Banach J. Math. Anal. (2023) 17:44
https://doi.org/10.1007/s43037-023-00269-y

Tusi Mathematical Research Group

ORIGINAL PAPER

Cohen strongly p-summing holomorphic mappings on Banach spaces

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Received: 6 December 2022 / Accepted: 11 April 2023

Abstract

Let E and F be complex Banach spaces, U be an open subset of E and $1 \leq p \leq \infty$. We introduce and study the notion of a Cohen strongly p-summing holomorphic mapping from U to F, a holomorphic version of a strongly p-summing linear operator. For such mappings, we establish both Pietsch Domination/Factorization Theorems and analyse their linearizations from $\mathcal{G}^\infty(U)$ (the canonical predual of $\mathcal{H}^\infty(U)$) and their transpositions on $\mathcal{H}^\infty(U)$. Concerning the space $\mathcal{D}_p^{\mathcal{H}^\infty}$ formed by such mappings and endowed with a natural norm $d_p^{\mathcal{H}^\infty}$, we show that it is a regular Banach ideal of bounded holomorphic mappings generated by composition with the ideal of strongly p-summing linear operators. Moreover, we identify the space $(\mathcal{D}_p^{\mathcal{H}^\infty}(U, F^*), d_p^{\mathcal{H}^\infty})$ with the dual of the completion of tensor product space $\mathcal{G}^\infty(U) \otimes F$ endowed with the Chevet–Saphar norm g_p.

Keywords Holomorphic mapping · p-Summing operator · Duality · Linearization

Mathematics Subject Classification 46E15 · 46G20 · 47B10

Communicated by Jari Taskinen.

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Published online: 07 May 2023

Introduction

The linear theory of absolutely summing operators between Banach spaces was initiated by Grothendieck [11] in 1950 with the introduction of the concept of 1-summing operator. In 1967, Pietsch [22] defined the class of absolutely p-summing operators for any $p > 0$ and established many of their fundamental properties.

The nonlinear theory for such operators started with Pietsch [23] in 1983. Since then, the idea of extending the theory of absolutely p-summing operators to other settings has been developed by various authors, namely, the polynomial, multilinear, Lipschitz and holomorphic settings (see, for example, [1, 2, 7, 8, 19, 27, 28]).

Summability for holomorphic mappings was first considered by Matos in a series of papers (see e.g. [13, 14]). Our approach in this paper is different from that of Matos. Moreover, strong p-summability in the sense of Dimant [7] was also addressed for subspaces of holomorphic mappings as polynomials and multilinear mappings under the name of factorable strongly p-summing (see [20, 24, 25]). In these papers, it was proved that the ideal of factorable strongly p-summing polynomials (multilinear mappings) coincides with the ideal formed by composition with p-summing linear operators. Ideals of polynomial mappings were also studied by Floret and García [9, 10].

In 1973, Cohen [5] introduced the concept of a strongly p-summing linear operator to characterize those operators whose adjoints are absolutely p^*-summing operators, where p^* denotes the conjugate index of $p \in (1, \infty]$. Influenced by this class of operators, we introduce and study a new concept of summability in the category of bounded holomorphic mappings, which yields the called *Cohen strongly p-summing holomorphic mappings*.

We now describe the contents of the paper. Let E and F be complex Banach spaces, U be an open subset of E and $1 \leq p \leq \infty$. We denote by $\mathcal{H}^\infty(U, F)$ the Banach space of all bounded holomorphic mappings from U to F, equipped with the supremum norm. In particular, $\mathcal{H}^\infty(U)$ stands for the space $\mathcal{H}^\infty(U, \mathbb{C})$. It is known that $\mathcal{H}^\infty(U)$ is a dual Banach space whose canonical predual, denoted $\mathcal{G}^\infty(U)$, is the norm-closed linear subspace of $\mathcal{H}^\infty(U)^*$ generated by the evaluation functionals at the points of U.

In Sect. 1, we fix the notation and recall some results on the space $\mathcal{H}^\infty(U, F)$, essentially, a remarkable linearization theorem due to Mujica [16] which is a key tool to establish our results.

In Sect. 2, we show that the space of all Cohen strongly p-summing holomorphic mappings denoted $\mathcal{D}_p^{\mathcal{H}^\infty}$ and equipped with a natural norm $d_p^{\mathcal{H}^\infty}$, is a regular Banach ideal of bounded holomorphic mappings. Furthermore, $\mathcal{D}_1^{\mathcal{H}^\infty} = \mathcal{H}^\infty$ with $d_1^{\mathcal{H}^\infty} = \|\cdot\|_\infty$.

The elements of the tensor product of two linear spaces can be viewed as linear mappings or bilinear forms (see [26, Section 1.3]). Following this idea, in Sect. 3 we introduce the tensor product $\Delta(U) \otimes F$ as a space of linear functionals on the space $\mathcal{H}^\infty(U, F^*)$, and equip this space with the known Chevet–Saphar norms g_p and d_p.

Section 4 addresses the duality theory: the space $(\mathcal{D}_p^{\mathcal{H}^\infty}(U, F^*), d_p^{\mathcal{H}^\infty})$ is canonically isometrically isomorphic to the dual of the completion of the tensor product space $\mathcal{G}^\infty(U) \otimes_{g_p} F$. In particular, we deduce that $\mathcal{H}^\infty(U, F^*)$ is a dual space.

Pietsch [22] established a Domination/Factorization Theorem for p-summing linear operators between Banach spaces. Characterizing previously the elements of the dual space of $\Delta(U) \otimes_{g_p} F$, we present for Cohen strongly p-summing holomorphic mappings both versions of Pietsch Domination Theorem and Pietsch Factorization Theorem in Sects. 5 and 6, respectively.

Moreover, in Sect. 5, we prove that a mapping $f : U \to F$ is Cohen strongly p-summing holomorphic if and only if Mujica's linearization $T_f : \mathcal{G}^\infty(U) \to F$ is a strongly p-summing operator. Several interesting applications of this fact are obtained.

In addition, we show that the ideal $\mathcal{D}_p^{\mathcal{H}^\infty}$ is generated by composition with the ideal $\mathcal{D}_p$ of strongly p-summing linear operators, that is, every mapping $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ admits a factorization in the form $f = T \circ g$, for some complex Banach space G, $g \in \mathcal{H}^\infty(U, G)$ and $T \in \mathcal{D}_p(G, F)$. Moreover, $d_p^{\mathcal{H}^\infty}(f)$ coincides with $\inf\{d_p(T)\, \|g\|_\infty\}$, where the infimum is extended over all such factorizations of f, and, curiously, this infimum is attained at Mujica's factorization of f. We also show that every $f \in \mathcal{D}_2^{\mathcal{H}^\infty}(U, F)$ factors through a Hilbert space whenever F is reflexive, and establish some inclusion and coincidence properties of spaces $\mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$.

These results represent advances in the research program initiated by Aron et al. [4] on the factorization of bounded holomorphic mappings in terms of an element of an operator ideal and a bounded holomorphic mapping.

Finally, we analyse holomorphic transposition of their elements and prove that every member of $\mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ has relatively weakly compact range that becomes relatively compact whenever F is reflexive. We thus contribute to the study of holomorphic mappings with relatively (weakly) compact range, begun by Mujica [16] and continued in [12].

1 Notation and preliminaries

Throughout this paper, unless otherwise stated, E and F will denote complex Banach spaces and U an open subset of E.

We first introduce some notation. As usual, B_E denotes the closed unit ball of E. For two vector spaces E and F, $L(E, F)$ stands for the vector space of all linear operators from E into F. In the case that E and F are normed spaces, $\mathcal{L}(E, F)$ represents the normed space of all bounded linear operators from E to F endowed with the canonical norm of operators. In particular, the algebraic dual $L(E, \mathbb{K})$ and the topological dual $\mathcal{L}(E, \mathbb{K})$ are denoted by E' and E^*, respectively. For each $e \in E$ and $e^* \in E'$, we frequently will write $\langle e^*, e\rangle$ instead of $e^*(e)$. We denote by κ_E the canonical isometric embedding of E into E^{**} defined by $\langle \kappa_E(e), e^*\rangle = \langle e^*, e\rangle$ for $e \in E$ and $e^* \in E^*$. For a set $A \subseteq E$, $\mathrm{co}(A)$ denotes the convex hull of A.

We now recall some concepts and results of the theory of holomorphic mappings on Banach spaces.

Theorem 1.1 (See [18, 7 Theorem] and [15, Theorem 8.7]) *Let E and F be complex Banach spaces and let U be an open set in E. For a mapping $f: U \to F$, the following conditions are equivalent*:

(i) *For each $a \in U$, there is an operator $T \in \mathcal{L}(E, F)$ such that*

$$\lim_{x \to a} \frac{f(x) - f(a) - T(x-a)}{\|x - a\|} = 0.$$

(ii) *For each $a \in U$, there exist an open ball $B(a, r) \subseteq U$ and a sequence of continuous m-homogeneous polynomials $(P_{m,a})_{m \in \mathbb{N}_0}$ from E into F such that*

$$f(x) = \sum_{m=0}^{\infty} P_{m,a}(x - a),$$

where the series converges uniformly for $x \in B(a, r)$.

(iii) *f is G-holomorphic (that is, for all $a \in U$ and $b \in E$, the map $\lambda \mapsto f(a + \lambda b)$ is holomorphic on the open set $\{\lambda \in \mathbb{C}\colon a + \lambda b \in U\}$) and continuous.* □

A mapping $f: U \to F$ is said to be *holomorphic* if it verifies the equivalent conditions in Theorem 1.1. The mapping T in condition (i) is uniquely determined by f and a, and is called the *differential of f at a* and denoted by $Df(a)$.

A mapping $f: U \to F$ is *locally bounded* if f is bounded on a suitable neighborhood of each point of U. Given a Banach space E, a subset $N \subseteq B_{E^*}$ is said to be *norming for E* if the function

$$N(x) = \sup\left\{\left|x^*(x)\right| : x^* \in N\right\} \quad (x \in E)$$

defines the norm on E.

If $U \subseteq E$ and $V \subseteq F$ are open sets, $\mathcal{H}(U, V)$ will represent the set of all holomorphic mappings from U to V. We will denote by $\mathcal{H}(U, F)$ the linear space of all holomorphic mappings from U into F and by $\mathcal{H}^\infty(U, F)$ the subspace of all $f \in \mathcal{H}(U, F)$ such that $f(U)$ is bounded in F. When $F = \mathbb{C}$, then we will write $\mathcal{H}^\infty(U, \mathbb{C}) = \mathcal{H}^\infty(U)$.

It is easy to prove that the linear space $\mathcal{H}^\infty(U, F)$, equipped with the supremum norm:

$$\|f\|_\infty = \sup\{\|f(x)\| : x \in U\} \qquad \left(f \in \mathcal{H}^\infty(U)\right),$$

is a Banach space. Let $\mathcal{G}^\infty(U)$ denote the norm-closed linear hull in $\mathcal{H}^\infty(U)^*$ of the set $\{\delta(x)\colon x \in U\}$ of *evaluation functionals* defined by

$$\langle \delta(x), f \rangle = f(x) \quad \left(f \in \mathcal{H}^\infty(U)\right).$$

In [16, 17], Mujica established the following properties of $\mathcal{G}^\infty(U)$.

Theorem 1.2 [16, Theorem 2.1] *Let E be a complex Banach space and let U be an open set in E.*

(i) *$\mathcal{H}^\infty(U)$ is isometrically isomorphic to $\mathcal{G}^\infty(U)^*$, via the evaluation mapping $J_U : \mathcal{H}^\infty(U) \to \mathcal{G}^\infty(U)^*$ given by*

$$\langle J_U(f), \gamma\rangle = \gamma(f) \quad \big(\gamma \in \mathcal{G}^\infty(U),\ f \in \mathcal{H}^\infty(U)\big).$$

(ii) *The mapping $g_U : U \to \mathcal{G}^\infty(U)$ defined by $g_U(x) = \delta(x)$ is holomorphic with $\|g_U(x)\| = 1$ for all $x \in U$.*

(iii) *For each complex Banach space F and each mapping $f \in \mathcal{H}^\infty(U, F)$, there exists a unique operator $T_f \in \mathcal{L}(\mathcal{G}^\infty(U), F)$ such that $T_f \circ g_U = f$. Furthermore, $\left\|T_f\right\| = \|f\|_\infty$.*

(iv) *The mapping $f \mapsto T_f$ is an isometric isomorphism from $\mathcal{H}^\infty(U, F)$ onto $\mathcal{L}(\mathcal{G}^\infty(U), F)$.*

(v) [16, Corollary 4.12] (*see also* [17, Theorem 5.1]). *$\mathcal{G}^\infty(U)$ consists of all functionals $\gamma \in \mathcal{H}^\infty(U)^*$ of the form $\gamma = \sum_{i=1}^\infty \lambda_i \delta(x_i)$ with $(\lambda_i)_{i\geq 1} \in \ell_1$ and $(x_i)_{i\geq 1} \in U^{\mathbb{N}}$. Moreover, $\|\gamma\| = \inf\left\{\sum_{i=1}^\infty |\lambda_i|\right\}$ where the infimum is taken over all such representations of γ.* □

2 Cohen strongly *p*-summing holomorphic mappings

Let E and F be Banach spaces and $1 \leq p \leq \infty$. Let us recall [6] that an operator $T \in \mathcal{L}(E, F)$ is *p-summing* if there exists a constant $C \geq 0$ such that, regardless of the natural number n and regardless of the choice of vectors $x_1, \ldots, x_n$ in E, we have the inequalities:

$$\left(\sum_{i=1}^n \|T(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^n \left|x^*(x_i)\right|^p\right)^{\frac{1}{p}} \qquad \text{if } 1 \leq p < \infty,$$

$$\max_{1\leq i\leq n} \|T(x_i)\| \leq C \sup_{x^* \in B_{E^*}} \left(\max_{1\leq i\leq n} \left|x^*(x_i)\right|\right) \qquad \text{if } p = \infty.$$

The infimum of such constants C is denoted by $\pi_p(T)$ and the linear space of all p-summing operators from E into F by $\Pi_p(E, F)$.

