8}$$

is a functional depending on the parameter s associated with the flow of the vector field X. Then, the variation of $\mathscr{S}$ induced by the tangent vector $\mathbb{X}_\gamma$ is defined to be:

$$\delta_{\mathbb{X}_\gamma}\mathscr{S}_\gamma \,:=\, \frac{d}{ds}\mathscr{S}_{\gamma_s}\Big|_{s=0} \,. \tag{3.9}$$

Thus:

$$\begin{aligned} \delta_{X(\gamma)}\mathscr{S}_\gamma \,&=\, \frac{d}{ds}\Big|_{s=0}\int_{\mathbb{I}}\left(j^1\gamma_s\right)^\star\lambda \,=\, \frac{d}{ds}\Big|_{s=0}\int_{\mathbb{I}}\left(j^1F_s^X\circ j^1\gamma\right)^\star\lambda \,=\\ &=\, \frac{d}{ds}\Big|_{s=0}\int_{\mathbb{I}}\left(j^1\gamma\right)^\star\circ\left(j^1F_s^X\right)^\star\lambda \,=\\ &=\,\int_{\mathbb{I}}\left(j^1\gamma\right)^\star\circ\left[\frac{d}{ds}\left(j^1F_s^X\right)^\star\Big|_{s=0}\lambda\right] \,=\, \int_{\mathbb{I}}\left(j^1\gamma\right)^\star\left[\mathcal{L}_{X^1}\lambda\right] \end{aligned} \tag{3.10}$$

where in the first step we used (2.185) for the morphism F_s^X, then the fact that $(f\circ g)^\star \,=\, g^\star\circ f^\star$, then the fact that $j^1F_s^X$ is a smooth function on $\mathbf{J}^1\pi$ which is $\mathcal{C}^1$ with respect to the parameter s (in order to put the derivative inside the integral) and finally the definition of the Lie derivative of a differential form along a vector field together with the definition of first order jet prolongation of a vector field on $\mathbb{Q}$ (see Appendix 2.A). Note that, since we are considering the pull-back via $j^1\gamma$ inside the integral, the variation of $\mathscr{S}$ induced by $\mathbb{X}_\gamma$ does not depend on the particular extension X chosen to define the variation of γ. Using the properties of the Lie derivative we get two terms out of the variation of $\mathscr{S}$:

$$\delta_{\mathbb{X}_\gamma}\mathscr{S}_\gamma \,=\, \int_{\mathbb{I}}\left(j^1\gamma\right)^\star i_{X^1}\mathrm{d}\lambda + \int_{\partial\mathbb{I}}\mathfrak{i}_{\partial\mathbb{I}}^\star\left(j^1\gamma\right)^\star i_{X^1}\lambda \,, \tag{3.11}$$

where the fact that the differential is a natural operator with respect to the pull-back and the Stokes' theorem were used and where $\mathfrak{i}_{\partial\mathbb{I}}$ is the canonical immersion of $\partial\mathbb{I}$ into $\mathbb{I}$. The previous equation has the structure of a "boundary term" plus an integral over the whole $\mathbb{I}$. In order to implement a variational principle in the sense of Schwinger-Weiss[11] and to extract from it a differential equation with an intrinsic procedure, we must "isolate" the boundary contribution, that is, we must be sure that the integral over $\mathbb{I}$ would not give rise to other boundary terms. By looking at the local expression of X^1 in (2.201) it is clear that this is not the case in general. Indeed, the terms containing the operator $\frac{d}{dt}$ in X^1 may give rise to boundary contributions after integrations by parts. However, recalling that the Lagrangian λ is only defined up to elements in the contact ideal, it is possible to use such a freedom in the definition of the Lagrangian to fix this situation. In order to avoid the emerging of boundary contributions from the first term in (3.11) we can fix the element in the contact ideal, say η, in such a way that:

$$i_{X^1}\,\mathrm{d}(\,\lambda+\eta\,) \tag{3.12}$$

does not depend on the π^1_0-vertical component of X^1 which is the one containing the total derivative operator. This is equivalent to ask:

$$(j^1\gamma)^\star\left[i_V\mathrm{d}(\lambda+\eta)\right] \,=\, 0 \tag{3.13}$$

[11]The variation only can depend on boundary terms.

for all π^1_0-vertical vector field V. A straightforward calculation (see [27] example 3.2.1 at page 43) shows that in order for this to happen, η must be:

$$\eta \;=\; \frac{\partial \mathscr{L}}{\partial v^j}\left(\mathrm{d}q^j - v^j\mathrm{d}t\right), \tag{3.14}$$

and, thus, that the Lagrangian, after fixing η reads:

$$\Theta_{\mathscr{L}} \;:=\; \lambda+\eta \;=\; \frac{\partial \mathscr{L}}{\partial v^j}\,\mathrm{d}q^j - \Bigg(\underbrace{v^j\frac{\partial \mathscr{L}}{\partial v^j}}_{\Delta\mathscr{L}} - \mathscr{L}\Bigg)\,\mathrm{d}t \;=\; \frac{\partial \mathscr{L}}{\partial v^j}\,\mathrm{d}q^j - E_{\mathscr{L}}\,\mathrm{d}t \;=:\; \pi^{\mathscr{L}}_j\,\mathrm{d}q^j - E_{\mathscr{L}}\,\mathrm{d}t\,, \tag{3.15}$$

where $\Delta = v^j\frac{\partial}{\partial v^j}$ is the (push-forward to $\mathbf{J}^1\pi$ of the) so-called PARTIAL LINEAR STRUCTURE of $\mathbf{T}\mathcal{Q}$. The differential form $\Theta_{\mathscr{L}}$ obtained by fixing the element in the contact ideal following the previous prescriptions is called a LEPAGEAN EQUIVALENT[12] of the Lagrangian λ and the previous result shows that the Lepage equivalent of a first order[13] Lagrangian is the well known POINCARÉ-CARTAN FORM. In terms of it, equation (3.11) becomes:

$$\delta_{\mathbb{X}_\gamma}\mathscr{S}_\gamma \;=\; \int_{\mathbb{I}}\left(j^1\gamma\right)^\star i_{X^1}\mathrm{d}\Theta_{\mathscr{L}} \;+\; \int_{\partial\mathbb{I}} \mathfrak{i}^\star_{\partial\mathbb{I}}\left(j^1\gamma\right)^\star i_{X^1}\,\Theta_{\mathscr{L}} \tag{3.16}$$

where now, the boundary contributions only come from the second term. Equation (3.16) is usually called in the literature FIRST VARIATIONAL FORMULA or FIRST FUNDAMENTAL FORMULA. The differential of $\Theta_{\mathscr{L}}$ is:

$$\mathrm{d}\Theta_{\mathscr{L}} \;=\; -\mathrm{d}\pi^{\mathscr{L}}_j\wedge\mathrm{d}q^j - \mathrm{d}E_{\mathscr{L}}\wedge\mathrm{d}t \;=:\; \omega_{\mathscr{L}} - \mathrm{d}E_{\mathscr{L}}\wedge\mathrm{d}t\,. \tag{3.17}$$

Now a Schwinger-Weiss variational principle can be implemented in an intrinsic way because the fact that the variation of $\mathscr{S}$ must depend only on boundary contributions is equivalent to the fact that the first term in (3.16) must vanish $\forall\, X$:

$$\int_{\mathbb{I}}\left(j^1\gamma\right)^\star i_{X^1}\mathrm{d}\Theta_{\mathscr{L}} \;=\; 0 \quad \forall\, X\in\mathfrak{X}^v(U^{(\gamma)}) \tag{3.18}$$

where $\mathfrak{X}^v(U^{(\gamma)})$ is the module of π-vertical vector fields on the open subset $U^{(\gamma)}$ of $\mathbb{Q}$. The fundamental lemma of the Calculus of Variations leads to the following equation:

$$\left(j^1\gamma\right)^\star i_{X^1}\mathrm{d}\Theta_{\mathscr{L}} \;=\; 0 \quad \forall\, X\in\mathfrak{X}^v(U^{(\gamma)}) \tag{3.19}$$

[12]An extensive literature exists on the subject. The Lepagean equivalent of a Lagrangian is uniquely defined by the previous arguments only for first order jet bundles and when the base manifold is 1-dimensional, i. e., for "Newtonian" Mechanics [25], [27]. Further (canonical) restrictions can be imposed in order to obtain a generalization of the Poincarè-Cartan form on higher order jet bundles when the base manifold is 1-dimensional but only for first order jet bundles in case the base manifold has dimension higher then 1 [33], [38], [80]–[82]. This means that a Poincarè-Cartan form can be canonically defined for Mechanics at any order but for Field Theories only in the first order case excluding higher order Field Theories like the Einstein-Hilbert formulation of Gravity (with this in mind Palatini's formulation acquires a particular important role from the geometrical point of view). For higher order Field Theories it is possible to uniquely define a Poincarè-Cartan form with the additional (not canonical) structure of a linear connection on the base space [83].

[13]In the sense that it is defined on a first order jet bundle.

which is a (generally implicit) differential equation whose solutions are given as critical mappings of a differential form, that is, mappings $j^1\gamma$ along which a differential form $(i_{X^1}\mathrm{d}\Theta_{\mathscr{L}})$ vanishes. The coordinate expression of (3.19) is the celebrated system of EULER-LAGRANGE EQUATIONS:

$$\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial v^j}(t,\gamma(t),\dot{\gamma}(t)) - \frac{\partial\mathscr{L}}{\partial q^j}(t,\gamma(t),\dot{\gamma}(t)) \,=\, 0\,, \tag{3.20}$$

where $\frac{d}{dt}$ represents the total derivative.

Note that the left hand side of (3.18), i.e., the first term in the right hand side of (3.16), is linear in X^1 and, thus, it can be seen as a differential 1-form on $\mathcal{F}_{\mathbb{Q}}$ following the definition below.

Definition 3.1.4 (EULER-LAGRANGE FORM). *Given a Lagrangian λ and the corresponding action functional $\mathscr{S}$, the* EULER-LAGRANGE FORM *is the differential 1-form on the space $\mathcal{F}_{\mathbb{Q}}$ on which $\mathscr{S}$ is defined, defined by the first term on the left hand side of* (3.16)*:*

$$\mathbb{EL} \;:\; \mathcal{F}_{\mathbb{Q}} \to \mathbf{T}^\star_\gamma\mathcal{F}_{\mathbb{Q}} \;:\; \gamma \mapsto \mathbb{EL}_\gamma\,, \tag{3.21}$$

such that:

$$i_{\mathbb{X}_\gamma}\mathbb{EL}_\gamma \,=\, \mathbb{EL}_\gamma(\mathbb{X}_\gamma) \,=\, \int_{\mathbb{I}}\left(j^1\gamma\right)^\star i_{X^1}\mathrm{d}\Theta_{\mathscr{L}}\,,\quad \forall\; \mathbb{X}_\gamma \in \mathbf{T}_\gamma\mathcal{F}_{\mathbb{Q}}\,, \tag{3.22}$$

where X^1 is the first order jet prolongation of any extension of $\mathbb{X}$ to a vector field on an open neighborhood of the image of γ in $\mathbb{Q}$.

Consequently to the latter definition, it is natural to define the space of solutions of the equations of motion of our system (the SOLUTION SPACE from now on), as follows.

Definition 3.1.5 (SOLUTION SPACE). *Given a dynamical system described by the Lagrangian λ, the* SOLUTION SPACE *associated to it is defined as the set of zeros of the Euler-Lagrange form associated to λ, say $\mathcal{EL}$:*

$$\begin{aligned}\mathcal{EL} \;:=&\; \{\,\gamma \in \mathcal{F}_{\mathbb{Q}} \;:\; \mathbb{EL}_\gamma \,=\, 0\,\} \,=\\ =&\; \left\{\,\gamma \in \mathcal{F}_{\mathbb{Q}} \;:\; \left(j^1\gamma\right)^\star i_{X^1}\mathrm{d}\Theta_{\mathscr{L}} \,=\, 0 \quad \forall\; X \in \mathfrak{X}(U^{(\gamma)})\,\right\} \,=\\ =&\; \{\,\gamma \in \mathcal{F}_{\mathbb{Q}} \;:\; \textit{solutions of } (3.20)\,\}\,.\end{aligned} \tag{3.23}$$

We will always assume $\mathcal{EL}$ to be a smooth immersed submanifold of $\mathcal{F}_{\mathbb{Q}}$, the immersion denoted by $\mathrm{i}_{\mathcal{EL}}$.

Remark 3.1.6. *In order to make contact with the usual description of classical Mechanics on the tangent bundle of a configuration manifold, let us first note that $\mathbf{J}^1\pi \simeq \mathbb{I}\times\mathbf{T}\mathcal{Q}$. Now, if $\mathscr{L}$ does not explicitly depend on t*[14]*, i.e., it is the pull-back*

[14] Systems of this type are called AUTONOMOUS

of a function on $\mathbf{T}\mathcal{Q}$*, say* $\bar{\mathscr{L}}$*, then* $\omega_{\mathscr{L}}$ *is the pull-back of the differential form* $\bar{\omega}_{\mathscr{L}}$ *on* $\mathbf{T}\mathcal{Q}$*:*

$$\bar{\omega}_{\mathscr{L}} \,=\, -\mathrm{d}\mathrm{d}_S\bar{\mathscr{L}}\,, \tag{3.24}$$

where:

$$S \,=\, \mathrm{d}q^j \otimes \frac{\partial}{\partial v^j} \tag{3.25}$$

is the so-called SOLDERING FORM *on* $\mathbf{T}\mathcal{Q}$ *and:*

$$\mathrm{d}_S\bar{\mathscr{L}} \,=\, S \circ \mathrm{d}\bar{\mathscr{L}} \,=\, \frac{\partial \bar{\mathscr{L}}}{\partial v^j}\mathrm{d}q^j\,, \tag{3.26}$$

is the so called NIJENHUIS DIFFERENTIAL *associated to the* $(1,\,1)$ *tensor* S *applied to the function* $\mathscr{L}$ *(see [84] for a review of differential calculus associated to* $(1,\,1)$ *tensors on manifolds). Furthermore, also the function* $E_{\mathscr{L}}$ *appearing in* (3.17) *turns out to be the pull-back of a function on* $\mathbf{T}\mathcal{Q}$*, say* $\bar{E}_{\mathscr{L}}$*, and, if* $\overline{\omega}_{\mathscr{L}}$ *is symplectic, solutions of* (3.19) *turn out to be integral curves of the second order vector field* $\bar{\Gamma}$ *on* $\mathbf{T}\mathcal{Q}$ *satisfying:*

$$i_{\bar{\Gamma}}\bar{\omega}_{\mathscr{L}} \,=\, \mathrm{d}\bar{E}_{\mathscr{L}}\,, \tag{3.27}$$

whereas, if $\overline{\omega}_{\mathscr{L}}$ *is only pre-symplectic, solutions turn out to be solutions of the pre-symplectic Hamiltonian system* $(\mathbf{T}\mathcal{Q},\,\overline{\omega}_{\mathscr{L}},\,\overline{E}_{\mathscr{L}})$*. The latter equation is the usual form of Euler-Lagrange equations on the tangent bundle.*

Let us stress that throughout the whole manuscript we will always consider dynamical systems of this type both within Mechanics and within field theories. Within field theories this choice is motivated by the fact that we are interested in fundamental theories, namely, those describing fundamental interactions for which the assumptions above are met.

It is interesting to stress that, for autonomous systems, a geometrical interpretation of solutions of Euler-Lagrange equations can be given within the so called TULCZYJEW TRIPLES *approach, where it can be proved that they are actually (Lagrangian) submanifolds of* $\mathbf{T}\mathbf{T}^\star\mathcal{Q}$ *[85]. Having in mind that an implicit (first order) differential equation over a manifold,* $\mathcal{M}$*, can be defined as a submanifold of its tangent bundle (see [86]), this underlines, once more, the fact that Euler-Lagrange equations are implicit equations (on* $\mathbf{T}^\star\mathcal{Q}$*).*

We conclude this section with an example in order to fix the ideas about the machinery introduced so far.

Example 3.1.7 (FREE PARTICLE ON THE LINE)**.** *Let us consider a free particle moving on a line. In this case the configuration manifold is* $\mathcal{Q} = \mathbb{R}$ *and the bundle* (3.1) *is* $\pi \;:\; \mathbb{Q} = \mathbb{I} \times \mathbb{R} \to \mathbb{I}$ *where* $\mathbb{I} = [a,\,b] \subset \mathbb{R}$*. We denote by* $\{\,t,\,q\}$ *a system of coordinates on* $\mathbb{Q}$*, where* t *is the coordinate on the interval* $\mathbb{I}$ *and by* γ *the sections of* π*. The first order jet bundle* $\mathbf{J}^1\pi$ *reads* $\mathbf{J}^1\pi \,=\, \mathbb{I} \times \mathbb{R}^2$ *where we consider* $\{\,t,\,q,\,v\,\}$ *as a system of coordinates. The Lagrangian describing the dynamical systems is:*

$$\lambda \,=\, \frac{1}{2}mv^2\,\mathrm{d}t\,, \tag{3.28}$$

m representing the mass of the particle, which gives rise to the following action functional:

$$\mathscr{S}_\gamma = \int_a^b \left(j^1\gamma\right)^\star \lambda = \int_a^b \frac{1}{2} m\dot{\gamma}^2(t)\, \mathrm{d}t\,. \tag{3.29}$$

The space $\Gamma(\pi)$ of smooth sections of π is not a Banach space. However, it can be equipped with the following norm:

$$\|\gamma\|_1 = \sup_{t\in\mathbb{I}}|\gamma(t)| + \sup_{t\in\mathbb{I}}|\dot{\gamma}(t)|\,, \tag{3.30}$$

and, as the following chain of inequalities proves[15]

$$\begin{aligned}\left|\mathscr{S}_\gamma - \mathscr{S}_{\tilde{\gamma}}\right| &\leq \frac{1}{2m}\int_a^b \left|\left(\dot{\gamma}(t)^2 - \dot{\tilde{\gamma}}(t)^2\right)\right| \mathrm{d}t \leq \\ &\leq \frac{1}{2m}\int_a^b \left[\left|\dot{\gamma}(t)(\dot{\gamma}(t) - \dot{\tilde{\gamma}}(t))\right| + \left|\dot{\tilde{\gamma}}(t)(\dot{\gamma}(t) - \dot{\tilde{\gamma}}(t))\right|\right] \leq \\ &\leq \frac{b-a}{2m}\left[\sup_{t\in\mathbb{I}}|\dot{\gamma}(t)|\sup_{t\in\mathbb{I}}|\dot{\gamma}(t) - \dot{\tilde{\gamma}}(t)| + \sup_{t\in\mathbb{I}}|\dot{\tilde{\gamma}}(t)|\sup_{t\in\mathbb{I}}|\dot{\gamma}(t) - \dot{\tilde{\gamma}}(t)|\right]\,,\end{aligned} \tag{3.31}$$

$\mathscr{S}$ is continuous with respect to the $\|\cdot\|_1$-norm defined above. Indeed, the last term of the chain of inequalities above vanishes as $\|\gamma - \tilde{\gamma}\|_1$ approaches zero. Consequently, $\mathscr{S}$ can be extended by continuity to the completion $\overline{\Gamma(\pi)}^{\|\cdot\|_1} =: \mathcal{F}_{\mathbb{Q}}$ which is, indeed, a Banach space. The contact form η in (3.14) is easily computed to be:

$$\eta = mv\,(\mathrm{d}q - v\mathrm{d}t)\,, \tag{3.32}$$

giving rise to the following Lepage equivalent:

$$\Theta_{\mathscr{L}} = mv\,\mathrm{d}q - \frac{1}{2}mv^2\mathrm{d}t\,. \tag{3.33}$$

The Lepage equivalent above gives rise to the following first fundamental formula:

$$\delta_{\mathbb{X}_\gamma}\mathscr{S}_\gamma = \int_a^b \left(j^1\gamma\right)^\star \left[\, i_{X^1}\left(\, m\mathrm{d}v\wedge\mathrm{d}q - mv\mathrm{d}v\wedge\mathrm{d}t\,\right)\right] + \left(j^1\gamma\right)^\star \left[i_{X^1}\left(mv\mathrm{d}q - \frac{1}{2}mv^2\mathrm{d}t\right)\right]\Bigg|_a^b\,, \tag{3.34}$$

where X^1 is the first order jet prolongation of a π-vertical vector field X on $\mathbb{Q}$ defined in a neighborhood of the image of γ, i.e., it is a vector field of the type:

$$X^1 = X_q\frac{\partial}{\partial q} + v\frac{\partial X_q}{\partial q}\frac{\partial}{\partial v}\,, \tag{3.35}$$

where X^q is a function on $\mathbb{Q} = \mathbb{I}\times\mathbb{R}$ defined for all $t\in\mathbb{I}$ and for q close[16] *to $\gamma(t)$. On the other hand, $\mathbb{X}_\gamma$ is the map:*

$$\mathbb{X}_\gamma \;:\; \mathbb{I}\to\mathbf{T}_{\gamma(t)}\mathbb{Q} \;:\; t\mapsto X(\gamma(t)) = X_q(t,\gamma(t))\frac{\partial}{\partial q}\bigg|_{\gamma(t)} =: \mathbb{X}_\gamma[\gamma]\frac{\delta}{\delta\gamma}\,, \tag{3.36}$$

[15] We are denoting by γ and $\tilde{\gamma}$ two different sections of π.

[16] In the sense of the sup-norm defined on $\mathcal{F}_{\mathbb{Q}}$.

following the notation (2.52)*. The first term on the left hand side is the (contraction of the tangent vector* $\mathbb{X}_\gamma$ *along the) Euler-Lagrange form associated to the Lagrangian describing our dynamical system. Equation* (3.19) *reads:*

$$\begin{aligned}\left(j^1\gamma\right)^\star\left[i^1_X \mathrm{d}\Theta_{\mathscr{L}}\right] \,&=\, \left(j^1\gamma\right)^\star\left[-m\,X_q\mathrm{d}v + mv\,\frac{\partial X_q}{\partial q}\mathrm{d}q - mv^2\,\frac{\partial X_q}{\partial q}\mathrm{d}t\right] \,= \\ &=\, \left[-\mathbb{X}_\gamma\,m\ddot{\gamma} + m\dot{\gamma}\,\frac{\delta\mathbb{X}_\gamma}{\delta\gamma}\dot{\gamma} - m\dot{\gamma}^2\,\frac{\delta\mathbb{X}_\gamma}{\delta\gamma}\right]\mathrm{d}t \,=\, -\mathbb{X}_\gamma\,m\ddot{\gamma}\,\mathrm{d}t \,=\, 0 \quad \forall\;\mathbb{X}_\gamma\,,\end{aligned} \tag{3.37}$$

which gives:

$$\ddot{\gamma}(t) \,=\, 0\,. \tag{3.38}$$

Let us stress that the Lagrangian considered is of the type that in Rem. 3.1.6 we called autonomous. Indeed, the function $\mathscr{L}$ *does not depend on* t *and, thus, it is the pull-back of a function* $\bar{\mathscr{L}}$ *on* $\mathbf{T}\mathcal{Q} \,=\, \mathbb{R}^2$ *having the same expression:*

$$\bar{\mathscr{L}} \,=\, \frac{1}{2}mv^2\,. \tag{3.39}$$

Consequently:

$$\omega_{\mathscr{L}} \,=\, m\mathrm{d}q\wedge\mathrm{d}v\,, \tag{3.40}$$

is the pull-back of the differential form $\bar{\omega}_{\mathscr{L}}$ *on* $\mathbf{T}\mathcal{Q} \,=\, \mathbb{R}^2$ *having the same expression:*

$$\bar{\omega}_{\mathscr{L}} \,=\, m\mathrm{d}q\wedge\mathrm{d}v\,, \tag{3.41}$$

where, as a straightforward computation shows:

$$\bar{\omega}_{\mathscr{L}} \,=\, -\mathrm{d}\mathrm{d}_S\bar{\mathscr{L}}\,. \tag{3.42}$$

On the other hand:

$$E_{\mathscr{L}} \,=\, v\frac{\partial\mathscr{L}}{\partial v} - \mathscr{L} \,=\, \frac{1}{2}mv^2\,, \tag{3.43}$$

is the pull-back of a function, $\bar{E}_{\mathscr{L}}$ *on* $\mathbf{T}\mathcal{Q} \,=\, \mathbb{R}^2$ *having the same expression:*

$$\bar{E}_{\mathscr{L}} \,=\, \frac{1}{2}mv^2\,. \tag{3.44}$$

The Hamiltonian equations associated to the Hamiltonian system $(\mathbf{T}\mathcal{Q},\,\bar{\omega}_{\mathscr{L}},\,\bar{E}_{\mathscr{L}})$ *read:*

$$i_{\bar{\Gamma}}\bar{\omega}_{\mathscr{L}} \,=\, \mathrm{d}\bar{E}_{\mathscr{L}}\,, \tag{3.45}$$

for the vector field $\bar{\Gamma}$ *on* $\mathbf{T}\mathcal{Q}$*:*

$$\bar{\Gamma} \,=\, \bar{\Gamma}_q\frac{\partial}{\partial q} + \bar{\Gamma}_v\frac{\partial}{\partial v} \tag{3.46}$$

that are:

$$-\bar{\Gamma}_v\,\mathrm{d}q + \bar{\Gamma}_q\,\mathrm{d}v \,=\, v\,\mathrm{d}v\,, \tag{3.47}$$

which give:

$$\bar{\Gamma}_v \,=\, 0\,, \qquad \bar{\Gamma}_q \,=\, v\,. \tag{3.48}$$

The integral curves of the vector field just defined, are curves $(\gamma(t),\, v(t))$ *on* $\mathbf{T}\mathcal{Q}$ *satisfying:*

$$\dot{\gamma}(t) \,=\, v(t), \qquad \dot{v}(t) \,=\, 0\,, \tag{3.49}$$

which are the tangent lift to $\mathbf{T}\mathcal{Q}$ *of the curves on* $\mathcal{Q}$ *satisfying:*

$$\ddot{\gamma}(t) \,=\, 0\,. \tag{3.50}$$

3.1.2. Hamiltonian formulation

The Hamiltonian formalism is settled on the reduced dual of $\mathbf{J}^1\pi$, i.e. what we called the COVARIANT PHASE SPACE in 2.4.1. Here, being $\pi \;:\; \mathbb{Q} \to \mathbb{I}$ the fibre bundle underlying the theory, we will denote the Covariant Phase Space by $\mathcal{P}(\mathbb{Q})$. Since we used $\{\, t,\, q^j,\, v^j \,\}_{j=1,\ldots,n}$ as local coordinates on $\mathbf{J}^1\pi$, we will use $\{\, t,\, q^j,\, p_j,\, p_0 \,\}_{j=1,\ldots,n}$ as local coordinates on the extended dual $\mathbf{J}^\dagger\pi$ and $\{\, t,\, q^j,\, p_j \,\}_{j=1,\ldots,n}$ as local coordinates on the Covariant Phase space, $\mathcal{P}(\mathbb{Q})$.

Definition 3.1.8 (HAMILTONIAN)**.** *A* HAMILTONIAN *is a (local, in general) section of the projection* κ *appearing in the diagram* (2.179)*, i.e., a (local, in general) map:*

$$H \;:\; \mathbf{J}^\star\pi \to \mathcal{P}(\mathbb{Q}) \;:\; (t,\, q^j,\, p_j) \mapsto \Big(t,\, q^j,\, p_j,\, H(t,\, q,\, p)\Big)\,. \tag{3.51}$$

Recalling the content of Sec. 2.4.2, the extended dual of π, say $\mathbf{J}^\dagger\pi$, has a canonical 1-semibasic n-form (actually a 1-form in this case), i.e.:

$$\boldsymbol{w} \,=\, p_j \mathrm{d}q^j + p_0 \mathrm{d}t \,=:\, \theta + p_0\mathrm{d}t\,, \tag{3.52}$$

where θ is the so called TAUTOLOGICAL FORM of $\mathbf{T}^\star\mathcal{Q}$. For any fixed Hamiltonian, it is possible to pull-back the canonical structure above to $\mathcal{P}(\mathbb{Q})$, in order to obtain a canonical structure on the latter space, which, under suitable conditions on H, turns to be a contact form [47], [48], in the sense that $\boldsymbol{w} \wedge \underbrace{\mathrm{d}\boldsymbol{w} \wedge \ldots \wedge \mathrm{d}\boldsymbol{w}}_{n \text{ times}} \neq 0$. The pull-back of (3.52) via $-H$[17] reads:

$$\Theta_H \,:=\, (\,-H\,)^\star\, \boldsymbol{w} \,=\, \theta - H\mathrm{d}t\,. \tag{3.53}$$

The latter is the Hamiltonian counterpart of the Lepagean equivalent introduced in the previous section.

In this section, elements of $\Gamma^{\text{SPLIT}}(\delta_1)$ defined in Sec. 2.4.2, will be denoted by $\xi \,=\, (\gamma,\, \varrho)$ where $q^j \circ \gamma \,=:\, \gamma^j$ and $p_j \circ \varrho \,=:\, \varrho_j$.

[17] The minus sign is just a matter of convention.

Definition 3.1.9 (Action functional)**.** *Given a Hamiltonian over $\mathcal{P}(\mathbb{Q})$, an* ACTION FUNCTIONAL *is a real-valued function on $\Gamma^{\text{SPLIT}}(\delta_1)$ given by:*

$$\mathscr{S} \;:\; \Gamma^{\text{SPLIT}}(\delta_1) \to \mathbb{R} \;:\; \xi \mapsto \mathscr{S}_\xi \;=\; \int_{\mathbb{I}} \xi^\star \, \Theta_H \,, \tag{3.54}$$

whose coordinate expression is:

$$\mathscr{S}_\xi \;=\; \int_{\mathbb{I}} \Big[\varrho_j(t)\, \dot{\gamma}^j(t) - H(\gamma(t), \varrho(t), t) \Big] \,\mathrm{d}t \,. \tag{3.55}$$

Remark 3.1.10. *As in the Lagrangian formalism, let us stress that $\Gamma^{\text{SPLIT}}(\delta_1)$ is not a Banach manifold, in general, and that we again assume to be able to perform a completion procedure of the type described in Rem. 3.1.3 in order to obtain a suitable Banach manifold that we will denote by $\mathcal{F}_{\mathcal{P}(\mathbb{Q})}$.*

The first variation of $\mathscr{S}$ in this case is described as follows. Consider a δ_1-vertical tangent vector to $\mathcal{F}_{\mathcal{P}(\mathbb{Q})}$ at ξ, say $\mathbb{X}_\xi$. Consider an extension of $\mathbb{X}_\xi$ to a (δ_1-vertical) vector field over an open neighborhood $U^{(\xi)}$ of the image of ξ in $\mathcal{P}(\mathbb{Q})$. We will denote it by X. Associated with X there exists a family of sections of δ_1 given by $\xi_s = F_s^X \circ \xi$. The fact that ξ_s is a family of sections of δ_1 is readily proven by noting that, being X δ_1-vertical, its flow projects onto the identity and the following chain of equalities holds:

$$\delta_1 \circ \xi_s \;=\; \underbrace{\delta_1 \circ F_s^X}_{\mathbb{1}_{\mathbb{I}} \circ \delta_1} \circ \xi \;=\; \mathbb{1}_{\mathbb{I}} \circ \underbrace{\delta_1 \circ \xi}_{=\mathbb{1}_{\mathbb{I}}} \;=\; \mathbb{1}_{\mathbb{I}} \,. \tag{3.56}$$

Then, the variation of the action functional induced by $\mathbb{X}_\xi$ can be defined as:

$$\delta_{\mathbb{X}_\xi} \mathscr{S}_\xi \;:=\; \frac{d}{ds}\bigg|_{s=0} \mathscr{S}_{\xi_s} \;=\; \frac{d}{ds}\bigg|_{s=0} \int_{\mathbb{I}} \xi_s^\star \Theta_H \,. \tag{3.57}$$

It reads:

$$\begin{aligned} \delta_{\mathbb{X}_\xi} \mathscr{S}_\xi \;&=\; \frac{d}{ds}\bigg|_{s=0} \mathscr{S}_{\xi_s} \;=\; \frac{d}{ds} \int_{\mathbb{I}} \xi_s^\star \, \Theta_H \bigg|_{s=0} \;=\; \frac{d}{ds} \int_{\mathbb{I}} \left(F_s^X \circ \xi \right)^\star \, \Theta_H \bigg|_{s=0} \;= \\ &=\; \frac{d}{ds} \int_{\mathbb{I}} \xi^\star \circ F_s^{X\star} \, \Theta_H \bigg|_{s=0} \;=\; \int_{\mathbb{I}} \xi^\star \left[\mathcal{L}_X \, \Theta_H \right] \,, \end{aligned} \tag{3.58}$$

where we used the fact that $(f \circ g)^* \;=\; g^* \circ f^*$ and the definition of the Lie derivative along the vector field X. Now, using the properties of the Lie derivative we get:

$$\delta_{\mathbb{X}_\xi} \mathscr{S}_\xi \;=\; \int_{\mathbb{I}} \xi^\star \left[i_X \mathrm{d}\Theta_H \right] + \int_{\partial \mathbb{I}} \mathrm{i}_{\partial \mathbb{I}}^\star \, \xi^\star \left[i_X \Theta_H \right] \,, \tag{3.59}$$

where, in this case, the form $\mathrm{d}\Theta_H$ reads:

$$\mathrm{d}\Theta_H \;=\; \omega + \mathrm{d}H \wedge \mathrm{d}t \,, \tag{3.60}$$

where ω is the pull-back to $\mathcal{P}(\mathbb{Q})$ of the symplectic structure of $\mathbf{T}^\star \mathcal{Q}$. The second term on the r.h.s. of the latter equation, emerging from the application of Stokes' theorem, is a "boundary term" in the sense that it depends only on the restriction of ξ to the boundary of $\mathbb{I}$, $\xi_{\partial\mathbb{I}} = \xi \circ \mathfrak{i}_{\partial\mathbb{I}}$ ($\mathfrak{i}_{\partial\mathbb{I}}$ denotes the canonical immersion of $\partial\mathbb{I}$ into $\mathbb{I}$). In the Hamiltonian case, since ξ is not a first order jet prolongation of a section, no additional boundary terms may appear from the first term on the r.h.s. and, thus, the problem of searching for a Lepage equivalent does not arise. Equation (3.59) is often called FIRST VARIATIONAL FORMULA or FIRST FUNDAMENTAL FORMULA. It is worth stressing that all the expressions in the integrals above only depend on the values assumed by the differential forms involved on the image of ξ because of the pull-back. Thus, they do not depend on the particular extension X of $\mathbb{X}_\xi$ chosen.

Now, following Schwinger-Weiss variational principle, the variation of $\mathscr{S}$ along any direction $\mathbb{X}_\xi$ can only depend on terms like the second one on the r.h.s. of the latter equation. Thus, the first term on the r.h.s. must vanish for any $\mathbb{X}_\xi$:

$$\int_{\mathbb{I}} \xi^\star \left[i_X \mathrm{d}\Theta_H \right] \,=\, 0 \quad \forall\; X \in \mathfrak{X}^v(U^{(\xi)}) \tag{3.61}$$

which, by virtue of the fundamental lemma of the Calculus of Variations, gives:

$$\xi^\star \left[i_X \mathrm{d}\Theta_H \right] \,=\, 0 \quad \forall\; X \in \mathfrak{X}^v(U^{(\xi)})\,. \tag{3.62}$$

The coordinate expression of (3.62) is the celebrated system of HAMILTON'S EQUATIONS:

$$\dot{\gamma}^j(t) \,=\, \frac{\partial H}{\partial p_j}(t,\, \gamma(t),\, \rho(t))\,, \qquad \dot{\rho}_j(t) \,=\, -\frac{\partial H}{\partial q^j}(t,\, \gamma(t),\, \rho(t))\,, \tag{3.63}$$

which is, differently from EULER-LAGRANGE EQUATIONS, a system of explicit first order differential equations.

Analogously to the Lagrangian case, the equations of motion can be written in terms of the following Euler-Lagrange form.

Definition 3.1.11 (EULER-LAGRANGE FORM). *Given a Hamiltonian H and the corresponding action functional $\mathscr{S}$, the* EULER-LAGRANGE FORM *is the differential 1-form on the space $\mathcal{F}_{\mathcal{P}(\mathbb{Q})}$ on which $\mathscr{S}$ is defined, defined by the first term on the left hand side of* (3.59)*:*

$$\mathbb{EL} \;:\; \mathcal{F}_{\mathcal{P}(\mathbb{Q})} \to \mathbf{T}^\star_\xi \mathcal{F}_{\mathcal{P}(\mathbb{Q})} \;:\; \xi \mapsto \mathbb{EL}_\xi\,, \tag{3.64}$$

such that:

$$i_{\mathbb{X}_\xi} \mathbb{EL}_\xi \,=\, \mathbb{EL}_\xi(\mathbb{X}_\xi) \,=\, \int_{\mathbb{I}} \xi^\star \left[i_X \mathrm{d}\Theta_{\mathscr{L}} \right]\,, \quad \forall\; \mathbb{X}_\xi \in \mathbf{T}_\xi \mathcal{F}_{\mathcal{P}(\mathbb{Q})}\,, \tag{3.65}$$

where X is any extension of $\mathbb{X}$ to a vector field on an open neighborhood of the image of ξ in $\mathcal{P}(\mathbb{Q})$.

Consequently, also in this case, it is natural to define the solution space as follows.

Definition 3.1.12 (Solution space)**.** *Given a dynamical system described by the Hamiltonian H, the* SOLUTION SPACE *associated to it is defined as the set of zeros of the Euler-Lagrange form associated to H, say $\mathcal{EL}$:*

$$\begin{aligned} \mathcal{EL} \, := & \left\{ \xi \in \mathcal{F}_{\mathcal{P}(\mathbb{Q})} \ : \ \mathbb{EL}_{\xi} \, = \, 0 \right\} \, = \\ = & \left\{ \xi \in \mathcal{F}_{\mathcal{P}(\mathbb{Q})} \ : \ \xi^{\star} \left[\, i_{X^1} \mathrm{d}\Theta_{\mathscr{L}} \, \right] \, = \, 0 \quad \forall \, X \in \mathfrak{X}(U^{(\xi)}) \right\} \, = \\ = & \left\{ \xi \in \mathcal{F}_{\mathcal{P}(\mathbb{Q})} \ : \ \textit{solutions of } (3.63) \right\} \, . \end{aligned} \tag{3.66}$$

We will always assume $\mathcal{EL}$ to be a smooth immersed submanifold of $\mathcal{F}_{\mathcal{P}(\mathbb{Q})}$, the immersion denoted by $\mathfrak{i}_{\mathcal{EL}}$.

Remark 3.1.13. *In order to make contact with the usual description of classical Mechanics on the cotangent bundle of a configuration space, let us stress that a direct computation shows that the previous equation has solutions ξ coinciding with the integral curves of the vector field Γ satisfying:*

$$i_{\Gamma} \mathrm{d}\Theta_H \, = \, 0 \, , \tag{3.67}$$

and such that Γ_t (the t-component of Γ) is equal to 1[18]*. With this last condition in mind Γ turns to be projectable onto $\mathbf{T}^{\star}\mathcal{Q}$ and, if H is actually the pull-back of a function $\bar{H}$ on $\mathbf{T}^{\star}\mathcal{Q}$, equation* (3.67) *can be seen as an equation on $\mathbf{T}^{\star}\mathcal{Q}$:*

$$i_{\bar{\Gamma}} \bar{\omega} \, = \, \mathrm{d}\bar{H} \tag{3.68}$$

where $\bar{\Gamma}$ is the projection of Γ onto $\mathbf{T}^{\star}\mathcal{Q}$ and $\bar{\omega}$ is the differential form on $\mathbf{T}^{\star}\mathcal{Q}$ whose pull-back to $\mathcal{P}(\mathbb{Q})$ is ω. The latter equation is the usual form of Hamilton equations on the cotangent bundle.

Again, we stress that along the entire manuscript we will consider only dynamical systems fulfilling the assumptions with the motivations explained in Rem. 3.1.6.

A direct computation shows that Γ is fixed by previous equation to be:

$$\Gamma \, = \, \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_j} \frac{\partial}{\partial q^j} - \frac{\partial H}{\partial q^j} \frac{\partial}{\partial p_j} \, , \tag{3.69}$$

and, thus, eventually, $\bar{\Gamma}$ reads:

$$\bar{\Gamma} \, = \, \frac{\partial \bar{H}}{\partial p_j} \frac{\partial}{\partial q^j} - \frac{\partial \bar{H}}{\partial q^j} \frac{\partial}{\partial p_j} \, . \tag{3.70}$$

Let us conclude the section by addressing the example 3.1.7 within the Hamiltonian formalism.

Example 3.1.14 (Free particle on the line)**.** *As in example 3.1.7, we consider a free particle moving on a line, thus, again, the configuration manifold is $\mathcal{Q} \, = \, \mathbb{R}$ and*

[18] This is required in order for ξ to be a section of δ_1.

the fibration (3.1) is $\pi \;:\; \mathbb{Q} = \mathbb{I} \times \mathbb{R} \to \mathbb{I}$ *where* $\mathbb{I} = [a, b] \subset \mathbb{R}$. *Again we will use the system of coordinates* $\{t, q\}$ *on* $\mathbb{Q}$ *where* t *represents the coordinate on* $\mathbb{I}$ *and we will denote by* γ *the sections of* π. *Here, the covariant phase space is* $\mathcal{P}(\mathbb{E}) = \mathbb{I} \times \mathbb{R}^2$ *where we consider* $\{t, q, p\}$ *as a system of coordinates. We will denote by* $\xi = (\gamma, \varrho)$ *elements of* $\Gamma^{\textit{SPLIT}}(\delta_1)$, *delta*$_1$ *denoting the projection* $\delta_1 \;:\; \mathbb{I} \times \mathbb{R}^2 \to \mathbb{I}$. *The Hamiltonian of the dynamical system is:*

$$H = \frac{p^2}{2m}, \tag{3.71}$$

m *representing the mass of the particle, which gives rise to the following action functional:*

$$\mathscr{S}_\xi = \int_a^b \xi^\star \left[p\mathrm{d}q - \frac{p^2}{2m}\mathrm{d}t \right] = \int_a^b \left[\varrho(t)\dot{\gamma}(t) - \frac{\varrho^2(t)}{2m} \right] \mathrm{d}t \,. \tag{3.72}$$

The space $\Gamma^{\textit{SPLIT}}(\delta_1)$ *is not a Banach space. However, it can be equipped with the following norm:*

$$\|\, \xi \,\|_{\sup} = \sup_{t\in\mathbb{I}} |\, \gamma(t) \,| + \sup_{t\in\mathbb{I}} |\, \varrho(t) \,| \,, \tag{3.73}$$

and, as the following chain of inequalities proves[19]

$$\begin{aligned}
\left| \mathscr{S}_\xi - \mathscr{S}_{\tilde{\xi}} \right| &\leq \int_a^b \left| \varrho(t)\dot{\gamma}(t) - \tilde{\varrho}(t)\dot{\tilde{\gamma}}(t) + \frac{\varrho(t)^2}{2m} - \frac{\tilde{\varrho}(t)^2}{2m} \right| \mathrm{d}t \leq \\
&\leq \int_a^b \Bigg[|\, (\varrho(t) - \tilde{\varrho}(t))\dot{\gamma}(t) \,| + \left| \tilde{\varrho}(t)(\dot{\gamma}(t) - \dot{\tilde{\gamma}}) \right| + \\
&\qquad + \frac{1}{2m} |\, \tilde{\varrho}(t)(\varrho(t) - \tilde{\varrho}(t)) \,| + \frac{1}{2m} |\, \varrho(t)(\varrho(t) - \tilde{\varrho}(t)) \,| \Bigg] \mathrm{d}t \leq \\
&\leq (b-a) \Bigg[\sup_{t\in\mathbb{I}} |\, \dot{\gamma}(t) \,| \sup_{t\in\mathbb{I}} |\, \varrho(t) - \tilde{\varrho}(t) \,| + \sup_{t\in\mathbb{I}} |\, \tilde{\varrho}(t) \,| \sup_{t\in\mathbb{I}} |\, \dot{\gamma}(t) - \dot{\tilde{\gamma}}(t) \,| + \\
&\qquad + \frac{1}{2m} \sup_{t\in\mathbb{I}} |\, \varrho(t) \,| \sup_{t\in\mathbb{I}} |\, \varrho(t) - \tilde{\varrho}(t) \,| + \frac{1}{2m} \sup_{t\in\mathbb{I}} |\, \tilde{\varrho}(t) \,| \sup_{t\in\mathbb{I}} |\, \varrho(t) - \tilde{\varrho}(t) \,| \Bigg] ,
\end{aligned} \tag{3.74}$$

$\mathscr{S}$ *is continuous with respect to the* sup*-norm defined above. Indeed, the last term of the chain of inequalities above vanishes as* $\|\, \xi - \tilde{\xi} \,\|_{\sup}$ *approaches zero. Consequently, it can be extended by continuity to the completion* $\overline{\Gamma^{\textit{SPLIT}}(\delta_1)}^{\|\cdot\|_{\sup}} =: \mathcal{F}_{\mathcal{P}(\mathbb{Q})}$ *which is, indeed, a Banach space. The first fundamental formula here reads:*

$$\delta_{\mathbb{X}_\xi} \mathscr{S}_\xi = \int_a^b \xi^\star \left[i_X \left(\mathrm{d}p \wedge \mathrm{d}q - \frac{p}{m} \mathrm{d}p \wedge \mathrm{d}t \right) \right] + \xi^\star \left[i_X \left(p\mathrm{d}q - \frac{p^2}{2m}\mathrm{d}t \right) \right] \Bigg|_a^b , \tag{3.75}$$

where X *is a* δ_1*-vertical vector field on* $\mathcal{P}(\mathbb{Q})$ *defined in a neighborhood of the image of* ξ, *i.e., it is a vector field of the type:*

$$X = X_q \frac{\partial}{\partial q} + X_p \frac{\partial}{\partial p} \,, \tag{3.76}$$

[19] We are denoting by ξ and $\tilde{\xi}$ two different elements of $\Gamma^{\mathrm{SPLIT}}(\delta_1)$.

where X_q and X_p are functions on $\mathcal{P}(\mathbb{Q}) = \mathbb{I} \times \mathbb{R}^2$ defined for all $t \in \mathbb{I}$ and for q and p close[20] *to $\gamma(t)$ and $\varrho(t)$ respectively. On the other hand, $\mathbb{X}_\xi$ is the map:*

$$\mathbb{X}_\xi \;:\; \mathbb{I} \to \mathbf{T}_{\xi(t)}\mathcal{P}(\mathbb{Q}) \;:\; t \mapsto X(\xi(t)) = X_q(t,\,\gamma(t),\,\varrho(t))\left.\frac{\partial}{\partial q}\right|_{\xi(t)} + X_p(t,\,\gamma(t),\,\varrho(t))\left.\frac{\partial}{\partial p}\right|_{\xi(t)} =: \\ =:\ \mathbb{X}_\gamma[\xi]\frac{\delta}{\delta\gamma} + \mathbb{X}_\varrho[\xi]\frac{\delta}{\delta\varrho}\,, \tag{3.77}$$

following the notation (2.52)*. The first term on the right hand side is the (contraction of the tangent vector $\mathbb{X}_\xi$ along the) Euler-Lagrange form associated to our dynamical system. Equation* (3.67) *reads:*

$$\xi^\star\left[\, i_X \mathrm{d}\Theta_H\,\right] = \xi^\star\left[-X_q\mathrm{d}p + X_p\mathrm{d}q - \frac{p}{m}X_p\mathrm{d}t\right] = \\ = \left[-\mathbb{X}_\gamma\,\dot\varrho + \mathbb{X}_\varrho\,\dot\gamma - \frac{\varrho}{m}\mathbb{X}_\varrho\right]\mathrm{d}t = \left[-\mathbb{X}_\gamma\dot\varrho + \mathbb{X}_\varrho\left(\dot\gamma - \frac{\varrho}{m}\right)\right]\mathrm{d}t = 0 \quad \forall\ \mathbb{X}_\gamma,\,\mathbb{X}_\varrho\,, \tag{3.78}$$

which gives:

$$\dot\gamma(t) = \frac{\varrho(t)}{m}\,, \qquad \dot\varrho(t) = 0\,. \tag{3.79}$$

Note that solutions of the latter equations coincide with the integral curves of the vector field Γ satisfying (3.67) *with t-component equal to one. Indeed, given a Γ of this type, namely:*

$$\Gamma = \frac{\partial}{\partial t} + X_q\frac{\partial}{\partial q} + X_p\frac{\partial}{\partial p}\,, \tag{3.80}$$

equation (3.67) *reads:*

$$i_\Gamma\left(\mathrm{d}p\wedge\mathrm{d}q - \frac{p}{m}\mathrm{d}p\wedge\mathrm{d}t\right) = X_p\mathrm{d}q - X_q\mathrm{d}p - \frac{p}{m}X_p\,\mathrm{d}t + \frac{p}{m}\mathrm{d}p = 0\,, \tag{3.81}$$

which gives:

$$X_t = 1\,, \qquad X_p = 0\,, \qquad X_q = \frac{p}{m}\,. \tag{3.82}$$

The integral curves of Γ are curves $(t(s),\,\gamma(s),\,\varrho(s))$ on $\mathcal{P}(\mathbb{Q})$ satisfying:

$$\dot t = 1\,, \qquad \dot\gamma = \frac{\varrho(t)}{m}\,, \qquad \dot\varrho(t) = 0\,, \tag{3.83}$$

which are equivalent to:

$$\dot\gamma(t) = \frac{\varrho(t)}{m}\,, \qquad \dot\varrho(t) = 0\,. \tag{3.84}$$

Let us stress that the Hamiltonian considered is of the type that in Rem. 3.1.13 we called autonomous. Indeed, the function H does not depend on t and, thus, it is the pull-back of a function $\bar H$ on $\mathbf{T}^\star\mathcal{Q} = \mathbb{R}^2$ having the same expression:

$$\bar H = \frac{p^2}{2m}\,. \tag{3.85}$$

[20] In the sense of the sup-norm defined on $\mathcal{F}_{\mathcal{P}(\mathbb{Q})}$.

Consequently, the equations of motion above are equivalent to the Hamiltonian equations associated to the Hamiltonian system $(\mathbf{T}^\star\mathcal{Q},\, \omega,\, \bar{H})$*, where:*

$$\omega \,=\, -\mathrm{d}\theta \,=\, -\mathrm{d}(p\mathrm{d}q) \,=\, \mathrm{d}q\wedge\mathrm{d}p\,, \tag{3.86}$$

is the symplectic structure of $\mathbf{T}^\star\mathcal{Q}$*. Hamiltonian equations associated to such a Hamiltonian system read:*

$$i_{\bar{\Gamma}}\omega \,=\, \mathrm{d}\bar{H}\,, \tag{3.87}$$

for the vector field $\bar{\Gamma}$ *on* $\mathbf{T}\mathcal{Q}$*:*

$$\bar{\Gamma} \,=\, \bar{\Gamma}_q\frac{\partial}{\partial q} + \bar{\Gamma}_p\frac{\partial}{\partial p} \tag{3.88}$$

that are:

$$-\bar{\Gamma}_p\,\mathrm{d}q + \bar{\Gamma}_q\,\mathrm{d}p \,=\, \frac{p}{m}\mathrm{d}p\,, \tag{3.89}$$

which give:

$$\bar{\Gamma}_p \,=\, 0\,, \qquad \bar{\Gamma}_q \,=\, \frac{p}{m}\,. \tag{3.90}$$

The integral curves of the vector field just defined, are curves $(\gamma(t),\, \varrho(t))$ *on* $\mathbf{T}^\star\mathcal{Q}$ *satisfying, again:*

$$\dot{\gamma}(t) \,=\, \frac{\varrho(t)}{m}, \qquad \dot{\varrho}(t) \,=\, 0\,, \tag{3.91}$$

that collectively says that γ *satisfies:*

$$\ddot{\gamma}(t) \,=\, 0\,. \tag{3.92}$$

3.1.3. Hamiltonian mechanical systems with constraints

As stressed many times in the previous sections, the main difference between the Lagrangian and the Hamiltonian formalism is that in the former the equations of motion appear as a system of implicit differential equations whereas in the latter they are explicit ones. This is related with the fact that the form $\omega_{\mathscr{L}}$ appearing in (3.17) is the pull-back to $\mathbf{J}^1\pi$ of a generally pre-symplectic structure on $\mathbf{T}\mathcal{Q}$ while the form ω appearing in (3.60) is the pull-back to $\mathcal{P}(\mathbb{Q})$ of a symplectic structure on $\mathbf{T}^\star\mathcal{Q}$. In the autonomous case, i.e. when the Lagrangian (resp. the Hamiltonian) does not depend explicitly on t, this results in the fact that solutions of (3.19) are solutions of a pre-symplectic Hamiltonian system (see (3.27)), whereas solutions of (3.67) are solutions of a Hamiltonian system (see (3.68)).

With this in mind, it is clear that the two formalisms are independent of each other, in general, and that only when some conditions are met, they can be related.

In order to outline such conditions, let us start by defining the so called FIBER DERIVATIVE.

Definition 3.1.15 (Fibre derivative). *Given a Lagrangian, $\lambda = \mathscr{L}\mathrm{d}t$, on $\mathbf{J}^1\pi$, the* fibre derivative *associated to $\mathscr{L}$, say $\mathrm{F}_{\mathscr{L}}$, is the morphism (over the identity) of fibre bundles from $\mathbf{J}^1\pi \to \mathbb{Q}$ to $\mathcal{P}(\mathbb{Q}) \to \mathbb{Q}$ such that:*

$$i_u\left[\,\mathrm{F}_{\mathscr{L}}(v)\,\right] \,=\, \frac{d}{ds}\mathscr{L}(t,\,q,\,v+su)\bigg|_{s=0}\,, \tag{3.93}$$

for all the v and u belonging to the same fibre of $\mathbf{J}^1\pi \to \mathbb{Q}$. The morphism of fibre bundles reads:

$$\mathrm{F}_{\mathscr{L}} \quad : \quad \mathbf{J}^1\pi \to \mathcal{P}(\mathbb{Q}) \quad : \quad (t,\,q^j,\,v^j) \mapsto \left(t,\,q^j,\,\frac{\partial \mathscr{L}}{\partial v^j}\right)\,. \tag{3.94}$$

If the fibre derivative above is a diffeomorphism of fibre bundles, then, given a Lagrangian, it is canonically defined a local function on $\mathcal{P}(\mathbb{Q})$, i.e. a Hamiltonian, by:

$$H \,=\, \mathrm{F}_{\mathscr{L}}^{-1\star}\left[\,p_j v^j - \mathscr{L}\,\right]\,, \tag{3.95}$$

the structure $\omega_{\mathscr{L}}$ of (3.17) turns out to be the pull-back of the structure ω of (3.60):

$$\omega_{\mathscr{L}} \,=\, \mathrm{d}\left(\frac{\partial \mathscr{L}}{\partial v^j}\right) \wedge \mathrm{d}q^j \,=\, \mathrm{d}\,(\,\mathrm{F}_{\mathscr{L}}^{\star} p_j\,) \wedge \mathrm{d}q^j \,=\, \mathrm{F}_{\mathscr{L}}^{\star}\omega\,, \tag{3.96}$$

and (3.19) is the pull-back via the diffeomorphism $\mathrm{F}_{\mathscr{L}}$ of (3.67), that is, the Lagrangian and the Hamiltonian formalism, being related by a diffeomorphism, are equivalent. In particular, in this case, also the Euler-Lagrange equations become explicit ones. On the other hand, when $\mathrm{F}_{\mathscr{L}}$ is not a diffeomorphism, the two formalisms are completely independent of each other.

It is relevant to stress, at this point, that, having in mind the previous discussion, it is clear that within the Lagrangian formalism the equations of motion may be explicit or not depending on the regularity of the Lagrangian (i.e., if it defines a fibre derivative which is a diffeomorphism or not), whereas within the Hamiltonian formalism the equations of motion always appear as a system of explicit differential equations. Indeed, in order to take into account the possibility of having, within the Hamiltonian formalism, equations of motion where, apart from the explicit "evolutionary" equations, also some constraint relations appear like the ones emerging within the Lagrangian formalism (when $\omega_{\mathscr{L}}$ is pre-symplectic), we will use a method related with the Lagrange multipliers theorem. In particular, we will introduce constraints within the Hamiltonian formulation by extremizing the action $\mathscr{S}$ not on the whole space $\mathcal{F}_{\mathcal{P}(\mathbb{Q})}$ but rather on a subset of it, say Ξ, obtaining, apart from the explicit Hamilton equations, also the constraint relations selecting Ξ inside $\mathcal{F}_{\mathcal{P}(\mathbb{Q})}$ as equations of motion.

To deal with this situation in the examples we will consider along the manuscript it is sufficient to recall the following version of the Lagrange multipliers theorem on Banach spaces.

