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ORIGINAL ARTICLE

Sublinear scalarizations for proper and approximate proper efficient points in nonconvex vector optimization

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Abstract
We show that under a separation property, a $\mathcal{Q}$-minimal point in a normed space is the minimum of a given sublinear function. This fact provides sufficient conditions, via scalarization, for nine types of proper efficient points; establishing a characterization in the particular case of Benson proper efficient points. We also obtain necessary and sufficient conditions in terms of scalarization for approximate Benson and Henig proper efficient points. The separation property we handle is a variation of another known property and our scalarization results do not require convexity or boundedness assumptions.

Keywords Scalarization · Proper efficiency · $\mathcal{Q}$-minimal point · Approximate proper efficiency · Nonconvex vector optimization · Nonlinear cone separation

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1 Introduction

Proper efficient points were introduced to eliminate efficient points exhibiting some abnormal properties. They can be described in terms of separations between the

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ordering cone and the considered set. Such points have been the object of many investigations, see for example (Benson 1979; Borwein and Zhuang 1993; Gong 2005; Guerraggio et al. 1994; Ha 2010; Hartley 1978; Henig 1982; Khan et al. 2015; Zheng 1997). In Ha (2010), the author presented the notion of $\mathcal{Q}$-minimal point and showed that several types of proper efficient points can be reduced in a unified form as $\mathcal{Q}$-minimal points. The following kinds of proper efficient points were studied in Ha (2010): Henig global proper efficient points, Henig proper efficient points, super efficient points, Benson proper efficient points, Hartley proper efficient points, Hurwicz proper efficient points, and Borwein proper efficient points; the latter three types considered for the first time. Optimality conditions for proper efficient points were obtained, and scalarization for $\mathcal{Q}$-minimal points was established. Since the scalar optimization theory is widely developed, scalarization turns out to be of great importance for the vector optimization theory (Ehrgott 2005; Ehrgott and Wiecek 2005; Jahn 2004; Khan et al. 2015; Luc 1989; Miettinen 1999; Pascoletti and Serafini 1984; Qiu 2008; Zheng 2000). In this work, we show that under a separation property called SSP, a $\mathcal{Q}$-minimal point in a normed space is the minimum of a given sublinear function. This fact provides sufficient conditions for the proper efficient points analysed in Ha (2010) and also for tangentially Borwein proper efficient points which were not considered there. The sufficient condition becomes a characterization in case of Benson proper efficient points. We note that SSP is a variation of a separation property introduced in Kasimbeyli (2010). On one hand, our results complement those obtained in Ha (2010) making use of a different scalar function and also establishing conditions for tangentially Borwein proper efficient points. In our results, for every type of proper efficient point, we apply the separation property to a fixed $\mathcal{Q}$-dilation of the ordering cone instead of to a sequence of ε-conic neigbhourhoods of the ordering cone (as it is done in Kasimbeyli 2010). This fact leads us to optimal conditions for nine types of proper efficient points (instead of two types in Kasimbeyli 2010) deriving scalarization results in the setting of normed spaces under weaker assumptions than those in Kasimbeyli (2010) and making use of the same scalar function. In addition, our characterization of Benson proper efficient points shed light to the last question stated in the conclusions of Guo and Zhang (2017).

Recently, several authors have been interested in introducing and studying approximate proper efficiency notions. The common idea in the concepts of approximate efficiency is to consider a set that approximates the ordering cone, that does not contain the origin in order to impose the approximate efficiency (or non-domination) condition. In Gutiérrez et al. (2012, 2016), the notions of approximate proper efficient points in the senses of Benson and Henig were introduced extending and improving the most important concepts of approximate proper efficiency given in the literature at the moment. In addition, the authors characterized such approximate efficient points through scalarization assuming generalized convexity conditions. In this manuscript, we adapt the approach followed to obtain optimal conditions for $\mathcal{Q}$-minimal points to establish new characterizations of Benson and Henig approximate proper efficient points through scalarizations. Again, our results are based on SSP and we do not impose any kind of convexity assumption. So, our results complement those in Gutiérrez et al. (2012, 2016).

The paper is organized as follows. We introduce preliminary terminology in Sect. 2. In Sect. 3, we introduce SSP and establish two separation theorems, Theorems 3.1 and 3.3. The first one provides an extension of (Kasimbeyli 2010, Theorem 4.3) to normed spaces in which some assumptions for the equivalence have been relaxed, and the latter, that is our main separation result, provides some optimal conditions for proper and approximate proper efficient points. Theorem 3.6 shows that under SSP, $\mathcal{Q}$-minimal points can be obtained minimizing a sublinear function explicitly defined. Corollary 3.7 particularizes the former necessary condition for each type of proper efficient point. Corollary 3.8 characterizes Benson proper efficient points via scalarization, and Corollaries 3.10 and 3.11 characterize, respectively, Henig global and tangentially Borwein proper efficient points under some extra assumptions. In Sect. 4, we recall the notions of approximate efficiency in the sense of Benson and of Henig. Theorems 4.1 and 4.2 provide, respectively, necessary and sufficient conditions through scalarization for approximate Benson proper efficient points; in a similar way but under extra assumptions, we establish Corollary 4.6 that characterizes approximate Henig proper points.

2 Notation and previous definitions

Throughout the paper X will denote a normed space, $\|\cdot\|$ the norm on X, X^* the dual space of X, $\|\cdot\|_*$ the norm on X^*, and 0_X the origin of X. By B_X (resp. B_X°) we denote the closed (resp. open) unit ball of X and by S_X we denote the unit sphere, i.e., $B_X := \{x \in X : \|x\| \leq 1\}$, $B_X^\circ = \{x \in X : \|x\| < 1\}$, and $S_X := \{x \in X : \|x\| = 1\}$. Given a subset $S \subset X$, we denote by $\overline{S}$ (resp. $\text{bd}(S)$, $\text{int}(S)$, $\text{co}(S)$, $\overline{\text{co}}(S)$) the closure (resp. the boundary, the interior, the convex hull, the closure of the convex hull) of S. Besides, for every $f \in X^*$, we will denote by $\sup_S f$ (resp. $\inf_S f$) the supremum (resp. infimum) of f on the set S. By $\mathbb{R}_+$ (resp. $\mathbb{R}_{++}$) we denote the set of non negative real numbers (resp. strictly positive real numbers). A subset $\mathcal{C} \subset X$ is said to be a cone if $\lambda x \in \mathcal{C}$ for every $\lambda \geq 0$ and $x \in \mathcal{C}$. Let $\mathcal{C} \subset X$ be a cone: $\mathcal{C}$ is said to be non-trivial if $\{0_X\} \subsetneq \mathcal{C} \subsetneq X$, $\mathcal{C}$ is said to be convex if it is a convex subset of X, $\mathcal{C}$ is said to be pointed if $(-\mathcal{C}) \cap \mathcal{C} = \{0_X\}$, and $\mathcal{C}$ is said to be solid if $\text{int}(\mathcal{C}) \neq \emptyset$. All cones in this manuscript are assumed to be non-trivial unless stated otherwise. From now on, A_0 denotes $A \cup \{0_X\}$ for every subset $A \subset X$. An open cone $\mathcal{Q} \subset X$ is an open set such that $\mathcal{Q}_0$ is a (non-trivial) cone. An open cone $\mathcal{Q}$ is said to be pointed (resp. convex) if $\mathcal{Q}_0$ is a pointed (resp. convex) cone on X. Fixed a subset $S \subset X$, we define the cone generated by S as $\text{cone}(S) := \{\lambda s : \lambda \geq 0,\ s \in S\}$ and $\overline{\text{cone}}(S)$ stands for the closure of $\text{cone}(S)$. A non-empty convex subset B of a convex cone $\mathcal{C}$ is said to be a base for $\mathcal{C}$ if $0_X \notin \overline{B}$ and for every $x \in \mathcal{C}\backslash\{0_X\}$ there exist unique $\lambda_x > 0$, $b_x \in B$ such that $x = \lambda_x b_x$. Given a cone $\mathcal{C} \subset X$, its dual cone is defined by $\mathcal{C}^* := \{f \in X^* : f(x) \geq 0,\ \forall x \in \mathcal{C}\}$ and the set of strictly positive functionals by $\mathcal{C}^\# := \{f \in X^* : f(x) > 0,\ \forall x \in \mathcal{C}, x \neq 0_X\}$. In general, $\text{int}(\mathcal{C}^*) \subset \mathcal{C}^\#$. It is known that a convex cone $\mathcal{C} \subset X$ has a base if and only if $\mathcal{C}^\# \neq \emptyset$, and $\mathcal{C}^\# \neq \emptyset$ implies that $\mathcal{C}$ is pointed. In particular, for every $f \in \mathcal{C}^\#$, the set $B := \{x \in \mathcal{C} : f(x) = 1\}$ is a base for $\mathcal{C}$. A convex cone $\mathcal{C}$ is said to have a bounded base if there exists a base B for $\mathcal{C}$

such that it is a bounded subset of X. It is known that $\mathcal{C}$ has a bounded base if and only if $\operatorname{int}(\mathcal{C}^*) \neq \emptyset$ if and only if 0_X is a denting point for $\mathcal{C}$ (see (Jamseson 1970, Theorem 3.8.4) for the first equivalence and García-Castaño et al. 2019; García-Castaño et al. 2015, 2021 for further information about dentability and optimization).

