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Influencing opinion networks - optimization and games

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Abstract

We consider a model of influence over a network with finite-horizon opinion dynamics. The network consists of agents that update their opinions via a trust structure as in the DeGroot dynamics (DeGroot (1974)). The model considers two potential external influencers that have fixed and opposite opinions. They aim to maximally impact the aggregate state of opinions at the end of the finite horizon by targeting one agent in one specific time period. In the case of only one influencer, we characterize optimal targets on the basis of two features: shift and amplification. Also, conditions are provided under which a specific target is optimal: the maximum-amplification target. In the case of two influencers, we focus on the existence and characterization of pure strategy equilibria in the corresponding two-person strategic zero-sum game. Roughly speaking, if the initial opinions are not too much in favour of either influencer, the influencers' equilibrium behaviour is also driven by the amplification of targets.

Keywords: Opinion dynamics - Networks - Influence - Targeting - Nash equilibria
JEL Classification: C72 - D72 - D85

1 Introduction

This paper studies the decision making of influencers of social networks. Social networks are instrumental in the performance of modern democracies, as they have become the main arena for the formation of public opinion and therefore the predominant playground for interference by actors in pursuit of political or commercial goals. Consequently, this is an active research area of institutes such as the EU Special Committee on Foreign Interference, which investigates the EU's main vulnerabilities and recommends steps to address them (cf. Russell (2022)). Identifying optimal targets in opinion networks is a crucial problem for those trying to impact societies or those trying to shield themselves from unwanted actors doing so. Many real-life situations concern actors who are not necessarily interested in the state of opinions in general, but who have an interest in influencing the state of opinions on a specific date, such as election day.

While most existing contributions have as their objective of investigation the long-term convergence of opinions in social networks, our aim is to characterize optimal strategies of influencers that wish to impact the state of opinions at a specific point in time: the end of a finite horizon. We develop a model for the opinion dynamics in a set of agents, which we can think of representing a society. To study optimal targets of influencers, we conduct an optimization analysis, where we first consider the case with a single influencer, and then a strategic situation, where we consider two influencers with opposite interests.

Our model takes as point of departure a set of agents and a finite time horizon, consisting of a number of discrete time instances. In each time period, every agent holds an opinion, which is a real number

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between 0 and 1. The agents start with a given opinion in the initial period. Then, from each period to the next, agents update their opinions through the word-of-mouth dynamics introduced in the seminal work of DeGroot (1974). Given the opinions in a period, an agent's opinion in the next period is a weighted average of their own and their neighbours' current opinions, where the weights remain fixed throughout the time horizon and reflect the level of trust that agents have in each other. At the end of the time horizon, we evaluate the aggregate state of opinions in the network. We consider two potential influencers that are not themselves agents of the society, but external actors. One influencer has the extreme opinion 1 and wants to maximize the aggregate state of opinions at the end of the finite horizon, and the other has opinion 0 and wants to minimize it. The act of influencing is modelled through the LINCAP model. Influencers have a budget and choose one specific agent in one specific period to target. Consequently, the opinion of this agent in this time period increases, in case of a target of the maximizing influencer, or decreases, in case of a target of the minimizing influencer. The size of the increase or decrease induced by a target is linear in the respective influencer's budget, which is indicated by the qualifier "LIN". The qualifier "CAP" indicates that targeted opinions are capped at 0 or 1, if the change induced by the target moved the opinions beyond these boundaries.

The work whose setup is the most similar to ours is Lever (2010). Motivated by the problem of campaign spending in elections, that paper studies a game with two persuaders (political parties) that allocate resources to sway voters. Their framework considers T rounds, where voters begin with a given initial opinion in round 0 and update their opinions until round T according to the DeGroot dynamics. In period T, an election takes place in which each voter casts a vote for one of the two persuaders, where the probability of voting for either party is dictated by their opinion in the final period T. What distinguishes our contribution from Lever (2010) is that we explicitly analyse a finite time horizon. Despite defining a final time period T, that paper lets T tend to infinity to derive results. Also, they allow the persuaders to spend their resources only in the first period, whereas we allow influencers to select any agent in any period of the time horizon. Moreover, they acknowledge that the persuaders' influencing actions are modelled with a functional form that is chosen for tractability. Our LINCAP model offers an intuitive alternative.

There are several other contributions that study a game in which players compete for prominence over the state of opinions in a society, and that take the DeGroot dynamics as a point of departure. Bimpikis et al. (2016) consider a network of consumers that hold a level awareness for two firms at each point in time of an discrete infinite time horizon. The two firms allocate marketing budgets to individuals to influence the word-of-mouth communication process. The firms wish to maximally impact the limiting behaviour of this system. Similar in spirit is the work by Grabisch et al. (2018), where two players with extreme opinions 0 and 1 seek to maximize their long-term influence in a society where opinions evolve according to the the DeGroot dynamics. To exert influence, each of Grabisch' players target exactly one agent by inserting an additional agent that is a neighbour to only the targeted agent. The inserted agent immutably holds the extreme opinion of the respective player, and thereby directly affects the opinion of the targeted agent in each updating step of the DeGroot dynamics. Two major differences between our work are that Bimpikis et al. (2016) and Grabisch et al. (2018) let the players modify the structure of the opinion dynamics, and that their players compete over the limiting behaviour of the system.

Goyal et al. (2014) also study a game where two players exert influence over a social network with opinion dynamics, but opt for a different type of dynamics. In their model, each agent in the network has a state (Red, Blue, or Uninfected). From each period to the next, the probability that an agent's state changes depends on the outcome of two functions that take as input their neighbours' states. Similar in spirit to the contribution of Lever (2010), the players have budgets with which they can "seed" the initial states, after which they let the system dynamics do its job and observe the outcome. Other types of opinion dynamics have been studied by Grabisch and Li (2020), who consider binary opinions, and Grabisch et al. (2019), who consider agents that are not organized in a network structure.

What most distinguishes our paper is the treatment of time. First, we consider opinion dynamics through a finite horizon. Second, we let the time horizon be an integral part of the influencers' strategy spaces. Other works either allow players to make time-invariant structural changes to the network (e.g., Bimpikis et al. (2016)), or allow players to exert influence at only one moment in the time horizon (e.g., Lever (2010), Goyal et al. (2014)). Consequently, the emphasis of their analyses is mostly in terms of the network structure and many of their results are expressed in notions of centrality. Several other game-theoretic studies do not contain a time element and instead have a primary focus on the structural properties of the network, often making use of centrality measures (e.g., Husslage et al. (2015), del Pozo et al. (2011)). Furthermore, an interested reader may wish to consult the introduction to models for dynamics of directed networks by Snijders et al. (2010).

In the case of only one influencer, we characterize the effect of an influencer's target on the aggregate state of opinions at the end of the time horizon, as the product of two features: the shift and amplification. The amplification of a target depends on the network structure and the target's place in the finite time horizon. The shift of a target depends on the the opinions at the start of the time horizon, as well as on the size of the influencer's budget. Consequently, we identify optimal targets in terms of shift and amplification. It turns out that, under certain conditions on the initial opinions, a certain type of target achieves the maximal effect: a target that attains the highest amplification. Loosely speaking, if the opinions at the start of the time horizon were not already too much in favor of the influencer's opinion, then targeting a maximum-amplification target is optimal.

In the case of two influencers with opposite interests, we identify the two-sided effect of a pair of targets by the two influencers. This effect will be the input for the payoff function of a corresponding strategic zero-sum game. To analyze this game, we decompose the two-sided effect into the respective players' one-sided effects and two well-interpretable interaction terms. We provide a sufficient condition under which there exists a specific type of Nash equilibrium in pure strategies, where both players choose a maximum-ampfication target. Roughly speaking, the existence of such an equilibrium depends on the initial opinions being not too much in favour of either influencer.