Theorem 3.1.16 (Lagrange multipliers theorem). *Let $\mathcal{M}$ be a Banach space and let $\mathscr{F}$ be a real-valued differentiable function on $\mathcal{M}$. Let $\mathcal{N}$ be a Banach space and Φ a smooth injective map from $\mathcal{N}$ to $\mathcal{C} = \Phi(\mathcal{N})$ with non-degenerate tangent map. Let $\mathcal{M}^\star$ denote the dual of $\mathcal{M}$ and let us define the real-valued function $\mathscr{F}^{\mathrm{ext}}$ on $\mathcal{M} \times \mathcal{M}^\star \times \mathcal{N}$:*

$$\mathscr{F}^{\mathrm{ext}}_{(m,\Lambda,n)} \;=\; \mathscr{F}_m + \langle \Lambda,\, m - \Phi(n)\rangle \tag{3.97}$$

where Λ represents a point in $\mathcal{M}^\star$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ represents the pairing between $\mathcal{M}$ and its dual.

Then, m is a critical point for $\mathscr{F}\Big|_{\mathcal{C}}$ iff (m, Λ, n) is a critical point of $\mathscr{F}^{\mathrm{ext}}$.

Proof. First, let us prove that if $m \in \mathcal{C}$ is a critical point for $\mathscr{F}$ then, there exist $\Lambda \in \mathcal{M}^\star$ and $n \in \mathcal{N}$ such that $(m,\,\Lambda,\,n)$ is a critical point for $\mathscr{F}^{\mathrm{ext}}$. The n is determined by the fact that, since Φ is injective and m is in the image of Φ, then there exists a unique $n \in \mathcal{N}$ such that $m = \Phi(n)$. What is more, since the tangent map of Φ is non-degenerate, for any $X_n \in \mathbf{T}_n\mathcal{N}$ there exists a $X_m \in \mathbf{T}_{\Phi(n)}\mathcal{C}$ such that $\Phi_\star X_n \;=\; X_m$. Now, since $m \in \mathcal{C}$ is critical for $\mathscr{F}$, then $\mathrm{d}\mathscr{F}_m\Big|_{\mathcal{C}} \;=\; 0$. On the other hand denoting by X_m, X_Λ and X_n the components of a tangent vector $X_{(m,\Lambda,n)} \in \mathbf{T}_{(m,\Lambda,n)}\mathcal{M} \times \mathcal{M}^\star \times \mathcal{N}$, $\mathrm{d}\mathscr{F}^{\mathrm{ext}}$ is computed to be:

$$\mathrm{d}\mathscr{F}^{\mathrm{ext}}_{(m,\Lambda,n)}(X_{(m,\Lambda,n)}) \;=\; \mathrm{d}\mathscr{F}_m(X_m) + \langle X_\Lambda,\, m - \Phi(n)\rangle + \langle \Lambda,\, X_m - \Phi_\star X_n\rangle\,. \tag{3.98}$$

Now, the three terms on the right hand side of the latter equation all vanish because $\mathrm{d}\mathscr{F}_m\Big|_{\mathcal{C}} \;=\; 0$, $m \;=\; \Phi(n)$ and $X_m \;=\; \Phi_\star X_n$ (i.e. X_m and X_n are Φ-related).

Now, let us prove the converse, that is, that if (m, Λ, n) is a critical point for $\mathscr{F}^{\mathrm{ext}}$ then m is a critical point for $\mathscr{F}$. Since (m, Λ, n) is a critical point for $\mathscr{F}^{\mathrm{ext}}$ then $\mathrm{d}\mathscr{F}^{\mathrm{ext}}_{(m,\Lambda,n)}(X_{(m,\Lambda,n)}) = 0$ for all $X_{(m,\Lambda,n)}$. In particular, if we consider a $X_{(m,\Lambda,n)}$ only having component X_Λ, then Eq. (3.98) gives $m = \Phi(n)$. This tells us that critical points of $\mathscr{F}^{\mathrm{ext}}$ are such that m is the image via Φ of some $n \in \mathcal{N}$, i.e., $m \in \mathcal{C}$. Now, since Φ is injective and its tangent map is non-degenerate then for any $X_n \in \mathbf{T}_n\mathcal{N}$ there exists $X_m \in \mathbf{T}_m\mathcal{C}$ such that $X_m \;=\; \Phi_\star X_n$. Therefore, tangent vectors to critical points of $\mathscr{F}^{\mathrm{ext}}$ are such that the components X_m and X_n are related by the equality $X_m \;=\; \Phi_\star X_n$. Consequently, by looking at Eq. (3.98) we get that if (m, Λ, n) is a critical point for $\mathscr{F}^{\mathrm{ext}}$, then $\mathrm{d}\mathscr{F}^{\mathrm{ext}}_{(m,\Lambda,n)} \;=\; \mathrm{d}\mathscr{F}_m \;=\; 0$. □

By looking at our situation, if $\mathcal{F}_{\mathcal{P}(\mathbb{Q})}$ is a Banach space (as it will be in the examples considered) and if the subset of fields Ξ where we want to search for extrema of $\mathscr{S}$ is the image into $\mathcal{F}_{\mathcal{P}(\mathbb{Q})}$ of a map Φ satisfying the hypothesis of the latter proposition, then we can search for the extrema of $\mathscr{S}$ restricted to Ξ by searching for the extrema of the functional $\mathscr{S}^{\mathrm{ext}}$ defined on $\mathcal{F}_{\mathcal{P}(\mathbb{Q})} \times \mathcal{F}^\star_{\mathcal{P}(\mathbb{Q})} \times \mathcal{N}$:

$$\mathscr{S}^{\mathrm{ext}}_{(\xi,\Lambda,n)} \;=\; \mathscr{S}_\xi + \langle \Lambda,\, \xi - \Phi(n)\rangle\,. \tag{3.99}$$

Explicitly, in the system of local coordinates chosen, the term $\langle \Lambda, \xi - \Phi(n) \rangle$ reads:

$$\langle \Lambda, \xi - \Phi(n) \rangle = \int_{\mathbb{I}} \Big[\Lambda_{\gamma_j} \Big(\gamma^j - \gamma^j \circ \Phi(n) \Big) + \Lambda_{\varrho}{}^j \left(\varrho_j - \varrho_j \circ \Phi(n) \right) \Big] \, \mathrm{d}t \,, \tag{3.100}$$

where $(\Lambda_{\gamma_j}, \Lambda_{\varrho}{}^j)$ is a system of local coordinates on $\mathcal{F}^{\star}_{\mathcal{P}(\mathbb{Q})}$. Consequently, $\mathscr{S}^{\mathrm{ext}}$ explicitly reads:

$$\mathscr{S}^{\mathrm{ext}}_{(\xi,\Lambda,n)} = \int_{\mathbb{I}} \Big[P^{\mu}_a \partial_\mu \phi^a - H(\xi) + \Lambda_{\gamma_j} \Big(\gamma^j - \gamma^j \circ \Phi(n) \Big) + \Lambda_{\varrho}{}^j \left(\varrho_j - \varrho_j \circ \Phi(n) \right) \Big] \, \mathrm{d}t \,. \tag{3.101}$$

3.2. Classical field theories

We now proceed with the multi-symplectic formulation of field theories, both in the Lagrangian and in the Hamiltonian formalism.

3.2.1. Lagrangian formulation

The Lagrangian formulation of first order field theories works similarly to that of Newtonian mechanical systems described in Sec. 3.1.1. The main ingredient here is an n-dimensional ($n := d + 1$) space-time $\mathscr{M}$ playing the role of the interval $\mathbb{I} \subset \mathbb{R}$ in Sec.3.1.1 and whose points (EVENTS) are used to parametrize the fields of the theory. We will denote local coordinates on an open neighborhood of $\mathscr{M}$ by $\{ x^\mu \}_{\mu=0,\ldots,d}$. From the mathematical point of view, the very aspect that will be used to carry on the theory is that $\mathscr{M}$ is orientable and, thus, that a volume form can be fixed on it, say $vol_{\mathscr{M}} = \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^d$ in the local chart considered. Actually in most of the examples considered $\mathscr{M}$ will be a space-time (i.e., a solution of EINSTEIN EQUATIONS) even if relevant examples (such as POISSON SIGMA MODELS) exist, in which $\mathscr{M}$ is just an orientable smooth manifold.

In general, $\mathscr{M}$ is allowed to be a smooth n-dimensional, orientable manifold with boundary. The boundary will be assumed to be a smooth $(n-1)$-dimensional, orientable manifold (not necessarily connected) properly embedded in $\mathscr{M}$ and, consequently, there exists a COLLAR around it, say $C_\epsilon = [0, \epsilon) \times \partial\mathscr{M}$ and an embedding $\mathfrak{i}_{C_\epsilon} : C_\epsilon \hookrightarrow \mathscr{M}$, such that $i(\{0\} \times \partial\mathscr{M}) = \partial\mathscr{M}$, and $\mathfrak{i}^{\star}_{C_\epsilon} vol_{\mathscr{M}} = \mathrm{d}x^0 \wedge vol_{\partial\mathscr{M}}$ where $vol_{\partial\mathscr{M}}$, is a volume form on $\partial\mathscr{M}$.

The fields of the theory are modelled as local sections of a bundle of the type (3.1), say:

$$\begin{tikzcd} \mathbb{E} \arrow[d, "\pi"] \\ \mathscr{M} \arrow[u, dashed, bend left, "\phi"] \end{tikzcd} \tag{3.102}$$

with typical fibre denoted by $\mathcal{E}$.

Remark 3.2.1. *Within the examples considered throughout the manuscript, $\mathbb{E}$ will be a vector bundle over $\mathscr{M}$. However, there exist examples in which $\mathcal{E}$ is a Poisson or a Jacobi manifold, like in Poisson and Jacobi sigma-models [87], or a Lie group, like in Quantum Mechanics where $\mathbb{E}$ represents the complex projective space of a complex separable Hilbert space which, for instance, when the Hilbert space is $\mathbb{C}^2$, is a principal bundle over it.*

We will denote local fibered coordinates on $\mathbb{E}$ by $\{\, x^\mu,\; u^a \,\}_{\mu=0,\dots,d;\, a=1,\dots,\dim\mathcal{E}}$.

As in Newtonian mechanical systems, the theory is settled on the first order jet bundle of $\mathbb{E}$, say:

$$\begin{tikzcd} \mathbf{J}^1\pi \arrow[d, "\pi^1_0"] \arrow[dd, bend right=60, "\pi_1"'] \\ \mathbb{E} \arrow[d, "\pi"] \\ \mathscr{M} \end{tikzcd} \tag{3.103}$$

equipped with the system of local coordinates $\{\, x^\mu,\; u^a,\; z^a_\mu \,\}_{\mu=0,\dots,d;\, a=1,\dots,\dim\mathcal{E}}$ adapted to the fibration π.

A Lagrangian and an action functional can be defined analogously to Sec. 3.1.1.

Definition 3.2.2 (Lagrangian)**.** *A Lagrangian is a π_1-horizontal 1-form on $\mathbf{J}^1\pi$, i. e., a differential 1-form on $\mathbf{J}^1\pi$, say λ, such that $i_V\lambda = 0 \quad \forall\, V$ being π_1-vertical. In the system of local coordinates chosen it reads:*

$$\lambda = \mathscr{L}(x, u, z)\, vol_{\mathscr{M}} \,, \tag{3.104}$$

where $\mathscr{L}$ is a smooth function on $\mathbf{J}^1\pi$.

Definition 3.2.3 (Action functional)**.** *Given a Lagrangian over $\mathbf{J}^1\pi$, an action functional is a real-valued function on the space of sections of π, $\Gamma(\pi)$, given by:*

$$\mathscr{S} \;:\; \Gamma(\pi) \to \mathbb{R} \;:\; \phi \mapsto \mathscr{S}_\phi = \int_{\mathscr{M}} \left(j^1\phi\right)^\star \lambda \tag{3.105}$$

whose coordinate expression is:

$$\mathscr{S}_\phi = \int_{\mathscr{M}} \mathscr{L}(x,\, \phi(x),\, \partial\phi(x))vol_{\mathscr{M}} \,, \tag{3.106}$$

and where $j^1\phi \in \Gamma(\pi_1)$ is the first order jet prolongation of the section ϕ defined in 2.4.1.

Remark 3.2.4. *As in Sec. 3.1.1, we will assume we are able to perform a suitable completion of the type described in Rem. 3.1.3 in order to define a suitable Banach manifold of fields on which* (3.105) *is well defined, that we will denote by $\mathcal{F}_{\mathbb{E}}$. The validity of such assumption will be verified case by case in the examples considered.*

As for the case of Mechanics, because of the form of the action, the Lagrangian is actually defined up to the addition of a contact form, i. e., a form such that:

$$\left(j^1\phi\right)^\star \eta \,=\, 0 \quad \forall \phi \,, \tag{3.107}$$

and that takes the following form in the chart considered:

$$\begin{aligned}\eta \,=&\, \eta^{(1)}_{a\mu_1\ldots\mu_d}\omega^a \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_d} + \eta^{(2)\ \ \ \ \ \ \ \ \nu}_{a\mu_1\ldots\mu_{d-1}b}\,\omega^a \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_{d-1}} \wedge \mathrm{d}z^b_\nu + \ldots \\ &+ \eta^{(d)\ \ \ \ \nu_1\ldots\nu_d}_{ab_1\ldots b_d}\,\omega^a \wedge \mathrm{d}z^{b_1}_{\nu_1} \wedge \ldots \wedge \mathrm{d}z^{b_d}_{\nu_d} + \text{TERMS OF HIGHER ORDER OF CONTACTNESS}\,,\end{aligned} \tag{3.108}$$

where $\omega^a \,=\, \mathrm{d}u^a - z^a_\mu \mathrm{d}x^\mu$ and by "terms of higher order of contactness" we mean all the other terms containing two or more ω^a. The terms involving one ω^a are called 1-contact forms whereas the terms involving wedge products of k ω^a are called k-contact forms.

The variation of the action functional is defined analogously to what we did in Sec. 3.1.1. Consider a tangent vector $\mathbb{X}_\phi \in \mathbf{T}_\phi \mathcal{F}_\mathbb{E}$ and an extension of it to a vector field X over $\mathbb{E}$ defined in an open neighborhood of the image of ϕ. Denote by F^X_s the flow of X and consider the following one-parameter family of sections of π:

$$\phi_s \,=\, F^X_s \circ \phi \,. \tag{3.109}$$

The same steps of equation (3.10) show that the variation of $\mathscr{S}$ along the direction $\mathbb{X}_\phi$ reads:

$$\delta_{\mathbb{X}_\phi}\mathscr{S}_\phi \,=\, \int_{\mathscr{M}} \left(j^1\phi\right)^\star i_{X^1}\mathrm{d}\lambda + \int_{\partial\mathscr{M}} \left(j^1\phi\right)^\star i_{X^1}\lambda \,, \tag{3.110}$$

where X^1 is the first order jet prolongation of X:

$$X^1 \,=\, X^\mu_x \frac{\partial}{\partial x^\mu} + X^a \frac{\partial}{\partial u^a} + \left(\frac{\partial}{\partial x^\mu} X^a_u - z^a_\mu d_\lambda X^\lambda_x\right) \frac{\partial}{\partial z^a_\mu} \,, \tag{3.111}$$

where $d_\lambda \,=\, \frac{\partial}{\partial x^\lambda} + z^a_\lambda \frac{\partial}{\partial u^a}$ is the total derivative operator. The latter formula is the analogous of equation (2.201). Again, because of the expression of X^1, integration by parts may give rise to boundary terms also from the first integral in the variation of $\mathscr{S}$. Therefore, as we did in Sec. 3.1.1, we fix the term η that we could add to λ in order to cancel all the possible boundary terms emerging from the first integral. Differently from the case of Mechanics, here the condition:

$$(j^1\phi)^\star \left[i_V \mathrm{d}(\lambda + \eta)\right] \,=\, 0 \,, \tag{3.112}$$

for all π^1_0-vertical vector field is not sufficient to univoquely define η. Indeed, all the terms of higher order of contactness vanish after the pull-back and remain completely undetermined. Following the approach of *Snyaticki et al.* (see [38], [80], [82]) a possibility to fix a unique η is the following supplementary condition:

$$i_V\, i_U\, \eta \,=\, 0 \,, \tag{3.113}$$

for all U and V π_1-vertical vector fields. It is easy to see that condition (3.113) fix $\eta^{(2)\ \ \ \ \ \nu}_{a\mu_1\dots\mu_{d-1}b}$, ..., $\eta^{(d)\ \ \ \nu_1\dots\nu_d}_{ab_1\dots b_d}$ together with the coefficients of the higher contact terms to vanish. At this point condition (3.112) is enough to fix η to be:

$$\eta \,=\, \eta^{(1)}_{a\mu_1\dots\mu_d}\omega^a \wedge \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_d} \tag{3.114}$$

where:

$$\eta^{(1)}_{a\mu_1\dots\mu_d} \,=\, \epsilon_{\lambda\mu_1\dots\mu_d}\frac{\partial \mathscr{L}}{\partial z^a_\lambda}\,, \tag{3.115}$$

where ϵ denotes the totally skew-symmetric symbol, and, thus, with a $\Theta_{\mathscr{L}}$ of the form:

$$\Theta_{\mathscr{L}} \,=\, \lambda + \eta \,=\, \mathscr{L}\, vol_{\mathscr{M}} + \frac{\partial \mathscr{L}}{\partial z^a_\lambda}\left(\mathrm{d}u^a - z^a_\nu \mathrm{d}x^\nu\right) \wedge i_\lambda vol_{\mathscr{M}}\,. \tag{3.116}$$

Again, this is the so called Lepagean equivalent of λ. Its differential reads:

$$\mathrm{d}\Theta_{\mathscr{L}} \,=\, \omega_{\mathscr{L}} + \mathrm{d}E_{\mathscr{L}} \wedge vol_{\mathscr{M}}\,, \tag{3.117}$$

where:

$$\omega_{\mathscr{L}} \,=\, \mathrm{d}\left(\frac{\partial \mathscr{L}}{\partial z^a_\mu}\right) \wedge \mathrm{d}u^a \wedge i_\mu vol_{\mathscr{M}}\,, \tag{3.118}$$

and:

$$E_{\mathscr{L}} \,=\, \mathscr{L} - z^a_\mu \frac{\partial \mathscr{L}}{\partial z^a_\mu}\,. \tag{3.119}$$

In terms of the Lepagean equivalent above, the first variational formula reads:

$$\delta_{\mathbb{X}_\phi}\mathscr{S}_\phi \,=\, \int_{\mathscr{M}}\left(j^1\phi\right)^\star i_{X^1}\mathrm{d}\Theta_{\mathscr{L}} + \int_{\partial\mathscr{M}}\left(j^1\phi\right)^\star i_{X^1}\Theta_{\mathscr{L}}\,, \tag{3.120}$$

where, now, the first integral on the right hand side does not give rise to any boundary term. Thus, in terms of $\Theta_{\mathscr{L}}$ the Schwinger-Weiss variational principle gives rise to the following equations of the motion:

$$\left(j^1\phi\right)^\star\left[i_{X^1}\mathrm{d}\Theta_{\mathscr{L}}\right] \,=\, 0 \quad \forall\ X \in \mathfrak{X}^v(\mathbb{E})\,, \tag{3.121}$$

which takes the following form in the chosen system of local coordinates:

$$d_\lambda \frac{\partial \mathscr{L}}{\partial z^a_\lambda}(x, \phi(x), \partial\phi(x)) - \frac{\partial \mathscr{L}}{\partial u^a}(x, \phi(x), \partial\phi(x)) \,=\, 0\,. \tag{3.122}$$

Remark 3.2.5. *An interpretation of the solutions of* (3.121) *as integral curves of a vector field like the one given in Rem. 3.1.6 require more work and will be given in Sec. 4.2.*

Let us conclude the section with a few examples in order to fix the ideas about the machinery introduced.

Example 3.2.6 (Free Klein-Gordon theory)**.** *Let us consider a free real Klein-Gordon field on the Minkowski space-time, i.e., a field describing a relativistic real boson with mass m evolving in absence of any external potential. In this case* *the space-time is the Minkowski space-time* $\mathscr{M} = (\mathbb{R}^4,\, \eta)^{21}$, *$\eta$ being the Minkowski metric, and the space on which the field takes values is $\mathcal{E} = \mathbb{R}$. Consequently, the bundle (3.102) reads $\pi \;:\; \mathbb{E} = \mathscr{M} \times \mathbb{R} \to \mathscr{M}$. We denote by $\{\, x^\mu,\, u \,\}_{\mu=0,\ldots,3}$ a system of coordinates on $\mathbb{E}$ where $\{\, x^\mu \,\}_{\mu=0,\ldots,3}$ is the system of coordinates chosen on $\mathscr{M}$. Furthermore, we denote by ϕ the sections of π. The first order jet bundle $\mathbf{J}^1\pi$ reads $\mathbf{J}^1\pi = \mathscr{M} \times \mathbb{R} \times \mathbb{R}^4$ where we consider $\{\, x^\mu,\, u,\, z^\mu \,\}_{\mu=0,\ldots,3}$ as a system of (global) coordinates. The Lagrangian describing the theory is:*

$$\lambda = \frac{1}{2} \left(\eta^{\mu\nu} z_\mu z_\nu - m^2 u^2 \right) vol_{\mathscr{M}} \,, \tag{3.123}$$

where $\eta^{\mu\nu}$ are the coefficients of the inverse of η and where m represents the mass of the boson. The action functional obtained is:

$$\mathscr{S}_\phi = \int_{\mathscr{M}} \left(j^1\phi \right)^\star \lambda = \int_{\mathscr{M}} \frac{1}{2} \left[\eta^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) - m^2 \phi^2(x) \right] vol_{\mathscr{M}} \,. \tag{3.124}$$

It is clear that such action functional is not well defined on the whole space of smooth sections of π. Indeed, it is clear that, in order for the integral to be well defined, the ϕ's should be at least square integrable (with respect to $vol_{\mathscr{M}}$) together with their first derivatives. Therefore, we will consider it to be defined on the subset of $\Gamma(\pi)$ given by smooth sections for which the norm:

$$\|\phi\|^2_{\mathcal{H}^1} = \int_{\mathscr{M}} \left[|\, \phi(x) \,|^2 + \sum_{\mu=0}^{3} |\, \partial_\mu \phi(x) \,|^2 \right] vol_{\mathscr{M}} \,, \tag{3.125}$$

is finite, say $\Gamma(\pi)_1$. The following chain of inequalities:

$$\begin{aligned}
\left| \mathscr{S}_\phi - \mathscr{S}_{\tilde\phi} \right| \leq & \frac{1}{2} \int_{\mathscr{M}} \left| \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \eta^{\mu\nu} \partial_\mu \tilde\phi \partial_\nu \tilde\phi + m^2 (\phi^2 - \tilde\phi^2) \right| vol_{\mathscr{M}} \leq \\
\leq & \frac{1}{2} \int_{\mathscr{M}} \Big[\left| \eta^{\mu\nu} \partial_\mu \phi (\partial_\nu \phi - \partial_\nu \tilde\phi) \right| + \left| \eta^{\mu\nu} \partial_\mu \tilde\phi (\partial_\nu \phi - \partial_\nu \tilde\phi) \right| + \\
& + m^2 \left| \phi (\phi - \tilde\phi) \right| + m^2 \left| \tilde\phi (\phi - \tilde\phi) \right| \Big] vol_{\mathscr{M}} \leq \\
\leq & \frac{1}{2} \Bigg[\sum_{\mu=0}^{3} \|\, \partial_\mu \phi \,\|_{\mathcal{L}^2} \|\, \partial_\mu \phi - \partial_\mu \tilde\phi \,\|_{\mathcal{L}^2} + \sum_{\mu=0}^{3} \|\, \partial_\mu \tilde\phi \,\|_{\mathcal{L}^2} \|\, \partial_\mu \phi - \partial_\mu \tilde\phi \,\|_{\mathcal{L}^2} + \\
& + m^2 \|\, \phi \,\|_{\mathcal{L}^2} \|\, \phi - \tilde\phi \,\|_{\mathcal{L}^2} + m^2 \|\, \tilde\phi \,\|_{\mathcal{L}^2} \|\, \phi - \tilde\phi \,\|_{\mathcal{L}^2} \Bigg] \,,
\end{aligned} \tag{3.126}$$

shows that $\mathscr{S}$ is continuous in the norm $\|\,\cdot\,\|_{\mathcal{H}^1}$. Indeed, if $\|\, \phi - \tilde\phi \,\|_{\mathcal{H}^1}$ approaches zero, then also $\|\, \phi - \tilde\phi \,\|_{\mathcal{L}^2}$ and $\|\, \partial_\mu \phi - \partial_\mu \tilde\phi \,\|_{\mathcal{L}^2}$ approach zero. Consequently, $\mathscr{S}$ can be extended by continuity to the completion $\overline{\Gamma(\pi)_1}^{\|\,\cdot\,\|_{\mathcal{H}^1}} = \mathcal{H}^1(\mathscr{M},\, vol_{\mathscr{M}}) =: \mathcal{F}_{\mathbb{E}}$. The contact form (3.115) here reads:

$$\eta = \epsilon_{\lambda\mu_1\mu_2\mu_3} \eta^{\mu\lambda} z_\mu \left(\mathrm{d}u - z_\rho \mathrm{d}x^\rho \right) \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \,, \tag{3.127}$$

[21]Here and in the rest of the manuscript we will adopt the signature $(+ - - -)$ for η.

and it gives rise to the following Lepage equivalent:

$$\Theta_{\mathscr{L}} \,=\, \frac{1}{2}\Big(\eta^{\mu\nu}z_\mu z_\nu - m^2u^2\Big)\,vol_{\mathscr{M}} + \eta^{\mu\lambda}z_\mu \mathrm{d}u \wedge i_\lambda vol_{\mathscr{M}}\,. \tag{3.128}$$

The Lepage equivalent above gives rise to the following first fundamental formula:

$$\delta_{\mathbb{X}_\phi}\mathscr{S}_\phi \,=\, \int_{\mathscr{M}}\Big(j^1\phi\Big)^\star\Big[i_{X^1}\Big((\eta^{\mu\nu}z_\mu \mathrm{d}z_\nu - m^2u\mathrm{d}u)\wedge vol_{\mathscr{M}} + \eta^{\mu\lambda}\mathrm{d}z_\mu\wedge\mathrm{d}u\wedge i_\lambda vol_{\mathscr{M}}\Big)\Big]\,, \tag{3.129}$$

where X^1 is the first order jet prolongation of a π-vertical vector field X on $\mathbb{E}$ defined in a neighborhood of the image of ϕ, i.e., it is a vector field of the type:

$$X^1 \,=\, X_u\frac{\partial}{\partial u} + z_\mu\frac{\partial X_u}{\partial u}\frac{\partial}{\partial z_\mu}\,, \tag{3.130}$$

where X_u is a function on $\mathscr{M}\times\mathbb{R}$ defined for all $x\in\mathscr{M}$ and for u close[22] *to $\phi(x)$. On the other hand, $\mathbb{X}_\phi$ is the map:*

$$\mathbb{X}_\phi \;:\; \mathscr{M}\to\mathbf{T}_{\phi(x)}\mathbb{E} \;:\; x\mapsto X(\phi(x)) \,=\, X_u(x,\,\phi(x))\frac{\partial}{\partial u}\bigg|_{\phi(x)} =:\, \mathbb{X}_\phi\frac{\delta}{\delta\phi}\,, \tag{3.131}$$

following the notation (2.52)*. The right hand side of the first fundamental formula above is the (contraction of the tangent vector $\mathbb{X}_\phi$ along the) Euler-Lagrange form associated to the Lagrangian describing our field theory. Equation* (3.121) *reads:*

$$\begin{aligned}\Big(j^1\phi\Big)^\star\left[i_{X^1}\mathrm{d}\Theta_{\mathscr{L}}\right] \,=\, \Big(j^1\phi\Big)^\star\Bigg[&\left(\eta^{\mu\nu}z_\mu z_\nu\frac{\partial X_u}{\partial u} - m^2uX_u\right)vol_{\mathscr{M}} + \\ &+\eta^{\mu\nu}\left(z_\mu\frac{\partial X_u}{\partial u}\mathrm{d}u - X_u\mathrm{d}z_\mu\right)\wedge i_\nu vol_{\mathscr{M}}\Bigg] = \\ =\, -\mathbb{X}_\phi&\Big(\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + m^2\phi\Big)\,vol_{\mathscr{M}} \,=\, 0 \quad\forall\,\mathbb{X}_\phi\,,\end{aligned} \tag{3.132}$$

which gives:

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + m^2\phi \,=\, 0\,, \tag{3.133}$$

being the celebrated Klein-Gordon equations.

Example 3.2.7 (Free Electrodynamics)**.** *Let us now consider a sourceless electromagnetic field on the Minkowski space-time $\mathscr{M} = (\mathbb{R}^4,\,\eta)$. This is a gauge field theory (the easiest example, being an Abelian one) whose structure group is $U(1)$ and, therefore, the fields of the theory are connection one-forms on the principal bundle $P = \mathscr{M}\times U(1)\to\mathscr{M}$ that represent the quadri-potential in the covariant formulation of classical Electrodynamics. These are represented as $\mathfrak{u}(1)$-valued 1-forms on $\mathscr{M}$ where $\mathfrak{u}(1) = i\mathbb{R}$ is the Lie algebra of the Lie group $U(1)$. Thus, they are sections of the bundle $\pi \;:\; \mathbb{E} = \mathbf{T}^\star\mathscr{M}\to\mathscr{M}$. We denote by $\{\,x^\mu,\,u_\mu\,\}_{\mu=0,\ldots,3}$ a*

[22] In the sense of the $\mathcal{H}^1$-norm defined on $\mathcal{F}_{\mathbb{E}}$.

system of coordinates on $\mathbb{E}$ *where* $\{ x^\mu \}_{\mu=0,\dots,3}$ *is the system of coordinates chosen on* $\mathscr{M}$*. Furthermore, we denote by* $A = A_\mu(x)\mathrm{d}x^\mu$ *the sections of* π*. The first order jet bundle* $\mathbf{J}^1\pi$ *is the trivial bundle over* $\mathscr{M}$ *whose typical fibre is* $\mathbf{T}^\star_m\mathscr{M} \times \left(\bigotimes^2 \mathbf{T}^\star_m\mathscr{M} \right)$ *where we consider* $\{ x^\mu, \, u_\mu, \, z_{\mu\nu} \}_{\mu,\nu=0,\dots,3}$ *as a system of coordinates. The Lagrangian describing the theory is:*

$$\lambda \; = \; -\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma} z_{[\mu\nu]} z_{[\rho\sigma]} \, vol_{\mathscr{M}} \, , \tag{3.134}$$

where the subscript $[\,\cdot\,]$ *denotes that we are considering the skewsymmetric part of the tensor. The action functional obtained is:*

$$\mathscr{S}_A \; = \; \int_{\mathscr{M}} \left(j^1 A \right)^\star \lambda \; = \; -\int_{\mathscr{M}} \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\partial_{[\mu}A_{\nu]}\partial_{[\rho}A_{\sigma]} \, vol_{\mathscr{M}} \; =: \; -\int_{\mathscr{M}} \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma} F_{\mu\nu}F_{\rho\sigma} \, . \tag{3.135}$$

Geometrically, the coefficients $F_{\mu\nu}$ *represents the components of the curvature of the connection* A*. It is clear that such action functional is not well defined on the whole space of smooth sections of* π*. Indeed, in order for the integral to be well defined, the* A*'s should be such that the functions* $F_{\mu\nu}$ *were at least square integrable. This is achieved by considering the* A_μ *to be* $\mathcal{H}^1$ *functions, as we are going to explain, but let us stress that this is not the minimal requirement in order to get* $F_{\mu\nu}$*'s being square integrable. Indeed, in order to get square integrable* $F_{\mu\nu}$*'s one should only require the first derivatives of the* A*'s to be square integrable but nothing is required, in principle, about the convergence properties of the* A*'s themselves. However, we are going to require the finiteness of the* $\mathcal{H}^1$*-norm because it will be useful in order to work with a well defined Hilbert space of fields, even if it is clear that from the physical point of view we are excluding some situations, namely all those (actually never observed in nature up to now) in which the potential* A_μ *is different from zero at infinity, such as the magnetic monopole. Therefore, we will consider the action functional to be defined on the subset of* $\Gamma(\pi)$ *given by smooth sections for which the norm:*

$$\| \, A \, \|^2_{\mathcal{H}^1} \; = \; \int_{\mathscr{M}} \left[\sum_{\mu=0}^{3} |A_\mu(x)|^2 + \sum_{\mu,\nu=0}^{3} |\partial_\mu A_\nu(x)|^2 \right] vol_{\mathscr{M}} \, , \tag{3.136}$$

is finite, say $\Gamma(\pi)_1$*. The following chain of inequalities*[23]

$$\begin{aligned} |\, \mathscr{S}_A - \mathscr{S}_{\tilde{A}} \,| \; &\leq \; \frac{1}{4}\int_{\mathscr{M}} \left| \, F_{\mu\nu}F^{\mu\nu} - \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \, \right| \, vol_{\mathscr{M}} \; \leq \\ &\leq \; \frac{1}{2}\left[\, \left| \, F_{\mu\nu}(\, F^{\mu\nu} - \tilde{F}^{\mu\nu} \,) \, \right| + \left| \, \tilde{F}_{\mu\nu}(\, F^{\mu\nu} - \tilde{F}^{\mu\nu} \,) \, \right| \, \right] vol_{\mathscr{M}} \; \leq \\ &\leq \; \frac{1}{2}\left[\sum_{\mu\nu} \| \, F_{\mu\nu} \, \|_{\mathcal{L}^2} \, \| \, F_{\mu\nu} - \tilde{F}_{\mu\nu} \, \|_{\mathcal{L}^2} + \sum_{\mu\nu} \| \, \tilde{F}_{\mu\nu} \, \|_{\mathcal{L}^2} \, \| \, F_{\mu\nu} - \tilde{F}_{\mu\nu} \, \|_{\mathcal{L}^2} \right] \, , \end{aligned} \tag{3.137}$$

[23]Here A and $\tilde{A}$ denote two different sections of π and $\tilde{F}_{\mu\nu} \; = \; \partial_{[\mu}\tilde{A}_{\nu]}$. Moreover, we raise and lower indices via the metric η.

shows that $\mathscr{S}$ is continuous in the norm $\|\cdot\|_{\mathcal{H}^1}$. Indeed, the last term of the chain of inequalities above vanishes as $\| A - \tilde{A} \|_{\mathcal{H}^1}$ approaches zero. Consequently, $\mathscr{S}$ can be extended by continuity to the completion $\overline{\Gamma(\pi)_1}^{\|\cdot\|_{\mathcal{H}^1}} = [\mathcal{H}^1(\mathscr{M}, vol_{\mathscr{M}})]^4 =: \mathcal{F}_{\mathbb{E}}$. The contact form (3.115) here reads:

$$\eta = -\eta^{\mu\rho}\eta^{\nu\sigma} z_{[\rho\sigma]}(\mathrm{d}u_\nu - z_{\tau\nu}\mathrm{d}x^\tau) \wedge i_\mu vol_{\mathscr{M}} , \tag{3.138}$$

and it gives rise to the following Lepage equivalent:

$$\Theta_\lambda = -\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma} z_{[\mu\nu]} z_{[\rho\sigma]} vol_{\mathscr{M}} - \eta^{\mu\rho}\eta^{\nu\sigma} z_{[\rho\sigma]}(\mathrm{d}u_\nu - z_{\tau\nu}\mathrm{d}x^\tau) \wedge i_\mu vol_{\mathscr{M}} . \tag{3.139}$$

The Lepage equivalent above gives rise to the following first fundamental formula:

$$\begin{aligned} \delta_{\mathbb{X}_A}\mathscr{S}_A = \int_{\mathscr{M}} \left(j^1 A\right)^\star \Bigg[i_{X^1}\Bigg(& -\frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma} z_{[\mu\nu]}\mathrm{d}z_{[\rho\sigma]} \wedge vol_{\mathscr{M}} + \\ & -\eta^{\mu\rho}\eta^{\nu\sigma}\mathrm{d}z_{[\rho\sigma]} \wedge \mathrm{d}u_\nu \wedge i_\mu vol_{\mathscr{M}} + \\ & +\eta^{\mu\rho}\eta^{\nu\sigma}(z_{\mu\nu}\mathrm{d}z_{[\rho\sigma]} + z_{[\rho\sigma]}\mathrm{d}z_{\mu\nu}) \wedge vol_{\mathscr{M}} \Bigg)\Bigg] , \end{aligned} \tag{3.140}$$

where X^1 is the first order jet prolongation of a π-vertical vector field X on $\mathbb{E}$ defined in a neighborhood of the image of A, i.e., it is a vector field of the type:

$$X^1 = X_{u_\mu}\frac{\partial}{\partial u_\mu} + z_{\mu\rho}\frac{\partial X_{u_\nu}}{\partial u_\rho}\frac{\partial}{\partial z_{\mu\nu}} , \tag{3.141}$$

where X_{u_μ} are functions on $\mathscr{M} \times \mathbb{R}^4$ defined for all $x \in \mathscr{M}$ and for u_μ close[24] *to $A_\mu(x)$. On the other hand, $\mathbb{X}_A$ is the map:*

$$\mathbb{X}_A \; : \; \mathscr{M} \to \mathbf{T}_{A(x)}\mathbb{E} \; : \; x \mapsto X(A(x)) = X_{u\mu}(x, A(x))\frac{\partial}{\partial u_\mu}\bigg|_{A(x)} = \mathbb{X}_{A\mu}\frac{\delta}{\delta A_\mu} , \tag{3.142}$$

following the notation (2.52). The right hand side of the latter equation is the (contraction of the tangent vector $\mathbb{X}_A$ along the) Euler-Lagrange form associated to the Lagrangian describing our field theory. Equation (3.121) reads:

$$\begin{aligned} \left(j^1 A\right)^\star [\, i_{X^1}\mathrm{d}\Theta_{\mathscr{L}} \,] = \left(j^1 A\right)^\star \Bigg[& -\frac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma} z_{[\mu\nu]} z_{[\rho\tau}\frac{\partial X_{u\sigma]}}{\partial u_\tau} \wedge vol_{\mathscr{M}} + \\ & -\eta^{\mu\rho}\eta^{\nu\sigma}\left(z_{[\rho\tau}\frac{\partial X_{u\sigma]}}{\partial u_\tau}\mathrm{d}u_\nu - X_{u\nu}\mathrm{d}z_{[\rho\sigma]} \right) \wedge i_\mu vol_{\mathscr{M}} + \\ & +\eta^{\mu\rho}\eta^{\nu\sigma}\left(z_{\mu\nu} z_{[\rho\tau}\frac{\partial X_{u\sigma]}}{\partial u_\tau} + z_{[\rho\sigma]} z_{\mu\tau}\frac{\partial X_{u\nu}}{\partial u_\tau} \right) vol_{\mathscr{M}} \Bigg] = \\ & = \mathbb{X}_{A\mu}\, \eta^{\mu\sigma}\eta^{\nu\rho}\partial_\nu F_{\rho\sigma} = 0 \quad \forall\, \mathbb{X}_{A\mu} , \end{aligned} \tag{3.143}$$

which gives:

$$\eta^{\nu\rho}\partial_\nu F_{\rho\sigma} = 0 , \tag{3.144}$$

being the celebrated covariant form of sourceless MAXWELL'S EQUATIONS *in vacuum.*

[24] In the sense of the $\mathcal{H}^1$-norm defined above.

Example 3.2.8 (Yang-Mills theories). *Let us now consider free Yang-Mills theories on the Minkowski space-time* $\mathscr{M} = (\mathbb{R}^4,\, \eta)$*, that is, gauge theories (without external sources) whose structure group is a generally semi-simple*[25] *Lie group*[26]*. Let us denote the structure group by* G *and its Lie algebra by* $\mathfrak{g}$*. In this case the fields of the theory are connection one forms on the bundle* $P = \mathscr{M} \times G \to \mathscr{M}$*, i.e.,* 1*-forms with values in* $\mathfrak{g}$*, namely, sections of* $\pi \;:\; \mathbb{E} = \mathbf{T}^\star\mathscr{M} \otimes \mathfrak{g} \to \mathscr{M}$*. We denote by* $\{\, x^\mu,\, u^a_\mu \,\}_{\mu=0,\ldots,3;a=1,\ldots,\dim\mathfrak{g}}$ *a system of coordinates on* $\mathbb{E}$*, where* $\{\, x^\mu \,\}_{\mu=0,\ldots,3}$ *is the system of coordinates chosen on* $\mathscr{M}$*. Furthermore, we denote by* $A = A^a_\mu(x)\mathrm{d}x^\mu \wedge \xi_a$ *sections of* π*, where* $\{\, \xi_a \,\}_{a=1,\ldots,\dim\mathfrak{g}}$ *is a basis of* $\mathfrak{g}$*. The first order jet bundle* $\mathbf{J}^1\pi$ *in this case is the trivial bundle over* $\mathscr{M}$ *whose typical is* $\mathbf{T}^\star_m\mathscr{M} \otimes \mathfrak{g} \times \left(\bigotimes^2 \mathbf{T}^\star_m\mathscr{M} \otimes \mathfrak{g} \right)$ *where we consider* $\{\, x^\mu,\, u^a_\mu,\, z^a_{\mu\nu} \,\}_{\mu=0,\ldots,3;a=1,\ldots,\dim\mathfrak{g}}$ *as a system of coordinates. The Lagrangian describing the theory here is:*

$$\lambda \,=\, -\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}G_{ab}\left(z^a_{[\mu\nu]} + \epsilon^a_{cd}u^c_\mu u^d_\nu \right)\left(z^b_{[\rho\sigma]} + \epsilon^b_{ef}u^e_\rho u^f_\sigma \right) vol_{\mathscr{M}}\,, \tag{3.145}$$

where G *denotes the Killing-Cartan metric on* $\mathfrak{g}$*. The action functional obtained is:*

$$\mathscr{S}_A \,=\, \int_{\mathscr{M}} \left(j^1 A\right)^\star \lambda \,=\, -\int_{\mathscr{M}} \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}G_{ab}F^a_{\mu\nu}F^b_{\rho\sigma}\, vol_{\mathscr{M}}\,, \tag{3.146}$$

where:

$$F^a_{\mu\nu} \,=\, \partial_{[\mu}A^a_{\nu]} + \epsilon^a_{bc}A^b_\mu A^c_\nu \,=\, (\,\nabla A\,)^a_{\mu\nu} \,=:\, \nabla_\mu A^a_\nu\,, \tag{3.147}$$

are the coefficients of the curvature of the connection A *and* ∇ *denotes the covariant derivative associated to the connection. It is clear that such action functional is not well defined on the whole space of smooth sections of* π*. In the next lines we will see that by considering the coefficients of the* A*'s to be* $\mathcal{H}^3$ *functions we get a well defined action functional on a well defined Hilbert space of fields, even if, as in the previous example it is worth mentioning that this is not the minimal requirement. Therefore, we will consider the action functional to be defined on the subset of* $\Gamma(\pi)$ *given by smooth sections for which the norm:*

$$\|\, A \,\|^2_{\mathcal{H}^3} \,=\, \int_{\mathscr{M}} \left[\sum_{\mu,\,a} |\, A^a_\mu(x)\,|^2 + \sum_{I,\mu,\,a} |\, D_I A^a_\mu(x)\,|^2 \right] vol_{\mathscr{M}}\,, \tag{3.148}$$

is finite, where I *is any multi-index of the type* $I = \mu_1\mu_2...$ *of length going from* 1 *to* 3 *and* $D_I = \partial_{\mu_1}\partial_{\mu_2}...$*. Denote such a subset by* $\Gamma(\pi)_1$*. The following inequalities*

[25] This assumption is made in order for the Killing-Cartan metric on the Lie algebra of the Lie group to be positive definite.

[26] Actually, those of physical interest are those for which the structure group is $U(1)$ (describing Electrodynamics considered in the previous example), $SU(2)$ (describing the theory of weak interactions), $SU(3)$ (describing Quantum Chromodynamics) and $U(1)\times SU(2)\times SU(3)$ (describing their unification).

hold:

$$
\begin{aligned}
\Big|\,\mathscr{S}_A - \mathscr{S}_{\tilde A}\,\Big| \;\le\; & \frac{1}{4}\int_{\mathscr{M}}\Big|\,(\partial_{[\mu}A^a_{\nu]} + \epsilon^a{}_{bc}A^b_\mu A^c_\nu)\,(\partial^{[\mu}A^{\nu]} + \epsilon_a{}^{de}A^\mu_d A^\nu_e) + \\
& \qquad - (\partial_{[\mu}\tilde A^a_{\nu]} + \epsilon^a{}_{bc}\tilde A^b_\mu \tilde A^c_\nu)\,(\partial^{[\mu}\tilde A^{\nu]} + \epsilon_a{}^{de}\tilde A^\mu_d \tilde A^\nu_e)\,\Big|\,vol_{\mathscr{M}} \;\le \\
\le\; & \underbrace{\frac{1}{4}\int_{\mathscr{M}}\Big|\,\partial_{[\mu}A^a_{\nu]}\partial^{[\mu}A^{\nu]}_a - \partial_{[\mu}\tilde A^a_{\nu]}\partial^{[\mu}\tilde A^{\nu]}_a\,\Big|\,vol_{\mathscr{M}}}_{=:\,\mathscr{I}_1} + \\
& + \underbrace{\frac{1}{4}\int_{\mathscr{M}}\Big|\,\epsilon^a{}_{bc}\epsilon_a{}^{de}(A^b_\mu A^c_\nu A^\mu_d A^\nu_e - \tilde A^b_\mu \tilde A^c_\nu \tilde A^\mu_d \tilde A^\nu_e)\,\Big|\,vol_{\mathscr{M}}}_{=:\,\mathscr{I}_2} + \\
& + \underbrace{\frac{1}{2}\int_{\mathscr{M}}\Big|\,\epsilon^a{}_{bc}(\partial^{[\mu}A^{\nu]}_a A^b_\mu A^c_\nu - \partial^{[\mu}\tilde A^{\nu]}_a \tilde A^b_\mu \tilde A^c_\nu)\,\Big|\,vol_{\mathscr{M}}}_{=:\,\mathscr{I}_3}\,.
\end{aligned}
\tag{3.149}
$$

First, a proof analogous to the previous example shows that:

$$
\mathscr{I}_1 \;\le\; 2\left[\sum_{\mu,\nu,a}\|\,\partial_{[\mu}A_{\nu]^a}\,\|_{\mathcal{L}^2}\,\|\,\partial_{[\mu}A^a_{\nu]} - \partial_{[\mu}\tilde A^a_{\nu]}\,\|_{\mathcal{L}^2} + \sum_{\mu,\nu,a}\|\,\partial_{[\mu}\tilde A_{\nu]^a}\,\|_{\mathcal{L}^2}\,\|\,\partial_{[\mu}A^a_{\nu]} - \partial_{[\mu}\tilde A^a_{\nu]}\,\|_{\mathcal{L}^2}\right].
\tag{3.150}
$$

Now, recall that, being $\mathscr{M}$ 4-dimensional, the space of $\mathcal{H}^3$ functions on it is a Banach algebra [88], which means that:

$$
\|\,A^b_\mu A^c_\nu\,\|_{\mathcal{H}^3} \;=\; \underbrace{\|\,A^b_\mu\,\|_{\mathcal{H}^3}}_{<\infty}\,\underbrace{\|\,A^\nu_c\,\|_{\mathcal{H}^3}}_{<\infty} \;<\; \infty\,,
\tag{3.151}
$$

that is, that $A^b_\mu A^\nu_c$ belongs to $\mathcal{H}^3(\mathscr{M},\, vol_{\mathscr{M}})$ $\quad\forall\ \mu,\,\nu,\,b,\,c$ and, a fortiori, that it belongs to $\mathcal{H}^{-3}(\mathscr{M},\, vol_{\mathscr{M}})$ which is the dual, with respect to the $\mathcal{L}^2$ scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{L}^2}$, of $\mathcal{H}^3$. This implies the following chain of inequalities:

$$
\begin{aligned}
\mathscr{I}_2 \;\le\; & \sum_{\mu,\nu,b,c,d,e}\int_{\mathscr{M}}\Big[\,\Big|\,A^b_\mu A^\mu_d(\tilde A^\nu_c\tilde A^e_\nu - A^\nu_c A^e_\nu)\,\Big| + \Big|\,\tilde A^b_\mu\tilde A^\mu_d(\tilde A^\nu_c\tilde A^e_\nu - A^\nu_c A^e_\nu)\,\Big|\,\Big]\,vol_{\mathscr{M}} \;\le \\
\le\; & \sum_{\mu,\nu,b,c,d,e}\Big[\,\|\,A^b_\mu A^\mu_d\,\|_{\mathcal{H}^{-3}}\,\|\,A^\nu_c A^e_\nu - \tilde A^\nu_c\tilde A^e_\nu\,\|_{\mathcal{H}^3} + \|\,\tilde A^b_\mu\tilde A^\mu_d\,\|_{\mathcal{H}^{-3}}\,\|\,A^\nu_c A^e_\nu - \tilde A^\nu_c\tilde A^e_\nu\,\|_{\mathcal{H}^3}\Big] \;\le \\
\le\; & \sum_{\mu,\nu,b,c,d,e}\Big[\Big(\|\,A^b_\mu A^\mu_d\,\|_{\mathcal{H}^{-3}} + \|\,\tilde A^b_\mu\tilde A^\mu_d\,\|_{\mathcal{H}^{-3}}\Big)\Big(\|\,A^\nu_c(A^e_\nu - \tilde A^e_\nu)\,\|_{\mathcal{H}^3} + \|\,\tilde A^\nu_c(A^e_\nu - \tilde A^e_\nu)\,\|_{\mathcal{H}^3}\Big) + \Big] \;\le \\
\le\; & \sum_{\mu,\nu,b,c,d,e}\Big[\Big(\|\,A^b_\mu A^\mu_d\,\|_{\mathcal{H}^{-3}} + \|\,\tilde A^b_\mu\tilde A^\mu_d\,\|_{\mathcal{H}^{-3}}\Big)\Big(\|\,A^\nu_c\,\|_{\mathcal{H}^3} + \|\,\tilde A^\nu_c\,\|_{\mathcal{H}^3}\Big)\,\|\,A^e_\nu - \tilde A^e_\nu\,\|_{\mathcal{H}^3}\Big].
\end{aligned}
\tag{3.152}
$$

Similar arguments allow to prove the following estimate:

$$
\mathscr{I}_3 \;\le\; \|\,\partial^{[\mu}A^{\nu]}\,\|_{\mathcal{H}^{-2}}\Big(\|\,A^b_\mu\,\|_{\mathcal{H}^3} + \|\,\tilde A^b_\mu\,\|_{\mathcal{H}^3}\Big)\,\|\,A^c_\nu - \tilde A^c_\nu\,\|_{\mathcal{H}^3} + \|\,\tilde A^b_\mu\tilde A^c_\nu\,\|_{\mathcal{L}^2}\,\|\,\partial^{[\mu}A^{\nu]}_a - \partial^{[\mu}\tilde A^{\nu]}_a\,\|_{\mathcal{L}^2}\,.
\tag{3.153}
$$

Estimates (3.150), (3.152) *and* (3.153), *shows that $\mathscr{I}_1$, $\mathscr{I}_2$ and $\mathscr{I}_3$ approach* 0 *when* $\|\,A - \tilde A\,\|_{\mathcal{H}^3}$ *approaches* 0, *that is, that $\mathscr{S}$ is continuous in the norm* (3.148). *Therefore,*

$\mathscr{S}$ can be extended by continuity to the completion $\overline{\Gamma(\pi)_1}^{\|\cdot\|_{\mathcal{H}^3}} = [\mathcal{H}^3(\mathscr{M}, vol_{\mathscr{M}})]^n =: \mathcal{F}_{\mathbb{E}}$, *where* $n = 4\dim\mathfrak{g}$. *The contact form* (3.115) *here reads:*

$$\eta = -\eta^{\mu\rho}\eta^{\nu\sigma}G_{ab}\,(z^a_{[\rho\sigma]} + \epsilon^a{}_{cd}u^c_\rho u^d_\sigma)\,(\mathrm{d}u^b_\nu - z^b_{\tau\nu}\mathrm{d}x^\tau) \wedge i_\mu vol_{\mathscr{M}}\,, \tag{3.154}$$

and it gives rise to the following Lepage equivalent:

$$\begin{aligned}\Theta_\lambda = & -\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}G_{ab}\left(z^a_{[\mu\nu]} + \epsilon^a_{cd}u^c_\mu u^d_\nu\right)\left(z^b_{[\rho\sigma]} + \epsilon^b_{ef}u^e_\rho u^f_\sigma\right) vol_{\mathscr{M}} + \\ & -\eta^{\mu\rho}\eta^{\nu\sigma}G_{ab}\,(z^a_{[\rho\sigma]} + \epsilon^a{}_{cd}u^c_\rho u^d_\sigma)\,(\mathrm{d}u^b_\nu - z^b_{\tau\nu}\mathrm{d}x^\tau) \wedge i_\mu vol_{\mathscr{M}}\,.\end{aligned} \tag{3.155}$$

The Lepage equivalent above gives rise to the following first fundamental formula:

$$\begin{aligned}\delta_{\mathbb{X}_A}\mathscr{S}_A = \int_{\mathscr{M}} \left(j^1A\right)^\star \Bigg[i_{X^1}\Bigg(& -\frac{1}{2}\eta^{\mu\rho}\eta^{\nu\sigma}G_{ab}\left(\mathrm{d}z^a_{[\mu\nu]} + \epsilon^a_{cd}\mathrm{d}u^c_\mu u^d_\nu + \epsilon^a_{cd}u^c_\mu \mathrm{d}u^d_\nu\right)\left(z^b_{[\rho\sigma]} + \epsilon^b_{ef}u^e_\rho u^f_\sigma\right) vol_{\mathscr{M}} + \\ & -\eta^{\mu\rho}\eta^{\nu\sigma}G_{ab}\,(\mathrm{d}z^a_{[\rho\sigma]} + \epsilon^a{}_{cd}\mathrm{d}u^c_\rho u^d_\sigma + \epsilon^a{}_{cd}u^c_\rho \mathrm{d}u^d_\sigma) \wedge (\mathrm{d}u^b_\nu - z^b_{\tau\nu}\mathrm{d}x^\tau) \wedge i_\mu vol_{\mathscr{M}} + \\ & +\eta^{\mu\rho}\eta^{\nu\sigma}G_{ab}\,(z^a_{[\rho\sigma]} + \epsilon^a{}_{cd}u^c_\rho u^d_\sigma)\,\mathrm{d}z^b_{\mu\nu} \wedge i_\mu vol_{\mathscr{M}} \Bigg)\Bigg]\,,\end{aligned} \tag{3.156}$$

where X^1 is the first order jet prolongation of a π-vertical vector field X on $\mathbb{E}$ defined in a neighborhood of the image of A, i.e., it is a vector field of the type:

$$X^1 = X_{u_\mu}{}^a \frac{\partial}{\partial u^a_\mu} + z^a_{\mu\rho}\frac{\partial X_{u_\nu}{}^b}{\partial u^a_\rho}\frac{\partial}{\partial z^b_{\mu\nu}}\,, \tag{3.157}$$

where $X_{u_\mu}{}^a$ are functions defined on points of $\mathbf{T}^\star\mathscr{M} \otimes \mathfrak{g}$ close[27] *to $A^a_\mu(x)$. On the other hand, $\mathbb{X}_A$ is the map:*

$$\mathbb{X}_A \;:\; \mathscr{M} \to \mathbf{T}_{A(x)}\mathbb{E} \;:\; x \mapsto X(A(x)) = X_{u_\mu}{}^a(x, A(x))\left.\frac{\partial}{\partial u^a_\mu}\right|_{A(x)} =: \mathbb{X}_{A_\mu}{}^a\frac{\delta}{\delta A^a_\mu}\,, \tag{3.158}$$

following the notation (2.52). *The right hand side of the latter equation is the (contraction of the tangent vector $\mathbb{X}_A$ along the) Euler-Lagrange form associated to the Lagrangian describing our field theory. Equation* (3.121) *reads:*

$$\begin{aligned}\left(j^1A\right)^\star[\,i_{X^1}\mathrm{d}\Theta_{\mathscr{L}}\,] = \left(j^1A\right)^\star\Bigg[& -\frac{1}{2}\eta^{\mu\rho}\eta^{\nu\sigma}G_{ab}\Big(z^d_{[\rho\sigma]}\epsilon^a{}_{bc}u^c_\nu\,X_{u_\mu}{}^b + z^d_{[\rho\sigma]}\epsilon^a{}_{bc}u^b_\mu\,X_{u_\nu}{}^c + \\ & +\epsilon^a{}_{bc}\epsilon^d{}_{ef}u^b_\mu u^e_\rho u^f_\sigma\,X_{u_\nu}{}^c + \epsilon^a{}_{bc}\epsilon^d{}_{ef}u^c_\nu u^e_\rho u^f_\sigma\,X_{u_\mu}{}^b\Big) + \\ & +\eta^{\mu\rho}\eta^{\nu\sigma}G_{ab}\,X_{u_\nu}{}^a\left(\mathrm{d}z^d_{[\rho\sigma]} + \epsilon^d{}_{ef}u^f_\sigma \mathrm{d}u^e_\rho + \epsilon^d{}_{ef}u^e_\rho \mathrm{d}u^f_\sigma\right)\Bigg)\Bigg] = \\ & 2\,\eta^{\mu\rho}\eta^{\nu\sigma}G_{ab}\mathbb{X}_{A_\nu}{}^a\,\nabla_\mu F^b_{\rho\sigma} = 0 \quad \forall\,\mathbb{X}_{A_\nu}{}^a\,,\end{aligned} \tag{3.159}$$

which gives:

$$\eta^{\mu\rho}\nabla_\mu F^a_{\rho\sigma} = 0\,, \tag{3.160}$$

being the celebrated YANG-MILLS EQUATIONS.