A convex cone $\mathcal{C}$ is said to have a (weak) compact base if there exists a base B of $\mathcal{C}$ which is a (weak) compact subset of X. A pointed cone admits a compact base if and only if it is locally compact if and only if the cone satisfies the strong property (π) if and only if there exists $f \in \mathcal{C}^\#$ such that $\mathcal{C} \cap \{f \leq \lambda\}$ is compact, $\forall \lambda > 0$. We refer the reader to (Köthe 1983, p. 338) for the first equivalence, to (Han 1994,Definition 2.1) for the definition of strong property (π) and to (Qiu 2001, Remark 2.1) for the last two equivalences. The following two sets are called augmented dual cones of a given cone $\mathcal{C}$ and they were introduced in Kasimbeyli (2010), $\mathcal{C}^{a*} := \{(f, \alpha) \in \mathcal{C}^\# \times \mathbb{R}_+ \colon f(x) - \alpha\|x\| \geq 0,\ \forall x \in \mathcal{C}\}$ and $\mathcal{C}^{a\#} := \{(f, \alpha) \in \mathcal{C}^\# \times \mathbb{R}_+ \colon f(x) - \alpha\|x\| > 0,\ \forall x \in \mathcal{C}, x \neq 0_X\}$. Clearly $\mathcal{C}^{a\#} \subset \mathcal{C}^{a*}$. Now, we introduce the following augmented dual cones $\mathcal{C}^{a*}_+ := \{(f, \alpha) \in \mathcal{C}^\# \times \mathbb{R}_{++} \colon f(x) - \alpha\|x\| \geq 0,\ \forall x \in \mathcal{C}\}$, $\mathcal{C}^{a\#}_+ := \{(f, \alpha) \in \mathcal{C}^\# \times \mathbb{R}_{++} \colon f(x) - \alpha\|x\| > 0,\ \forall x \in \mathcal{C}, x \neq 0_X\}$. It is clear that $\mathcal{C}^{a\#}_+ \subset \mathcal{C}^{a*}_+$.

Let $\mathcal{C} \subset X$ be a pointed convex cone, then $\mathcal{C}$ provides a partial order on X, say $\leq$, in the following way, $x \leq y \Leftrightarrow y - x \in \mathcal{C}$. In this situation, we say that X is a partially ordered normed space and $\mathcal{C}$ is the ordering cone. Let X be a partially ordered normed space, $\mathcal{C} \subset X$ the ordering cone, and $A \subset X$ a subset. We say that $x_0 \in A$ is an efficient (or Pareto minimal) point of A, written $x_0 \in \operatorname{Min}(A, \mathcal{C})$, if $(x_0 - \mathcal{C}) \cap A = \{x_0\}$. Next, we define some types of proper efficient points. Note that (i)–(iii) and (v)–(viii) are taken from (Ha 2010, Definition 21.3) (see also (Khan et al. 2015, Definition 2.4.4)), (iv) is taken from Makarov and Rachkovski (1999) adapting from maximal to minimal proper efficient point, and (ix) is taken from Eichfelder and Kasimbeyli (2014). The latter was obtained adapting (Borwein 1977, Definition 2) and was called Borwein proper efficient point, but we have changed the name to distinguish (v) and (ix) below. Recall that fixed a subset $A \subset X$ and a point $\bar{x} \in \overline{A}$, the contingent cone to A at $\bar{x}$ is defined by $T(A, \bar{x}) := \{\lim_n \lambda_n(x_n - \bar{x}) \in X \colon (\lambda_n) \subset \mathbb{R}_{++}, (x_n) \subset A, \text{ and } \lim_n x_n = \bar{x}\}$, see (Aubin and Frankowska 1990) for details.

Definition 1 Let X be a partially ordered normed space, $\mathcal{C}$ the ordering cone, and $A \subset X$ a subset.

(i) $x_0 \in A$ is called a positive proper efficient point of A, $x_0 \in \operatorname{Pos}(A, \mathcal{C})$, if there exists $f \in \mathcal{C}^\#$ such that $f(x_0) = \inf_A f$.

(ii) $x_0 \in A$ is called a Hurwicz proper efficient point of A, $x_0 \in \operatorname{Hu}(A, \mathcal{C})$, if $\overline{\operatorname{co}}(\mathcal{K}) \cap (-\mathcal{C}) = \{0_X\}$ for $\mathcal{K} = \operatorname{cone}((A - x_0) \cup \mathcal{C})$.

(iii) $x_0 \in A$ is called a Benson proper efficient point of A, $x_0 \in \operatorname{Be}(A, \mathcal{C})$, if $0_X \in \operatorname{Min}(\overline{\operatorname{cone}}(A + \mathcal{C} - x_0), \mathcal{C})$.

(iv) $x_0 \in A$ is called a Hartley proper efficient point of A, $x_0 \in \operatorname{Ha}(A, \mathcal{C})$, if $x_0 \in \operatorname{Min}(A, \mathcal{C})$ and there exists $M > 0$ such that if $f \in \mathcal{C}^*$ and $f(x - x_0) < 0$ for some $x \in A$, then there exists $g \in \mathcal{C}^*$, $g \neq 0$, satisfying $\|g\| f(x - x_0) \geq -\|f\| M g(x - x_0)$.

(v) $x_0 \in A$ is called a Borwein proper efficient point of A, $x_0 \in \operatorname{Bo}(A, \mathcal{C})$, if $\overline{\operatorname{cone}}(A - x_0) \cap (-\mathcal{C}) = \{0_X\}$.

(vi) $x_0 \in A$ is called a Henig global proper efficient point of A, $x_0 \in \mathrm{GHe}(A, \mathcal{C})$, if $x_0 \in \mathrm{Min}(A, \mathcal{K})$ for some convex cone $\mathcal{K}$ such that $\mathcal{C} \setminus \{0_X\} \subset \mathrm{int}(\mathcal{K})$.

(vii) $x_0 \in A$ is called a Henig proper efficient point of A, $x_0 \in \mathrm{He}(A, \mathcal{C})$, if $\mathcal{C}$ has a base B and there exists $\varepsilon > 0$ such that $\overline{\mathrm{cone}}(A - x_0) \cap (-\overline{\mathrm{cone}}(B + \varepsilon B_X)) = \{0_X\}$.

(viii) $x_0 \in A$ is called a super efficient point of A, $x_0 \in \mathrm{SE}(A, \mathcal{C})$, if there exists $\rho > 0$ such that $\overline{\mathrm{cone}}(A - x_0) \cap (B_X - \mathcal{C}) \subset \rho B_X$.

(ix) $x_0 \in A$ is a tangentially Borwein proper efficient point of A, written $x_0 \in \mathrm{TBo}(A, \mathcal{C})$, if $T(A + \mathcal{C}, x_0) \cap (-\mathcal{C}) = \{0_X\}$.

Proper efficient points were introduced for two main reasons: first, to eliminate certain anomalous minimal points; second, to establish some equivalent scalar problems whose solutions provide at least most of the minimal points. We have the following chain of inclusions (see Ha 2010, Proposition 21.4) and (Khan et al 2015, Proposition 2.4.6). $\mathrm{Pos}(A, \mathcal{C}) \subset \mathrm{Hu}(A, \mathcal{C}) \subset \mathrm{Be}(A, \mathcal{C})$, $\mathrm{Pos}(A, \mathcal{C}) \subset \mathrm{GHe}(A, \mathcal{C})$, $\mathrm{SE}(A, \mathcal{C}) \subset \mathrm{Ha}(A, \mathcal{C}) \subset \mathrm{Be}(A, \mathcal{C}) \subset \mathrm{Bo}(A, \mathcal{C})$, $\mathrm{SE}(A, \mathcal{C}) \subset \mathrm{GHe}(A, \mathcal{C}) \subset \mathrm{Be}(A, \mathcal{C})$, $\mathrm{SE}(A, \mathcal{C}) \subset \mathrm{He}(A, \mathcal{C})$. In addition, by Jahn (2004, Theorem 3.44) if we have $T(A + \mathcal{C}, x_0) \subset \overline{\mathrm{cone}}(A + \mathcal{C}, x_0)$, then $\mathrm{Be}(A, \mathcal{C}) \subset \mathrm{TBo}(A, \mathcal{C})$. Furthermore, under extra assumptions on A or $\mathcal{C}$, we can find more inclusions (again Ha 2010; Khan et al. 2015).