Section 2 defines networks with finite-horizon opinion dynamics. The one-sided LINCAP model is defined in Section 3, where we study a single influencer's optimization problem. We formulate the shift and amplification concepts to analyse the effect of a target. In Section 4, we consider two influencers and define the two-sided LINCAP model. We derive a decomposition result of the two-sided effect and use it to derive a specific type of pure Nash equilibrium for the corresponding strategic game. Section 5 concludes.

2 Networks with finite-horizon opinion dynamics

We denote a network with finite-horizon opinion dynamics by

$$P = \Big(N, W, x^0, T\Big).$$

Here the set $N = \{1, ..., n\}$ corresponds to agents that make up a social network. The $n \times n$ matrix W captures the trust structure in the social network. For every pair of agents $(i, j) \in N \times N$, the weight $W_{ij} \in [0, 1]$ specifies how much agent i values the opinion of agent j. The trust matrix W collects these weights, that add up to 1 for each agent i, so that

$$\sum_{j \in N} W_{ij} = 1, \quad \text{for all } i \in N.$$

The set of agents N and the trust matrix W correspond to a weighted directed graph $G = (N, A, w)$, where A represents the set of all possible arcs, given by

$$A = \{(i, j) : i \in N, j \in N\},$$

and the mapping $w : A \longrightarrow [0, 1]$ assigns the weight $W_{ij}(= w_{ij})$ to each arc $(i, j) \in A$. In the description of P, period $T \in \mathbb{N}$ defines the end of a discrete time horizon, where the time periods are represented by $0, 1, ..., T$. Finally, x^0 is a vector of opinions in which $x_i^0 \in [0, 1]$ is the initial opinion of agent $i \in N$.

We assume the opinions evolve through the horizon according to the DeGroot dynamics (DeGroot (1974)). We denote the opinions at times 0,1,...,T by a series of vectors $x^0, x^1, ..., x^T$. Agent $i \in N$ recursively updates his opinion by considering a weighted combination of the opinions in the previous period of the agents that he trusts:

$$x_i^t = \sum_{j \in N} W_{ij} \cdot x_j^{t-1} \quad \text{for all } i \in N, t \in \{1, ..., T\}.$$

We will address the evolution of the initial opinions through the time horizon as the natural opinions.

Definition 2.1. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics. The *natural opinions* of P are given by[1]

$$x^t = W^t x^0 \qquad \text{for } t = 0, ..., T.$$

At the end of the horizon, in period T, we assess the state of opinions in the network. In particular, throughout this paper, the sum of final opinions $\sum_{i \in N} x_i^T$ will be used as an evaluation criterion.

Example 2.1. Consider a network with finite-horizon opinion dynamics $P = (N, W, x^0, T)$ with agents $N = \{1, 2, 3\}$, trust matrix

$$W = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{bmatrix},$$

initial opinions $x^0 = (0.5, 0.7, 0.3)$ and $T = 2$. Using the weights in the trust matrix W we construct a graph G that captures the social network and its internal trust structure. Figure 1 depicts G.

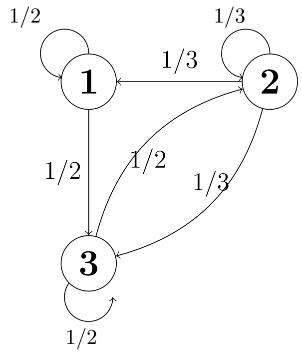

Figure 1: The graph G corresponding to trust matrix W.

In period $t = 0$, the sum of opinions is $\sum_{i \in N} x_i^0 = 1.5$. The natural opinions evolve through periods $t = 0, 1, 2$ as follows:

$$x^0 = \begin{bmatrix} 0.5 \\ 0.7 \\ 0.3 \end{bmatrix},$$

$$x^1 = Wx^0 = \begin{bmatrix} 0.4 \\ 0.5 \\ 0.5 \end{bmatrix},$$

$$x^2 = W^2 x^0 = \begin{bmatrix} 0.45 \\ 0.47 \\ 0.5 \end{bmatrix}.$$

In period $T = 2$, the sum of opinions equals $\sum_{i \in N} x_i^2 = 1.42$.

◁

3 One-sided influence: optimization

We consider a network with finite-horizon opinion dynamics $P = (N, W, x^0, T)$ and we introduce one influencer. For now, we think of the influencer as a person or organization that wants to maximize the opinions in period T; later, we will consider a minimizing influencer. The influencer targets an agent $j \in N$ in a period $\tau \in \{0, 1, ..., T\}$. We denote the action of targeting agent j in period τ by $\mu = (\tau, j)$.

[1]Throughout this paper, we adopt the convention that $W^0 = I$, where I denotes the $n \times n$ identity matrix.

Definition 3.1. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics. A *target* is a tuple $\mu = (\tau, j)$, with period $\tau \in \{0, ..., T\}$ and agent $j \in N$. We denote the set of all targets by

$$M = \{0, 1, ..., T\} \times N \ .$$

Notice in particular that opinions can also be targeted in the first period $\tau = 0$ and in last period $\tau = T$.

The influencer is assumed to have a budget $\delta > 0$ and uses it to increase the targeted opinion. When targeted by $\mu = (\tau, j)$, the natural opinion x_j^τ is increased by δ and is capped at 1 if $x_j^\tau + \delta$ exceeds 1. In other words, the amount by which x_j^τ is increased is either the full budget δ, if the natural opinion leaves enough room to be increased by δ, or by $1 - x_j^\tau$, which is the room available for increase.

Definition 3.2. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics and consider a maximizing influencer with the target $\mu = (\tau, j) \in M$ and budget $\delta > 0$. The *shift* of opinion x_j^τ due to target μ is given by

$$shift(\mu) = \min\{\delta, 1 - x_j^\tau\} \ .$$

To obtain the influenced opinions in period T, we trace the opinions through the finite time horizon. The opinions in the periods leading up to period τ are not affected by target $\mu = (\tau, j)$, so that they coincide with the natural opinions. In period τ, the j^{th} entry of the opinion vector is increased by $shift(\mu)$. In following periods, the opinions again evolve through time according to the opinion dynamics. We define this evolution of opinions under one-sided influence as the LINCAP model, where LINCAP stands for "linear with capping". To reflect that the opinions are influenced by μ, we denote the opinion vectors throughout the horizon by $\tilde{x}^1(\mu), ..., \tilde{x}^T(\mu)$.

Definition 3.3. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics and consider a maximizing influencer with target $\mu = (\tau, j)$ and budget $\delta > 0$. The *one-sided LINCAP* model defines the influenced opinions as[2]

$$\tilde{x}^t(\mu) = \begin{cases} W^t x^0 & \text{if } t < \tau \\ W^\tau x^0 + shift(\mu) \cdot e^j & \text{if } t = \tau \\ W^{t-\tau} \tilde{x}^\tau(\mu) & \text{if } t > \tau \end{cases} \qquad \text{for } t = 0, 1, ..., T.$$

In order to analyze the impact of a target $\mu \in M$, it is not sufficient to consider only the vector $\tilde{x}^T(\mu)$ of influenced opinions in period T, because they contain the natural opinions as well as an additional amount of opinion due to the target. The effect of a target measures the amount by which the sum of opinions in period T is increased as a result of the target, by subtracting the natural opinions.

Definition 3.4. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics and let $\mu \in M$ be a target. The *effect* of μ is given by

$$Effect(\mu) \ = \ \sum_{i \in N} \tilde{x}_i^T(\mu) - \sum_{i \in N} x_i^T.$$

It is easily verified that the effect of a target of a maximizing influencer is always nonnegative.

The maximizing influencer aims to choose a target μ^* that maximizes the effect.