[27] With respect to the $\mathcal{H}^3$-norm defined above.

3.2.2. Hamiltonian formulation

The Hamiltonian formulation of first order field theories follows the lines of Sec. 3.1.2 with the obvious generalizations. It is settled on the COVARIANT PHASE SPACE associated with the fibration $\pi \; : \; \mathbb{E} \to \mathscr{M}$, say $\mathcal{P}(\mathbb{E})$. As in Sec. 2.4.2 we will denote local coordinates on the extended dual of $\mathbf{J}^1\pi$ by $\{\, x^\mu,\, u^a,\, \rho^\mu_a,\, \rho_0 \,\}_{\mu=0,\dots,d;\, a=1,\dots,\dim\mathcal{E}}$ and local coordinates on $\mathcal{P}(\mathbb{E})$ by $\{\, x^\mu,\, u^a,\, \rho^\mu_a \,\}_{\mu=0,\dots,d;\, a=1,\dots,\dim\mathcal{E}}$.

Definition 3.2.9 (HAMILTONIAN). *A HAMILTONIAN is a section of the projection κ appearing in the diagram* (2.179)*, i.e., a local map:*

$$H \;\; : \;\; \mathbf{J}^\star\pi \to \mathcal{P}(\mathbb{E}) \;\; : \;\; (x^\mu,\, u^a,\, \rho^\mu_a) \mapsto (x^\mu,\, u^a,\, \rho^\mu_a,\, H(x,\, u,\, \rho)) \,. \tag{3.161}$$

Recalling the content of Sec. 2.4.2, the extended dual of π, $\mathbf{J}^\dagger\pi$, has a canonical 1-semibasic $(n+1)$-form, i.e.:

$$\boldsymbol{w} \;=\; \rho^\mu_a \mathrm{d}u^a \wedge i_\mu vol_{\mathscr{M}} + \rho_0 vol_{\mathscr{M}} \,. \tag{3.162}$$

For any fixed Hamiltonian, the pull-back of the canonical multi-symplectic structure (3.162) to $\mathcal{P}(\mathbb{E})$, gives a canonical structure on the latter space which reads:

$$\Theta_H \;:=\; (\, -H \,)^\star \,\boldsymbol{w} \;=\; \rho^\mu_a \mathrm{d}u^a \wedge vol_{\mathscr{M}} - H vol_{\mathscr{M}} \,. \tag{3.163}$$

Its differential reads:

$$\mathrm{d}\Theta_H \;=\; \omega - \mathrm{d}H \wedge vol_{\mathscr{M}} \,, \tag{3.164}$$

where:

$$\omega \;=\; \mathrm{d}\rho^\mu_a \wedge \mathrm{d}u^a \wedge vol_{\mathscr{M}} \,. \tag{3.165}$$

As we discussed in Sec. 3.1.2, on $\mathcal{P}(\mathbb{E})$, since the counterpart of the first order jet prolongation of sections of π does not exist, we will work with elements of $\Gamma^{\text{SPLIT}}(\delta_1)$, namely pairs, denoted by $\chi \;=\; (\phi,\, P)$, of sections of π and δ^1_0 respectively.

Definition 3.2.10 (ACTION FUNCTIONAL). *Given a Hamiltonian over $\mathcal{P}(\mathbb{E})$, an ACTION FUNCTIONAL is a real-valued function on $\Gamma^{\text{SPLIT}}(\delta_1)$ given by:*

$$\mathscr{S} \;\; : \;\; \Gamma^{\text{SPLIT}}(\delta_1) \to \mathbb{R} \;\; : \;\; \chi \mapsto \mathscr{S}_\chi \;=\; \int_{\mathbb{I}} \chi^\star\, \Theta_H \,, \tag{3.166}$$

whose coordinate expression is:

$$\mathscr{S}_\chi \;=\; \int_{\mathscr{M}} [\, P^\mu_a(x) \partial_\mu \phi^a(x) \; - H(x,\, \phi(x), P(x,\, \phi(x))) \,]\; vol_{\mathscr{M}} \,. \tag{3.167}$$

Remark 3.2.11. *Also in this case we are going to assume the possibility of performing a completion of the type of Rem. 3.1.3 in order to obtain a suitable Banach manifold of fields on which the action functional is well defined and that we will denote by $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$. We will refer to elements of $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$ as DYNAMICAL FIELDS. We will take care of the validity of the assumption above case by case in the examples.*

Following the same steps made in Sec. 3.1.2 one gets the FIRST VARIATIONAL FORMULA:

$$\delta_{\mathbb{X}_\chi} \mathscr{S}_\chi \,=\, \int_{\mathscr{M}} \chi^\star \left[\, i_X \mathrm{d}\Theta_H \,\right] + \int_{\partial\mathscr{M}} \mathfrak{i}^\star_{\partial\mathscr{M}}\, \chi^\star \left[\, i_X \Theta_H \,\right] . \tag{3.168}$$

As in the Lagrangian formalism, the second term on the r.h.s. is a "boundary term" in the sense that it depends only on the restriction of χ to the boundary of $\mathscr{M}$, $\chi_{\partial\mathscr{M}} = \chi \circ \mathfrak{i}_{\partial\mathscr{M}}$ ($\mathfrak{i}_{\partial\mathscr{M}}$ denotes the canonical immersion of $\partial\mathscr{M}$ into $\mathscr{M}$). As in the mechanical case, since χ is not a first order jet prolongation of a section, no additional boundary terms may appear from the first term on the r.h.s. and, thus, the problem of searching for a Lepage equivalent does not arise.

In this case the Schwinger-Weiss variational principle gives the following equations of motion:

$$\chi^\star \,\left[\, i_X \mathrm{d}\Theta_H \,\right] \,=\, 0 \quad \forall\; X \in \mathfrak{X}^v(U^{(\chi)})\,, \tag{3.169}$$

whose coordinate expression is:

$$\frac{\partial \phi^a}{\partial x^\mu} \,=\, \left.\frac{\partial H}{\partial \rho^\mu_a}\right|_\chi , \qquad \frac{\partial P^\mu_a}{\partial x^\mu} \,=\, -\left.\frac{\partial H}{\partial u^a}\right|_\chi , \tag{3.170}$$

that are the so called COVARIANT HAMILTON EQUATIONS or DE DONDER-WEYL EQUATIONS.

Remark 3.2.12. *Also in this case, an interpretation of the solutions of De Donder-Weyl equations as integral curves of a vector field like the one given in Rem. 3.1.13 require more work and will be given in Sec. 4.2.*

Example 3.2.13 (FREE KLEIN-GORDON THEORY)**.** *As in example 3.2.6, we consider a free, massive, boson field on the Minkowski space-time. Therefore, here, the bundle* (3.102) *is* $\pi \;:\; \mathbb{E} = \mathscr{M} \times \mathbb{R} \to \mathscr{M}$ *where* $\mathscr{M} = (\mathbb{R}^4,\, \eta)$*,* η *being the Minkowski metric. Again, we denote by* $\{\, x^\mu,\, u \,\}_{\mu=0,\dots,3}$ *a system of (global) coordinates on* $\mathbb{E}$ *where* $\{\, x^\mu \,\}_{\mu=0,\dots,3}$ *is the system of (global) coordinates chosen on* $\mathscr{M}$ *and, again, we denote by* ϕ *the sections of* π*. Here, the covariant phase space is* $\mathcal{P}(\mathbb{E}) = \mathscr{M} \times \mathbb{R} \times \mathbb{R}^4$ *where we consider* $\{\, x^\mu,\, u,\, \rho^\mu \,\}_{\mu=0,\dots,3}$ *as a system of (global, again) coordinates. We will denote by* $\chi = (\phi,\, P) = (\phi,\, P^\mu)$ *elements of* $\Gamma^{\text{SPLIT}}(\delta_1)$*,* δ_1 *denoting the projection* $\mathscr{M} \times \mathbb{R} \times \mathbb{R}^4 \to \mathscr{M}$*. The Hamiltonian of the theory is:*

$$H \,=\, \frac{1}{2} \left(\, \eta_{\mu\nu} \rho^\mu \rho^\nu + m^2 u^2 \,\right) , \tag{3.171}$$

m representing the mass of the boson field. The action functional obtained is:

$$\begin{aligned} \mathscr{S}_\chi \,&=\, \int_{\mathscr{M}} \chi^\star \left[\, \rho^\mu \mathrm{d}u \wedge i_\mu vol_{\mathscr{M}} - \frac{1}{2}\left(\, \eta_{\mu\nu}\rho^\mu\rho^\nu + m^2 u^2 \,\right) vol_{\mathscr{M}} \,\right] \,= \\ &=\, \int_{\mathscr{M}} \left[\, P^\mu(x) \partial_\mu \phi(x) - \frac{1}{2} \eta_{\mu\nu} P^\mu(x) P^\nu(x) - \frac{1}{2} m^2 \phi^2(x) \,\right] vol_{\mathscr{M}} \,. \end{aligned} \tag{3.172}$$

Again, it is clear that such action functional is not well defined on the whole space of smooth splitting sections $\Gamma^{\text{SPLIT}}(\delta_1)$*. Indeed, in order for the integral to be well* *defined, the ϕ's should be at least square integrable (with respect to $vol_{\mathscr{M}}$) as well as the P's, and the product of the P's with the first derivatives of ϕ's should be integrable. Thus, the first derivatives of the ϕ's should be square integrable as well. For technical reasons that will be clear in Sec. 4.2, we will actually ask for more regularity, and we will consider $\mathscr{S}$ to be defined on the subset of $\Gamma^{\text{SPLIT}}(\delta_1)$ given by smooth splitting sections for which the norm:*

$$\|\chi\|^2 \,=\, \|\phi\|^2_{\mathcal{H}^2} + \sum_{\mu=0}^{3} \|P^\mu\|^2_{\mathcal{H}^1}\,, \tag{3.173}$$

is finite, say $\Gamma^{\text{SPLIT}}(\delta_1)_1$. The following chain of inequalities[28]

$$\begin{aligned} |\mathscr{S}_\chi - \mathscr{S}_{\tilde{\chi}}| \leq & \int_{\mathscr{M}} \Big| P^\mu\partial_\mu\phi - \tilde{P}^\mu\partial_\mu\tilde{\phi} - \frac{1}{2}P^\mu P_\mu + \frac{1}{2}\tilde{P}^\mu\tilde{P}_\mu - \frac{1}{2}m^2\phi^2 + \frac{1}{2}m^2\tilde{\phi}^2 \Big| vol_{\mathscr{M}} \leq \\ \leq & \int_{\mathscr{M}} \Big[\Big| P^\mu(\partial_\mu\phi - \partial_\mu\tilde{\phi}) \Big| + \Big| (P^\mu - \tilde{P}^\mu)\partial_\mu\tilde{\phi} \Big| + \\ & + \frac{1}{2}\Big| P^\mu(P_\mu - \tilde{P}_\mu) \Big| + \frac{1}{2}\Big| \tilde{P}^\mu(P_\mu - \tilde{P}_\mu) \Big| + \\ & + \frac{1}{2}m^2\Big| \phi(\phi - \tilde{\phi}) \Big| + \frac{1}{2}m^2\Big| \tilde{\phi}(\phi - \tilde{\phi}) \Big| \Big] vol_{\mathscr{M}} \leq \\ \leq & \sum_{\mu=0}^{3} \| P^\mu \|_{\mathcal{L}^2} \| \partial_\mu(\phi - \tilde{\phi}) \|_{\mathcal{L}^2} + \sum_{\mu=0}^{3} \| P^\mu - \tilde{P}^\mu \|_{\mathcal{L}^2} \| \partial_\mu\phi \|_{\mathcal{L}^2} + \\ & + \sum_{\mu=0}^{3} \| P^\mu \|_{\mathcal{L}^2} \| P^\mu - \tilde{P}^\mu \|_{\mathcal{L}^2} + \sum_{\mu=0}^{3} \| \tilde{P}^\mu \|_{\mathcal{L}^2} \| P^\mu - \tilde{P}^\mu \|_{\mathcal{L}^2} + \\ & + \frac{1}{2}m^2 \| \phi \|_{\mathcal{L}^2} \| \phi - \tilde{\phi} \|_{\mathcal{L}^2} + \frac{1}{2}m^2 \| \tilde{\phi} \|_{\mathcal{L}^2} \| \phi - \tilde{\phi} \|_{\mathcal{L}^2}\,, \end{aligned} \tag{3.174}$$

shows that $\mathscr{S}$ is continuous in the norm defined above. Indeed, the last term in the chain of inequalities above vanishes as $\| \chi - \tilde{\chi} \|$ approaches zero. Consequently, $\mathscr{S}$ can be extended by continuity to the completion $\overline{\Gamma^{\text{SPLIT}}(\delta_1)_1}^{\|\cdot\|} \,=\, \mathcal{H}^2(\mathscr{M},\, vol_{\mathscr{M}}) \times \left[\, \mathcal{H}^1(\mathscr{M},\, vol_{\mathscr{M}}) \right]^4 =: \mathcal{F}_{\mathcal{P}(\mathbb{E})}$. The first fundamental formula here reads:

$$\delta_{\mathbb{X}_\chi}\mathscr{S}_\chi \,=\, \int_{\mathscr{M}} \chi^\star \left[i_X \left(\mathrm{d}\rho^\mu \wedge \mathrm{du} \wedge i_\mu vol_{\mathscr{M}} - (\eta_{\mu\nu}\rho^\mu \mathrm{d}\rho^\nu + m^2 u \mathrm{du}) \wedge vol_{\mathscr{M}} \right) \right]\,, \tag{3.175}$$

where X is a δ_1-vertical vector field on $\mathcal{P}(\mathbb{E})$ defined in a neighborhood of the image of χ, i.e., it is a vector field of the type:

$$X \,=\, X_u\frac{\partial}{\partial u} + X_\rho{}^\mu\frac{\partial}{\partial\rho^\mu}\,, \tag{3.176}$$

where X_u and $X_\rho{}^\mu$ are functions on $\mathcal{P}(\mathbb{E}) \,=\, \mathscr{M} \times \mathbb{R} \times \mathbb{R}^4$ defined for all $x \in \mathscr{M}$ and for u and ρ^μ close[29] *to $\phi(x)$ and $P^\mu(x)$ respectively. On the other hand, $\mathbb{X}_\chi$ is the*

[28]Here $\chi \,=\, (\phi,\, P^\mu)$ and $\tilde{\chi} \,=\, (\tilde{\phi},\, \tilde{P}^\mu)$ denote two different sections of δ_1.

[29]In the sense of the norm defined above on $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$.

map:

$$\mathbb{X}_\chi \;:\; \mathscr{M} \to \mathbf{T}_{\chi(x)}\mathcal{P}(\mathbb{E}) \;:\; x \mapsto X(\chi(x)) = X_u(x,\, \phi(x),\, P(x))\frac{\partial}{\partial u}\bigg|_{\chi(x)} + \\ + X_\rho{}^\mu(x,\, \phi(x),\, P(x))\frac{\partial}{\partial \rho^\mu}\bigg|_{\chi(x)} =: \\ =: \mathbb{X}_\phi[\chi]\frac{\delta}{\delta\phi} + \mathbb{X}_P{}^\mu[\chi]\frac{\delta}{\delta P^\mu}\,, \tag{3.177}$$

following the notation (2.52)*. The right hand side is the (contraction of the tangent vector* $\mathbb{X}_\chi$ *along the) Euler-Lagrange form associated to our field theory. Equation* (3.169) *reads:*

$$\begin{aligned} \chi^\star\left[\, i_X \mathrm{d}\Theta_H \,\right] &= \chi^\star\left[\, (\, X_\rho{}^\mu \mathrm{d}u - X_u \mathrm{d}\rho^\mu\,) \wedge i_\mu vol_{\mathscr{M}} - \left(\eta_{\mu\nu}\rho^\mu X_\rho{}^\nu + m^2 u X_u\right) vol_{\mathscr{M}}\,\right] = \\ &= \left[\, \mathbb{X}_P{}^\mu \partial_\mu \phi - \mathbb{X}_\phi \partial_\mu P^\mu - \eta_{\mu\nu} P^\mu \mathbb{X}_P{}^\nu + m^2\phi \mathbb{X}_\phi \,\right] vol_{\mathscr{M}} = \\ &= \left[\, -\mathbb{X}_\phi \left(\partial_\mu P^\mu + m^2\phi\right) + \mathbb{X}_P{}^\mu \left(\,\partial_\mu\phi - \eta_{\mu\nu}P^\mu\,\right)\right] vol_{\mathscr{M}} = 0 \quad \forall\, \mathbb{X}_\phi,\, \mathbb{X}_P{}^\mu\,, \end{aligned} \tag{3.178}$$

which gives:

$$\partial_\mu P^\mu + m^2\phi = 0\,, \qquad \partial_\mu\phi = \eta_{\mu\nu}P^\mu\,, \tag{3.179}$$

that collectively says that the field ϕ *obeys the celebrated Klein-Gordon equation:*

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi(x) + m^2\phi(x) = 0\,. \tag{3.180}$$

3.2.3. Hamiltonian theories with constraints

The content of Sec. 3.1.3 can be generalized in a straightforward way to the case of field theories.

In this case the morphism (3.94) is a bundle morphism between $\mathbf{J}^1\pi \to \mathbb{E}$ and $\mathcal{P}(\mathbb{E})$ that reads:

$$\mathrm{F}_{\mathscr{L}} \;:\; \mathbf{J}^1\pi \to \mathcal{P}(\mathbb{E}) \;:\; (x^\mu,\, u^a,\, z^a_\mu) \mapsto \left(x^\mu,\, u^a,\, \frac{\partial \mathscr{L}}{\partial z^a_\mu}\right)\,. \tag{3.181}$$

Again, if the fibre derivative above is a diffeomorphism of fibre bundles, then, given a Lagrangian, it is canonically defined a local function on $\mathcal{P}(\mathbb{E})$, i.e. a Hamiltonian, by:

$$H = \mathrm{F}_{\mathscr{L}}^{-1\star}\left[\, \rho^\mu_a z^a_\mu - \mathscr{L}\,\right]\,, \tag{3.182}$$

and the structure $\mathrm{d}\Theta_{\mathscr{L}}$ of (3.117) turns out to be the pull-back of the structure $\mathrm{d}\Theta$ of (3.164).

Also the formulation of Hamiltonian theories with constraints goes along the lines of Sec. 3.1.3. After proving Thm. 3.1.16, one can find the extrema of $\mathscr{S}$ within the subset Ξ of $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$. Again, if $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$ is a Banach space (as it will be in all the examples considered) and if the subset of fields Ξ where we want to search for extrema of $\mathscr{S}$ is the image into $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$ of a map Φ satisfying the hypothesis of the latter proposition, then we can search for the extrema of $\mathscr{S}$ restricted to Ξ by searching for the extrema of the functional $\mathscr{S}^{\text{ext}}$ defined on $\mathcal{F}_{\mathcal{P}(\mathbb{E})} \times \mathcal{F}^{\star}_{\mathcal{P}(\mathbb{E})} \times \mathcal{N}$:

$$\mathscr{S}^{\text{ext}}_{(\chi,\Lambda,n)} \,=\, \mathscr{S}_{\chi} + \langle \Lambda,\, \chi - \Phi(n)\rangle \,. \tag{3.183}$$

Here, in the system of local coordinates chosen, $\langle \Lambda,\, \chi - \Phi(n)\rangle$ is:

$$\langle \Lambda,\, \chi - \Phi(n)\rangle \,=\, \int_{\mathscr{M}} \Big[\, \Lambda_{\phi_a} \left(\phi^a - \phi^a \circ \Phi(n)\right) + \Lambda_{P^{\mu}_a} \Big(P^a_{\mu} - P^a_{\mu} \circ \Phi(n)\Big) \Big]\, vol_{\mathscr{M}} \,, \tag{3.184}$$

where $(\Lambda_{\phi_a},\, \Lambda_{P^{\mu}_a})$ is a system of local coordinates on $\mathcal{F}^{\star}_{\mathcal{P}(\mathbb{E})}$. Consequently, $\mathscr{S}^{\text{ext}}$ explicitly reads:

$$\mathscr{S}^{\text{ext}}_{(\chi,\Lambda,n)} \,=\, \int_{\mathscr{M}} \Big[\, P^{\mu}_a \partial_{\mu}\phi^a - H(\xi) + \Lambda_{\phi_a} \left(\phi^a - \phi^a \circ \Phi(n)\right) + \Lambda_{P^{\mu}_a} \left(P^{\mu}_a - P^{\mu}_a \circ \Phi(n)\right) \Big]\, vol_{\mathscr{M}} \,. \tag{3.185}$$

Example 3.2.14 (Free Electrodynamics)**.** *As in example 3.2.7, we consider a sourceless electromagnetic field on the Minkowski space-time* $\mathscr{M} \,=\, (\mathbb{R}^4,\, \eta)$*. As we argued in example 3.2.7, the bundle underlying the theory is* $\pi \;:\; \mathbb{E} \,=\, \mathbf{T}^{\star}\mathscr{M} \to \mathscr{M}$ *whose sections, denoted by* $A \,=\, A_{\mu}(x)\mathrm{d}x^{\mu}$*, represent the quadri-potential in terms of which the covariant description of classical Electrodynamics is given. On* $\mathbb{E}$ *we denote by* $\{\, x^{\mu},\, u_{\mu}\,\}_{\mu=0,\dots,3}$ *a system of coordinates where* $\{\, x^{\mu}\,\}_{\mu=0,\dots,3}$ *is the system of coordinates chosen on* $\mathscr{M}$*. The covariant phase space* $\mathcal{P}(\mathbb{E})$ *is the reduced dual of the bundle* $\mathbf{J}^1\pi$ *constructed in example 3.2.7, namely the trivial bundle over* $\mathscr{M}$ *whose typical fibre is* $\mathbf{T}^{\star}_m\mathscr{M} \times \left(\bigotimes^2 \mathbf{T}_m\mathscr{M}\right)$ *where we consider* $\{\, x^{\mu},\, u_{\mu},\, \rho^{\mu\nu}\,\}_{\mu,\,\nu=0,\dots,3}$ *as a system of coordinates. We will denote by* $\chi \,=\, (A,\, P) \,=\, (A_{\mu},\, P^{\mu\nu})$ *elements of* $\Gamma^{\text{SPLIT}}(\delta_1)$*,* δ_1 *denoting the projection* $\mathcal{P}(\mathbb{E}) \to \mathscr{M}$*. The Hamiltonian of the theory is:*

$$H \,=\, \frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma}\rho^{\mu\nu}\rho^{\rho\sigma} \,. \tag{3.186}$$

The action functional obtained is:

$$\begin{aligned} \mathscr{S}_{\chi} \,&=\, \int_{\mathscr{M}} \chi^{\star} \left[\, \rho^{\mu\nu}\, \mathrm{d}u_{\mu} \wedge i_{\nu} vol_{\mathscr{M}} - \frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma}\rho^{\mu\nu}\rho^{\rho\sigma}\, vol_{\mathscr{M}} \right] \,= \\ &=\, \int_{\mathscr{M}} \left[\, P^{\mu\nu}(x)\partial_{\nu}A_{\mu} - \frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma}P^{\mu\nu}(x)P^{\rho\sigma}(x) \right]\, vol_{\mathscr{M}} \,. \end{aligned} \tag{3.187}$$

As in example 3.2.7, it is clear that such action functional is not well defined on the whole space of smooth splitting sections, $\Gamma^{\text{SPLIT}}(\delta_1)$*. Indeed, in order for the integral to be well defined, the* P*'s should be at least square integrable and the product of the* P*'s with the first derivatives of the* A*'s should be integrable. Thus, the first derivatives of the* A*'s should be square integrable as well. Actually, for technical reasons that will be clear in Sec. 4.2, we will ask for even more regularity and we will consider those* A*'s being* $\mathcal{H}^2$ *functions and those* P*'s being* $\mathcal{H}^1$ *functions and this will allow us* *to work with well defined Hilbert spaces of fields. However, as in example 3.2.7, it should be stressed that this is not the minimal requirement in order for $\mathscr{S}$ to be well defined and this choice may exclude some physical systems (even if it seems that only systems never observed in nature up to now are excluded). We will consider $\mathscr{S}$ to be defined on the subset of $\Gamma^{\textit{SPLIT}}(\delta_1)$ given by smooth splitting sections for which the norm:*

$$\| \chi \|^2 \,=\, \sum_{\mu=0}^{3} \| A_\mu \|^2_{\mathcal{H}^2} + \sum_{\mu,\,\nu=0}^{3} \| P^{\mu\nu} \|^2_{\mathcal{H}^1} \,, \tag{3.188}$$

is finite, say $\Gamma^{\textit{SPLIT}}(\delta_1)_1$. The following chain of inequalities:

$$\begin{aligned} \left|\mathscr{S}_\chi - \mathscr{S}_{\tilde{\chi}}\right| \leq & \int_{\mathscr{M}} \left| P^{\mu\nu}\partial_\nu A_\mu - \tilde{P}^{\mu\nu}\partial_\nu \tilde{A}_\mu + \frac{1}{2}P^{\mu\nu}P_{\mu\nu} - \frac{1}{2}\tilde{P}^{\mu\nu}\tilde{P}_{\mu\nu}\right| vol_{\mathscr{M}} \,\leq \\ & \leq \int_{\mathscr{M}} \bigg[\left| P^{\mu\nu}(\partial_\nu A_\mu - \partial_\nu \tilde{A}_\mu)\right| + \left| \partial_\nu A_\mu (P^{\mu\nu} - \tilde{P}^{\mu\nu})\right| + \\ & \qquad + \left| P^{\mu\nu}(P_{\mu\nu} - \tilde{P}_{\mu\nu})\right| + \left| \tilde{P}^{\mu\nu}(P_{\mu\nu} - \tilde{P}_{\mu\nu})\right| \bigg] vol_{\mathscr{M}} \,\leq \\ & \leq \sum_{\mu,\nu} \| P^{\mu\nu} \|_{\mathcal{L}^2} \| \partial_\nu A_\mu - \partial_\nu \tilde{A}_\mu \|_{\mathcal{L}^2} + \sum_{\mu,\nu} \| \partial_\nu A_\mu \|_{\mathcal{L}^2} \| P^{\mu\nu} - \tilde{P}^{\mu\nu} \|_{\mathcal{L}^2} + \\ & + \frac{1}{2}\sum_{\mu,\nu}\Big(\| P^{\mu\nu} \|_{\mathcal{L}^2} + \| \tilde{P}^{\mu\nu} \|_{\mathcal{L}^2} \Big) \| P_{\mu\nu} - \tilde{P}_{\mu\nu} \|_{\mathcal{L}^2} \,, \end{aligned} \tag{3.189}$$

shows that $\mathscr{S}$ is continuous in the norm defined above. Indeed, the last term in the chain of inequalities above vanishes as $\| \chi - \tilde{\chi} \|$ approaches zero. Consequently, $\mathscr{S}$ can be extended by continuity to the completion $\overline{\Gamma^{\textit{SPLIT}}(\delta_1)_1}^{\|\cdot\|} \,=\, [\,\mathcal{H}^2(\mathscr{M},\, vol_{\mathscr{M}})\,]^4 \times [\,\mathcal{H}^1(\mathscr{M},\, vol_{\mathscr{M}})\,]^{16} \,=:\, \mathcal{F}_{\mathcal{P}(\mathbb{E})}$.

As we will show, the correct dynamics of a sourceless electromagnetic field in vacuum is described via a variational principle applied to the action functional defined above, constrained to the image of the following map:

$$\Phi \;:\; \mathcal{F}_{\mathcal{P}(\mathbb{E})} \to \mathcal{F}_{\mathcal{P}(\mathbb{E})} \;:\; (A_\mu,\, P^{\mu\nu}) \mapsto \Phi\,[\,(A_\mu,\, P^{\mu\nu})\,] \,=\, (A_\mu,\, P^{[\mu\nu]}) \,, \tag{3.190}$$

where by $P^{[\mu\nu]}$ we mean the skew-symmetric part of $P^{\mu\nu}$. Thus, we can use the theory developed in the current section in order to find extrema of $\mathscr{S}$ restricted to the image of Φ and we can assert that they coincide with extrema of the functional:

$$\begin{aligned} \mathscr{S}^{\text{ext}}_{(\chi,\,\Lambda)} \,&=\, \int_{\mathscr{M}} \chi^\star \Theta_H + \int_{\mathscr{M}} \Lambda_{P_{\mu\nu}} \Big(P^{\mu\nu} - P^{[\mu\nu]} \Big) \, vol_{\mathscr{M}} \,= \\ &=\, \int_{\mathscr{M}} \chi^\star \Theta_H + \int_{\mathscr{M}} \Lambda_{P_{\mu\nu}}(x) P^{(\mu\nu)}(x) \, vol_{\mathscr{M}} \,, \end{aligned} \tag{3.191}$$

where $P^{(\mu\nu)}$ denotes the symmetric part of $P^{\mu\nu}$, the manifold $\mathcal{N}$ in this case coincides with $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$ and we are denoting by $\Lambda \,=\, ({\Lambda_A}^\mu,\, {\Lambda_P}^{\mu\nu})$ elements of $\mathcal{F}^\star_{\mathcal{P}(\mathbb{E})}$ which, in this case, coincide with $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$ itself, since $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$ is a Hilbert space and, thus, it is isomorphic to its dual space. Here the first fundamental formula reads:

$$\delta_{\mathbb{X}_{(\chi,\Lambda)}} \mathscr{S}^{\text{ext}} \,=\, \int_{\mathscr{M}} \chi^\star\, [\, i_X \mathrm{d}\Theta_H\,] + \int_{\mathscr{M}} \Big[\, \mathbb{X}_{\Lambda_{P\,\mu\nu}} P^{(\mu\nu)}(x) + \Lambda_{P_{\mu\nu}} {\mathbb{X}_P}^{(\mu\nu)} \Big] \, vol_{\mathscr{M}} \,, \tag{3.192}$$

where X is a δ_1-vertical vector field on $\mathcal{P}(\mathbb{E})$ defined in a neighborhood of the image of χ, i.e., it is a vector field of the type:

$$X \,=\, X_{u_\mu}\frac{\partial}{\partial u_\mu} + X_\rho{}^{\mu\nu}\frac{\partial}{\partial \rho^{\mu\nu}}\,, \tag{3.193}$$

where X_{u_μ} and $X_\rho{}^{\mu\nu}$ are functions on $\mathcal{P}(\mathbb{E})$ defined for all $x \in \mathscr{M}$ and for u_μ and $\rho^{\mu\nu}$ close[30] *to $A_\mu(x)$ and $P^{\mu\nu}(x)$ respectively. On the other hand, by looking at the proof of Prop. 3.1.16, $\mathbb{X}_{(\chi,\Lambda)}$ is a tangent vector to $\mathcal{F}_{\mathcal{P}(\mathbb{E})}\times\mathcal{F}^\star_{\mathcal{P}(\mathbb{E})}$ which is Φ-related to the tangent vector $\mathbb{X}_\chi$ (namely, it preserves the constraints):*

$$\begin{aligned}\mathbb{X}_\chi \;:\; \mathscr{M}\to\mathbf{T}_{\chi(x)}\mathcal{P}(\mathbb{E}) \;:\; x\mapsto X(\chi(x)) \,=& X_{u_\mu}(x,\,A(x),P(x))\frac{\partial}{\partial u_\mu}\bigg|_{\chi(x)} +\\ &+X_\rho{}^{\mu\nu}(x,A(x),P(x))\frac{\partial}{\partial\rho^{\mu\nu}}\bigg|_{\chi(x)} =:\\ &=: \mathbb{X}_A[\chi]_\mu\frac{\delta}{\delta A_\mu}+\mathbb{X}_P{}^{\mu\nu}[\chi]\frac{\delta}{\delta P^{\mu\nu}}\,,\end{aligned} \tag{3.194}$$

following the notation (2.52)*. Thus, $\mathbb{X}_{(\chi,\,\Lambda)}$ reads:*

$$\mathbb{X}_{(\chi,\,\Lambda)} \,=\, \mathbb{X}_A[\chi]_\mu\frac{\delta}{\delta A_\mu}+\mathbb{X}_{P^{[\mu\nu]}}[\chi]\frac{\delta}{\delta P^{[\mu\nu]}}+\mathbb{X}_{\Lambda_{\mu\nu}}[(\chi,\Lambda)]\frac{\delta}{\delta\Lambda_{P\,\mu\nu}}\,. \tag{3.195}$$

In this case, the equations of motion read:

$$\begin{aligned}\chi^\star\left[\,i_X\mathrm{d}\Theta_H\,\right] + \mathbb{X}_{\Lambda_{P\,\mu\nu}}P^{(\mu\nu)}(x) \,=\,& \chi^\star\Big[\,X_\rho{}^{\mu\nu}\mathrm{d}u_\mu\wedge i_\nu vol_{\mathscr{M}}+\\ &-X_{u_\mu}\mathrm{d}\rho^{\mu\nu}\wedge i_\nu vol_{\mathscr{M}}-\eta_{\mu\rho}\eta_{\nu\sigma}\rho^{\mu\nu}X_\rho{}^{\rho\sigma}vol_{\mathscr{M}}\,\Big]+\\ &+\mathbb{X}_{\Lambda_{P\,\mu\nu}}P^{(\mu\nu)} \,=\\ &=\mathbb{X}_{P^{[\mu\nu]}}\left(\,\partial_\nu A_\mu-\eta_{\mu\rho}\eta_{\nu\sigma}P^{\rho\sigma}\,\right)-\mathbb{X}_{A_\mu}\,\partial_\nu P^{\mu\nu}+\mathbb{X}_{\Lambda_{P\,\mu\nu}}P^{(\mu\nu)} \,=\\ &=\,-\,\mathbb{X}_{P^{[\mu\nu]}}\Big(\,F_{\mu\nu}+\eta_{[\mu\rho}\eta_{\nu]\sigma}P^{\rho\sigma}\,\Big)-\mathbb{X}_{A_\mu}\,\partial_\nu P^{\mu\nu}+\mathbb{X}_{\Lambda_{P\,(\mu\nu)}}P^{(\mu\nu)} \,=\, 0\\ &\quad\forall\;\mathbb{X}_{A_\mu},\,\mathbb{X}_P{}^{\mu\nu},\,\mathbb{X}_{\Lambda_{\mu\nu}}\,,\end{aligned} \tag{3.196}$$

where $F_{\mu\nu} \,=\, \partial_{[\mu}A_{\nu]}$ and which gives:

$$F_{\mu\nu}+\eta_{[\mu\rho}\eta_{\nu]\sigma}P^{\rho\sigma} \,=\, 0\,,\qquad \partial_\nu P^{\mu\nu} \,=\, 0\,,\qquad P^{(\mu\nu)} \,=\, 0\,, \tag{3.197}$$

that collectively amounts to:

$$\eta^{\mu\nu}\partial_\mu F_{\nu\rho} \,=\, 0\,, \tag{3.198}$$

being the celebrated covariant form of sourceless MAXWELL'S EQUATIONS *in vacuum.*

A formulation of the equations of motion in terms of a (pre-symplectic) Hamiltonian system similar to that given at the end of example 3.1.14 will be given in Sec. 4.2.5.

[30]With respect to the Sobolev norm defined above.

Example 3.2.15 (YANG-MILLS THEORIES). *As in example 3.2.8, we consider free Yang-Mills theories on the Minkowski space-time* $\mathscr{M} = (\mathbb{R}^4, \eta)$. *As we argued in example 3.2.8, the bundle underlying the theory is* $\pi : \mathbb{E} = \mathbf{T}^\star\mathscr{M} \otimes \mathfrak{g} \to \mathscr{M}$, *where* $\mathfrak{g}$ *is the Lie algebra of a semi-simple Lie group* G. *The sections of* π, *denoted by* $A = A^a_\mu(x)\mathrm{d}x^\mu \otimes \xi_a$, *are Lie algebra valued* 1*-forms representing the analogue of the quadri-potential of the previous example. On* $\mathbb{E}$ *we denote by* $\{ x^\mu, u^a_\mu \}_{\mu=0,\dots,3;a=1,\dots,\dim\mathfrak{g}}$ *a system of coordinates, where* $\{ x^\mu \}_{\mu=0,\dots,3}$ *is the system of coordinates chosen on* $\mathscr{M}$. *The covariant phase space* $\mathcal{P}(\mathbb{E})$, *is the reduced dual of the bundle* $\mathbf{J}^1\pi$ *constructed in example 3.2.8, namely the trivial bundle over* $\mathscr{M}$ *whose typical fibre is* $\mathbf{T}^\star_m \otimes \mathfrak{g}\mathscr{M} \times \left(\bigotimes^2 \mathbf{T}_m\mathscr{M} \otimes \mathfrak{g}^\star\right)$ *where we consider* $\{ x^\mu, u^a_\mu, \rho^{\mu\nu}_a \}_{\mu,\nu=0,\dots,3;a=1,\dots,\dim\mathfrak{g}}$ *as a system of coordinates. We will denote by* $\chi = (A, P) = (A^a_\mu, P^{\mu\nu}_a)$ *elements of* $\Gamma^{\textit{SPLIT}}(\delta_1)$, δ_1 *denoting the projection* $\mathcal{P}(\mathbb{E}) \to \mathscr{M}$. *The Hamiltonian of the theory is:*

$$H = \frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma}G^{ab}\rho^{\mu\nu}_a\rho^{\rho\sigma}_b + \epsilon^a{}_{bc}\rho^{\mu\nu}_a u^b_\mu u^c_\nu\,, \tag{3.199}$$

G *denoting the Killing-Cartan metric on* $\mathfrak{g}$. *The action functional obtained is:*

$$\begin{aligned}\mathscr{S}_\chi &= \int_{\mathscr{M}} \chi^\star\left[\rho^{\mu\nu}_a\,\mathrm{d}u^a_\mu \wedge i_\nu vol_{\mathscr{M}} - \left(\frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma}G^{ab}\rho^{\mu\nu}_a\rho^{\rho\sigma}_b + \epsilon^a{}_{bc}\rho^{\mu\nu}_a u^b_\mu u^c_\nu\right) vol_{\mathscr{M}}\right] = \\ &= \int_{\mathscr{M}}\left[P^{\mu\nu}_a(x)\partial_\nu A^a_\mu - \left(\frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma}G^{ab}P^{\mu\nu}_a(x)P^{\rho\sigma}_b(x) + \epsilon^a{}_{bc}P^{\mu\nu}_a A^b_\mu A^c_\nu\right)\right] vol_{\mathscr{M}}\,.\end{aligned} \tag{3.200}$$

Also in this case, it is clear that such action functional is not well defined on the whole space of smooth splitting sections, $\Gamma^{\textit{SPLIT}}(\delta_1)$. *Indeed, in order for the integral to be well defined, the* P*'s should be at least square integrable and the product of the* P*'s with the first derivatives of the* A*'s should be integrable. Thus, the first derivatives of the* A*'s should be square integrable as well. Actually, for technical reasons that will be clear in Sec. 4.2, we will ask for even more regularity, namely, we will consider those* A*'s being* $\mathcal{H}^3$ *functions and those* P*'s being* $\mathcal{H}^2$ *functions and this will allow us to work with well defined Hilbert spaces of fields. However, as in the previous example, it should be stressed that this is not the minimal requirement in order for* $\mathscr{S}$ *to be well defined and that this choice may exclude some physical system (even if it seems that only systems never observed in nature up to now are excluded). We will consider* $\mathscr{S}$ *to be defined on the subset of* $\Gamma^{\textit{SPLIT}}(\delta_1)$ *given by smooth splitting sections for which the norm:*

$$\|\,\chi\,\|^2 = \sum_{\mu,a}\|\,A^a_\mu\,\|^2_{\mathcal{H}^3} + \sum_{\mu,\nu,a}\|\,P^{\mu\nu}_a\,\|^2_{\mathcal{H}^2}\,, \tag{3.201}$$

is finite, say $\Gamma^{\textit{SPLIT}}(\delta_1)_1$. *The same techniques used in example 3.2.8 and 3.2.14 allow to prove that* $\mathscr{S}$ *is continuous in the norm defined above and, thus, it can be extended by continuity to the completion* $\overline{\Gamma^{\textit{SPLIT}}(\delta_1)_1}^{\|\cdot\|} = \left[\mathcal{H}^3(\mathscr{M}, vol_{\mathscr{M}})\right]^n \times \left[\mathcal{H}^2(\mathscr{M}, vol_{\mathscr{M}})\right]^m =: \mathcal{F}_{\mathcal{P}(\mathbb{E})}$ *where* $n = 4\dim\mathfrak{g}$ *and* $m = 16\dim\mathfrak{g}$.

As we will show, the correct dynamics of a free Yang-Mills field is described via a variational principle applied to the action functional defined above, constrained to *the image of the following map:*

$$\Phi \;:\; \mathcal{F}_{\mathcal{P}(\mathbb{E})} \to \mathcal{F}_{\mathcal{P}(\mathbb{E})} \;:\; (A^a_\mu,\, P^{\mu\nu}_a) \mapsto \Phi\Big[(A^a_\mu,\, P^{\mu\nu}_a)\Big] \;=\; (A^a_\mu,\, P^{[\mu\nu]}_a)\,, \tag{3.202}$$

where by $P^{[\mu\nu]}_a$ *we mean the skew-symmetric part of* $P^{\mu\nu}_a$*. Thus, we can use the theory developed in the current section in order to find extrema of* $\mathscr{S}$ *restricted to the image of* Φ *and we can assert that they coincide with extrema of the functional:*

$$\begin{aligned}\mathscr{S}^{\mathrm{ext}}_{(\chi,\,\Lambda)} \;&=\; \int_{\mathscr{M}} \chi^\star\Theta_H + \int_{\mathscr{M}} \Lambda_{P^{\;a}_{\mu\nu}}\Big(P^{\mu\nu}_a - P^{[\mu\nu]}_a\Big)\, vol_{\mathscr{M}} \;= \\ &=\; \int_{\mathscr{M}} \chi^\star\Theta_H + \int_{\mathscr{M}} \Lambda_{P^{\;a}_{\mu\nu}}(x) P^{(\mu\nu)}_a(x)\, vol_{\mathscr{M}}\,,\end{aligned} \tag{3.203}$$

where $P^{(\mu\nu)}_a$ *denotes the symmetric part of* $P^{\mu\nu}_a$*, the manifold* $\mathcal{N}$ *in this case coincides with* $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$ *and we are denoting by* $\Lambda = (\Lambda_{A^{\;\mu}_a},\, \Lambda_{P^{\mu\nu}_a})$ *elements of* $\mathcal{F}^\star_{\mathcal{P}(\mathbb{E})}$ *which, in this case, coincide with* $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$ *itself, since* $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$ *is a Hilbert space and, thus, it is isomorphic to its dual space. Here the first fundamental formula reads:*

$$\delta_{\mathbb{X}_{(\chi,\Lambda)}}\mathscr{S}^{\mathrm{ext}}_{(\chi,\Lambda)} \;=\; \int_{\mathscr{M}} \chi^\star\left[\, i_X \mathrm{d}\Theta_H \,\right] + \int_{\mathscr{M}} \Big[\, \mathbb{X}_{\Lambda_{P^{\;a}_{\mu\nu}}} P^{(\mu\nu)}_a(x) + \Lambda_{P^{\;a}_{\mu\nu}} \mathbb{X}_{P^{(\mu\nu)}_a} \Big]\, vol_{\mathscr{M}}\,, \tag{3.204}$$

where X *is a* δ_1*-vertical vector field on* $\mathcal{P}(\mathbb{E})$ *defined in a neighborhood of the image of* χ*, i.e., it is a vector field of the type:*

$$X \;=\; X_{u^{\;a}_\mu}\frac{\partial}{\partial u^a_\mu} + X_{\rho^{\mu\nu}_a}\frac{\partial}{\partial \rho^{\mu\nu}_{\;\;\;a}}\,, \tag{3.205}$$

where $X_{u^{\;a}_\mu}$ *and* $X_{\rho^{\mu\nu}_a}$ *are functions on* $\mathcal{P}(\mathbb{E})$ *defined for all* $x \in \mathscr{M}$ *and for* u^a_μ *and* $\rho^{\mu\nu}_a$ *close*[31] *to* $A^a_\mu(x)$ *and* $P^{\mu\nu}_a(x)$ *respectively. On the other hand,* $\mathbb{X}_{(\chi,\Lambda)}$ *is a tangent vector to* $\mathcal{F}_{\mathcal{P}(\mathbb{E})} \times \mathcal{F}^\star_{\mathcal{P}(\mathbb{E})}$ *which is* Φ*-related to the tangent vector* $\mathbb{X}_\chi$*:*

$$\begin{aligned}\mathbb{X}_\chi \;:\; \mathscr{M} \to \mathbf{T}_{\chi(x)}\mathcal{P}(\mathbb{E}) \;:\; x \mapsto X(\chi(x)) \;&= X_{u^{\;a}_\mu}(x,\, A(x), P(x))\frac{\partial}{\partial u^a_\mu}\bigg|_{\chi(x)} + \\ &+ X_{\rho^{\mu\nu}_a}(x, A(x), P(x))\frac{\partial}{\partial \rho^{\mu\nu}_a}\bigg|_{\chi(x)} =: \\ &=: \mathbb{X}_{A^{\;a}_\mu}[\chi]\frac{\delta}{\delta A^a_\mu} + \mathbb{X}_{P^{\mu\nu}_a}[\chi]\frac{\delta}{\delta P^{\mu\nu}_a}\,,\end{aligned} \tag{3.206}$$

following the notation (2.52)*. Thus,* $\mathbb{X}_{(\chi,\,\Lambda)}$ *reads:*

$$\mathbb{X}_{(\chi,\,\Lambda)} \;=\; \mathbb{X}_{A^{\;a}_\mu}[\chi]\frac{\delta}{\delta A^a_\mu} + \mathbb{X}_{P^{[\mu\nu]}_a}[\chi]\frac{\delta}{\delta P^{[\mu\nu]}} + \mathbb{X}_{\Lambda^{\;a}_{\mu\nu}}[(\chi,\Lambda)]\frac{\delta}{\delta \Lambda_{P^{\;a}_{\mu\nu}}}\,. \tag{3.207}$$

[31]With respect to the Sobolev norm defined above.

In this case, the equations of motion read:

$$
\begin{aligned}
\chi^\star\left[\, i_X \mathrm{d}\Theta_H \,\right] + \mathbb{X}_{\Lambda_P\,\mu\nu} P^{(\mu\nu)}(x) \,=\, \chi^\star\Big[\, & X_{\rho_a}^{\ \mu\nu} \mathrm{d}u_\mu^a \wedge i_\nu vol_{\mathscr{M}} - X_{u_\mu^a}\mathrm{d}\rho_a^{\mu\nu} \wedge i_\nu vol_{\mathscr{M}} + \\
& -(\eta_{\mu\rho}\eta_{\nu\sigma}\rho_a^{\mu\nu} G^{ab} X_{\rho_b}^{\ \rho\sigma} + \epsilon^a{}_{bc} X_{\rho_a}^{\ \mu\nu} u_\mu^b u_\nu^c + \\
& +\epsilon^a{}_{bc}\rho_a^{\mu\nu} u_\nu^c X_{u_\mu^b} + \epsilon^a{}_{bc}\rho_a^{\mu\nu} u_\mu^b X_{u_\nu^c})\, vol_{\mathscr{M}} \,\Big] + \\
& + \mathbb{X}_{\Lambda_P\,\mu\nu} P^{(\mu\nu)} \,= \\
& = \mathbb{X}_{P_a^{[\mu\nu]}} \Big(\partial_\nu A_\mu^a - \epsilon^a{}_{bc} A_\mu^b A_\nu^c - \eta_{\mu\rho}\eta_{\nu\sigma} G^{ab} P_b^{\rho\sigma} \Big) + \\
& - \mathbb{X}_{A_\mu^a} \nabla_\nu P_a^{\mu\nu} + \mathbb{X}_{\Lambda_P\,\mu\nu}^{\ \ a} P_a^{(\mu\nu)} \,= \\
& = -\, \mathbb{X}_{P_a^{[\mu\nu]}} \Big(F_{\mu\nu}^a + \eta_{[\mu\rho}\eta_{\nu]\sigma} G^{ab} P_b^{\rho\sigma} \Big) + \\
& - \mathbb{X}_{A_\mu^a} \nabla_\nu P_a^{\mu\nu} + \mathbb{X}_{\Lambda_P\,(\mu\nu)}^{\ \ a} P_a^{(\mu\nu)} \,=\, 0 \\
& \quad \forall\, \mathbb{X}_{A_\mu^a},\, \mathbb{X}_{P_a^{\mu\nu}},\, \mathbb{X}_{\Lambda_{\mu\nu}^a}\,,
\end{aligned}
\tag{3.208}
$$

where $F_{\mu\nu}^a = \partial_{[\mu}A_{\nu]} + \epsilon^a{}_{bc}A_\mu^b A_\nu^c =: \nabla_\mu A_\nu^a$ *and* $\nabla_\nu P_a^{\mu\nu} = \partial_\nu P_a^{\mu\nu} - \epsilon^c{}_{ba} P_c^{\nu\mu} A_\nu^b - \epsilon^b{}_{ac} P_b^{\mu\nu} A_\nu^c$ *are the covariant derivatives of* A *and* P *with respect to the connection* A*, and which gives:*

$$
F_{\mu\nu}^a + \eta_{[\mu\rho}\eta_{\nu]\sigma} G^{ab} P_b^{\rho\sigma} \,=\, 0\,, \qquad \nabla_\nu P_a^{\mu\nu} \,=\, 0\,, \qquad P_a^{(\mu\nu)} \,=\, 0\,, \tag{3.209}
$$

that collectively amounts to:

$$
\eta^{\mu\nu}\nabla_\mu F_{\nu\rho}^a \,=\, 0\,, \tag{3.210}
$$

being the celebrated Yang-Mills equations.

A formulation of the equations of motion in terms of a (pre-symplectic) Hamiltonian system similar to that given at the end of example 3.1.14 will be given in Sec. 4.2.5.

Example 3.2.16 (Palatini's Gravity)**.** *In this example we consider General Relativity in the so-called* Palatini's approach*. General Relativity can be described as a field theory over a space-time* $(\mathscr{M},\, g)$ *where the metric* g *is the configuration field of the theory. Historically, the first variational formulation of the theory is due to A. Einstein and D. Hilbert. It can be proved [89, Chap. 3], that Einstein equations can be obtained as Euler-Lagrange equations for the following action functional:*

$$
\mathscr{S}_{\text{E-H}g} \,=\, \int_{\mathscr{M}} \mathcal{R}\, \epsilon\, vol_{\mathscr{M}}\,, \tag{3.211}
$$

which is called the Einstein-Hilbert action, where $\mathcal{R}$ *is the scalar curvature of* g*,* $\epsilon = \sqrt{-\mathrm{det}g}$ *and* $vol_{\mathscr{M}} = \mathrm{d}x^0 \wedge \dots \wedge \mathrm{d}x^d$*.*

We briefly recall how $\mathcal{R}$ *is defined in order to introduce the notation we will use when we will pass to the tetradic formalism. Being* $\mathbf{T}\mathscr{M}$ *a vector bundle, a linear connection can always be defined on it [90, Appendix 11.4]. It can be encoded into a vertical valued 1-form on* $\mathbf{T}\mathscr{M}$ *of the type:*

$$
A \,=\, \Big(\mathrm{d}v^\mu - A_\nu{}^\mu{}_\rho v^\rho \mathrm{d}x^\nu \Big) \otimes \frac{\partial}{\partial v^\mu}\,, \tag{3.212}
$$

where $\{\, x^\mu,\, v^\mu \,\}_{\mu=0,\ldots,d}$ is a local coordinate system on $\mathbf{T}\mathscr{M}$ and $A_\nu{}^\mu{}_\rho$ are the so-called CONNECTION COEFFICIENTS. *The* CURVATURE *of the connection Γ is defined to be the Frolicher-Nijenhuis bracket between A and itself:*

$$R \,:=\, [A,\, A]_{F-N} \,=\, R_{\lambda\mu}{}^\nu{}_\rho v^\rho \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \otimes \frac{\partial}{\partial v^\nu}\,, \tag{3.213}$$

where:

$$R_{\lambda\mu}{}^\nu{}_\rho \,=\, \frac{1}{2}\left(\partial_\lambda A_\mu{}^\nu{}_\rho - \partial_\mu A_\lambda{}^\nu{}_\rho + A_\lambda{}^\sigma{}_\rho A_\mu{}^\nu{}_\sigma - A_\mu{}^\sigma{}_\rho A_\lambda{}^\nu{}_\sigma\right)\,. \tag{3.214}$$

The TORSION *of the connection is defined as the Frolicher-Nijenhuis bracket between R and the soldering form:*

$$S \,=\, \mathrm{d}x^\mu \otimes \frac{\partial}{\partial v^\mu}\,, \tag{3.215}$$

of $\mathbf{T}\mathscr{M}$, i.e.:

$$T \,=\, [R,\, S]_{F-N} \,=\, T_\mu{}^\nu{}_\rho \mathrm{d}x^\rho \wedge \mathrm{d}x^\mu \otimes \frac{\partial}{\partial x^\nu}\,, \tag{3.216}$$

where:

$$T_\mu{}^\nu{}_\rho \,=\, T_\mu{}^\nu{}_\rho - T_\nu{}^\mu{}_\rho\,. \tag{3.217}$$

The Ricci tensor, $\mathscr{R}$, of the connection A is the $(0,2)$-tensor on $\mathscr{M}$ whose coefficients are $\mathscr{R}_{\mu\nu} \,=\, R_{\mu\rho}{}^\rho{}_\nu$ and the scalar curvature $\mathcal{R}$ is:

$$\mathcal{R} \,=\, g^{\mu\nu}\mathscr{R}_{\mu\nu}\,. \tag{3.218}$$

When a metric is defined on $\mathscr{M}$, a particular connection on $\mathscr{M}$ can always be defined, i.e., the so called LEVI-CIVITA CONNECTION, *which is the (unique) connection on $\mathbf{T}\mathscr{M}$ such that $T \,=\, 0$ and such that the covariant derivative of g with respect to the connection vanishes. The scalar curvature appearing in the Einstein-Hilbert action above is the scalar curvature constructed from the Levi-Civita connection associated to the metric g. For this reason, in the Einstein-Hilbert formulation, General Relativity is a second order theory. Indeed, being $\mathcal{R}$ the scalar curvature of the Levi-Civita connection of the metric, it depends on g and its first and second order derivatives. Consequently, the action functional depends on the fields and their derivatives up to second order. Even if a quite solid geometrical theory to work with higher order field theories exists (see [74] and references therein), it is much easier to work with first order theories when possible. Indeed, General Relativity can be put into a first order theory by considering the metric g and the connection A as independent objects. This is the so-called Palatini's variational formulation of General Relativity.*

Palatini's action reads [89]:

$$\mathscr{S}_{P(g,A)} \,=\, \int_{\mathscr{M}} g^{\mu\nu}\mathscr{R}_{\mu\nu}\,\epsilon\,\mathrm{vol}_{\mathscr{M}}\,, \tag{3.219}$$

where $\mathscr{R}$ is the Ricci curvature of the connection A.