3 Separation theorems and scalarization for Q-minimal points

We begin this section introducing a separation property of cones called strict separation property (SSP for short). Later, we establish two theorems which separate two cones in a normed space by a sublevel set of a sublinear function. The second separation theorem will be key to determining optimality conditions for the proper efficient points introduced in Sect. 2.

Throughout the paper, we denote by A_0 the set $A \cup \{0_X\}$, for any $A \subset X$. On the other hand, every cone $\mathcal{C} \subset X$ we consider is assumed non-trivial, i.e., $\{0_X\} \subsetneq \mathcal{C} \subsetneq X$. For any cone $\mathcal{C}$, we define the following convex sets $\mathcal{C}_\wedge := \mathrm{co}(\mathcal{C} \cap S_X)$ and $\mathcal{C}_\vee := \mathrm{co}((\mathrm{bd}(\mathcal{C}) \cap S_X)_0)$ to be used in the following separation property.

Definition 2 Let X be a normed space and $\mathcal{C}$, $\mathcal{K}$ cones on X. We say that the pair of cones $(\mathcal{C}, \mathcal{K})$ has the strict separation property (SSP for short) if $0_X \notin \overline{\mathcal{C}_\wedge - \mathcal{K}_\vee}$.

Remark 1 Given two cones $\mathcal{C}, \mathcal{K} \subset X$ we have the following.

(i) $(\mathcal{C}, \mathcal{K})$ has SSP $\Leftrightarrow$ $(\overline{\mathcal{C}}, \mathcal{K})$ has SSP $\Leftrightarrow$ $(\mathcal{C}, \overline{\mathcal{K}})$ has SSP $\Leftrightarrow$ $(\overline{\mathcal{C}}, \overline{\mathcal{K}})$ has SSP.

(ii) Since $\mathrm{bd}(A) = \mathrm{bd}(X \setminus A)$ for every subset $A \subset X$ and $\mathrm{bd}(\mathcal{K}) = \mathrm{bd}(\mathcal{K} \setminus \{0_X\})$ for every non-trivial cone $\mathcal{K} \subset X$, it follows that $(\mathcal{C}, \mathcal{K})$ has SSP if and only if $(-\mathcal{C}, -\mathcal{K})$ has SSP if and only if $(\mathcal{C}, (X \setminus \mathcal{K})_0)$ has SSP.

In (Kasimbeyli 2010, Definition 4.1) the author introduces the separation property SP for closed cones under the condition $0_X \notin \overline{\mathcal{C}_\wedge} - \overline{\mathcal{K}_\vee}$. It is clear that, for closed cones, SSP implies SP. The reverse is true in reflexive Banach spaces because on such spaces the closure of the difference of two bounded convex sets equals the difference of the closures of the sets.

The following result provides a version of (Kasimbeyli 2010, Theorem 4.3) in the setting of general normed spaces. It is worth pointing out that we do not need extra assumptions to obtain an equivalence (such as the cones to be convex and closed or the space to be of finite dimension).

Theorem 3.1 *Let X be a normed space and $\mathcal{C}$, $\mathcal{K}$ cones on X. The following assertions are equivalent.*

(i) $(\mathcal{C}, \mathcal{K})$ *has SSP.*
(ii) *There exist* $\delta_2 > \delta_1 > 0$ *and* $f \in X^*$ *such that* $(f, \alpha) \in \mathcal{C}_+^{a\#}$ *and* $0 < f(y) + \alpha\|y\|$ *for every* $\alpha \in (\delta_1, \delta_2)$ *and* $y \in bd(-\mathcal{K})$, $y \neq 0_X$.
(iii) *There exist* $\delta_2 > \delta_1 > 0$ *and* $f \in X^*$ *such that* $(f, \alpha) \in \mathcal{C}_+^{a\#}$ *and* $f(x)+\alpha\|x\| < 0 < f(y)+\alpha\|y\|$ *for every* $\alpha \in (\delta_1, \delta_2)$, $x \in -\overline{co}(\mathcal{C})$, $x \neq 0_X$, *and* $y \in bd(-\mathcal{K})$, $y \neq 0_X$.

Proof (i) $\Rightarrow$ (ii) Since $0_X \notin \overline{\mathcal{C}_\wedge - \mathcal{K}_\vee}$, by Fabian (2011,Theorem 2.12) there exist $f \in X^*$ and $\beta_1, \beta_2 \in \mathbb{R}$ such that $f(0) = 0 < \beta_1 < \beta_2 < f(c-k) = f(c) - f(k)$ for every $c \in \mathcal{C}_\wedge$, $k \in \mathcal{K}_\vee$. Then $f(k) < \beta_1 + f(k) < \beta_2 + f(k) < f(c)$, for every $c \in \mathcal{C}_\wedge$, $k \in \mathcal{K}_\vee$. Denote $S := \sup_{\mathcal{K}_\vee} f$. Then $0 \leq S < +\infty$ because $0_X \in \mathcal{K}_\vee$ and the set $\mathcal{K}_\vee$ is bounded. Furthermore, for every $c \in \mathcal{C}_\wedge$, $k \in \mathcal{K}_\vee$ we have $f(k) \leq S < \beta_1 + S < \beta_2 + S \leq f(c)$. Therefore, denoting $\delta_1 := \beta_1 + S$ and $\delta_2 := \beta_2 + S$ we get that $f(k) < \alpha < f(c)$ for every $c \in \mathcal{C}_\wedge$, $k \in \mathcal{K}_\vee$ and $\alpha \in (\delta_1, \delta_2)$. On the other hand, since $0_X \in \mathcal{K}_\vee$, it follows that $f(c) > 0$ for every $c \in \mathcal{C}_\wedge$. Thus $f \in C^\#$. Now, fix arbitrary $\alpha \in (\delta_1, \delta_2)$ and $y \in \text{bd}(\mathcal{K}) \cap S_X$. As $f(y) < \alpha$ and $\|y\| = 1$, we get that $f(y)-\alpha\|y\| < 0$, which leads to $f(-y)+\alpha\|y\| > 0$. Therefore $f(y)+\alpha\|y\| > 0$ for every $y \in -\text{bd}(\mathcal{K})\backslash\{0_X\}$. Fix now $\alpha \in (\delta_1, \delta_2)$ and $x \in (-\mathcal{C}) \cap S_X$. Again, $0_X \in \mathcal{K}_\vee$ implies $f(0_X) = 0 < \alpha < f(-x)$. Since $\|-x\| = 1$, we get that $0 < f(-x)-\alpha\|-x\|$, which leads to $f(x)+\alpha\|x\| < 0$. Consequently, $f(x)+\alpha\|x\| < 0$ for every $x \in -\mathcal{C}$, $x \neq 0_X$. Clearly $(f, \alpha) \in \mathcal{C}_+^{a\#}$.

(ii) $\Rightarrow$ (iii) Consider $(f, \alpha) \in \mathcal{C}_+^{a\#}$ for some $\alpha \in (\delta_1, \delta_2)$. Then $f(x) + \alpha\|x\| < 0$ for every $x \in -\mathcal{C}$, $x \neq 0_X$. Now, sublinearity of $f(\cdot)+\alpha\|\cdot\|$ yields $f(x)+\alpha\|x\| < 0$ for every $x \in -\text{co}(\mathcal{C})$, $x \neq 0_X$. Next, we will prove the precedent inequality for $x \in -\overline{\text{co}}(\mathcal{C})$, $x \neq 0_X$. Assume, contrary to our claim, that there exist some $\bar{x} \in -(\overline{\text{co}}(\mathcal{C})\backslash\text{co}(\mathcal{C}))$, $\bar{x} \neq 0_X$, such that $f(\bar{x}) + \alpha\|\bar{x}\| = 0$. Then, there exists a sequence $\{x_n'\} \subset -\text{co}(\mathcal{C})$ such that $\lim_{n\to\infty} x_n' = \bar{x}$. Now, we fix some $\hat{\alpha} \in (\delta_1, \delta_2)$ such that $\alpha < \hat{\alpha}$. Then, $\lim_{n\to\infty} f(x_n')+\hat{\alpha}\|x_n'\| = \lim_{n\to\infty} f(x_n')+\alpha\|x_n'\|+\lim_{n\to\infty}(\hat{\alpha}-\alpha)\|x_n'\| = (\hat{\alpha}-\alpha)\|\bar{x}\| > 0$, because $\bar{x} \neq 0_X$. Hence, there exists some $n_0 \in \mathbb{N}$ such that $f(x_{n_0}') + \alpha\|x_{n_0}'\| > 0$, which is impossible because $x_{n_0}' \in -\text{co}(\mathcal{C})$, $x_{n_0}' \neq 0_X$.

