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Duality in Financial Networks

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Abstract

This article introduces the concept of duality in financial networks. In bankruptcy problems, in which a bankrupt entity divides its non-negative assets among a group of claimants, duality of bankruptcy rules entails the division of losses versus gains. Financial networks generalize bankruptcy problems by allowing for multiple agents with individual assets interconnected by mutual claims. We show that allowing for negative assets is imperative to adequately formulate dual financial networks and dual bankruptcy problems. We show that there is a one-to-one correspondence between payment schemes based on bankruptcy rules in a financial network and payment schemes based on the dual of those bankruptcy rules in the dual financial network. Moreover, dual financial networks enable us to define dual transfer rules and dual allocation rules. We show that transfer rules based on self-dual bankruptcy rules need not necessarily be self-dual, whereas allocation rules based on self-dual bankruptcy rules are always self-dual.

Keywords: duality, bankruptcy problems, financial networks, transfer rules, allocation rules.

JEL Classification Number: D74, G10, G33.

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1 Introduction

There exists a large body of literature on the formal analysis of problems in which a non-negative amount is to be allocated among a finite set of claimants that each have a claim on the estate. Formally, the analysis restricts itself to the case in which the estate is insufficient to honor all claims. Problems of this kind are often referred to as *bankruptcy problems*.[1] Solutions to bankruptcy problems are formalized by *bankruptcy rules* which, for each bankruptcy problem, prescribe a division of the estate among the claimants. For an excellent survey of this literature, we refer the reader to Thomson (2019).

Bankruptcy problems and the associated bankruptcy rules form the basis for the financial networks that are the focus of this article. We follow the seminal article of Eisenberg and Noe (2001) in the sense that a financial network consists of a finite set of agents, e.g., financial institutions, in which each agent has an initial estate and in which agents may have mutual claims on each other.[2] Note that financial networks encompass the class of bankruptcy problems as a financial network may have exactly one agent that is in debt to the remaining agents. In Eisenberg and Noe (2001), agents pay in accordance with the proportional bankruptcy rule, i.e., payments are in proportion to the claims. We follow the convention that bankruptcy rules dictate the payments between the agents in the financial network, although we allow for general agent-specific bankruptcy rules (cf. Csóka and Herings (2018); Ketelaars and Borm (2021); Csóka and Herings (2023)).

The 2007-2008 financial crisis led to a surge in empirical and theoretical research on financial contagion and financial stability of which the framework of Eisenberg and Noe (2001) has been the foundation. The model of Eisenberg and Noe (2001) has been extended in various ways, e.g., by applying the model to cases where a shortfall of payments is not the root cause of contagion (Cifuentes, Ferrucci, & Shin, 2005), by allowing for different seniority of claims (Elsinger, 2009), or by allowing for bankruptcy costs (Rogers & Veraart, 2013). For excellent surveys on contagion in financial networks, we refer the reader to Glasserman and Young (2016), and Jackson and Pernoud (2021).

A well-studied notion in the context of bankruptcy rules is the notion of *duality*, which concerns the allocation of the amount that is missing instead of the amount that is available. More specifically, the dual of a bankruptcy rule prescribes that claimants first receive their full claim after which the excess amount, which is equal to the sum of claims minus the estate, is allocated in accordance with the given bankruptcy rule based on the same claims. Aumann and Maschler (1985) defines a bankruptcy rule to be *self-dual* if it coincides with its corresponding dual bankruptcy rule.

Interestingly, it is seen in this article that allowing agents to have a negative estate is imperative to adequately formulate dual bankruptcy problems and, more generally, dual financial networks. In fact, negative estates in this article are of a different nature than those arising in the literature on contagion in financial networks (see, among others, Elsinger, Lehar, and Summer (2006), Glasserman and Young (2016), and Demange (2023)), where the initial estate of an agent equals the difference between outside-network assets, e.g., stocks

[1] Alternatively, depending on the context, problems of this kind may also be called taxation problems (cf. Young (1988)), rationing problems (cf. Moulin (2000)), or claims problems (cf. Thomson (2019)).

[2] However, the model of Eisenberg and Noe (2001) is similar to the model of Elimam, Girgis, and Kotob (1996).

or loans, and outside-network liabilities, e.g., debts or deposits. If the outside liabilities exceed outside assets, the estate of an agent is negative, so a negative estate is interpreted as the amount that an agent is in debt to agents outside those modeled in the network. Hence, inside-network liabilities will be paid once outside-network liabilities are paid. In the current context of duality however, negative estates correspond to the amount used for own consumption before creditors in the network are paid.

Furthermore, as bilateral transfers between agents can take place in a financial network, an agent that initially does not have enough to pay off all its debts could be able to do so after receiving sufficiently high payments from the other agents. In other words, the funds an agent has at its disposal to pay off its creditors may exceed its total debts, which is outside the scope of bankruptcy problems. For this reason, we consider a larger class of bankruptcy problems, which we call *claims problems*, in which the estate may be negative or may exceed the sum of the claims. Correspondingly, bankruptcy rules are superseded by so-called *claims rules* which, for each claims problem, prescribe a division of the estate among the agents.