There are several good reasons for not taking g as the fundamental field of the theory, but rather the so-called tetrad fields[32]*. A tetrad at a point $m \in \mathscr{M}$ is defined to be a map from a basis of the tangent space of $\mathscr{M}$ at m to elements of the tangent space of a flat n-dimensional*[33] *manifold (which is isomorphic to $\mathbb{R}^n$) equipped with a Lorentzian metric:*

$$e(x) \;:\; \mathbf{T}_m\mathscr{M} \to \mathbb{R}^n \;:\; \frac{\partial}{\partial x^\mu} \mapsto e(x)\left(\frac{\partial}{\partial x^\mu}\right) = e^I_\mu(x)\xi_I\,, \quad I = 1, ..., n\,, \tag{3.220}$$

where ξ_I is a basis of the vector space $\mathbb{R}^n$. Therefore, $e(x)$ can be thought of as a $(1,\,1)$ *tensor on $\mathbf{T}_m\mathscr{M} \otimes \mathbb{R}^n$ of the type:*

$$e(x) = e^I_\mu(x)\mathrm{d}x^\mu \otimes \xi_I\,, \tag{3.221}$$

which satisfies:

$$g_{\mu\nu}e^\mu_I e^\nu_J = \eta_{IJ}\,, \tag{3.222}$$

η being the Minkowski metric which $\mathbb{R}^n$ is equipped with. The existence of such a map is ensured by the existence around any point m of $\mathscr{M}$ of normal (or Gauss) coordinates in which the metric g is the Minkowski metric and its first derivatives in m vanish (see [92]). Now, a tetrad field is defined to be a local section e of the bundle $\mathbf{T}\mathscr{M} \otimes \mathbb{R}^n \to \mathscr{M}$ satisfying, at each point, a condition of the type (3.222)*.*

The duals of the tetrad fields are defined by considering the following map:

$$e^\star(x) \;:\; \mathbf{T}^\star_m\mathscr{M} \to \mathbb{R}^n \;:\; \mathrm{d}x^\mu \mapsto e^\star(x)\,(\mathrm{d}x^\mu) = e^\mu_I(x)\xi^I\,, \quad I = 1, ..., n\,, \tag{3.223}$$

where $\left\{\,\xi^I\,\right\}_{I=1,\dots,n}$ is the dual basis of $\{\,\xi_I\,\}_{I=1,\dots,n}$. $e^\star(x)$ can be thought of as a $(1,\,1)$ *tensor on $\mathbf{T}^\star_m\mathscr{M} \otimes \mathbb{R}^n$ of the type:*

$$e^\star(x) = e^\mu_I(x)\frac{\partial}{\partial x^\mu} \otimes \xi^I\,, \tag{3.224}$$

which satisfies:

$$g^{\mu\nu}e^I_\mu e^J_\nu = \eta^{IJ}\,. \tag{3.225}$$

Therefore, a dual tetrad field is defined to be a local section of the bundle $\mathbf{T}^\star\mathscr{M} \otimes \mathbb{R}^n \to \mathscr{M}$ satisfying, at each point, the condition (3.225)*.*

Now, the action $\mathscr{S}_P$ can be expressed in terms of the connection A and the tetrad fields in the following way. The curvature (3.213) *can be expressed in the basis of the ξ_I's by taking the pull-back of R via the inverse of the map $e^\star$:*

$$\left(e^{\star -1}\right)^\star R = R_{\lambda\mu}{}^\nu{}_\rho e^\lambda_I e^\mu_J v^\rho \xi^I \wedge \xi^J \otimes \frac{\partial}{\partial v^\nu} =: R_{IJ}{}^\nu{}_\rho v^\rho \xi^I \wedge \xi^J \otimes \frac{\partial}{\partial v^\nu}\,. \tag{3.226}$$

[32] The use of the e's instead of g as fundamental field of the theory is widespread nowadays. Indeed, within the most common approaches to Quantum Gravity the use of tetrads is necessary to describe fermions. Moreover, as we will see in the next lines, tetrads describe very well the idea of the gravitational field as a deviation of the space-time from being flat and, therefore, many authors prefer to use e instead of g also from the point of view of the sake of Physical conceptual clearness (see [91]).

[33] Actually, even if the definition does not depend on the dimension, the name *tetrad* is devoted to the 4-dimensional case, which is the case we will address.

Consequently, the Ricci tensor can be expressed in terms of the tetrad fields as follows:

$$\mathscr{R}_{\mu\nu} \,=\, R_{\mu\sigma}{}^{\sigma}{}_{\nu} \,=\, R_{IJ}{}^{\sigma}{}_{\nu} e^I_\mu e^J_\sigma \,, \tag{3.227}$$

and the scalar curvature reads:

$$\mathcal{R} \,=\, g^{\mu\nu}\mathscr{R}_{\mu\nu} \,=\, \eta^{IJ} e^\mu_I e^\nu_J R_{KL}{}^{\sigma}{}_{\nu} e^K_\mu e^L_\sigma \,=\, -e^\nu_K e^\sigma_L R^{KL}{}_{\nu\sigma} \,. \tag{3.228}$$

Therefore, in terms of tetrad fields, the action 3.219 reads:

$$\mathscr{S}_{\mathrm{P(e,A)}} \,=\, -\int_{\mathscr{M}} \epsilon e^\mu_I e^\nu_J R^{IJ}{}_{\mu\nu} vol_{\mathscr{M}} \,. \tag{3.229}$$

which is the so-called TETRADIC PALATINI'S ACTION.

It is worth noting that $R^{IJ}{}_{\mu\nu}$ are the coefficient of a 2-form on $\mathscr{M}$ with values in $(\mathbb{R}^n \wedge \mathbb{R}^n, \eta)$. When $n = 4$, $(\mathbb{R}^n \wedge \mathbb{R}^4, \eta)$ is isomorphic with the Lie algebra of the orthogonal group $O(1,3)$. Therefore, the indices IJ can be considered as a collective index $a = IJ = 1, ..., \dim \mathfrak{o}(1,3)$ and $R^{IJ}{}_{\mu\nu}$ can be seen as the coefficients of the curvature of a connection one-form on $\mathscr{M}$ with values in $\mathfrak{o}(1,3)$.

Now we are ready to see how to develop the multi-symplectic formulation of the tetradic Palatini's action. In particular, we will see that (3.229) can be regarded as a Yang-Mills action in a suitable limit, the so-called TOPOLOGICAL LIMIT, *and constrained to a suitable subset of fields. For this reason we will necessitate the theory developed in Sect. 3.2.3 to deal with such a theory.*

Let us consider the Yang-Mills action written in the previous example with a dimensional constant multiplying the quadratic term in the momenta fields:

$$\mathscr{S}_{\text{Y-M}\chi} \,=\, -\int_{\mathscr{M}} \left[P^{\mu\nu}_a F^a_{\mu\nu} + \frac{1}{4g} P^{\mu\nu}_a P^a_{\mu\nu} \right] vol_{\mathscr{M}} \,. \tag{3.230}$$

Its topological limit[34] *is defined to be:*

$$\mathscr{S}^0_{\text{Y-M}\chi} \,=\, \lim_{g\to\infty} \mathscr{S}_{\text{Y-M}\chi} \,=\, -\int_{\mathscr{M}} P^{\mu\nu}_a F^a_{\mu\nu}\, vol_{\mathscr{M}} \,. \tag{3.231}$$

Note that such a topological limit is obtained by considering the limit for $g \to \infty$ of the Yang-Mills Hamiltonian of the previous example with a dimensional constant g:

$$H \,=\, \frac{1}{4g} \rho^{\mu\nu}_a \rho^a_{\mu\nu} + \frac{1}{2} \epsilon^a_{bc} \rho^{\mu\nu}_a \alpha^b_\mu \alpha^c_\nu \,. \tag{3.232}$$

Having in mind what we said a few lines ago about the indices IJ appearing in the tetradic Palatini's action, the action (3.229) can be seen as the topological limit of the action of a Yang-Mills theory with structure group $O(1,3)$ with the identification of the momenta of the theory $P^{\mu\nu}_a$ with the expression $\epsilon e^\mu_I e^\nu_J$ appearing in Eq. (3.219).

[34]The reason for this name is that the theory it will give rise is a topological one, in the sense that the dynamics of the theory will lie entirely into the kernel of a pre-symplectic structure, as it will be clear in Sec. 4.2.

Let us make this last claim more precise. Indeed, the internal indices of a $O(1,3)$ Yang-Mills configuration fields can be written as $a = IJ$ where I, J runs from 0 to 3 and IJ must be considered as a collective index running on $\{0, ..., 3\} \wedge \{0, ..., 3\}$. Therefore, here we denote Yang-Mills configuration fields as:

$$A = A(x)^{IJ}_{\mu} \mathrm{d}x^{\mu} \otimes \xi_I \wedge \xi_J \,, \tag{3.233}$$

where $\{\, \xi_I \,\}_{I=0,...,3}$ represents a basis of $\mathbb{R}^4$, $\{\, \xi_I \wedge \xi_J \,\}_{I,J=0,...,3}$ represents a basis of $\mathfrak{o}(1,3)$. On the other hand, momenta fields will be denoted as:

$$P = P^{\mu\nu}_{IJ}(x) \frac{\partial}{\partial x^{\mu}} \wedge \frac{\partial}{\partial x^{\nu}} \otimes \xi^I \wedge \xi^J \,, \tag{3.234}$$

where $\{\, \xi^I \,\}_{I=0,...,3}$ is a basis of $\mathbb{R}^4$, $\{\, \xi^I \wedge \xi^J \,\}_{I,J=0,...,3}$ represents a basis of $\mathfrak{o}(1,3)^{\star}$. As usual, dynamical fields of the theory are denoted by $\chi = (A,\, P) \in \mathcal{F}_{\mathcal{P}(\mathbb{E})}$. On the other hand, we saw that tetrad fields are defined as sections of the bundle $\mathbf{T}\mathscr{M} \otimes \mathbb{R}^4 \to \mathscr{M}$ which reads:

$$e = e^{\mu}_{I}(x) \frac{\partial}{\partial x^{\mu}} \otimes \xi^I \,. \tag{3.235}$$

Let us denote by $\mathscr{E}$ the space of tetrad fields. Then, the following map can be defined:

$$\mathscr{P} \;:\; \mathcal{F}_{\mathbb{E}} \times \mathscr{E} \to \mathcal{F}_{\mathcal{P}(\mathbb{E})} \;:\; (A,\, e) \mapsto \left(A,\, P = \epsilon e^{\mu}_{I} e^{\nu}_{J} \frac{\partial}{\partial x^{\mu}} \wedge \frac{\partial}{\partial x^{\nu}} \otimes \xi^I \wedge \xi^J \right) . \tag{3.236}$$

We will refer to it as Palatini map *or* Palatini constraint*. Then, the tetradic Palatini's action is nothing but the pull-back via $\mathscr{P}$ of the topological limit of the Yang-Mills action:*

$$\mathscr{S}_{\mathrm{P}} = \mathscr{P}^{\star} \mathscr{S}^{0}_{\text{Y-M}} \,, \tag{3.237}$$

whose extrema are the extrema of $\mathscr{S}^{0}_{\text{Y-M}}$ constrained along the image of $\mathscr{P}$. Therefore, being $\mathscr{P}$ a map satisfying the hypothesis of Prop. 3.1.16, we are in the situation depicted in Sec. 3.2.3 where $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$ is the space of dynamical fields of a $O(1,3)$ Yang-Mills theory and Ξ is the subset of $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$ being the image of $\mathscr{E}$ via the map $\Phi = \mathscr{P}$.

Therefore, now we will find extrema of $\mathscr{S}_{\mathrm{P}}$ by using the theory developed in Sect. 3.2.3. In this case the manifold $\mathcal{N}$ of Prop. 3.1.16 does not coincide with $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$, and it reads $\mathcal{N} = \mathcal{F}_{\mathbb{E}} \times \mathscr{E}$. Thus, the extended action functional is defined on $\mathcal{F}_{\mathcal{P}(\mathbb{E})} \times \mathcal{F}_{\mathcal{P}(\mathbb{E})}{}^{\star} \times \mathscr{E}$, whose elements will be denoted by $(\chi,\, \Lambda,\, e)$. Here, the analogue of (3.203) *reads:*

$$\mathscr{S}^{\text{ext}}_{(\chi, \Lambda, e)} = \int_{\mathscr{M}} \chi^{\star} \Theta^{0}_{H} + \int_{\mathscr{M}} \Lambda_{P^{IJ}_{\mu\nu}} \left(P^{\mu\nu}_{IJ} - \epsilon\, e^{[\mu}_{I} e^{\nu]}_{J} \right) vol_{\mathscr{M}} \,, \tag{3.238}$$

where:

$$\Theta^{0}_{H} = \rho^{\mu\nu}_{IJ} \mathrm{d}u^{IJ}_{\mu} \wedge i_{\nu} vol_{\mathscr{M}} - \epsilon^{IJ}{}_{KLMN} \rho^{\mu\nu}_{IJ} u^{KL}_{\mu} u^{MN}_{\nu} \, vol_{\mathscr{M}} \,, \tag{3.239}$$

is the topological limit of the Θ_H *of the previous example. Consequently:*

$$\delta_{\mathbb{X}_{(\chi,\Lambda,e)}}\mathscr{S}^{\mathrm{ext}}_{(\chi,\Lambda,e)} \,=\, \int_{\mathscr{M}} \chi^\star\left[\, i_X \mathrm{d}\Theta^0_H\,\right] + \int_{\mathscr{M}} \Big[\, \mathbb{X}_{\Lambda_P}{}^{IJ}_{\mu\nu}\left(\, P^{\mu\nu}_{IJ} - \epsilon\, e^{[\mu}_I e^{\nu]}_J\,\right) + \\ + \Lambda_P{}^{\mu\nu}_{IJ}\left(\, \mathbb{X}_{P}{}^{IJ}_{\mu\nu} - 2\epsilon e^{[\mu}_I (e^\rho_K e^{\nu]}_J\, \mathbb{X}_{e}{}^{\rho}_K + \mathbb{X}_{e}{}^{\nu]}_J)\,\right)\Big]\, vol_{\mathscr{M}}\,, \tag{3.240}$$

where the second term in the second integral vanishes because of the fact that the tangent vector

$$\mathbb{X}_{(\chi,\Lambda,e)} \,=\, \mathbb{X}_{A}{}^{IJ}_{\mu}\frac{\delta}{\delta A^{IJ}_\mu} + \mathbb{X}_{P}{}^{\mu\nu}_{IJ}\frac{\delta}{\delta P^{\mu\nu}_{IJ}} + \mathbb{X}_{\Lambda_A}{}^{\mu}_{IJ}\frac{\delta}{\delta \Lambda_A{}^{\mu}_{IJ}} + \mathbb{X}_{\Lambda_P}{}^{IJ}_{\mu\nu}\frac{\delta}{\delta \Lambda_P{}^{IJ}_{\mu\nu}} + \mathbb{X}_{e}{}^{\mu}_I\frac{\delta}{\delta e^\mu_I}\,, \tag{3.241}$$

must be tangent to the image of $\mathscr{P}$*, as explained in Prop. 3.1.16 and, thus, is such that:*

$$\mathbb{X}_{P}{}^{\mu\nu}_{IJ} \,=\, 2\epsilon e^{[\mu}_I (e^\rho_K e^{\nu]}_J\, \mathbb{X}_{e}{}^{\rho}_K + \mathbb{X}_{e}{}^{\nu]}_J)\,. \tag{3.242}$$

Following the same steps of the previous example, one sees that the equations of motion read:

$$\chi^\star[\, i_X \mathrm{d}\Theta_H\,] + \mathbb{X}_{\Lambda_P}{}^{IJ}_{\mu\nu}\left(\, P^{\mu\nu}_{IJ} - \epsilon\, e^{[\mu}_I e^{\nu]}_J\,\right) \,=\, \left(\, e^K_\rho e^{\nu]}_J \mathbb{X}_{e}{}^{\rho}_K + \mathbb{X}_{e}{}^{\nu]}_J\,\right) 2\epsilon e^{\mu]}_I F^{IJ}_{\mu\nu} + \\ - \mathbb{X}_{A}{}^{IJ}_{\mu} \nabla_\nu P^{\mu\nu}_{IJ} + \\ + \mathbb{X}_{\Lambda_P}{}^{IJ}_{\mu\nu}\left(\, P^{\mu\nu}_{IJ} - \epsilon\, e^{[\mu}_I e^{\nu]}_J\,\right) \,=\, 0 \quad \forall\ \mathbb{X}_{A}{}^{IJ}_{\mu},\, \mathbb{X}_{P}{}^{\mu\nu}_{IJ},\, \mathbb{X}_{e}{}^{\mu}_I\,, \tag{3.243}$$

which gives:

$$e^\mu_I F^{IJ}_{\mu\nu} \,=\, 0\,, \qquad \nabla_\nu P^{\mu\nu}_{IJ} \,=\, 0\,, \qquad P^{\mu\nu}_{IJ} \,=\, \epsilon\, e^{[\mu}_I e^{\nu]}_J\,, \tag{3.244}$$

that collectively amounts to:

$$e^\mu_I F^{IJ}_{\mu\nu} \,=\, 0\,, \qquad \nabla_\nu\left(\, \epsilon\, e^{[\mu}_I e^{\nu]}_J\,\right) \,=\, 0\,, \tag{3.245}$$

being, respectively, Einstein's equations *in vacuum and the* torsionless condition *for the connection associated to the metric.*

4. THE GEOMETRY OF THE SOLUTION SPACE

In this chapter, which is the core of the manuscript, we pass to our main aim, namely the study of the geometry of the space of solutions of the equations of motion (solution space, for short) geometrically formulated in the previous chapter and, in particular, the possibility of equipping it with a Poisson structure. The chapter is organized as follows. In Sec. 4.1 we will show how from the intrinsic variational formulation of dynamical system described in the previous chapter, a canonical 2-form defined on the space of solutions of the equations of motion emerges. Then, in Sec. 4.2 we will show the role of such canonical structure in formulating, at least locally around particular 1-codimension hypersurfaces of the space-time of the theory, the dynamical system as a pre-symplectic Hamiltonian system. We will discover how this approach allows to algorithmically find the Cauhcy data space for our system that turns out to be automatically equipped with a canonical 2-form as well. Finally, in Sec. 4.3 we will show how to consistently use such a canonical structure on the space of Cauchy data to induce a Poisson bracket structure on the solution space of the theory.

4.1. The canonical structure on the solution space

We start by showing the emergence of a canonical 2-form on the solution space from the intrinsic variational principle defined in the previous chapter. As in the previous chapter we will do it both within mechanical systems and within field theories in the Lagrangian formulation as well as in the Hamiltonian one and in the Hamiltonian theories with additional constraints.

4.1.1. Lagrangian Mechanics

Let us start this section by noting that also the second term on the right hand side of (3.16) can be interpreted as the contraction of a differential form on $\mathcal{F}(\mathbb{Q})$ along a tangent vector, even if in this case more comments are needed. Indeed, since the second term on the right hand side of (3.16), i.e.:

$$\int_{\partial \mathbb{I}} \mathfrak{i}^{\star}_{\partial \mathbb{I}} \left(j^1 \gamma \right)^{\star} i_{X^1} \Theta_{\mathscr{L}} \,, \tag{4.1}$$

only depends on the restriction of $j^1\gamma$ to the boundary $\partial \mathbb{I}$, it can be seen only as the pull-back of a differential form on the space of restrictions of elements $j^1\gamma$ to $\partial \mathbb{I}$. Recall that the γs belong to a suitable completion of the space of sections of π. We assume, from now on, that, given the norm with respect to which $\mathcal{F}(\mathbb{Q})$ is constructed, a norm on the space of jets of sections restricted to $\partial \mathbb{I}$ is naturally induced and which allows to define a smooth Banach manifold structure. We will denote the smooth Banach manifold obtained by $\mathcal{F}_{\mathbb{Q}}^{\partial\mathbb{I}}$ and we will often refer to it as the space of restrictions of jets of sections to $\partial\mathbb{I}$. Denote by $\Pi_{\partial\mathbb{I}}$ the map:

$$\Pi_{\partial\mathbb{I}} \;:\; \mathcal{F}_{\mathbb{Q}} \to \mathcal{F}_{\mathbb{Q}}^{\partial\mathbb{I}} \;:\; \gamma \mapsto j^1\gamma\Big|_{\partial\mathbb{I}} =: j^1\gamma_{\partial\mathbb{I}}\,, \tag{4.2}$$

Then, the second term in the right hand side of (3.16) can be seen as the pull-back (via $\Pi_{\partial\mathbb{I}}$) to $\mathcal{F}_{\mathbb{Q}}$ of a differential 1-form, say $\alpha^{\partial\mathbb{I}}$, on $\mathcal{F}_{\mathbb{Q}}^{\partial\mathbb{I}}$ contracted along a tangent vector:

$$i_{\mathbb{X}_\gamma}\left[\Pi_{\partial\mathbb{I}}^\star\alpha^{\partial\mathbb{I}}\right] = \Pi_{\partial\mathbb{I}}^\star\alpha^{\partial\mathbb{I}}(\mathbb{X}_\gamma) = \int_{\partial\mathbb{I}} \mathfrak{i}_{\partial\mathbb{I}}^\star\left(j^1\gamma\right)^\star i_{X^1}\Theta_{\mathscr{L}} \quad \forall\, \mathbb{X}_\gamma \in \mathbf{T}_\gamma\mathcal{F}_{\mathbb{Q}}\,, \tag{4.3}$$

where X^1 is the first order jet prolongation of any extension of $\mathbb{X}_\gamma$ to an open neighborhood of the image of γ into $\mathbb{Q}$. Note that in terms of such a differential form and the Euler-Lagrange form, the first fundamental formula reads:

$$\mathrm{d}\mathscr{S}_\gamma = \mathbb{EL}_\gamma + \Pi_{\partial\mathbb{I}}^\star{\alpha^{\partial\mathbb{I}}}_\gamma\,. \tag{4.4}$$

The goal of this section is to show that, starting from $\Pi_{\partial\mathbb{I}}^\star\alpha^{\partial\mathbb{I}}$, it is possible to define a canonical differential 2-form on the solution space of the theory.

First, let us recall that $\mathbb{I}$ is a differential manifold with boundary. What is more, it is an orientable differential manifold on which we could fix the orientation to be the positive (outer) one, for instance. It means that at each point of the boundary, i.e. the two extrema of the interval, say a and b, we are considering the versor pointing outside the interval, thus, having opposite directions in the two cases. The boundary is the following disconnected manifold made by two points, $\partial\mathbb{I} = \{\,a\,\}\cup\{\,b\,\}$ which is in turn a topological manifold, actually an orientable one with the orientation inherited from $\mathbb{I}$. Thus, the differential one-form defined above reads:

$$\begin{aligned}\Pi_{\partial\mathbb{I}}^\star{\alpha^{\partial\mathbb{I}}}_\gamma\,(\,\mathbb{X}_\gamma\,) &= \int_{\partial\mathbb{I}} \mathfrak{i}_{\partial\mathbb{I}}^\star\left(j^1\gamma\right)^\star i_{X^1}\Theta_{\mathscr{L}} = \left(j^1\gamma\right)^\star\left[\,i_{X^1}\Theta_{\mathscr{L}}\,\right]\Big|_b - \left(j^1\gamma\right)^\star\left[\,i_{X^1}\Theta_{\mathscr{L}}\,\right]\Big|_a = \\ &=: \Pi_b^\star{\alpha^b}_\gamma\,(\,\mathbb{X}_\gamma\,) - \Pi_a^\star{\alpha^a}_\gamma\,(\,\mathbb{X}_\gamma\,)\;,\end{aligned} \tag{4.5}$$

where α^a (resp. α^b) is a differential form on the space of restrictions of elements of $\mathcal{F}_{\mathbb{Q}}$ to a (resp. b), say $\mathcal{F}_{\mathbb{Q}}^a$ (resp. $\mathcal{F}_{\mathbb{Q}}^b$) and Π_a (resp. Π_b) is the corresponding restriction map analogous to $\Pi_{\partial\mathbb{I}}$. Referring to the system of local coordinates chosen on $\mathbf{J}^1\pi$, the differential form above reads:

$$\Pi_{\partial\mathbb{I}}^\star{\alpha^{\partial\mathbb{I}}}_\gamma = \Pi_b^\star{\alpha^b}_\gamma - \Pi_a^\star{\alpha^a}_\gamma = \left(j^1\gamma\right)^\star\left[\pi_{\mathscr{L}j}\mathrm{d}q^j\right]\Big|_b - \left(j^1\gamma\right)^\star\left[\pi_{\mathscr{L}j}\mathrm{d}q^j\right]\Big|_a. \tag{4.6}$$

Its differential reads:

$$\Pi_{\partial\mathbb{I}}^\star{\Omega^{\partial\mathbb{I}}}_\gamma := -\mathrm{d}\Pi_{\partial\mathbb{I}}^\star{\alpha^{\partial\mathbb{I}}}_\gamma = -\mathrm{d}\Pi_b^\star{\alpha^b}_\xi + \mathrm{d}\Pi_a^\star{\alpha^a}_\gamma = \Pi_b^\star\,\mathrm{d}{\alpha^b}_\gamma - \Pi_a^\star\,\mathrm{d}{\alpha^a}_\gamma =: -\Pi_b^\star{\Omega^b}_\gamma + \Pi_a^\star{\Omega^a}_\gamma\,. \tag{4.7}$$

where, again Ω^a (resp Ω^b) is a differential form on $\mathcal{F}^a_{\mathbb{Q}}$ (resp. $\mathcal{F}^b_{\mathbb{Q}}$). A direct computation shows the explicit expression of the latter 2-form, i.e.:

$$\Pi^\star_{\partial\mathbb{I}}{\Omega^{\partial\mathbb{I}}}_\gamma\,(\,\mathbb{X}_\gamma,\,\mathbb{Y}_\gamma\,)\;=\;-\int_{\partial\mathbb{I}}\mathfrak{i}^\star_{\partial\mathbb{I}}\,\left(j^1\gamma\right)^\star[\,i_{Y^1}\,i_{X^1}\,\mathrm{d}\Theta_{\mathscr{L}}\,]\;, \tag{4.8}$$

where X^1 and Y^1 are first order jet prolongations of two arbitrary extensions of $\mathbb{X}_\gamma$ and $\mathbb{Y}_\gamma$ to an open neighborhood of the image of γ into $\mathbb{Q}$. Consequently, $\Pi^\star_a{\Omega^a}_\gamma$ and $\Pi^\star_b{\Omega^b}_\gamma$ read:

$$\Pi^\star_a{\Omega^a}_\gamma(\mathbb{X}_\gamma,\,\mathbb{Y}_\gamma)\;=\;-\left(j^1\gamma\right)^\star[\,i_{Y^1}\,i_{X^1}\,\mathrm{d}\Theta_{\mathscr{L}}\,]\Big|_a \tag{4.9}$$

$$\Pi^\star_b{\Omega^b}_\gamma(\mathbb{X}_\gamma,\,\mathbb{Y}_\gamma)\;=\;-\left(j^1\gamma\right)^\star[\,i_{Y^1}\,i_{X^1}\,\mathrm{d}\Theta_{\mathscr{L}}\,]\Big|_b \tag{4.10}$$

which, in the system of local coordinates chosen, are:

$$\Pi^\star_a{\Omega^a}_\gamma\;=\;\left(j^1\gamma\right)^\star\left[\mathrm{d}q^j\wedge\mathrm{d}\pi_{\mathscr{L}j}\right]\Big|_a\,, \tag{4.11}$$

$$\Pi^\star_b{\Omega^b}_\gamma\;=\;\left(j^1\gamma\right)^\star\left[\mathrm{d}q^j\wedge\mathrm{d}\pi_{\mathscr{L}j}\right]\Big|_b\,. \tag{4.12}$$

For any $t\in\mathbb{I}$, it is possible to define the 2-form:

$$\Pi^\star_t{\Omega^t}_\gamma(\mathbb{X}_\gamma,\,\mathbb{Y}_\gamma)\;=\;\left(j^1\gamma\right)^\star[\,i_{Y^1}\,i_{X^1}\,\mathrm{d}\Theta_{\mathscr{L}}\,]\Big|_t \tag{4.13}$$

The goal of the rest of the section is to prove that the structure $\Pi^\star_t\Omega^t$ at each point $t\in\mathbb{I}$ is the same if evaluated at $\gamma\in\mathcal{EL}$, i. e., $\Pi^\star_t\Omega^t$ is a canonical structure on the solution space of the theory.

Let us start by recalling that, provided $\mathcal{EL}$ is an immersed Banach submanifold of $\mathcal{F}(\mathbb{Q})$, as we assumed, the following fact holds.

Proposition 4.1.1. *$\mathcal{EL}$ is an isotropic manifold for the differential 2-form $\Pi^\star_{\partial\mathbb{I}}\Omega^{\partial\mathbb{I}}$.*

Proof. Consider (4.4) and take the differential of both sides:

$$\underbrace{\mathrm{d}\,\mathrm{d}}_{=0}\mathscr{S}_\gamma\;=\;\mathrm{d}\mathbb{EL}_\gamma+\mathrm{d}\Pi^\star_{\partial\mathbb{I}}\alpha^{\partial\mathbb{I}}_\gamma\;=\;\mathrm{d}\mathbb{EL}_\gamma-\Pi^\star_{\partial\mathbb{I}}\Omega^{\partial\mathbb{I}}_\gamma\,, \tag{4.14}$$

i.e.:

$$\mathrm{d}\mathbb{EL}_\xi-\Pi^\star_{\partial\mathbb{I}}\Omega^{\partial\mathbb{I}}_\gamma\;=\;0\,. \tag{4.15}$$

Consider the pull-back of the left hand side of the latter equation via $\mathfrak{i}_{\mathcal{EL}}$. The pull-back acts naturally with respect to d and $\mathcal{EL}$ is the space of zeroes of $\mathbb{EL}$, thus, $\mathfrak{i}^\star_{\mathcal{EL}}\mathrm{d}\mathbb{EL}\;=\;\mathrm{d}\mathfrak{i}^\star_{\mathcal{EL}}\mathbb{EL}\;=\;0$. Therefore:

$$\mathfrak{i}^\star_{\mathcal{EL}}\,\Pi^\star_{\partial\mathbb{I}}{\Omega^{\partial\mathbb{I}}}_\gamma\;=\;0\,. \tag{4.16}$$

□

In particular, by looking at (4.7), the previous proposition gives:

$$\Pi^\star_b{\Omega^b}_\gamma\;=\;\Pi^\star_a{\Omega^a}_\gamma\,,\quad\forall\,\gamma\in\mathcal{EL}\,. \tag{4.17}$$

Also the following, apparently obvious, fact holds.

Proposition 4.1.2. *Consider a (connected) closed subinterval of $\mathbb{I}$, say $\tilde{\mathbb{I}}$. Denote by $\tilde{\mathcal{E}\mathcal{L}}$ the solution space related with the variational principle formulated on $\tilde{\mathbb{I}}$ instead of $\mathbb{I}$. Then, the restrictions of the elements in $\mathcal{E}\mathcal{L}$ to $\tilde{\mathbb{I}}$ belong to $\tilde{\mathcal{E}\mathcal{L}}$.*

Proof. Let us denote by $\tilde{\mathrm{i}}$ the canonical immersion of $\tilde{\mathbb{I}}$ into $\mathbb{I}$. Given $\tilde{\mathrm{i}}$, the space $\tilde{\mathbb{Q}} = \tilde{\mathbb{I}} \times \mathcal{Q}$ is canonically immersed into $\mathbb{Q}$ via the map $\tilde{\mathrm{i}}_{\mathbb{Q}} := \tilde{\mathrm{i}} \times \mathbb{1}_{\mathcal{Q}}$ and $\mathbf{J}^{\tilde{\mathbf{1}}}\pi$ (i.e. the first order jet bundle of $\tilde{\mathbb{Q}} \to \tilde{\mathbb{I}}$) is canonically immersed into $\mathbf{J}^1\pi$ via $j^1\tilde{\mathrm{i}}_{\mathbb{Q}}$. We will denote by $\tilde{\gamma}$ and by γ sections of $\tilde{\pi} \ : \ \tilde{\mathbb{Q}} \to \tilde{\mathbb{I}}$ and of $\pi \ : \ \mathbb{Q} \to \mathbb{I}$ respectively. Denote by $\tilde{\pi}^1_0$ and $\tilde{\pi}_1$ the projections of $\mathbf{J}^{\tilde{\mathbf{1}}}\pi$ onto $\tilde{\mathbb{Q}}$ and $\tilde{\mathbb{I}}$ respectively. For all γ there exists a $\tilde{\gamma}$ (its restriction $\gamma|_{\tilde{\mathbb{I}}}$) such that the following diagram commutes:

$$
\begin{tikzcd}
\mathbf{J}^{\tilde{\mathbf{1}}}\pi \arrow[r, hook, "j^1\tilde{\mathrm{i}}_{\mathbb{Q}}"] \arrow[d, "\tilde{\pi}^1_0"] & \mathbf{J}^1\pi \arrow[d, "\pi^1_0"'] \\
\tilde{\mathbb{Q}} \arrow[r, hook, "\tilde{\mathrm{i}}_{\mathbb{Q}}"] & \mathbb{Q} \\
\tilde{\mathbb{I}} \arrow[r, hook, "\tilde{\mathrm{i}}"] \arrow[u, dashed, "\tilde{\gamma}"'] \arrow[uu, dashed, bend left, "j^1\tilde{\gamma}"] & \mathbb{I} \arrow[u, dashed, "\gamma"] \arrow[uu, dashed, bend right, "j^1\gamma"']
\end{tikzcd}
\tag{4.18}
$$

The variational principle formulated on $\mathbf{J}^{\tilde{\mathbf{1}}}\pi$ gives the following equations of motion:

$$
\left(j^1\tilde{\gamma}\right)^\star \left[i_{\tilde{X}^1} \left[(j^1\tilde{\mathrm{i}}_{\mathbb{Q}})^\star \mathrm{d}\Theta_{\mathscr{L}} \right] \right] = 0 \quad \forall \ \tilde{X} \in \mathfrak{X}^{\tilde{\pi}}(U^{(\tilde{\gamma})}) \,, \tag{4.19}
$$

where $\mathfrak{X}^{\tilde{\pi}}(U^{(\tilde{\gamma})})$ denotes the module of vertical (with respect to $\tilde{\pi}$) vector fields defined on an open neighborhood of the image of $\tilde{\gamma}$ in $\tilde{\mathbb{Q}}$. The left hand side of the previous equation can be rewritten as:

$$
\begin{aligned}
\left(j^1\tilde{\gamma}\right)^\star \left[i_{\tilde{X}^1} \left[(j^1\tilde{\mathrm{i}}_{\mathbb{Q}})^\star \mathrm{d}\Theta_{\mathscr{L}} \right] \right] &= \left(j^1\tilde{\gamma}\right)^\star \left[(j^1\tilde{\mathrm{i}}_{\mathbb{Q}})^\star \, i_{X^1} \mathrm{d}\Theta_{\mathscr{L}} \right] = \left(j^1\tilde{\mathrm{i}}_{\mathbb{Q}} \circ j^1\tilde{\gamma}\right)^\star (\, i_{X^1} \mathrm{d}\Theta_{\mathscr{L}} \,) = \\
&= \left(j^1\gamma \circ \tilde{\mathrm{i}}\right)^\star (\, i_{X^1} \mathrm{d}\Theta_{\mathscr{L}} \,) = \tilde{\mathrm{i}}^\star \left[\left(j^1\gamma\right)^\star i_X \mathrm{d}\Theta_{\mathscr{L}} \right] \,,
\end{aligned}
\tag{4.20}
$$

for some X which is $\tilde{\mathrm{i}}_{\mathbb{Q}}$-related with $\tilde{X}$ and where γ is one of the sections of π_1 that restrict to $\tilde{\gamma}$. The previous sequence of equalities clearly shows that if γ is a solution for the variational problem on $\mathbb{I}$, i. e., if $(\, j^1\gamma \,)^\star (\, i_{X^1} \mathrm{d}\Theta_{\mathscr{L}} \,) = 0$, then its restriction $\tilde{\gamma}$ is a solution of the variational principle on $\tilde{\mathbb{I}}$. □

A combination of the previous results allows to prove the following.

Proposition 4.1.3. *The structure $\Pi_t^\star\Omega^t$ does not depend on the particular t chosen if evaluated on γ belonging to the solution space $\mathcal{E}\mathcal{L}$.*

Proof. Let us fix the extremum a of $\mathbb{I}$ and consider all the possible subintervals of $\mathbb{I}$ of the type $[a, \, t]$. Then, a straightforward application of the previous two propositions gives:

$$
\Pi_a^\star\Omega^a{}_\gamma = \Pi_t^\star\Omega^t{}_\gamma \,, \quad \forall \ t \in \mathbb{I} \tag{4.21}
$$

if $\gamma \in \tilde{\mathcal{EL}}$, where $\tilde{\mathcal{EL}}$ is the solution space of the variational problem formulated on $[a,\, t]$. In other words, the structure $\Pi_t^\star {\Omega^t}_\gamma$ is constant along $\mathbb{I}$ if evaluated on solutions of the equations of the motion, and, therefore, we will refer to it as the CANONICAL STRUCTURE on the solution space of the equations of the motion. $\square$

4.1.2. Hamiltonian Mechanics

The construction of the canonical structure on the solution space within the Hamiltonian formalism goes along the lines of the previous section.

In this case the differential 1-form defined on the space $\mathcal{F}_{\mathcal{P}(\mathbb{Q})}$ appearing in the first fundamental formula reads:

$$i_{\mathbb{X}_\xi} \Pi_{\partial\mathbb{I}}^\star {\alpha^{\partial\mathbb{I}}}_\xi \;=\; \Pi_{\partial\mathbb{I}}^\star {\alpha^{\partial\mathbb{I}}}_\xi(\mathbb{X}_\xi) \;=\; \int_{\partial\mathbb{I}} \mathrm{i}_{\partial\mathbb{I}}^\star\, \xi^\star i_X \Theta_H \quad \forall\, \mathbb{X}_\xi \in \mathbf{T}_\xi \mathcal{F}_{\mathcal{P}(\mathbb{Q})}\,, \tag{4.22}$$

where X is any extension of $\mathbb{X}_\xi$ to an open neighborhood of the image of ξ into $\mathcal{P}(\mathbb{Q})$. In terms of it and of the Euler-Lagrange form, the first fundamental formula reads:

$$\mathrm{d}\mathscr{S}_\xi \;=\; \mathbb{EL}_\xi + \Pi_{\partial\mathbb{I}}^\star {\alpha^{\partial\mathbb{I}}}_\xi\,. \tag{4.23}$$

Starting from the latter differential 1-form, it is possible to define a 2-form on $\mathcal{F}_{\mathcal{P}(\mathbb{Q})}$ for any $t \in \mathbb{I}$ analogously to how it was done in the previous section:

$$\Pi_t^\star {\Omega^t}_\xi(\mathbb{X}_\xi,\, \mathbb{Y}_\xi) \;=\; \xi^\star \left[\, i_Y\, i_X\, \mathrm{d}\Theta_H \,\right]\Big|_t \tag{4.24}$$

which, in the system of local coordinates chosen on $\mathcal{P}(\mathbb{Q})$, reads:

$$\Pi_t^\star {\Omega^t}_\xi \;=\; \xi^\star \left[\mathrm{d}p_j \wedge \mathrm{d}q^j\right]\Big|_t\,, \tag{4.25}$$

and it is possible to prove that, if evaluated on $\xi \in \mathcal{EL}$, it does not depend on the particular $t \in \mathbb{I}$ chosen. In particular, first, it is possible to prove the analogous of Prop. 4.1.1.

Proposition 4.1.4. *$\mathcal{EL}$ is an isotropic manifold for the differential 2-form $\Pi_{\partial\mathbb{I}}^\star \Omega^{\partial\mathbb{I}}$.*

Then, with the obvious modifications, it is possible to prove the analogue of Prop. 4.1.2.

Proposition 4.1.5. *Consider a (connected) subinterval of $\mathbb{I}$, say $\tilde{\mathbb{I}}$. Denote by $\tilde{\mathcal{EL}}$ the solution space related with the variational principle formulated on $\tilde{\mathbb{I}}$ instead of $\mathbb{I}$. Then, the restrictions of the elements in $\mathcal{EL}$ to $\tilde{\mathbb{I}}$ belong to $\tilde{\mathcal{EL}}$.*

Proof. The proof is analogue to that of Prop. 4.1.2 with the obvious modification. In this case the analogue of diagram (4.18) is:

$$\begin{tikzcd} \tilde{\mathcal{P}(\mathbb{Q})} \arrow[r, hook, "\tilde{\mathrm{i}}_{\mathcal{P}}"] & \mathcal{P}(\mathbb{Q}) \\ \tilde{\mathbb{I}} \arrow[u, dashed, "\tilde{\xi}"'] \arrow[r, hook, "\tilde{\mathrm{i}}"] & \mathbb{I} \arrow[u, dashed, "\xi"'] \end{tikzcd} \tag{4.26}$$

where $\tilde{\mathfrak{i}}_{\mathcal{P}}$ is:

$$\tilde{\mathfrak{i}}_{\mathcal{P}} \;:\; \mathcal{P}(\tilde{\mathbb{Q}}) \simeq \tilde{\mathbb{I}} \times \mathbf{T}^\star \mathcal{Q} \to \mathcal{P}(\mathbb{Q}) \simeq \mathbb{I} \times \mathbf{T}^\star \mathcal{Q} \;:\; \tilde{\mathfrak{i}}_{\mathcal{P}} \,=\, \tilde{\mathfrak{i}} \times \mathbb{1}_{\mathbf{T}^\star \mathcal{Q}} \,. \tag{4.27}$$

In this case, the variational principle formulated on $\mathcal{P}(\tilde{\mathbb{Q}})$ gives the following equations of motion:

$$\tilde{\xi}^\star \left(i_{\tilde{X}}\, \tilde{\mathfrak{i}}^\star_{\mathcal{P}} \Omega_{\mathrm{H}} \right) \,=\, 0 \quad \forall\, \tilde{X} \in \mathfrak{X}^{\tilde{\delta_1}}(U^{(\tilde{\xi})})\,, \tag{4.28}$$

where $\mathfrak{X}^{\tilde{\delta_1}}(U^{(\tilde{\gamma})})$ denotes the module of vertical (with respect to $\tilde{\delta_1}$) vector fields defined on an open neighborhood of the image of $\tilde{\xi}$ in $\mathcal{P}(\tilde{\mathbb{Q}})$. The chain of equalities (4.20) in this case reads:

$$\tilde{\xi}^\star \left(i_{\tilde{X}}\, \tilde{\mathfrak{i}}^\star_{\mathcal{P}} \Omega_{\mathrm{H}} \right) \,=\, \tilde{\xi}^\star \left(\tilde{\mathfrak{i}}^\star_{\mathcal{P}}\, i_X \Omega_{\mathrm{H}} \right) \,=\, \left(\tilde{\mathfrak{i}}_{\mathcal{P}} \circ \tilde{\xi} \right)^\star (\, i_X \Omega_{\mathrm{H}} \,) \,=\, \left(\xi \circ \tilde{\mathfrak{i}} \right)^\star (\, i_X \Omega_{\mathrm{H}} \,) \,=\, \tilde{\mathfrak{i}}^\star \left[\xi^\star (\, i_X \Omega_{\mathrm{H}} \,) \right] , \tag{4.29}$$

for some X which is $\tilde{\mathfrak{i}}_{\mathcal{P}}$-related with $\tilde{X}$ and where ξ is one of the sections of δ_1 that restrict to $\tilde{\xi}$. Again, the sequence of equalities above shows that if ξ is a solution for the variational problem on $\mathbb{I}$, i. e., if $\xi^\star i_{X^1} \mathrm{d}\Theta_H \,=\, 0$, then its restriction $\tilde{\xi}$ is a solution of the variational principle on $\tilde{\mathbb{I}}$. □

Also in this case, a straightforward application of Prop. 4.1.4 and 4.1.5 allows to prove the independence of $\Pi_t^\star \Omega^t$ on the particular t chosen if evaluated on elements of the solution space.

Proposition 4.1.6. *The structure $\Pi_t^\star \Omega^t$ does not depend on the particular t chosen if evaluated on ξ belonging to the solution space $\mathcal{EL}$.*

4.1.3. Hamiltonian systems with constraints

Within Hamiltonian mechanical systems with additional constrains of the type introduced in Sec. 3.1.3, the canonical structure has the same expression as for Hamiltonian theories without additional constraints. Indeed, as we saw in Sec. 3.1.3, extrema of $\mathscr{S}$ restricted to some $\Xi \,=\, \Phi(\mathcal{N}) \,\subset\, \mathcal{F}_{\mathcal{P}(\mathbb{Q})}$ are in one-to-one correspondence with extrema of $\mathscr{S}^{\mathrm{ext}} \,=\, \mathscr{S} + \langle \Lambda,\, \xi - \Phi(n) \rangle$. What is more, by looking at Eq. (3.98) we see that the additional term appearing in the variation of $\mathscr{S}^{\mathrm{ext}}$ reads:

$$\langle X_\Lambda,\, m - \Phi(n) \rangle + \langle \Lambda,\, X_m - \Phi_\star X_n \rangle \,, \tag{4.30}$$

which is not a boundary term and, thus, does not contribute to the 1-form $\Pi_t^\star \alpha^t$. This means that the canonical structure $\Pi_t^\star \omega$ associated to $\mathscr{S}^{\mathrm{ext}}$ has the same expression as the one associated with $\mathscr{S}$ even if it is defined on the enlarged space of fields $\mathcal{F}_{\mathcal{P}(\mathbb{Q})} \times \mathcal{F}^\star_{\mathcal{P}(\mathbb{Q})} \times \mathcal{N}$.

4.1.4. First order Lagrangian field theories

Most of the results of section 4.1.1 can be straightforwardly generalized to the realm of first order Lagrangian field theories and we will omit the proof for them, while we focus our attention to aspects that are peculiar of field theories.

The first obvious generalization is that, assuming $\partial\mathscr{M}$ to be the disjoint union of k connected codimension-1 hypersurfaces, say $\Sigma_1, ..., \Sigma_k$, the analogue of the structure $\Pi^\star_{\partial\mathbb{I}}\alpha^{\partial\mathbb{I}}$ on $\mathcal{F}_{\mathbb{Q}}$, that we denote by $\Pi^\star_{\partial\mathscr{M}}\alpha^{\partial\mathscr{M}}$, is the structure on $\mathcal{F}_{\mathbb{E}}$ given by:

$$\Pi^\star_{\partial\mathscr{M}}\alpha^{\partial\mathscr{M}}_\phi\,(\,\mathbb{X}_\phi\,) \,=\, \sum_{j=1}^{k}\,\mathfrak{o}_j\,\Pi^\star_{\Sigma_j}\alpha^{\Sigma_j}_\phi\,(\,\mathbb{X}_\phi\,)\,, \quad \forall\,\mathbb{X}_\phi \in \mathbf{T}_\phi\mathcal{F}_{\mathbb{E}}\,, \tag{4.31}$$

where $\mathfrak{o}_j$ is the verse chosen for the orientation inherited by Σ_j, $\Pi^\star_{\Sigma_j}\alpha^{\Sigma_j}_\phi\,(\,\mathbb{X}_\phi\,)$ is:

$$\Pi^\star_{\Sigma_j}\alpha^{\Sigma_j}_\phi\,(\,\mathbb{X}_\phi\,) \,=\, \int_{\Sigma_j} \mathfrak{i}^\star_{\partial\mathscr{M}}\left(j^1\phi\right)^\star [\,i_{X^1}\Theta_{\mathscr{L}}\,]\,, \tag{4.32}$$

and X^1 is the first order jet prolongation of any extension of $\mathbb{X}_\phi$ to an open neighborhood of the image of ϕ in $\mathbb{E}$. In a system of local coordinates in which Σ_j is the hypersurface given by $x^0 \,=\, x^0_{\Sigma_j} \,=\, const$, and using the notation introduced in Sec. 2.1.4 it reads:

$$\Pi^\star_{\Sigma_j}\alpha^{\Sigma_j}_\phi \,=\, \left(j^1\phi\right)^\star \frac{\partial\mathscr{L}}{\partial z^a_0}\delta\phi^a\bigg|_{\Sigma_j} \,=\, \frac{\delta\mathscr{L}}{\delta\dot{\varphi}^a}\,(\varphi^a,\,\dot{\varphi}^a,\,\partial_k\varphi^a)\,\delta\varphi^a\,, \tag{4.33}$$

where φ^a denotes the restriction of ϕ^a to Σ_j, $\dot{\varphi}^a$ denotes $\partial_0\phi^a$ restricted to Σ_j and $\partial_k\varphi^a \,=\, \frac{\partial\varphi^a}{\partial x^j}$. This is a straightforward generalization of (4.5). The differential of the form above gives the following differential 2-form on $\mathcal{F}_{\mathbb{E}}$:

$$\Pi^\star_{\partial\mathscr{M}}\Omega^{\partial\mathscr{M}}_\phi\,(\,\mathbb{X}_\phi,\,\mathbb{Y}_\phi\,) \,=\, -\int_{\partial\mathscr{M}} \mathfrak{i}^\star_{\partial\mathscr{M}}\left(j^1\phi\right)^\star [\,i_{Y^1}\,i_{X^1}\,\mathrm{d}\Theta_{\mathscr{L}}\,]\,, \tag{4.34}$$

that can be equivalently defined for any 1-codimension hypersurface of $\mathscr{M}$, say Σ:

$$\Pi^\star_{\Sigma}\Omega^{\Sigma}_\phi\,(\,\mathbb{X}_\phi,\,\mathbb{Y}_\phi\,) \,=\, -\int_{\Sigma} \mathfrak{i}^\star_{\Sigma}\left(j^1\phi\right)^\star [\,i_{Y^1}\,i_{X^1}\,\mathrm{d}\Theta_{\mathscr{L}}\,]\,, \tag{4.35}$$

where $\mathfrak{i}_\Sigma$ denotes the immersion of Σ into $\mathscr{M}$. In particular, from now on, we are going to assume that Σ is a SLICE of the space-time $\mathscr{M}$, that is, that it splits the space-time into two regions, say $\mathscr{M}^+$ and $\mathscr{M}^-$ being space-times themselves. In a system of local coordinates for which Σ is the hypersurface given by $x^0 \,=\, x^0_\Sigma \,=\, const$ and using the notation introduced in Sec. 2.1.4, the two form $\Pi^\star_\Sigma\Omega^\Sigma_\phi$ reads:

$$\Pi^\star_\Sigma\Omega^\Sigma_\phi \,=\, \frac{\delta^2\mathscr{L}}{\delta\varphi^b\delta\dot{\varphi}^a}\delta\varphi^a\wedge\delta\varphi^b + \frac{\delta^2\mathscr{L}}{\delta\dot{\varphi}^b\delta\dot{\varphi}^a}\delta\varphi^a\wedge\delta\dot{\varphi}^b + \frac{\delta^2\mathscr{L}}{\delta\partial_j\varphi^b\delta\dot{\varphi}^a}\delta\varphi^a\wedge\partial_j\delta\varphi^b\,. \tag{4.36}$$

Provided $\mathcal{EL}$ is a smooth immersed Banach submanifold of $\mathcal{F}(\mathbb{E})$, as we assumed, Proposition 4.1.1, being a direct consequence of the first fundamental formula, can be generalized as follows.

Proposition 4.1.7. *$\mathcal{E}\mathscr{L}_{\mathscr{M}}$ is an isotropic manifold for the differential 2-form $\Pi^{\star}_{\partial\mathscr{M}}\Omega^{\partial\mathscr{M}}$.*

On the other hand, proposition 4.1.2 can be generalized to the following proposition.

Proposition 4.1.8. *Consider a closed (connected) submanifold of $\mathscr{M}$, say $\tilde{\mathscr{M}}$, having the same dimension of $\mathscr{M}$. Denote by $\mathcal{E}\tilde{\mathscr{L}}$ the solution space related with the variational principle formulated on $\tilde{\mathscr{M}}$ instead of $\mathscr{M}$. Then the restrictions of elements in $\mathcal{E}\mathscr{L}$ to $\tilde{\mathscr{M}}$ belong to $\mathcal{E}\tilde{\mathscr{L}}$.*

And, again, the combination of these two results allows for proving the following proposition.

Proposition 4.1.9. *The structure $\Pi^{\star}_{\Sigma}\Omega^{\Sigma}$ only depends on the homology classe of the slice Σ chosen, if evaluated on ϕ belonging to the solution space $\mathcal{E}\mathscr{L}$.*

Proof. Consider two arbitrary slices, Σ_1 and Σ_2 such that they select a region of the space-time, say $\mathscr{M}_{12}$, being a submanifold of the type considered in Prop. 4.1.8 and whose boundary is made of the two slices considered, both carrying the orientation pointing outside the region. Then (4.31) and Prop. 4.1.7 and 4.1.8 gives:

$$\Pi^{\star}_{\Sigma_1}\Omega^{\Sigma^1}_{\phi} = \Pi^{\star}_{\Sigma_2}\Omega^{\Sigma^2}_{\phi} , \tag{4.37}$$

if $\phi \in \mathcal{E}\tilde{\mathscr{L}}$, i.e., the solution space associated with the variational principle formulated on $\mathscr{M}_{12}$. Note that any slice defines a class of slices made by all those slices whose union with Σ is the boundary of a region of $\mathscr{M}$ of the type of Prop. 4.1.8. Thus, from Eq. (4.37) we conclude that $\Pi^{\star}_{\Sigma}\Omega^{\Sigma}$ is the same on any slice Σ belonging to the same class.

The thesis follows from the observation that given any couple of slices belonging to different classes, there exists another one belonging to both the previous classes. □

4.1.5. First order Hamiltonian field theories

The content of the previous section can be adapted to the setting of first order Hamiltonian field theories straightforwardly.

The analogue of the structure appearing in (4.31) is the 1-form on $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$ given by:

$$\Pi^{\star}_{\partial\mathscr{M}}\alpha^{\partial\mathscr{M}}_{\chi}\left(\mathbb{X}_{\chi}\right) = \sum_{j=1}^{k} \mathfrak{o}_j\, \Pi^{\star}_{\Sigma_j}\alpha^{\Sigma_j}_{\chi}\left(\mathbb{X}_{\chi}\right) , \quad \forall\, \mathbb{X}_{\chi} \in \mathbf{T}_{\phi}\mathcal{F}_{\mathcal{P}(\mathbb{E})} , \tag{4.38}$$

where $\partial\mathscr{M}$ is the disjoint union of $\Sigma_1, ..., \Sigma_k$, $\mathfrak{o}_j$ is the orientation chosen on Σ_j, $\Pi^{\star}_{\Sigma_j}\alpha^{\Sigma_j}_{\chi}\left(\mathbb{X}_{\chi}\right)$ is:

$$\Pi^{\star}_{\Sigma_j}\alpha^{\Sigma_j}_{\chi}\left(\mathbb{X}_{\chi}\right) = \int_{\Sigma_j} \mathfrak{i}^{\star}_{\partial\mathscr{M}}\chi^{\star}\left[\, i_X\Theta_H \,\right] , \tag{4.39}$$

and X is any extension of $\mathbb{X}_\chi$ to an open neighborhood of the image of χ in $\mathcal{P}(\mathbb{E})$. In a system of local coordinates in which Σ_j is the hypersurface given by $x^0 = x^0_{\Sigma_j} = const$, and using the notation introduced in Sec. 2.1.4 it reads:

$$\Pi^\star_{\Sigma_j} \alpha^{\Sigma_j}_\chi = \chi^\star \left(\rho^0_a \right) \delta\phi^a \Big|_{\Sigma_j} = p_a \delta\varphi^a \,, \tag{4.40}$$

where φ^a and p_a denote the restrictions of ϕ^a and P^0_a to Σ_0 respectively.

The differential of the form above gives the differential 2-form on $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$:

$$\Pi^\star_{\partial\mathscr{M}} \Omega^{\partial\mathscr{M}}_\chi \left(\mathbb{X}_\chi, \mathbb{Y}_\chi \right) = - \int_{\partial\mathscr{M}} \mathfrak{i}^\star_{\partial\mathscr{M}} \chi^\star \left[i_Y \, i_X \, \mathrm{d}\Theta_H \right] \,, \tag{4.41}$$

that can be equivalently defined for any slice Σ of $\mathscr{M}$:

$$\Pi^\star_\Sigma \Omega^\Sigma_\chi \left(\mathbb{X}_\chi, \mathbb{Y}_\chi \right) = - \int_\Sigma \mathfrak{i}^\star_\Sigma \chi^\star \left[i_Y \, i_X \, \mathrm{d}\Theta_H \right] \,, \tag{4.42}$$

where $\mathfrak{i}_\Sigma$ denotes the immersion of Σ into $\mathscr{M}$. In a system of local coordinates for which Σ is the hypersurface given by $x^0 = x^0_\Sigma = const$ and using the notation introduced in Sec. 2.1.4, the two form $\Pi^\star_\Sigma \Omega^\Sigma_\chi$ reads:

$$\Pi^\star_\Sigma \Omega^\Sigma_\chi = \delta\varphi^a \wedge \delta p_a \,. \tag{4.43}$$

Propositions 4.1.7, 4.1.8 and 4.1.9 can be straightforwardly adapted to the following propositions with analogous proofs.