The analogous notion for holomorphic mappings could be introduced as follows.

Definition 2.1 Let E and F be complex Banach spaces, let U be an open subset of E, and let $1 \leq p \leq \infty$. A holomorphic mapping $f : U \to F$ is said to be *p-summing* if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in U$, we have

$$\left(\sum_{i=1}^n \|f(x_i)\|^p\right)^{\frac{1}{p}} \leq C \sup_{g \in B_{\mathcal{H}^\infty(U)}} \left(\sum_{i=1}^n |g(x_i)|^p\right)^{\frac{1}{p}} \qquad \text{if } 1 \leq p < \infty,$$

$$\max_{1\leq i\leq n} \|f(x_i)\| \leq C \sup_{g \in B_{\mathcal{H}^\infty(U)}} \left(\max_{1\leq i\leq n} |g(x_i)|\right) \qquad \text{if } p = \infty.$$

We denote by $\pi_p^{\mathcal{H}^\infty}(f)$ the infimum of such constants C, and by $\Pi_p^{\mathcal{H}^\infty}(U, F)$ the set of all p-summing holomorphic mappings from U into F.

p-Summing holomorphic mappings are of little interest to us as $\Pi_p^{\mathcal{H}^\infty}(U, F) = \mathcal{H}^\infty(U, F)$ with $\pi_p^{\mathcal{H}^\infty}(f) = \|f\|_\infty$ for all $f \in \Pi_p^{\mathcal{H}^\infty}(U, F)$, and furthermore the subclass of p-summing holomorphic mappings that we will study in this paper includes this case.

Let $1 \leq p \leq \infty$ and let p^* denote the *conjugate index of* p given by

$$p^* = \begin{cases} \frac{p}{p-1} & \text{if } 1 < p < \infty, \\ \infty & \text{if } p = 1, \\ 1 & \text{if } p = \infty. \end{cases}$$

In [5], Cohen introduced the following subclass of p-summing operators between Banach spaces: an operator $T \in \mathcal{L}(E, F)$ is *strongly p-summing* if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in E$ and $y_1^*, \ldots, y_n^* \in F^*$, we have

$$\sum_{i=1}^n \left|\langle y_i^*, T(x_i)\rangle\right| \leq C \left(\sum_{i=1}^n \|x_i\|\right) \sup_{y^{**}\in B_{F^{**}}} \left(\max_{1\leq i\leq n} \left|y^{**}(y_i^*)\right|\right) \qquad \text{if} \quad p = 1,$$

$$\sum_{i=1}^n \left|\langle y_i^*, T(x_i)\rangle\right| \leq C \left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}} \sup_{y^{**}\in B_{F^{**}}} \left(\sum_{i=1}^n \left|y^{**}(y_i^*)\right|^{p^*}\right)^{\frac{1}{p^*}} \qquad \text{if} \quad 1 < p < \infty,$$

$$\sum_{i=1}^n \left|\langle y_i^*, T(x_i)\rangle\right| \leq C \left(\max_{1\leq i\leq n} \|x_i\|\right) \sup_{y^{**}\in B_{F^{**}}} \left(\sum_{i=1}^n \left|y^{**}(y_i^*)\right|\right) \qquad \text{if} \quad p = \infty.$$

The infimum of such constants C is denoted by $d_p(T)$, and the space of all strongly p-summing operators from E into F by $\mathcal{D}_p(E, F)$. If $p = 1$, we have $\mathcal{D}_1(E, F) = \mathcal{L}(E, F)$.

We now introduce a version of this concept in the setting of holomorphic mappings.

Definition 2.2 Let E and F be complex Banach spaces, let U be an open subset of E, and let $1 \leq p \leq \infty$. A holomorphic mapping $f : U \to F$ is said to be *Cohen strongly p-summing* if there exists a constant $C \geq 0$ such that for all $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $x_1, \ldots, x_n \in U$ and $y_1^*, \ldots, y_n^* \in F^*$, we have

$$\sum_{i=1}^n |\lambda_i| \left|\langle y_i^*, f(x_i)\rangle\right| \leq C \left(\sum_{i=1}^n |\lambda_i|\right) \sup_{y^{**}\in B_{F^{**}}} \left(\max_{1\leq i\leq n} \left|y^{**}(y_i^*)\right|\right) \qquad \text{if} \quad p = 1,$$

$$\sum_{i=1}^n |\lambda_i| \left|\langle y_i^*, f(x_i)\rangle\right| \leq C \left(\sum_{i=1}^n |\lambda_i|^p\right)^{\frac{1}{p}} \sup_{y^{**}\in B_{F^{**}}} \left(\sum_{i=1}^n \left|y^{**}(y_i^*)\right|^{p^*}\right)^{\frac{1}{p^*}} \qquad \text{if} \quad 1 < p < \infty,$$

$$\sum_{i=1}^n |\lambda_i| \left|\langle y_i^*, f(x_i)\rangle\right| \leq C \left(\max_{1\leq i\leq n} |\lambda_i|\right) \sup_{y^{**}\in B_{F^{**}}} \left(\sum_{i=1}^n \left|y^{**}(y_i^*)\right|\right) \qquad \text{if} \quad p = \infty.$$

We denote by $d_p^{\mathcal{H}^\infty}(f)$ the infimum of such constants C, and by $\mathcal{D}_p^{\mathcal{H}^\infty}(U,F)$ the set of all Cohen strongly p-summing holomorphic mappings from U into F.

The introduction of the scalars λ_i in the previous definition is justified by the assertion (v) of Theorem 1.2. Proposition 2.5 shows that $\mathcal{D}_1^{\mathcal{H}^\infty} = \mathcal{H}^\infty$.

The concept of an ideal of bounded holomorphic mappings is inspired by the analogous one for bounded linear operators between Banach spaces [26, Section 8.2].

Definition 2.3 An *ideal of bounded holomorphic mappings* (or simply, a *bounded-holomorphic ideal*) is a subclass $\mathcal{I}^{\mathcal{H}^\infty}$ of the class $\mathcal{H}^\infty$ of all bounded holomorphic mappings such that for each complex Banach space E, each open subset U of E and each complex Banach space F, the components

$$\mathcal{I}^{\mathcal{H}^\infty}(U,F) := \mathcal{I}^{\mathcal{H}^\infty} \cap \mathcal{H}^\infty(U,F)$$

satisfy the following properties:

(I1) $\mathcal{I}^{\mathcal{H}^\infty}(U,F)$ is a linear subspace of $\mathcal{H}^\infty(U,F)$,
(I2) For any $g \in \mathcal{H}^\infty(U)$ and $y \in F$, the mapping $g \cdot y \colon x \mapsto g(x)y$ from U to F is in $\mathcal{I}^{\mathcal{H}^\infty}(U,F)$,
(I3) *The ideal property*: If H, G are complex Banach spaces, V is an open subset of H, $h \in \mathcal{H}(V,U)$, $f \in \mathcal{I}^{\mathcal{H}^\infty}(U,F)$ and $S \in \mathcal{L}(F,G)$, then $S \circ f \circ h$ is in $\mathcal{I}^{\mathcal{H}^\infty}(V,G)$.

A bounded-holomorphic ideal $\mathcal{I}^{\mathcal{H}^\infty}$ is said to be *normed (Banach)* if there exists a function $\|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty}} : \mathcal{I}^{\mathcal{H}^\infty} \to \mathbb{R}_0^+$ such that for every complex Banach space E, every open subset U of E and every complex Banach space F, the following conditions are satisfied:

(N1) $(\mathcal{I}^{\mathcal{H}^\infty}(U,F), \|\cdot\|_{\mathcal{I}^{\mathcal{H}^\infty}})$ is a normed (Banach) space with $\|f\|_\infty \le \|f\|_{\mathcal{I}^{\mathcal{H}^\infty}}$ for all $f \in \mathcal{I}^{\mathcal{H}^\infty}(U,F)$,
(N2) $\|g \cdot y\|_{\mathcal{I}^{\mathcal{H}^\infty}} = \|g\|_\infty \|y\|$ for every $g \in \mathcal{H}^\infty(U)$ and $y \in F$,
(N3) If H, G are complex Banach spaces, V is an open subset of H, $h \in \mathcal{H}(V,U)$, $f \in \mathcal{I}^{\mathcal{H}^\infty}(U,F)$ and $S \in \mathcal{L}(F,G)$, then $\|S \circ f \circ h\|_{\mathcal{I}^{\mathcal{H}^\infty}} \le \|S\| \|f\|_{\mathcal{I}^{\mathcal{H}^\infty}}$.

A normed bounded-holomorphic ideal $\mathcal{I}^{\mathcal{H}^\infty}$ is said to be *regular* if for any $f \in \mathcal{H}^\infty(U,F)$, we have that $f \in \mathcal{I}^{\mathcal{H}^\infty}(U,F)$ with $\|f\|_{\mathcal{I}^{\mathcal{H}^\infty}} = \|\kappa_F \circ f\|_{\mathcal{I}^{\mathcal{H}^\infty}}$ whenever $\kappa_F \circ f \in \mathcal{I}^{\mathcal{H}^\infty}(U,F^{**})$.

The following class of bounded holomorphic mappings appears involved in Definition 2.3.

Lemma 2.4 *Let $g \in \mathcal{H}^\infty(U)$ and $y \in F$. The mapping $g \cdot y \colon U \to F$, given by $(g \cdot y)(x) = g(x)y$, belongs to $\mathcal{H}^\infty(U,F)$ with $\|g \cdot y\|_\infty = \|g\|_\infty \|y\|$.* □

We are now ready to establish the main result of this section.

Proposition 2.5 *$(\mathcal{D}_p^{\mathcal{H}^\infty}, d_p^{\mathcal{H}^\infty})$ is a regular Banach ideal of bounded holomorphic mappings. Furthermore, $\mathcal{D}_1^{\mathcal{H}^\infty} = \mathcal{H}^\infty$ with $d_1^{\mathcal{H}^\infty} = \|\cdot\|_\infty$.*

Proof We will only prove the case $1 < p < \infty$. The cases $p = 1$ and $p = \infty$ follow similarly.

(N1) We first show that $\mathcal{D}_p^{\mathcal{H}^\infty}(U, F) \subseteq \mathcal{H}^\infty(U, F)$ with $\|f\|_\infty \le d_p^{\mathcal{H}^\infty}(f)$ for all $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$. Indeed, given $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$, we have

$$\left|\langle y^*, f(x)\rangle\right| \le d_p^{\mathcal{H}^\infty}(f) \sup_{y^{**}\in B_{F^{**}}} \left|y^{**}(y^*)\right| = d_p^{\mathcal{H}^\infty}(f)\,\|y^*\|$$

for all $x \in U$ and $y^* \in F^*$. By Hahn–Banach Theorem, we obtain that $\|f(x)\| \le d_p^{\mathcal{H}^\infty}(f)$ for all $x \in U$. Hence $f \in \mathcal{H}^\infty(U, F)$ with $\|f\|_\infty \le d_p^{\mathcal{H}^\infty}(f)$.

Let $f_1, f_2 \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$. Given $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $x_1, \ldots, x_n \in U$ and $y_1^*, \ldots, y_n^* \in F^*$, we have

$$\sum_{i=1}^n |\lambda_i|\left|\langle y_i^*, f_1(x_i)\rangle\right| \le d_p^{\mathcal{H}^\infty}(f_1)\left(\sum_{i=1}^n |\lambda_i|^p\right)^{\frac{1}{p}} \sup_{y^{**}\in B_{F^{**}}}\left(\sum_{i=1}^n \left|y^{**}(y_i^*)\right|^{p^*}\right)^{\frac{1}{p^*}},$$

$$\sum_{i=1}^n |\lambda_i|\left|\langle y_i^*, f_2(x_i)\rangle\right| \le d_p^{\mathcal{H}^\infty}(f_2)\left(\sum_{i=1}^n |\lambda_i|^p\right)^{\frac{1}{p}} \sup_{y^{**}\in B_{F^{**}}}\left(\sum_{i=1}^n \left|y^{**}(y_i^*)\right|^{p^*}\right)^{\frac{1}{p^*}}.$$

Using the two inequalities above, we obtain

$$\begin{aligned}\sum_{i=1}^n |\lambda_i|\left|\langle y_i^*, (f_1+f_2)(x_i)\rangle\right| &\le \sum_{i=1}^n |\lambda_i|\left|\langle y_i^*, f_1(x_i)\rangle\right| + \sum_{i=1}^n |\lambda_i|\left|\langle y_i^*, f_2(x_i)\rangle\right| \\ &\le \left(d_p^{\mathcal{H}^\infty}(f_1) + d_p^{\mathcal{H}^\infty}(f_2)\right)\left(\sum_{i=1}^n |\lambda_i|^p\right)^{\frac{1}{p}} \sup_{y^{**}\in B_{F^{**}}}\left(\sum_{i=1}^n \left|y^{**}(y_i^*)\right|^{p^*}\right)^{\frac{1}{p^*}}.\end{aligned}$$

This tells us that $f_1 + f_2 \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ with $d_p^{\mathcal{H}^\infty}(f_1 + f_2) \le d_p^{\mathcal{H}^\infty}(f_1) + d_p^{\mathcal{H}^\infty}(f_2)$.

Let $\lambda \in \mathbb{C}$ and $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$. Given $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $x_i \in U$ and $y_i^* \in F^*$ for $i = 1, \ldots, n$, we have

$$\begin{aligned}\sum_{i=1}^n |\lambda_i|\left|\langle y_i^*, (\lambda f)(x_i)\rangle\right| &= |\lambda|\sum_{i=1}^n |\lambda_i|\left|\langle y_i^*, f(x_i)\rangle\right| \\ &\le |\lambda|\, d_p^{\mathcal{H}^\infty}(f)\left(\sum_{i=1}^n |\lambda_i|^p\right)^{\frac{1}{p}} \sup_{y^{**}\in B_{F^{**}}}\left(\sum_{i=1}^n \left|y^{**}(y_i^*)\right|^{p^*}\right)^{\frac{1}{p^*}},\end{aligned}$$

and thus $\lambda f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ with $d_p^{\mathcal{H}^\infty}(\lambda f) \le |\lambda| d_p^{\mathcal{H}^\infty}(f)$. This implies that $d_p^{\mathcal{H}^\infty}(\lambda f) = 0 = |\lambda|\, d_p^{\mathcal{H}^\infty}(f)$ if $\lambda = 0$. For $\lambda \neq 0$, we have $d_p^{\mathcal{H}^\infty}(f) = d_p^{\mathcal{H}^\infty}(\lambda^{-1}(\lambda f)) \le |\lambda|^{-1}\, d_p^{\mathcal{H}^\infty}(\lambda f)$, hence $|\lambda|\, d_p^{\mathcal{H}^\infty}(f) \le d_p^{\mathcal{H}^\infty}(\lambda f)$, and so $d_p^{\mathcal{H}^\infty}(\lambda f) = |\lambda|\, d_p^{\mathcal{H}^\infty}(f)$.