Definition 3.5. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics and consider a maximizing influencer with budget $\delta > 0$. A target $\mu^* \in M$ is *optimal* if

$$Effect(\mu^*) \geq Effect(\mu) \qquad \text{for all targets } \mu \in M.$$

The key technique that we employ to characterize optimal targets is to decompose the effect of a target into the product of its shift and amplification. The shift was given in Definition 3.2 and measures the immediate impact of the target $\mu = (\tau, j)$, which is the increment in the targeted opinion x_j^τ. Between the period τ in which the target brings about the increase given by $shift(\mu)$ and the final period T in which the sum of opinions in the network is evaluated, the impact of the target trickles down to other agents in the network. During these periods $\tau, ..., T$, the total impact of the target can either augment or diminish. The amplification of the target will be the factor by which the shift is multiplied to measure the total impact of the target on the opinions in period T. The amplification of a target is the column sum of the trust matrix W raised to the number of remaining periods until the end of the horizon $(T - \tau)$, where the column corresponds to the targeted agent j.

[2]We denote the j^{th} unit vector of length n by e^j, where $e_k^j = 1$ if $k = j$ and $e_k^j = 0$ if $k \neq j$.

Definition 3.6. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics and let $\mu = (\tau, j)$ be a target. The *amplification* of μ is given by

$$ampl(\mu) = \sum_{i \in N} (W^{T-\tau})_{ij} \ .$$

For conciseness, and to reflect more visibly the period τ and agent j of the target, we interchangeably write the amplification of a target as

$$ampl(\mu) = a_j^\tau \qquad \text{for all targets } \mu = (\tau, j).$$

Moreover, observe in particular that in period T, the amplification a_i^T of targeting any agent $i \in N$ always equals 1. If an agent is targeted at the end of the horizon, the shift does not trickle down to other agents and is immediately evaluated into the sum of final opinions.

It turns out that the effect of a target is the product of its shift and amplification.

Lemma 3.1. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics and let $\mu \in M$ be a target of a maximizing influencer. Then, the effect of μ is given by

$$Effect(\mu) = shift(\mu) \cdot ampl(\mu) \ .$$

Proof. Let $\mu = (\tau, j)$. We first rewrite the effect of μ in vector notation and develop the influenced opinions of period T according to the one-sided LINCAP model as follows:[3]

$$\begin{aligned} Effect(\mu) &= \textstyle\sum_{i \in N} \tilde{x}_i^T(\mu) - \sum_{i \in N} x_i^T \\ &= e^\intercal \left(\tilde{x}^T(\mu) \qquad\qquad\qquad\qquad -x^T \right) \\ &= e^\intercal \left(W^{T-\tau} \tilde{x}^\tau(\mu) \qquad\qquad\qquad -x^T \right) \\ &= e^\intercal \left(W^{T-\tau} \left(W^\tau x^0 + shift(\mu) \cdot e^j \right) -x^T \right) \\ &= e^\intercal \left(x^T + shift(\mu) \cdot W^{T-\tau} e^j \qquad -x^T \right) \\ &= shift(\mu) \cdot e^\intercal W^{T-\tau} e^j \\ &= shift(\mu) \cdot ampl(\mu) \ . \end{aligned}$$

Observe that in Definition 3.3, the first case ($t < \tau$) is superfluous if the target takes place in period $\tau = 0$ and the last case ($t > \tau$) is superfluous if $\tau = T$. For such targets, the latter derivation could be written more concisely as well. In particular, we note that for $\tau = T$ we could skip several steps by writing that

$$\tilde{x}^T(\mu) = x^T + shift(\mu) \cdot e^j.$$

□

In the following example, we illustrate the notions of effect, shift and amplification.

Example 3.1. Consider a network with finite-horizon opinion dynamics $P = (N, W, x^0, T)$ and a maximizing influencer with budget $\delta = 0.2$. The set of agents is $N = \{1, 2, 3\}$ and the trust-matrix is

$$W = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{bmatrix} . \tag{1}$$

The initial opinions are $x^0 = (0.1, 0.9, 0.6)$ and the time horizon is $T = 2$. The shift, amplification and effect are provided in Table 1.

[3] We denote the all-ones vector of length n by e.

Shift

τ \ j	1	2	3
0	0.20	0.10	0.20
1	0.20	0.20	0.20
2	0.20	0.20	0.20

Amplification

τ \ j	1	2	3
0	0.69	0.94	1.36
1	0.83	0.83	1.33
2	1.00	1.00	1.00

Effect

τ \ j	1	2	3
0	0.14	0.09	0.27
1	0.17	0.17	0.26
2	0.20	0.20	0.20

Table 1: The shift, amplification and effect of all targets $\mu = (\tau, j) \in M$.

Observe that the target $\mu^* = (\tau^*, j^*)$ with $\tau^* = 0$ and $j^* = 3$ is optimal, because it attains the maximum effect.

As an illustration, we explicitly compute the effect 0.17 of target $\mu = (\tau, j)$ with $\tau = 1$ and $j = 2$. First we compute the sum of natural opinions in period T and the sum of influenced opinions in period T. The natural opinions are

$$x^0 = \begin{bmatrix} 0.10 \\ 0.90 \\ 0.60 \end{bmatrix}, \quad x^1 = \begin{bmatrix} 0.35 \\ 0.53 \\ 0.75 \end{bmatrix}, \quad x^2 = \begin{bmatrix} 0.55 \\ 0.54 \\ 0.64 \end{bmatrix}.$$

Hence, the sum of natural opinions in period $T = 2$ is

$$\sum_{i \in N} x_i^2 = 1.73.$$

Next, we compute the influenced opinions using the one-sided LINCAP model:

$$\tilde{x}^0(\mu) = W^0 x^0 = \begin{bmatrix} 0.1 \\ 0.9 \\ 0.6 \end{bmatrix},$$

$$\tilde{x}^1(\mu) = W^1 x^0 + shift(\mu) \cdot e_2 = \begin{bmatrix} 0.35 \\ 0.53 \\ 0.75 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.35 \\ 0.73 \\ 0.75 \end{bmatrix},$$

$$\tilde{x}^2(\mu) = W^1 \tilde{x}^1(\mu) = \begin{bmatrix} 0.55 \\ 0.61 \\ 0.74 \end{bmatrix}.$$

Hence, the sum of affected opinions in period $T = 2$ is

$$\sum_{i \in N} \tilde{x}_i^2(\mu) = 1.90.$$

Indeed, the effect of target $\mu = (1, 2)$ is therefore

$$Effect(\mu) = 1.90 - 1.73 = 0.17.$$

$\lhd$

As a direct consequence of the decomposition of the effect given in Lemma 3.1, targets with maximum amplification are logical candidates for optimality.

Definition 3.7. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics. A target $\hat{\mu}$ is a *maximum-amplification target (MAT)* if

$$ampl(\hat{\mu}) \geq ampl(\mu) \qquad \text{for all targets } \mu \in M.$$

The main result of this section is that a MAT $\mu = (\tau, j)$ is optimal if the corresponding natural opinion x_j^τ is such that the influencer can choose this target and spend the full budget δ effectively. It follows directly from Lemma 3.1 and is stated without proof.

Theorem 3.1. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics and consider a maximizing influencer with budget $\delta > 0$. Let target $\mu^* = (\tau^*, j^*)$ be a MAT. If the corresponding natural opinion $x_{j^*}^{\tau^*}$ satisfies

$$x_{j^*}^{\tau^*} \leq 1 - \delta\,, \tag{2}$$

then μ^* is optimal for the maximizing influencer.

In the previous Example 3.1, one can verify that condition (2) holds and the MAT $\hat{\mu} = (0, 3)$ is optimal. The following example provides a case where condition (2) is violated and the MAT is not optimal.