(iii) $\Rightarrow$ (i) For any $x \in (-\mathcal{C}) \cap S_X$, $y \in (-\text{bd}(\mathcal{K})) \cap S_X$, $y \neq 0_X$, and $\alpha \in (\delta_1, \delta_2)$ we have $f(x) < -\alpha < f(y)$. The former inequalities also hold for $x \in \text{co}((-\mathcal{C})\cap S_X)$ and $y \in \text{co}(-(\text{bd}(\mathcal{K})\cap S_X)_0)$. Indeed, fix $\alpha \in (\delta_1, \delta_2)$ and $x \in -\mathcal{C}_\wedge = \text{co}((-\mathcal{C})\cap S_X)$. Then $x = \sum_{i=1}^n \lambda_i x_i$ for $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$, $x_i \in (-\mathcal{C}) \cap S_X$, and $f(x_i) < -\alpha$. Therefore, $f(x) = \sum_{i=1}^n \lambda_i f(x_i) < -\alpha$. For the same reason, $-\alpha < f(y)$ for every $y \in -\mathcal{K}_\vee = \text{co}(-(\text{bd}(\mathcal{K}) \cap S_X)_0)$. This implies that $f(y) - f(x) > \frac{\delta_2-\delta_1}{2}$ for every $x \in -\mathcal{C}_\wedge$ and $y \in -\mathcal{K}_\vee$. To obtain a contradiction, suppose that $(\mathcal{C}, \mathcal{K})$ does not have SSP. Then $(-\mathcal{C}, -\mathcal{K})$ does not have SSP either, i.e., $0_X \in \overline{-\mathcal{C}_\wedge - (-\mathcal{K}_\vee)}$. Then, there exist two sequences $(x_n)_n \subset -\mathcal{C}_\wedge$ and $(y_n)_n \subset -\mathcal{K}_\vee$ such that $0 = \lim_n \|x_n - y_n\|$. Then, by continuity of f, we have $\lim_n f(y_n - x_n) = 0$. As a consequence, there exists $n_0 \in \mathbb{N}$ such that $| f(y_n) - f(x_n) | < \frac{\delta_2-\delta_1}{2}$ for every $n > n_0$, a contradiction. □

Next, a direct consequence of the precedent result.

Corollary 3.2 *Let X be a normed space and $\mathcal{C}$, $\mathcal{K}$ cones on X. Then $(\mathcal{C}, \mathcal{K})$ has SSP if and only if $(\overline{co}(\mathcal{C}), \mathcal{K})$ has SSP.*

Theorem 3.1 establishes that the results in Kasimbeyli (2010) obtained for reflexive Banach spaces can be extended to general normed spaces under SSP. On the other hand, the following result is our main separation theorem and it will provide optimal conditions for the proper minimal points introduced in Sect. 2 and for the approximate proper efficient points in Sect. 4.

Theorem 3.3 *Let X be a normed space and $\mathcal{C}$, $\mathcal{K}$ cones on X such that $\overline{co}(\mathcal{C}) \cap \mathcal{K} \neq \{0_X\}$. If $(\mathcal{C}, \mathcal{K})$ has SSP, then $int(\mathcal{K}) \neq \emptyset$ and there exist $\delta_2 > \delta_1 > 0$ and $f \in X^*$ such that $(f, \alpha) \in \mathcal{C}_+^{a\#}$ and $f(x)+\alpha\|x\| < 0 < f(y)+\alpha\|y\|$, for every $\alpha \in (\delta_1, \delta_2)$, $x \in -\overline{co}(\mathcal{C})$, $x \neq 0_X$, and $y \in X\backslash int(-\mathcal{K})$, $y \neq 0_X$.*

Proof Assume that $(\mathcal{C}, \mathcal{K})$ has SSP. By Theorem 3.1, there exist $\delta_2 > \delta_1 > 0$ and $f \in X^*$ such that $(f, \alpha) \in \mathcal{C}_+^{a\#}$ and $f(x) + \alpha\|x\| < 0 < f(y) + \alpha\|y\|$, for every $\alpha \in (\delta_1, \delta_2)$, $x \in -\overline{\mathrm{co}}(\mathcal{C})$, $x \neq 0_X$, and $y \in \mathrm{bd}(-\mathcal{K})$, $y \neq 0_X$. First, we check that $\mathrm{int}(\mathcal{K}) \neq \emptyset$. Fix an arbitrary $\bar{x} \in (-\overline{\mathrm{co}}(\mathcal{C})) \cap (-\mathcal{K})$ such that $\bar{x} \neq 0_X$. Then $f(\bar{x}) + \alpha\|\bar{x}\| < 0$. On the other hand, since $\bar{x} \in \mathrm{bd}(-\mathcal{K})$ implies $f(\bar{x}) + \alpha\|\bar{x}\| > 0$, we have $\bar{x} \notin \mathrm{bd}(-\mathcal{K})$, so $\bar{x} \in \mathrm{int}(-\mathcal{K})$. Thus $\mathrm{int}(\mathcal{K}) \neq \emptyset$. Next, we will prove that $0 < f(y)+\alpha\|y\|$ for every $y \in X\backslash\mathrm{int}(-\mathcal{K})$, $y \neq 0_X$. Assume, contrary to our claim, that there exists $\bar{y} \in X \setminus \mathrm{int}(-\mathcal{K})$ such that $\bar{y} \neq 0_X$ and satisfying $f(\bar{y}) + \alpha\|\bar{y}\| \leq 0$. It is clear that $\bar{y} \notin \mathrm{bd}(-\mathcal{K})$. Hence $\bar{y} \in X\backslash(-\overline{\mathcal{K}})$. We consider again the former $\bar{x} \in \mathrm{int}(-\mathcal{K})$ and there exists $\lambda_0 \in (0, 1)$ such that $x_0 = \lambda_0\bar{x} + (1 - \lambda_0)\bar{y} \in \mathrm{bd}(-\mathcal{K})$. As a consequence, $f(x_0) + \alpha\|x_0\| = f(\lambda_0\bar{x} + (1 - \lambda_0)\bar{y}) + \alpha\|\lambda_0\bar{x} + (1 - \lambda_0)\bar{y}\| \leq \lambda_0(f(\bar{x}) + \alpha\|\bar{x}\|) + (1 - \lambda_0)(f(\bar{y}) + \alpha\|\bar{y}\|) < 0$, which contradicts $x_0 \in \mathrm{bd}(-\mathcal{K})$. □

The following result shows the relative position of a pair of cones having SSP.

Corollary 3.4 *Let X be a normed space and $\mathcal{C}$, $\mathcal{K}$ cones on X. If $\mathcal{C}$ and $\mathcal{K}$ have SSP, then either $\overline{co}(\mathcal{C})\backslash\{0_X\} \subset int(\mathcal{K})$ or $\overline{co}(\mathcal{C})\backslash\{0_X\} \subset int(X\backslash\mathcal{K})$.*

Proof If $(\mathcal{C}, \mathcal{K})$ has SSP, by Corollary 3.2 we get that $(\overline{\mathrm{co}}(\mathcal{C}), \mathcal{K})$ has SSP, and then $(-\overline{\mathrm{co}}(\mathcal{C}), -\mathcal{K})$ has SSP, too. By Theorem 3.1, we have $(-\overline{\mathrm{co}}(\mathcal{C})\backslash\{0_X\}) \cap \mathrm{bd}(-\mathcal{K}) = \emptyset$, hence $(\overline{\mathrm{co}}(\mathcal{C})\backslash\{0_X\}) \cap \mathrm{bd}(\mathcal{K}) = \emptyset$. Now, suppose that the assertion of the corollary is false. Then, there exist $c_1, c_2 \in \overline{\mathrm{co}}(\mathcal{C})\backslash\{0_X\}$ such that $c_1 \in \mathrm{int}(\mathcal{K})$ and $c_2 \in \mathrm{int}(X\backslash\mathcal{K})$. As $c_1 \in \overline{\mathrm{co}}(\mathcal{C}) \cap \mathcal{K}$, Theorem 3.3 applies. Hence there exists $(f, \alpha) \in \mathcal{C}_+^{a\#}$ such that $f(x) + \alpha\|x\| < 0 < f(y) + \alpha\|y\|$, for every $x \in -\overline{\mathrm{co}}(\mathcal{C})$, $x \neq 0_X$, and $y \in X\backslash\mathrm{int}(-\mathcal{K})$, $y \neq 0_X$. Now, on one hand, $f(-c_2) + \alpha\| - c_2\| < 0$ because $c_2 \in \overline{\mathrm{co}}(\mathcal{C})$. But on the other hand, since $c_2 \in \mathrm{int}(X\backslash\mathcal{K}) \subset X\backslash\mathcal{K} \subset X\backslash\mathrm{int}(\mathcal{K})$, it follows that $0 < f(-c_2) + \alpha\| - c_2\|$, a contradiction. □

We next introduce the notion of $\mathcal{Q}$-minimal point (from Ha 2010; Khan et al. 2015) to derive scalarizations for proper efficient points in a unified way.