Moreover, if $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ and $d_p^{\mathcal{H}^\infty}(f) = 0$, then $\|f\|_\infty = 0$ by (N1), and so $f = 0$. Thus, $\left(\mathcal{D}_p^{\mathcal{H}^\infty}(U, F), d_p^{\mathcal{H}^\infty}\right)$ is a normed space.

To prove that $\left(\mathcal{D}_p^{\mathcal{H}^\infty}(U, F), d_p^{\mathcal{H}^\infty}\right)$ is complete, it suffices to prove that every absolutely convergent series is convergent. So let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ such that $\sum_{n\in\mathbb{N}} d_p^{\mathcal{H}^\infty}(f_n)$ is convergent. Since $\|f_n\|_\infty \le d_p^{\mathcal{H}^\infty}(f_n)$ for all $n \in \mathbb{N}$ and $(\mathcal{H}^\infty(U, F), \|\cdot\|_\infty)$ is a Banach space, then $\sum_{n\in\mathbb{N}} f_n$ converges in $(\mathcal{H}^\infty(U, F), \|\cdot\|_\infty)$ to a function $f \in \mathcal{H}^\infty(U, F)$. Given $m \in \mathbb{N}$, $x_1, \dots, x_m \in U$, $y_1^*, \dots, y_m^* \in F^*$ and $\lambda_1, \dots, \lambda_m \in \mathbb{C}$, we have

$$
\begin{aligned}
&\sum_{k=1}^m |\lambda_k| \left|\left\langle y_k^*, \sum_{i=1}^n f_i(x_k)\right\rangle\right| \\
&\quad \le d_p^{\mathcal{H}^\infty}\left(\sum_{i=1}^n f_i\right)\left(\sum_{k=1}^m |\lambda_k|^p\right)^{\frac{1}{p}} \sup_{y^{**}\in B_{F^{**}}} \left(\sum_{k=1}^m \left|y^{**}(y_k^*)\right|^{p^*}\right)^{\frac{1}{p^*}} \\
&\quad \le \left(\sum_{i=1}^n d_p^{\mathcal{H}^\infty}(f_i)\right)\left(\sum_{k=1}^m |\lambda_k|^p\right)^{\frac{1}{p}} \sup_{y^{**}\in B_{F^{**}}} \left(\sum_{k=1}^m \left|y^{**}(y_k^*)\right|^{p^*}\right)^{\frac{1}{p^*}}
\end{aligned}
$$

for all $n \in \mathbb{N}$, and by taking limits with $n \to \infty$ yields

$$
\sum_{k=1}^m |\lambda_k| \left|\left\langle y_k^*, f(x_k)\right\rangle\right| \le \left(\sum_{n=1}^\infty d_p^{\mathcal{H}^\infty}(f_n)\right)\left(\sum_{k=1}^m |\lambda_k|^p\right)^{\frac{1}{p}} \sup_{y^{**}\in B_{F^{**}}} \left(\sum_{k=1}^m \left|y^{**}(y_k^*)\right|^{p^*}\right)^{\frac{1}{p^*}}.
$$

Hence $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ with $\pi_p^{\mathcal{H}^\infty}(f) \le \sum_{n=1}^\infty d_p^{\mathcal{H}^\infty}(f_n)$. Moreover, we have

$$
d_p^{\mathcal{H}^\infty}\left(f - \sum_{i=1}^n f_i\right) = d_p^{\mathcal{H}^\infty}\left(\sum_{i=n+1}^\infty f_i\right) \le \sum_{i=n+1}^\infty d_p^{\mathcal{H}^\infty}(f_i)
$$

for all $n \in \mathbb{N}$, and thus f is the $d_p^{\mathcal{H}^\infty}$-limit of the series $\sum_{n\in\mathbb{N}} f_n$.

(N2) Let $g \in \mathcal{H}^\infty(U)$ and $y \in F$. If $y = 0$, there is nothing to prove. Assume $y \ne 0$. By Lemma 2.4, $g \cdot y \in \mathcal{H}^\infty(U, F)$. Given $n \in \mathbb{N}$, $x_1, \dots, x_n \in U$, $y_1^*, \dots, y_n^* \in F^*$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, we have

$$
\begin{aligned}
&\sum_{i=1}^n |\lambda_i| \left|\left\langle y_i^*, (g\cdot y)(x_i)\right\rangle\right| \\
&\quad = \sum_{i=1}^n |\lambda_i|\, |g(x_i)| \left|\left\langle y_i^*, y\right\rangle\right| \\
&\quad \le \|g\|_\infty \|y\| \sum_{i=1}^n |\lambda_i| \left|\left\langle y_i^*, \frac{y}{\|y\|}\right\rangle\right|
\end{aligned}
$$

$$\le \|g\|_\infty \|y\| \left(\sum_{i=1}^n |\lambda_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n \left|\left\langle y_i^*, \frac{y}{\|y\|}\right\rangle\right|^{p^*}\right)^{\frac{1}{p^*}}$$

$$= \|g\|_\infty \|y\| \left(\sum_{i=1}^n |\lambda_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n \left|\left\langle \kappa_F\left(\frac{y}{\|y\|}\right), y_i^*, \right\rangle\right|^{p^*}\right)^{\frac{1}{p^*}}$$

$$\le \|g\|_\infty \|y\| \left(\sum_{i=1}^n |\lambda_i|^p\right)^{\frac{1}{p}} \sup_{y^{**}\in B_{F^{**}}} \left(\sum_{i=1}^n \left|y^{**}(y_i^*)\right|^{p^*}\right)^{\frac{1}{p^*}}$$

by applying the Hölder inequality, and therefore $g\cdot y \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ with $d_p^{\mathcal{H}^\infty}(g\cdot y) \le \|g\|_\infty \|y\|$. Conversely, by applying what was proved in (N1), we have $\|g\|_\infty \|y\| = \|g\cdot y\|_\infty \le d_p^{\mathcal{H}^\infty}(g\cdot y)$.

(N3) Let H, G be complex Banach spaces, V be an open subset of H, $h \in \mathcal{H}(V, U)$, $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ and $S \in \mathcal{L}(F, G)$. We can suppose $S \neq 0$. Given $n \in \mathbb{N}$, $x_1, \ldots, x_n \in U$, $y_1^*, \ldots, y_n^* \in G^*$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, we have

$$\sum_{i=1}^n |\lambda_i| \left|\left\langle y_i^*, S(f(h(x_i)))\right\rangle\right|$$

$$= \sum_{i=1}^n |\lambda_i| \left|\left\langle y_i^* \circ S, f(h(x_i))\right\rangle\right|$$

$$\le d_p^{\mathcal{H}^\infty}(f) \left(\sum_{i=1}^n |\lambda_i|^p\right)^{\frac{1}{p}} \sup_{y^{**}\in B_{F^{**}}} \left(\sum_{i=1}^n \left|y^{**}(y_i^* \circ S)\right|^{p^*}\right)^{\frac{1}{p^*}}$$

$$= \|S\|\, d_p^{\mathcal{H}^\infty}(f) \left(\sum_{i=1}^n |\lambda_i|^p\right)^{\frac{1}{p}} \sup_{y^{**}\in B_{F^{**}}} \left(\sum_{i=1}^n \left|\left(y^{**} \circ \frac{S^*}{\|S\|}\right)(y_i^*)\right|^{p^*}\right)^{\frac{1}{p^*}}$$

$$\le \|S\|\, d_p^{\mathcal{H}^\infty}(f) \left(\sum_{i=1}^n |\lambda_i|^p\right)^{\frac{1}{p}} \sup_{z^{**}\in B_{G^{**}}} \left(\sum_{i=1}^n \left|z^{**}(y_i^*)\right|^{p^*}\right)^{\frac{1}{p^*}}$$

and therefore $S \circ f \circ h \in \mathcal{D}_p^{\mathcal{H}^\infty}(V, G)$ with $d_p^{\mathcal{H}^\infty}(S \circ f \circ h) \le \|S\|\, d_p^{\mathcal{H}^\infty}(f)$.

We now prove that the ideal $\mathcal{D}_p^{\mathcal{H}^\infty}$ is regular. Let $f \in \mathcal{H}^\infty(U, F)$ and assume that $\kappa_F \circ f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F^{**})$. Given $n \in \mathbb{N}$, $x_1, \ldots, x_n \in U$, $y_1^*, \ldots, y_n^* \in F^*$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, we have

$$\sum_{i=1}^n |\lambda_i| \left|\left\langle y_i^*, f(x_i)\right\rangle\right|$$

$$= \sum_{i=1}^n |\lambda_i| \left|\left\langle \kappa_F(f(x_i)), y_i^*\right\rangle\right|$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} |\lambda_i| \left|\left\langle \kappa_{F^*}(y_i^*), \kappa_F(f(x_i))\right\rangle\right| \\
&\leq d_p^{\mathcal{H}^\infty}(\kappa_F \circ f) \left(\sum_{i=1}^{n} |\lambda_i|^p\right)^{\frac{1}{p}} \sup_{y^{****} \in B_{F^{****}}} \left(\sum_{i=1}^{n} \left|y^{****}(\kappa_{F^*}(y_i^*))\right|^{p^*}\right)^{\frac{1}{p^*}} \\
&\leq d_p^{\mathcal{H}^\infty}(\kappa_F \circ f) \left(\sum_{i=1}^{n} |\lambda_i|^p\right)^{\frac{1}{p}} \sup_{y^{**} \in B_{F^{**}}} \left(\sum_{i=1}^{n} \left|y^{**}(y_i^*)\right|^{p^*}\right)^{\frac{1}{p^*}},
\end{aligned}
$$

and thus $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ with $d_p^{\mathcal{H}^\infty}(f) \leq d_p^{\mathcal{H}^\infty}(\kappa_F \circ f)$. The reverse inequality follows from (N3).

Finally, we have seen in (N1) that $\mathcal{D}_1^{\mathcal{H}^\infty}(U, F) \subseteq \mathcal{H}^\infty(U, F)$ with $\|f\|_\infty \leq d_1^{\mathcal{H}^\infty}(f)$ for all $f \in \mathcal{D}_1^{\mathcal{H}^\infty}(U, F)$. For the converse, let $f \in \mathcal{H}^\infty(U, F)$. If $f = 0$, there is nothing to prove. Assume $f \neq 0$. Given $n \in \mathbb{N}$, $x_1, \ldots, x_n \in U$, $y_1^*, \ldots, y_n^* \in F^*$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, we have

$$
\begin{aligned}
\sum_{i=1}^{n} |\lambda_i| \left|\left\langle y_i^*, f(x_i)\right\rangle\right| &= \|f\|_\infty \sum_{i=1}^{n} |\lambda_i| \left|\left\langle \kappa_F\left(\frac{f(x_i)}{\|f\|_\infty}\right), y_i^*\right\rangle\right| \\
&\leq \|f\|_\infty \left(\sum_{i=1}^{n} |\lambda_i|\right) \max_{1\leq i\leq n} \left(\sup_{y^{**} \in B_{F^{**}}} \left|y^{**}(y_i^*)\right|\right) \\
&= \|f\|_\infty \left(\sum_{i=1}^{n} |\lambda_i|\right) \sup_{y^{**} \in B_{F^{**}}} \left(\max_{1\leq i\leq n} \left|y^{**}(y_i^*)\right|\right),
\end{aligned}
$$

and therefore $f \in \mathcal{D}_1^{\mathcal{H}^\infty}(U, F)$ with $d_1^{\mathcal{H}^\infty}(f) \leq \|f\|_\infty$. □

3 The tensor product $\Delta(U) \otimes F$

We introduce $\Delta(U) \otimes F$ as a space of linear functionals on $\mathcal{H}^\infty(U, F^*)$.

Definition 3.1 Let E and F be complex Banach spaces and let U be an open subset of E. For each $x \in U$, let $\delta(x) \colon \mathcal{H}^\infty(U) \to \mathbb{C}$ be the linear functional defined by

$$\langle \delta(x), f \rangle = f(x) \quad \left(f \in \mathcal{H}^\infty(U)\right).$$

Let $\Delta(U)$ be the linear subspace of $\mathcal{H}^\infty(U)'$ spanned by the set $\{\delta(x) \colon x \in U\}$.

For any $x \in U$ and $y \in F$, let $\delta(x) \otimes y \colon \mathcal{H}^\infty(U, F^*) \to \mathbb{C}$ be the linear functional given by

$$(\delta(x) \otimes y)(f) = \langle f(x), y \rangle \qquad \left(f \in \mathcal{H}^\infty(U, F^*)\right).$$

We define the tensor product $\Delta(U) \otimes F$ as the linear subspace of $\mathcal{H}^\infty(U, F^*)'$ spanned by the set

$$\{\delta(x) \otimes y \colon x \in U,\ y \in F\}\,.$$

We say that $\delta(x) \otimes y$ is an *elementary tensor* of $\Delta(U) \otimes F$. Note that each element u in $\Delta(U) \otimes F$ is of the form $u = \sum_{i=1}^n \lambda_i (\delta(x_i) \otimes y_i)$, where $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $x_i \in U$ and $y_i \in F$ for $i = 1, \ldots, n$. This representation of u is not unique. It is worth noting that each element u of $\Delta(U) \otimes F$ can be represented as $u = \sum_{i=1}^n \delta(x_i) \otimes y_i$ since $\lambda(\delta(x) \otimes y) = \delta(x) \otimes (\lambda y)$.