Proposition 4.1.10. *$\mathcal{EL}_\mathscr{M}$ is an isotropic manifold for the differential 2-form $\Pi^\star_{\partial\mathscr{M}} \Omega^{\partial\mathscr{M}}$.*

Proposition 4.1.11. *Consider a (connected) submanifold of $\mathscr{M}$, say $\tilde{\mathscr{M}}$, open into $\mathscr{M}$ and having the same dimension of $\mathscr{M}$. Denote by $\tilde{\mathcal{EL}}$ the solution space related with the variational principle formulated on $\tilde{\mathscr{M}}$ instead of $\mathscr{M}$. Then the restrictions of elements in $\mathcal{EL}$ to $\tilde{\mathscr{M}}$ belong to $\tilde{\mathcal{EL}}$.*

Proposition 4.1.12. *The structure $\Pi^\star_\Sigma \Omega^\Sigma$ does not depend on the particular slice Σ chosen if evaluated on χ belonging to the solution space $\mathcal{EL}$.*

4.1.6. First order Hamiltonian field theories with constraints

Within first order Hamiltonian field theories with additional constrains of the type introduced in Sec. 3.2.3, the canonical structure has the same expression as for Hamiltonian theories without additional constraints. Indeed, as we saw in Sec. 3.2.3, extrema of $\mathscr{S}$ restricted to some $\Xi = \Phi(\mathcal{N}) \subset \mathcal{F}_{\mathcal{P}(\mathbb{E})}$ are in one-to-one correspondence with extrema of $\mathscr{S}^{\mathrm{ext}} = \mathscr{S} + \langle \Lambda, \, \chi - \Phi(n) \rangle$. What is more, by looking at Eq. (3.98) we see that the additional term appearing in the variation of $\mathscr{S}^{\mathrm{ext}}$ reads:

$$\langle X_\Lambda, \, m - \Phi(n) \rangle + \langle \Lambda, \, X_m - \Phi_\star X_n \rangle \,, \tag{4.44}$$

which is not a boundary term and, thus, does not contribute to the 1-form $\Pi^\star_\Sigma \alpha^\Sigma$. This means that the canonical structure $\Pi^\star_\Sigma \Omega^\Sigma$ associated to $\mathscr{S}^{\mathrm{ext}}$ has the same expression as the one associated with $\mathscr{S}$ even if it is defined on the enlarged space of fields $\mathcal{F}_{\mathcal{P}(\mathbb{E})} \times \mathcal{F}^\star_{\mathcal{P}(\mathbb{E})} \times \mathcal{N}$.

4.2. Pre-symplectic formalism near a slice and the pre-symplectic constraint algorithm

In this section we show how the canonical structure on the solution space constructed in the previous section can be used to model, for any slice of the space-time of the theory[35], the solution space as the space of solutions of a pre-symplectic Hamiltonian system. As usual we will perform the construction both within mechanical systems and within field theories in the Lagrangian formalism as well as in the Hamiltonian and Hamiltonian+constraints ones. We also end this section by discussing, in Sec. 4.2.7, how this formulation in terms of pre-symplectic Hamiltonian systems can be consistently used to deal with the correspondence between symmetries and conserved quantities within field theories.

4.2.1. Lagrangian Mechanics

As we said in Rem. 3.1.6, within Lagrangian Mechanics we always consider autonomous dynamical systems for which the dynamics is described as a (generally pre-symplectic) Hamiltonian system $(\mathbf{T}\mathcal{Q},\, \omega_{\mathscr{L}},\, E_{\mathscr{L}})$. It is worth stressing that $\mathbf{T}\mathcal{Q}$ coincides with the space of jets of sections[36] of π restricted to any $t \in \mathbb{I}$, say $\mathcal{F}_{\mathbb{Q}}^t$ and $\omega_{\mathscr{L}}$ coincides with the structure Ω^t from which the canonical structure $\Pi_t^\star \Omega^t$ comes from. Indeed, $\mathcal{F}_{\mathbb{Q}}^t$ is a space of maps from the space $\{\, t \,\}$, made by a single point, to $\mathbf{T}\mathcal{Q}$ and, thus, it coincides with the space of values, namely, $\mathbf{T}\mathcal{Q}$. On the other hand, the canonical structure $\Pi_t^\star \Omega^t$ reads:

$$\Pi_t^\star {\Omega^t}_\gamma(\mathbb{X}_\gamma,\, \mathbb{Y}_\gamma) \,=\, \Big(j^1\gamma\Big)^\star \left[\, i_{X^1} i_{Y^1} \mathrm{d}\Theta_{\mathscr{L}} \,\right]\Big|_t \,, \tag{4.45}$$

which, recalling that:

$$\mathrm{d}\Theta_{\mathscr{L}} \,=\, \omega_{\mathscr{L}} + \mathrm{d}E_{\mathscr{L}} \wedge \mathrm{d}t \,, \tag{4.46}$$

and taking into account that $\mathbb{X}_\gamma$ and $\mathbb{Y}_\gamma$ are vertical with respect to π_1, gives:

$$\Pi_t^\star {\Omega^t}_\gamma(\mathbb{X}_\gamma,\, \mathbb{Y}_\gamma) \,=\, \Big(j^1\gamma\Big)^\star \left[\, i_{X^1} i_{Y^1} \omega_{\mathscr{L}} \,\right]\Big|_t \,=\, \omega_{\mathscr{L}}(X^1,\, Y^1)\Big|_{j^1\gamma(t)} \,, \tag{4.47}$$

which says that the canonical structure $\Pi_t^\star \Omega^t$ is the pull-back to $\mathcal{F}_{\mathbb{Q}}$ of the structure $\omega_{\mathscr{L}}$ on $\mathbf{T}\mathcal{Q}$ and, thus, that $\Omega^t \,=\, \omega_{\mathscr{L}} \,, \quad \forall\, t \in \mathbb{I}$.

Now, being $\omega_{\mathscr{L}}$ pre-symplectic in general, one should apply the pre-symplectic constraint algorithm described in Sec. 2.3.2 in order to obtain well defined dynamical equations on the so-called stable manifold of the algorithm, denoted by $\mathcal{M}_\infty$. They take the form:

$$i_{\Gamma_\infty} \omega_{\mathscr{L}\infty} \,=\, \mathrm{d}E_{\mathscr{L}\infty} \,, \tag{4.48}$$

[35] That within mechanical systems seen as field theories over a $0+1$-dimensional space-time, amounts to a single point on the time interval.

[36] Whose structure of smooth Banach manifold is assumed to be obtained as described in the beginning of Sec. 4.1.1

where $\omega_{\mathscr{L}\infty}$ and $E_{\mathscr{L}\infty}$ are the pull-back of $\omega_{\mathscr{L}}$ and $E_{\mathscr{L}}$ respectively to $\mathcal{M}_\infty$ which is assumed to be a smooth immersed submanifold of $\mathbf{T}\mathcal{Q}$, the immersion map denoted by $\mathfrak{i}_\infty$. As a matter of fact $\omega_{\mathscr{L}\infty}$ may be symplectic or pre-symplectic, depending on the particular Lagrangian considered.

Let us consider for a moment the former case, namely, that $\omega_{\mathscr{L}\infty}$ is non-degenerate. Then, for any smooth function on $\mathcal{M}_\infty$ the Hamiltonian vector field is uniquely defined and, in particular, this is true for $E_{\mathscr{L}\infty}$. Consequently Eq. (4.48) defines a unique smooth vector field Γ_∞ whose integral curves, immersed into $\mathbf{T}\mathcal{Q}$ via $\mathfrak{i}_\infty$, are elements of $\mathcal{F}_{\mathbb{Q}}$ being solutions of the Hamiltonian system. Being Γ_∞ a smooth vector field, to each point $m_\infty \in \mathcal{M}_\infty$ it is associated a unique curve on $\mathcal{M}_\infty$, the integral curve, γ_∞, of Γ_∞ passing through m_∞, which is obtained by "evolving" m_∞ via the flow of Γ_∞:

$$\gamma_\infty(s) \;=\; F_s^{\Gamma_\infty} \cdot m_\infty \,. \tag{4.49}$$

Furthermore, to such a curve γ_∞, it is associated a unique solution of the Hamiltonian system, say $\gamma \in \mathcal{E}\mathscr{L}$:

$$\gamma(s) \;=\; \mathfrak{i}_\infty \left[F_s^{\Gamma_\infty} \cdot m_\infty \right] \;:=\; \Psi \cdot m_\infty \,, \tag{4.50}$$

and the correspondence between $m_\infty \in \mathcal{M}$ and $\gamma \in \mathcal{E}\mathscr{L}$ is one-to-one. Indeed, Ψ is injective since $F_s^{\Gamma_\infty}$ is a diffeomorphism and $\mathfrak{i}_\infty$ is a smooth immersion. Moreover, Ψ is surjective onto $\mathcal{E}\mathscr{L}$ since by definition $\mathcal{E}\mathscr{L}$ is made by those elements of $\mathcal{F}_{\mathbb{Q}}$ which are in the image of $\mathfrak{i}_\infty$. Therefore, Ψ is a bijection between $\mathcal{M}_\infty$ and $\mathcal{E}\mathscr{L}$ (and, being smooth, can be used to induce a smooth manifold structure on $\mathcal{E}\mathscr{L}$). For this reason the manifold $\mathcal{M}_\infty$ can be considered that space of Cauchy data for the equations of motion.

The situation may be resumed via the following diagram:

$$\begin{tikzcd}
\mathcal{E}\mathscr{L} \arrow[d, "\Pi_t"] \arrow[dd, bend right=50, "\Psi^{-1}"'] & \Pi_t^\star\omega_{\mathscr{L}} \arrow[l, squiggly] \\
\mathbf{T}\mathcal{Q} & \omega_{\mathscr{L}} \arrow[l, squiggly] \\
\mathcal{M}_\infty \arrow[u, "\mathfrak{i}_\infty"'] \arrow[uu, bend left=70, "\Psi"] & \omega_{\mathscr{L}\infty} \arrow[l, squiggly]
\end{tikzcd} \tag{4.51}$$

from which, noting that $\Pi_t \;=\; \mathfrak{i}_\infty \circ \Psi^{-1}$, it is also clear the relation between $\Pi_t^\star\omega_{\mathscr{L}}$ and $\omega_{\mathscr{L}\infty}$:

$$\Pi_t^\star\omega_{\mathscr{L}} \;=\; \left(\mathfrak{i}_\infty \circ \Psi^{-1}\right)^\star \omega_{\mathscr{L}} \;=\; \Psi^{-1\star}\mathfrak{i}_\infty^\star\omega_{\mathscr{L}} \;=\; \Psi^{-1\star}\omega_{\mathscr{L}\infty} \,. \tag{4.52}$$

The discussion above can be resumed by saying that when the structure $\omega_{\mathscr{L}\infty}$ emerging from the pre-symplectic constraint algorithm applied to the pre-symplectic Hamiltonian system describing our Lagrangian dynamical system is symplectic, then the solution space $\mathcal{E}\mathscr{L}$ is diffeomorphic to the stable manifold emerging from the algorithm and, thus, it is a symplectic manifold. Being the two manifolds diffeomorphic, all the geometric structures on the former are related via a diffeomorphism to those on the latter, and working on $\mathcal{EL}$ or on $\mathcal{M}_\infty$ is completely equivalent. The convenience in working directly on $\mathcal{M}_\infty$ rather than on $\mathcal{EL}$ is that the latter solution space is determined by solving the equations of motion whereas the former is determined via an algorithmic geometrical procedure that does not require one to actually solve the equations of motion. For this reason from now on, we will often refer directly to $\mathcal{M}_\infty$ as the solution space of our system.

On the other hand, when $\omega_{\mathscr{L}}$ is pre-symplectic (and with constant rank, as we will assume from now on), the vector field Γ_∞ is determined up to elements in the kernel of $\omega_{\mathscr{L}}$. This means that, for any point $m_\infty \in \mathcal{M}_\infty$, the curve γ_∞ solution of (4.48) is not unique, but rather is of the type:

$$\gamma_\infty(s) \, = \, F_s^{\Gamma_\infty} \cdot m_\infty \,, \tag{4.53}$$

modulo the action of F^W, for any W belonging, at each point, to the kernel of $\omega_{\mathscr{L}\infty}$. Thus, in this case, solutions of (4.48) passing through m_∞ are parametrized by the elements W in the kernel of $\omega_{\mathscr{L}}$. Solutions associated to different W are said to be GAUGE EQUIVALENT and are indistinguishable from the physical point of view. Therefore, the usual approach of theoretical physicists is to collect all the them into equivalence classes and to refer to the space of equivalence classes as the solution space. Instead, also in this case we will refer to the whole $\mathcal{M}_\infty$ as the solution space, since it contains all the solutions of the equations of motion, even if some of them are gauge equivalent, and this will allow to classify theories being gauge theories or not as those theories for which $\mathcal{M}_\infty$ is pre-symplectic or symplectic.

Example 4.2.1 (FREE PARTICLE ON THE LINE)**.** *In example 3.1.7 we already saw that the solutions of the equations of motion of a free particle moving on a line ($\mathcal{Q} = \mathbb{R}$) coincide with the integral curves of the vector field Γ satisfying:*

$$i_\Gamma \omega_{\mathscr{L}} \, = \, \mathrm{d}E_{\mathscr{L}} \,, \tag{4.54}$$

where:

$$\omega_{\mathscr{L}} \, = \, -\mathrm{d}\mathrm{d}_S \mathscr{L} \,, \tag{4.55}$$

with:

$$\mathscr{L} \, = \, \frac{1}{2} m v^2 \,, \tag{4.56}$$

and:

$$E_{\mathscr{L}} \, = \, \frac{1}{2} m v^2 \,. \tag{4.57}$$

This gives an explicit example of what we said in the present section, namely, that the dynamics of a free particle moving on a line can be formulated in terms of the (symplectic, in this case) Hamiltonian system ($\mathbf{T}\mathbb{R}$, $\omega_{\mathscr{L}}$, $E_{\mathscr{L}}$). Since in this case the 2-form:

$$\omega_{\mathscr{L}} \, = \, m \, \mathrm{d}q \wedge \mathrm{d}v \,, \tag{4.58}$$

is symplectic, there is no need of performing the pre-symplectic constraint algorithm and the space of Cauchy data $\mathcal{M}_\infty$ coincide with $\mathbf{T}\mathbb{R}$ itself. Moreover $\mathcal{M}_\infty = \mathbf{T}\mathbb{R}$ is diffeomorphic to the solution space $\mathscr{E}\mathscr{L}$ as it can be seen by explicitly writing the solutions of the equations of motion:

$$\gamma(t) \,=\, q + v(t - t_0)\,, \tag{4.59}$$

which shows how, for any point $(q,\, v) \in \mathbf{T}\mathbb{R} \,=\, \mathcal{M}_\infty$ there exists a unique solution $\gamma(t) \in \mathscr{E}\mathscr{L}$ and vice-versa.

4.2.2. Hamiltonian Mechanics

Within Hamiltonian Mechanics, as it was stressed in Rem. 3.1.13, we considered dynamical systems for which the dynamics is modelled as the Hamiltonian system $(\mathbf{T}^\star\mathcal{Q},\, \omega,\, H)$, where, again $\mathbf{T}^\star\mathcal{Q}$ coincides with the space of elements of $\mathcal{F}_{\mathcal{P}(\mathbb{Q})}$ restricted to $\{\, t \,\} \subset \mathbb{I}$, say $\mathcal{F}^t_{\mathcal{P}(\mathbb{Q})}$, the canonical structure $\Pi_t^\star\Omega^t$ is:

$$\Pi_t^\star {\Omega^t}_\xi(\mathbb{X}_\xi,\, \mathbb{Y}_\xi) \,=\, \omega(X,\, Y)\Big|_{\xi(t)}\,, \tag{4.60}$$

that is, $\Omega^t \,=\, \omega\,, \quad \forall\; t \in \mathbb{I}$, and H is the function on $\mathbf{T}^\star\mathcal{Q}$ that in Remark 3.1.13 we denoted by $\bar{H}$. However, in this case, since ω is the canonical symplectic structure of $\mathbf{T}^\star\mathcal{Q}$, there is no need to apply the pre-symplectic constraint algorithm and the final stable manifold, i.e., the space of Cauchy data $\mathcal{M}_\infty$, coincides with $\mathbf{T}^\star\mathcal{Q}$ itself. The situation is resumed in the following diagram:

$$\begin{tikzcd}
\mathscr{E}\mathscr{L} \arrow[d, "\Pi_t"] \arrow[dd, bend right=60, "\Psi^{-1}"'] & \Pi_t^\star\omega \arrow[l, rightsquigarrow] \\
\mathbf{T}^\star\mathcal{Q} \arrow[d, equal] & \omega \arrow[l, rightsquigarrow] \arrow[d, equal] \\
\mathcal{M}_\infty \arrow[uu, bend left=80, "\Psi"] & \omega \arrow[l, rightsquigarrow]
\end{tikzcd} \tag{4.61}$$

where, now, Ψ is simply the flow of the Hamiltonian vector field Γ_H satisfying:

$$i_{\Gamma_H}\omega \,=\, \mathrm{d}H\,, \tag{4.62}$$

i.e. $\Psi \,=\, F_s^{\Gamma_H}$ and $\Psi^{-1} \,=\, \Pi_t$. In this case Eq. (4.52) reads:

$$\Pi_t^\star\Omega^t \,=\, {\Psi^{-1}}^\star\omega\,. \tag{4.63}$$

Thus, here we lie in the symplectic case of the previous section for which the solution space $\mathscr{E}\mathscr{L}$ is diffeomorphic to the space of Cauchy data $\mathcal{M}_\infty$ which coincides with the symplectic manifold $(\mathbf{T}^\star\mathcal{Q},\, \omega)$.

Example 4.2.2 (Free particle on the line). *In example 3.1.14 we already saw that the solutions of the equations of motion of a free particle moving on a line ($\mathcal{Q} = \mathbb{R}$), within the Hamiltonian formalism, coincide with the integral curves of the vector field Γ satisfying:*

$$i_\Gamma \omega \,=\, \mathrm{d}H\,, \tag{4.64}$$

where:

$$\omega \,=\, \mathrm{d}q \wedge \mathrm{d}p\,, \tag{4.65}$$

is the canonical symplectic structure of $\mathbf{T}^\star\mathbb{R}$ and

$$H \,=\, \frac{p^2}{2m}\,. \tag{4.66}$$

This gives an explicit example of what we said in the present section, namely, that the dynamics of a free particle moving on a line can be formulated in terms of the Hamiltonian system ($\mathbf{T}^\star\mathbb{R}$, ω, H). As said above, there is no need of performing the pre-symplectic constraint algorithm and the space of Cauchy data $\mathcal{M}_\infty$ coincide with $\mathbf{T}\mathbb{R}$ itself. Moreover $\mathcal{M}_\infty = \mathbf{T}\mathbb{R}$ is diffeomorphic to the solution space $\mathcal{EL}$ as it can be seen by explicitly writing the solutions of the equations of motion:

$$\gamma(t) \,=\, q + \frac{p}{m}(t - t_0)\,, \qquad \varrho(t) \,=\, p\,, \tag{4.67}$$

which shows how, for any point $(q,\, p) \in \mathbf{T}^\star\mathbb{R} = \mathcal{M}_\infty$ there exists a unique solution $\xi(t) \,=\, (\gamma(t),\, \varrho(t)) \in \mathcal{EL}$ and vice-versa.

4.2.3. Hamiltonian systems with constraints

Within Hamiltonian mechanical systems with additional constraints of the type considered in Sec. 3.1.3, we saw that the canonical structure on the solution space has the same expression as for theories without additional constraints.

On the other hand, as it is clear from Eq. (3.101), the additional term $\langle \Lambda,\, \chi - \Phi(n)\rangle$ in the modified action, $\mathscr{S}^{\mathrm{ext}}$, on the extended space of fields, $\mathcal{F}_{\mathcal{P}(\mathbb{Q})} \times \mathcal{F}^\star_{\mathcal{P}(\mathbb{Q})} \times \mathcal{N}$, has the net result of subtracting to the Hamiltonian H, the function $\left[\Lambda^\gamma_j\,(\gamma^j - \gamma^j \circ \Phi(n)) + \Lambda^{\varrho j}\,(\varrho_j - \varrho_j \circ \Phi(n))\right]_t$. Consequently, the pre-symplectic Hamiltonian system which is associated to the modified action, $\mathscr{S}^{\mathrm{ext}}$, is ($\mathbf{T}^\star\mathcal{Q} \times {\mathcal{F}^\star_{\mathcal{P}(\mathbb{Q})}}^\Sigma \times \mathcal{N}$, ω^{ext}, H^{ext}), where ${\mathcal{F}^\star_{\mathcal{P}(\mathbb{Q})}}^\Sigma$ is made by the restrictions to Σ of the elements in the dual of $\mathcal{F}_{\mathcal{P}(\mathbb{Q})}$, $\omega^{\mathrm{ext}} = \tau^\star\omega$, τ being the projection $\tau \;:\; \mathbf{T}^\star\mathcal{Q} \times {\mathcal{F}^\star_{\mathcal{P}(\mathbb{Q})}}^\Sigma \times \mathcal{N} \to \mathbf{T}^\star\mathcal{Q}$ and:

$$H^{\mathrm{ext}} \,=\, H - \left[\Lambda^\gamma_j\left(\gamma^j - \gamma^j \circ \Phi(n)\right) + \Lambda^{\varrho j}\,(\varrho_j - \varrho_j \circ \Phi(n))\right]_t \tag{4.68}$$

for any $t \in \mathbb{I}$. Thus, the following proposition holds.

Proposition 4.2.3. *Extrema of the functional $\mathscr{S}^{\mathrm{ext}}$ in* (3.101) *are the solutions of the pre-symplectic system* ($\mathbf{T}^\star\mathcal{Q} \times {\mathcal{F}^\star_{\mathcal{P}(\mathbb{Q})}}^\Sigma \times \mathcal{N}$, ω^{ext}, H^{ext}).

4.2.4. First order Lagrangian field theories

Within first order Lagrangian field theories, the fact that, at least locally around a slice $\Sigma \subset \mathscr{M}$, the dynamics can be described in terms of a pre-symplectic Hamiltonian system, is a consequence of the observation that, at least locally around Σ, the space of fields $\mathcal{F}_{\mathbb{E}}$ is diffeomorphic to the space of curves on the space of restrictions to Σ, say $\mathcal{F}_{\mathbb{E}}^{\Sigma}$.

More precisely, consider a slice $\Sigma \subset \mathscr{M}$. By means of the IMMERSION THEOREM [57], there always exists a system of local coordinates $\{ x^{\mu} \}_{\mu=0,\ldots,d}$ on $\mathscr{M}$ such that $\Sigma = \{ x^0 = x^0_{\Sigma} = const \}$. Sometimes we will refer to x^0 as the TRANSVERSAL COORDINATE to Σ to say that x^0 together with a system of coordinates on Σ provides a system of coordinates for the whole $\mathscr{M}$ in a neighborhood of Σ. Consider the space of fields restricted to Σ, say $\mathcal{F}_{\mathbb{E}}^{\Sigma}$:

$$\mathcal{F}_{\mathbb{E}}^{\Sigma} = \{ \phi \mid_{\Sigma} =: \varphi =: \phi_{\Sigma} \} \,, \tag{4.69}$$

that, again, we assume to be a smooth Banach manifold. Moreover, we will denote by $\dot{\varphi}$ the restriction to Σ of $\partial_0 \phi$ and by $\partial_j \varphi$ the restriction of $\partial_j \phi$ to Σ. Consider the collar $C_{\epsilon}^{\Sigma} = [x^0_{\Sigma},\, x^0_{\Sigma} + \epsilon) \times \Sigma$ close to Σ, such that the volume reads $vol_{C_{\epsilon}^{\Sigma}} = \mathrm{d}x^0 \wedge vol_{\Sigma}$. Denote by $\mathfrak{i}_{\epsilon}$ the immersion of C_{ϵ}^{Σ} into $\mathscr{M}$, by $\mathscr{S}^{\epsilon} := \mathfrak{i}_{\epsilon}^{\star} \mathscr{S}$ and by $\mathcal{F}_{\mathbb{E}}^{\epsilon}$ the space of dynamical fields restricted to C_{ϵ}^{Σ}, that, again, is assumed to be a smooth Banach manifold. $\mathcal{F}_{\mathbb{E}}^{\epsilon}$ is diffeomorphic to the space of curves on $\mathcal{F}_{\mathbb{E}}^{\Sigma}$, whose elements are denoted by $\sigma_s = \varphi_s$. The isomorphism, say ϖ, reads:

$$\varphi^a_s\left(\underline{x}\right) = \phi^a\left(x^0 = s,\, \underline{x}\right) \,, \tag{4.70}$$

with $s \in [x^0_{\Sigma},\, x^0_{\Sigma} + \epsilon)$. In other words, ϖ is the identification of the coordinate transversal to Σ with the evolution parameter describing the curve on $\mathcal{F}_{\mathbb{E}}^{\Sigma}$. Therefore, it is also continuous and differentiable together with its inverse and, thus, it can be used to induce a smooth manifold structure on $\Gamma(\mathcal{F}_{\mathbb{E}}^{\Sigma})$. Consequently, ϖ is a diffeomorphism. We will denote by $\Gamma\left(\mathcal{F}_{\mathbb{E}}^{\Sigma}\right) = \varpi\left(\mathcal{F}_{\mathbb{E}}^{\epsilon}\right)$ the space of curves $\sigma_{(\cdot)}$. With the above notations in mind, the pull-back of $\mathscr{S}$ to $\Gamma\left(\mathcal{F}_{\mathcal{P}(\mathbb{E})}^{\Sigma}\right)$ reads:

$$\left(\mathscr{S}^{\epsilon}_{\varpi}\right)_{\sigma} := \left(\varpi^{-1\star}\mathscr{S}^{\epsilon}\right)_{\sigma} = \int_{x^0_{\Sigma}}^{x^0_{\Sigma}+\epsilon} \mathrm{d}s \int_{\Sigma} \mathscr{L}(\varphi^a_s,\, \dot{\varphi}^a_s,\, \partial_j \varphi^a_s) vol_{\Sigma} =: \int_{x^0_{\Sigma}}^{x^0_{\Sigma}+\epsilon} L(\sigma_s,\, \dot{\sigma}_s)\mathrm{d}s \,, \tag{4.71}$$

where the dot denotes the derivative with respect to s and where we are interpreting the integrand as a Lagrangian function L on $\mathbf{T}(\mathcal{F}_{\mathbb{E}}^{\Sigma})$ which, evaluated along the tangent lift of σ_s, namely, $t\sigma_s = (\sigma_s,\, \dot{\sigma}_s)$, reads:

$$L\left(\varphi, v_{\varphi}\right) \mid_{t\sigma_s} = \int_{\Sigma} \mathscr{L}(\varphi^a_s,\, \dot{\varphi}^a_s,\, \partial_j \varphi^a_s) vol_{\Sigma} \,, \tag{4.72}$$

where $\{ \varphi \}$ is a system of local coordinates on $\mathcal{F}_{\mathbb{E}}^{\Sigma}$ and $\{ \varphi, v_{\varphi} \}$ is a system of local coordinates on $\mathbf{T}\mathcal{F}_{\mathbb{E}}^{\Sigma}$.

First note that, as a direct consequence of proposition 4.1.11, extrema of $\mathscr{S}^\epsilon$ are in one-to-one correspondence, via $\mathfrak{i}_\epsilon$[37], with extrema of $\mathscr{S}$ for x^0 in $[x^0_\Sigma,\, x^0_\Sigma + \epsilon)$. What is more, the above discussion shows that extrema of $\mathscr{S}^\epsilon$ are in one-to-one correspondence (via ϖ) with extrema of $\varpi^{-1\star}\mathscr{S}^\epsilon \;=:\; \mathscr{S}^\epsilon_\varpi$. On the other hand, as we recalled in Rem. 3.1.6, extrema of an action functional written in terms of a Lagrangian function on some tangent bundle of the type (4.72) are the solutions of a pre-symplectic Hamiltonian system where the pre-symplectic form reads:

$$\omega_L \;=\; -\mathrm{d}\,\mathrm{d}_S L\,, \tag{4.73}$$

S being the soldering tensor on the tangent bundle and where the Hamiltonian is:

$$E_L \;=\; \Delta(L) - L\,. \tag{4.74}$$

In particular, in our case, the Lagrangian L is defined on the tangent bundle of $\mathcal{F}^\Sigma_{\mathbb{E}}$, i.e. $\mathbf{T}\mathcal{F}^\Sigma_{\mathbb{E}}$, whose soldering form reads (using the notation introduced in Sec. 2.1.4 for vector fields and 1-forms):

$$S \;=\; \delta\varphi^a \otimes \frac{\delta}{\delta\dot{\varphi}^a}\,, \tag{4.75}$$

and whose partial linear structure reads:

$$\Delta \;=\; \dot{\varphi}^a \frac{\delta}{\delta\dot{\varphi}^a}\,. \tag{4.76}$$

Consequently, a direct computation gives:

$$\omega_L \;=\; -\mathrm{d}\mathrm{d}_S L \;=\; \frac{\delta^2\mathscr{L}}{\delta\varphi^b\delta\dot{\varphi}^a}\delta\varphi^a\wedge\delta\varphi^b + \frac{\delta^2\mathscr{L}}{\delta\dot{\varphi}^b\delta\dot{\varphi}^a}\delta\varphi^a\wedge\delta\dot{\varphi}^b + \frac{\delta^2\mathscr{L}}{\delta\partial_j\varphi^b\delta\dot{\varphi}^a}\delta\varphi^a\wedge\partial_j\delta\varphi^b\,, \tag{4.77}$$

which coincides with the evaluation of the 2-form on $\mathcal{F}^\Sigma_{\mathbb{E}}$ appearing in (4.36) on the tangent lift of the curve φ_s. The discussion above amounts to say that extrema of $\mathscr{S}^\epsilon_\varpi$ are solutions of the pre-symplectic Hamiltonian system $(\mathbf{T}\mathcal{F}^\Sigma_{\mathbb{E}},\, \Omega^\Sigma,\, E_L)$. What is more, such solutions are in one-to-one correspondence with extrema of the action functional restricted to the collar C^Σ_ϵ, i.e. $\mathscr{S}^\epsilon$.

Thus, the following holds.

Proposition 4.2.4. *Extrema of the functional $\mathscr{S}^\epsilon$, i.e., solutions restricted to the collar C^Σ_ϵ around a slice $\Sigma \subset \mathscr{M}$, are in one-to-one correspondence with solutions of the pre-symplectic Hamiltonian system $(\mathbf{T}\mathcal{F}^\Sigma_{\mathbb{E}},\, \Omega^\Sigma,\, E_L)$.*

Example 4.2.5 (Free Klein-Gordon theory). *As first example, let us consider the theory developed in example 3.2.6, that is the free real Klein-Gordon field on the Minkowski space-time. As explained in example 3.2.6, the space of fields reads $\mathcal{F}_{\mathbb{E}} \;=\; \mathcal{H}^1(\mathscr{M},\, vol_{\mathscr{M}})$. Now, without loss of generality, consider a slice that locally*

[37] Restricted to its range.

is of the type $\Sigma = \{ x^0 = x^0_\Sigma \}$[38] *such that, around* Σ, $vol_{\mathscr{M}} = \mathrm{d}x^0 \wedge vol_\Sigma$ *where* vol_Σ *is a volume form on* Σ. *Following the notation introduced in the present section, the space* $\mathcal{F}^\Sigma_{\mathbb{E}} = \mathcal{F}_{\mathbb{E}}\Big|_\Sigma$, *reads, by means of the* TRACE THEOREM *(see [93]),* $\mathcal{F}^\Sigma_{\mathbb{E}} = \mathcal{H}^{\frac{1}{2}}(\Sigma, vol_\Sigma)$ *and its elements will be denoted by* $\varphi(\underline{x}) \in \mathcal{H}^{\frac{1}{2}}(\Sigma, vol_\Sigma)$, $\underline{x}$ *denoting a point in* Σ. $\mathcal{H}^{\frac{1}{2}}(\Sigma, vol_\Sigma)$ *is a Hilbert space and, thus, it is isomorphic to its tangent space. Consequently* $\mathbf{T}\mathcal{F}^\Sigma_{\mathbb{E}} = \left[\mathcal{H}^{\frac{1}{2}}(\Sigma, vol_\Sigma) \right]^2$ *whose elements will be denoted by* $(\varphi(\underline{x}), \dot{\varphi}(\underline{x}))$. *Recalling that the Lagrangian describing the theory is* (3.123), *the 2-form* Ω^Σ *on* $\mathbf{T}\mathcal{F}^\Sigma_{\mathbb{E}}$ *is readily computed according to* (4.77) *and reads:*

$$\Omega^\Sigma = \delta\dot{\varphi} \wedge \delta\varphi \,. \tag{4.78}$$

On the other hand, the function E_L *is computed, according to* (4.74), *to be:*

$$E_L = -\frac{1}{2} \left(\dot{\varphi}^2 + \delta^{jk}\partial_j\varphi\partial_k\varphi + m^2\varphi^2 \right) \tag{4.79}$$

The 2-form Ω^Σ *has an empty kernel and, since* $\mathbf{T}\mathcal{F}^\Sigma_{\mathbb{E}}$ *is a Hilbert space, it is strongly symplectic. Therefore, there is no need for applying the pre-symplectic constraint algorithm and the Hamiltonian system* $(\mathbf{T}\mathcal{F}^\Sigma_{\mathbb{E}}, \Omega^\Sigma, E_L)$ *gives the following Hamiltonian equations:*

$$i_{\mathbb{\Gamma}}\Omega^\Sigma = \mathrm{d}E_L \,, \tag{4.80}$$

where $\mathbb{\Gamma}$ *is a vector field on* $\mathbf{T}\mathcal{F}^\Sigma_{\mathbb{E}}$ *which reads:*

$$\mathbb{\Gamma} = \mathbb{X}_\varphi \frac{\delta}{\delta\varphi} + \mathbb{X}_{\dot{\varphi}} \frac{\delta}{\delta\dot{\varphi}} \,. \tag{4.81}$$

A straightforward computation gives:

$$\mathbb{X}_{\dot{\varphi}} = \delta^{jk}\partial_j\varphi\partial_k\varphi - m^2\varphi^2 \,, \qquad \mathbb{X}_\varphi = \dot{\varphi} \,, \tag{4.82}$$

that collectively gives:

$$\frac{d^2}{ds}\varphi - \delta^{jk}\partial_j\varphi\partial_k\varphi = -m^2\varphi \,, \tag{4.83}$$

which is, provided with the identification of the transversal coordinate to Σ (x^0) *and the parameter* s, *Klein-Gordon equation obtained in example 3.2.6.*

Example 4.2.6 (FREE ELECTRODYNAMICS). *Within free Electrodynamics on the Minkowski space-time, in example 3.2.7, we saw that the space of fields reads* $\mathcal{F}_{\mathbb{E}} = \left[\mathcal{H}^1(\mathscr{M}, vol_{\mathscr{M}}) \right]^4$. *Again, without loss of generality, consider a slice that locally is of the type* $\Sigma = \{ x^0 = x^0_\Sigma \}$ *such that, around* Σ, $vol_{\mathscr{M}} = \mathrm{d}x^0 \wedge vol_\Sigma$ *where* vol_Σ *is a volume form on* Σ. *Following the notation introduced in the present section, the space* $\mathcal{F}^\Sigma_{\mathbb{E}} = \mathcal{F}_{\mathbb{E}}\Big|_\Sigma$, *reads* $\mathcal{F}^\Sigma_{\mathbb{E}} = \left[\mathcal{H}^{\frac{1}{2}}(\Sigma, vol_\Sigma) \right] 4$ *and its elements will be denoted by* $a_\mu(\underline{x}) \in \left[\mathcal{H}^{\frac{1}{2}}(\Sigma, vol_\Sigma) \right]$, $\underline{x}$ *denoting a point in* Σ. $\left[\mathcal{H}^{\frac{1}{2}}(\Sigma, vol_\Sigma) \right]$ *is a Hilbert space and, thus, it is isomorphic to its tangent space. Consequently* $\mathbf{T}\mathcal{F}^\Sigma_{\mathbb{E}} =$

[38]Since Σ is an embedded submanifold of $\mathscr{M}$, a system of local coordinates such that Σ is of this type always exists.

$\left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^4 \times \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^4$ *whose elements will be denoted by* $(a_\mu(\underline{x}),\, \dot{a}_\mu(\underline{x}))$. *Recalling that the Lagrangian describing the theory is* (3.134)*, the* 2*-form* Ω^Σ *on* $\mathbf{T}\mathcal{F}_{\mathbb{E}}^{\Sigma}$ *is readily computed according to* (4.77) *and reads:*

$$\Omega^\Sigma \,=\, 2\delta^{jk}\delta a_j \wedge (\, \partial_k \delta a_0 - \delta \dot{a}_k\,)\ . \tag{4.84}$$

On the other hand, the function E_L *is computed, according to* (4.74)*, to be:*

$$E_L \,=\, \frac{1}{2}\delta^{jk}\,(\, \dot{a}_j - \partial_j a_0\,)\,(\, \partial_k a_0 - \dot{a}_k\,) - \frac{1}{4}\delta^{jl}\delta^{km}\,(\, \partial_j a_k - \partial_k a_j\,)\,(\, \partial_l a_m - \partial_m a_l\,)\ . \tag{4.85}$$

In this case the 2*-form* Ω^Σ *is pre-symplectic and it can be seen that its kernel is, at each point:*

$$\mathrm{ker}\Omega^\Sigma \,=\, \langle \left\{ \psi\frac{\delta}{\delta a_0} + \partial_k\psi\frac{\delta}{\delta \dot{a}_k} \right\}\rangle\,, \tag{4.86}$$

for any $\psi \in \mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)$ *whose gradient lies in* $\left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^3$*. Therefore, the first manifold* $\mathfrak{i}_1\,(\,\mathcal{M}_1\,)$ *of the pre-symplectic constraint algorithm is obtained out of the following compatibility condition:*

$$i_{\mathbb{V}}\mathrm{d}E_L \,=\, 0 \quad \forall\ \mathbb{V} \in \mathrm{ker}\Omega^\Sigma\,, \tag{4.87}$$

which explicitly reads:

$$2\psi\delta^{jk}\partial_k\,(\, \dot{a}_j - \partial_j a_0\,) \,=\, 0\,. \tag{4.88}$$

Note that, since physically the a_μ *represent the components of the quadri-potential restricted to* Σ*, this is the constraint saying that the divergence of the electric field associated to* a_μ *vanishes on* Σ*. By looking at* Ω^Σ*, it can be readily seen that* $\mathbf{T}_{(a,\dot{a})}\mathcal{M}_1^\perp$ *is:*

$$\mathbf{T}_{(a,\dot{a})}\mathcal{M}_1^\perp \,=\, \langle \left\{ \partial_k\zeta\frac{\delta}{\delta a_k} \right\}\rangle \oplus \mathrm{ker}\Omega^\Sigma_{(a,\dot{a})}\Big|_{\mathcal{M}_1}\,, \tag{4.89}$$

where ζ *is any function in* $\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)$[39]*. A straightforward computation shows that:*

$$\mathfrak{i}_1^\star\left(i_{\partial_k\zeta\frac{\delta}{\delta a_k}}\mathrm{d}E_L \right) \,=\, 0\,, \tag{4.90}$$

and, thus, $\mathcal{M}_1$ *is the final manifold of the algorithm. Here, we are in the case in which the pre-symplectic constraint algorithm ends up still with a pre-symplectic manifold since the vector field:*

$$\partial_k\zeta\frac{\delta}{\delta a_k}\,, \tag{4.91}$$

lies in the kernel of $\Omega^\Sigma_\infty \,=\, \mathfrak{i}_1^\star\Omega^\Sigma \,=\, \mathfrak{i}_\infty^\star\Omega^\Sigma$*. Thus* $\mathrm{ker}\Omega^\Sigma_\infty \,=\, \mathrm{grad}\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)$*, that is, the image of the Hilbert space* $\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)$ *into* $\left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^3$ *via the* grad *operator. Consequently, on the final manifold, the vector field* $\mathbb{\Gamma}_\infty$ *satisfying:*

$$i_{\mathbb{\Gamma}_\infty}\Omega^\Sigma_\infty \,=\, \mathrm{d}E_{L\infty}\,, \tag{4.92}$$

[39]This is in order for its gradient to belong to $\left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^3$.

with $E_{L\infty} = \mathfrak{i}^\star_\infty E_L$ is determined up to the addition of any element of $\ker\Omega^\Sigma_\infty$ *and reads:*

$$\mathbb{\Gamma}_\infty = \mathbb{\Gamma}_{\infty a_\mu}\frac{\delta}{\delta a_\mu} + \mathbb{\Gamma}_{\infty \dot{a}_\mu}\frac{\delta}{\delta \dot{a}_\mu} \quad (\textit{up to the addition of any element of } \ker\Omega^\Sigma_\infty)\,, \tag{4.93}$$

where:

$$\mathbb{\Gamma}_{\infty a_j} = -\partial_j a_0\,, \qquad \partial_k \mathbb{\Gamma}_{\infty a_j} = 0\,, \qquad \mathbb{\Gamma}_{\infty \dot{a}_k} = -\frac{1}{2}(\partial_j \dot{a}_0 - \ddot{a}_j) + \frac{1}{2}\delta^{lm}\partial_l\partial_m a_j\,, \tag{4.94}$$

whereas $\mathbb{\Gamma}_{\infty a_0}$ and $\mathbb{\Gamma}_{\infty \dot{a}_0}$ remain completely undetermined as a consequence of the fact that we are dealing with a gauge theory. Therefore, solutions of our pre-symplectic system are integral curves of any of the $\mathbb{\Gamma}_\infty$ above, i.e., solutions of:

$$\frac{d}{ds}a_j = -\partial_j a_0\,, \qquad \frac{d}{ds}\delta^{jk}\partial_k a_j = 0\,, \qquad \frac{d^2}{ds^2}a_j = \Delta a_j - \partial_j \dot{a}_0\,, \tag{4.95}$$

immersed into the original manifold, namely, provided they obey the constraint:

$$\delta^{jk}\partial_k\,(\,\dot{a}_j - \partial_j a_0\,) = 0\,, \tag{4.96}$$

and such that a_0 is completely arbitrary. Arbitrarily fixing a_0 amounts to a gauge choice. For instance, if we fix it to be 0*, we are left with the following set of equations:*

$$\frac{d^2}{ds^2}a_j = \Delta a_j\,, \qquad \frac{d}{ds}\delta^{jk}\partial_k a_j = 0\,, \tag{4.97}$$

being Maxwell's equations in vacuum and without sources written in example 3.2.7 when the scalar potential a_0 is fixed to be 0*.*

4.2.5. First order Hamiltonian field theories

Differently to what happened in Sec. 4.2.2, within first order Hamiltonian field theories, the space of fields restricted to a slice $\Sigma \subset \mathscr{M}$ is not a cotangent bundle. However, at least locally around Σ, one is able to model the theory as a pre-symplectic Hamiltonian system where the pre-symplectic structure is again the structure Ω^Σ from which the canonical structure $\Pi^\star_\Sigma\Omega^\Sigma$ comes from. More precisely, close enough to Σ, we will formulate our theory as the pre-symplectic Hamiltonian system $(\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})},\, \Omega^\Sigma,\, \mathcal{H})$, where $\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})}$ is the space of fields in $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$ restricted to Σ and $\mathcal{H}$ is a suitable Hamiltonian.

Consider a slice Σ. By means of the IMMERSION THEOREM [57], there always exists a system of local coordinates $\{\,x^\mu\,\}_{\mu=0,\dots,d}$ on $\mathscr{M}$ such that locally $\Sigma = \{\,x^0 = x^0_\Sigma = const\,\}$. Sometimes we will refer to x^0 as the TRANSVERSAL COORDINATE to Σ to say that x^0 together with a system of coordinates on Σ provides a system of coordinates for the whole $\mathscr{M}$ in a neighborhood of Σ. Consider the space of dynamical fields restricted to Σ, say $\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})}$:

$$\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})} = \Big\{\,\chi\,|_\Sigma = (\,\phi^a\,|_\Sigma,\, P^\mu_a\,|_\Sigma\,) = \Big(\,\phi^a\,|_\Sigma,\, P^0_a\,|_\Sigma,\, P^j_a\,|_\Sigma\,\Big) =: \Big(\,\varphi^a,\, p_a,\, \beta^j_a\,\Big) =: \chi_\Sigma\,\Big\} =: \mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E}),0}\times\mathscr{B} \tag{4.98}$$

where we isolated the component P^0_a of the momentum field transversal to Σ and where $(\varphi^a,\, p_a) \in \mathcal{F}^{\Sigma}_{\mathcal{P}(\mathbb{E}),\,0}$ and $\beta^j_a \in \mathscr{B}$.

Consider the collar $C^{\Sigma}_{\epsilon} = [x^0_{\Sigma},\, x^0_{\Sigma} + \epsilon) \times \Sigma$ close to Σ, such that the volume reads $vol_{C^{\Sigma}_{\epsilon}} = \mathrm{d}x^0 \wedge vol_{\Sigma}$. Denote by $\mathfrak{i}_{\epsilon}$ the immersion of C^{Σ}_{ϵ} into $\mathscr{M}$, by $\mathscr{S}^{\epsilon} := \mathfrak{i}^{\star}_{\epsilon}\mathscr{S}$ and by $\mathcal{F}^{\epsilon}_{\mathcal{P}(\mathbb{E})}$ the space of dynamical fields restricted to C^{Σ}_{ϵ}. $\mathcal{F}^{\epsilon}_{\mathcal{P}(\mathbb{E})}$ is diffeomorphic to the space of curves on $\mathcal{F}^{\Sigma}_{\mathcal{P}(\mathbb{E})}$, whose elements are denoted by $\sigma_s = (\,\varphi_s,\, p_s,\, \beta_s\,)$. The isomorphism, say ϖ, reads:

$$
\begin{aligned}
\varphi^a_s\left(\underline{x}\right) &= \phi^a\left(x^0 = s,\, \underline{x}\right)\,, \\
p_{a\,s}\left(\underline{x}\right) &= P^0_a\left(x^0 = s,\, \underline{x}\right)\,, \\
\beta^j_{a\,s}\left(\underline{x}\right) &= P^j_a\left(x^0 = s,\, \underline{x}\right)\,,
\end{aligned}
\tag{4.99}
$$

with $s \in [x^0_{\Sigma},\, x^0_{\Sigma} + \epsilon)$. In other words, ϖ is the identification of the coordinate transversal to Σ with the evolution parameter describing the curve on $\mathcal{F}^{\Sigma}_{\mathcal{P}(\mathbb{E})}$. For the same reasons of the previous section ϖ is a diffeomorphism. Let $\Gamma\left(\mathcal{F}^{\Sigma}_{\mathcal{P}(\mathbb{E})}\right) = \varpi\left(\mathcal{F}^{\epsilon}_{\mathcal{P}(\mathbb{E})}\right)$ denote the space of curves $\sigma_{(\cdot)}$. The pull-back of $\mathscr{S}$ to $\Gamma\left(\mathcal{F}^{\Sigma}_{\mathcal{P}(\mathbb{E})}\right)$ is:

$$
\left(\mathscr{S}^{\epsilon}_{\varpi}\right)_{\sigma} := \left(\varpi^{-1\star}\mathscr{S}^{\epsilon}\right)_{\sigma} = \int_{x^0_{\Sigma}}^{x^0_{\Sigma}+\epsilon} \mathrm{d}s \int_{\Sigma} \left(p_{a\,s}\dot{\varphi}^a_s + \beta^k_{a\,s}\partial_k\varphi^a_s - H(\sigma_s)\right) vol_{\Sigma} =: \int_{x^0_{\Sigma}}^{x^0_{\Sigma}+\epsilon} L(\sigma_s,\, \dot{\sigma}_s)\mathrm{d}s\,,
\tag{4.100}
$$

where the dot denotes the derivative with respect to s and where we are interpreting the integrand as a Lagrangian function L on $\mathbf{T}(\mathcal{F}^{\Sigma}_{\mathcal{P}(\mathbb{E})})$ which, evaluated along the tangent lift of σ_s, namely, $t\sigma_s = (\sigma_s,\, \dot{\sigma}_s)$, reads:

$$
\begin{aligned}
L\left(\varphi, v_{\varphi}, p, v_p, \beta, v_{\beta}\right)|_{t\sigma_s} &= \int_{\Sigma}\left(p_{a\,s}\dot{\varphi}^a_s + \beta^k_{a\,s}\partial_k\varphi^a - H(\gamma_s)\right) vol_{\Sigma} = \\
&=: \langle p, \dot{\varphi}\rangle + \langle \beta, \mathrm{d}_{\Sigma}\varphi\rangle - \int_{\Sigma} H(\varphi_s, p_s, \beta_s)\mathrm{vol}_{\Sigma} =: \langle p, \dot{\varphi}\rangle - \mathcal{H}(\gamma_s)\,,
\end{aligned}
\tag{4.101}
$$

where $\{\,\varphi, p, \beta\,\}$ is a system of local coordinates on $\mathcal{F}^{\Sigma}_{\mathcal{P}(\mathbb{E})}$ and $\{\,\varphi, v_{\varphi}, p, v_p, \beta, v_{\beta}\,\}$ is a system of local coordinates on $\mathbf{T}\mathcal{F}^{\Sigma}_{\mathcal{P}(\mathbb{E})}$. In the previous formula d_{Σ} denotes the differential over the smooth 3-manifold Σ, the symbol $\langle\cdot,\cdot\rangle$ denotes both the integration over Σ and the contraction over all the indices of the fields, whereas the functional $\mathcal{H}$ is:

$$
\mathcal{H}(\gamma_s) = \langle \beta, \mathrm{d}_{\Sigma}\varphi\rangle - \int_{\Sigma} H(\gamma_s) vol_{\Sigma}\,.
\tag{4.102}
$$

It is a direct consequence of proposition 4.1.11 that extrema of $\mathscr{S}^{\epsilon}$ are in one-to-one correspondence, via $\mathfrak{i}_{\epsilon}$[40], with extrema of $\mathscr{S}$ for x^0 in $[x^0_{\Sigma},\, x^0_{\Sigma} + \epsilon)$. What is more, the above discussion shows that extrema of $\mathscr{S}^{\epsilon}$ are in one-to-one correspondence (via ϖ) with extrema of $\varpi^{-1\star}\mathscr{S}^{\epsilon} =: \mathscr{S}^{\epsilon}_{\varpi}$.

Now, it is possible to prove [37, Theorem 3.1] the result mentioned in the first lines of the section.

[40]Restricted to its range.

Proposition 4.2.7. *Extrema of the functional $\mathscr{S}^{\epsilon}_{\varpi}$, say $\mathcal{EL}^{\epsilon}_{\varpi}$, are the solutions of the pre-symplectic system* $\left(\mathcal{F}^{\Sigma}_{\mathcal{P}(\mathbb{E})},\, \Omega^{\Sigma},\, \mathcal{H}\right)$.

Proof. By looking at (4.42), the structure Ω^{Σ} in the system of local coordinates chosen, reads:

$$\Omega^{\Sigma}(\mathbb{X}_{\chi_\Sigma},\, \mathbb{Y}_{\chi_\Sigma}) \,=\, \int_{\Sigma} \chi^{\star}_{\Sigma}\left[\, i_X i_Y \mathrm{d}\Theta_H \,\right] \,=\, \int_{\Sigma} \left[\, \mathbb{X}^a_{\varphi}\mathbb{Y}_{p_a} - \mathbb{X}_{p_a}\mathbb{Y}^a_{\varphi} \,\right] vol_{\Sigma}\,, \tag{4.103}$$

where we used the notation (2.52) for the tangent vectors $\mathbb{X}_{\chi_\Sigma}$ and $\mathbb{Y}_{\chi_\Sigma}$ to $\mathcal{F}^{\Sigma}_{\mathcal{P}(\mathbb{E})}$. It is evident from the latter expression that $\mathrm{ker}\Omega^{\Sigma}$ is made by tangent vectors only having components along β^j_a, i.e.:

$$\mathrm{ker}\Omega^{\Sigma} \,=\, \mathrm{span}\left\{ \frac{\delta}{\delta\beta^j_a} \right\}. \tag{4.104}$$

The first step of the pre-symplectic constrain algorithm consists in imposing elements of $\mathrm{ker}\Omega^{\Sigma}$ to belong to $\mathrm{kerd}\mathcal{H}$. A direct computation shows that this imposes the following compatibility condition:

$$\frac{\delta\mathcal{H}}{\delta\beta^j_a} \,=\, \frac{\partial\varphi^a}{\partial x^j} - \frac{\partial H}{\partial\rho^j_a}(\varphi,\, p,\, \beta) \,=\, 0\,. \tag{4.105}$$

If one is able to express the β's in terms of the φ's and the p's (as it will be the case in all the examples considered in the sequel), then the algorithm stops at the first step and the stable manifold reads $\mathcal{M}_{\infty} \,=\, \mathcal{F}^{\Sigma}_{\mathcal{P}(\mathbb{E}),\,0} \,=\, \{\, (\varphi^a,\, p_a)\,\}$ which is a smooth immersed submanifold of $\mathcal{F}^{\Sigma}_{\mathcal{P}(\mathbb{E})}$ whose immersion is given by the expression of the β's in terms of φ's and p's. The structure Ω^{Σ}_{∞} reads:

$$\Omega^{\Sigma}_{\infty}(\mathbb{X}_{m_\infty},\, \mathbb{Y}_{m_\infty}) \,=\, \int_{\Sigma} \left[\, \mathbb{X}^a_{\varphi}\mathbb{Y}_{p_a} - \mathbb{X}_{p_a}\mathbb{Y}^a_{\varphi} \,\right] vol_{\Sigma}\,, \tag{4.106}$$

where $m_{\infty} \,=\, (\varphi^a,\, p_a)$ is a point in $\mathcal{M}_{\infty}$ and, now, $\mathbb{X}^a_{\varphi}$, $\mathbb{Y}_{p_a}$, $\mathbb{X}_{p_a}$ and $\mathbb{Y}^a_{\varphi}$ denote, following the notation (2.52), the components (along φ^a and p_a) of the tangent vectors $\mathbb{X}_{m_\infty}$ and $\mathbb{Y}_{m_\infty}$ to $\mathcal{M}_{\infty}$ at m_{∞}. Evidently, Ω^{Σ}_{∞} is symplectic[41] and the Hamiltonian vector field $\mathbb{X}_{\mathcal{H}_\infty}$ associated to $\mathcal{H}_{\infty} \,=\, \mathfrak{i}^{\star}_{\infty}\mathcal{H}$ reads:

$$\begin{aligned} \mathbb{X}_{\mathcal{H}_{\infty}}{}^a_{\varphi} \,&=\, \frac{d\varphi^a_s}{ds} \,=\, \frac{\delta\mathcal{H}}{\delta p_a} \,=\, -\frac{\partial H}{\partial\rho^0_a}(\varphi_s,\, p_s,\, \beta_s)\,, \\ \mathbb{X}_{\mathcal{H}_{\infty} p_a} \,&=\, \frac{dp_{a\,s}}{ds} \,=\, -\frac{\delta\mathcal{H}}{\delta\varphi^a} \,=\, \frac{\partial\beta^k_{a\,s}}{\partial x^k} + \frac{\partial H}{\partial u^a}(\varphi_s,\, p_s,\, \beta_s)\,. \end{aligned} \tag{4.107}$$

The thesis of the proposition follows from the fact that the latter equations, together with (4.105) formally coincide with De Donder-Weyl equations (3.170) provided the parameter s is identified with the transversal coordinate x^0. □

Thus, we obtained that extrema of $\mathscr{S}^{\epsilon}$, i. e., the elements of the solution space restricted to the collar C^{Σ}_{ϵ}, are in one-to-one correspondence, via ϖ, with solutions of the pre-symplectic system $(\mathcal{F}^{\Sigma}_{\mathcal{P}(\mathbb{E})},\, \Omega^{\Sigma},\, \mathcal{H})$.

[41]That this is actually strongly symplectic should be proved case by case.

Remark 4.2.8. *Notice that all the previous arguments are local in the space-time, in the sense that they are all valid for the parameter ϵ small enough. The possibility of being global depends on the particular space-time considered. In particular, in the cases where $\mathscr{M}$ has a preassigned metric, the space $\mathcal{E}\mathscr{L}^\epsilon$ can be substituted with the whole $\mathcal{E}\mathscr{L}$ in the previous discussion when $\mathscr{M}$ is diffeomorphic to $\Sigma \times \mathbb{R}$ for some codimension-one hypersurface Σ, i.e., for* GLOBALLY HYPERBOLIC *space-times. They are a large class of physically interesting space-times and in the examples considered in this manuscript, only space-times of this kind will appear.*

Having formulated the dynamical content of our theory as a pre-symplectic Hamiltonian system, now the story proceeds like in Sec. 4.2.1, i.e., we should apply the pre-symplectic constraint algorithm in order to obtain the space of Cauchy data $\mathcal{M}_\infty$ which we again assume to be a smooth immersed submanifold of $\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})}$, $\mathfrak{i}_\infty$ denoting the immersion map. As well as in Sec. 4.2.1, if the structure $\Omega^\Sigma_\infty = \mathfrak{i}_\infty^\star \Omega^\Sigma$ is (strongly) symplectic we will end up with the situation depicted by the following diagram:

$$
\begin{tikzcd}
\mathcal{E}\mathscr{L}^\epsilon_\varpi \arrow[r, shift left, "\varpi^{-1}"] \arrow[d, "\Pi_\Sigma"] \arrow[dd, bend right=60, "\Psi"'] & \mathcal{E}\mathscr{L}^\epsilon \arrow[l, shift left, "\varpi"] & \Pi_\Sigma^\star \Omega^\Sigma \arrow[l, squiggly] \\
\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})} & \Omega^\Sigma \arrow[l, squiggly] & \\
\mathcal{M}_\infty \arrow[u, "\mathfrak{i}_\infty"'] \arrow[uu, bend left=40, "\Psi^{-1}"] & \Omega^\Sigma_\infty \arrow[l, squiggly] &
\end{tikzcd}
\tag{4.108}
$$

where Ψ is a diffeomorphism. From the diagram above one is able to prove the analogue of (4.52), namely:

$$
\Pi_\Sigma^\star \Omega^\Sigma = \left(\Psi^{-1} \circ \varpi \right)^\star \Omega^\Sigma_\infty \,, \tag{4.109}
$$

which shows how it is equivalent to work with $\mathcal{E}\mathscr{L}$ equipped with $\Pi_\Sigma^\star \Omega^\Sigma$ or with $\mathcal{M}_\infty$ equipped with Ω^Σ_∞. Therefore, we will often refer to $\mathcal{M}_\infty$ itself as "solution space" of the theory. Again, also in case Ω^Σ_∞ is only pre-symplectic we will refer to the whole $\mathcal{M}_\infty$ (without quotienting with respect to the kernel of Ω^Σ_∞) as "solution space" and we will distinguish theories with or without gauge symmetries as those for which the solution space is a (strongly) symplectic or a pre-symplectic manifold.