Definition 3 Let X be a normed space, $\mathcal{Q} \subset X$ an open cone, and $A \subset X$ a subset. We say that $x_0 \in A$ is a $\mathcal{Q}$-minimal point of A, $x_0 \in \mathcal{Q}\mathrm{Min}(A)$, if $(A - x_0) \cap (-\mathcal{Q}) = \emptyset$.

The following notion is directly related to $\mathcal{Q}$-minimal points.

Definition 4 Let X be a partially ordered normed space, $\mathcal{C} \subset X$ the ordering cone, and $\mathcal{Q} \subset X$ an open cone. We say that $\mathcal{Q}$ dilates $\mathcal{C}$ (or $\mathcal{Q}$ is a dilation of $\mathcal{C}$) if $\mathcal{C} \setminus \{0_X\} \subset \mathcal{Q}$.

The following result shows that the proper efficient points introduced in Definition 1 are $\mathcal{Q}$-minimal points with $\mathcal{Q}$ being appropiately chosen cones. Assertions (i)–(viii) below are Assertions (iii)–(x) from (Ha 2010, Theorem 21.7) respectively, under minor adaptations (see also (Khan et al. 2015, Theorem 2.4.11)). On the other hand, assertion (ix) below can be proved in a similar way as (ii) and (iii) in (Ha 2010, Theorem 21.7) so we omit the proof. We need more terminology. Let X be a partially ordered normed space, $\mathcal{C} \subset X$ the ordering cone, and B a base of $\mathcal{C}$. Let $\delta_B := \inf\{\|b\| : b \in B\} > 0$, for every $0 < \eta < \delta_B$ we define a convex, pointed and open cone $V_\eta(B) := \mathrm{cone}(B + \eta B_X^\circ)$. On the other hand, given $0 < \varepsilon < 1$ we define another open cone $\mathcal{C}(\varepsilon) := \{x \in X : d(x, \mathcal{C}) < \varepsilon d(x, -\mathcal{C})\}$.

Theorem 3.5 *Let X be a partially ordered normed space, $\mathcal{C}$ the ordering cone, $A \subset X$ a subset, and $x_0 \in A$. The following statements hold.*

(i) $x_0 \in Pos(A, \mathcal{C})$ *if and only if* $x_0 \in \mathcal{Q}Min(A)$ *for* $\mathcal{Q} = \{x \in X : f(x) > 0\}$ *and some* $f \in \mathcal{C}^{\#}$.
(ii) $x_0 \in Hu(A, \mathcal{C})$ *if and only if* $x_0 \in \mathcal{Q}Min(A)$ *for* $\mathcal{Q} = -X \setminus \overline{co}(\mathcal{K})$ *and* $\mathcal{K} := cone((A - x_0) \cup \mathcal{C})$.
(iii) $x_0 \in Be(A, \mathcal{C})$ *if and only if* $x_0 \in \mathcal{Q}Min(A)$ *for* $\mathcal{Q} = -X \setminus \overline{cone}(A - x_0 + \mathcal{C})$.
(iv) $x_0 \in Ha(\mathcal{A}, \mathcal{C})$ *if and only if* $x_0 \in \mathcal{Q}Min(A)$ *for* $\mathcal{Q} = \mathcal{C}(\varepsilon)$ *and* $\varepsilon > 0$.
(v) $x_0 \in Bo(\mathcal{A}, \mathcal{C})$ *if and only if* $x_0 \in \mathcal{Q}Min(A)$ *for* Q *an open cone dilating* $\mathcal{C}$.
(vi) $x_0 \in GHe(\mathcal{A}, \mathcal{C})$ *if and only if* $x_0 \in \mathcal{Q}Min(A)$ *for* Q *a pointed convex open cone dilating* $\mathcal{C}$.
(vii) $x_0 \in He(\mathcal{A}, \mathcal{C})$ *if and only if* $x_0 \in \mathcal{Q}Min(A)$ *for* $Q = V_\eta(B)$, $0 < \eta < \delta_B$, *and a base* B *of* $\mathcal{C}$.
(viii) $x_0 \in SE(\mathcal{A}, \mathcal{C})$ *if and only if* $x_0 \in \mathcal{Q}Min(A)$ *for* $Q = V_\eta(B)$, $0 < \eta < \delta_B$, *and* B *a bounded base of* $\mathcal{C}$.
(ix) $x_0 \in TBo(A, \mathcal{C})$ *if and only if* $x_0 \in \mathcal{Q}Min(A)$ *for* $\mathcal{Q} = -X \setminus T(A + \mathcal{C}, x_0)$.

In Ha (2010), the author provides some necessary and sufficient conditions for a $\mathcal{Q}$-minimal point to be a solution for a scalar optimization problem. In the following result, we provide a necessary condition in terms of SSP for $\mathcal{Q}$-minimal points to be a solution of another scalar problem.

Theorem 3.6 *Let X be a partially ordered normed space, $\mathcal{C}$ the ordering cone, $\mathcal{Q} \subset X$ an open cone, and $x_0 \in A \subset X$. Assume that $x_0 \in \mathcal{Q}Min(A)$ and $\mathcal{C} \cap \mathcal{Q} \neq \emptyset$. If $(\mathcal{C}, \mathcal{Q}_0)$ has SSP, then there exists $(f, \alpha) \in \mathcal{C}_+^{a\#}$ such that*

$$\min_{x \in A} f(x - x_0) + \alpha \|x - x_0\|$$

is attained only at x_0.

Proof It is not restrictive to assume that $x_0 = 0_X$ (by translation we obtain the general case). Since $(\mathcal{C},\ \mathcal{Q}_0)$ has SSP, Theorem 3.3 applies and there exists $(f,\alpha) \in \mathcal{C}_+^{a\#}$ such that $f(c) + \alpha\|c\| < 0 < f(x) + \alpha\|x\|$ for every $c \in -\mathcal{C}\backslash\{0_X\}$ and $x \in X\backslash(-\mathcal{Q})$. Since $A \subset X\backslash(-\mathcal{Q})$, it follows that $0 < f(x) + \alpha\|x\|$ for every $x \in A$, $x \neq 0_X$. □

In the following result, we establish a particular version of the former one for each type of proper efficient point introduced in Definition 1.

Corollary 3.7 *Let* X *be a partially ordered normed space,* $\mathcal{C}$ *the ordering cone, and* $x_0 \in A \subset X$. *Assume that either of the following statements holds.*

(i) $x_0 \in Pos(A,\mathcal{C})$ *and take* $g \in \mathcal{C}^{\#}$ *such that* $x_0 \in \mathcal{Q}Min(A,\mathcal{C})$ *for* $\mathcal{Q} = \{x \in X : g(x) > 0\}$.
(ii) $x_0 \in Hu(A,\mathcal{C})$ *and take* $\mathcal{Q} = -X\backslash\overline{co}(\mathcal{K})$ *for* $\mathcal{K} := cone((A - x_0) \cup \mathcal{C})$.
(iii) $x_0 \in Be(A,\mathcal{C})$ *and take* $\mathcal{Q} = -X\backslash\overline{cone}(A - x_0 + \mathcal{C})$.
(iv) $x_0 \in Ha(\mathcal{A},\mathcal{C})$ *and take* $\mathcal{Q} = C(\varepsilon)$ *for some* $\varepsilon > 0$.
(v) $x_0 \in Bo(\mathcal{A},\mathcal{C})$ *and take* Q *an open cone dilating* $\mathcal{C}$.
(vi) $x_0 \in GHe(\mathcal{A},\mathcal{C})$ *and take* Q *a pointed convex open cone dilating* $\mathcal{C}$.
(vii) $x_0 \in He(\mathcal{A},\mathcal{C})$ *and take* $Q = V_\eta(B)$, $0 < \eta < \delta_B$, *for some base* B *of* $\mathcal{C}$.
(viii) $x_0 \in SE(\mathcal{A},\mathcal{C})$ *and take* $Q = V_\eta(B)$, $0 < \eta < \delta_B$, *for some bounded base* B *of* $\mathcal{C}$.
(ix) $x_0 \in TBo(A,\mathcal{C})$ *and take* $\mathcal{Q} = -X\backslash T(A + \mathcal{C}, x_0)$.

If $(\mathcal{C},\ \mathcal{Q}_0)$ *has SSP, then there exists* $(f,\alpha) \in \mathcal{C}_+^{a\#}$ *such that*

$$\min_{x\in A}\ f(x - x_0) + \alpha\|x - x_0\| \tag{1}$$

is attained only at x_0.

Proof We begin noting the inclusions $\mathcal{C} \setminus \{0_X\} \subset \mathcal{Q}$ for (i)–(ix). The inclusion for the cases (i) and (iv)–(ix) is trivial. For case (ii), we apply the following chain of equivalences: $\overline{\text{co}}(\mathcal{K}) \cap (-\mathcal{C}) = \{0_X\} \Leftrightarrow \overline{\text{co}}(\mathcal{K}) \cap (-\mathcal{C}\backslash\{0_X\}) = \emptyset \Leftrightarrow -\mathcal{C}\backslash\{0_X\} \subset X\backslash\overline{\text{co}}(\mathcal{K}) = -\mathcal{Q}$. For case (iii) we apply $\overline{\mathcal{K}} \cap (-\mathcal{C}) = \{0_X\} \Leftrightarrow \overline{\mathcal{K}} \cap (-\mathcal{C}\backslash\{0_X\}) = \emptyset \Leftrightarrow -\mathcal{C}\backslash\{0_X\} \subset X\backslash\overline{\mathcal{K}} = \mathcal{Q}$.