As a straightforward consequence from Definition 3.1, we describe the action of a tensor u in $\Delta(U) \otimes F$ on a function f in $\mathcal{H}^\infty(U, F^*)$:

Lemma 3.2 *Let $u = \sum_{i=1}^n \lambda_i \delta(x_i) \otimes y_i \in \Delta(U) \otimes F$ and $f \in \mathcal{H}^\infty(U, F^*)$. Then*

$$u(f) = \sum_{i=1}^n \lambda_i \left\langle f(x_i), y_i \right\rangle .$$

□

The following characterization of the zero tensor of $\Delta(U) \otimes F$ follows immediately from [26, Proposition 1.2].

Proposition 3.3 *If $u = \sum_{i=1}^n \delta(x_i) \otimes y_i \in \Delta(U) \otimes F$, the following are equivalent:*

(i) $u = 0$.
(ii) $\sum_{i=1}^n g(x_i)\phi(y_i) = 0$ *for every $g \in B_{\mathcal{H}^\infty(U)}$ and $\phi \in B_{F^*}$.* □

By Definition 3.1, $\Delta(U) \otimes F$ is a linear subspace of $\mathcal{H}^\infty(U, F^*)'$. In fact, we have:

Proposition 3.4 $\langle \Delta(U) \otimes F, \mathcal{H}^\infty(U, F^*) \rangle$ *forms a dual pair, where the bilinear form $\langle \cdot, \cdot \rangle$ associated to the dual pair is given by*

$$\langle u, f \rangle = \sum_{i=1}^n \lambda_i \left\langle f(x_i), y_i \right\rangle$$

for $u = \sum_{i=1}^n \lambda_i \delta(x_i) \otimes y_i \in \Delta(U) \otimes F$ and $f \in \mathcal{H}^\infty(U, F^*)$.

Proof Since $\langle u, f \rangle = u(f)$ by Lemma 3.2, it is immediate that $\langle \cdot, \cdot \rangle$ is a well-defined bilinear map from $(\Delta(U) \otimes F) \times \mathcal{H}^\infty(U, F^*)$ to $\mathbb{C}$. On the one hand, if $u \in \Delta(U) \otimes F$ and $\langle u, f \rangle = 0$ for all $f \in \mathcal{H}^\infty(U, F^*)$, then $u = 0$ follows easily from Proposition 3.3, and thus $\mathcal{H}^\infty(U, F^*)$ separates points of $\Delta(U) \otimes F$. On the other hand, if $f \in \mathcal{H}^\infty(U, F^*)$ and $\langle u, f \rangle = 0$ for all $u \in \Delta(U) \otimes F$, then $\langle f(x), y \rangle = \langle \delta(x) \otimes y, f \rangle = 0$ for all $x \in U$ and $y \in F$, this means that $f = 0$ and thus $\Delta(U) \otimes F$ separates points of $\mathcal{H}^\infty(U, F^*)$. □

Since $\langle \Delta(U) \otimes F, \mathcal{H}^\infty(U, F^*) \rangle$ is a dual pair, we can identify $\mathcal{H}^\infty(U, F^*)$ with a linear subspace of $(\Delta(U) \otimes F)'$ as follows.

Corollary 3.5 *For each $f \in \mathcal{H}^\infty(U, F^*)$, the functional $\Lambda_0(f)\colon \Delta(U) \otimes F \to \mathbb{C}$, given by*

$$\Lambda_0(f)(u) = \sum_{i=1}^{n} \lambda_i \langle f(x_i), y_i \rangle$$

for $u = \sum_{i=1}^{n} \lambda_i \delta(x_i) \otimes y_i \in \Delta(U) \otimes F$, is linear. We will say that $\Lambda_0(f)$ is the linear functional on $\Delta(U) \otimes F$ associated to f. Furthermore, the map $f \mapsto \Lambda_0(f)$ is a linear monomorphism from $\mathcal{H}^\infty(U, F^*)$ into $(\Delta(U) \otimes F)'$.

Proof Let $f \in \mathcal{H}^\infty(U, F^*)$. Note that $\Lambda_0(f)(u) = \langle u, f \rangle$ for all $u \in \Delta(U) \otimes F$. It is immediate that $\Lambda_0(f)$ is a well-defined linear functional on $\Delta(U) \otimes F$ and that $f \mapsto \Lambda_0(f)$ from $\mathcal{H}^\infty(U, F^*)$ into $(\Delta(U) \otimes F)'$ is a well-defined linear map. Finally, let $f \in \mathcal{H}^\infty(U, F^*)$ and assume that $\Lambda_0(f) = 0$. Then $\langle u, f \rangle = 0$ for all $u \in \Delta(U) \otimes F$. Since $\Delta(U) \otimes F$ separates points of $\mathcal{H}^\infty(U, F^*)$ by Proposition 3.4, it follows that $f = 0$ and this proves that Λ_0 is one-to-one. □

Given two linear spaces E and F, the tensor product space $E \otimes F$ equipped with a norm α will be denoted by $E \otimes_\alpha F$, and the completion of $E \otimes_\alpha F$ by $E \widehat{\otimes}_\alpha F$. If E and F are normed spaces, a *cross-norm* on $E \otimes F$ is a norm α such that $\alpha(x \otimes y) = \|x\| \, \|y\|$ for all $x \in E$ and $y \in F$.

Given two normed spaces E and F, the projective norm π on $E \otimes F$ (see [26, Chapter 2]) takes the following form on $\Delta(U) \otimes F$:

$$\pi(u) = \inf \left\{ \sum_{i=1}^{n} |\lambda_i| \, \|y_i\| : u = \sum_{i=1}^{n} \lambda_i \delta(x_i) \otimes y_i \right\} \quad (u \in \Delta(U) \otimes F),$$

where the infimum is taken over all the representations of u as above.

We next see that, on the space $\Delta(U) \otimes F$, the projective norm and the norm induced by the dual norm of the supremum norm of $\mathcal{H}^\infty(U, F^*)$ coincide.

Theorem 3.6 *The linear space $\Delta(U) \otimes F$ is contained in $\mathcal{H}^\infty(U, F^*)^*$. Moreover, $\pi(u) = H(u)$ for all $u \in \Delta(U) \otimes F$, where H is the norm on $\Delta(U) \otimes F$ defined by*

$$H(u) = \sup \left\{ |u(f)| : f \in \mathcal{H}^\infty(U, F^*), \ \|f\|_\infty \leq 1 \right\} \quad (u \in \Delta(U) \otimes F).$$

Proof Let $\lambda \in \mathbb{C}$, $x \in U$ and $y \in F$. Since $\lambda\delta(x) \otimes y$ is a linear map on $\mathcal{H}^\infty(U, F^*)$ and

$$|(\lambda\delta(x) \otimes y)(f)| = |\lambda \langle f(x), y \rangle| \leq |\lambda| \, \|f(x)\| \, \|y\| \leq |\lambda| \, \|f\|_\infty \, \|y\|$$

for all $f \in \mathcal{H}^\infty(U, F^*)$, then $\lambda\delta(x) \otimes y \in \mathcal{H}^\infty(U, F^*)^*$ with $\|\lambda\delta(x) \otimes y\| \leq |\lambda| \, \|y\|$, and thus $\Delta(U) \otimes F \subseteq \mathcal{H}^\infty(U, F^*)^*$.

Let $u \in \Delta(U) \otimes F$ and let $\sum_{i=1}^{n} \lambda_i \delta(x_i) \otimes y_i$ be a representation of u. Since u is linear and

$$|u(f)| = \left| \sum_{i=1}^{n} \lambda_i \langle f(x_i), y_i \rangle \right| \leq \sum_{i=1}^{n} |\lambda_i| \, \|f(x_i)\| \, \|y_i\| \leq \|f\|_\infty \sum_{i=1}^{n} |\lambda_i| \, \|y_i\|$$

for all $f \in \mathcal{H}^\infty(U, F^*)$, we deduce that $H(u) \leq \sum_{i=1}^{n} |\lambda_i| \, \|y_i\|$. Since this holds for each representation of u, it follows that $H(u) \leq \pi(u)$. Hence, $H \leq \pi$. To prove that the reverse inequality, suppose by contradiction that $H(u_0) < 1 < \pi(u_0)$ for some $u_0 \in \Delta(U) \otimes F$. Denote $B = \{u \in \Delta(U) \otimes F : \pi(u) \leq 1\}$. Clearly, B is a closed and convex set in $\Delta(U) \otimes_\pi F$. Applying the Hahn–Banach Separation Theorem to B and $\{u_0\}$, we obtain a functional $\eta \in (\Delta(U) \otimes_\pi F)^*$ such that

$$1 = \|\eta\| = \sup\{\mathrm{Re}\, \eta(u) : u \in B\} < \mathrm{Re}\, \eta(u_0).$$

Define $f : U \to F^*$ by $\langle f(x), y \rangle = \eta\,(\delta(x) \otimes y)$ for all $y \in F$ and $x \in U$. It is easy to prove that f is well defined and $f \in \mathcal{H}^\infty(U, F^*)$ with $\|f\|_\infty \leq 1$. Moreover, $u(f) = \eta(u)$ for all $u \in \Delta(U) \otimes F$. Therefore $H(u_0) \geq |u_0(f)| \geq \mathrm{Re}\, u_0(f) = \mathrm{Re}\, \eta(u_0)$, so $H(u_0) > 1$ and this is a contradiction. □

We now will define the Chevet–Saphar norms on the tensor product $E \otimes F$. Let E and F be normed spaces and let $1 \leq p \leq \infty$. Given $u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F$, denote

$$\|(x_1, \ldots, x_n)\|_{\ell_p^n(E)} = \begin{cases} \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \max_{1 \leq i \leq n} \|x_i\| & \text{if } p = \infty, \end{cases}$$

and

$$\|(y_1, \ldots, y_n)\|_{\ell_p^{n,w}(F)} = \begin{cases} \sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^{n} |y^*(y_i)|^p\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \sup_{y^* \in B_{F^*}} \left(\max_{1 \leq i \leq n} |y^*(y_i)|\right) & \text{if } p = \infty. \end{cases}$$

If $E = F = \mathbb{C}$, we write $\ell_p^n(E) = \ell_p^n$ and $\ell_{p^*}^{n,w}(F) = \ell_{p^*}^{n,w}$. According to [26, Section 6.2], the Chevet–Saphar norms are defined on $E \otimes F$ by

$$d_p(u) = \inf\left\{\|(x_1, \ldots, x_n)\|_{\ell_{p^*}^{n,w}(E)} \, \|(y_1, \ldots, y_n)\|_{\ell_p^n(F)}\right\},$$
$$g_p(u) = \inf\left\{\|(x_1, \ldots, x_n)\|_{\ell_p^n(E)} \, \|(y_1, \ldots, y_n)\|_{\ell_{p^*}^{n,w}(F)}\right\},$$

the infimum being taken over all representations of u as $u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F$.

Since $\|\delta(x)\| = 1$ for all $x \in U$, the norm g_p on $\Delta(U) \otimes F$ takes the form:

$$g_p(u) = \inf\left\{\|(\lambda_1, \ldots, \lambda_n)\|_{\ell_p^n} \, \|(y_1, \ldots, y_n)\|_{\ell_{p^*}^{n,w}(F)} : u = \sum_{i=1}^{n} \lambda_i \delta(x_i) \otimes y_i\right\}.$$

Notice that g_p is a cross-norm on $\Delta(U) \otimes F$.

We next show that g_1 on $\Delta(U) \otimes F$ is just the projective tensor norm π.

Proposition 3.7 $g_1(u) = \pi(u)$ *for all* $u \in \Delta(U) \otimes F$.

Proof Let $u \in \Delta(U) \otimes F$ and let $\sum_{i=1}^{n} \lambda_i \delta(x_i) \otimes y_i$ be a representation of u. We have

$$\pi(u) \leq \sum_{i=1}^{n} |\lambda_i| \, \|y_i\| = \sum_{i=1}^{n} |\lambda_i| \left(\sup_{y^* \in B_{F^*}} \left| y^*(y_i) \right| \right)$$
$$\leq \sum_{i=1}^{n} |\lambda_i| \max_{1 \leq i \leq n} \left(\sup_{y^* \in B_{F^*}} \left| y^*(y_i) \right| \right) = \|(\lambda_1, \ldots, \lambda_n)\|_{\ell_1^n} \|(y_1, \ldots, y_n)\|_{\ell_\infty^{n,w}(F)},$$

and taking the infimum over all representations of u gives $\pi(u) \leq g_1(u)$. For the reverse inequality, notice that $g_1(\lambda\delta(x) \otimes y) \leq |\lambda| \, \|y\|$ for all $\lambda \in \mathbb{C}$, $x \in U$ and $y \in F$. Since g_1 is a norm on $\Delta(U) \otimes F$, it follows that

$$g_1(u) = g_1 \left(\sum_{i=1}^{n} \lambda_i \delta(x_i) \otimes y_i \right) \leq \sum_{i=1}^{n} g_1 \left(\lambda_i \delta(x_i) \otimes y_i \right) \leq \sum_{i=1}^{n} |\lambda_i| \, \|y_i\|$$

and taking the infimum over all representations of u yields $g_1(u) \leq \pi(u)$. □

4 Duality for Cohen strongly *p*-summing holomorphic mappings

We now show that the duals of the tensor product $\mathcal{G}^\infty(U) \widehat{\otimes}_{g_p} F$ can be canonically identified as spaces of Cohen strongly p-summing holomorphic mappings.

Theorem 4.1 *Let* $1 \leq p \leq \infty$. *Then* $\mathcal{D}_p^{\mathcal{H}^\infty}(U, F^*)$ *is isometrically isomorphic to* $(\mathcal{G}^\infty(U) \widehat{\otimes}_{g_p} F)^*$, *via the mapping* $\Lambda \colon \mathcal{D}_p^{\mathcal{H}^\infty}(U, F^*) \to (\mathcal{G}^\infty(U) \widehat{\otimes}_{g_p} F)^*$ *defined by*

$$\Lambda(f)(u) = \sum_{i=1}^{n} \lambda_i \left\langle f(x_i), y_i \right\rangle$$

for $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F^*)$ *and* $u = \sum_{i=1}^{n} \lambda_i \delta(x_i) \otimes y_i \in \Delta(U) \otimes F$. *Furthermore, its inverse is given by*

$$\left\langle \Lambda^{-1}(\varphi)(x), y \right\rangle = \left\langle \varphi, \delta(x) \otimes y \right\rangle$$

for $\varphi \in (\mathcal{G}^\infty(U) \widehat{\otimes}_{g_p} F)^*$, $x \in U$ *and* $y \in F$.