Example 4.2.9 (FREE KLEIN-GORDON THEORY)**.** *Here we consider, as a first example, the theory developed in example 3.2.13, namely the free real Klein-Gordon field on the Minkowski space-time. As explained in example 3.2.13, the space of dynamical fields reads $\mathcal{F}_{\mathcal{P}(\mathbb{E})} = \mathcal{H}^2(\mathscr{M}, vol_{\mathscr{M}}) \times \left[\mathcal{H}^1(\mathscr{M}, vol_{\mathscr{M}}) \right]^4$. Again, without loss of generality, consider a slice $\Sigma = \{ x^0 = \mathscr{X}^0_\Sigma \}$ such that, around Σ, $vol_{\mathscr{M}} = \mathrm{d}x^0 \wedge vol_\Sigma$ where vol_Σ is a volume form on Σ. Following the notation introduced in the present section, the space $\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})} = \mathcal{F}_{\mathcal{P}(\mathbb{E})}\big|_\Sigma$, reads $\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})} = \mathcal{H}^{\frac{3}{2}}(\Sigma, vol_\Sigma) \times \left[\mathcal{H}^{\frac{1}{2}} \right]^4$ and its elements will be denoted by $\chi_\Sigma = \left(\phi\big|_\Sigma, P^0\big|_\Sigma, P^j\big|_\Sigma \right) =: (\varphi(\underline{x}), p(\underline{x}), \beta^j(\underline{x})) \in$*

$\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma) \times \mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma) \times \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^3$, *where* $\underline{x}$ *denotes a point in* Σ *and we separated the tangent components of the momenta fields to* Σ *and the transversal one. By looking at* (4.42), Ω^Σ *is computed to be:*

$$\Omega^\Sigma \,=\, \delta\varphi \wedge \delta p\,. \tag{4.110}$$

On the other hand the Hamiltonian functional (4.102) *is computed to be:*

$$\mathcal{H} \,=\, \int_\Sigma \left[\beta^j \partial_j \varphi - \frac{1}{2}\left(p^2 - \delta_{jk}\partial_j\varphi\partial_k\varphi + m^2\varphi^2\right)\right] vol_\Sigma\,. \tag{4.111}$$

As it was pointed out in the proof of Prop. 4.2.7, the kernel of Ω^Σ *is:*

$$\mathrm{ker}\Omega^\Sigma \,=\, \langle\left\{\frac{\delta}{\delta\beta^\mu}\right\}\rangle\,. \tag{4.112}$$

The first manifold $\mathfrak{i}_1\left(\mathcal{M}_1\right)$ *of the pre-symplectic constraint algorithm is obtained out of the following compatibility condition:*

$$i_{\frac{\delta}{\delta\beta^\mu}}\mathrm{d}\mathcal{H} \,=\, 0\,, \tag{4.113}$$

which explicitly reads:

$$\beta_j \,=\, -\partial_j\varphi\,. \tag{4.114}$$

Consequently, since the β*'s can be expressed in terms of the* φ*'s, as it was pointed out in Prop. 4.2.7 the manifold* $\mathcal{M}_1$ *reads:*

$$\mathcal{M}_1 \,=\, \mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E}),\,0} \,=\, \mathcal{H}^{\frac{2}{2}}(\Sigma,\, vol_\Sigma) \times \mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma) \ni (\varphi,\, p)\,, \tag{4.115}$$

whose immersion into $\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})}$ *is given by the condition* (4.114). $\mathbf{T}_{\chi_\Sigma}\mathcal{M}_1^\perp$ *is seen to coincide with the kernel of* Ω^Σ *restricted to* $\mathcal{M}_1$. *Therefore, no additional constraints emerge and* $\mathcal{M}_1 \,=\, \mathcal{M}_\infty$. *The structure:*

$$\Omega^\Sigma_\infty \,=\, \mathfrak{i}^\star_\infty\Omega^\Sigma \,=\, \mathfrak{i}^\star_1\Omega^\Sigma \,=\, \delta\varphi \wedge \delta p\,, \tag{4.116}$$

is symplectic and the pull-back of Hamiltonian functional to $\mathcal{M}_\infty$ *reads:*

$$\mathcal{H}_\infty \,=\, \mathfrak{i}^\star_\infty\mathcal{H} \,=\, -\frac{1}{2}\int_\Sigma \left[\delta^{jk}\partial_j\varphi\partial_k\varphi + p^2 + m^2\varphi^2\right] vol_\Sigma\,. \tag{4.117}$$

Therefore, the vector field $\mathbb{\Gamma}_\infty$ *satisfying:*

$$i_{\mathbb{\Gamma}_\infty}\Omega^\Sigma_\infty \,=\, \mathrm{d}\mathcal{H}_\infty\,, \tag{4.118}$$

is:

$$\mathbb{\Gamma}_\infty \,=\, \mathbb{\Gamma}_{\infty\varphi}\frac{\delta}{\delta\varphi} + \mathbb{\Gamma}_{\infty p}\frac{\delta}{\delta p}\,, \tag{4.119}$$

where:

$$\mathbb{\Gamma}_{\infty\varphi} \,=\, -p\,, \qquad \mathbb{\Gamma}_{\infty p} \,=\, -m^2\varphi + \delta^{jk}\partial_j\partial_k\varphi\,, \tag{4.120}$$

Therefore, solutions of our pre-symplectic Hamiltonian system are integral curves of $\mathbb{\Gamma}_\infty$, i.e., solutions of:

$$\frac{d}{ds}\dot{\varphi}_s \,=\, -p_s\,, \qquad \frac{d}{ds}p_s \,=\, -m^2\varphi_s + \delta^{jk}\partial_j\partial_k\varphi_s\,, \tag{4.121}$$

immersed into the original manifold, namely, provided they obey:

$$\beta_{sj} \,=\, -\partial_j\varphi_s\,. \tag{4.122}$$

The equations above, provided we identify the parameter s with the coordinate transversal to Σ, collectively amounts to:

$$\partial_\mu P^\mu + m^2\phi \,=\, 0\,, \qquad \partial_\mu\phi \,=\, \eta_{\mu\nu}P^\mu\,, \tag{4.123}$$

and, consequently, to the Klein-Gordon equation:

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi(x) + m^2\phi(x) \,=\, 0\,, \tag{4.124}$$

obtained in the end of example 3.2.13.

4.2.6. First order Hamiltonian field theories with constraints

Within first order Hamiltonian field theories with additional constraints of the type considered in Sec. 3.2.3, we saw that the canonical structure on the solution space has the same expression as for theories without additional constraints.

On the other hand, as it is clear from Eq. (3.185), the additional term $\langle\Lambda,\,\chi - \Phi(n)\rangle$ in the modified action on the extended space of fields, $\mathscr{S}^{\mathrm{ext}}$, has the net result of subtracting to the Hamiltonian H, the function $\Lambda^\phi_a\,(\phi^a - u^a\circ\Phi(n)) + \Lambda^{P^a}{}_\mu\,(P^\mu_a - \rho^\mu_a\circ\Phi(n))$. Consequently, the Hamiltonian functional (4.102) obtained from $\mathscr{S}^{\mathrm{ext}}$ turns out to be modified in the following way:

$$\begin{aligned}\mathcal{H}^{\mathrm{ext}}(\chi_\Sigma,\,\Lambda_\Sigma,n) \,=\, &\int_\Sigma\Bigg\{\,\beta^k_a\partial_k\varphi^a - H(\chi_\Sigma) + \\ &+\Big[\lambda^\varphi_a\left(\varphi^a - u^a\circ\Phi(n)\Big|_\Sigma\right) + \lambda^{P^a}{}_0\left(p_a - \rho^0_a\circ\Phi(n)\Big|_\Sigma\right) + \lambda^{P^a}{}_k\left(\beta^k_a - \rho^k_a\circ\Phi(n)\Big|_\Sigma\right)\Big]\Bigg\}vol_\Sigma\end{aligned} \tag{4.125}$$

where $\lambda^\varphi_a \,:=\, \Lambda^\phi_a\Big|_\Sigma$ and $\lambda^{P^\mu}{}_a \,=:\, \Lambda^{P^\mu}{}_a\Big|_\Sigma$. Therefore, in this case the pre-symplectic Hamiltonian system associated to the modified action, $\mathscr{S}^{\mathrm{ext}}$, is $\left(\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})}\times\mathcal{F}^{\Sigma\,\star}_{\mathcal{P}(\mathbb{E})}\times\mathcal{N},\,\Omega^{\Sigma^{\mathrm{ext}}},\,\mathcal{H}^{\mathrm{ext}}\right)$ where $\Omega^{\Sigma^{\mathrm{ext}}} \,=\, \tau^\star\Omega^\Sigma$, τ being the projection $\tau\,:\,\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})}\times\mathcal{F}^{\Sigma\,\star}_{\mathcal{P}(\mathbb{E})}\times\mathcal{N}\to\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})}$. With this in mind, Prop. 4.2.3, in this case is generalized as follows.

Proposition 4.2.10. *Extrema of the functional $\mathscr{S}^{\mathrm{ext}\,\epsilon}{}_\varpi$, say $\mathcal{EL}^{\mathrm{ext}\,\epsilon}{}_\varpi$, are the solutions of the pre-symplectic system $\left(\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})}\times\mathcal{F}^{\Sigma\,\star}_{\mathcal{P}(\mathbb{E})}\times\mathcal{N},\,\Omega^{\Sigma^{\mathrm{ext}}},\,\mathcal{H}^{\mathrm{ext}}\right)$.*

Example 4.2.11 (Free Electrodynamics). *Within free Electrodynamics, as explained in example 3.2.14, we considered the space of dynamical fields* $\mathcal{F}_{\mathcal{P}(\mathbb{E})} = \left[\mathcal{H}^2(\mathscr{M},\, vol_{\mathscr{M}})\right]^4 \times \left[\mathcal{H}^1(\mathscr{M},\, vol_{\mathscr{M}})\right]^{16}$*. Again, without loss of generality, consider a slice* $\Sigma = \{\, x^0 = \mathbb{X}^0_\Sigma \,\}$ *such that, around* Σ, $vol_{\mathscr{M}} = \mathrm{d}x^0 \wedge vol_\Sigma$ *where* vol_Σ *is a volume form on* Σ*. Following the notation introduced in the present section, the space* $\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})} = \mathcal{F}_{\mathcal{P}(\mathbb{E})}\Big|_\Sigma$*, reads* $\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})} = \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\right]^4 \times \left[\mathcal{H}^{\frac{1}{2}}\right]^{16}$ *and its elements will be denoted by* $\left(A_\mu\Big|_\Sigma,\, P^{00}\Big|_\Sigma,\, P^{k0}\Big|_\Sigma,\, P^{0k}\Big|_\Sigma,\, P^{jk}\Big|_\Sigma\right) =: (a_\mu(\underline{x}), p^0(\underline{x}),\, p^k(\underline{x}),\, \tilde{p}^k(\underline{x}),\, \beta^{jk}(\underline{x})) \in \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\right]^4 \times \mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma) \times \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^3 \times \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^3 \times \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^9$*, where* $\underline{x}$ *denotes a point in* Σ *and we separated the tangent components of the momenta fields to* Σ *and the transversal one. As discussed in the present section, the fact that we are considering a theory with constraints does not affect the expression of the 2-form* $\Omega^{\Sigma\text{ext}}$*, which is explicitly computed to be:*

$$\Omega^{\Sigma\text{ext}} = \delta a_\mu \wedge \delta p^\mu \,. \tag{4.126}$$

On the other hand, the Hamiltonian functional (4.125) *reads:*

$$\mathcal{H}^{\text{ext}}_{\overline{\chi}_\Sigma} = \int_\Sigma \left[p^\mu \partial_\mu a_0 + \beta^{jk}\partial_k a_j - \eta_{\mu\nu}p^\mu p^\nu + \eta_{jl}\eta_{\mu\nu}\beta^{j\mu}\beta^{l\nu} + \lambda_{p_k}(p^k + \tilde{p}^k) + \lambda_{p_0}p^0 + \lambda_{\beta_{jk}}\beta^{(jk)} \right] vol_{\mathscr{M}} \,, \tag{4.127}$$

where $\lambda_{p_0} \in \mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)^\star = \mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)$, $\lambda_{p_k},\, \lambda_{\tilde{p}_k} \in \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)^\star\right]^3 = \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^3$ *and* $\lambda_{\beta_{jk}} \in \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)^\star\right]^9 = \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^9$ *and* $\overline{\chi}_\Sigma = (a_\mu,\, p^\mu,\, \tilde{p}^k,\, \beta^{jk},\, \lambda_{p_\mu},\, \lambda_{\beta_{jk}})$ *denotes a point in the extended space of dynamical fields. The kernel of* Ω^Σ *is readily seen to be:*

$$\ker\Omega^{\Sigma\text{ext}} = \langle \left\{ \frac{\delta}{\delta\tilde{p}^k},\, \frac{\delta}{\delta\beta^{jk}},\, \frac{\delta}{\delta\lambda_{p_\mu}},\, \frac{\delta}{\delta\lambda_{\beta_{jk}}} \right\} \rangle \,. \tag{4.128}$$

Consequently, the first manifold of the pre-symplectic constraint algorithm is obtained by imposing:

$$i_{\mathbb{V}}\Omega^{\Sigma\text{ext}} = 0 \quad \forall\ \mathbb{V} \in \ker\Omega^{\Sigma\text{ext}} \,, \tag{4.129}$$

which explicitly gives the following constraints:

$$\begin{aligned} \lambda_{p_k} &= 0, & \partial_k a_j + 2\delta_{jl}\delta_{km}\beta^{lm} + \lambda_{\beta_{jk}} &= 0\,, \\ p^0 &= 0\,, \quad p^k = -\tilde{p}^k\,, & \beta^{(jk)} &= 0\,. \end{aligned} \tag{4.130}$$

Thus:

$$\mathcal{M}_1 = \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\right]^4 \times \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^3 \times \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^{12} \ni (a_\mu,\, p^k,\, \lambda_{\beta_{jk}})\,, \tag{4.131}$$

whose immersion into the original manifold is given by the relations (4.130)*.* $\mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}_1^\perp$ *is seen to be:*

$$\mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}_1^\perp = \langle \left\{ \frac{\delta}{\delta a_0} \right\} \rangle \oplus \ker\Omega^{\Sigma\text{ext}}_{\overline{\chi}_\Sigma}\Big|_{\mathcal{M}_1} \,, \tag{4.132}$$

and, thus, the second manifold of the algorithm is obtained imposing:

$$\mathfrak{i}_1^\star \left[i_{\frac{\delta}{\delta a_0}}\Omega^{\Sigma\text{ext}} \right] = 0\,, \tag{4.133}$$

which explicitly reads:

$$\partial_k p^k \;=\; 0\,. \tag{4.134}$$

Therefore, the second manifold of the algorithm reads:

$$\begin{aligned}\mathcal{M}_2 \;=\; & \left[\,\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\right]^4 \times \left[\,\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma, \mathrm{div}0)\right]^3 \times \left[\,\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^9 \ni \\ & \ni (a_\mu,\, p^k,\, \lambda_{\beta\, jk})\,,\end{aligned} \tag{4.135}$$

where by $\left[\,\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma, \mathrm{div}0)\right]^3$ *we mean the elements* p^k *of* $\left[\,\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^3$ *such that* $\partial_k p^k \;=\; 0$*. As we will prove in the next section* $\left[\,\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma, \mathrm{div}0)\right]^3$ *is a closed (Hilbert) subspace of* $\left[\,\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^3$ *and, thus, it is canonically immersed into it via an immersion denoted by* $\mathfrak{i}_2$*. By virtue of this last constraint,* $\mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}_2^\perp$ *is seen to be:*

$$\mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}_2^\perp \;=\; \langle\left\{\,\partial_k\psi\frac{\delta}{\delta a_k}\,\right\}\rangle \oplus \mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}_1^\perp\Big|_{\mathcal{M}_2}\,. \tag{4.136}$$

A direct computation shows that:

$$\mathfrak{i}_2^\star\left[\,i_{\partial_k\psi\frac{\delta}{\delta a_k}}\mathcal{H}^{\mathrm{ext}}\,\right] \;=\; 0\,, \tag{4.137}$$

thus, no other constraints emerge and $\mathcal{M}_2 \;=\; \mathcal{M}_\infty$*. The structure:*

$$\Omega^{\Sigma\mathrm{ext}}_\infty \;=\; \mathfrak{i}_\infty^\star\Omega^{\Sigma\mathrm{ext}} \;=\; \mathfrak{i}_2^\star\Omega^{\Sigma\mathrm{ext}} \;=\; \delta a_k \wedge \delta p^k\,, \tag{4.138}$$

is still pre-symplectic whose kernel is:

$$\ker\Omega^{\Sigma\mathrm{ext}}_\infty \;=\; \langle\left\{\,\frac{\delta}{\delta a_0},\, \partial_k\psi\frac{\delta}{\delta a_k},\, \frac{\delta}{\delta\lambda_{\beta\, jk}}\,\right\}\rangle\,. \tag{4.139}$$

Consequently, the vector field $\mathbb{\Gamma}_\infty$ *satisfying:*

$$i_{\mathbb{\Gamma}_\infty}\Omega^{\Sigma\mathrm{ext}}_\infty \;=\; \mathrm{d}\mathcal{H}^{\mathrm{ext}}_\infty\,, \tag{4.140}$$

where:

$$\mathcal{H}_\infty \;=\; \mathfrak{i}_\infty^\star\mathcal{H} \;=\; \int_\Sigma \left[\,\frac{1}{4}\delta^{jl}\delta^{km}\partial_{[j}a_{k]}\partial_{[l}a_{m]} - \delta_{jk}p^jp^k\,\right] vol_\Sigma\,, \tag{4.141}$$

is determined up to elements in $\ker\Omega^\Sigma_\infty$ *and reads:*

$$\mathbb{\Gamma}_\infty \;=\; \mathbb{\Gamma}_{\infty a k}\frac{\delta}{\delta a_k} + \mathbb{\Gamma}_{\infty p}{}^k\frac{\delta}{\delta p^k}\ (\textit{up to the addition of elements in } \ker\Omega^\Sigma_\infty)\,, \tag{4.142}$$

where:

$$\mathbb{\Gamma}_{\infty a k} \;=\; 2\delta_{jk}p^j\,, \qquad \mathbb{\Gamma}_{\infty p}{}^k \;=\; \frac{1}{2}\delta^{km}\delta^{jl}\partial_j\partial_l a_m\,. \tag{4.143}$$

Thus, we obtain that solutions of our pre-symplectic Hamiltonian system are the integral curves of any of the $\mathbb{\Gamma}_\infty$ *above, i.e., solutions of:*

$$\frac{d}{ds}a_k \;=\; 2\delta_{jk}p^j\,, \qquad \frac{d}{ds}p^k \;=\; \frac{1}{2}\delta^{km}\delta^{jl}\partial_j\partial_l a_m\,, \tag{4.144}$$

with a_0, λ_{p_0} and $\lambda_{\beta_{jk}}$ remaining completely undetermined, immersed back into $\mathcal{F}^{\Sigma}_{\mathcal{P}(\mathbb{E})}$, i.e., provided with the constraint:

$$\partial_k p^k \,=\, 0\,. \tag{4.145}$$

If we fix arbitrary $a_0 \,=\, 0$ as in example 4.2.6 and also λ_{p_0} and $\lambda_{\beta_{jk}}$ to be zero (as we are allowed to) the equations above collectively give:

$$\frac{d^2}{ds^2} a_j \,=\, \Delta a_j\,, \qquad \frac{d}{ds}\delta^{jk}\partial_k a_j \,=\, 0\,, \tag{4.146}$$

being the equations written in the end of example 4.2.6, namely, Maxwell's equations in vacuum and without sources of example 3.2.7 and 3.2.14 when the scalar potential a_0 is fixed to be 0.

Example 4.2.12 (Yang-Mills theories)**.** *Let us now consider Yang-Mills theories. As discussed in example 3.2.15, here the space of dynamical fields reads $\mathcal{F}_{\mathcal{P}(\mathbb{E})} \,=\, [\,\mathcal{H}^3(\mathscr{M},\, vol_{\mathscr{M}})\,]^n \times [\,\mathcal{H}^2(\mathscr{M}, vol_{\mathscr{M}})\,]^m$ with $n \,=\, 4\mathrm{dim}\mathfrak{g}$ and $m \,=\, 16\mathrm{dim}\mathfrak{g}$. Again, without loss of generality, consider a slice $\Sigma \,=\, \{\, x^0 \,=\, \mathbb{X}^0_\Sigma \,\}$ such that, around Σ, $vol_{\mathscr{M}} \,=\, \mathrm{d}x^0 \wedge vol_\Sigma$ where vol_Σ is a volume form on Σ. Consequently, the space of dynamical fields restricted to Σ reads $\mathcal{F}^{\Sigma}_{\mathcal{P}(\mathbb{E})} \,=\, \left[\,\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)\,\right]^n \times \left[\,\mathcal{H}^{\frac{3}{2}}(\Sigma, vol_\Sigma)\,\right]^m$ and its elements will be denoted by $\left(\, A^a_\mu\big|_\Sigma,\, P^{00}_a\big|_\Sigma,\, P^{k0}_a\big|_\Sigma,\, P^{0k}_a\big|_\Sigma,\, P^{jk}_a\big|_\Sigma \,\right) \,=:\, (a^a_\mu(\underline{x}), p^0_a(\underline{x}),\, p^k_a(\underline{x}),\, \tilde{p}^k_a(\underline{x}),\, \beta^{jk}_a($ $\left[\,\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)\,\right]^n \times \left[\,\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\,\right]^{\mathrm{dim}\mathfrak{g}} \times \left[\,\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\,\right]^{3\mathrm{dim}\mathfrak{g}} \times \left[\,\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\,\right]^{3\mathrm{dim}\mathfrak{g}} \times \left[\,\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\,\right]^{9\mathrm{dim}\mathfrak{g}}$, where $\underline{x}$ denotes a point in Σ and we separated the tangent components of the momenta fields to Σ and the transversal one. Here, the 2-form $\Omega^{\Sigma^{\mathrm{ext}}}$ reads:*

$$\Omega^{\Sigma^{\mathrm{ext}}} \,=\, \delta a^a_\mu \wedge \delta p^\mu_a\,. \tag{4.147}$$

On the other hand, the Hamiltonian functional (4.125) *is:*

$$\begin{aligned}\mathcal{H}^{\mathrm{ext}}_{\overline{\chi}_\Sigma} \,=\, \int_\Sigma \Big[\, & \tilde{p}^k_a \nabla_k a^a_0 + \beta^{jk}_a \nabla_k a^a_j - \frac{1}{2} G^{ab}\eta_{\mu\nu} p^\mu_a p^\nu_b - \frac{1}{2} G^{ab}\delta_{jk}\tilde{p}^j_a \tilde{p}^j_b - \frac{1}{2} G^{ab}\delta_{jl}\delta_{km}\beta^{jk}_a \beta^{lm}_b + \\ & + p^\mu_a [a_\mu,\, a_0]_a + \lambda_{p_k}{}^a (p^k_a + \tilde{p}^k_a) + \lambda_{p_0}{}^a p^0 + \lambda_{\beta_{jk}}{}^a \beta^{(jk)}_a \,\Big] vol_\Sigma\,,\end{aligned} \tag{4.148}$$

where $\lambda_{p_0}{}^a \in \left[\,\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)^\star\,\right]^{\mathrm{dim}\mathfrak{g}} \,=\, \left[\,\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\,\right]^{\mathrm{dim}\mathfrak{g}}$, $\lambda_{p_k}{}^a$ and $\lambda_{\tilde{p}_k}{}^a \in \left[\,\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)^\star\,\right]^{3\mathrm{dim}\mathfrak{g}} \,=\, \left[\,\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\,\right]^{3\mathrm{dim}\mathfrak{g}}$ and $\lambda_{\beta_{jk}}{}^a \in \left[\,\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)^\star\,\right]^{9\mathrm{dim}\mathfrak{g}} \,=\, \left[\,\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\,\right]^{9\mathrm{dim}\mathfrak{g}}$ and:

$$\overline{\chi}_\Sigma \,=\, \left(\, a^a_\mu,\, p^\mu_a,\, \tilde{p}^k_a,\, \beta^{jk}_a,\, \lambda_{p_\mu}{}^a,\, \lambda_{\tilde{p}_k}{}^a,\, \lambda_{\beta_{jk}}{}^a \,\right)\,, \tag{4.149}$$

denotes a point in the extended space of fields. The kernel of $\Omega^{\Sigma^{\mathrm{ext}}}$ is readily seen to be:

$$\mathrm{ker}\Omega^{\Sigma^{\mathrm{ext}}} \,=\, \langle \left\{\, \frac{\delta}{\delta \tilde{p}^k_a},\, \frac{\delta}{\delta \beta^{jk}_a},\, \frac{\delta}{\delta \lambda_{p_\mu}{}^a},\, \frac{\delta}{\delta \lambda_{\tilde{p}_k}{}^a},\, \frac{\delta}{\delta \lambda_{\beta_{jk}}{}^a} \,\right\} \rangle\,. \tag{4.150}$$

Consequently, the first manifold of the pre-symplectic constraint algorithm is obtained by imposing:

$$i_{\mathbb{V}}\Omega^{\Sigma^{\mathrm{ext}}} \,=\, 0 \quad \forall\; \mathbb{V} \in \mathrm{ker}\Omega^{\Sigma^{\mathrm{ext}}}\,, \tag{4.151}$$

which explicitly gives:

$$\lambda_{p_k^a} + \nabla_k a_0^a = \delta_{jk} G^{ab} \tilde{p}_b^j\,, \qquad \nabla_k a_j^a = \delta_{jk} G^{ab} \beta_b^{jk}\,, \\ p_a^0 = 0\,, \qquad \tilde{p}_a^k = -p_a^k\,, \qquad \beta_a^{(jk)} = 0\,. \tag{4.152}$$

Thus:

$$\mathcal{M}_1 = \left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)\right]^n \times \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\right]^{3\dim\mathfrak{g}} \times \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\right]^{9\dim\mathfrak{g}} \ni \left(a_\mu^a,\, p_a^k,\, \lambda_{\beta\,jk}^{\;a}\right)\,, \tag{4.153}$$

whose immersion into original manifold is given by the relations (4.152)*.* $\mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}_1^\perp$ *is seen to be:*

$$\mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}_1^\perp = \langle\left\{ \frac{\delta}{\delta a_0^a} \right\}\rangle \oplus \ker\Omega^{\Sigma\text{ext}}_{\overline{\chi}_\Sigma}\Big|_{\mathcal{M}_1}\,, \tag{4.154}$$

and, thus, the second manifold of the algorithm is obtained by imposing:

$$\mathfrak{i}_\infty^\star\left[i_{\frac{\delta}{\delta a_0^a}} \Omega^{\Sigma\text{ext}} \right] = 0\,, \tag{4.155}$$

which explicitly reads:

$$\nabla_k p_a^k = 0\,. \tag{4.156}$$

Therefore, the second manifold of the algorithm reads:

$$\mathcal{M}_2 = \left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)\right]^n \times \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma, \nabla 0)\right]^{3\dim\mathfrak{g}} \times \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\right]^{9\dim\mathfrak{g}} \ni \\ \ni (a_\mu^a,\, p_a^k,\, \lambda_{\beta\,jk}^{\;a})\,, \tag{4.157}$$

where by $\left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma, \nabla 0)\right]^{3\dim\mathfrak{g}}$ *we mean elements* p_a^k *of* $\left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\right]^{3\dim\mathfrak{g}}$ *such that* $\nabla_k p_a^k = 0$*. As we will prove in the next section* $\left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma, \nabla 0)\right]^{3\dim\mathfrak{g}}$ *is a closed (Hilbert) subspace of* $\left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\right]^{3\dim\mathfrak{g}}$ *and, thus, it is canonically immersed into it via an immersion denoted by* $\mathfrak{i}_2$*. By virtue of this last constraint,* $\mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}_2^\perp$ *is seen to be:*

$$\mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}_2^\perp = \langle\left\{ \mathbb{V}_\psi = \nabla_k\psi^a \frac{\delta}{\delta a_k} + [p^k,\, \psi]_a \frac{\delta}{\delta p_a^k} \right\}\rangle \oplus \mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}_1^\perp\Big|_{\mathcal{M}_2}\,, \tag{4.158}$$

for any ψ^a *belonging to* $\left[\mathcal{H}^{\frac{7}{2}}\right]^{\dim\mathfrak{g}}$[42]*. A direct computation shows that:*

$$\mathfrak{i}_2^\star\left[i_{\mathbb{V}_\psi}\mathcal{H}^{\text{ext}} \right] = 0\,, \tag{4.161}$$

---

[42]We are choosing $\psi^a \in \left[\mathcal{H}^{\frac{7}{2}}(\Sigma,\, vol_\Sigma)\right]^{\dim\mathfrak{g}}$ so that its covariant derivative is still $\left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)\right]^{3\dim\mathfrak{g}}$ function. Indeed, since $\mathcal{H}^{\frac{5}{2}}(\Sigma, vol_\Sigma)$ is a Banach algebra (see [88]), the following inequality holds:

$$\sum_{k,a}\|\nabla_k\psi^a\|_{\mathcal{H}^{\frac{5}{2}}} \leq \sum_{k,a}\|\partial_k\psi^a\|_{\mathcal{H}^{\frac{5}{2}}} + \sum_{k,a}|\epsilon^a{}_{bc}|\|a_k^b\psi^c\|_{\frac{5}{2}} \leq 3\sum_{k,a}\|\psi^a\|_{\mathcal{H}^{\frac{7}{2}}} + \sum_{k,a}|\epsilon^a{}_{bc}|B_{b,c,k}\|a_k^b\|_{\mathcal{H}^{\frac{5}{2}}}\|\psi^c\|_{\frac{5}{2}}\,, \tag{4.159}$$

for some constants $B_{b,c,k}$, where the inequality $\sum_{k,a}\|\partial_k\psi^a\|_{\mathcal{H}^{\frac{5}{2}}} \leq 3\sum_{k,a}\|\psi^a\|_{\mathcal{H}^{\frac{7}{2}}}$ is due to

$$\sum_{k,a}\|\partial_k\psi^a\|_{\mathcal{H}^{\frac{5}{2}}} = \sum_{k,a}\left\||k|^{\frac{5}{2}}k_k\tilde{\psi}^a\right\|_{\mathcal{L}^2} \leq 3\sum_a\left\||k|^{\frac{7}{2}}\tilde{\psi}^a\right\|_{\mathcal{L}^2} = 3\sum_a\|\psi^a\|_{\mathcal{H}^{\frac{7}{2}}}\,, \tag{4.160}$$

where $\tilde{\psi}^a$ is the Fourier transform of ψ^a.

thus, no other constraints emerge and $\mathcal{M}_2 = \mathcal{M}_\infty$. Actually, as we will see, a_0^a and $\lambda_{\beta}{}^{a}_{jk}$ will not appear in $\mathcal{H}_\infty^{\mathrm{ext}}$ and, thus, they turn out to be not dynamical, in the sense that they do not appear at all in the equations of motion. Thus, they can be fixed once and for all without affecting the rest of the discussion. For this reason we consider $\mathcal{M}_\infty$ to be actually:

$$\begin{aligned}\mathcal{M}_\infty \;=\; & \left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)\right]^{3\dim\mathfrak{g}} \times \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma, \nabla 0)\right]^{3\dim\mathfrak{g}} \ni \\ & \ni (a_k^a,\, p_a^k)\,,\end{aligned} \tag{4.162}$$

The structure:

$$\Omega_\infty^{\Sigma\mathrm{ext}} \;=\; \mathfrak{i}_\infty^\star \Omega^{\Sigma\mathrm{ext}} \;=\; \mathfrak{i}_2^\star \Omega^{\Sigma\mathrm{ext}} \;=\; \delta a_k^a \wedge \delta p_a^k\,, \tag{4.163}$$

is still pre-symplectic whose kernel is:

$$\ker\Omega_\infty^{\Sigma\mathrm{ext}} \;=\; \langle \left\{ \frac{\delta}{\delta a_0},\, \mathbb{V}_\psi,\, \frac{\delta}{\delta \lambda_{\beta_{jk}}} \right\} \rangle\,. \tag{4.164}$$

Consequently, the vector field $\mathbb{\Gamma}_\infty$ satisfying:

$$i_{\mathbb{\Gamma}_\infty} \Omega_\infty^{\Sigma\mathrm{ext}} \;=\; \mathrm{d}\mathcal{H}_\infty^{\mathrm{ext}}\,, \tag{4.165}$$

where $\mathcal{H}_\infty = \mathfrak{i}_\infty^\star \mathcal{H}$, is determined up to elements in $\ker\Omega_\infty^\Sigma$ and reads:

$$\mathbb{\Gamma}_\infty \;=\; \mathbb{\Gamma}_{\infty}{}_{a}{}^{a}_{k} \frac{\delta}{\delta a_k^a} + \mathbb{\Gamma}_{\infty}{}_{p}{}^{k}_{a} \frac{\delta}{\delta p_a^k} \;\textit{(up to the addition of elements in } \ker\Omega_\infty^\Sigma)\,, \tag{4.166}$$

where:

$$\mathbb{\Gamma}_{\infty}{}_{a}{}^{a}_{k} \;=\; 2\delta_{jk} G^{ab} p_b^j\,, \qquad \mathbb{\Gamma}_{\infty}{}_{p}{}^{k}_{a} \;=\; \frac{1}{2}\delta^{km}\delta^{jl} G_{ab} \nabla_j \nabla_l a_m^b\,. \tag{4.167}$$

Thus, we obtain that solutions of our pre-symplectic Hamiltonian system are the integral curves of any of the $\mathbb{\Gamma}_\infty$ above, i.e., solutions of:

$$\frac{d}{ds} a_k^a \;=\; 2\delta_{jk} G^{ab} p_b^j\,, \qquad \frac{d}{ds} p^k \;=\; \frac{1}{2}\delta^{km}\delta^{jl} G_{ab} \nabla_j \nabla_l a_m^b\,, \tag{4.168}$$

with a_0 and $\lambda_{\beta_{jk}}$ remaining completely undetermined, immersed back into $\mathcal{F}_{\mathcal{P}(\mathbb{E})}^\Sigma$, i.e., provided with the constraints (4.152) *and* (4.156). *It is immediate to see that the combination of the equations above together with the constraints are equivalent, provided one identify the parameter s with the coordinate transversal to Σ, to Yang-Mills equations we wrote in the end of example 3.2.15.*

Example 4.2.13 (Palatini's Gravity)**.** *Within Palatini's Gravity developed in example 3.2.16, we saw that the extended space of dynamical fields used to describe it as a Hamiltonian theory with additional constraints is $\mathcal{F}_{\mathcal{P}(\mathbb{E})} \times \mathcal{F}_{\mathcal{P}(\mathbb{E})}^\star \times \mathscr{E}$, where $\mathcal{F}_{\mathcal{P}(\mathbb{E})}$ is the space of dynamical fields of a Yang-Mills theory with gauge group $O(1,3)$ and $\mathscr{E}$ is the space of tetrads on the space-time. As usual, we consider a slice $\Sigma = \{\, x^0 = \mathbb{X}_\Sigma^0 \,\}$ such that, around Σ, $vol_{\mathscr{M}} = \mathrm{d}x^0 \wedge vol_\Sigma$ where vol_Σ is a volume form on Σ. The extended space of fields restricted to Σ reads $\mathcal{F}_{\mathcal{P}(\mathbb{E})}^\Sigma \times \mathcal{F}_{\mathcal{P}(\mathbb{E})}^\Sigma \times \mathscr{E}^\Sigma$ whose elements are* *denoted by* $\overline{\chi}_\Sigma = \Big(A^{IJ}_\mu\Big|_\Sigma,\, P^{\mu 0}_{IJ}\Big|_\Sigma,\, P^{0k}_{IJ}\Big|_\Sigma,\, P^{jk}_{IJ}\Big|_\Sigma,\, \Lambda_{A}{}^{\mu}_{IJ}\Big|_\Sigma,\, \Lambda_{P_{0\mu}}{}^{IJ}\Big|_\Sigma,\, \Lambda_{P_{jk}}{}^{IJ}\Big|_\Sigma,\, e^\mu_I\Big|_\Sigma \Big) = \Big(a^{IJ}_\mu(\underline{x}),\, p^\mu_{IJ}(\underline{x}),\, \tilde{p}^k(\underline{x}),\, \beta^{jk}_{IJ}(\underline{x}),\, \lambda_{p_{0\mu}}{}^{IJ}(\underline{x}),\, \lambda_{\beta_{jk}}{}^{IJ}(\underline{x}),\, e^\mu_I(\underline{x}) \Big)$ *where* $\underline{x}$ *denotes a point in* Σ *and with a slight abuse of notation we still denote tetrads restricted to* Σ *by* e*. The* 2*-form* Ω^Σ *is computed to be:*

$$\Omega^{\Sigma\mathrm{ext}} \,=\, \delta a^{IJ}_\mu \wedge \delta p^\mu_{IJ}\,, \tag{4.169}$$

whereas the Hamiltonian functional (4.125) *is:*

$$\begin{aligned}\mathcal{H}^{\mathrm{ext}}_{\overline{\chi}_\Sigma} \,=\, \int_\Sigma \Big[& p^0_{IJ}\partial_0 a^{IJ}_0 + p^k_{IJ}\nabla_k a^{IJ}_0 + 2\beta^{IJ}_{kj}\nabla_j a^{IJ}_k - 2p^0_{IJ}[a_0,\, a_0]^{IJ} + \lambda_{p_\mu}{}^{IJ}(p^\mu_{IJ} - \epsilon e^{[\mu}_I e^{0]}_J) + \\ & + \lambda_{p_0}{}^{IJ} p^0_{IJ} + \lambda_{\tilde{p}_k}{}^{IJ}(\tilde{p}^k_{IJ} - \epsilon e^{[0}_I e^{k]}_J) + \lambda_{\beta_{jk}}{}^{IJ}(\beta^{jk}_{IJ} - \epsilon e^{[j}_I e^{k]}_J) \Big] vol_\Sigma\,,\end{aligned} \tag{4.170}$$

where $[\,\cdot\,,\,\cdot\,]$ *denotes the Lie product of the Lie algebra of the structure group. The kernel of* $\Omega^{\Sigma\mathrm{ext}}$ *is seen to be:*

$$\ker\Omega^{\Sigma\mathrm{ext}} \,=\, \langle \left\{ \frac{\delta}{\delta\tilde{p}^k_{IJ}},\, \frac{\delta}{\delta\beta^{jk}_{IJ}},\, \frac{\delta}{\delta e^0_I},\, \frac{\delta}{\delta e^k_I},\, \frac{\delta}{\delta\lambda_{p_\mu}{}^{IJ}},\, \frac{\delta}{\delta\lambda_{\tilde{p}_k}{}^{IJ}},\, \frac{\delta}{\delta\lambda_{\beta_{jk}}{}^{IJ}} \right\} \rangle\,. \tag{4.171}$$

The first manifold of the pre-symplectic constraint algorithm is obtained by imposing:

$$i_{\mathbb{V}}\mathrm{d}\mathcal{H}^{\mathrm{ext}} \,=\, 0 \quad \forall\ \mathbb{V} \in \ker\Omega^{\Sigma\mathrm{ext}}\,, \tag{4.172}$$

which explicitly gives the following conditions:

$$\begin{aligned} \nabla_k a_0 \,&=\, -\lambda_{\tilde{p}_k}{}^{IJ}\,, \qquad \nabla_k a^{IJ}_j \,=\, -\lambda_{\beta_{jk}}{}^{IJ}\,, \\ e^k_J\lambda_{\tilde{p}_k}{}^{IJ} \,&=\, 0\,, \qquad e^0_J(\lambda_{p_j}{}^{IJ} - \lambda_{\tilde{p}_j}{}^{IJ}) + 2e^k_J\lambda_{\beta_{jk}}{}^{IJ} \,=\, 0\,, \\ p^\mu_{IJ} \,&=\, \epsilon e^{[\mu}_I e^{0]}_J\,, \qquad \tilde{p}^k_{IJ} \,=\, \epsilon e^{[0}_I e^{k]}_J\,, \qquad \beta^{jk}_{IJ} \,=\, \epsilon e^{[j}_I e^{k]}_J\,. \end{aligned} \tag{4.173}$$

These constraints allows to eliminate the fields p^μ_{IJ}*,* $\tilde{p}^k_{IJ}$*,* β^{jk}_{IJ}*,* $\lambda_{\tilde{p}_j}{}^{IJ}$ *and* $\lambda_{\beta_{jk}}{}^{IJ}$ *and, therefore, the manifold* $\mathcal{M}_1$ *is:*

$$\begin{aligned}\mathcal{M}_1 \,=\, \Big\{ \Big(a^{IJ}_\mu,\, e^\mu_I,\, \lambda_{p_\mu}{}^{IJ} \Big) \ :\ & \nabla_k a^{IJ}_0 \,=\, \lambda_{p_k}{}^{IJ},\, \nabla_j a^{IJ}_k \,=\, \lambda_{\beta_{jk}}{}^{IJ}, \\ & e^k_J\lambda_{p_k}{}^{IJ} \,=\, 0\,, e^0_J\lambda_{p_j}{}^{IJ} + e^k_J\lambda_{\beta_{jk}}{}^{IJ} \,=\, 0 \Big\}\,,\end{aligned} \tag{4.174}$$

and its immersion into $\mathcal{M}$ *is given by the remaining conditions:*

$$p^\mu_{IJ} \,=\, \epsilon e^{[\mu}_I e^{0]}_J\,, \qquad \tilde{p}^k_{IJ} \,=\, \epsilon e^{[0}_I e^{k]}_J\,, \qquad \beta^{jk}_{IJ} \,=\, \epsilon e^{[j}_I e^{k]}_J\,. \tag{4.175}$$

$\mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}_1^\perp$ *is computed to be:*

$$\mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}_1^\perp \,=\, \langle \left\{ \frac{\delta}{\delta a^{IJ}_0} \right\} \rangle \oplus \ker\Omega^{\Sigma\mathrm{ext}}\Big|_{\mathcal{M}_1}\,, \tag{4.176}$$

and, consequently, the second manifold $\mathcal{M}_2$ *is obtained by imposing:*

$$\mathfrak{i}_1^\star \left[i_{\frac{\delta}{\delta a^{IJ}_0}} \Omega^{\Sigma\mathrm{ext}} \right] \,=\, 0\,, \tag{4.177}$$

which explicitly reads:

$$\nabla_k(\epsilon e_I^{[k} e_J^{0]}) \,=\, 0\,. \tag{4.178}$$

Thus:

$$\begin{aligned}\mathcal{M}_2 \,=\, \Big\{ \Big(a_\mu^{IJ},\, e_I^\mu,\, \lambda_{p_k}^{\;IJ}\Big) \;:\; & \nabla_k a_0^{IJ} \,=\, \lambda_{p_k}^{\;IJ},\, \nabla_j a_k^{IJ} \,=\, \lambda_{\beta_{jk}}^{\;IJ}, \\ & e_J^k \lambda_{p_k}^{\;IJ} \,=\, 0\,,\;\; e_J^0 \lambda_{p_j}^{\;IJ} + e_J^k \lambda_{\beta_{jk}}^{\;IJ} \,=\, 0 \\ & \nabla_k(\epsilon e_I^{[k} e_J^{0]}) \,=\, 0 \Big\}\,.\end{aligned} \tag{4.179}$$

The vector fields:

$$(\nabla_k a_0^{IJ} - \lambda_{p_k}^{\;IJ}) \frac{\delta}{\delta a_k^{IJ}}, \qquad -\Big(\nabla_j(\epsilon e_I^{[j} e_J^{k]}) + [p^k, a_0]_{IJ}\Big) \frac{\delta}{\delta p^k_{IJ}} \tag{4.180}$$

are computed to belong to $\mathbf{T}_{\overline{\mathbb{X}}_\Sigma}\mathcal{M}_2^\perp$ but, since they preserve the Hamiltonian functional, no other constraints emerge and $\mathcal{M}_2 \,=\, \mathcal{M}_\infty$. It is a matter of direct computation to show that $\mathfrak{i}_2^\star \mathcal{H}^{\text{ext}} \,=\, 0$. Therefore, on the final manifold the canonical equation reads:

$$\Big(i_{\mathbb{\Gamma}} \Omega^{\Sigma^{\text{ext}}} - \mathrm{d}\mathcal{H}^{\text{ext}} \Big)_{\mathcal{M}_\infty} \,=\, 0\,, \tag{4.181}$$

which, since, as we said above, $\mathfrak{i}_\infty^\star \mathcal{H}^{\text{ext}} \,=\, 0$, reduce to:

$$\Big(i_{\mathbb{\Gamma}} \Omega^{\Sigma^{\text{ext}}} \Big)_{\mathcal{M}_\infty} \,=\, 0\,. \tag{4.182}$$

This means that the dynamics lies, at each point, entirely in $\mathbf{T}_{\overline{\mathbb{X}}_\Sigma}\mathcal{M}_\infty^\perp$. Therefore, solutions of the original pre-symplectic system are the integral curves (restricted to $\mathcal{M}_\infty$) of a vector field $\mathbb{\Gamma}$ that, at each point of $\mathcal{M}$ lies in $\mathbf{T}_{\overline{\mathbb{X}}_\Sigma}\mathcal{M}_\infty^\perp$.

Now, $\mathbf{T}_{\overline{\mathbb{X}}_\Sigma}\mathcal{M}_\infty^\perp$ contains a part of vectors being tangent to $\mathcal{M}_\infty$ and a part of vectors being tangent to $\mathcal{M}$ but not tangent to $\mathcal{M}_\infty$, we will denote them respectively by:

$$\mathbf{T}_{\overline{\mathbb{X}}_\Sigma}\mathcal{M}_\infty^\perp \,=\, \mathbf{T}_{\overline{\mathbb{X}}_\Sigma}\mathcal{M}_\infty^{\perp\,\parallel} \bigcup \mathbf{T}_{\overline{\mathbb{X}}_\Sigma}\mathcal{M}_\infty^{\perp\,\perp}\,. \tag{4.183}$$

We already found the "transversal" part, because it is spanned by the tangent vectors in (4.180)*. The integral curves of a vector field which at each point of $\mathcal{M}$ coincides with $\mathbb{X}$ are the solutions of:*

$$\begin{aligned}\frac{da_k^{IJ}}{ds} \,&=\, \nabla_k a_0^{IJ} - \lambda_{p_k}^{\;IJ}\,, \\ \frac{dp^k_{IJ}}{ds} \,&=\, -\nabla_j \Big(\epsilon e^j_{[I} e^k_{J]}\Big) - \Big[p^k,\, a_0\Big]_{IJ}\,.\end{aligned} \tag{4.184}$$

Their restriction to $\mathcal{M}_\infty$ are the solutions of the combination of (4.184) *and the constraints selecting $\mathcal{M}_\infty$, which reads:*

$$\nabla_\mu \Big(\epsilon e^\mu_{[I} e^\nu_{J]} \Big) \,=\, 0\,, \qquad e_I^\mu F_{\mu\nu}^{IJ} \,=\, 0\,, \tag{4.185}$$

where $F_{\mu\nu}^{IJ} \,=\, \nabla_\mu A_\nu^{IJ}$. The first of the latter equations is the condition for the metric g associated with e to be torsion-less, whereas the second set of equations are Einstein equations in vacuum (see [91]).

Regarding the remaining part of $\mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}_\infty^\perp$, say the tangent one $\mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}_\infty^{\perp\,\|}$, we will show how it is made by the generators of gauge transformations of the theory. It is a matter of direct computation to prove that the kernel of $\Omega^{\Sigma^{\rm ext}}_\infty$ is spanned, at each point, by the infinity (parametrized by Lie algebra-valued functions ψ^{IJ}) of tangent vectors $\mathbb{X}^G_\psi$ with components:

$$\left(\mathbb{X}^G_\psi\right)_{ak}^{IJ} = \nabla_k\psi^{IJ}\,, \qquad \left(\mathbb{X}^G_\psi\right)_{eI}^{\mu} = \psi^K_I e^\mu_K\,, \tag{4.186}$$

and by the infinity (parametrized by vector fields on Σ) of tangent vectors $\mathbb{X}^D$ with components:

$$\left(\mathbb{X}^D_\xi\right)_{ak}^{IJ} = \xi^j\nabla_k a^{IJ}_j\,, \qquad \left(\mathbb{X}^D_\xi\right)_{eI}^{0} = -\xi^j\nabla_j e^0_K\,, \qquad \left(\mathbb{X}^D_\xi\right)_{eI}^{k} = -\xi^j\nabla_j e^k_I\,, \tag{4.187}$$

where ξ^j are the components of a vector field on Σ. The vector fields which, at each point, coincide with the tangent vectors $\mathbb{X}^G$ satisfy:

$$[\mathbb{X}^G_\psi,\, \mathbb{X}^G_\phi] = \mathbb{X}^G_{[\psi,\phi]} \tag{4.188}$$

and, therefore, they are a representation of the Lie algebra $\mathfrak{o}(1,3)$ on $\mathcal{M}_\infty$ given by the following action:

$$\begin{aligned} a^{IJ}_\mu &\mapsto \psi^I_K a^{KL}_\mu {\psi^{-1}}^{LJ} + {\psi^{-1}}^J{}_K\partial_\mu\psi^{KJ}\,, \\ e^\mu_I &\mapsto \psi^K_I e^\mu_K\,, \end{aligned} \tag{4.189}$$

which agree with the gauge transformation written in [91, Chap. 2, page 41] (this justifies the notation $\mathbb{X}^G$). On the other hand, the vector fields which, at each point, coincide with the tangent vectors $\mathbb{X}^D$ satisfy:

$$[\mathbb{X}^D_\xi,\, \mathbb{X}^D_\zeta] = \mathbb{X}_{[\xi,\zeta]}\,, \tag{4.190}$$

provided that ξ and ζ are divergenceless and, therefore, they are a representation of the group of volume-preserving diffeomorphisms (which justifies the notation $\mathbb{X}^D$) of Σ on $\mathcal{M}_\infty$ given by the following action:

$$\begin{aligned} a^{IJ}_k &\mapsto \left(\mathscr{T}^\xi_\nabla a\right)^{IJ}_k\,, \\ e^\mu_I &\mapsto -\left(\mathscr{T}^\xi_\nabla e\right)^\mu_I\,, \end{aligned} \tag{4.191}$$

where $\mathscr{T}^\xi_\nabla$ is the parallel transport along the flow of ξ associated with the connection ∇. The tangent vectors $\mathbb{X}^G$ and $\mathbb{X}^D$ are $\mathfrak{i}_\infty$-related with the tangent vectors to the original manifold of the pre-symplectic constraint algorithm having, respectively, components:

$$\left(\mathbb{X}^G_\psi\right)_{ak}^{IJ} = \nabla_k\psi^{IJ}\,, \qquad \left(\mathbb{X}^G_\psi\right)_{pIJ}^{k} = \epsilon\left(U^L_{[I}\psi^K_L e^0_K e^k_{J]} - W^{kL}_{j[J}e^0_{I]}\psi^K_L e^j_K\right) \tag{4.192}$$

and:

$$\left(\mathbb{X}^D_\xi\right)_{ak}^{IJ} = \xi^j F^{IJ}_{kj}\,, \qquad \left(\mathbb{X}^D_\xi\right)_{pIJ}^{k} = -\left(U^K_{[I}e^k_{J]}\xi^j\nabla_j e^0_K - W^{kK}_{j[J}e^0_{I]}\xi^l\nabla_l e^j_K\right)\,, \tag{4.193}$$

where $U^L_{[I} = \delta^L_{[I} - e^0_{[I} e^L_0$, $W^{kL}_{j[J} = \delta^k_j \delta^L_{[J} + e^L_j e^k_{[J}$ *and being actually tangent to* $\mathcal{M}_\infty$*. The latter tangent vectors span, at each point,* $\mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}^{\perp\,\|}_\infty$ *and, therefore, the whole* $\mathbf{T}_{\overline{\chi}_\Sigma}\mathcal{M}^\perp_\infty$ *is now characterized.*

To resume the above discussion, we saw that solutions of the pre-symplectic Hamiltonian system $\left(\mathcal{F}^\Sigma_{\mathcal{P}(\mathbb{E})} \times \mathcal{F}^{\Sigma\,\star}_{\mathcal{P}(\mathbb{E})} \times \mathscr{E},\, \Omega^{\Sigma\mathrm{ext}},\, \mathcal{H}^{\mathrm{ext}}\right)$*, which, at least locally, coincide with extrema of the tetradic Palatini's action, are integral curves of the vector field* (4.180) *restricted to the final manifold* $\mathcal{M}_\infty$ *of the PCA, that is, torsionless solutions of Einstein's equations (see* (4.185)*), up to gauge transformations* (4.189) *and* (4.191) *obtained as the tangent part of the kernel of the pre-symplectic form.*

4.2.7. Symmetries and momentum maps on the solution space

Since we have shown that, at least locally around a slice Σ of the space-time, the dynamical content of field theories is encoded into a pre-symplectic Hamiltonian system, we can now apply the theory of Sec. 2.2.3 and 2.3.5 to show the correspondence between symmetries and momentum maps (that is, conserved quantities) within this setting. In particular, to show the procedure both in the symplectic and in the pre-symplectic case, we will apply it to the case of Klein-Gordon theory and to free Electrodynamics in the Hamiltonian formalism.