Now assume that either of (i)–(ix) holds. By Theorem 3.5, $x_0 \in \mathcal{Q}\text{Min}(A,\mathcal{C})$ and by the former paragraph, $\mathcal{C} \cap \mathcal{Q} \neq \emptyset$. Then Theorem 3.6 applies. □

The following result characterizes Benson proper minimal points via (1).

Corollary 3.8 *Let* X *be a partially ordered normed space,* $\mathcal{C}$ *the ordering cone, and* $x_0 \in A \subset X$. *Assume that* $(-\mathcal{C},\ \overline{cone}(A - x_0 + \mathcal{C}))$ *has SSP. Then* $x_0 \in Be(A,\mathcal{C})$ *if and only if there exists* $(f,\alpha) \in \mathcal{C}_+^{a\#}$ *such that* $\min_{x\in A} f(x - x_0) + \alpha\|x - x_0\|$ *is attained only at* x_0.

Proof It is not restrictive to assume that $x_0 = 0_X$. $\Rightarrow$ As $(-\mathcal{C},\ \overline{\text{cone}}(A + \mathcal{C}))$ has SSP, then $(\mathcal{C},\ -(X\backslash\overline{\text{cone}}(A + \mathcal{C}))_0)$ has SSP too. Therefore, Remark 1 yields that

$(-\mathcal{C},\ (X\backslash\overline{\text{cone}}(A+\mathcal{C}))_0)$ has SSP as well. Now, apply Theorem 3.5 (iii) and Corollary 3.7 (iii). $\Leftarrow$ Fix $(f,\alpha)\in\mathcal{C}_+^{a\#}$ such that $\min_{x\in A} f(x)+\alpha\|x\|$ is attained only at 0_X. By (Gasimov 2001, Theorem 1) $0_X\in \text{Be}(A,\mathcal{C})$. □

(Kasimbeyli 2010, Theorem 5.8) also characterizes Benson proper efficient points via (1), but under more restrictive assumptions than Corollary 3.8 and, in addition, applying the separation property to a sequence of ε-conic neighbourhoods instead of to an only cone $-\overline{\text{cone}}(A-x_0+\mathcal{C})$.

In the following, we study sufficient conditions to have $\text{GHe}(A,\mathcal{C})=\text{Be}(A,\mathcal{C})$. Such equality will lead to a characterization for Henig global proper efficient points via (1). The set $\text{GHe}(A,\mathcal{C})$ is contained in the set $\text{Be}(A,\mathcal{C})$ whenever $\mathcal{C}$ is a closed, convex, and pointed cone (see Guerraggio et al. 1994). The next result establishes the equality of such sets under some extra assumptions.

Theorem 3.9 *Let X be a partially ordered normed space, $\mathcal{C}$ the ordering cone, and $A\subset X$ a subset such that $A+\mathcal{C}$ is convex. If $\mathcal{C}$ has a weakly compact base, then $GHe(A,\mathcal{C})=Be(A,\mathcal{C})$.*

Proof $\subset$ is provided by Khan et al. (2015, Proposition 2.4.6 (i)) for separated topological vector spaces and non closed cones. $\supset$ Fix an arbitrary $x_0\in\text{Be}(A,\mathcal{C})$. It is not restrictive to assume that $x_0=0_X$. Then $0_X\in\text{Min}(A,\mathcal{C})$ and $0_X\in\text{Min}(\overline{\text{cone}}(A+\mathcal{C}),\mathcal{C})$. Hence $(-\mathcal{C})\cap\overline{\text{cone}}(A+\mathcal{C})=\{0_X\}$. On the other hand, $A+\mathcal{C}$ is convex. Then $\text{cone}(A+\mathcal{C})$ is convex, implying that $\overline{\text{cone}}(A+\mathcal{C})$ is weak closed. Now, Kasimbeyli (2010, Theorem 5.2) applies and there exists a convex cone $\mathcal{K}$ such that $-\mathcal{C}\backslash\{0_X\}\subset\text{int}(\mathcal{K})$ and $\mathcal{K}\cap\overline{\text{cone}}(A+\mathcal{C})=\{0_X\}$. Then $0_X\in\text{GHe}(A,\mathcal{C})$. □

The following result is a direct consequence of Corollary 3.8 and Theorem 3.9.

Corollary 3.10 *Let X be a partially ordered normed space, $\mathcal{C}$ the ordering cone, and $x_0\in A\subset X$ such that $A+\mathcal{C}$ is convex. Assume that $\mathcal{C}$ has a weakly compact base. If $(-\mathcal{C},\ \overline{cone}(A-x_0+\mathcal{C}))$ has SSP, then $x_0\in GHe(A,\mathcal{C})$ if and only if there exists $(f,\alpha)\in\mathcal{C}_+^{a\#}$ such that $\min_{x\in A}\{f(x-x_0)+\alpha\|x-x_0\|\}$ is attained only at x_0.*

Corollary 3.8 provides a sufficient condition to find elements in $\text{TBo}(A,\mathcal{C})$. In the following, we will establish that it becomes a characterization if we assume the following geometric condition on $x_0\in A$. It is said that a set $A\subset X$ is starshaped at some $x_0\in A$, if $\lambda x+(1-\lambda)x_0\in A$ for every $x\in A$ and $\lambda\in[0,1]$. Since (Jahn 2004, Corollary 3.46) establishes that $T(A,x_0)=\overline{\text{cone}}(A-x_0)$ whenever A is starshaped at $x_0\in A$, it follows the equivalence $x_0\in\text{Be}(A,\mathcal{C})\Leftrightarrow x_0\in\text{TBo}(A,\mathcal{C})$ whenever $A+\mathcal{C}$ is starshaped at $x_0\in A$. Consequently, we obtain the following characterization for tangentially Borwein proper efficient points.

Corollary 3.11 *Let X be a partially ordered normed space, $\mathcal{C}$ the ordering cone, and $x_0\in A\subset X$. Assume that $A+\mathcal{C}$ is starshaped at x_0 and that $\mathcal{C}$ has a weakly compact base. If $(-\mathcal{C},\ \overline{cone}(A-x_0+\mathcal{C}))$ has SSP, then $x_0\in TBo(A,\mathcal{C})$ if and only if there exists $(f,\alpha)\in\mathcal{C}_+^{a\#}$ such that $\min_{x\in A}\{f(x-x_0)+\alpha\|x-x_0\|\}$ is attained only at x_0.*

We finish this section with the following problem for future research.

Problem 3.12 *Is it possible to characterize $\mathcal{Q}$-minimal points via SSP assuming any extra conditions?*

In Theorem 3.6 we provide necessary conditions for $\mathcal{Q}$-minimal points. On the other hand, in Corollaries 3.8, 3.10, and 3.11, we establish characterizations of Benson, global Henig, and tangentially Borwein proper efficient points, respectively, answering the former problem for such particular kinds of $\mathcal{Q}$-minimal points. So, it is of interest to solve Problem 3.12 for any of the other types of $\mathcal{Q}$-minimal points.

4 Scalarization for approximate proper efficient points

In this section, we obtain optimal conditions through scalarization for approximate proper efficient points in the senses of Benson and Henig. We obtain our results after extending the approach for Benson and Henig proper efficient points in the precedent section.

Let us introduce the terminology of approximate proper efficiency. From now on, the ordering cone $\mathcal{C} \subset X$ is assumed to be closed, convex, and pointed. The notions of approximate efficiency are defined replacing the ordering cone $\mathcal{C}$ by a non-empty set D that approximates it. For a non-empty set $D \subset X \backslash \{0_X\}$, we define the set $D(\varepsilon) := \varepsilon D$, for $\varepsilon > 0$, and $D(0) := \text{cone}(D) \setminus \{0_X\}$. We also introduce the family of sets $\overline{\mathcal{H}} := \{\emptyset \neq D \subset X\backslash\{0_X\} : \overline{\text{cone}}(D) \cap (-\mathcal{C}) = \{0_X\}\}$. Notice that $D(\varepsilon) \in \overline{\mathcal{H}}$, for every $D \in \overline{\mathcal{H}}$ and $\varepsilon \geq 0$. Now, fixed any $D \in \overline{\mathcal{H}}$, we introduce the family $\mathcal{G}(D) := \{\mathcal{C}' \subset X : \mathcal{C}' \text{ is an open convex cone}, \mathcal{C}\backslash\{0_X\} \subset \mathcal{C}',\ D \cap (-\mathcal{C}') = \emptyset\}$. Note that $\mathcal{G}(D(\varepsilon)) = \mathcal{G}(D)$ for every $\varepsilon \geq 0$. The following notion was introduced by Gutiérrez et al. (2012) for locally convex spaces.