Proof We prove it for $1 < p \leq \infty$. The case $p = 1$ is similarly proved.

Let $f \in \mathcal{D}_p^{\mathcal{H}_\infty}(U, F^*)$ and let $\Lambda_0(f)\colon \Delta(U) \otimes F \to \mathbb{C}$ be its associate linear functional. We claim that $\Lambda_0(f) \in (\Delta(U) \otimes_{g_p} F)^*$ with $\|\Lambda_0(f)\| \leq d_p^{\mathcal{H}^\infty}(f)$. Indeed, given $u = \sum_{i=1}^n \lambda_i \delta(x_i) \otimes y_i \in \Delta(U) \otimes F$, we have

$$\begin{aligned}
|\Lambda_0(f)(u)| &= \left|\sum_{i=1}^n \lambda_i \langle f(x_i), y_i\rangle\right| \leq \sum_{i=1}^n |\lambda_i| \, |\langle \kappa_F(y_i), f(x_i)\rangle| \\
&\leq d_p^{\mathcal{H}^\infty}(f) \, \|(\lambda_1, \ldots, \lambda_n)\|_{\ell_p^n} \sup_{y^{***} \in B_{F^{***}}} \left(\sum_{i=1}^n \left|y^{***}(\kappa_F(y_i))\right|^{p^*}\right)^{\frac{1}{p^*}} \\
&\leq d_p^{\mathcal{H}^\infty}(f) \, \|(\lambda_1, \ldots, \lambda_n)\|_{\ell_p^n} \sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^n \left|y^*(y_i)\right|^{p^*}\right)^{\frac{1}{p^*}} \\
&= d_p^{\mathcal{H}^\infty}(f) \, \|(\lambda_1, \ldots, \lambda_n)\|_{\ell_p^n} \, \|(y_1, \ldots, y_n)\|_{\ell_{p^*}^{n,w}(F)},
\end{aligned}$$

and taking infimum over all the representations of u, we deduce that $|\Lambda_0(f)(u)| \leq d_p^{\mathcal{H}^\infty}(f) g_p(u)$. Since u was arbitrary, then $\Lambda_0(f)$ is continuous on $\Delta(U) \otimes_{g_p} F$ with $\|\Lambda_0(f)\| \leq d_p^{\mathcal{H}^\infty}(f)$, as claimed.

Since $\Delta(U)$ is a norm-dense linear subspace of $\mathcal{G}^\infty(U)$ and g_p is a cross-norm on $\mathcal{G}^\infty(U) \otimes F$, then $\Delta(U) \otimes F$ is a dense linear subspace of $\mathcal{G}^\infty(U) \otimes_{g_p} F$ and therefore also of its completion $\mathcal{G}^\infty(U)\widehat{\otimes}_{g_p} F$. Hence there is a unique continuous mapping $\Lambda(f)$ from $\mathcal{G}^\infty(U)\widehat{\otimes}_{g_p} F$ to $\mathbb{C}$ that extends $\Lambda_0(f)$. Further, $\Lambda(f)$ is linear and $\|\Lambda(f)\| = \|\Lambda_0(f)\|$.

Let $\Lambda\colon \mathcal{D}_p^{\mathcal{H}_\infty}(U, F^*) \to (\mathcal{G}^\infty(U)\widehat{\otimes}_{g_p} F)^*$ be the mapping so defined. Since the mapping $\Lambda_0\colon \mathcal{D}_p^{\mathcal{H}_\infty}(U, F^*) \to (\Delta(U) \otimes F)'$ is a linear monomorphism by Corollary 3.5, it follows easily that Λ is so. To prove that Λ is a surjective isometry, let $\varphi \in (\mathcal{G}^\infty(U)\widehat{\otimes}_{g_p} F)^*$ and define $f_\varphi\colon U \to F^*$ by

$$\left\langle f_\varphi(x), y\right\rangle = \varphi(\delta(x) \otimes y) \quad (x \in U,\ y \in F).$$

Given $x \in U$, the linearity of both φ and the product tensor in the second variable yields that the functional $f_\varphi(x)\colon F \to \mathbb{C}$ is linear, and since

$$\left|\left\langle f_\varphi(x), y\right\rangle\right| = |\varphi(\delta(x) \otimes y)| \leq \|\varphi\| \, g_p(\delta(x) \otimes y) \leq \|\varphi\| \, \|y\|$$

for all $y \in F$, we deduce that $f_\varphi(x) \in F^*$ with $\|f_\varphi(x)\| \leq \|\varphi\|$. Since x was arbitrary, we have that f_φ is bounded with $\left\|f_\varphi\right\|_\infty \leq \|\varphi\|$.

We now prove that $f_\varphi\colon U \to F^*$ is holomorphic. To this end, we first claim that, for every $y \in F$, the function $f_y\colon U \to \mathbb{C}$ defined by

$$f_y(x) = \varphi(\delta(x) \otimes y) \quad (x \in U)$$

is holomorphic. Let $a \in U$. Since $g_U : U \to \mathcal{G}^\infty(U)$ is holomorphic by Theorem 1.2, there exists $Dg_U(a) \in \mathcal{L}(E, \mathcal{G}^\infty(U))$ such that

$$\lim_{x \to a} \frac{\delta(x) - \delta(a) - Dg_U(a)(x-a)}{\|x-a\|} = 0.$$

Consider the function $T(a) \colon E \to \mathbb{C}$ given by

$$T(a)(x) = \varphi(Dg_U(a)(x) \otimes y) \quad (x \in E).$$

Clearly, $T(a) \in E^*$ and since

$$\begin{aligned} & f_y(x) - f_y(a) - T(a)(x-a) \\ & \quad = \varphi(\delta(x) \otimes y) - \varphi(\delta(a) \otimes y) - \varphi(Dg_U(a)(x-a) \otimes y) \\ & \quad = \varphi\left((\delta(x) - \delta(a) - Dg_U(a)(x-a)) \otimes y\right), \end{aligned}$$

it follows that

$$\begin{aligned} \lim_{x \to a} \frac{f_y(x) - f_y(a) - T(a)(x-a)}{\|x-a\|} &= \lim_{x \to a} \frac{\varphi\left((\delta(x) - \delta(a) - Dg_U(a)(x-a)) \otimes y\right)}{\|x-a\|} \\ &= \lim_{x \to a} \varphi\left(\frac{\delta(x) - \delta(a) - Dg_U(a)(x-a)}{\|x-a\|} \otimes y\right) \\ &= \varphi(0 \otimes y) = \varphi(0) = 0. \end{aligned}$$

Hence, f_y is holomorphic at a with $Df_y(a) = T(a)$, and this proves our claim. Now, notice that the set $\{\kappa_F(y) \colon y \in B_F\} \subseteq B_{F^{**}}$ is norming for F^* since

$$\left\|y^*\right\| = \sup\left\{\left|y^*(y)\right| : y \in B_F\right\} = \sup\left\{\left|\kappa_F(y)(y^*)\right| : y \in B_F\right\}$$

for every $y^* \in F^*$, and that $\kappa_F(y) \circ f_\varphi = f_y$ for every $y \in F$ since

$$(\kappa_F(y) \circ f_\varphi)(x) = \kappa_F(y)(f_\varphi(x)) = \left\langle f_\varphi(x), y\right\rangle = \varphi(\delta(x) \otimes y) = f_y(x)$$

for all $x \in U$.

We are now ready to show that $f_\varphi \colon U \to F^*$ is holomorphic. Indeed, let $a \in U$ and $b \in E$. Denote $V = \{\lambda \in \mathbb{C} \colon a + \lambda b \in U\}$. Clearly, the mapping $h \colon V \to U$ given by $h(\lambda) = a + \lambda b$ is holomorphic. Since $f_\varphi \circ h$ is locally bounded and $\kappa_F(y) \circ (f_\varphi \circ h) = f_y \circ h$ is holomorphic on the open set $V \subseteq \mathbb{C}$ for all $y \in F$, Proposition A.3 in [3] assures that $f_\varphi \circ h$ is holomorphic. This means that f_φ is G-holomorphic but since it is also locally bounded, we deduce that f_φ is continuous by [15, Proposition 8.6]. Now, we conclude that f_φ is holomorphic by Theorem 1.1.

We now prove that $f_\varphi \in \mathcal{D}_p^{H^\infty}(U, F^*)$. To see this, take $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $x_i \in U$ and $y_i^{**} \in F^{**}$ for $i = 1, \ldots, n$. Let $\varepsilon > 0$ and consider the finite-dimensional subspaces $V = \mathrm{lin}\{y_1^{**}, \ldots, y_n^{**}\} \subseteq F^{**}$ and $W = \mathrm{lin}\{f_\varphi(x_1), \ldots, f_\varphi(x_n)\} \subseteq F^*$. The principle of local reflexivity [6, Theorem 8.16] gives us a bounded linear operator $T_{(\varepsilon, V, W)} \colon V \to F$ such that

(i) $T_{(\varepsilon,V,W)}(y^{**}) = y^{**}$ for every $y^{**} \in V \cap \kappa_F(F)$,
(ii) $(1-\varepsilon)\,\|y^{**}\| \leq \left\|T_{(\varepsilon,V,W)}(y^{**})\right\| \leq (1+\varepsilon)\,\|y^{**}\|$ for every $y^{**} \in V$,
(iii) $\left\langle y^*, T_{(\varepsilon,V,W)}(y^{**})\right\rangle = \langle y^{**}, y^*\rangle$ for every $y^{**} \in V$ and $y^* \in W$.

Using (iii) and taking $y_i = T_{(\varepsilon,V,W)}(y_i^{**})$, we first have

$$\begin{aligned}
\left|\sum_{i=1}^n \lambda_i \left\langle y_i^{**}, f_\varphi(x_i)\right\rangle\right| &= \left|\sum_{i=1}^n \lambda_i \left\langle f_\varphi(x_i), T_{(\varepsilon,V,W)}(y_i^{**})\right\rangle\right| \\
&= \left|\sum_{i=1}^n \lambda_i \left\langle f_\varphi(x_i), y_i\right\rangle\right| \\
&= \left|\varphi\left(\sum_{i=1}^n \lambda_i \delta(x_i) \otimes y_i\right)\right| \\
&\leq \|\varphi\|\, g_p\left(\sum_{i=1}^n \lambda_i \delta(x_i) \otimes y_i\right) \\
&\leq \|\varphi\|\, \|(\lambda_1, \ldots, \lambda_n)\|_{\ell_p^n}\, \|(y_1, \ldots, y_n)\|_{\ell_{p^*}^{n,w}(F)}.
\end{aligned}$$

Since

$$\begin{aligned}
\|(y_1, \ldots, y_n)\|_{\ell_{p^*}^{n,w}(F)} &= \sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^n \left|y^*(y_i)\right|^{p^*}\right)^{\frac{1}{p^*}} \\
&= \sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^n \left|\left\langle y^*, T_{(\varepsilon,V,W)}(y_i^{**})\right\rangle\right|^{p^*}\right)^{\frac{1}{p^*}} \\
&= \sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^n \left|\left\langle \kappa_F(T_{(\varepsilon,V,W)}(y_i^{**})), y^*\right\rangle\right|^{p^*}\right)^{\frac{1}{p^*}} \\
&\leq \left\|\kappa_F \circ T_{(\varepsilon,V,W)}\right\| \sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^n \left|\left\langle y_i^{**}, y^*\right\rangle\right|^{p^*}\right)^{\frac{1}{p^*}} \\
&= \left\|T_{(\varepsilon,V,W)}\right\| \sup_{y^* \in B_{F^*}} \left(\sum_{i=1}^n \left|\left\langle \kappa_{F^*}(y^*), y_i^{**}\right\rangle\right|^{p^*}\right)^{\frac{1}{p^*}} \\
&\leq (1+\varepsilon) \sup_{y^{***} \in B_{F^{***}}} \left(\sum_{i=1}^n \left|y^{***}(y_i^{**})\right|^{p^*}\right)^{\frac{1}{p^*}} \\
&= (1+\varepsilon)\left\|(y_1^{**}, \ldots, y_n^{**})\right\|_{\ell_{p^*}^{n,w}(F^{**})},
\end{aligned}$$

it follows that

$$\left|\sum_{i=1}^{n} \lambda_i \left\langle y_i^{**}, f_\varphi(x_i)\right\rangle\right| \leq \|\varphi\| \left\|(\lambda_1, \ldots, \lambda_n)\right\|_{\ell_p^n} (1+\varepsilon) \left\|(y_1^{**}, \ldots, y_n^{**})\right\|_{\ell_{p^*}^{n,w}(F^{**})}.$$

By the arbitrariness of ε, we deduce that

$$\left|\sum_{i=1}^{n} \lambda_i \left\langle y_i^{**}, f_\varphi(x_i)\right\rangle\right| \leq \|\varphi\| \left\|(\lambda_1, \ldots, \lambda_n)\right\|_{\ell_p^n} \left\|(y_1^{**}, \ldots, y_n^{**})\right\|_{\ell_{p^*}^{n,w}(F^{**})},$$

and this implies that $f_\varphi \in \mathcal{D}_p^{H^\infty}(U, F^*)$ with $d_p^{\mathcal{H}^\infty}(f_\varphi) \leq \|\varphi\|$.