Example 3.2. Consider a network with finite-horizon opinion dynamics $P = (N, W, x^0, T)$ and a maximizing player with budget $\delta = 0.2$. The set of agents is $N = \{1, 2, 3\}$ and the trust-matrix is given in equation (1). The initial opinions are $x^0 = (0.1, 0.5, 0.9)$ and the time horizon is $T = 2$. The shift, amplification and effect of all targets are given in the following Table 2 (rounded to two decimals). Notice that we obtain the same the amplification values as in Example 3.1, because the trust matrix is the same.

Shift			
τ \ j	1	2	3
0	0.20	0.20	0.10
1	0.20	0.20	0.20
2	0.20	0.20	0.20

Amplification			
τ \ j	1	2	3
0	0.69	0.94	1.36
1	0.83	0.83	1.33
2	1.00	1.00	1.00

Effect			
τ \ j	1	2	3
0	0.14	0.19	0.14
1	0.17	0.17	0.27
2	0.20	0.20	0.20

Table 2: The shift, amplification and effect of all targets $\mu = (\tau, j) \in M$.

The amplification values indicate that the unique MAT is $\hat{\mu} = (0, 3)$. However, the effect values imply that the unique optimal target is $\mu^* = (1, 3)$.

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So far, we have considered a maximizing influencer. For a minimizing influencer, we follow a similar line of reasoning and obtain similar results.

The crucial dissimilarity is a different definition of the shift a target, which is negative for a minimizing influencer. Henceforth, to distinguish the shifts caused by influencers with opposite objective, we denote the shift of a target μ of a maximizing and a minimizing influencer by $shift_+(\mu)$ and $shift_-(\mu)$, respectively. Given a minimizing player with budget $\delta > 0$ and target $\mu = (\tau, j)$, the shift is given by

$$shift_-(\mu) = -\min\{\delta, x_j^\tau\} .$$

The definitions of the LINCAP model and the effect remain unchanged, apart from the substitution of $shift_-(\mu)$. For targets of the minimizing influencer, the effect is always nonpositive. The minimizing influencer aims to minimize the effect, for which we have a decomposition result that is similar to Lemma 3.1.

Lemma 3.2. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics and let $\mu \in M$ be a target of a minimizing influencer. Then, the effect of μ is given by

$$Effect(\mu) = shift_-(\mu) \cdot ampl(\mu) .$$

Consequently, a MAT is optimal if it admits a minimizing influencer to spend the full budget δ effectively.

Theorem 3.2. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics and consider a minimizing influencer with budget $\delta > 0$. Let target $\mu^* = (\tau^*, j^*)$ be a MAT. If the corresponding natural opinion $x_{j^*}^{\tau^*}$ satisfies

$$x_{j^*}^{\tau^*} \geq \delta ,$$

then μ^* is optimal for the minimizing influencer.

In the next section, we examine the strategic interaction that emerges when a maximizing influencer and a minimizing influencer operate in the same network with finite-horizon opinion dynamics.

4 Two-sided influence: games

We consider a network with finite-horizon opinion dynamics $P = (N, W, x^0, T)$ with a maximizing and a minimizing influencer, whom we will now address as players 1 and 2, respectively. We introduce the corresponding *influencing game* in which the two players, with budgets $\delta_1 > 0$ and $\delta_2 > 0$, want to influence the sum of opinions in period T. Both players choose a target independently and simultaneously and we assume that the players choose their target before the start of the time horizon. In other words, the players are non-reactive and engage in a silent duel. Due to the target $\mu_1 = (\tau_1, j_1)$ of player 1 the opinion of agent j_1 in period τ_1 increases. Due to the target $\mu_2 = (\tau_2, j_2)$ of player 2 the opinion of agent j_2 in period τ_2 decreases. The opinions evolve from period 0 up to period T under influence of the two targets. At the end of the horizon, in period T, we evaluate the sum of opinions, which player 1 aims to maximize and player 2 aims to minimize.

The influencing game is a strategic zero-sum game. The possible pure strategies of both players are all targets, so that the strategy sets of player 1 and player 2 are given by

$$M_1 = M_2 = \{0, ..., T\} \times N \ .$$

Given a target combination $(\mu_1, \mu_2) \in M_1 \times M_2$, we compute the sum of opinions in period T

$$\sum_{i \in N} \tilde{x}_i^T(\mu_1, \mu_2) \ .$$

We will later define the influenced opinions $(\tilde{x}^t(\mu_1, \mu_2))_{t \in \{0,...,T\}}$ exactly. The influenced opinion vector $\tilde{x}^T(\mu_1, \mu_2)$ in period T comprises the natural opinions and the impact of the targets μ_1 and μ_2. To separate the influence of the targets from the natural opinions we consider the two-sided effect

$$Effect(\mu_1, \mu_2) = \sum_{i \in N} \tilde{x}_i^T(\mu_1, \mu_2) - \sum_{i \in N} x_i^T \ .$$

Given the target combination (μ_1, μ_2), the $Effect(\mu_1, \mu_2)$ is the payoff that player 1 receives and that player 2 pays. For a network with finite-horizon opinion dynamics P and budget vector $\delta = (\delta_1, \delta_2)$, we denote the corresponding influencing game by $\Gamma_{P,\delta}$ and we capture the effect of all target combinations $(\mu_1, \mu_2) \in M_1 \times M_2$ in a matrix $E_{P,\delta}$. For an influencing game $\Gamma_{P,\delta}$, a Nash equilibrium (Nash (1951)) is a strategy combination $(\mu_1^*, \mu_2^*) \in M_1 \times M_2$ whose targets μ_1 and μ_2 are each others' unilateral best response, i.e.,

$$\begin{aligned} Effect(\mu_1^*, \mu_2^*) &\geq Effect(\mu_1, \mu_2^*) && \text{for all } \mu_1 \in M_1 \ , \\ Effect(\mu_1^*, \mu_2^*) &\leq Effect(\mu_1^*, \mu_2) && \text{for all } \mu_2 \in M_2 \ . \end{aligned}$$

The *two-sided LINCAP* model defines the influenced opinions $(\tilde{x}^t(\mu_1, \mu_2))_{t \in \{0,...,T\}}$ throughout the finite time horizon. From each period t to the next period $t+1$, for $t \in \{0, ..., T-1\}$, the opinions evolve according to the dynamics in the network P, that is, through weighted combinations prescribed by the trust matrix W. Before the first target, the opinions coincide with the natural opinions $(x^t)_{t \in \{0,...,T\}}$. In period τ_1 of target $\mu_1 = (\tau_1, j_1)$, opinion $\tilde{x}_{j_1}^{\tau_1}$ is increased by δ_1 and capped at 1 if necessary. In period τ_2 of target $\mu_2 = (\tau_2, j_2)$, opinion $\tilde{x}_{j_2}^{\tau_2}$ is decreased by δ_2 and capped at 0 if necessary. We distinguish four cases: (i) target μ_1 occurs in an earlier period than target μ_2; (ii) target μ_1 occurs in a later period than target μ_2; (iii) targets μ_1 and μ_2 occur in the same period and they influence different agents; (iv) targets μ_1 and μ_2 influence the same agent in the same period.