Example 4.2.14 (Klein-Gordon theory)**.** *We saw that, given a slice* Σ *of the Minkowski space-time* $\mathscr{M}$*, Klein-Gordon theory can be formulated in terms of the pre-symplectic Hamiltonian system* $(\mathcal{M}_\infty,\, \Omega^\Sigma_\infty,\, \mathcal{H}_\infty)$*, where:*

$$\mathcal{M}_\infty \;=\; \mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma) \times \mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\,, \tag{4.194}$$

whose elements are denoted by $(\varphi,\, p)$*,*

$$\Omega^\Sigma_\infty \;=\; \delta\varphi \wedge \delta p\,, \tag{4.195}$$

and:

$$\mathcal{H}_\infty \;=\; -\frac{1}{2}\int_\Sigma \left[\, \delta^{jk}\partial_j\varphi\partial_k\varphi + p^2 + m^2\varphi^2 \,\right]\, vol_\Sigma\,. \tag{4.196}$$

We consider the group of symmetries of the space-time $\mathscr{M}$*, that is, the Poincaré group* $\mathscr{P} = \tau \rtimes \mathcal{L}$*, which contains the subgroup of translations, denoted by* τ*, spatial rotations, denoted by* $\mathscr{R}$ *and the Lorentz boosts, denoted by* $\mathscr{B}$*.* $\mathscr{P}$ *can be represented via the following action on* $\mathscr{M}$ *by means of* 5×5 *matrices:*

$$\Phi^{\mathscr{M}}_\tau \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 & a^0 \\ 0 & 1 & 0 & 0 & a^1 \\ 0 & 0 & 1 & 0 & a^2 \\ 0 & 0 & 0 & 1 & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\,, \tag{4.197}$$

$$\Phi^{\mathscr{M}}_{\mathscr{R}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \mathscr{R}_{11} & \mathscr{R}_{12} & \mathscr{R}_{13} & 0 \\ 0 & \mathscr{R}_{21} & \mathscr{R}_{22} & \mathscr{R}_{23} & 0 \\ 0 & \mathscr{R}_{31} & \mathscr{R}_{32} & \mathscr{R}_{33} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} , \tag{4.198}$$

$$\Phi^{\mathscr{M}}_{\mathscr{B}} = \begin{bmatrix} \mathscr{B}_{00} & \mathscr{B}_{01} & \mathscr{B}_{02} & \mathscr{B}_{03} & 0 \\ \mathscr{B}_{10} & \mathscr{B}_{11} & \mathscr{B}_{12} & \mathscr{B}_{13} & 0 \\ \mathscr{B}_{20} & \mathscr{B}_{21} & \mathscr{B}_{22} & \mathscr{B}_{23} & 0 \\ \mathscr{B}_{30} & \mathscr{B}_{31} & \mathscr{B}_{32} & \mathscr{B}_{33} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} , \tag{4.199}$$

obtained by taking the first 4 components of the action of the elements in the Poincaré group on the point $(x^0,\, x^1,\, x^2,\, x^3,\, 1)$*:*

$$\Phi^{\mathscr{M}}_{\tau} \cdot m = \begin{bmatrix} 1 & 0 & 0 & 0 & a^0 \\ 0 & 1 & 0 & 0 & a^1 \\ 0 & 0 & 1 & 0 & a^2 \\ 0 & 0 & 0 & 1 & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{bmatrix} = \begin{bmatrix} x^0 + a^0 \\ x^1 + a^1 \\ x^2 + a^2 \\ x^3 + a^3 \\ 1 \end{bmatrix} , \tag{4.200}$$

$$\Phi^{\mathscr{M}}_{\mathscr{R}} \cdot m = \begin{bmatrix} 1 & \mathbf{0} & 0 \\ 0 & \mathscr{R}^j_k & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^k \\ 1 \end{bmatrix} = \begin{bmatrix} x^0 \\ \mathscr{R}^j_k x^k \\ 1 \end{bmatrix} , \tag{4.201}$$

where $\mathscr{R}^j_k$ *is a matrix in* $O(3)$*, and:*

$$\Phi^{\mathscr{M}}_{\mathscr{B}} \cdot m = \begin{bmatrix} \mathscr{B}^{\mu}_{\nu} & 0 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} x^{\mu} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathscr{B}^{\mu}_{\nu} x^{\nu} \\ 1 \end{bmatrix} , \tag{4.202}$$

where $\mathscr{B}^{\mu}_{\nu}$ *is a matrix in* $SO(1,3)$*. The actions above can be lifted to the bundle* $\mathbb{E} \to \mathscr{M}$*, to* $\mathcal{P}(\mathbb{E})$ *(see [94]), and to* $\mathcal{EL}$ *in the following way:*

$$\Phi^{\mathcal{P}(\mathbb{E})}_{\tau} \;:\; \mathcal{P}(\mathbb{E}) \to \mathcal{P}(\mathbb{E}) \;:\; (x^{\mu},\, u,\, \rho^{\mu}) \mapsto \Phi^{\mathcal{P}(\mathbb{E})}_{\tau}(x^{\mu},\, u,\, \rho^{\mu}) = (x^{\mu} + a^{\mu},\, u,\, \rho^{\mu}) , \tag{4.203}$$

$$\Phi^{\mathcal{P}(\mathbb{E})}_{\mathscr{R}} \;:\; \mathcal{P}(\mathbb{E}) \to \mathcal{P}(\mathbb{E}) \;:\; (x^{\mu},\, u,\, \rho^{\mu}) \mapsto \Phi^{\mathcal{P}(\mathbb{E})}_{\mathscr{R}}(x^{\mu},\, u,\, \rho^{\mu}) = \left(x^0,\, \mathscr{R}^j_k x^k,\, u,\, \rho^0,\, \mathscr{R}^j_k \rho^k\right) , \tag{4.204}$$

$$\Phi^{\mathcal{P}(\mathbb{E})}_{\mathscr{B}} \;:\; \mathcal{P}(\mathbb{E}) \to \mathcal{P}(\mathbb{E}) \;:\; (x^{\mu},\, u,\, \rho^{\mu}) \mapsto \Phi^{\mathcal{P}(\mathbb{E})}_{\mathscr{B}}(x^{\mu},\, u,\, \rho^{\mu}) = (\mathscr{B}^{\mu}_{\nu} x^{\nu},\, u,\, \mathscr{B}^{\mu}_{\nu} \rho^{\nu}) , \tag{4.205}$$

$$\begin{aligned} \Phi^{\mathcal{EL}}_{\tau} \;:\; \mathcal{EL} \to \mathcal{EL} \;:\; (x^{\mu},\, \phi(x^{\mu}),\, P^{\mu}(x^{\mu})) &\mapsto \Phi^{\mathcal{EL}}_{\tau}(x^{\mu},\, \phi(x^{\mu}),\, P^{\mu}(x^{\mu})) = \\ &= (x^{\mu} + a^{\mu},\, \phi(x^{\mu} + a^{\mu}),\, P^{\mu}(x^{\mu} + a^{\mu})) , \end{aligned} \tag{4.206}$$

$$\begin{aligned} \Phi^{\mathcal{EL}}_{\mathscr{R}} \;:\; \mathcal{EL} \to \mathcal{EL} \;:\; (x^{\mu},\, \phi(x^{\mu}),\, P^{\mu}(x^{\mu})) &\mapsto \Phi^{\mathcal{EL}}_{\mathscr{R}}(x^{\mu},\, \phi(x^{\mu}),\, P^{\mu}(x^{\mu})) = \\ &= \left(x^0,\, \mathscr{R}^j_k x^k,\, \phi(x^0,\, \mathscr{R}^j_k x^k),\, P^0(x^0,\, \mathscr{R}^j_k P^k(x^0,\, \mathscr{R}^j_k x^k),\, \mathscr{R}^j_k x^k)\right) , \end{aligned} \tag{4.207}$$

$$\Phi^{\mathcal{EL}}_{\mathscr{B}} \;:\; \mathcal{EL} \to \mathcal{EL} \;:\; (x^\mu,\, \phi(x^\mu),\, P^\mu(x^\mu)) \mapsto \Phi^{\mathcal{EL}}_{\mathscr{B}}(x^\mu,\, \phi(x^\mu),\, P^\mu(x^\mu)) = \\ = (\mathscr{B}^\mu_\nu x^\nu,\, \phi(\mathscr{B}^\mu_\nu x^\nu),\, \mathscr{B}^\mu_\nu P^\nu(\mathscr{B}^\mu_\nu x^\nu))\,. \tag{4.208}$$

In particular we are interested in the action on $\mathcal{EL}$. That they represent symmetries for our pre-symplectic Hamiltonian system can be verified in two ways. The first is by computing the generators of the transformations above, that are vector fields on $\mathcal{EL}$, by taking their push-forward via $\Psi^{-1} \circ \varpi$, that are vector fields on $\mathcal{M}_\infty$ and by proving that they preserve $\mathcal{H}_\infty$ and the pre-symplectic form. The second way, which is the more viable is to take the pull-back of $\mathcal{H}_\infty$ and the pre-symplectic form to $\mathcal{EL}$ via $\varpi^{-1} \circ \Psi$ and by checking the invariance under the generators on $\mathcal{EL}$. By looking at diagram (4.108) *this is the same as taking the pull-back of $\mathcal{H}$ via Π_Σ and checking its invariance. Straightforward calculations show that this is the case for all the transformations above.*

To compute the momentum maps associated to the symmetries above one could proceed, again, in two ways. First, by pushing-forward the generators of the symmetries on $\mathcal{EL}$ to $\mathcal{M}_\infty$ and, then, by computing the momentum maps associated to the symplectic structure Ω^Σ_∞. The second one is to pull-back Ω^Σ_∞ to $\mathcal{EL}$, which gives the canonical structure $\Pi^\star_\Sigma \Omega^\Sigma$, and computing the momentum maps associated with it. We will use this second option in the present example.

Consider $\xi = \xi_{\tau^0}$, i.e, the element in the Lie algebra $\mathfrak{g} = \mathbb{R}^4 \rtimes \mathfrak{so}(1,3)$ of the Poincaré group $\mathscr{P}$, generating the translation $x^\mu \mapsto (x^0 + a^0,\, x^j)$. The generator associated reads:

$$\mathbb{X}_{\xi_{\tau^0}} = \frac{d}{da^0}\left[\Phi_{e^{a^0 \xi_{\tau^0}}} \cdot \chi\right]_{a^0=0} = -P^0 \frac{\delta}{\delta\phi} - (m^2\phi + \Delta\phi)\frac{\delta}{\delta P^0} - \frac{\partial P^0}{\partial x^{(j)}}\frac{\delta}{\delta P^{(j)}}\,, \tag{4.209}$$

where (j) means that no sums must be taken over the index. Consequently:

$$\Pi^\star_\Sigma \Omega^\Sigma(\mathbb{X}_{\xi_{\tau^0}}, \cdot\,) = -P^0 \delta P^0 + (m^2\phi + \Delta\phi)\delta\phi\,. \tag{4.210}$$

Therefore, the function $J_{\xi_{\tau^0}}$ satisfying:

$$\Pi^\star_\Sigma \Omega^\Sigma(\mathbb{X}_{\xi_{\tau^0}}, \cdot\,) = \mathrm{d}J_{\xi_{\tau^0}}\,, \tag{4.211}$$

can be straightforwardly computed to be:

$$J_{\xi_{\tau^0}} = \int_\Sigma \frac{1}{2}\left[m^2\phi^2 - \delta^{jk}\frac{\partial\phi}{\partial x^j}\frac{\partial\phi}{\partial x^k} - {P^0}^2\right]_\Sigma vol_\Sigma\,. \tag{4.212}$$

A similar computation can be performed for the elements $\xi_{\tau^k} \in \mathfrak{g}$ generating the translation $x^\mu \mapsto (x^0,\, x^j + \delta^j_k a^k)$, obtaining the function $J_{\xi_{\tau^k}}$:

$$J_{\xi_{\tau^k}} = \int_\Sigma P^0 \frac{\partial\phi}{\partial x^k}\bigg|_\Sigma vol_\Sigma\,. \tag{4.213}$$

Note that $J_{\xi_{\tau^0}}$ and $J_{\xi_{\tau^k}}$ coincide with the charges associated with the energy-momentum tensor of the Klein-Gordon theory. The momentum map is the map between $\mathcal{EL}$ and $\mathfrak{g}^\star$ satisfying:

$$\mathbb{J} \;:\; \mathcal{EL} \to \mathfrak{g}^\star \;:\; \langle\, \mathbb{J}(\chi)\,,\, \xi\,\rangle \,=\, J_\xi\,. \tag{4.214}$$

Let us compute it in the case $\xi \,=\, \xi_{\tau^0}$. The element $\xi_{\tau^0} \in \mathfrak{g}$ such that $e^{a^0 \xi_{\tau^0}} \,=\, \Phi^{\mathscr{M}}_{\tau^0}$ is represented as:

$$\Phi_{\xi_{\tau^0}} \,=\, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\,. \tag{4.215}$$

Therefore, if we represent $\mathbb{J}(\chi)$ as:

$$\Phi_{\mathbb{J}} \,=\, \begin{bmatrix} \mathbb{J}_{00}(\chi) & \cdots & \mathbb{J}_{04}(\chi) \\ \vdots & & \vdots \\ \mathbb{J}_{40}(\chi) & \cdots & \mathbb{J}_{44}(\chi) \end{bmatrix}\,, \tag{4.216}$$

we get:

$$\langle\, \mathbb{J}(\chi)\,,\, \xi_{\tau^0}\,\rangle \,=\, \mathrm{Tr}\left[\, {\Phi_{\mathbb{J}}}^\dagger\, \Phi^{\mathscr{M}}_{\xi_{\tau^0}}\,\right] \,=\, \mathbb{J}_{04}(\chi)\,, \tag{4.217}$$

and, consequently:

$$\mathbb{J}_{04}(\chi) \,=\, J_{\xi_{\tau^0}}\,. \tag{4.218}$$

A similar computation for $\xi \,=\, \xi_{\tau^k}$ gives:

$$\mathbb{J}_{0(4-k)}(\chi) \,=\, J_{\xi_{\tau^k}}\,. \tag{4.219}$$

Analogous computations can be done to compute the momentum maps associated to the other transformations in the Poincarè group. Here we report the results for the sake of completeness. The conserved quantities associated to the three spatial rotations are:

$$\begin{aligned} J_{\xi_{\mathscr{R}^3}} \,&=\, \int_\Sigma \left[P^0 \left(x^2 \frac{\partial \phi}{\partial x^1} - x^1 \frac{\partial \phi}{\partial x^2} \right) \right]_\Sigma vol_\Sigma\,, \\ J_{\xi_{\mathscr{R}x^2}} \,&=\, \int_\Sigma \left[P^0 \left(x^1 \frac{\partial \phi}{\partial x^3} - x^3 \frac{\partial \phi}{\partial x^1} \right) \right]_\Sigma vol_\Sigma\,, \\ J_{\xi_{\mathscr{R}x^1}} \,&=\, \int_\Sigma \left[P^0 \left(x^2 \frac{\partial \phi}{\partial x^3} - x^3 \frac{\partial \phi}{\partial x^2} \right) \right]_\Sigma vol_\Sigma\,. \end{aligned} \tag{4.220}$$

Note that $J_{\xi_{\mathscr{R}x^j}}$ are the j-components (j going from 1 to 3) of the divergence of the generalized angular momentum of the Klein-Gordon field. Regarding the momentum maps, by using the definition one gets:

$$\mathbb{J}(\chi)_j \,=\, J_{\xi_{\mathscr{R}x^j}} \quad j \,=\, 1, 2, 3\,. \tag{4.221}$$

Regarding the three Lorentz boosts, one gets the following conserved functions :

$$\begin{aligned} J_{\xi_{\mathscr{B}x^1}} &= -\int_\Sigma \left[P^0 \left(x^1 \frac{\partial\phi}{\partial x^0} + x^0 \frac{\partial\phi}{\partial x^1} \right) + P^1 \phi \right]_\Sigma vol_\Sigma \\ J_{\xi_{\mathscr{B}x^2}} &= -\int_\Sigma \left[P^0 \left(x^2 \frac{\partial\phi}{\partial x^0} + x^0 \frac{\partial\phi}{\partial x^2} \right) + P^2 \phi \right]_\Sigma vol_\Sigma \\ J_{\xi_{\mathscr{B}x^3}} &= -\int_\Sigma \left[P^0 \left(x^3 \frac{\partial\phi}{\partial x^0} + x^0 \frac{\partial\phi}{\partial x^3} \right) + P^3 \phi \right]_\Sigma vol_\Sigma \,, \end{aligned} \tag{4.222}$$

which are the remaining three independent components of the divergence of the generalized angular momentum usually found in Field Theory textbooks. The components of the momentum map in this case are:

$$\mathbb{J}(\chi)_j \,=\, J_{\xi_{\mathscr{B}x^j}} \quad j \,=\, 1, 2, 3\,. \tag{4.223}$$

Example 4.2.15 (Free Electrodynamics). *As it was shown in example 4.2.11, given a slice Σ of the Minkowski space-time $\mathscr{M}$, free Electrodynamics can be formulated in terms of the pre-symplectic Hamiltonian system $(\mathcal{M}_\infty,\, \Omega_\infty^\Sigma,\, \mathcal{H}_\infty)$, where:*

$$\mathcal{M}_\infty \,=\, \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma) \right]^3 \times \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma;\, \mathrm{div0}) \right]^3 \,, \tag{4.224}$$

whose elements are denoted by $(a_k,\, p^k)$,

$$\Omega_\infty^\Sigma \,=\, \delta a_k \wedge \delta p^k \,, \tag{4.225}$$

and:

$$\mathcal{H}_\infty \,=\, \int_\Sigma \left[\frac{1}{4} \delta^{jl} \delta^{km} \partial_{[j} a_{k]} \partial_{[l} a_{m]} - \delta_{jk} p^j p^k \right] vol_\Sigma \tag{4.226}$$

The kernel of Ω_∞^Σ consists of vector fields of the type:

$$\mathbb{V}_\psi \,=\, \partial_k \psi \frac{\delta}{\delta a_k} \,, \tag{4.227}$$

for some $\psi \in \mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)$. They generate (via their flow) the following action on $\mathcal{M}_\infty$:

$$\Phi_{\mathbb{V}_\psi} \;:\; \mathcal{M}_\infty \to \mathcal{M}_\infty \;:\; m_\infty \mapsto \Phi_{\mathbb{V}_\psi} \cdot m_\infty \,=\, (a_k + \partial_k \psi,\, p^k)\,. \tag{4.228}$$

As we will prove in the next section, the tangent space of $\mathcal{M}_\infty$ at some point m_∞ splits as:

$$\mathbf{T}_{m_\infty} \mathcal{M}_\infty \simeq \mathcal{M}_\infty \,=\, \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma;\, \mathrm{div0}) \right]^3 \oplus \mathrm{grad} \mathcal{H}^{\frac{5}{2}} \times \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma;\, \mathrm{div0}) \right]^3 \,, \tag{4.229}$$

the second term representing the kernel of Ω_∞^Σ, say K. On the bundle $\mathcal{M}_\infty \to \mathcal{M}_\infty / K$ we consider the connection represented by the following $(1,\,1)$ tensor field:

$$P \,=\, P_k \otimes \frac{\delta}{\delta a_k} \,, \tag{4.230}$$

where:

$$P_k \,=\, \partial_k \int_\Sigma G_\Delta(\underline{x},\, \underline{y}) \delta^{jl} \partial_j \delta a_l(\underline{y})\, \mathrm{d}^3 y\,, \tag{4.231}$$

with G_Δ being the Green's function of the Laplacian operator, $\underline{x}$, $\underline{y}$ points in Σ and $\{\, \delta a_j \,\}_{j=1,2,3}$ a dual basis of $\left\{\, \frac{\delta}{\delta a_j} \,\right\}_{j=1,2,3}$. Note that the complement of $\mathrm{ker}\Omega^\Sigma_\infty \,=\, \mathrm{grad}\mathcal{H}^{\frac{5}{2}}$ represents exactly the horizontal distribution associated with the connection chosen.

The coisotropic embedding theorem in this case leads to the following symplectic manifold, which is actually a Hilbert space (and, thus isomorphic to its tangent space):

$$\begin{aligned}\tilde{\mathcal{M}} \,\simeq \mathbf{T}_{\tilde{m}} \tilde{\mathcal{M}} \,&=\, \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma;\, \mathrm{div}0) \right]^3 \oplus \mathrm{grad}\mathcal{H}^{\frac{5}{2}} \times \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma;\, \mathrm{div}0) \right]^3 \times \mathrm{grad}\mathcal{H}^{\frac{5}{2}^\star} \,\simeq \\ &\simeq\, \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma;\, \mathrm{div}0) \right]^3 \oplus \mathrm{grad}\mathcal{H}^{\frac{5}{2}} \times \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma;\, \mathrm{div}0) \right]^3 \times \mathrm{grad}\mathcal{H}^{\frac{5}{2}}\,,\end{aligned} \tag{4.232}$$

We denote points in $\tilde{\mathcal{M}}$ by $\tilde{m}_\infty \,=\, (\tilde{a}_k,\, \partial_k\phi,\, p^k,\, \mu^k)$. The symplectic structure $\tilde{\Omega^\Sigma}_\infty$ reads:

$$\tilde{\Omega}(\mathbb{X},\, \mathbb{Y}) \,=\, \underbrace{\int_\Sigma \left(\tilde{\mathbb{X}}_{ak} \mathbb{Y}_p{}^k - \mathbb{X}_p{}^k \tilde{\mathbb{X}}_{ak} \right) vol_\Sigma}_{=\Omega_W(\mathbb{X},\, \mathbb{Y})} + \underbrace{\mathbb{X}_\mu{}^k \partial_k \mathbb{Y}_\psi - \partial_k \mathbb{X}_\psi \mathbb{Y}_\mu{}^k}_{=\Omega_{K\oplus K^\star}(\mathbb{X},\, \mathbb{Y})}\,. \tag{4.233}$$

where $\mathbb{X}^k_\mu$ is the component of $\mathbb{X}$ along the dual of the kernel of Ω^Σ_∞.

Now, as in the previous example, let us consider the group of symmetries of the space-time $\mathscr{M}$, i.e., the Poincaré group. In particular, let us focus on the group of spatial rotations. In this case, since we are working on the manifold $\tilde{\mathcal{M}}$ obtained via the coisotropic embedding theorem applied to $\mathcal{M}_\infty$, we will lift their action to $\mathcal{M}_\infty$ and, then, by using the theory developed in Sec. 2.3.5, to $\tilde{\mathcal{M}}$. Spatial rotations act on $\mathscr{M}$ by means of $SO(3)$ matrices:

$$\Phi^{\mathscr{M}}_{\mathscr{R}} \quad : \quad \mathscr{M} \to \mathscr{M} \quad : \quad x^k \mapsto \Phi^{\mathscr{M}}_{\mathscr{R}}(x)^k \,=\, \mathscr{R}^k_j x^j\,, \tag{4.234}$$

where $\mathscr{R}^k_j$ represent the matrix elements of the $SO(3)$ matrix $\Phi^{\mathscr{M}}_{\mathscr{R}}$. Since the configuration fields, a_k, are 1-forms on $\mathscr{M}$ and the momenta fields, p^k, are contravariant tensors on $\mathscr{M}$, they transform via the pull-back (via $\Phi^{\mathscr{M}}_{\mathscr{R}^{-1}}$) and via the push-forward (via $\Phi^{\mathscr{M}}_{\mathscr{R}}$) respectively. Therefore, the action of $\mathscr{R}$ is lifted to $\mathcal{M}_\infty$ as:

$$\begin{aligned}\Phi^{\mathcal{M}_\infty}_{\mathscr{R}} \quad : \quad \mathcal{M}_\infty \to \mathcal{M}_\infty \quad &: \; (\tilde{a}_k(x),\, \partial_k\phi(x),\, p^k(x)) \mapsto \\ &\mapsto \left({\mathscr{R}^{-1}}^j{}_k \tilde{a}_j \left(\Phi^{\mathscr{M}}_{\mathscr{R}} \cdot x \right),\, {\mathscr{R}^{-1}}^j{}_k \partial_j \phi \left(\Phi^{\mathscr{M}}_{\mathscr{R}} \cdot x \right),\, \mathscr{R}^k_j p^j (\Phi^{\mathscr{M}}_{\mathscr{R}} \cdot x) \right)\,.\end{aligned} \tag{4.235}$$

Following the theory of Sec. 2.3.5 this action is lifted to $\tilde{\mathcal{M}}$ as:

$$\begin{aligned}\Phi^{\mathcal{P}}_{\mathscr{R}} \quad : \quad \mathcal{P} \to \mathcal{P} \quad &: \; \left(\tilde{a}_k(x),\, \partial_k\phi(x),\, p^k(x),\, \mu^k(x) \right) \mapsto \\ &\mapsto \left({\mathscr{R}^{-1}}^j{}_k \tilde{a}_j \left(\Phi^{\mathscr{M}}_{\mathscr{R}} \cdot x \right),\, {\mathscr{R}^{-1}}^j{}_k \partial_j \phi \left(\Phi^{\mathscr{M}}_{\mathscr{R}} \cdot x \right),\, \mathscr{R}^k_j p^j (\Phi^{\mathscr{M}}_{\mathscr{R}} \cdot x),\, \mathscr{R}^k_j \mu^j \left(\Phi^{\mathscr{M}}_{\mathscr{R}} \cdot x \right) \right)\,.\end{aligned} \tag{4.236}$$

Let us focus for a moment on the case in which $\mathscr{R}$ is a rotation around the x^3-axis, denoted, for short, $\mathscr{R}^3$. We prove that the action above is canonical by explicitly constructing the momentum map. The element of the Lie algebra generating rotations around the x^3-axis is $\xi_{\mathscr{R}^3}$ and is an element in $\mathfrak{so}(3)$ represented by the matrix:

$$\Phi_{\xi_{\mathscr{R}^3}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} . \tag{4.237}$$

The generator associated, $\mathbb{X}_{\xi_{\mathscr{R}^3}}$, is:

$$\begin{aligned} \mathbb{X}_{\xi_{\mathscr{R}^3}} = \frac{d}{d\theta}\left[\Phi^{\mathcal{P}}_{e^{\theta\xi_{\mathscr{R}^3}}} \cdot \tilde{m}_\infty\right]_{\theta=0} &= \left(\mathbb{X}_{\xi_{\mathscr{R}^3}}\right)_{\tilde{a}_k} \frac{\delta}{\delta \tilde{a}_k} + \left(\mathbb{X}_{\xi_{\mathscr{R}^3}}\right)_{\partial\psi_k} \frac{\delta}{\delta\partial_k\psi} + \\ &+ \left(\mathbb{X}_{\xi_{\mathscr{R}^3}}\right)^k_p \frac{\delta}{\delta p^k} + \left(\mathbb{X}_{\xi_{\mathscr{R}^3}}\right)^k_\mu \frac{\delta}{\delta \mu^k} , \end{aligned} \tag{4.238}$$

where:

$$\left(\mathbb{X}_{\xi_{\mathscr{R}^3}}\right)_{\tilde{a}_k} = \begin{bmatrix} (x^2\partial_1 - x^1\partial_2)\tilde{a}_1 + \tilde{a}_2 \\ -\tilde{a}_1 + (x^2\partial_1 - x^1\partial_2)\tilde{a}_2 \\ (x^2\partial_1 - x^1\partial_2)\tilde{a}_3 \end{bmatrix} \tag{4.239}$$

$$\left(\mathbb{X}_{\xi_{\mathscr{R}^3}}\right)_{\partial\phi_k} = \begin{bmatrix} (x^2\partial_1 - x^1\partial_2)\partial_1\phi + \partial_2\phi \\ -\partial_1\phi + (x^2\partial_1 - x^1\partial_2)\partial_2\phi \\ (x^2\partial_1 - x^1\partial_2)\partial_3\phi \end{bmatrix} , \tag{4.240}$$

$$\left(\mathbb{X}_{\xi_{\mathscr{R}^3}}\right)^k_p = \begin{bmatrix} (x^2\partial_1 - x^1\partial_2)p^1 + p^2 \\ -p^1 + (x^2\partial_1 - x^1\partial_2)p^2 \\ (x^2\partial_1 - x^1\partial_2)p^3 \end{bmatrix} \tag{4.241}$$

$$\left(\mathbb{X}_{\xi_{\mathscr{R}^3}}\right)^k_\mu = \begin{bmatrix} (x^2\partial_1 - x^1\partial_2)\mu^1 + \mu^2 \\ -\mu^1 + (x^2\partial_1 - x^1\partial_2)\mu^2 \\ (x^2\partial_1 - x^1\partial_2)\mu^3 \end{bmatrix} . \tag{4.242}$$

With this in hand, the function $J_{\xi_{\mathscr{R}^3}}$ satisfying:

$$\tilde{\Omega}(\mathbb{X}_{\xi_{\mathscr{R}^3}}, \cdot) = \mathrm{d}J_{\xi_{\mathscr{R}^3}} , \tag{4.243}$$

is straightforwardly computed to be:

$$\begin{aligned} &J_{\xi_{\mathscr{R}^3}}(\tilde{a}_k, \partial_k\phi, p^k, \mu^k) = \\ &= \int_\Sigma \left[(x^2\partial_1 - x^1\partial_2)\tilde{a}_1 p^1 + \tilde{a}_2 p^1 + (x^2\partial_1 - x^1\partial_2)\tilde{a}_2 p^2 - \tilde{a}_1 p^2 + (x^2\partial_1 - x^1\partial_2)\tilde{a}_3 p^3\right] vol_\Sigma \\ &+ \langle \mu^1 , (x^2\partial_1 - x^1\partial_2)\partial_1\phi - \partial_2\phi\rangle + \langle \mu^2 , (x^2\partial_1 - x^1\partial_2)\partial_2\phi + \partial_1\phi\rangle + \langle \mu^3 , (x^2\partial_1 - x^1\partial_2)\partial_3\phi\rangle . \end{aligned} \tag{4.244}$$

Let us note that the quantities in the right arguments of the pairings are the vertical components of $\mathbb{X}_{\xi_{\mathscr{R}^3}}$ with respect to the connection on $\tilde{\mathcal{M}}$ that extends (constantly along $K^\star$) the connection P chosen. With this in mind the function (4.244) has *exactly the expression* (2.157). *The currents* $J_{\xi_{\mathscr{R}x^1}}$ *and* $J_{\xi_{\mathscr{R}x^2}}$ *can be computed in the same way obtaining:*

$$
\begin{aligned}
J_{\xi_{\mathscr{R}x^1}} = & \int_\Sigma \left[(x^1\partial_3 - x^3\partial_1)\tilde{a}_1 p^1 + (x^1\partial_3 - x^3\partial_1)\tilde{a}_2 p^2 + \tilde{a}_3 p^2 + (x^1\partial_3 - x^3\partial_1)\tilde{a}_3 p^3 - \tilde{a}_2 p^3 \right] vol_\Sigma + \\
& + \langle \mu^1, (x^1\partial_3 - x^3\partial_1)\partial_1\phi \rangle + \langle \mu^2, (x^1\partial_3 - x^3\partial_1)\partial_2\phi - \partial_3\phi \rangle + \langle \mu^3, (x^1\partial_3 - x^3\partial_1)\partial_3\phi + \partial_2\phi \rangle \,,
\end{aligned} \tag{4.245}
$$

and:

$$
\begin{aligned}
J_{\xi_{\mathscr{R}x^2}} = & \int_\Sigma \left[(x^3\partial_2 - x^2\partial_3)\tilde{a}_1 p^1 + \tilde{a}_3 p^1 + (x^3\partial_2 - x^2\partial_3)\tilde{a}_2 p^2 + (x^3\partial_2 - x^2\partial_3)\tilde{a}_3 p^3 - \tilde{a}_1 p^3 \right] vol_\Sigma + \\
& + \langle \mu^1 \,, (x^3\partial_2 - x^2\partial_3)\partial_1\phi - \partial_3\phi \rangle + \langle \mu^2 \,, (x^3\partial_2 - x^2\partial_3)\partial_2\phi \rangle + \langle \mu^3 \,, (x^3\partial_2 - x^2\partial_3)\partial_3\phi + \partial_1\phi \rangle \,.
\end{aligned} \tag{4.246}
$$

Now, as we said in Sec. 2.3.5, the three currents computed so far can be pulled back to the physical space $\mathcal{M}_\infty$ *by putting* $\mu = 0$ *obtaining the standard components* $0jk$ *of the generalized angular momentum of the Electromagnetic field.*

4.3. Covariant Poisson brackets

We end this chapter, and the manuscript, by showing how the canonical structure on the space of Cauchy data defined in the previous section can be used to equip the solution space with a Poisson bracket structure. We will distinguish two cases:

- The case in which the structure is symplectic, of which we will give some examples (for instance the free particle as mechanical system and the Klein-Gordon theory as an example of field theory), in which the construction of the Poisson bracket is straightforward, being the standard one described in Sec. 2.2.1.
- The case in which the structure is pre-symplectic where we will use the regularization procedure related to the coisotropic embedding theorem described in Sec. 2.3.4 and already used in Sec. 4.2.7, to define a Poisson bracket. Within this approach the Poisson bracket turns out to be defined, in general, on an enlargement of the solution space and, thus, we will identify two sub-cases of this situation. The first (exemplified by free Electrodynamics), occurring when the connection we fix during the coisotropic embedding procedure is flat, in which the Poisson bi-vector field defined on the enlarged manifold is projectable onto the solution space giving rise, consequently, to a Poisson bracket on it. The second one (exemplified by free Yang-Mills theories), occurring when a flat connection can not be fixed during the coisotropic embedding procedure, in which the Poisson bi-vector field on the enlarged space can not be projected onto the solution space. In this case, in order to have a Poisson bracket structure, one is forced to work on such enlarged space where the additional degrees

of freedom can be interpreted in terms of the well known concept of GHOST FIELDS necessary to quantize non-Abelian gauge theories.

4.3.1. Covariant Poisson brackets in Lagrangian Mechanics

Let us start by focusing on the case where the solution space $\mathcal{EL} \simeq \mathcal{M}_\infty$ is a symplectic manifold, that is, the structure $\omega_{\mathscr{L}\infty}$ on $\mathcal{M}_\infty$ coming from the pre-symplectic constraint algorithm applied to the pre-symplectic Hamiltonian system describing our dynamical system, is symplectic and, thus, the canonical structure $\Pi_t^\star \omega_{\mathscr{L}} = \Psi^{-1\star}\omega_{\mathscr{L}\infty}$ on $\mathcal{EL}$ related to the latter via the diffeomorphism Ψ^{-1} is symplectic.

In this case, how to define a Poisson bracket on the solution space is a straightforward application of the discussion before and after Eq. (2.87). Indeed, given two functions on $\mathcal{EL}$, say F and G, their Poisson bracket is given by:

$$\{G,\, F\} \,=\, \Pi_t^\star\omega_{\mathscr{L}}(\mathbb{X}_G,\, \mathbb{X}_F) \,=\, i_{\mathbb{X}_F}\mathrm{d}G\,, \tag{4.247}$$

where $\mathbb{X}_F$ (resp. $\mathbb{X}_G$) is the unique solution of:

$$\Pi_t^\star\omega_{\mathscr{L}}(\mathbb{X}_F,\, \cdot\,) \,=\, \mathrm{d}F\,. \tag{4.248}$$

Equivalently, if we consider the pull-back of F and G to $\mathcal{M}_\infty$ via Ψ (which amounts to considering F and G as functions of the unique Cauchy datum in $\mathcal{M}_\infty$ giving rise to the solution on which F and G depend), say:

$$f \,:=\, \Psi^\star F\,, \qquad g \,:=\, \Psi^\star G\,, \tag{4.249}$$

then, a Poisson bracket between f and g can be immediately defined via the symplectic structure $\omega_{\mathscr{L}\infty}$ as:

$$\{g,\, f\} \,=\, \omega_{\mathscr{L}\infty}(\mathbb{X}_f,\, \mathbb{X}_g) \,=\, i_{\mathbb{X}_f}\mathrm{d}g\,, \tag{4.250}$$

where $\mathbb{X}_f$ (resp. $\mathbb{X}_g$) is the unique solution of:

$$\omega_{\mathscr{L}\infty}(\mathbb{X}_f,\, \cdot\,) \,=\, \mathrm{d}f(\,\cdot\,)\,. \tag{4.251}$$

As we stressed several times, being $\mathcal{M}_\infty$ and $\mathcal{EL}$ diffeomorphic, all the geometric structures on the former are equivalent to the geometric structures on the latter, in the sense that the former can be pulled-back to the latter via the diffeomorphism Ψ relating them. Of course, this also happens for the Poisson brackets just defined. Indeed, as a consequence of the relation (4.52) between $\Pi_t^\star\omega_{\mathscr{L}}$ and $\omega_{\mathscr{L}\infty}$, the following relation can be proved between the brackets above:

$$\begin{aligned}\{G,\, F\}_\gamma \,&=\, \left(\Pi_t^\star\omega_{\mathscr{L}}\right)_\gamma(\mathbb{X}_G,\, \mathbb{X}_F) \,=\, \left[\Psi^{-1\star}\omega_{\mathscr{L}\infty}\right]_\gamma(\mathbb{X}_G,\, \mathbb{X}_F) \,= \\ &=\, \left[\omega_{\mathscr{L}\infty}\right]_{\Psi^{-1}(\gamma)}\left({\Psi^{-1}}_\star\mathbb{X}_G,\, {\Psi^{-1}}_\star\mathbb{X}_F\right) \,= \\ &=\, \left[\omega_{\mathscr{L}\infty}\right]_{\Psi^{-1}(\gamma)}(\mathbb{X}_g,\, \mathbb{X}_f) \,=\, \{g,\, f\}_{\Psi^{-1}(\gamma)}\,,\end{aligned} \tag{4.252}$$

which says that the brackets are equivalent in the sense that the diagram below is commutative:

$$\begin{tikzcd} (F,\, G) \in \mathcal{C}^\infty(\mathcal{EL}) \arrow[r, "\Pi_t^\star \omega_{\mathscr{L}}"] \arrow[d, "\Psi^\star"] & \{G,\, F\}_{\Pi_t^\star \omega_{\mathscr{L}}} \arrow[d, "\Psi^\star"] \\ (f,\, g) \in \mathcal{C}^\infty(\mathcal{M}_\infty) \arrow[r, "\omega_{\mathscr{L}\infty}"] & \{g,\, f\}_{\omega_{\mathscr{L}\infty}} \end{tikzcd} \tag{4.253}$$

As in Sec. 4.1.1, because of this equivalence, we will always prefer to work directly on the manifold $\mathcal{M}_\infty$ in order to construct the Poisson bracket on the solution space.

Within the cases in which $\mathcal{M}_\infty$ turns out to be pre-symplectic we will use the trick of the coisotropic embedding theorem presented in Sec. 2.3.4 in order to define a Poisson bracket. As explained in Sec.2.3.4, given the pre-symplectic manifold $(\mathcal{M}_\infty,\, \omega_{\mathscr{L}\infty})$, for which we assume the kernel of $\omega_{\mathscr{L}\infty}$ to be of constant dimension, given any connection on the bundle $\mathbf{K} \to \mathcal{M}_\infty$ with typical fibre given by K, it is possible to construct a canonical symplectic manifold, say $(\tilde{\mathcal{M}},\, \tilde{\omega})$, being an enlargement of it and such that $(\mathcal{M}_\infty,\, \omega_{\mathscr{L}\infty})$ is a coisotropic submanifold. By looking at Eq. (2.129), denoting by[43]

$$P \;=\; P^j \otimes V_j\,, \tag{4.254}$$

the connection and by $\{\, \mu_j \,\}_{j=1,\dots,\mathrm{dim}K^\star}$ a suitable choice of coordinates on $K^\star$, the symplectic structure reads:

$$\tilde{\omega} \;=\; \tau^\star \omega_{\mathscr{L}\infty} + \mathrm{d}\mu_j \wedge P^j + \mu_j \mathrm{d}P^j\,, \tag{4.255}$$

where τ denotes the projection from $\tilde{\mathcal{M}}$ to $\mathcal{M}_\infty$ and which, splitting the differential of P^j into its vertical and horizontal components:

$$\mathrm{d}P^j \;=\; \mathrm{d}_H P^j + \mathrm{d}_V P^j \;=\; \mathrm{d}P^j\,(\,(\mathbb{1} - P)(\,\cdot\,),\, (\mathbb{1} - P)(\,\cdot\,)\,) + \mathrm{d}P^j\,(\,P(\,\cdot\,),\, P(\,\cdot\,)\,)\;, \tag{4.256}$$

can be written as:

$$\tilde{\omega} \;=\; \tilde{\omega}^\alpha_W \oplus \tilde{\omega}^\alpha_{K\oplus K^\star}\,, \tag{4.257}$$

where:

$$\tilde{\omega}^\alpha_W \;=\; \tau^\star \omega_{\mathscr{L}\infty} + \mu_j \mathrm{d}_H P^j \tag{4.258}$$

has only components on the horizontal space W defined by the connection, and:

$$\tilde{\omega}^\alpha_{K\oplus K^\star} \;=\; \mathrm{d}\mu_j \wedge P^j + \mu_j \mathrm{d}_V P^j\,, \tag{4.259}$$

has only components on $K \oplus K^\star$. As discussed in Sec. 2.3.4, if the connection chosen is such that at least $\mathrm{d}_H P^j$ vanishes for all j (i.e. the connection is flat), then the inverse (restricted to W) of the 2-form $\tilde{\omega}^\alpha_W$ is a Poisson bi-vector field $\tilde{\lambda}_W$ on $\tilde{\mathcal{M}}$ which is projectable onto $\mathcal{M}_\infty$ via τ to the Poisson bi-vector field:

$$\lambda_W \;=\; \tau_\star \tilde{\lambda}_W \tag{4.260}$$

[43]Using the notation of Sec. 2.3.4, V_j are the elements of a basis of K and P^j are 1-forms defining the connection.

which gives rise to the Poisson bracket:

$$\{g,\, f\} \;=\; \lambda_W(\mathrm{d}g,\, \mathrm{d}f)\,, \tag{4.261}$$

on $\mathcal{M}_\infty$.

On the other hand, if $\mathrm{d}_H P^j \neq 0$, a Poisson bracket can only be defined on the enlarged manifold $\tilde{\mathcal{M}}$.

We will see, in the following paragraphs, examples of both situations.

Free particle

To show the procedure of constructing the Poisson bracket on the solution space outlined in the present section, let us apply it to the example 3.1.7, 4.2.1. In this case we saw that the solution space $\mathcal{EL}$ is diffeomorphic to the space of Cauchy data $\mathcal{M}_\infty$ which is $\mathbf{T}\mathcal{Q} \;=\; \mathbf{T}\mathbb{R} \;\simeq\; \mathbb{R}^2$ where we used $\{\,q,\, v\,\}$ as a system of coordinates. The structure $\omega_{\mathscr{L}\infty}$ reads:

$$\omega_{\mathscr{L}\infty} \;=\; m\mathrm{d}q \wedge \mathrm{d}v\,, \tag{4.262}$$

which is symplectic. Thus it gives rise to a Poisson bracket in the usual way without using the coisotropic embedding procedure. Consider two functions on $\mathcal{EL}$, say:

$$F[\gamma] \;=\; \gamma(t_1) \;=\; q + v(t_1 - t_0), \qquad G[\gamma] \;=\; \gamma(t_2) \;=\; q + v(t_2 - t_0)\,. \tag{4.263}$$

Following the procedure outlined above, their pull-back to $\mathcal{M}_\infty \;=\; \mathbb{R}^2$ read:

$$f(q,\, v) \;=\; \Psi^\star F[\gamma] \;=\; q+v(t_1-t_0)\,, \qquad g(q,\, v) \;=\; \Psi^\star G[\gamma] \;=\; q+v(t_2-t_0)\,, \tag{4.264}$$

namely, they have the same expression but now they are seen as functions of q and v. The Poisson bracket reads:

$$\{g,\, f\} \;=\; X_f(g), \tag{4.265}$$

where X_f is determined by:

$$i_{X_f}\omega_{\mathscr{L}\infty} \;=\; \mathrm{d}f\,, \tag{4.266}$$

and reads:

$$X_f \;=\; \frac{t_1 - t_0}{m}\frac{\partial}{\partial q} - \frac{1}{m}\frac{\partial}{\partial v}\,. \tag{4.267}$$

Consequently:

$$\{g,\, f\}_{(q,v)} \;=\; \frac{t_1 - t_2}{m}\,, \tag{4.268}$$

and

$$\{G,\, F\}_\gamma \;=\; \frac{t_1 - t_2}{m}\,, \tag{4.269}$$

that, again, have the same expression but should be understood as functions on $\mathbb{R}^2$ and on $\mathcal{EL}$ respectively.

4.3.2. Covariant Poisson brackets in Hamiltonian Mechanics

As we stressed in Sec. 4.2.2, within Hamiltonian mechanical systems, the structure emerging from the pre-symplectic constraint algorithm applied to the pre-symplectic Hamiltonian system describing our dynamical system is the canonical symplectic structure ω of the cotangent bundle of the configuration manifold, $\mathbf{T}^\star\mathcal{Q}$. Therefore, within Hamiltonian mechanical systems we always lie in the symplectic case of the previous section, where the Poisson bracket on $\mathcal{EL}$ reads:

$$\{G,\, F\} \,=\, \Pi_t^\star\omega(\mathbb{X}_G,\, \mathbb{X}_F) \,=\, i_{\mathbb{X}_F}\mathrm{d}G\,, \tag{4.270}$$

where F and G are functions on $\mathcal{EL}$ and $\mathbb{X}_F$ (resp. $\mathbb{X}_G$) is the unique solution of:

$$\Pi_t^\star\omega(\mathbb{X}_F,\, \cdot\,) \,=\, \mathrm{d}F\,. \tag{4.271}$$

On the other hand, on $\mathcal{M}_\infty \,=\, \mathbf{T}\mathcal{Q}$, the Poisson bracket is:

$$\{g,\, f\} \,=\, \omega(\mathbb{X}_f,\, \mathbb{X}_g) \,=\, i_{\mathbb{X}_f}\mathrm{d}g\,, \tag{4.272}$$

where $\mathbb{X}_f$ (resp. $\mathbb{X}_g$) is the unique solution of:

$$\omega(\mathbb{X}_f,\, \cdot\,) \,=\, \mathrm{d}f(\,\cdot\,)\,, \tag{4.273}$$

and, again, if

$$f \,:=\, \Psi^\star F\,, \qquad g \,:=\, \Psi^\star G\,, \tag{4.274}$$

the relation between the two brackets is:

$$\{G,\, F\}_\xi \,=\, \{g,\, f\}_{\Psi^{-1}(\xi)} \tag{4.275}$$

that is, they are equivalent in the sense that the following diagram:

$$\begin{array}{ccc} (F,\, G) \in \mathcal{C}^\infty(\mathcal{EL}) & \xrightarrow{\Pi_t^\star\omega} & \{G,\, F\}_{\Pi_t^\star\omega} \\ \big\downarrow{\scriptstyle \Psi^\star} & & \big\downarrow{\scriptstyle \Psi^\star} \\ (f,\, g) \in \mathcal{C}^\infty(\mathcal{M}_\infty) & \xrightarrow{\quad\omega\quad} & \{g,\, f\}_\omega \end{array} \tag{4.276}$$

commutes.

Free particle

Let us consider the example 3.1.14, 4.2.2 to show the procedure just outlined to construct a Poisson bracket on the solution space within Hamiltonian mechanical systems. In this case we saw that $\mathcal{EL}$ is isomorphic to the space of Cauchy data $\mathcal{M}_\infty$ which is $\mathbf{T}^\star\mathcal{Q} \,=\, \mathbb{T}^\star\mathbb{R} \,\simeq\, \mathbb{R}^2$ where we used $\{\, q,\, p\,\}$ as a system of coordinates. The structure ω here is the canonical one of $\mathbf{T}^\star\mathbb{R}$, namely:

$$\omega \,=\, \mathrm{d}q \wedge \mathrm{d}p\,, \tag{4.277}$$

and the Poisson bracket is computed straightforwardly. Consider two functions on $\mathscr{E}\mathscr{L}$:

$$F[\xi] \,=\, \gamma(t_1) \,=\, q + \frac{p}{m}(t_1 - t_0), \qquad G[\xi] \,=\, \varrho(t_2) \,=\, p\,, \tag{4.278}$$

and take their pull-back to $\mathcal{M}_\infty \simeq \mathbb{R}^2$:

$$f(q,\,p) \,=\, \Psi^\star[\xi] \,=\, q + \frac{p}{m}(t_1 - t_0)\,, \qquad g(q,\,p) \,=\, \Psi^\star G[\xi] \,=\, p\,, \tag{4.279}$$

which, again, have the same expression but should be considered as functions of q and p. The Poisson bracket reads:

$$\{g,\,f\} \,=\, X_f(g)\,, \tag{4.280}$$

where X_f is determined by:

$$i_{X_f}\omega \,=\, \mathrm{d}f\,, \tag{4.281}$$

and reads:

$$X_f \,=\, \frac{t_1 - t_0}{m}\frac{\partial}{\partial q} - \frac{\partial}{\partial p}\,. \tag{4.282}$$

Consequently:

$$\{g,\,f\}_{(q,p)} \,=\, -1\,, \tag{4.283}$$

and:

$$\{G,\,F\}_\xi \,=\, -1\,, \tag{4.284}$$

that, again, have the same expression but should be understood as functions on $\mathbb{R}^2$ and on $\mathscr{E}\mathscr{L}$ respectively.

4.3.3. Covariant Poisson brackets in Hamiltonian systems with constraints

In this case, as stressed in Sec. 4.2.3, the dynamics our dynamical system is described in terms of the pre-symplectic Hamiltonian system $(\mathbf{T}^\star\mathcal{Q} \times \mathcal{F}^\star_{\mathcal{P}(\mathbb{Q})}{}^\Sigma \times \mathcal{N},\, \omega^{\mathrm{ext}},\, H^{\mathrm{ext}})$ and nothing is said about the fact that the final stable manifold of the pre-symplectic constraint algorithm, $\mathcal{M}_\infty$, (if it exists) is a symplectic one. Indeed, here we lie, in general, in the pre-symplectic case described in Sec. 4.3.1, namely, the structure on the Cauchy data space $\mathcal{M}_\infty$ emerging from the pre-symplectic constraint algorithm, say $\omega^{\mathrm{ext}}_\infty = \mathfrak{i}^\star_\infty\omega^{\mathrm{ext}}$ ($\mathfrak{i}_\infty$ denoting, as usual, the immersion of $\mathcal{M}_\infty$ into $\mathbf{T}^\star\mathcal{Q}\times\mathcal{F}^\star_{\mathcal{P}(\mathbb{Q})}{}^\Sigma\times\mathcal{N}$) is generally pre-symplectic.

Therefore, in this case, as explained in Sec. 4.3.1, given any connection on the bundle $\mathbf{K} \to \mathcal{M}_\infty$ whose typical fibre is the kernel, K, of $\omega^{\mathrm{ext}}_\infty$, say:

$$P \,=\, P^j \otimes V_j\,, \tag{4.285}$$

$\{\, V_j \,\}_{j=1,\dots,\mathrm{dim}K}$ denoting a basis of K, it is defined a symplectic manifold $(\tilde{\mathcal{M}},\, \tilde{\omega}^{\mathrm{ext}}_\infty)$ with:

$$\tilde{\omega}^{\mathrm{ext}}_\infty \,=\, \tau^\star\omega^{\mathrm{ext}}_\infty + \mathrm{d}\mu^j \wedge P^j + \mu_j \mathrm{d}P^j \,=\, \tilde{\omega}^{\mathrm{ext}\,\alpha}_{\infty\ W} \oplus \tilde{\omega}^{\mathrm{ext}\,\alpha}_{\infty\ K\oplus K^\star}\,, \tag{4.286}$$

where τ denotes the projection of $\tilde{\mathcal{M}}$ onto $\mathcal{M}_\infty$, $\{ \mu_j \}_{j=1,\dots,\dim K^\star}$ is a suitable system of coordinates on $K^\star$ and:

$$\tilde{\omega}^{\text{ext}\,\alpha}_{\infty\; W} \,=\, \tau^\star \tilde{\omega}^{\text{ext}}_\infty + \mu_j \mathrm{d}_H P^j \tag{4.287}$$

has only components on the horizontal space W defined by the connection, whereas:

$$\tilde{\omega}^{\text{ext}\,\alpha}_{\infty\; K\oplus K^\star} \,=\, \mathrm{d}\mu_j \wedge P^j + \mu_j \mathrm{d}_V P^j \,, \tag{4.288}$$

has only components on $K \oplus K^\star$. With $\tilde{\omega}^{\text{ext}}_\infty$ in hand, a Poisson bracket on $\tilde{\mathcal{M}}$ is defined by:

$$\{g,\, f\} \,=\, \tilde{\omega}^{\text{ext}}_\infty(X_g,\, X_f) \,=\, \tilde{\lambda}(\mathrm{d}g,\, \mathrm{d}f)\,, \tag{4.289}$$

for any couple of functions f and g on $\mathcal{M}_\infty$, where X_f and X_g denote the Hamiltonian vector fields of f and g with respect to the symplectic structure $\tilde{\omega}^{\text{ext}}_\infty$ and $\tilde{\lambda}$ is the inverse of $\tilde{\omega}^{\text{ext}}_\infty$. As discussed in Sec. 4.3.1, if the connection is flat, the inverse of $\tilde{\omega}^{\text{ext}\,\alpha}_{\infty\; K\oplus K^\star}$, say $\tilde{\lambda}_W$, is projectable to $\mathcal{M}_\infty$ to a Poisson bivector field, say λ_W giving rise to the following Poisson bracket on $\mathcal{M}_\infty$:

$$\{g,\, f\} \,=\, \lambda_W(\mathrm{d}g,\, \mathrm{d}f)\,. \tag{4.290}$$

4.3.4. Covariant Poisson brackets within symplectic field theories

In Sec. 4.2.4 and 4.2.5 we defined, both in the Lagrangian and in the Hamiltonian formalism, theories with or without gauge symmetries as those theories for which the final manifold resulting from the application of the pre-symplectic constraint algorithm to the pre-symplectic Hamiltonian system describing the field theory is pre-symplectic or symplectic respectively. In this section we will deal with theories without gauge symmetries, i.e., those for which the stable manifold $\mathcal{M}_\infty$ is symplectic, and we will refer sometimes to them as SYMPLECTIC FIELD THEORIES.

Basically, both in the Lagrangian and in the Hamiltonian setting, if $\mathcal{M}_\infty$ is symplectic, all the constructions given in the beginning of Sec. 4.3.1 and in Sec. 4.3.2 can be reproduced.

More precisely, referring to the Hamiltonian formalism, the structure Ω^Σ_∞ on $\mathcal{M}_\infty$ turns out to be symplectic and, thus, the canonical structure $\Pi^\star_\Sigma\Omega^\Sigma \,=\, \Psi^{-1} \circ \varpi^\star \Omega^\Sigma_\infty$ on $\mathcal{EL}^\epsilon$ related to the latter via the diffeomorphism $(\Psi^{-1} \circ \varpi)$ is symplectic as well. On $\mathcal{EL}^\epsilon$ the Poisson bracket associated to $\Pi^\star_\Sigma\Omega^\Sigma$ reads:

$$\{G,\, F\} \,=\, \Pi^\star_\Sigma\Omega^\Sigma(\mathbb{X}_G,\, \mathbb{X}_F) \,=\, i_{\mathbb{X}_F}\mathrm{d}G\,, \tag{4.291}$$

where F and G are functions on $\mathcal{EL}^\epsilon$ and $\mathbb{X}_F$ (resp. $\mathbb{X}_G$) is the unique solution of:

$$\Pi^\star_\Sigma\Omega^\Sigma(\mathbb{X}_F,\, \cdot\,) \,=\, \mathrm{d}F\,. \tag{4.292}$$

On the other hand, on $\mathcal{M}_\infty$, the Poisson bracket is:

$$\{g,\, f\} \,=\, \Omega^\Sigma_\infty(\mathbb{X}_f,\, \mathbb{X}_g) \,=\, i_{\mathbb{X}_f}\mathrm{d}g\,, \tag{4.293}$$

where $\mathbb{X}_f$ (resp. $\mathbb{X}_g$) is the unique solution of:

$$\Omega^{\Sigma}_{\infty}(\mathbb{X}_f,\,\cdot\,) \;=\; \mathrm{d}f(\,\cdot\,)\,, \tag{4.294}$$

and, if

$$f \;:=\; \left(\Psi \circ \varpi^{-1}\right)^{\star} F\,, \qquad g \;:=\; \left(\Psi \circ \varpi^{-1}\right)^{\star} G\,, \tag{4.295}$$

the relation between the two brackets is:

$$\{G,\, F\}_{\chi} \;=\; \{g,\, f\}_{(\Psi\circ\varpi^{-1})(\xi)} \tag{4.296}$$

that is, they are equivalent in the sense that the following diagram:

$$\begin{tikzcd}
(F,\,G) \in \mathcal{C}^{\infty}(\mathcal{EL}^{\epsilon}) \arrow[r, "\Pi^{\star}_{\Sigma}\Omega^{\Sigma}"] \arrow[d, "(\Psi\circ\varpi^{-1})^{\star}"] & \{G,\,F\}_{\Pi^{\star}_{\Sigma}\Omega^{\Sigma}} \arrow[d, "(\Psi\circ\varpi^{-1})^{\star}"] \\
(f,\,g) \in \mathcal{C}^{\infty}(\mathcal{M}_{\infty}) \arrow[r, "\Omega^{\Sigma}_{\infty}"] & \{g,\,f\}_{\Omega^{\Sigma}_{\infty}}
\end{tikzcd} \tag{4.297}$$

commutes.