For any $u = \sum_{i=1}^{n} \lambda_i \delta(x_i) \otimes y_i \in \Delta(U) \otimes F$, we get

$$\Lambda(f_\varphi)(u) = \sum_{i=1}^{n} \lambda_i \left\langle f_\varphi(x_i), y_i\right\rangle = \sum_{i=1}^{n} \lambda_i \varphi(\delta(x_i) \otimes y_i) = \varphi\left(\sum_{i=1}^{n} \lambda_i \delta(x_i) \otimes y_i\right) = \varphi(u).$$

Hence $\Lambda(f_\varphi) = \varphi$ on a dense subspace of $\mathcal{G}^\infty(U)\widehat{\otimes}_{g_p} F$ and, consequently, $\Lambda(f_\varphi) = \varphi$, which shows the last statement of the theorem. Moreover, $d_p^{\mathcal{H}^\infty}(f_\varphi) \leq \|\varphi\| = \left\|\Lambda(f_\varphi)\right\|$ and the theorem holds. □

In particular, in view of Theorem 4.1 and taking into account Propositions 2.5, 3.6 and 3.7, we can identify the space $\mathcal{H}^\infty(U, F^*)$ with the dual space of $\mathcal{G}^\infty(U)\widehat{\otimes}_H F$.

Corollary 4.2 *$\mathcal{H}^\infty(U, F^*)$ is isometrically isomorphic to $(\mathcal{G}^\infty(U)\widehat{\otimes}_H F)^*$, via the mapping $\Lambda\colon \mathcal{H}^\infty(U, F^*) \to (\mathcal{G}^\infty(U)\widehat{\otimes}_H F)^*$ given by*

$$\Lambda(f)(u) = \sum_{i=1}^{n} \lambda_i \left\langle f(x_i), y_i\right\rangle$$

for $f \in \mathcal{H}^\infty(U, F^*)$ and $u = \sum_{i=1}^{n} \lambda_i \delta(x_i) \otimes y_i \in \Delta(U) \otimes F$. Furthermore, its inverse is given by

$$\left\langle \Lambda^{-1}(\varphi)(x), y\right\rangle = \left\langle \varphi, \delta(x) \otimes y\right\rangle$$

for $\varphi \in (\mathcal{G}^\infty(U)\widehat{\otimes}_H F)^*$, $x \in U$ and $y \in F$. □

Remark 4.3 It is known (see [26, p. 24]) that if E and F are Banach spaces, then $\mathcal{L}(E, F^*)$ is isometrically isomorphic to $(E\widehat{\otimes}_\pi F)^*$, via $\Phi\colon \mathcal{L}(E, F^*) \to (E\widehat{\otimes}_\pi F)^*$ given by

$$\left\langle \Phi(T), \sum_{i=1}^{n} x_i \otimes y_i \right\rangle = \sum_{i=1}^{n} \left\langle T(x_i), y_i\right\rangle$$

for $T \in \mathcal{L}(E, F^*)$ and $\sum_{i=1}^{n} x_i \otimes y_i \in E \otimes F$. Notice that the identification Λ in Corollary 4.2 is just $\Phi \circ \Phi_0$, where $\Phi_0 \colon f \mapsto T_f$ is the isometric isomorphism from $\mathcal{H}^\infty(U, F^*)$ onto $\mathcal{L}(\mathcal{G}^\infty(U), F)$ given in Theorem 1.2.

5 Pietsch domination for Cohen strongly *p*-summing holomorphic mappings

In [22], Pietsch established a domination theorem for p-summing linear operators between Banach spaces. To present a version of this theorem for Cohen strongly p-summing holomorphic mappings on Banach spaces, we first characterize the elements of the dual space of $\Delta(U) \otimes_{g_p} F$.

Theorem 5.1 *Let* $\varphi \in (\Delta(U) \otimes F)'$, $C > 0$ *and* $1 < p \leq \infty$. *The following conditions are equivalent*:

(i) $|\varphi(u)| \leq C g_p(u)$ *for all* $u \in \Delta(U) \otimes F$.

(ii) *For any representation* $\sum_{i=1}^{n} \lambda_i \delta(x_i) \otimes y_i$ *of* $u \in \Delta(U) \otimes F$, *we have*

$$\sum_{i=1}^{n} |\varphi(\lambda_i \delta(x_i) \otimes y_i)| \leq C g_p(u).$$

(iii) *There exists a Borel regular probability measure* μ *on* B_{F^*} *such that*

$$|\varphi(\lambda \delta(x) \otimes y)| \leq C\, |\lambda|\, \|y\|_{L_{p^*}(\mu)}$$

for all $\lambda \in \mathbb{C}$, $x \in U$ *and* $y \in F$, *where*

$$\|y\|_{L_{p^*}(\mu)} = \left(\int_{B_{F^*}} \left| y^*(y) \right|^{p^*} \, \mathrm{d}\mu(y^*) \right)^{\frac{1}{p^*}}.$$

Proof (i) $\Rightarrow$ (ii): Let $u \in \Delta(U) \otimes F$ and let $\sum_{i=1}^{n} \lambda_i \delta(x_i) \otimes y_i$ be a representation of u. It is elementary that the function $T \colon \mathbb{C}^n \to \mathbb{C}$ defined by

$$T(t_1, \ldots, t_n) = \sum_{i=1}^{n} t_i \varphi(\lambda_i \delta(x_i) \otimes y_i), \quad \forall (t_1, \ldots, t_n) \in \mathbb{C}^n$$

is linear and continuous on $(\mathbb{C}^n, \|\cdot\|_{\ell_\infty^n})$ with $\|T\| = \sum_{i=1}^{n} |\varphi(\lambda_i \delta(x_i) \otimes y_i)|$.

For any $(t_1, \ldots, t_n) \in \mathbb{C}^n$ with $\|(t_1, \ldots, t_n)\|_{\ell_\infty^n} \leq 1$, by (i) we have

$$\begin{aligned} |T(t_1, \ldots, t_n)| &= \left| \sum_{i=1}^{n} t_i \varphi(\lambda_i \delta(x_i) \otimes y_i) \right| = \left| \varphi\left(\sum_{i=1}^{n} t_i \lambda_i \delta(x_i) \otimes y_i \right) \right| \\ &\leq C g_p \left(\sum_{i=1}^{n} t_i \lambda_i \delta(x_i) \otimes y_i \right) \end{aligned}$$

$$\leq C \left\|(t_1\lambda_1, \ldots, t_n\lambda_n)\right\|_{\ell_p^n} \left\|(y_1, \ldots, y_n)\right\|_{\ell_{p^*}^{n,w}(F)}$$
$$\leq C \left\|(\lambda_1, \ldots, \lambda_n)\right\|_{\ell_p^n} \left\|(y_1, \ldots, y_n)\right\|_{\ell_{p^*}^{n,w}(F)},$$

and, therefore,

$$\sum_{i=1}^{n} |\varphi(\lambda_i\delta(x_i) \otimes y_i)| \leq C \left\|(\lambda_1, \ldots, \lambda_n)\right\|_{\ell_p^n} \left\|(y_1, \ldots, y_n)\right\|_{\ell_{p^*}^{n,w}(F)}.$$

Taking infimum over all the representations of u, we deduce that

$$\sum_{i=1}^{n} |\varphi(\lambda_i\delta(x_i) \otimes y_i)| \leq C g_p(u).$$

(ii) $\Rightarrow$ (iii): Let $\mathcal{P}$ be the set of all Borel regular probability measures μ on B_{F^*}. Clearly, it is a convex compact subset of $(C(B_{F^*})^*, w^*)$. Assume first $1 < p < \infty$. Let M be set of all functions from $\mathcal{P}$ to $\mathbb{R}$ of the form

$$f_{((\lambda_i)_{i=1}^n,(x_i)_{i=1}^n,(y_i)_{i=1}^n)}(\mu) = \sum_{i=1}^{n} |\varphi(\lambda_i\delta_U(x_i) \otimes y_i)|$$
$$-\left(\frac{C}{p}\left\|(\lambda_i)_{i=1}^n\right\|_{\ell_p^n}^p + \frac{C}{p^*}\sum_{i=1}^{n}\|y_i\|_{L_{p^*}(\mu)}^{p^*}\right),$$

where $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $x_i \in U$ and $y_i \in F$ for $i = 1, \ldots, n$.

It is easy check that M satisfies the three conditions of Ky Fan's Lemma (see [6, 9.10]):

(a) Each $f_{((\lambda_i)_{i=1}^n,(x_i)_{i=1}^n,(y_i)_{i=1}^n)} \in M$ is convex and lower semicontinuous.
(b) If $g \in \mathrm{co}(M)$, there is $f_{((\lambda_i)_{i=1}^n,(x_i)_{i=1}^n,(y_i)_{i=1}^n)} \in M$ with $g(\mu) \leq f_{((\lambda_i)_{i=1}^n,(x_i)_{i=1}^n,(y_i)_{i=1}^n)}(\mu)$ for all $\mu \in \mathcal{P}$.
(c) Each $f_{((\lambda_i)_{i=1}^n,(x_i)_{i=1}^n,(y_i)_{i=1}^n)} \in M$ has a value less or equal than 0.

By Ky Fan's Lemma, there is a $\mu \in \mathcal{P}$ such that $f(\mu) \leq 0$ for all $f \in M$. In particular, we have

$$f_{(t\lambda,x,t^{-1}y)}(\mu) = \left|\varphi(t\lambda\delta_U(x) \otimes t^{-1}y)\right| - \frac{C}{p}t^p|\lambda|^p - \frac{C}{p^*}t^{-p^*}\|y\|_{L_{p^*}(\mu)}^{p^*} \leq 0$$

for all $t \in \mathbb{R}^+$, $\lambda \in \mathbb{C}$, $x \in U$ and $y \in F$. It follows that

$$|\varphi(\lambda\delta_U(x) \otimes y)| \leq C\left(\frac{t^p|\lambda|^p}{p} + \frac{t^{-p^*}\|y\|_{L_{p^*}(\mu)}^{p^*}}{p^*}\right),$$

and, applying again the aforementioned identity, we conclude that

$$|\varphi(\lambda\delta_U(x)\otimes y)| \le C\,|\lambda|\,\|y\|_{L_{p^*}(\mu)}\,.$$

The case $p=\infty$ is similarly proved but without applying the cited identity and taking $C/p=0$ and $p^*=1$.

(iii) $\Rightarrow$ (i): Let $u\in\Delta(U)\otimes F$ and let $\sum_{i=1}^n \lambda_i\delta(x_i)\otimes y_i$ be a representation of u. Using (iii) and the Hölder inequality, we obtain

$$\begin{aligned}
|\varphi(u)| &\le \sum_{i=1}^n |\varphi\left(\lambda_i\delta(x_i)\otimes y_i\right)| \le C\sum_{i=1}^n |\lambda_i|\,\|y_i\|_{L_{p^*}(\mu)}\\
&\le C\,\|(\lambda_1,\ldots,\lambda_n)\|_{\ell_p^n}\left(\sum_{i=1}^n \|y_i\|_{L_{p^*}(\mu)}^{p^*}\right)^{\frac{1}{p^*}}\\
&= C\,\|(\lambda_1,\ldots,\lambda_n)\|_{\ell_p^n}\left(\int_{B_{F^*}}\sum_{i=1}^n \left|y^*(y_i)\right|^{p^*}\,d\mu(y^*)\right)^{\frac{1}{p^*}}\\
&\le C\,\|(\lambda_1,\ldots,\lambda_n)\|_{\ell_p^n}\left(\sup_{y^*\in B_{F^*}}\sum_{i=1}^n \left|y^*(y_i)\right|^{p^*}\right)^{\frac{1}{p^*}}\\
&= C\,\|(\lambda_1,\ldots,\lambda_n)\|_{\ell_p^n}\,\|(y_1,\ldots,y_n)\|_{\ell_{p^*}^{n,w}(F)}\,,
\end{aligned}$$

and taking infimum over all the representations of u, we conclude that $|\varphi(u)|\le Cg_p(u)$. □

We are now ready to present the announced result. Compare to [5, Theorem 2.3.1].

Theorem 5.2 (Pietsch Domination) *Let* $1<p\le\infty$ *and* $f\in\mathcal{H}^\infty(U,F)$. *The following conditions are equivalent*:

(i) *f is Cohen strongly p-summing holomorphic.*
(ii) *For any* $\sum_{i=1}^n \lambda_i\delta(x_i)\otimes y_i^*\in\Delta(U)\otimes F^*$, *we have*

$$\left|\sum_{i=1}^n \lambda_i\left\langle y_i^*, f(x_i)\right\rangle\right| \le d_p^{\mathcal{H}^\infty}(f)\,\|(\lambda_1,\ldots,\lambda_n)\|_{\ell_p^n}\,\left\|(y_1^*,\ldots,y_n^*)\right\|_{\ell_{p^*}^{n,w}(F^*)}\,.$$

(iii) *There is a constant $C>0$ and a Borel regular probability measure μ on $B_{F^{**}}$ such that*

$$\left|\left\langle y^*, f(x)\right\rangle\right| \le C\left\|y^*\right\|_{L_{p^*}(\mu)}$$

for all $x\in U$ and $y^*\in F^*$, where

$$\left\|y^*\right\|_{L_{p^*}(\mu)} = \left(\int_{B_{F^{**}}}\left|y^{**}(y^*)\right|^{p^*}d\mu(y^{**})\right)^{\frac{1}{p^*}}.$$

In this case, $d_p^{\mathcal{H}^\infty}(f)$ is the minimum of all constants $C > 0$ satisfying the preceding inequality.

Proof (i) $\Rightarrow$ (ii) is immediate from Definition 2.2.