Definition 4.1. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics, let $\mu_1 = (\tau_1, j_1) \in M_1$ with budget $\delta_1 > 0$ be a target of player 1, and let $\mu_2 = (\tau_2, j_2) \in M_2$ with budget $\delta_2 > 0$ be a target of player 2. Then, the *two-sided LINCAP model* defines the influenced opinions $(\tilde{x}^t(\mu_1, \mu_2))_{t \in \{0,...,T\}}$ as follows.
If $\tau_1 < \tau_2$,

$$\tilde{x}^t(\mu_1, \mu_2) = \begin{cases} x^t & \text{if } t < \tau_1 \\ x^{\tau_1} + \min\{\delta_1, 1 - x_{j_1}^{\tau_1}\} \cdot e^{j_1} & \text{if } t = \tau_1 \\ W\tilde{x}^{t-1}(\mu_1, \mu_2) & \text{if } \tau_1 < t < \tau_2 \\ W\tilde{x}^{\tau_2 - 1}(\mu_1, \mu_2) - \min\left\{\delta_2, \ (e^{j_2})^\intercal \, W\tilde{x}^{\tau_2 - 1}(\mu_1, \mu_2)\right\} \cdot e^{j_2} & \text{if } t = \tau_2 \\ W\tilde{x}^{t-1}(\mu_1, \mu_2) & \text{if } t > \tau_2 \end{cases} \quad \text{for } t = 0, ..., T.$$

If $\tau_1 > \tau_2$,

$$\tilde{x}^t(\mu_1,\mu_2) = \begin{cases} x^t & \text{if } t < \tau_2 \\ x^{\tau_2} - \min\{\delta_2, x^{\tau_2}_{j_2}\} \cdot e^{j_2} & \text{if } t = \tau_2 \\ W\tilde{x}^{t-1}(\mu_1,\mu_2) & \text{if } \tau_2 < t < \tau_1 \\ W\tilde{x}^{\tau_1-1}(\mu_1,\mu_2) + \min\left\{\delta_1,\ 1-(e^{j_1})^\intercal\, W\tilde{x}^{\tau_1-1}(\mu_1,\mu_2)\right\} \cdot e^{j_1} & \text{if } t = \tau_1 \\ W\tilde{x}^{t-1}(\mu_1,\mu_2) & \text{if } t > \tau_1 \end{cases} \quad \text{for } t = 0,...,T.$$

If $\tau_1 = \tau_2 = \tau$ and $j_1 \neq j_2$,

$$\tilde{x}^t(\mu_1,\mu_2) = \begin{cases} x^t & \text{if } t < \tau \\ x^{\tau} + \min\{\delta_1, 1 - x^{\tau}_{j_1}\} \cdot e^{j_1} & \\ \quad - \min\{\delta_2, x^{\tau}_{j_2}\} \cdot e^{j_2} & \text{if } t = \tau \\ W\tilde{x}^{t-1}(\mu_1,\mu_2) & \text{if } t > \tau \end{cases} \quad \text{for } t = 0,...,T.$$

If $(\tau_1, j_1) = (\tau_2, j_2) = (\tau, j)$,

$$\tilde{x}^t(\mu_1,\mu_2) = \begin{cases} x^t & \text{if } t < \tau \\ x^{\tau} + \min\{\delta_1 - \delta_2, 1 - x^{\tau}_{j}\} \cdot e^{j} & \text{if } t = \tau \text{ and } \delta_1 \geq \delta_2 \\ x^{\tau} - \min\{\delta_2 - \delta_1, x^{\tau}_{j}\} \cdot e^{j} & \text{if } t = \tau \text{ and } \delta_1 < \delta_2 \\ W\tilde{x}^{t-1}(\mu_1,\mu_2) & \text{if } t > \tau \end{cases} \quad \text{for } t = 0,...,T.$$

The following is a example of an influencing game. For two target combinations the payoff is computed explicitly.

Example 4.1. Consider a network with opinion dynamics $P = (N, W, x^0, T)$ with agents $N = \{1, 2, 3\}$, initial opinions $x^0 = (0.1, 0.9, 0.6)$, time horizon $T = 1$, and trust matrix

$$W = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

Moreover, consider players 1 and 2 with budget vector $\delta = (\delta_1, \delta_2) = (0.3, 0.2)$.
Then, the corresponding influencing game can be described by the matrix $E_{P,\delta}$ given by

$$E_{P,\delta} = \begin{array}{c} \\ (0,1) \\ (0,2) \\ (0,3) \\ (1,1) \\ (1,2) \\ (1,3) \end{array} \begin{array}{c} \begin{array}{cccccc} (0,1) & (0,2) & (0,3) & (1,1) & (1,2) & (1,3) \end{array} \\ \begin{bmatrix} 0.09 & 0.09 & -0.01 & 0.05 & 0.05 & 0.05 \\ 0 & 0.09 & -0.18 & -0.11 & -0.11 & -0.11 \\ 0.32 & 0.24 & 0.14 & 0.20 & 0.20 & 0.20 \\ 0.22 & 0.14 & 0.04 & 0.10 & 0.10 & 0.10 \\ 0.22 & 0.14 & 0.04 & 0.10 & 0.10 & 0.10 \\ 0.17 & 0.14 & 0.04 & 0.05 & 0.05 & 0.10 \end{bmatrix} \end{array}.$$

For any target combination $(\mu_1, \mu_2) \in M_1 \times M_2$, the computation of the $Effect(\mu_1, \mu_2)$ requires the sum of natural opinions in period $T = 1$. Here, the natural opinions and the sum of natural opinions in period $T = 1$ are given by

$$x^1 = Wx^0 = \begin{bmatrix} 0.35 \\ 0.53 \\ 0.75 \end{bmatrix} \quad \text{and} \quad \sum_{i \in N} x^1_i = 1.63\,. \tag{3}$$

Now we consider the strategy combination $(\mu_1, \mu_2) = ((0,3), (1,1))$. The two-sided LINCAP model defines the opinions $\left(\tilde{x}^t(\mu_1,\mu_2)\right)_{t \in \{0,1\}}$ as follows:

$$\tilde{x}^0(\mu_1,\mu_2) = x^0 \underbrace{+ \min\{\delta_1, 1 - x^0_3\} \cdot e^3}_{\mu_1} = \begin{bmatrix} 0.10 \\ 0.90 \\ 0.60 \end{bmatrix} + \min\{0.30, 1 - 0.60\} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.10 \\ 0.90 \\ 0.90 \end{bmatrix},$$

$$\tilde{x}^1(\mu_1,\mu_2) = W\tilde{x}^0(\mu_1,\mu_2) \underbrace{- \min\left\{\delta_2, (e^1)^\intercal W\tilde{x}^0(\mu_1,\mu_2)\right\} \cdot e^1}_{\mu_2} = \begin{bmatrix} 0.50 \\ 0.63 \\ 0.90 \end{bmatrix} - \min\{0.20, 0.50\} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.30 \\ 0.63 \\ 0.90 \end{bmatrix}.$$

Hence, the sum of influenced opinions in period $T = 1$ is given by

$$\sum_{i \in N} \tilde{x}_i^1(\mu_1, \mu_2) = 1.83 \; . \tag{4}$$

By subtracting the sum of natural opinions (3) from the influenced opinions (4), the cell $(E_{P,\delta})_{(\mu_1,\mu_2)}$ for $(\mu_1, \mu_2) = ((0,3),(1,1))$ is therefore given by

$$Effect(\mu_1, \mu_2) = \sum_{i \in N} \tilde{x}_i^1(\mu_1, \mu_2) - \sum_{i \in N} x_i^1 \; = 1.83 - 1.63 = 0.20 \; .$$

Next, we consider the strategy combination $(\mu_1, \mu_2) = ((0,2),(0,2))$. We compute the opinions $\big(\tilde{x}^t(\mu_1, \mu_2)\big)_{t \in \{0,1\}}$ with the two-sided LINCAP model as follows:

$$\tilde{x}^0(\mu_1, \mu_2) = x^0 \underbrace{+ \min\{\delta_1 - \delta_2, 1 - x_2^0\} \cdot e^2}_{\mu_1,\, \mu_2} = \begin{bmatrix} 0.10 \\ 0.90 \\ 0.60 \end{bmatrix} + \min\{0.30 - 0.20, 1 - 0.90\} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.10 \\ 1.00 \\ 0.60 \end{bmatrix} \; , \tag{5}$$

$$\tilde{x}^1(\mu_1, \mu_2) = W \tilde{x}^0(\mu_1, \mu_2) = \begin{bmatrix} 0.30 \\ 0.57 \\ 0.80 \end{bmatrix} \; . \tag{6}$$

Hence, the sum of influenced opinions in period $T = 1$ is given by

$$\sum_{i \in N} \tilde{x}_i^1(\mu_1, \mu_2) = 1.72 \; , \tag{7}$$

and the cell $(E_{P,\delta})_{(\mu_1,\mu_2)}$ is therefore given by

$$Effect(\mu_1, \mu_2) = \sum_{i \in N} \tilde{x}_i^1(\mu_1, \mu_2) - \sum_{i \in N} x_i^1 \; = 1.72 - 1.63 = 0.09 \; .$$

◁

Now we establish a decomposition of the two-sided effect into the one-sided effects of the two players in combination with two player-specific interaction terms.