Klein-Gordon theory

As an example of construction of the Poisson bracket on the solution space of a symplectic field theory, let us consider the Klein-Gordon theory within the Lagrangian formulation developed in examples 3.2.6, 4.2.5. In this case we showed that the solution space $\mathcal{EL}$ is isomorphic to the space of Cauchy data $\mathcal{M}_{\infty}$ which reads:

$$\mathcal{M}_{\infty} \;=\; \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_{\Sigma})\right]^2 \,\ni\, (\varphi,\, \dot{\varphi})\,. \tag{4.298}$$

The symplectic structure on $\mathcal{M}_{\infty}$ was computed to be:

$$\Omega^{\Sigma}_{\infty} \;=\; \delta\varphi \wedge \delta\dot{\varphi}\,,. \tag{4.299}$$

In order to compute the Poisson bracket, let us consider the following two functions on $\mathcal{EL}$:

$$F_{x_1}[\phi] \;=\; \phi(x_1)\,, \qquad G_{x_2}[\phi] \;=\; \partial_0\phi(x_2)\,, \tag{4.300}$$

where ϕ is a solution of Klein-Gordon equation and x_1 and x_2 are two points in the Minkowski space-time $\mathscr{M}$. In order to compute the pull-back of F_{x_1} and G_{x_2} via $(\,\Psi \circ \varpi^{-1}\,)$, we must explicitly write the solution ϕ in terms of the Cauchy data φ and $\dot{\varphi}$ on Σ which reads:

$$\phi(x) \;=\; \int_{\Sigma\times\Sigma} \left(\varphi(\underline{x}')\cos\left[\omega_k(x^0 - x^0_{\Sigma})\right] - \dot{\varphi}(\underline{x}')\frac{\sin\left[\omega_k(x^0 - x^0_{\Sigma})\right]}{\omega_k}\right) e^{i\underline{k}\cdot(\underline{x}-\underline{x}')}\,\mathrm{d}^3k\,\mathrm{d}^3x'\,, \tag{4.301}$$

where $\underline{x}'$ and $\underline{k}$ are two points in two copies of Σ and $\omega_k = \sqrt{|\underline{k}|^2 + m^2}$. With this explicit solution in mind, the functions f_{x_1} and g_{x_2} are computed to be:

$$
\begin{aligned}
f_{x_1}[\varphi,\, \dot{\varphi}] \,&=\, \left(\Psi \circ \varpi^{-1}\right)^{\star} F_{x_1}[\phi] \,= \\
&= \int_{\Sigma\times\Sigma} \left(\varphi(\underline{x}')\cos\left[\omega_k(x_1^0 - x_\Sigma^0)\right] - \dot{\varphi}(\underline{x}')\frac{\sin\left[\omega_k(x_1^0 - x_\Sigma^0)\right]}{\omega_k}\right) e^{i\underline{k}\cdot(\underline{x}_1 - \underline{x}')}\,\mathrm{d}^3k\,\mathrm{d}^3x'\,, \\
g_{x_2}[\varphi,\, \dot{\varphi}] \,&=\, \left(\Psi \circ \varpi^{-1}\right)^{\star} G_{x_2}[\phi] \,= \\
&= \int_{\Sigma\times\Sigma} \left(\varphi(\underline{x}')\omega_k\sin\left[\omega_k(x_2^0 - x_\Sigma^0)\right] + \dot{\varphi}(\underline{x}')\cos\left[\omega_k(x_2^0 - x_\Sigma^0)\right]\right) e^{i\underline{k}\cdot(\underline{x}_2 - \underline{x}')}\,\mathrm{d}^3k\,\mathrm{d}^3x'\,.
\end{aligned}
\tag{4.302}
$$

Then, the bracket between f_{x_1} and g_{x_2} is given by:

$$
\{g_{x_2},\, f_{x_1}\} \,=\, \mathbb{X}_{f_{x_1}}(g)\,, \tag{4.303}
$$

where $\mathbb{X}_{f_{x_1}}$ is determined by:

$$
i_{\mathbb{X}_{f_{x_1}}}\Omega_\infty^\Sigma \,=\, \mathrm{d}f_{x_1}\,, \tag{4.304}
$$

and is computed to be:

$$
\mathbb{X}_{f_{x_1}} \,=\, -\int_\Sigma \frac{\sin\left[\omega_k(x_1^0 - x_\Sigma^0)\right]}{\omega_k}\, e^{i\underline{k}\cdot(\underline{x}_1 - \underline{x})}\,\mathrm{d}^3k\,\frac{\delta}{\delta\varphi} + \int_\Sigma \cos\left[\omega_k(x_1^0 - x_\Sigma^0)\right] e^{i\underline{k}\cdot(\underline{x}_1 - \underline{x})}\,\mathrm{d}^3k\,\frac{\delta}{\delta\dot{\varphi}}\,, \tag{4.305}
$$

using the notation (2.53). Then the bracket between f_{x_1} and g_{x_2} is:

$$
\{g_{x_2},\, f_{x_1}\}_{(\varphi,\,\dot{\varphi})} \,=\, \mathbb{X}_f(g) \,=\, \int_\Sigma \cos\left[\omega_k(x_1^0 - x_2^0)\right] e^{i\underline{k}\cdot(\underline{x}_1 - \underline{x}_2)}\,\mathrm{d}^3k\,, \tag{4.306}
$$

and, consequently, the bracket between F_{x_1} and G_{x_2} is its pull-back via $\varpi \circ \Psi^{-1}$, i.e.:

$$
\{G_{x_2},\, F_{x_1}\}_\phi \,=\, \int_\Sigma \cos\left[\omega_k(x_1^0 - x_2^0)\right] e^{i\underline{k}\cdot(\underline{x}_1 - \underline{x}_2)}\,\mathrm{d}^3k\,, \tag{4.307}
$$

which has, again, the same expression as the bracket on $\mathcal{M}_\infty$ but it should be thought of as a function on $\mathcal{EL}$.

4.3.5. Covariant Poisson bracket within gauge theories

Here we deal with the case in which the stable manifold obtained out of the pre-symplectic constraint algorithm is still pre-symplectic. Just to fix the ideas let us focus on the Hamiltonian formalism, in which the stable manifold reads $(\mathcal{M}_\infty,\, \Omega_\infty^\Sigma)$, but the discussion can be performed in the same way both within Lagrangian field theories and within Hamiltonian theories with additional constraints where the stable manifold reads $(\mathcal{M}_\infty,\, \Omega_\infty^\Sigma)$ and $(\mathcal{M}_\infty,\, \Omega_\infty^{\Sigma^{\mathrm{ext}}})$ respectively.

Again, here we lie in the pre-symplectic case of Sec. 4.3.1 where, given a connection:

$$
P \,=\, P^j \otimes V_j\,, \tag{4.308}
$$

on the bundle $\mathbf{K} \to \mathcal{M}_\infty$ with typical fibre given by K, we get a symplectic manifold out of the coisotroipic embedding theorem equipped with the symplectic structure:

$$\tilde{\Omega^{\Sigma}}_\infty = \tau^\star \Omega^\Sigma_\infty + \mathrm{d}\mu_j \wedge P^j + \mu_j \mathrm{d}P^j \,, \tag{4.309}$$

$\{ \mu_j \}_{j=1,\dots,\dim K^\star}$ denoting a suitable coordinate system on $K^\star$. Again, $\tilde{\Omega^{\Sigma}}_\infty$ splits locally as follows:

$$\tilde{\Omega^{\Sigma}}_\infty = \tilde{\Omega^{\Sigma}}^\alpha \infty_W \oplus \tilde{\Omega^{\Sigma}}^\alpha_{\infty K \oplus K^\star} \,, \tag{4.310}$$

where:

$$\tilde{\Omega^{\Sigma}}^\alpha_{\infty W} = \tau^\star \tilde{\Omega^{\Sigma}}_\infty + \mu_j \mathrm{d}_H P^j \,, \tag{4.311}$$

is the term which only has components on W and:

$$\tilde{\Omega^{\Sigma}}^\alpha_{\infty K \oplus K^\star} = \mathrm{d}\mu_j \wedge P^j + \mu_j \mathrm{d}_V P^j \,, \tag{4.312}$$

is the term which only has components along $K \oplus K^\star$.

As discussed in Sec. 2.3.4, if P is such that at least $\mathrm{d}_H P^j$ vanishes for all j (i.e. the connection is flat), then the inverse (restricted to W) of $\tilde{\Omega^{\Sigma}}^\alpha_{\infty W}$ is a Poisson bi-vector field $\tilde{\lambda}_W$ on $\tilde{\mathcal{M}}$ which is projectable onto $\mathcal{M}_\infty$ via τ to the Poisson bi-vector field:

$$\lambda_W = \tau_\star \tilde{\lambda}_W \tag{4.313}$$

which gives rise to a the Poisson bracket:

$$\{g,\, f\} = \lambda_W(\mathrm{d}g,\, \mathrm{d}f) \,, \tag{4.314}$$

on $\mathcal{M}_\infty$.

On the other hand, if $\mathrm{d}_H P^j \neq 0$, a Poisson bracket can only be defined on the enlarged manifold $\tilde{\mathcal{M}}$. Note that the number of additional degrees of freedom emerging by working on $\tilde{\mathcal{M}}$ instead of $\mathcal{M}_\infty$ coincide with the dimension of the kernel of Ω^Σ_∞ and, thus, with the number of GHOST FIELDS used within the BRST approach to quantize non-Abelian gauge theories (see, for instance, [55]). Moreover, they are dual to the generators of gauge transformations, as the ghost fields defined in [55]. This means that the additional degrees of freedom we are forced to use in order to defined a Poisson bracket structure in this case can be interpreted in terms of ghost fields.

As examples of both situations, we will perform explicitly the construction within free Electrodynamics and within Yang-Mills theories in the Hamiltonian formalism.

Free Electrodynamics

As discussed in example 4.2.11, within free Electrodynamics the formulation of the theory in terms of a pre-symplectic Hamiltonian system around a slice Σ of the space-time $\mathscr{M}$ leads to the following final stable manifold:

$$\mathcal{M}_\infty = \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma) \right]^3 \times \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma;\, \mathrm{div}0) \right]^3 \,, \tag{4.315}$$

whose points are denoted, in a system of local coordinates, by $(a_k,\, p^k)$. The latter space turns out to be a Hilbert space, since one proves that $\left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma;\, \mathrm{div}0)\right]^3$ is a closed subspace of $\left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma)\right]^3$. Indeed, the following holds.

Proposition 4.3.1. *The gradient operator* grad : $\mathcal{H}^{\frac{3}{2}}(\Sigma, vol_\Sigma) \to \mathcal{H}^{\frac{1}{2}}(\Sigma, vol_\Sigma)$ *is closed.*

Proof. Consider a sequence $\{f_n\}_{n\in\mathbb{N}}$ with $f_n \in \mathcal{H}^{\frac{3}{2}}(\Sigma, vol_\Sigma)$ such that

$$\lim_{n\to\infty} f_n \;=\; f$$

in $\mathcal{H}^{\frac{3}{2}}(\Sigma, vol_\Sigma)$. The following inequalities:

$$\begin{aligned}\sum_j \|\partial_j(f_n - f)\|_{\mathcal{H}^{\frac{1}{2}}} \;&=\; \sum_{j,k}\int_{\mathbb{R}^3} |\tilde{f}_n - \tilde{f}|^2 k_j\, k_k\, |k|\, \mathrm{d}^3k \\ &\leq\; 9\int_{\mathbb{R}^3} |\tilde{f}_n - \tilde{f}|^2\, |k|^3 \mathrm{d}^3k \;=\; 9\|f_n - f\|_{\mathcal{H}^{\frac{3}{2}}}\end{aligned} \tag{4.316}$$

hold, thus proving that the operator grad maps closed sets into closed sets. □

Consequently, $\left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma;\, \mathrm{div}0)\right]^3$, being the kernel of the div operator (which is the adjoint of grad), is – recall the closed range theorem – a closed subspace in $\left[\mathcal{H}^{\frac{1}{2}}(\Sigma, vol_\Sigma)\right]^3$, whose complement is the range of the action of the grad operator into $\mathcal{H}^{\frac{1}{2}}(\Sigma, vol_\Sigma)$, that is the following decomposition into closed (and, thus, Hilbert) subspaces holds:

$$\left[\mathcal{H}^{\frac{1}{2}}(\Sigma, vol_\Sigma)\right]^3 \;=\; \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_\Sigma;\, \mathrm{div}0)\right]^3 \oplus \mathrm{grad}\mathcal{H}^{\frac{3}{2}}(\Sigma, vol_\Sigma)\,. \tag{4.317}$$

On the other hand, as we saw in example 4.2.11, the 2-form Ω_∞^Σ reads:

$$\Omega_\infty^\Sigma \;=\; \delta a_k \wedge \delta p^k\,, \tag{4.318}$$

whose kernel, at each point, is:

$$\ker\Omega_\infty^\Sigma \;=\; \langle \left\{ \partial_k\zeta \frac{\delta}{\delta a_k} \right\} \rangle\,, \tag{4.319}$$

for some $\zeta \in \mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)$. Notice that a proof similar to that of Prop. 4.3.1 shows that the following decomposition:

$$\left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\right]^3 \;=\; \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma;\, \mathrm{div}0)\right]^3 \oplus \mathrm{grad}\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma) \tag{4.320}$$

holds, where the first term is the space of $\mathcal{H}^{\frac{3}{2}}$ functions with zero divergence and the second term is the range under the action of the gradient operator upon the set of $\mathcal{H}^{\frac{5}{2}}$ functions.

In order to perform the coisotropic embedding procedure to construct the Poisson bracket on $\mathcal{M}_\infty$ let us consider a connection on the bundle $\mathcal{M}_\infty \to \mathcal{M}_\infty/K$, K

denoting the characteristic distribution of Ω^{Σ}_{∞}. We consider, as in the previous section, the connection represented by the following (1, 1) tensor field:

$$P \,=\, P_k \otimes \frac{\delta}{\delta a_k}\,, \tag{4.321}$$

where:

$$P_k \,=\, \partial_k \int_{\Sigma} G_{\Delta}(\underline{x},\, \underline{y}) \delta^{jl} \partial_j \delta a_l(\underline{y})\, \mathrm{d}^3 y\,, \tag{4.322}$$

with G_{Δ} being the Green's function of the Laplacian operator, $\underline{x}$, $\underline{y}$ points in Σ and $\{\, \delta a_j \,\}_{j=1,2,3}$ a dual basis of $\left\{ \frac{\delta}{\delta a_j} \right\}_{j=1,2,3}$. Such a connection, gives rise to the following splitting of $\mathbf{T}_{m_\infty}\mathcal{M}_\infty \simeq \mathcal{M}_\infty$:

$$\mathbf{T}_{m_\infty}\mathcal{M}_\infty \simeq \mathcal{M}_\infty \,=\, \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_{\Sigma};\, \mathrm{div0}) \right]^3 \oplus \mathrm{grad}\mathcal{H}^{\frac{5}{2}} \times \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_{\Sigma};\, \mathrm{div0}) \right]^3 , \tag{4.323}$$

where the complement of $\mathrm{ker}\Omega^{\Sigma}_{\infty} \,=\, \mathrm{grad}\mathcal{H}^{\frac{5}{2}}$ represents the horizontal distribution associated with the connection.

The enlarged manifold obtained extending $\mathcal{M}_\infty$ via the dual of K_{m_∞} via the procedure explained in the present section is:

$$\begin{aligned} \tilde{\mathcal{M}} \,\simeq \mathbf{T}_{\tilde{m}}\tilde{\mathcal{M}} \,&=\, \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_{\Sigma};\, \mathrm{div0}) \right]^3 \oplus \mathrm{grad}\mathcal{H}^{\frac{5}{2}} \times \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_{\Sigma};\, \mathrm{div0}) \right]^3 \times \mathrm{grad}\mathcal{H}^{\frac{5}{2}\star} \simeq \\ &\simeq \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_{\Sigma};\, \mathrm{div0}) \right]^3 \oplus \mathrm{grad}\mathcal{H}^{\frac{5}{2}} \times \left[\mathcal{H}^{\frac{1}{2}}(\Sigma,\, vol_{\Sigma};\, \mathrm{div0}) \right]^3 \times \mathrm{grad}\mathcal{H}^{\frac{5}{2}}\,, \end{aligned} \tag{4.324}$$

where, since we proved that $\mathrm{grad}\mathcal{H}^{\frac{5}{2}}$ is a closed (and, thus, Hilbert) subspace of $\left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_{\Sigma}) \right]^3$, it is isomorphic to its dual and where a point will be denoted, in a system of local coordinates, by $(\tilde{a}_k,\, \partial_k\psi,\, p^k,\, \mu^k)$.

In this case, a direct computation shows that:

$$\mathrm{d}P_k \,=\, 0 \tag{4.325}$$

and, thus, we are in the case of Sect. 2.3.4. Consequently, denoting an element of $\mathbf{T}_{\tilde{m}}\tilde{\mathcal{M}}$ by $\mathbb{X} \,=\, (\tilde{\mathbb{X}}_{ak},\, \partial_k\mathbb{X}_{\psi},\, {\mathbb{X}_p}^k,\, {\mathbb{X}_\mu}^k)$ the symplectic structure on $\tilde{\mathcal{M}}_\infty$ is:

$$\tilde{\Omega}(\mathbb{X},\, \mathbb{Y}) \,=\, \underbrace{\int_{\Sigma} \left(\tilde{\mathbb{X}}_{ak} {\mathbb{Y}_p}^k - {\mathbb{X}_p}^k \tilde{\mathbb{X}}_{ak} \right) vol_{\Sigma}}_{=\Omega_W(\mathbb{X},\, \mathbb{Y})} + \underbrace{{\mathbb{X}_\mu}^k \partial_k \mathbb{Y}_{\psi} - \partial_k\mathbb{X}_{\psi} {\mathbb{Y}_\mu}^k}_{=\Omega_{K\oplus K^\star}(\mathbb{X},\, \mathbb{Y})}\,. \tag{4.326}$$

The inverse of $\tilde{\Omega}$ reads:

$$\tilde{\Lambda} \,=\, \Lambda_W \oplus \Lambda_{K\oplus K^\star}\,, \tag{4.327}$$

where:

$$\Lambda_W \,=\, \frac{\delta}{\delta\tilde{a}_k} \wedge \frac{\delta}{\delta p^k}\,, \tag{4.328}$$

using the notation (2.53). Such a bivector field is a Poisson bivector field being the inverse of a closed and non-degenerate form on W and it is projectable to the Poisson bivector field over $\mathcal{M}_\infty$:

$$\lambda \,=\, \frac{\delta}{\delta\tilde{a}_k} \wedge \frac{\delta}{\delta p^k}\,, \tag{4.329}$$

giving rise to the following Poisson bracket between any two functions on $\mathcal{M}_\infty$:

$$\{ f, g \} = \int_\Sigma \left(\frac{\delta f}{\delta \tilde{a}_k} \frac{\delta g}{\delta p^k} - \frac{\delta f}{\delta p^k} \frac{\delta g}{\delta \tilde{a}_k} \right) vol_\Sigma . \tag{4.330}$$

As an explicit example one can consider the following two functions on $\mathcal{EL}$:

$$F[\chi] = A_{k_1}(x_1) , \qquad G[\chi] = A_{k_2}(x_2) . \tag{4.331}$$

Their pull-back via $(\Psi \circ \varpi^{-1})$ is obtained by explicitly writing the solutions of the equations of motion and expressing them in terms of the Cauchy data $(a_k, p^k) \in \mathcal{M}_\infty$. In this case the Hamiltonian vector field $\mathbb{\Gamma}_\infty$ associated to $\mathcal{H}_\infty$ via Ω^Σ_∞ is determined up to elements in the kernel of Ω^Σ_∞, i.e., up to gauge transformations. However, since the Poisson bi-vector field constructed on $\mathcal{M}_\infty$ belongs to the horizontal distribution of the connection considered, the vertical part of $\mathbb{\Gamma}_\infty$ will not contribute in any way to the bracket associated to λ. Thus we could fix the (undetermined) vertical part of $\mathbb{\Gamma}_\infty$ to be zero, which means, by noting that the horizontal distribution of the connection chosen is made by divergence-less a's and p's, that we are lead with the equations of motion (4.146) where the second one is trivial since we are restricting to divergenceless a's. Thus, we only need a solution of:

$$\frac{d^2}{ds^2} a_j = \Delta a_j , \tag{4.332}$$

which reads:

$$\begin{aligned} a_{k,s}(\underline{x}) = \frac{1}{4\pi} \Bigg[& \int_\Sigma \Big(a_k(\underline{y}) + |s - x^0_\Sigma| \delta_{kj} p^j_\Sigma(\underline{y}) \Big) \tilde{G}_{\underline{x}, |s - x^0_\Sigma|}(\underline{y}) \, vol^y_\Sigma \\ & + \int_\Sigma |s - x^0_\Sigma| a_k(\underline{y}) \frac{\partial}{\partial s} \tilde{G}_{\underline{x},(s-x^0_\Sigma)}(\underline{y}) (\Theta(s - x^0_\Sigma) - \Theta(x^0_\Sigma - s)) \, vol^y_\Sigma \Bigg] , \end{aligned} \tag{4.333}$$

where $\tilde{G}_{\underline{x},a}(\underline{y})$ is the characteristic function of the surface of the sphere centered at $\underline{x}$ with radius a and Θ is the Heaviside function. Consequently, the bracket is computed to be:

$$\begin{aligned} \{g, f\}_{(a,p)} = \frac{\delta_{k_1 k_2}}{16\pi^2} \int_\Sigma \Bigg\{ & \Big(|x^0_2 - x^0_\Sigma| - |x^0_1 - x^0_\Sigma| \Big) \tilde{G}_{\underline{x}_1, |x^0_1 - x^0_\Sigma|}(\underline{y}) \tilde{G}_{\underline{x}_2, |x^0_2 - x^0_\Sigma|}(\underline{y}) \\ & + |x^0_1 - x^0_\Sigma| |x^0_2 - x^0_\Sigma| \Bigg[\frac{\partial}{\partial s} \tilde{G}_{\underline{x}_1,(s-x^0_\Sigma)}(\underline{y}) \Big|_{s=x^0_1} \Big(\Theta(x^0_1 - x^0_\Sigma) - \Theta(x^0_\Sigma - x^0_1) \Big) \tilde{G}_{\underline{x}_2, |x^0_2 - x^0_\Sigma|}(\underline{y}) \\ & - \frac{\partial}{\partial s} \tilde{G}_{\underline{x}_2,(s-x^0_\Sigma)}(\underline{y}) \Big|_{s=x^0_2} \Big(\Theta(x^0_2 - x^0_\Sigma) - \Theta(x^0_\Sigma - x^0_2) \Big) \tilde{G}_{\underline{x}_1, |x^0_1 - x^0_\Sigma|}(\underline{y}) \Bigg] \Bigg\} vol^y_\Sigma . \end{aligned} \tag{4.334}$$

The bracket between F and G is obtained, again, by pulling-back the latter function to $\mathcal{EL}$, which gives, again, a function with the same expression but that should be thought of as a function on $\mathcal{EL}$.

Yang-Mills theories

Within Yang-Mills theories, we saw, in example 4.2.12 that the final manifold obtained out of the pre-symplectic constraint algorithm is:

$$\begin{aligned}\mathcal{M}_\infty &= \left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)\right]^{3\dim\mathfrak{g}} \times \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma, \nabla 0)\right]^{3\dim\mathfrak{g}} \ni \\ &\ni (a_k^a,\, p_a^k)\,,\end{aligned} \tag{4.335}$$

That this is a Hilbert space, is due to the fact that $\left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)\right]^{3\dim\mathfrak{g}}$ is a Hilbert space and that $\left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma;\, \nabla 0)\right]^{3\dim\mathfrak{g}}$ is a closed (and, thus, Hilbert) subspace of $[\,\mathcal{H}^1(\Sigma, vol_\Sigma)\,]^{3\dim\mathfrak{g}}$. Indeed, ∇ acts as a linear operator between the Hilbert spaces $\left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)\right]^{\dim\mathfrak{g}}$ and $\left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\right]^{3\dim\mathfrak{g}}$:

$$\nabla \quad : \quad \left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)\right]^{\dim\mathfrak{g}} \to \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\right]^{3\dim\mathfrak{g}} \quad : \quad f^a \mapsto \nabla_k f^a\,. \tag{4.336}$$

Its adjoint is an operator from $\left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)^\star\right]^{3\dim\mathfrak{g}}$ (that can be identified with $\left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\right]^{3\dim\mathfrak{g}}$) itself, to $\left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)^\star\right]^{\dim\mathfrak{g}}$ (that can be identified with $\left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)\right]^{\dim\mathfrak{g}}$):

$$\nabla^\star \quad : \quad \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\right]^{3\dim\mathfrak{g}} \to \left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)\right]^{\dim\mathfrak{g}} \quad : \quad p_a^k \mapsto \nabla_k p_a^k\,. \tag{4.337}$$

The following holds.

Proposition 4.3.2 (Closedness of ∇ in $\mathcal{H}^{\frac{3}{2}}$)**.** *The operator ∇ is a closed operator from $\left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)\right]^{\dim\mathfrak{g}}$ to $\left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma)\right]^{3\dim\mathfrak{g}}$.*

Proof. Consider a sequence of functions in $\left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)\right]^{\dim\mathfrak{g}}$, say $\{\, f_n^a \,\}_{n\in\mathbb{N}}$ converging to some $f^a \in \left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma)\right]^{\dim\mathfrak{g}}$ in the $\mathcal{H}^{\frac{5}{2}}$-norm. Then $\nabla_k f_n^a$ converges to $\nabla_k f^a$ in the $\mathcal{H}^{\frac{3}{2}}$-norm. Indeed:

$$\begin{aligned}\sum_{k,a} \|\nabla_k f_n^a - \nabla_k f^a\|_{\mathcal{H}^{\frac{3}{2}}} &= \sum_{k,a} \|\nabla_k (f_n^a - f^a)\|_{\mathcal{H}^{\frac{3}{2}}} = \sum_{k,a} \|\partial_k (f_n^a - f^a) + \epsilon^a{}_{bc} a_k^b (f_n^c - f^c)\|_{\mathcal{H}^{\frac{3}{2}}} = \\ &\le \sum_{k,a} \|\partial_k (f_n^a - f^a)\|_{\mathcal{H}^{\frac{3}{2}}} + \sum_{k,a} + \sum_{k,a} |\epsilon^a{}_{bc}| \|a_k^b (f_n^c - f^c)\|_{\mathcal{H}^{\frac{3}{2}}} \le \\ &\le 3 \sum_a \|f_n^a - f^a\|_{\mathcal{H}^{\frac{5}{2}}} + \sum_{k,a} |\epsilon^a{}_{bc}| \|a_k^b\|_{\mathcal{H}^{\frac{5}{2}}} \|f_n^c - f^c\|_{\mathcal{H}^{\frac{5}{2}}}\,,\end{aligned} \tag{4.338}$$

where the last inequality is due to the content of footnote 6 and to the fact that $\mathcal{H}^{\frac{5}{2}}(\Sigma, vol_\Sigma)$ is a Banach algebra. Because of the latter inequality, $\sum_{k,a} \|\nabla_k f_n^a - \nabla_k f^a\|_{\mathcal{H}^{\frac{3}{3}}}$ approaches zero when $\|f_n^c - f^c\|_{\mathcal{H}^{\frac{5}{2}}}$ approaches zero. Thus, by definition of closed operator, ∇ is closed. $\square$

Therefore, by means of the closed range theorem, the kernel of the adjoint of ∇, i.e.:

$$\left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_\Sigma;\, \nabla 0)\right]^{3\dim\mathfrak{g}}\,, \tag{4.339}$$

is a closed split subspace of $\left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_{\Sigma})\right]^{3\dim\mathfrak{g}}$ whose orthogonal complement coincide with the image of ∇, say $\nabla\left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_{\Sigma})\right]^{\dim\mathfrak{g}}$. That is, the following splitting into closed (and, thus, Hilbert) subspaces exists:

$$\left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_{\Sigma})\right]^{3\dim\mathfrak{g}} \,=\, \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_{\Sigma};\, \nabla 0)\right]^{3\dim\mathfrak{g}} \oplus \nabla\left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_{\Sigma})\right]^{\dim\mathfrak{g}}\,, \tag{4.340}$$

and the p^k_a's of $\mathcal{M}_\infty$ lie exactly in the first component of such splitting. Being $\mathcal{M}_\infty$ a Hilbert space is is isomorphic to its tangent space at each point:

$$\mathbf{T}_{(a,p)}\mathcal{M}_\infty \,=\, \left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_{\Sigma})\right]^{3\dim\mathfrak{g}} \times \left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_{\Sigma},\nabla 0)\right]^{3\dim\mathfrak{g}}\,. \tag{4.341}$$

Such tangent space also coincide with the space of solutions of the linearization of the constraint $\nabla_k p^k_a \,=\, 0$, i.e., with the space of functions $\mathbb{X}_{a\,k}^{\;a}$ and $\mathbb{X}_{p_a}^{\;k}$ (representing the components of the tangent vector) satisfying:

$$\nabla^\star_k \mathbb{X}_{p_a}^{\;k} \,=\, [p^k,\, \mathbb{X}_{ak}]_a\,, \tag{4.342}$$

as the following proposition proves.

Proposition 4.3.3. *The space of solutions of:*

$$\nabla_k \mathbb{X}_{p_a}^{\;k} \,=\, [p^k,\, \mathbb{X}_{ak}]_a\,, \tag{4.343}$$

is an affine space modelled over the vector space $\left[\mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_{\Sigma};\, \nabla 0)\right]^{3\dim\mathfrak{g}}$.

Proof. As we proved above, $\mathcal{H}^{\frac{3}{2}}(\Sigma, vol_{\Sigma})$ splits as $\mathcal{H}^{\frac{3}{2}}(\Sigma, vol_{\Sigma}) \,=\, \mathcal{H}^{\frac{3}{2}}(\Sigma,\, vol_{\Sigma};\, \nabla 0)\oplus \nabla\mathcal{H}^{\frac{5}{2}}(\Sigma, vol_{\Sigma})$. Let us denote by $\tilde{\mathbb{X}}_{p_a}^{\;k}$ and $\nabla_k \mathbb{X}^a_\psi$ the components of the p_k-component of $\mathbb{X}$ in such a splitting. Then, equation (4.343) reads:

$$\Delta \mathbb{X}^a_\psi \,=\, [p^k,\, \mathbb{X}_{ak}]_a\,, \tag{4.344}$$

where Δ is the covariant Laplacian. The last equation has a unique solution for any fixed $\mathbb{X}_{a\,k}^{\;a}$ given by the action of the Green function of Δ on the right hand side. This means that solutions of (4.343) are parametrized by all the $\tilde{\mathbb{X}}_{p_a}^{\;k}$ (belonging to $\left[\mathcal{H}^{\frac{3}{2}}(\Sigma\,, vol_{\Sigma};\, \nabla 0)\right]^{3\dim\mathfrak{g}}$) and by a particular solution of (4.344), i.e., it is an affine space modelled over the vector space $\left[\mathcal{H}^{\frac{3}{2}}(\Sigma\,, vol_{\Sigma};\, \nabla 0)\right]^{3\dim\mathfrak{g}}$. □

Now, following what we said in the present section, we will use Ω^Σ_∞ to define a Poisson bracket on the solution space of the theory. Again we lie in the case in which Ω^Σ_∞ is pre-symplectic, therefore, in order to define a Poisson bracket on $\mathcal{M}_\infty$, we will use again the coisotropic embedding theorem. In this case the connection we will fix on the bundle $\mathcal{M}_\infty \to \mathcal{M}_\infty/\ker\Omega^\Sigma_\infty$, is the one introduced in [95], [96]. Such a connection is represented by the following (1, 1) tensor field over $\mathcal{M}_\infty$:

$$P \,=\, P^a_k \otimes \frac{\delta}{\delta a^a_k} + [p^k, \mathscr{G}]_a \otimes \frac{\delta}{\delta p^k_a} \in \mathscr{T}^1_1(\mathcal{M}_\infty)\,, \tag{4.345}$$

following the notation (2.53), where:

$$P^a_k \;=\; \nabla_k \int_\Sigma G_\Delta \eta^{lj} \nabla_l \delta a^a_j vol_\Sigma \,, \qquad \mathscr{G}^a \;=\; \int_\Sigma G_\Delta \eta^{lj} \nabla_l \delta a^a_j \, vol_\Sigma \,, \tag{4.346}$$

with G_Δ being the Green function of the covariant Laplacian opertator $\Delta \,=\, \eta^{jk}\nabla_j \nabla_k$ and $\left\{ \delta a^a_j,\, \delta p^j_a \right\}$ being a basis of differential one forms dual to the basis of vector fields $\left\{ \frac{\delta}{\delta a^a_k},\, \frac{\delta}{\delta p^k_a} \right\}_{a=1,\dots,dim\mathfrak{g},k=1,2,3}$. Note that, (4.345) is actually a connection on the bundle $\mathcal{M}_\infty \to \mathcal{M}_\infty / \mathrm{ker}\Omega^\Sigma_\infty$ because it is the identity on vertical tangent vectors, i.e.:

$$P(\mathbb{V}_\psi) \;=\; \mathbb{V}_\psi \,, \tag{4.347}$$

for $\mathbb{V}_\psi \in \mathrm{ker}\Omega^\Sigma_\infty$ and, as it is proven in [95], [96], it is equivariant with respect to the vertical automorphisms of the bundle. We will denote by $R \,:=\, \mathbb{1} - P$ the projector over horizontal vector fields. The latters are indeed defined to be the image of R, say $\mathrm{Im}R$.

Now, in order to apply the coisotropic embedding procedure, we should identify the dual of the vector space spanned at each point of $\mathcal{M}_\infty$ by $\mathbb{V}_\psi$. By looking at the expression of $\mathbb{V}_\psi$, the subspace of $\mathbf{T}_{(a,p)}\mathcal{M}_\infty$ spanned by $\mathbb{V}_\psi$ at (a,p), is parametrized by the $\psi^a \in \left[\mathcal{H}^{\frac{7}{2}}(\Sigma,\, vol_\Sigma) \right]^{\mathrm{dim}\mathfrak{g}}$ and is the subspace of $\mathbf{T}_{(a,p)}\mathcal{M}_\infty$ given by the image of the operator $\nabla \oplus [p^k,\, \cdot\,]$ acting on $\left[\mathcal{H}^{\frac{7}{2}}(\Sigma,\, vol_\Sigma) \right]^{\mathrm{dim}\mathfrak{g}}$:

$$\mathscr{V} \;:=\; \prod_{k,a} \nabla \left[\mathcal{H}^{\frac{7}{2}}(\Sigma,\, vol_\Sigma) \right]^{\mathrm{dim}\mathfrak{g}} \times \left[p^k,\, \left[\mathcal{H}^{\frac{7}{2}}(\Sigma,\, vol_\Sigma) \right]^{\mathrm{dim}\mathfrak{g}} \right] \,, \tag{4.348}$$

where ∇ is the covariant derivative associated with the fixed connection a of the point $(a,p) \in \mathcal{M}_\infty$. That $\mathscr{V}$ is a Hilbert space itself, is a consequence of the fact that $\nabla \oplus [p^k,\, \cdot\,]$ is a closed operator acting on $\left[\mathcal{H}^{\frac{7}{2}}(\Sigma,\, vol_\Sigma) \right]^{\mathrm{dim}\mathfrak{g}}$. Indeed, the following two propositions hold.

Proposition 4.3.4 (Closedness of ∇ in $\mathcal{H}^{\frac{5}{2}}$). ∇ *is a closed operator from* $\left[\mathcal{H}^{\frac{7}{2}}(\Sigma,\, vol_\Sigma) \right]^{\mathrm{dim}\mathfrak{g}}$ *into* $\left[\mathcal{H}^{\frac{5}{2}}(\Sigma,\, vol_\Sigma) \right]^{3\mathrm{dim}\mathfrak{g}}$.

Proof. The proof is analogous to the of Prop. 4.3.2. □

With the same techniques, the following can be proved.

Proposition 4.3.5 (Closedness of $[p^k,\, \cdot\,]$). $[p^k,\, \cdot\,]$ *is a closed operator from* $\left[\mathcal{H}^{\frac{7}{2}}(\Sigma,\, vol_\Sigma) \right]^{\mathrm{dim}\mathfrak{g}}$ *to* $\left[\mathcal{H}^{\frac{7}{2}}(\Sigma,\, vol_\Sigma;\, \nabla 0) \right]^{\mathrm{dim}\mathfrak{g}}$.

Being $\mathscr{V}$ a Hilbert space, it has a well defined dual space (isomorphic with $\mathscr{V}$ itself). Let us denote it by $\mathscr{V}^\star$ let us denote by $\mathbb{X}_{\mu^k_a}$ its elements.

In this case, since $\mathrm{d}P^a_k$ is different from zero for the connection chosen here, we are in the case described in Sec. 2.3.4. The symplectic manifold one constructs here is a tubular neinghborhood of the zero section of the bundle $\mathbf{K}^\star$ over $\mathcal{M}_\infty$ with typical fibre $\mathscr{V}^\star$. Denote by $\tilde{\mathcal{M}}$ such a manifold and by $(a^a_k,\, p^k_a,\, \mu^k_a)$ a system of coordinates. Then the symplectic structure on $\tilde{\mathcal{M}}$ reads:

$$\tilde{\Omega} \;=\; \tau^\star \Omega^\Sigma_\infty\Big|_{\mathrm{Im}\mathcal{R}} + \mathrm{d}\mu_{a_a}{}^k \wedge P^a_k + \mathrm{d}\mu_{p_a}{}^k \wedge [p^k,\, \mathscr{G}]_a + \mu^k_a \mathrm{d}P^a_k\,, \tag{4.349}$$

where τ is the projection from $\tilde{\mathcal{M}}$ to $\mathcal{M}_\infty$. Then, given two functions on $\mathcal{M}_\infty$, say f and g, the structure $\tilde{\Omega}$ above, allows to write the Poisson bracket between $\tilde{f} := \tau^\star f$ and $\tilde{g} := \tau^\star g$, which reads:

$$\{\,\tilde{f},\, \tilde{g}\,\} \;=\; \tilde{\Omega}(\mathbb{X}_{\tilde{f}},\, \mathbb{X}_{\tilde{g}})\,, \tag{4.350}$$

where $\mathbb{X}_{\tilde{f}}$ and $\mathbb{X}_{\tilde{g}}$ are the Hamiltonian vector fields associated with $\tilde{f}$ and $\tilde{g}$ via $\tilde{\Omega}$.

For the lack of absence of an analytic solution for Yang-Mills equations, here, as a specific example, we consider the following functions on $\mathcal{M}_\infty$:

$$f \;=\; p^{k_1}_{a_1}(\underline{x}_1) \qquad g \;=\; p^{k_2}_{a_2}(\underline{x}_2)\,. \tag{4.351}$$

Their pull-back to $\tilde{\mathcal{M}}$ reads:

$$\tilde{f} \;=\; p^{k_1}_{a_1}(\underline{x}_1) \qquad \tilde{g} \;=\; p^{k_2}_{a_2}(\underline{x}_2)\,. \tag{4.352}$$

Then, a direct computation shows that:

$$\begin{aligned}\left\{\tilde{g}, \tilde{f}\right\} \;=\; \mathbb{X}_{\tilde{f}}(\tilde{g}) \;=\; -2\int_\Sigma \Bigg[\, & \mu_{a_a}{}^k(\underline{y}) \nabla^y_k G_\Delta(\underline{y}, \underline{x}) \delta^{k_1 k_2} \epsilon^a{}_{a_1 a_2} \delta(\underline{x}, \underline{x}_1) \delta(\underline{x}, \underline{x}_2) + \\ & + \mu_{p_k}{}^a(\underline{y}) G_\Delta(\underline{y}, \underline{x}) p^k_b(\underline{y}) \delta^{k_1 k_2} \epsilon_a{}^{bc} \epsilon_{c a_1 a_2} \delta(\underline{x}, \underline{x}_1) \delta(\underline{x}, \underline{x}_2)\Bigg] \mathrm{d}^3x \mathrm{d}^3y\,,\end{aligned} \tag{4.353}$$

where one sees that, even if the functions we started with are pull-back of functions on $\mathcal{M}_\infty$, that is, they do not depend on the additional degrees of freedom μ_a and μ_p, the latters do appear in the Poisson bracket.

5. CONCLUSIONS

In the present manuscript we concluded the program, partially developed in [1]–[4], [47], [48], of constructing a Poisson bracket on the space of the solutions of the equations of motion (always referred to as SOLUTION SPACE) of a large class of classical field theories, namely, the classical counterpart of quantum field theories describing fundamental interactions.

To resume, we showed how, within the multi-symplectic formulation of field theories, the solution space is canonically equipped with a pre-symplectic 2-form.

In particular we saw that for some theories (those not exhibiting gauge symmetries) it is strongly symplectic and, thus, it gives rise to a Poisson bracket expressed in terms of the bivector field being the inverse of the symplectic structure.

On the other hand, for those theories exhibiting a gauge symmetry, the canonical 2-form has instead a non-trivial kernel. Here, we saw that by means of the coisotropic embedding theorem a symplectic, and, thus, a Poisson, structure on a suitable enlargement of the solution space can be always defined. Moreover, we saw that for those theories for which a flat connection can be fixed within the coisotropic embedding procedure (that amounts to the possibility of performing a global gauge fixing in the space of fields), such Poisson structure projects to a Poisson structure on the solution space. When this is not the case (this happens for theories exhibiting Gribov's ambiguities) the use of additional degrees of freedom, interpreted as the BRST ghost fields, is necessary in order to define a Poisson structure.

BIBLIOGRAPHY

[1] F. M. Ciaglia, F. Di Cosmo, L. A. Ibort, G. Marmo, L. Schiavone, and A. Zampini, "The geometry of the solution space of first order Hamiltonian field theories I: from particle dynamics to Electrodynamics," *arXiv:2208.14136*,

[2] F. M. Ciaglia, F. Di Cosmo, L. A. Ibort, G. Marmo, L. Schiavone, and A. Zampini, "The geometry of the solution space of first order Hamiltonian field theories II: non-Abelian gauge theories," *arXiv:2208.14155*,

[3] F. M. Ciaglia, F. Di Cosmo, L. A. Ibort, G. Marmo, L. Schiavone, and A. Zampini, "Symmetries and Covariant Poisson Brackets on Presymplectic Manifolds," *Symmetry*, vol. 14, no. 70, pp. 1–28, 2022.

[4] L. Schiavone, "Poisson brackets on the space of solutions of first order Hamiltonian field theories," in *Proceedings of the 3rd BYMAT Conference*, 2021, pp. 191–194.

[5] N. Bohr, "On the Quantum Theory of Line Spectra," *D. Kgl. Danske Vidensk. Selsk. Skrifter, Naturvidensk. og Mathem.*, vol. 8, no. 1, 1918.

[6] G. Wentzel, "Eine Verallgemeinerung der Quantenbedingungen fur die Zwecke der Wellenmechanik," *Zeitschrift fur Physik*, vol. 38, no. 6 - 7, pp. 518–529, 1926.

[7] H. A. Kramers, "Wellenmechanik und halbzahlige Quantisierung," *Zeitschrift fur Physik*, vol. 39, no. 10-11, pp. 828–840, 1926.

[8] L. Brillouin, "La mécanique ondulatoire de Schrodinger: une méthode générale de resolution par approximations successives," *Comptes Rendus de l'Académie des Sciences*, vol. 183, pp. 24–26, 1926.

[9] J. E. Moyal and M. S. Bartlett, "Quantum mechanics as a statistical theory," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 45, no. 1, pp. 99–124, 1949.

[10] H. J. Groenewold, "On the principles of elementary quantum mechanics," *Physica*, vol. 12, no. 7, pp. 405–460, 1946.

[11] R. J. Glauber, "Coherent and Incoherent States of the Radiation Field," *Physical Review*, vol. 131, no. 6, pp. 2766–2788, 1963.

[12] E. C. G. Sudarshan, "Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams," *Physical Review Letters*, vol. 10, no. 7, pp. 277–279, 1963.

[13] B. Helffer, A. Martinez, and D. Robert, "Ergodicité et limite semi-classique," *Comm. Math, Phys.*, vol. 109, no. 2, pp. 313–326, 1987.

[14] A. Figalli, M. Ligabò, and T. Paul, "Semiclassical Limit for Mixed States with Singular and Rough Potentials," *Indiana University Mathematics Journal*, vol. 61, no. 1, pp. 192–222, 2012.

[15] P. A. M. Dirac, "The fundamental equations of Quantum Mechanics," *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, vol. 109, pp. 642–653, 1925.

[16] R. E. Peierls, "The commutation laws of relativistic field theory," *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, vol. 214, no. 1117, pp. 143–157, 1952.

[17] B. S. DeWitt, *The Global approach to Quantum Field Theory*. Oxford: Claredon Press, 2003.

[18] B. S. DeWitt, "Invariant commutators for the Quantized Gravitational Field," *Physical Review Letters*, vol. 4, no. 6, pp. 317–320, 1960.

[19] B. S. DeWitt, *Dynamical theory of groups and fields*. New York: Gordon and Breach, 1965.

[20] J. Kijowski, "A finite-dimensional canonical formalism in the Classical Field Theory," *Communications in Mathematical Physics*, vol. 30, no. 2, pp. 99–128, 1973.

[21] J. Kijowski and W. Szczyrba, "A canonical structure for classical field theories," *Communications in Mathematical Physics*, vol. 46, no. 2, pp. 183–206, 1976.

[22] W. M. Tulczyjew and J. Kijowski, *A Symplectic Framework for Field Theories*. Berlin Heidelberg: Springer, 1979.

[23] P. L. García and A. Pérez-Rendón, "Symplectic approach to the theory of quantized fields. I," *Communications in Mathematical Physics*, vol. 13, no. 1, pp. 24–44, 1969.

[24] P. L. García and A. Pérez-Rendón, "Symplectic approach to the theory of quantized fields. II," *Archive for Rational Mechanics and Analysis*, vol. 43, no. 2, pp. 101–124, 1971.

[25] D. Krupka, O. Krupkova, and D. J. Saunders, "The cartan form and its generalizations in the calculus of variations," *International Journal of Geometric Methods in Modern Physics*, vol. 7, no. 4, pp. 631–654, 2010.

[26] D. Krupka, O. Krupkova, and D. J. Saunders, "Cartan-Lepage forms in geometric mechanics," *International Journal of Non-Linear Mechanics*, vol. 47, no. 10, pp. 1154–1160, 2012.

[27] O. Krupkova, *The geometry of ordinary variational equations*. Berlin: Springer-Verlag, Berlin, 1997, p. 251.

[28] L. Fatibene and M. Francaviglia, *Natural and Gauge Natural Formalism for Classical Field Theories*. Heidelberg: Springer Netherlands, 2003.

[29] H. Goldschmidt and S. Sternberg, “The Hamilton-Cartan formalism in the calculus of variations,” *Annales de l’institut Fourier*, vol. 23, no. 1, pp. 203–267, 1973.

[30] N. Román-Roy, “Multisymplectic Lagrangian and Hamiltonian formalisms of classical field theories,” *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)*, vol. 5, no. 100, pp. 1–25, 2009.

[31] J. Berra-Montiel, A. Molgado, and A. Rodriguez-Lopez, “A review on geometric formulations for classical field theory: the Bonzom-Livine model for gravity,” *Classical and Quantum Gravity*, 2021.

[32] J. Margalef-Bentabol and E. J. S. Villasenor, “Geometric formulation of the covariant phase space methods with boundaries,” *Physical Review D*, vol. 103, no. 2, 2021.

[33] M. J. Gotay, “A Multisymplectic Framework for Classical Field Theory and the Calculus of Variations I. Covariant Hamiltonian Formalism,” in *Mechanics, Analysis and Geometry: 200 Years after Lagrange*, M. Francaviglia, Ed., 1991, pp. 203–235.

[34] M. J. Gotay, “A multisymplectic framework for classical field theory and the calculus of variations II: space + time decomposition,” *Differential Geometry and its Applications*, vol. 1, no. 4, pp. 375–390, 1991.

[35] G. Giachetta, L. Mangiarotti, and G. Sardanashvily, *New Lagrangian and Hamiltonian Methods in Field Theory*. Signapore: World Scientific Publishing Co., 1997.

[36] J. F. Cariñena, M. Crampin, and L. A. Ibort, “On the multisymplectic formalism for first order field theories,” *Differential Geometry and its Applications*, vol. 1, no. 4, pp. 345–374, 1991.

[37] L. A. Ibort and A. Spivak, “Covariant Hamiltonian field theories on manifolds with boundary: Yang-Mills theories,” *Journal of Geometric Mechanics*, vol. 9, no. 1, pp. 47–82, 2017.

[38] E. Binz, J. Sniatycki, and H. Fischer, *Geometry of classical fields*. Amsterdam: North-Holland, 1988, p. 450.

[39] A. Echeverria-Enríquez, M. C. Muñoz-Lecanda, and N. Román-Roy, “Geometry of multisymplectic Hamiltonian first-order field theories,” *Journal of Mathematical Physics*, vol. 41, no. 11, pp. 7402–7444, 2000.

[40] A. Echeverría-Enríquez, M. C. Muñoz-Lecanda, and N. Román-Roy, “Geometry of Lagrangian First-order Classical Field Theories,” *Fortschritte der Physik/Progress of Physics*, vol. 44, no. 3, pp. 235–280, 1996.

[41] J. E. Marsden, R. Montgomery, P. J. Morrison, and W. B. Thompson, “Covariant poisson brackets for classical fields,” *Annals of Physics*, vol. 169, no. 1, pp. 29–47, 1986.

[42] M. Forger and S. V. Romero, "Covariant poisson brackets in geometric field theory," *Communications in Mathematical Physics*, vol. 256, no. 2, pp. 375–410, 2005.

[43] M. Forger and M. O. Salles, "On covariant poisson brackets in classical field theory," *Journal of Mathematical Physics*, vol. 56, no. 10, p. 102 901, 2015.

[44] I. Khavkine, "Covariant phase space, constraints, gauge and the Peierls formula," *International Journal of Modern Physics A*, vol. 29, no. 5, 2014.

[45] M. Asorey, F. M. Ciaglia, F. Di Cosmo, L. A. Ibort, and G. Marmo, "Covariant Jacobi brackets for test particles," *Modern Physics Letters A*, vol. 32, no. 23, 2017.

[46] M. Asorey, F. M. Ciaglia, F. Di Cosmo, and L. A. Ibort, "Covariant brackets for particles and fields," *Modern Physics Letters A*, vol. 32, no. 19, 2017.

[47] F. M. Ciaglia, F. Di Cosmo, L. A. Ibort, G. Marmo, and L. Schiavone, "Covariant variational evolution and Jacobi brackets: Particles," *Modern Physics Letters A*, vol. 35, no. 23, 1 –17 –10.1142/S0217732320200011, 2020.

[48] F. M. Ciaglia, F. Di Cosmo, L. A. Ibort, G. Marmo, and L. Schiavone, "Covariant Variational Evolution and Jacobi brackets: Fields," *Modern Physics Letters A*, vol. 35, no. 23, 1 –16 –10.1142/S0217732320502065, 2020.

[49] F. Gieres, "Covariant canonical formulations of classical field theories," *ArXiv:2109.07330*, 2021.

[50] J. M. Souriau, *Structure of Dynamical Systems : a Symplectic View of Physics*. Boston: Birkhauser, 1997, p. 444.

[51] G. J. Zuckerman, "Action principles and Global Geometry," in *Advanced Series in Mathematical Physics, Mathematical Aspects of String Theory*, S. T. Yau, Ed., World Scientific Pub Co., 1987, pp. 259–284.

[52] C. Crnkovic and E. Witten, "Covariant description of canonical formalism in geometrical theories," in *Three hundred years of Gravitation*, S. W. Hawking and W. Israel, Eds., 1986, pp. 676–684.

[53] C. Crnkovic, "Symplectic Geometry of the Covariant Phase Space," *Classical and Quantum Gravity*, vol. 5, no. 12, p. 1557, 1988.

[54] L. Vitagliano, "Secondary calculus and the covariant phase space," *Journal of Geometry and Physics*, vol. 59, no. 4, pp. 426–447, 2009.

[55] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*. Princeton: Princeton University Press, 1994.

[56] F. M. Ciaglia, F. Di Cosmo, L. A. Ibort, G. Marmo, and L. Schiavone, "Covariant reduction of classical Hamiltonian Field Theories: From D'Alembert to Klein-Gordon and Schrödinger," *Modern Physics Letters A*, vol. 35, no. 23, pp. 1–14, 2020.

[57] R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, Tensor Analysis, and Applications* (Applied Mathematical Sciences). New York: Springer, 1988.

[58] P. W. Michor, I. Kolár, and J. Slovàk, *Natural operations in Differential Geometry*. Berlin Heidelberg: Springer-Verlag, 1993.

[59] P. W. Michor, *Manifolds of Differentiable Mappings*. Shiva Publishing Limited, Kent, 1980.

[60] S. Lang, *Differential Manifolds*. New York: Springer-Verlag, 1985.

[61] D. Beltita, T. Golinski, and A. B. Tumpach, "Queer Poisson brackets," *Journal of Geometry and Physics*, vol. 132, pp. 358–362, 2018.

[62] P. W. Michor and A. Kriegl, *The Convenient Setting of Global Analysis*. American Mathematical Society, Mathematical Surveys and Monographs, 1997, pp. 1–608.

[63] R. Abraham, J. E. Marsden, T. Ratiu, and R. Cushman, *Foundations of Mechanics*. Redwood City: Addison-Wesley Publishing Company, Inc., 1978, p. 806.

[64] V. I. Arnold, *Mathematical methods of Classical Mechanics*. New York: Springer-Verlag, 1989, p. 508.

[65] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*. Cambridge University Press, 1990, p. 468.

[66] J. P. Ortega and T. Ratiu, *Momentum Maps and Hamiltonian Reduction*. Basel: Birkhauser, 2004.

[67] W. M. Tulczyjew, "Poisson brackets and canonical manifolds," *Bull. Acad. Pol. Sci. (Math. and Astronomy)*, vol. 22, pp. 931–934, 1974.

[68] M. J. Gotay, "Presymplectic Manifolds, Geometric Constraint Theory and the Dirac-Bergmann Theory of Constraints," Ph.D. dissertation, University of Maryland College Park, 1979.

[69] M. J. Gotay, "On coisotropic imbeddings of presymplectic manifolds," *Proceedings of the American Mathematical Society*, vol. 84, no. 1, pp. 111–114, 1982.

[70] Y. G. Oh and J. S. Park, "Deformations of coisotropic submanifolds and strong homotopy Lie algebroids," *Inventiones Mathematicae*, vol. 161, pp. 287–360, 2005.

[71] B. A. Dubrovin, M. Giordano, G. Marmo, and A. Simoni, "Poisson brackets on presymplectic manifolds," *International Journal of Modern Physics A*, vol. 08, no. 21, pp. 3747–3771, 1993.

[72] D. J. Saunders, *The Geometry of Jet Bundles*. Cambridge: Cambridge University Press, 1989.

[73] O. Rossi and D. J. Saunders, “Dual jet bundles, Hamiltonian systems and connections,” *Differential Geometry and its Applications*, vol. 35, pp. 178–198, 2014.

[74] D. Krupka, *Introduction to global variational geometry.* Atlantis Press, 2015, pp. 1–354.

[75] D. Krupka, “The contact ideal,” *Differential Geometry and its Applications*, vol. 5, no. 3, pp. 257–276, 1995.

[76] V. Aldaya and J. A. Azcarraga, “Geometric formulation of Classical Mechanics and Field Theory,” *Riv. Nuovo Cimento*, vol. 3, no. 10, pp. 1–66, 1980.

[77] G. Giachetta, L. Mangiarotti, and G. Sardanashvily, *Advanced classical field theory.* Singapore: World Scientific Publishing Co., 2009, pp. 1–382.

[78] G. Marmo, E. Saletan, A. Simoni, and B. Vitale, *Dynamical systems : a differential geometric approach to symmetry and reduction.* Chichester: John Wiley & Sons Inc, 1985, p. 390.

[79] F. M. Ciaglia, G. Marmo, and L. Schiavone, “From classical trajectories to quantum commutation relations,” in *Springer Proceedings in Physics*, vol. 229, Springer, 2019, pp. 163–185.

[80] J. Sniatycki, “On the Geometric Structure of Classical Field Theory in Lagrangian Formulation,” *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 68, no. 2, pp. 475–484, 1970.

[81] M. Horák and I. Kolár, “On the higher order Poincaré-Cartan forms,” *Czechoslovak Mathematical Journal*, vol. 33, no. 3, pp. 467–475, 1983.

[82] J. Sniatycki and R. Segev, “De Donder Construction for Higher Jets,” *http://arxiv.org/abs/1808.0* 2018.

[83] I. Kolár, “A geometrical version of the higher order Hamilton formalism in fibred manifolds,” *Journal of Geometry and Physics*, vol. 1, no. 2, pp. 127–137, 1984.

[84] G. Morandi, C. Ferrario, G. Lo Vecchio, G. Marmo, and C. Rubano, “The inverse problem in the calculus of variations and the geometry of the tangent bundle,” *Physics Reports*, vol. 188, no. 3-4, pp. 147–284, 1990.

[85] W. M. Tulczyjew, *Geometric Formulations of Physical Theories*, Lecture no. Bibliopolis, 1989, pp. 1–129.

[86] G. Mendella, G. Marmo, and W. M. Tulczyjew, “Integrability of implicit differential equations,” *Journal of Physics A: Mathematical and General*, vol. 28, no. 1, pp. 149–163, 1995.

[87] F. Bascone, F. Pezzella, and P. Vitale, “Jacobi sigma models,” *Journal of High Energy Physics*, vol. 110, pp. 1–20, 2021.

[88] R. A. Adams, *Sobolev spaces.* New York: Academic Press, 1975.

[89] J. Baez and J. P. Muniain, *Gauge Fields, Knots and Gravity* (Series on Knots and Everything). Singapore: World Scientific, 1994.

[90] G. Giachetta, L. Mangiarotti, and G. Sardanashvily, *Geometric formulation of Classical and Quantum Mechanics*. Singapore: World Scientific Publishing Co., 2010, pp. 1–392.

[91] C. Rovelli, *Quantum Gravity*. Cambridge: Cambridge University Press, 2004.

[92] M. Spivak, *A comprensive introduction to Differential Geometry*. Houston: Publish or Perish, Inc., 1999.

[93] R. Dautray and J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology - vol 2*. Berlin Heidelberg: Springer-Verlag, 1990.

[94] M. Asorey, L. A. Ibort, and A. Spivak, "Admissible boundary conditions for Hamiltonian field theories," *International Journal of Geometric Methods in Modern Physics*, vol. 8, no. 1740006, pp. 1–17, 2017.

[95] M. S. Narasimhan and T. R. Ramadas, "Geometry of SU(2) gauge fields," *Comm. Math, Phys.*, vol. 67, no. 2, pp. 121–136, 1979.

[96] I. M. Singer, "Some remarks on Gribov Ambiguity," *Communications in Mathematical Physics*, vol. 60, pp. 7–12, 1978.