(ii) $\Rightarrow$ (iii): Clearly, $\kappa_F \circ f \in \mathcal{H}^\infty(U, F^{**})$. Appealing to Corollary 3.5, consider its associate linear functional $\Lambda_0(\kappa_F \circ f)\colon \Delta(U) \otimes F^* \to \mathbb{C}$. Given $u = \sum_{i=1}^n \lambda_i \delta(x_i) \otimes y_i^* \in \Delta(U) \otimes F^*$, we have

$$|\Lambda_0(\kappa_F \circ f)(u)| = \left|\sum_{i=1}^n \lambda_i \left\langle (\kappa_F \circ f)(x_i), y_i^* \right\rangle\right| = \left|\sum_{i=1}^n \lambda_i \left\langle y_i^*, f(x_i)\right\rangle\right|$$
$$\le d_p^{\mathcal{H}^\infty}(f) \, \|(\lambda_1, \ldots, \lambda_n)\|_{\ell_p^n} \left\|(y_1^*, \ldots, y_n^*)\right\|_{\ell_{p^*}^{n,w}(F^*)}$$

by (ii). Since it holds for each representation of u, we deduce that

$$|\Lambda_0(\kappa_F \circ f)(u)| \le d_p^{\mathcal{H}^\infty}(f) g_p(u).$$

By Theorem 5.1, there exists a Borel regular probability measure μ on $B_{F^{**}}$ such that

$$\left|\langle y^*, f(x)\rangle\right| = \left|\Lambda_0(\kappa_F \circ f)(\delta(x) \otimes y^*)\right| \le d_p^{\mathcal{H}^\infty}(f) \left(\int_{B_{F^{**}}} \left|y^{**}(y^*)\right|^{p^*} \, d\mu(y^{**})\right)^{\frac{1}{p^*}}$$

for all $x \in U$ and $y^* \in F^*$. Moreover, $d_p^{\mathcal{H}^\infty}(f)$ belongs to the set of all constants $C > 0$ satisfying the inequality in (iii).

(iii) $\Rightarrow$ (i): Given $x \in U$ and $y^* \in F^*$, we have

$$\left|\Lambda_0(\kappa_F \circ f)(\delta(x) \otimes y^*)\right| = \left|\langle y^*, f(x)\rangle\right| \le \left\|y^*\right\|_{L_{p^*}(\mu)}$$

by applying (iii). Now, Theorem 5.1 tells us that for any representation $\sum_{i=1}^n \lambda_i \delta(x_i) \otimes y_i^*$ of $u \in \Delta(U) \otimes F^*$, we have

$$\sum_{i=1}^n |\lambda_i| \left|\langle y_i^*, f(x_i)\rangle\right| = \sum_{i=1}^n |\lambda_i| \left|\langle (\kappa_F \circ f)(x_i), y_i^*\rangle\right| = \sum_{i=1}^n \left|\Lambda_0(\kappa_F \circ f)(\lambda_i \delta(x_i) \otimes y_i^*)\right|$$
$$\le C g_p(u) \le C \, \|(\lambda_1, \ldots, \lambda_n)\|_{\ell_p^n} \left\|(y_1^*, \ldots, y_n^*)\right\|_{\ell_{p^*}^{n,w}(F^*)}.$$

Hence $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ with $d_p^{\mathcal{H}^\infty}(f) \le C$. This also shows the last assertion of the statement. □

Remark 5.3 Theorem 5.2 is mainly a particular case of Theorem 4.6 in [21] since a Cohen strongly p-summing holomorphic mapping $(1 < p < \infty)$ is an $R_1, R_2 - S$-abstract (p, p^*)-summing mapping for $R_1\colon [0,1] \times U \times \mathbb{C} \to [0, \infty)$ defined by

$$R_1(t, x, \lambda) = |\lambda|,$$

$R_2\colon B_{F^{**}} \times U \times F^* \to [0, \infty)$ given by

$$R_2(y^{**}, x, y^*) = |y^{**}(y^*)|,$$

and $S\colon \mathcal{H}^\infty(U, F) \times U \times \mathbb{C} \times F^* \to [0, \infty)$ defined by

$$S(f, x, \lambda, y^*) = |\lambda||\langle y^*, f(x)\rangle|.$$

This unified abstract version of Pietsch Domination Theorem has been used by several authors whenever trying to get a domination result in a very short way. Our proof is also short and appeals directly to Ky Fan's Lemma as it was made to establish such an abstract version.

We now study the relationship between a Cohen strongly p-summing holomorphic mapping from U to F and its associate linearization from $\mathcal{G}^\infty(U)$ to F.

Theorem 5.4 *Let $1 < p \leq \infty$ and $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent*:

(i) *$f: U \to F$ is Cohen strongly p-summing holomorphic.*
(ii) *$T_f: \mathcal{G}^\infty(U) \to F$ is strongly p-summing.*

In this case, $d_p(T_f) = d_p^{\mathcal{H}^\infty}(f)$. Furthermore, the mapping $f \mapsto T_f$ is an isometric isomorphism from $(\mathcal{D}_p^{\mathcal{H}^\infty}(U, F), d_p^{\mathcal{H}^\infty})$ onto $(\mathcal{D}_p(\mathcal{G}^\infty(U), F), d_p)$.

Proof (i) $\Rightarrow$ (ii): Assume that $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$. By Theorem 5.2, there is a constant $C > 0$ and a Borel regular probability measure μ on $B_{F^{**}}$ such that $|\langle y^*, f(x)\rangle| \leq C \|y^*\|_{L_{p^*}(\mu)}$ for all $x \in U$ and $y^* \in F^*$.

Let $y^* \in F^*$ and $\gamma \in \mathcal{G}^\infty(U)$. By Theorem 1.2, given $\varepsilon > 0$, we can take a representation $\sum_{i=1}^\infty \lambda_i \delta(x_i)$ of γ such that $\sum_{i=1}^\infty |\lambda_i| \leq \|\gamma\| + \varepsilon$. We have

$$\begin{aligned} \left|\langle y^*, T_f(\gamma)\rangle\right| &= \left|\left\langle y^*, \sum_{i=1}^\infty \lambda_i T_f(\delta_U(x_i))\right\rangle\right| = \left|\left\langle y^*, \sum_{i=1}^\infty \lambda_i f(x_i)\right\rangle\right| \\ &\leq \sum_{i=1}^\infty |\lambda_i| \left|\langle y^*, f(x_i)\rangle\right| \\ &\leq C \left\|y^*\right\|_{L_{p^*}(\mu)} \sum_{i=1}^\infty |\lambda_i| \leq C \left\|y^*\right\|_{L_{p^*}(\mu)} (\|\gamma\| + \varepsilon). \end{aligned}$$

As ε was arbitrary, it follows that

$$\left|\langle y^*, T_f(\gamma)\rangle\right| \leq C \left\|y^*\right\|_{L_{p^*}(\mu)} \|\gamma\|.$$

Taking infimum over all such constants C, we have

$$\left|\langle y^*, T_f(\gamma)\rangle\right| \leq d_p^{\mathcal{H}^\infty}(f) \left\|y^*\right\|_{L_{p^*}(\mu)} \|\gamma\|$$

by Theorem 5.2. It follows that

$$\sup\left\{\left|\left\langle y^*, T_f(\gamma)\right\rangle\right| : y^* \in F^*,\ \left\|y^*\right\|_{L_{p^*}(\mu)} \leq 1\right\} \leq d_p^{\mathcal{H}^\infty}(f)\,\|\gamma\|$$

for all $\gamma \in \mathcal{G}^\infty(U)$. Therefore $T_f \in \mathcal{D}_p(\mathcal{G}^\infty(U), F)$ with $d_p(T_f) \leq d_p^{\mathcal{H}^\infty}(f)$ by Pietsch Domination Theorem for strongly p-summing operators [5, Theorem 2.3.1].

(ii) $\Rightarrow$ (i): Assume that $T_f \in \mathcal{D}_p(\mathcal{G}^\infty(U), F)$. Given $x \in U$ and $y^* \in F^*$, we have

$$\left|\left\langle y^*, f(x)\right\rangle\right| = \left|\left\langle y^*, T_f(\delta_U(x)\right\rangle\right| \leq d_p(T_f)\left\|y^*\right\|_{L_{p^*(\mu)}}\|\delta_U(x)\| = d_p(T_f)\left\|y^*\right\|_{L_{p^*(\mu)}}$$

by [5, Theorem 2.3.1] for some Borel regular probability measure μ on $B_{F^{**}}$. It follows that $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ with $d_p^{\mathcal{H}^\infty}(f) \leq d_p(T_f)$ by Theorem 5.2.

Since $d_p(T_f) = d_p^{\mathcal{H}^\infty}(f)$ for all $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$, to prove the last assertion of the statement, it suffices to show that the mapping $f \mapsto T_f$ from $\mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ to $\mathcal{D}_p(\mathcal{G}^\infty(U), F)$ is surjective. Indeed, take $T \in \mathcal{D}_p(\mathcal{G}^\infty(U), F)$ and then $T = T_f$ for some $f \in \mathcal{H}^\infty(U, F)$ by Theorem 1.2. Hence $T_f \in \mathcal{D}_p(\mathcal{G}^\infty(U), F)$, and thus $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ by the above proof. □

The equivalence (i) $\Leftrightarrow$ (iii) of Theorem 5.2 admits the following reformulation.

Corollary 5.5 *Let $1 < p \leq \infty$ and $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent*:

(i) *$f : U \to F$ is Cohen strongly p-summing holomorphic.*
(ii) *There exists a complex Banach space G and an operator $S \in \mathcal{D}_p(G, F)$ such that*

$$\left|\left\langle y^*, f(x)\right\rangle\right| \leq \left\|S^*(y^*)\right\| \quad (x \in U,\ y^* \in F^*).$$

In this case, $d_p^{\mathcal{H}^\infty}(f)$ is the infimum of all $d_p(S)$ with S satisfying (ii), *and this infimum is attained at T_f* (Mujica's linearization of f).

Proof (i) $\Rightarrow$ (ii): If $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$, then $T_f \in \mathcal{D}_p(\mathcal{G}^\infty(U), F)$ with $d_p^{\mathcal{H}^\infty}(f) = d_p(T_f)$ by Theorem 5.4. From Theorem 1.2, we infer that

$$\left|\left\langle y^*, f(x)\right\rangle\right| = \left|\left\langle y^*, T_f(\delta_U(x)\right\rangle\right| = \left|\left\langle (T_f)^*(y^*), \delta_U(x)\right\rangle\right| \leq \left\|(T_f)^*(y^*)\right\|$$

for all $x \in U$ and $y^* \in F^*$.

(ii) $\Rightarrow$ (i): Assume that (ii) holds. Then $S^* \in \Pi_{p^*}(F^*, G^*)$ with $\pi_{p^*}(S^*) = d_p(S)$ by [5, Theorem 2.2.2]. By Pietsch Domination Theorem for p-summing linear operators (see [6, Theorem 2.12]), there is a Borel regular probability measure μ on $B_{F^{**}}$ such that

$$\left\|S^*(y^*)\right\| \leq \pi_{p^*}(S^*)\left\|y^*\right\|_{L_{p^*}(\mu)}$$

for all $y^* \in F^*$. For any $x \in U$ and $y^* \in F^*$, it follows that

$$\left|\langle y^*, f(x)\rangle\right| \leq \left\|S^*(y^*)\right\| \leq \pi_{p^*}(S^*) \left\|y^*\right\|_{L_{p^*}(\mu)}.$$

Hence, $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ with $d_p^{\mathcal{H}^\infty}(f) \leq \pi_{p^*}(S^*) = d_p(S)$ by Theorem 5.2. □

As a consequence of Theorem 5.4, an application of [4, Theorem 3.2] shows that the Banach ideal $\mathcal{D}_p^{\mathcal{H}^\infty}$ is generated by composition with the Banach operator ideal $\mathcal{D}_p$, but we prefer to give here a proof to complete the information.

Corollary 5.6 *Let $1 < p \leq \infty$ and $f \in \mathcal{H}^\infty(U, F)$. The following conditions are equivalent*:

(i) *$f : U \to F$ is Cohen strongly p-summing holomorphic.*
(ii) *There is a complex Banach space G, $g \in \mathcal{H}^\infty(U, G)$ and $T \in \mathcal{D}_p(G, F)$ so that $f = T \circ g$.*

In this case, $d_p^{\mathcal{H}^\infty}(f) = \inf\{d_p(T) \|g\|_\infty\}$, where the infimum is taken over all factorizations of f as in (ii), *and this infimum is attained at $T_f \circ g_U$* (Mujica's factorization of f).

Proof (i) $\Rightarrow$ (ii): If $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$, we have $f = T_f \circ g_U$, where $\mathcal{G}^\infty(U)$ is a complex Banach space, $T_f \in \mathcal{D}_p(\mathcal{G}^\infty(U), F)$ and $g_U \in \mathcal{H}^\infty(U, \mathcal{G}^\infty(U))$ by Theorems 1.2 and 5.4. Moreover,

$$\inf\left\{d_p(T) \|g\|_\infty\right\} \leq d_p(T_f) \|g_U\|_\infty = d_p^{\mathcal{H}^\infty}(f).$$

(ii) $\Rightarrow$ (i): Assume $f = T \circ g$ with G, g and T being as in (ii). Since $g = T_g \circ g_U$ by Theorem 1.2, it follows that $f = T \circ T_g \circ g_U$ which implies that $T_f = T \circ T_g$, and thus $T_f \in \mathcal{D}_p(\mathcal{G}^\infty(U), F)$ by the ideal property of $\mathcal{D}_p$. By Theorem 5.4, we obtain that $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ with

$$d_p^{\mathcal{H}^\infty}(f) = d_p(T_f) = d_p(T \circ T_g) \leq k_p(T) \left\|T_g\right\| = d_p(T) \|g\|_\infty,$$

and so $d_p^{\mathcal{H}^\infty}(f) \leq \inf\{d_p(T) \|g\|_\infty\}$ by taking the infimum over all factorizations of f. □

When F is reflexive, every $f \in \mathcal{D}_2^{\mathcal{H}^\infty}(U, F)$ factors through a Hilbert space as we see below.

Corollary 5.7 *Let F be a reflexive complex Banach space. If $f \in \mathcal{D}_2^{H^\infty}(U, F)$, then there exist a Hilbert space H, an operator $T \in \mathcal{D}_2(H, F)$ and a mapping $g \in \mathcal{H}^\infty(U, H)$ such that $f = T \circ g$.*

Proof Assume that $f \in \mathcal{D}_2^{H^\infty}(U, F)$. By Theorem 5.4, $T_f \in \mathcal{D}_2(\mathcal{G}^\infty(U), F)$. Hence $(T_f)^* \in \Pi_2(F^*, \mathcal{G}^\infty(U)^*)$ by [5, Theorem 2.2.2]. By [6, Corollary 2.16 and Examples 2.9 (b)], there exist a Hilbert space H and operators $T_1 \in \Pi_2(F^*, H)$ and $T_2 \in \mathcal{L}(H, \mathcal{G}^\infty(U)^*)$ such that $(T_f)^* = T_2 \circ T_1$.