Proposition 4.1. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics and consider players 1 and 2 with targets $\mu_1 = (\tau_1, j_1) \in M_1$ and $\mu_2 = (\tau_2, j_2) \in M_2$ and budgets $\delta_1 > 0$ and $\delta_2 > 0$. The two-sided effect can be written as

$$Effect(\mu_1, \mu_2) = Effect(\mu_1) + \Omega_1(\mu_1, \mu_2) + Effect(\mu_2) + \Omega_2(\mu_1, \mu_2) \; ,$$

where, if $\mu_1 = \mu_2$, the interaction terms are given by

$$\Omega_1(\mu_1, \mu_2) = \begin{cases} 0 & \text{if } x_{j_1}^{\tau_1} \leq 1 - \delta_1 \\ \min\{x_{j_1}^{\tau_1} + \delta_1 - 1, \delta_2\} \cdot a_{j_1}^{\tau_1} & \text{if } x_{j_1}^{\tau_1} > 1 - \delta_1 \end{cases} \; , \tag{8}$$

$$\Omega_2(\mu_1, \mu_2) = \begin{cases} 0 & \text{if } x_{j_2}^{\tau_2} \geq \delta_2 \\ -\min\{\delta_2 - x_{j_2}^{\tau_2}, \delta_1\} \cdot a_{j_2}^{\tau_2} & \text{if } x_{j_2}^{\tau_2} < \delta_2 \end{cases} \; , \tag{9}$$

and, if $\mu_2 \neq \mu_1$, the interaction terms are given by

$$\Omega_1(\mu_1, \mu_2) = \begin{cases} 0 & \text{if } \tau_1 \leq \tau_2 \\ \min\big\{ \max\{x_{j_1}^{\tau_1} + \delta_1 - 1, 0\}, & \\ \qquad \min\{\delta_2, x_{j_2}^{\tau_2}\} \cdot W_{j_1,j_2}^{\tau_1 - \tau_2} \;\big\} \cdot a_{j_1}^{\tau_1} & \text{if } \tau_1 > \tau_2 \end{cases} \; , \tag{10}$$

$$\Omega_2(\mu_1, \mu_2) = \begin{cases} 0 & \text{if } \tau_2 \leq \tau_1 \\ -\min\big\{ \max\{\delta_2 - x_{j_2}^{\tau_2}, 0\}, & \\ \qquad \min\{\delta_1, 1 - x_{j_1}^{\tau_1}\} \cdot W_{j_2,j_1}^{\tau_2 - \tau_1} \;\big\} \cdot a_{j_2}^{\tau_2} & \text{if } \tau_1 < \tau_2 \end{cases} \; . \tag{11}$$

The proof of Proposition 4.1 requires several straightforward case distinctions and is uninformative. Therefore, it is omitted.

Several observations will help us interpret the interaction terms. First, it is useful to observe that the interaction term with respect to player 1 is always nonnegative and the interaction term with respect to player 2 is always nonpositive. This can be verified directly by reviewing the possible evaluations of the minimum and maximum operators in equations (8), (9), (10) and (11).

Lemma 4.1. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics with players 1 and 2 with budget vector $\delta = (\delta_1, \delta_2)$. Then, for all target combinations $(\mu_1, \mu_2) \in M_1 \times M_2$, we have that

$$\Omega_1(\mu_1, \mu_2) \geq 0 \;,$$
$$\Omega_2(\mu_1, \mu_2) \leq 0 \;.$$

Consequently, recalling that the one-sided effects of players 1 and 2 can also be only nonnegative and nonpositive, respectively, we observe that for all target combinations $(\mu_1, \mu_2) \in M_1 \times M_2$ the two-sided effect consists of a nonnegative and nonpositive part, i.e.,

$$Effect(\mu_1, \mu_2) = \underbrace{Effect(\mu_1) + \Omega_1(\mu_1, \mu_2)}_{\geq 0} + \underbrace{Effect(\mu_2) + \Omega_2(\mu_1, \mu_2)}_{\leq 0} \;.$$

The latter expression clarifies the strategic interaction in the influencing game. The evaluation criterion is the sum of opinions in period T, upon which the target μ_1 applies an upward force, the target μ_2 applies a downward force, and the two-sided effect measures the net upward or downward force. Apparently, the total upward force that target μ_1 generates is the sum of two elements, namely, the one-sided $Effect(\mu_1)$ that target μ_1 could generate individually without target μ_2, and the additional thrust $\Omega_1(\mu_1, \mu_2)$ that target μ_1 is able to produce due to the presence of target μ_2. Hence, we can interpret the interaction term $\Omega_1(\mu_1, \mu_2)$ of player 1 as how much target μ_2 'helps' target μ_1 achieve an additional upward force in the influencing game, as compared to the one-sided optimization setting. Similarly, we can interpret the interaction term $\Omega_2(\mu_1, \mu_2)$ of player 2 as how much target μ_1 causes target μ_2 to be more effective. It follows from equations (8), (9), (10) and (11) that a target can only acquire such additional force if it does not occur earlier in the time horizon than the other target. This could drive the players to choose a target that occurs later in the time horizon. On the other hand, high amplification values tend to occur at earlier stages in the time horizon, incentivizing players to choose a target that occurs earlier in the time horizon.

To see how one target can help the other target become more effective, we first explain the interaction term $\Omega_1(\mu_1, \mu_2)$ of player 1 (equation (8)). Suppose that the players choose equal targets $\mu_1 = \mu_2$. The one-sided effect of a target $\mu_1 = (\tau_1, j_1)$ can be curbed by capping if the natural opinion $x_{j_1}^{\tau_1}$ is too high. If $x_{j_1}^{\tau_1} > 1 - \delta_1$, then the targeted opinion is raised by only $1 - x_{j_1}^{\tau_1}$, instead of the full budget δ_1. Now, with $\mu_1 = \mu_2$ in the two-sided setting, μ_2 decreases the same opinion that μ_1 targets, so that μ_1 has more room to bring about a larger shift and thus be more effective. If $x_{j_1}^{\tau_1} \leq 1 - \delta_1$, then player 1's target is already maximally effective in the one-sided setting, so a decrease in $x_{j_1}^{\tau_1}$ due to μ_2 does not make μ_1 additionally effective in the two-sided setting, so that $\Omega_1(\mu_1, \mu_2) = 0$. However, if $x_{j_1}^{\tau_1} > 1 - \delta_1$, then the amount $x_{j_1}^{\tau_1} + \delta_1 - 1$ of player 1's budget is lost in the one-sided setting due to capping. Hence, if μ_2 decreases $x_{j_1}^{\tau_1}$ by δ_2, then player 1 can achieve an additional shift of at most δ_2, of which the effect in period T is obtained by multiplication with the amplification $a_{j_1}^{\tau_1}$, so that equation (8) states that

$$\Omega_1(\mu_1, \mu_2) = \min\{x_{j_1}^{\tau_1} + \delta_1 - 1, \delta_2\} \cdot a_{j_1}^{\tau_1} \qquad \text{if } \; \mu_1 = \mu_2 \; \text{ and } \; x_{j_1}^{\tau_1} > 1 - \delta_1 \;.$$