Definition 5 Let X be a partially ordered normed space, $\mathcal{C}$ the ordering cone, $A \subset X$ a subset, $\varepsilon \geq 0$, and $D \in \overline{\mathcal{H}}$. We say that $x_0 \in A$ is a Benson (D, ε)-efficient point of A, written $x_0 \in \text{Be}(A, \mathcal{C}, D, \varepsilon)$, if $0_X \in \text{Min}(\overline{\text{cone}}(A + D(\varepsilon) - x_0), \mathcal{C})$.

For the notion of approximate Henig efficiency we take the characterization (Gutiérrez et al. 2016, Theorem 3.3 (c)) (adapted to normed spaces) as a definition instead of the original (Gutiérrez et al. 2016, Definition 3.1).

Definition 6 Let X be a partially ordered normed space, $\mathcal{C}$ the ordering cone, $A \subset X$ a subset, $\varepsilon \geq 0$, and $D \in \overline{\mathcal{H}}$. We say that $x_0 \in A$ is a Henig (D, ε)-efficient point of A, written $x_0 \in \text{He}(A, \mathcal{C}, D, \varepsilon)$, if there exists $\mathcal{C}_{D,\varepsilon} \in \mathcal{G}(D)$ such that $\overline{\text{cone}}(A + D(\varepsilon) - x_0) \cap (-\mathcal{C}_{D,\varepsilon}) = \emptyset$.

It is clear that $\text{He}(A, \mathcal{C}, D, \varepsilon) \subset \text{Be}(A, \mathcal{C}, D, \varepsilon)$. We begin our analysis determining two necessary conditions for approximate proper efficiency in the sense of Benson. Following Gutiérrez et al. (2006), we denote by $\text{AMin}(g, A, \varepsilon)$ the set of ε-approximate solutions of the scalar optimization problem $\text{Min}_{x \in A}\, g(x)$, i.e., $\text{AMin}(g, A, \varepsilon) = \{x \in A : g(x) - \varepsilon \leq g(z),\ \forall z \in A\}$, where $g : X \to \mathbb{R}$, $A \subset X$, $A \neq \emptyset$, and $\varepsilon > 0$. By means of approximate solutions were derived necessary and sufficient conditions for ε-efficient solutions in Gutiérrez et al. (2006). On the other hand, for every $(f, \alpha) \in \mathcal{C}_+^{a\#}$ and $x_0 \in A$, we denote by $g_{(f,\alpha,x_0)}$ the mapping defined by $g_{(f,\alpha,x_0)}(x) = f(x -$

$x_0) + \alpha\|x - x_0\|$ for every $x \in X$. For simplicity of notation, we write $g_{(f,\alpha)}$ instead of the sublinear map $g_{(f,\alpha,0_X)}$.

Theorem 4.1 *Let X be a partially ordered normed space, $\mathcal{C}$ the ordering cone, $x_0 \in A \subset X$, $\varepsilon \geq 0$, and $D \in \overline{\mathcal{H}}$. Assume that $(-\mathcal{C}, \overline{cone}(A - x_0 + D(\varepsilon)))$ has SSP. If $x_0 \in Be(A, \mathcal{C}, D, \varepsilon)$, then there exists $(f, \alpha) \in \mathcal{C}_+^{a\#}$ such that:*

(i) $\min\limits_{x \in (A+D(\varepsilon)) \cup \{x_0\}} f(x - x_0) + \alpha\|x - x_0\|$ *is attained only at x_0.*
(ii) $x_0 \in AMin(g_{(f,\alpha,x_0)}, A, \lambda)$ *for* $\lambda = \inf\limits_{d \in D(\varepsilon)} f(d) + \alpha\|d\|$.

Proof It is not restrictive to assume that $x_0 = 0_X$. Since $(-\mathcal{C}, \overline{\text{cone}}(A + D(\varepsilon)))$ has SSP, then $(\mathcal{C}, -\overline{\text{cone}}(A + D(\varepsilon)))$ has SSP too. Therefore, by Remark 1, $(\mathcal{C}, -(X \setminus \overline{\text{cone}}(A + D(\varepsilon))_0))$ has SSP as well. As $0_X \in \text{Be}(A, \mathcal{C}, D, \varepsilon)$ implies $\mathcal{C}\setminus\{0_X\} \subset -X\setminus\overline{\text{cone}}(A + D(\varepsilon))$, Theorem 3.3 applies and there exists $(f, \alpha) \in \mathcal{C}_+^{a\#}$ such that $f(c)+\alpha\|c\| < 0 < f(x)+\alpha\|x\|$ for every $c \in -\mathcal{C}, c \neq 0_X$, and $x \in \overline{\text{cone}}(A+D(\varepsilon))$, $x \neq 0_X$. Since $A+D(\varepsilon) \subset \overline{\text{cone}}(A+D(\varepsilon))$, it follows that $0 < f(x)+\alpha\|x\|$ for every $x \in A + D(\varepsilon), x \neq 0_X$. Then, we have (i). Let us prove (ii). Since $g_{(f,\alpha)}$ is a sublinear map, we have $0 < f(x) + \alpha\|x\| + f(d) + \alpha\|d\|$ for every $x \in A$, $x \neq 0_X$, and $d \in D(\varepsilon)$. Fixing $x = 0_X$, we get that $\lambda := \inf\limits_{d \in D(\varepsilon)} f(d) + \alpha\|d\| \geq 0$. Consequently, $0 \leq f(x) + \alpha\|x\| + \lambda$ for every $x \in A$, i.e., $g_{(f,\alpha,0_X)}(0_X) - \lambda \leq g_{(f,\alpha,0_X)}(x)$ for every $x \in A$. Then $0_X \in \text{AMin}(g_{(f,\alpha,x_0)}, A, \lambda)$. □

Let us recall that a function $g : X \to \mathbb{R}$ is strongly monotonically increasing if for each $x, y \in X$, $y - x \in \mathcal{C}\setminus\{0_X\} \Rightarrow g(x) < g(y)$. It is clear that $g_{(f,\alpha)}$ is strongly monotonically increasing for every $(f, \alpha) \in \mathcal{C}_+^{a\#}$. Monotonicity will be used in the proof of the following result showing that the necessary condition (i) in Theorem 4.1 is also sufficient.

Theorem 4.2 *Let X be a partially ordered normed space, $\mathcal{C}$ the ordering cone, $x_0 \in A \subset X$, $\varepsilon \geq 0$, and $D \in \overline{\mathcal{H}}$. If there exists $(f, \alpha) \in \mathcal{C}_+^{a\#}$ such that $f(x - x_0) + \alpha\|x - x_0\| \geq 0$ for every $x \in A + D(\varepsilon)$, then $x_0 \in Be(A, \mathcal{C}, D, \varepsilon)$.*

Proof This proof is an adaptation of (Gasimov et al. 2001,Theorem 1). It is not restrictive to assume that $x_0 = 0_X$. Fix $(f, \alpha) \in \mathcal{C}_+^{a\#}$ such that $f(x) + \alpha\|x\| \geq 0$ for every $x \in A + D(\varepsilon)$. Then $f(y) + \alpha\|y\| \geq 0$ for every $y \in A + D(\varepsilon)$. We will show that $0_X \in \text{Be}(A, \mathcal{C}, D, \varepsilon)$. Clearly, $f(y) + \alpha\|y\| \geq 0$ for every $y \in \text{cone}(A + D(\varepsilon))$ and, by continuity, $f(y) + \alpha\|y\| \geq 0$ for every $y \in \overline{\text{cone}}(A + D(\varepsilon))$. Now, assume that $0_X \notin \text{Be}(A, \mathcal{C}, D, \varepsilon)$. Then there exists $\bar{y} \in \overline{\text{cone}}(A + D(\varepsilon)) \cap (-\mathcal{C}\setminus\{0_X\})$. But strongly monotonicity of $g_{(f,\alpha)}$ (Kasimbeyli 2010, Theorem 3.5) implies that $f(\bar{y}) + \alpha\|\bar{y}\| < 0$, a contradiction. □

As a consequence of the former result and Theorem 4.1 (i) we obtain the following characterization for approximate proper efficiency in the sense of Benson.

Corollary 4.3 *Let X be a partially ordered normed space, $\mathcal{C}$ the ordering cone, $x_0 \in A \subset X$, $\varepsilon \geq 0$, and $D \in \overline{\mathcal{H}}$. Assume that $(-\mathcal{C}, \overline{cone}(A - x_0 + D(\varepsilon)))$ has SSP. Then $x_0 \in Be(A, \mathcal{C}, D, \varepsilon)$ if and only if there exists $(f, \alpha) \in \mathcal{C}_+^{a\#}$ such that* $\min\limits_{x \in (A+D(\varepsilon)) \cup \{x_0\}} f(x - x_0) + \alpha\|x - x_0\|$ *is attained only at x_0.*

Unfortunately, the necessary condition (ii) in Theorem 4.1 is not sufficient, as the following example shows.