On the one hand, we have $(T_f)^{**} = (T_1)^* \circ (T_2)^*$, where $(T_1)^* \in \mathcal{D}_2(H, F^{**})$ by [5, Theorem 2.2.2]. On the other hand, we have $(T_f)^{**} \circ \kappa_{\mathcal{G}^\infty(U)} = \kappa_F \circ T_f$ with κ_F being bijective (since F is reflexive). Consequently, we obtain $f = T \circ g$, where $T = (\kappa_F)^{-1} \circ (T_1)^* \in \mathcal{D}_2(H, F)$ and $g = (T_2)^* \circ \kappa_{\mathcal{G}^\infty(U)} \circ g_U \in \mathcal{H}^\infty(U, H)$. □

Applying Theorem 5.4 and [5, Theorem 2.4.1], we get useful inclusion relations.

Corollary 5.8 *Let $1 < p_1 \le p_2 \le \infty$. If $f \in \mathcal{D}_{p_2}^{\mathcal{H}^\infty}(U, F)$, then $f \in \mathcal{D}_{p_1}^{\mathcal{H}^\infty}(U, F)$ and $d_{p_1}^{\mathcal{H}^\infty}(f) \le d_{p_2}^{\mathcal{H}^\infty}(f)$.* □

These inclusion relations can become coincidence relations when F^* has cotype 2 (see [6, pp. 217–221] for definitions and results on this class of spaces). Compare to [6, Corollary 11.16].

Corollary 5.9 *Let $2 < p \le \infty$. If F^* has cotype 2, then $\mathcal{D}_p^{\mathcal{H}^\infty}(U, F) = \mathcal{D}_2^{\mathcal{H}^\infty}(U, F)$ and $d_p^{\mathcal{H}^\infty}(f) = d_2^{\mathcal{H}^\infty}(f)$ for all $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$.*

Proof By Corollary 5.8, we have $\mathcal{D}_p^{\mathcal{H}^\infty}(U, F) \subseteq \mathcal{D}_2^{\mathcal{H}^\infty}(U, F)$ with $d_2^{\mathcal{H}^\infty}(f) \le d_p^{\mathcal{H}^\infty}(f)$ for all $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$.

For the converse, let $f \in \mathcal{D}_2^{\mathcal{H}^\infty}(U, F)$. Then $T_f \in \mathcal{D}_2(\mathcal{G}^\infty(U), F)$ with $d_2(T_f) = d_2^{\mathcal{H}^\infty}(f)$ by Theorem 5.4. Hence $(T_f)^* \in \Pi_2(F^*, \mathcal{G}^\infty(U)^*)$ with $\pi_2((T_f)^*) = d_2(T_f)$ by [5, Theorem 2.2.2]. Then, by [6, Corollary 11.16], $(T_f)^* \in \Pi_1(F^*, \mathcal{G}^\infty(U)^*)$ with $\pi_1((T_f)^*) = \pi_2((T_f)^*)$. Hence, $(T_f)^* \in \Pi_{p^*}(F^*, \mathcal{G}^\infty(U)^*)$ with $\pi_{p^*}((T_f)^*) \le \pi_1((T_f)^*)$ by [6, Theorem 2.8]. Then, by [5, Theorem 2.2.2], $T_f \in \mathcal{D}_p(\mathcal{G}^\infty(U), F)$ with $d_p(T_f) = \pi_{p^*}((T_f)^*)$.

Finally, $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F)$ with $d_p^{\mathcal{H}^\infty}(f) = d_p(T_f)$ by Theorem 5.4, and therefore $d_p^{\mathcal{H}^\infty}(f) \le d_2^{\mathcal{H}^\infty}(f)$. □

Given $f \in \mathcal{H}^\infty(U, F)$, the *transpose* of f is the mapping $f^t \colon F^* \to \mathcal{H}^\infty(U)$ defined by

$$f^t(y^*) = y^* \circ f \quad (y^* \in F^*).$$

It is known (see [12, Proposition 1.6]) that $f^t \in \mathcal{L}(F^*, \mathcal{H}^\infty(U))$ with $\|f^t\| = \|f\|_\infty$. Furthermore, $f^t = J_U^{-1} \circ (T_f)^*$ with $J_U \colon \mathcal{H}^\infty(U) \to \mathcal{G}^\infty(U)^*$ being the identification established in Theorem 1.2.

The next result establishes the relation of a Cohen strongly p-summing holomorphic mapping $f \colon U \to F$ and its transpose $f^t \colon F^* \to \mathcal{H}^\infty(U)$. Compare to [5, Theorem 2.2.2].

Theorem 5.10 *Let $1 < p \le \infty$ and $f \in \mathcal{H}^\infty(U, F)$. Then $f \in \mathcal{D}_p^{\mathcal{H}_\infty}(U, F^*)$ if and only if $f^t \in \Pi_{p^*}(F^*, \mathcal{H}^\infty(U))$. In this case, $d_p^{\mathcal{H}^\infty}(f) = \pi_{p^*}(f^t)$.*

Proof Applying Theorem 5.4, [5, Theorem 2.2.2] and [6, 2.4 and 2.5], respectively, we have

$$f \in \mathcal{D}_p^{\mathcal{H}_\infty}(U, F^*) \Leftrightarrow T_f \in \mathcal{D}_p(\mathcal{G}^\infty(U), F)$$

$$\Leftrightarrow (T_f)^* \in \Pi_{p^*}(F^*, \mathcal{G}^\infty(U)^*)$$
$$\Leftrightarrow f^t = J_U^{-1} \circ (T_f)^* \in \Pi_{p^*}(F^*, \mathcal{H}^\infty(U)).$$

In this case, $d_p^{\mathcal{H}^\infty}(f) = d_p(T_f) = \pi_{p^*}((T_f)^*) = \pi_{p^*}(J_U \circ f^t) = \pi_{p^*}(f^t)$. □

The study of holomorphic mappings with relatively (weakly) compact range was initiated by Mujica [16] and followed in [12].

Corollary 5.11 *Let* $1 < p \leq \infty$.

(i) *Every Cohen strongly* p*-summing holomorphic mapping* $f : U \to F$ *has relatively weakly compact range.*
(ii) *If* F *is reflexive, then every Cohen strongly* p*-summing holomorphic mapping* $f : U \to F$ *has relatively compact range.*

Proof If $f \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F^*)$, then $f^t \in \Pi_{p^*}(F^*, \mathcal{H}^\infty(U))$ by Theorem 5.10. Hence the linear operator f^t is weakly compact and completely continuous by [6, 2.17]. Since f^t is weakly compact, this means that f has relatively weakly compact range by [12, Theorem 2.7]. Since f^t is completely continuous and F^* is reflexive, it is known that f^t is compact and, equivalently, f has relatively compact range by [12, Theorem 2.2]. □

6 Pietsch factorization for Cohen strongly *p*-summing holomorphic mappings

We devote this section to the analogue of Pietsch Factorization Theorem for p-summing linear operators [6, Theorem 2.13] for the class of Cohen strongly p-summing holomorphic mappings. Recall that, for every Banach space F, we have the canonical isometric injections $\kappa_F : F \to F^{**}$ and $\iota_F : F \to C(B_{F^*})$ defined, respectively, by

$$\left\langle \kappa_F(y), y^* \right\rangle = y^*(y) \quad \left(y \in F,\ y^* \in F^*\right),$$
$$\left\langle \iota_F(y), y^* \right\rangle = y^*(y) \quad \left(y \in F,\ y^* \in B_{F^*}\right).$$

Moreover, if μ is a regular Borel measure on $(B_{F^{**}}, w^*)$, j_p denotes the canonical map from $C(B_{F^*})$ to $L_p(\mu)$.

Theorem 6.1 (Pietsch Factorization) *Let* $1 < p \leq \infty$ *and* $f \in \mathcal{H}^\infty(U, F)$. *The following conditions are equivalent*:

(i) $f : U \to F$ *is Cohen strongly* p*-summing holomorphic.*
(ii) *There exist a regular Borel probability measure* μ *on* $(B_{F^{**}}, w^*)$, *a closed subspace* S_{p^*} *of* $L_{p^*}(\mu)$ *and a bounded holomorphic mapping* $g : U \to (S_{p^*})^*$ *such that*

the following diagram commutes:

$$\begin{array}{ccccc} (S_{p^*})^* & & \xrightarrow{(j_{p^*})^*} & & (\iota_{F^*}(F^*))^* \\ g\uparrow & & & & \downarrow (\iota_{F^*})^* \\ U & \xrightarrow{f} & F & \xrightarrow{\kappa_F} & F^{**} \end{array}$$

In this case, $d_p^{\mathcal{H}^\infty}(f) = \|g\|_\infty$.

Proof (i) $\Rightarrow$ (ii): Let $f \in \mathcal{D}_p^{\mathcal{H}_\infty}(U, F)$. Then $f^t \in \Pi_{p^*}(F^*, \mathcal{H}^\infty(U))$ by Theorem 5.10. By [6, Theorem 2.13], there exist a regular Borel probability measure μ on $(B_{F^{**}}, w^*)$, a subspace $S_{p^*} := \overline{j_{p^*}(i_{F^*}(F^*))}$ of $L_{p^*}(\mu)$, and an operator $T \in \mathcal{L}(S_{p^*}, \mathcal{H}^\infty(U))$ with $\|T\| = \|f^t\|$ such that the following diagram commutes:

$$\begin{array}{ccc} \iota_{F^*}(F^*) & \xrightarrow{j_{p^*}} & S_{p^*} \\ \iota_{F^*}\uparrow & & \downarrow T \\ F^* & \xrightarrow{f^t} & \mathcal{H}^\infty(U) \end{array}$$

Dualizing, we obtain

$$\begin{array}{ccc} U & \overset{f}{\to} & F \\ \delta_U \downarrow & & \downarrow \kappa_F \\ \mathcal{H}^\infty(U)^* & \overset{(f^t)^*}{\to} & F^{**} \\ T^* \downarrow & & (\iota_{F^*})^* \uparrow \\ (S_{p^*})^* & \overset{(j_{p^*})^*}{\to} & (\iota_{F^*}(F^*))^* \end{array}$$

Let $g := T^* \circ g_U$. Clearly, $g \in \mathcal{H}^\infty(U, (S_{p^*})^*)$ with $\|g\|_\infty \le \|T\|$, and thus

$$\|g\|_\infty \le \|f^t\| = \|f\|_\infty \le d_p^{\mathcal{H}^\infty}(f).$$

Moreover, since $f^t = T \circ j_{p^*} \circ \iota_{F^*}$, we have

$$\kappa_F \circ f = (f^t)^* \circ g_U = (\iota_{F^*})^* \circ (j_{p^*})^* \circ T^* \circ g_U = (\iota_{F^*})^* \circ (j_{p^*})^* \circ g.$$

(ii) $\Rightarrow$ (i): Since $\kappa_F \circ f = (\iota_{F^*})^* \circ (j_{p^*})^* \circ g$, it follows that $f^t \circ (\kappa_F)^* = ((\iota_{F^*})^* \circ (j_{p^*})^* \circ g)^t$. As $(\kappa_F)^* \circ \kappa_{F^*} = \mathrm{id}_{F^*}$, we obtain that

$$f^t = ((\iota_{F^*})^* \circ (j_{p^*})^* \circ g)^t \circ \kappa_{F^*}.$$

Since $j_{p^*} \in \Pi_{p^*}(\iota_{F^*}(F^*), S_{p^*})$ (see [6, Examples 2.9]), then

$$(j_{p^*})^* \in \mathcal{D}_p((S_{p^*})^*, (i_{F^*}(F^*))^*)$$

by [5, Theorem 2.2.2]. Hence $(\iota_{F^*})^* \circ (j_{p^*})^* \circ g \in \mathcal{D}_p^{\mathcal{H}^\infty}(U, F^{**})$ with

$$d_p^{\mathcal{H}^\infty}((\iota_{F^*})^* \circ (j_{p^*})^* \circ g) \leq d_p((\iota_{F^*})^* \circ (j_{p^*})^*) \, \|g\|_\infty = \pi_{p^*}(j_{p^*} \circ \iota_{F^*}) \, \|g\|_\infty$$

by the ideal property of $\mathcal{D}_p$, Corollary 5.6 and [5, Theorem 2.2.2]. Applying Theorem 5.10 and the ideal property of Π_p, we deduce that $f^t = ((\iota_{F^*})^* \circ (j_{p^*})^* \circ g)^t \circ \kappa_{F^*} \in \Pi_{p^*}(F^*, \mathcal{H}_\infty(U))$. Again, Theorem 5.10 gives that $f \in \mathcal{D}_p^{\mathcal{H}_\infty}(U, F)$ with $d_p^{\mathcal{H}^\infty}(f) = \pi_{p^*}(f^t)$. Moreover,

$$\begin{aligned} d_p^{\mathcal{H}^\infty}(f) &= \pi_{p^*}(((\iota_{F^*})^* \circ (j_{p^*})^* \circ g)^t \circ \kappa_{F^*}) \\ &\leq \pi_{p^*}(((\iota_{F^*})^* \circ (j_{p^*})^* \circ g)^t) \, \|\kappa_{F^*}\| \\ &\leq d_p^{\mathcal{H}^\infty}((\iota_{F^*})^* \circ (j_{p^*})^* \circ g) \\ &\leq \pi_{p^*}(j_{p^*} \circ \iota_{F^*}) \, \|g\|_\infty \\ &\leq \pi_{p^*}(j_{p^*}) \, \|\iota_{F^*}\| \, \|g\|_\infty \leq \|g\|_\infty . \end{aligned}$$

□

Acknowledgements The authors would like to thank the referees for their valuable comments that have improved considerably this paper. Research of A. Jiménez-Vargas was partially supported by project UAL-FEDER grant UAL2020-FQM-B1858, by Junta de Andalucía grants P20_00255 and FQM194, and by grant PID2021-122126NB-C31 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe". J. M. Sepulcre was also supported by PGC2018-097960-B-C22 (MCIU/AEI/ERDF, UE).

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

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