One can explain the interaction $\Omega_1(\mu_1, \mu_2)$ under unequal targets $\mu_1 \neq \mu_2$ in equation (10) similarly. If $\tau_1 \leq \tau_2$, then μ_2 clearly does not reduce the opinion $x_{j_1}^{\tau_1}$ before it is targets by μ_1, so that the effectiveness of μ_1 is not heightened due to μ_2 and thus $\Omega_1(\mu_1, \mu_2) = 0$. If $\tau_1 > \tau_2$, then μ_2 may reduce the opinion targeted by μ_1. Due to μ_2, the opinion $x_{j_2}^{\tau_2}$ is reduced by $(\min\{\delta_2, x_{j_2}^{\tau_2}\})$, which propagates through the periods $(\tau_2+1), ..., \tau_1$ to reduce the opinion targeted by μ_1 by $(\min\{\delta_2, x_{j_2}^{\tau_2}\} \cdot W_{j_1,j_2}^{\tau_1-\tau_2})$. This gives player 1 more room than in the one-sided setting to raise the opinion of agent j_1 in period τ_1. The amount by which capping curbs the shift of μ_1 in the one-sided setting is given by $(\max\{x_{j_1}^{\tau_1} + \delta_1 - 1, 0\})$, so that the additional effect of μ_1 due to μ_2 is given by

$$\Omega_1(\mu_1, \mu_2) = \min\big\{ \max\{x_{j_1}^{\tau_1} + \delta_1 - 1, 0\}, \\ \min\{\delta_2, x_{j_2}^{\tau_2}\} \cdot W_{j_1,j_2}^{\tau_1-\tau_2} \big\} \cdot a_{j_1}^{\tau_1} \qquad \text{if } \; \tau_1 > \tau_2 \; \text{ and } \; x_{j_1}^{\tau_1} > 1 - \delta_1 \;.$$

The interaction terms $\Omega_2(\mu_1, \mu_2)$ in equations (9) and (11) can be explained analogously.

Having observed that the capping mechanism is the driver of strategic interaction in the influencing game, we now show that if the natural opinions are distant enough from both extreme opinions 0 and 1, then both players can choose any target without losing any of their budget through capping. In such cases, the players of the influencing game will choose their targets identically as in the one-sided optimization setting.

Theorem 4.1. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics with players 1 and 2 with budget vector $\delta = (\delta_1, \delta_2)$. If

$$\delta_2 \leq x_i^0 \leq 1 - \delta_1 \tag{12}$$

for all $i \in N$, then

$$Effect(\mu_1, \mu_2) = Effect(\mu_1) + Effect(\mu_2) \ .$$

for all target combinations $(\mu_1, \mu_2) \in M_1 \times M_2$.

Proof. Let P and $\delta = (\delta_1, \delta_2)$ be such that condition (12) holds for all $i \in N$. Recall that the natural opinions in periods $t \in \{1, ..., T\}$ are defined recursively as

$$x_i^t = \sum_{j \in N} W_{ij} \cdot x_j^{t-1}$$

for all $i \in N, t \in \{1, ..., T\}$. Since the natural opinions in periods $t \in \{1, ..., T\}$ are convex combinations of the natural opinions in the previous period $t-1$, it can be proven inductively that if condition (12) holds for the initial opinions, then we also have

$$\delta_2 \leq x_i^t \leq 1 - \delta_1$$

for the periods $t \in \{1, ..., T\}$, for all $i \in N$. Hence, by equations (8), (9), (10) and (11), we have for all target combinations $(\mu_1, \mu_2) \in M_1 \times M_2$ that

$$\Omega_1(\mu_1, \mu_2) = \Omega_2(\mu_1, \mu_2) = 0 \ ,$$

and therefore, by Proposition 4.1,

$$Effect(\mu_1, \mu_2) = Effect(\mu_1) + Effect(\mu_2) \ .$$

□

Of particular interest is the situation in which both players 1 and 2 target a maximum-amplification target (MAT), so that $\mu_1 = \mu_2 = \mu^*$, where μ^* is a MAT. If the natural opinions admit both players to spend their full budgets by targeting this MAT, then (μ^*, μ^*) is a Nash equilibrium.

Theorem 4.2. Let $P = (N, W, x^0, T)$ be a network with finite-horizon opinion dynamics with players 1 and 2 with budget vector $\delta = (\delta_1, \delta_2)$ and let $\Gamma_{P,\delta}$ be the associated strategic influencing game. Let target $\mu_1^* = \mu_2^* = (\tau^*, j^*)$ be a MAT. If the natural opinion $x_{j^*}^{\tau^*}$ satisfies

$$\delta_2 \leq x_{j^*}^{\tau^*} \leq 1 - \delta_1 \ , \tag{13}$$

then (μ_1^*, μ_2^*) is a Nash equilibrium of $\Gamma_{P,\delta}$.

Proof. Assume that P is such that condition (13) is satisfied. We assume that player 2 chooses target μ_2^* and show that

$$Effect(\mu_1, \mu_2^*) \leq Effect(\mu_1^*, \mu_2^*) \quad \text{for all } \mu_1 \in M_1.$$

If player 1 chooses target μ_1^*, then he obtains the payoff

$$\begin{aligned} Effect(\mu_1^*, \mu_2^*) &= Effect(\mu_1^*) + \underbrace{\Omega_1(\mu_1^*, \mu_2^*)}_{=0} + Effect(\mu_2^*) + \underbrace{\Omega_2(\mu_1^*, \mu_2^*)}_{=0} \\ &= \delta_1 \cdot a_{j^*}^{\tau^*} - \delta_2 \cdot a_{j^*}^{\tau^*} \ . \end{aligned}$$

If player chooses any other target $\mu_1 \neq \mu_1^*$, then he obtains at most this payoff $Effect(\mu_1^*, \mu_2^*)$. For targets $\mu_1 = (\tau_1, j)$ with $\tau_1 \leq \tau^*$, or with $\tau_1 > \tau^*$ and $x_{j_1}^{\tau_1} \leq 1 - \delta_1$, we have by Theorem 3.1, Proposition 4.1 and Lemma 4.1 that

$$\begin{aligned} Effect(\mu_1, \mu_2^*) &= Effect(\mu_1) + \underbrace{\Omega_1(\mu_1, \mu_2^*)}_{=0} + Effect(\mu_2^*) + \underbrace{\Omega_2(\mu_1, \mu_2^*)}_{\leq 0} \\ &\leq \delta_1 \cdot a_{j^*}^{\tau^*} - \delta_2 \cdot a_{j^*}^{\tau^*} . \end{aligned}$$