Example 1 Take $X = \mathbb{R}^2$, the norm $\|(x, y)\| = \sqrt{x^2 + y^2}$, $\mathcal{C} = \text{cone}(\{(1, 1)\})$, $A = \{(x, 0) \in \mathbb{R}^2 : -1 \leq x \leq 0\}$, and $D = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\| = 1, x \geq 0, y \leq 0\}$. Fix $\varepsilon = 1$, $x_0 = (0, 0) \in A$, $f = (1, 1) \in X^*$, and $\alpha = \frac{4}{3} > 0$. Then $(0, 0) \in \text{AMin}(g_{(f,\alpha,x_0)}, A, \lambda)$ but $(0, 0) \notin \text{Be}(A, \mathcal{C}, D, \varepsilon)$.

Proof If $\lambda = \inf_{d \in D(\varepsilon)} f(d) + \alpha\|d\| = \frac{1}{3}$ and $g_{(f,\alpha,x_0)}(x, 0) = x + \dfrac{4}{3} \mid x \mid = -\frac{1}{3}x \geq 0 > -\frac{1}{3} = -\lambda$, for every $-1 \leq x \leq 0$, then $(0, 0) \in \text{AMin}(g_{(f,\alpha,x_0)}, A, \lambda)$. However, $(-1, -1) \in \overline{\text{cone}}(A + D(\varepsilon))$ because $(-1, -1) = (-1, 0) + (0, -1)$, $(-1, 0) \in A$, and $(0, -1) \in D(\varepsilon)$. Therefore, $(-1, -1) \in (-\mathcal{C}) \cap \overline{\text{cone}}(A + D(\varepsilon))$, which implies that $(0, 0) \notin \text{Be}(A, \mathcal{C}, D, \varepsilon)$. ◻

The preceding example leads us to the following natural question.

Problem 4.4 *Is it possible to characterize approximate Benson proper efficient points via ε-approximate solutions assuming any extra conditions?*

We devote the rest of this section to study approximate Henig proper efficiency. An easy adaptation of the proof of Theorem 4.1 gives the following necessary conditions for approximate proper solution in the sense of Henig.

Theorem 4.5 *Let X be a partially ordered normed space, $\mathcal{C}$ the ordering cone, $A \subset X$ a subset, $\varepsilon \geq 0$, and $D \in \overline{\mathcal{H}}$. Let $x_0 \in He(A, \mathcal{C}, D, \varepsilon)$ and the corresponding $\mathcal{C}_{D,\varepsilon} \in \mathcal{G}(D)$. If $(-\mathcal{C}_{D,\varepsilon}, \overline{cone}(A + D(\varepsilon) - x_0))$ has SSP, then there exists $(f, \alpha) \in (\mathcal{C}_{D,\varepsilon})^{a\#}_{+} \subset \mathcal{C}^{a\#}_{+}$ such that:*

(i) $\min_{x \in (A + D(\varepsilon)) \cup \{x_0\}} f(x - x_0) + \alpha\|x - x_0\|$ *is attained only at x_0.*
(ii) $x_0 \in AMin(g_{(f,\alpha,x_0)}, A, \lambda)$ *for* $\lambda = \inf_{d \in D(\varepsilon)} f(d) + \alpha\|d\|$.

Since $\text{He}(A, \mathcal{C}, D, \varepsilon) \subset \text{Be}(A, \mathcal{C}, D, \varepsilon)$, Theorem 4.1 also provides necessary conditions for Henig approximate proper solutions. Furthermore, when the former inclusion becomes a set equality, Corollary 4.3 provides a characterization Henig approximate proper solutions. This leads to the last results in the work. Before stating them, we introduce the notion of approximating family of cones.

Definition 7 Let X be a partially ordered normed space and $\mathcal{C}$ the ordering cone.

(i) Let $\mathcal{F} = \{\mathcal{C}_n \subset X : n \in \mathbb{N}\}$ be a family of decreasing (with respect to the inclusion) solid, closed, and pointed convex cones. We say that $\mathcal{F}$ approximates $\mathcal{C}$ if $\mathcal{C} \setminus \{0_X\} \subset \text{int}(\mathcal{C}_n)$ eventually (i.e., there exists $n_0 \in \mathbb{N}$ such that $\mathcal{C}\setminus\{0_X\} \subset \text{int}(\mathcal{C}_n)$ for every $n \geq n_0$) and $\mathcal{C} = \cap_n \mathcal{C}_n$.
(ii) Let $\mathcal{F}$ be an approximating family of cones for $\mathcal{C}$. We say that $\mathcal{F}$ separates $\mathcal{C}$ from a closed cone $\mathcal{K} \subset X$ if $\mathcal{C} \cap \mathcal{K} = \{0_X\} \Rightarrow \mathcal{C}_n \cap \mathcal{K} = \{0_X\}$ eventually.

Given $D \subset X\setminus\{0_X\}$, $\varepsilon > 0$, and $x \in X$, we denote by $\mathcal{S}(D(\varepsilon), x)$ the set of all families of cones that approximate $\mathcal{C}$ and separate $\mathcal{C}$ from the cone $-\overline{\text{cone}}(A - x + D(\varepsilon))$.

Corollary 4.6 *Let X be a partially ordered normed space, $\mathcal{C}$ the ordering cone, $x_0 \in A \subset X$, $\varepsilon \geq 0$, and $D \in \overline{\mathcal{H}}$. Assume that $(-\mathcal{C}, \overline{cone}(A - x_0 + D(\varepsilon)))$ has SSP and $\mathcal{S}(D(\varepsilon), x_0) \neq \emptyset$. Then $x_0 \in He(A, \mathcal{C}, D, \varepsilon)$ if and only if there exists $(f, \alpha) \in \mathcal{C}_+^{a\#}$ such that $\min\limits_{x \in (A+D(\varepsilon)) \cup \{x_0\}} f(x - x_0) + \alpha\|x - x_0\|$ is attained only at x_0.*

Proof By Gutiérrez et al. (2019, Theorem 3.1) $x_0 \in \mathrm{He}(A, \mathcal{C}, D, \varepsilon) \Leftrightarrow x_0 \in \mathrm{Be}(A, \mathcal{C}, D, \varepsilon)$. Now, Corollary 4.3 applies. □

Note that $\mathcal{S}(D(\varepsilon), x_0) \neq \emptyset$ whenever X is finite-dimensional (Henig 1982, Theorem 2.1) or if $\mathcal{C}$ has a weakly compact base and $\overline{\mathrm{cone}}(A - x_0 + D(\varepsilon))$ is weakly closed (Gutiérrez et al. 2019, Theorem 2.3). Therefore, we have the following.

Corollary 4.7 *Let X be a partially ordered normed space, $\mathcal{C}$ the ordering cone, $x_0 \in A \subset X$, $\varepsilon \geq 0$, and $D \in \overline{\mathcal{H}}$. Assume that $(-\mathcal{C}, \overline{cone}(A - x_0 + D(\varepsilon)))$ has SSP and at least one of the following assertions hold:*

(i) X has finite dimension.
(ii) $\mathcal{C}$ has a weakly compact base and $\overline{cone}(A - x_0 + D(\varepsilon))$ is weakly closed.

Then $x_0 \in He(A, \mathcal{C}, D, \varepsilon)$ if and only if there exists $(f, \alpha) \in \mathcal{C}_+^{a\#}$ such that

$$\min_{x \in (A+D(\varepsilon)) \cup \{x_0\}} f(x - x_0) + \alpha\|x - x_0\|,$$

is attained only at x_0.

5 Conclusions

In this work, we provide optimal sufficient conditions with a sublinear function for Henig global proper efficient points, Henig proper efficient points, super efficient points, Benson proper efficient points, Hartley proper efficient points, Hurwicz proper efficient points, Borwein proper efficient points, and tangentially Borwein proper efficient points; in the case of Benson proper efficiency the optimal condition becomes a characterization. The approach is done in a unified way considering such proper efficient points as $\mathcal{Q}$-minimal points. For every type of proper efficient point we apply a separation property to a fixed $\mathcal{Q}$-dilation of the ordering cone. For future research we ask if it is possible to characterize $\mathcal{Q}$-minimal points in general via SSP assuming any extra conditions. In the last part of the work, we adapt our arguments to obtain new characterizations of Benson and Henig approximate proper efficient points through scalarizations. We also provide necessary conditions for approximate Benson proper efficient points via ε-approximate solutions and we ask if it is possible to extend such a result to a characterization assuming any extra condition. Our results are established in the setting of normed spaces and they do not impose any kind of convexity and boundedness assumption.

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Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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