For targets $\mu_1 = (\tau_1, j_1)$ with $\tau_1 > \tau^*$ and $x_{j_1}^{\tau_1} > 1 - \delta_1$, the one-sided effect and the interaction term of player 1 satisfy

$$\begin{aligned} Effect(\mu_1) &= (1 - x_{j_1}^{\tau_1}) \cdot a_{j_1}^{\tau_1} \\ &\leq (1 - x_{j_1}^{\tau_1}) \cdot a_{j^*}^{\tau^*} , \\ \Omega_1(\mu_1, \mu_2^*) &= \min\left\{ x_{j_1}^{\tau_1} + \delta_1 - 1,\ \min\{\delta_2, x_{j_2}^{\tau_2}\} \cdot W_{j_1,j_2}^{\tau_1 - \tau_2} \right\} \cdot a_{j_1}^{\tau_1} \\ &\leq (x_{j_1}^{\tau_1} + \delta_1 - 1) \cdot a_{j^*}^{\tau^*} , \end{aligned}$$

so that

$$\begin{aligned} Effect(\mu_1, \mu_2^*) &= Effect(\mu_1) + \Omega_1(\mu_1, \mu_2^*) + Effect(\mu_2^*) + \underbrace{\Omega_2(\mu_1, \mu_2^*)}_{=0} \\ &\leq (1 - x_{j_1}^{\tau_1}) \cdot a_{j^*}^{\tau^*} + (x_{j_1}^{\tau_1} + \delta_1 - 1) \cdot a_{j^*}^{\tau^*} - \delta_2 \cdot a_{j^*}^{\tau^*} \\ &= \delta_1 \cdot a_{j^*}^{\tau^*} - \delta_2 \cdot a_{j^*}^{\tau^*} . \end{aligned}$$

Similarly, it can be shown that

$$Effect(\mu_1^*, \mu_2^*) \leq Effect(\mu_1^*, \mu_2) \quad \text{for all } \mu_2 \in M_2.$$

Therefore, (μ_1^*, μ_2^*) is a pure Nash equilibrium of $\Gamma_{P,\delta}$. □

The following example illustrates Theorem 4.2.

Example 4.2. Reconsider the influencing game of Example 4.1. To find the MAT, we compute the amplification for all targets, provided in Table 3.

τ \ j	1	2	3
0	0.83	0.83	1.33
1	1	1	1

Table 3: The amplification of all targets $\mu = (\tau, j)$.

From the amplification values we conclude that the MAT of this example is given by $\mu^* = (\tau^*, j^*) = (0, 3)$ with $a_3^0 = 1.33$. The corresponding natural opininion is given by $x_3^0 = 0.6$ and the budget vector is given by $\delta = (\delta_1, \delta_2) = (0.3, 0.2)$. Hence, condition (13) of Theorem 4.2 is satisfied and the strategy combination (μ^*, μ^*) is a pure Nash equilibrium of $E_{P,\delta}$. Indeed, this can be verified with Example 4.1.

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If condition (13) of Theorem 4.2 is violated, then a (pure) equilibrium may not exist in the corresponding influencing game. This is illustrated in Example 4.3.

Example 4.3. Reconsider the same network with opinion dynamics $P = (N, W, x^0, T)$ as in Example 4.1 with players 1 and 2 with budget vector $\delta = (\delta_1, \delta_2) = (0.3, 0.2)$. We redefine the initial opinions as

$x^0 = (0.1, 0.6, 0.9)$. Then, the corresponding influencing game $\Gamma_{P,\delta}$ is given by the matrix

$$E_{P,\delta} = \begin{array}{c} (0,1) \\ (0,2) \\ (0,3) \\ (1,1) \\ (1,2) \\ (1,3) \end{array} \overset{\begin{array}{cccccc} (0,1) & (0,2) & (0,3) & (1,1) & (1,2) & (1,3) \end{array}}{\begin{bmatrix} 0.09 & 0.09 & -0.01 & 0.05 & 0.05 & 0.05 \\ 0.17 & 0.09 & -0.01 & 0.05 & 0.05 & 0.05 \\ 0.05 & -0.03 & 0.14 & -0.06 & -0.06 & -0.06 \\ 0.22 & 0.14 & 0.04 & 0.10 & 0.10 & 0.10 \\ 0.22 & 0.14 & 0.04 & 0.10 & 0.10 & 0.10 \\ 0.17 & 0.14 & 0.04 & 0.05 & 0.05 & 0.10 \end{bmatrix}}.$$

One can readily verify that this influencing game has no pure Nash equilibrium.

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5 Concluding Remarks

In this paper we study the strategic behaviour of opinion network influencers. Our starting point is the DeGroot (1974) dynamics where agents iteratively update their opinions as weighted combinations of other agents whom they trust. We trace the opinions through a finite horizon, during which one or two influencers target the opinion of one specific agent in one specific period. The influencers' capacity for impact is limited by a given budget, which they aim to spend in a manner that maximally affects the state of opinions at the end of the horizon. If we consider only one influencer, this induces an optimization problem, for which we characterize optimal targets for minimizing and maximizing influencers. In the case of two influencers, we provide conditions under which a special Nash equilibrium of the corresponding strategic influencing game exists, where both players target the time-agent combination where an opinion shift is maximally amplified up to the end of the horizon.

The model we propose most naturally resembles elections, in which political parties with conflicting opinions compete for dominance over the opinions held on a specific date: election day. It would be interesting to investigate empirically if political campaigners' strategies resemble those characterized by our results.

In this paper we have adopted the sum of opinions in the final period as an evaluation criterium. Alternatively, if one wished to model real-life elections more closely, one could adopt a discrete evaluation criterion of the state of opinions at the end of the time horizon. Then the agents of the social network, holding real-valued opinions between 0 and 1 throughout the time horizon, would convert their opinion into a discrete vote on election day. Undoubtedly, in such a model, the influencers would place more importance on the 'swing voters' that can, in principle, be attracted to vote for either side with relative ease.

Alternatively, one could loosen the assumption that influencers have to spend their entire budget on a single target, and instead undertake a multiple-target approach. If one allowed the influencers to divide their budgets among as many targets as they wish, then some of our results would be maintained. In particular, if the opinions in the network are such that the influencers' full budget can be spent on the maximum-amplification target, then doing so will dominate any multiple-target strategy. However, in other cases, optimization becomes far more complex, as is illustrated in Example 5.1.

Example 5.1. Consider the network with finite-horizon opinion dynamics $P = (N, W, x^0, T)$ with agents $N = \{1, 2, 3\}$, trust matrix

$$W = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{bmatrix},$$

initial opinions $x^0 = (0.1, 0.9, 0.6)$ and time horizon $T = 3$. Then, we have the natural opinions and amplification values as provided in Table 4 (rounded to three decimals).

Natural opinions			
t \ i	1	2	3
0	0.100	0.900	0.600
1	0.350	0.533	0.750
2	0.550	0.544	0.642
3	0.596	0.579	0.593

Amplification			
τ \ j	1	2	3
0	0.662	0.995	1.342
1	0.694	0.944	1.361
2	0.833	0.833	1.333
3	1.000	1.000	1.000

Table 4: The natural opinions and amplification values for all time-agent combinations.

We consider a maximizing influencer with budget $\delta = 0.5$ and allow her to divide the budget among multiple targets. In the spirit of our single-target characterization of optimal targets, it seems prudent to spend 0.25 of the budget on the maximum-amplification target (MAT) given by $\hat{\mu} = (1, 3)$, and the remaining 0.25 on the target $\mu^I = (0, 3)$ with the second highest amplification. Then, she would achieve the effect

$$Effect(\hat{\mu}_{(0.25)}, \mu^I_{(0.25)}) = 0.506,$$

where we specify the budget spent on targets in their respective subscript. However, she could notice that the shift of the target μ^I hampers the shift of the MAT $\hat{\mu}$ and thus decide to relocate the budget of 0.25 of target $\hat{\mu}$ to another target, for instance $\mu^{II} = (3, 1)$. If she spends 0.25 on μ^I and 0.25 on μ^{II}, she would achieve the higher effect

$$Effect(\mu^{II}_{(0.25)}, \mu^{III}_{(0.25)}) = 0.585.$$

We observe that allocating part of the budget to preceding targets may hamper the MAT's effectiveness.

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