In this article, a financial network is represented by a *mutual claims problem* that is characterized by an estates vector, possibly with entries that are zero or negative, containing the individual assets of the agents and a claims matrix containing the non-negative mutual claims between the agents. We introduce to each mutual claims problem a *dual mutual claims problem* in which the claims matrix is the same as in the original mutual claims problem, but in which the *loss* of each agent, being equal to its liabilities to the other agents minus its estate and claims on the other agents, serves as the estate of this agent. For the special case of a claims problem, the estate in the associated dual claims problem is defined as the sum of the claims minus the estate and the claims are identical to the original claims. We use this notion of a dual mutual claims problem to analyze duality of the transfers between agents, which are formalized by transfer rules, and duality of the transfer allocation to the agents, which are formalized by allocation rules. The results of this analysis are described in more detail below.

The payments of agents are in accordance with claims rules. These rules are represented by a claims rules vector ϕ in which each component specifies the agent-specific claims rule an agent uses. The basis for our analysis will be *ϕ-transfer schemes* that contain consistent payments of agents in the sense that, for each agent, the payments to other agents follow from the allocation of its estate plus incoming payments in accordance with its claims rule. The set of ϕ-transfer schemes is shown to be a complete lattice. Therefore, there exists a bottom ϕ-transfer scheme and a top ϕ-transfer scheme. We show a one-to-one correspondence between transfer schemes in a mutual claims problem and in its corresponding dual problem. That is, a payment matrix is a ϕ-transfer scheme for a mutual claims problem if and only if the claims matrix minus this payment matrix is a ϕ^*-transfer scheme for the dual mutual claims problem, where the components of the vector of claims rules ϕ^* are the associated dual claims rules of the claims rules in ϕ. In fact, we show that the relationship is more explicit: the bottom (resp. top) ϕ-transfer scheme of a mutual claims problem is equal to the claims matrix minus the top (resp. bottom) ϕ^*-transfer scheme of the corresponding dual mutual claims problem.

So-called *ϕ-based transfer rules* prescribe, for each mutual claims problem, how payments between agents should take place by selecting exactly one ϕ-transfer scheme. As ϕ-transfer schemes need not be unique, there is a choice to be made. Given a ϕ-based transfer rule, its corresponding dual rule first settles all mutual claims, but as there may exist agents for which the sum of the estate and all outstanding claims on other agents falls short of paying off all debts, settling all mutual claims may be infeasible, so a loss has to be repaid in accordance with the given ϕ-based transfer rule. In other words, a dual ϕ-based transfer rule prescribes a payment matrix which is equal to the claims matrix minus the ϕ-transfer scheme prescribed by the ϕ-based transfer rule with respect to the dual mutual claims problem. We show that the dual of a ϕ-based transfer rule prescribes a ϕ^*-based transfer scheme and as such is a ϕ^*-based transfer rule.

So-called *ϕ-based allocation rules* prescribe, for each mutual claims problem, a reallocation of the total estate, where the allocation to each agent is equal to its initial estate plus its net payments in accordance with a ϕ-transfer scheme. In contrast to ϕ-based transfer rules for which a choice among the set of ϕ-transfer schemes has to be made, a ϕ-based allocation rule prescribes a unique allocation. In other words, any choice of ϕ-transfer scheme, as prescribed by a ϕ-based transfer rule, results in the same ϕ-based transfer allocation. We show that, given a ϕ-based allocation rule, the corresponding dual rule prescribes a reallocation of the total estate on the basis of a ϕ^*-transfer scheme. Therefore, the dual of a ϕ-based allocation rule is a ϕ^*-based allocation rule.

Finally, we analyze *self-duality* of ϕ-based transfer rules and ϕ-based allocation rules in terms of the payment matrices they prescribe and the allocations they prescribe, respectively. We define a ϕ-based transfer rule and a ϕ-based allocation rule to be self-dual if they coincide with their corresponding dual rule. We show that self-duality of claims rules in ϕ carries over to self-duality of ϕ-based allocation rules, and vice versa. However, one should tread carefully with respect to ϕ-based transfer rules because self-duality of such rules is a stronger requirement. Although self-duality of a ϕ-based transfer rule implies self-duality of the associated ϕ-based allocation rule, the reverse statement need not necessarily hold true. In particular, we show that self-duality of a ϕ-based transfer rule is not guaranteed if all claims rules in ϕ are self-dual. As ϕ-transfer schemes need not be unique, one should choose ϕ-transfer schemes in an appropriate way as to make a ϕ-based transfer rule self-dual.

The article is organized as follows. Section 2 discusses duality in claims problems. Section 3 introduces duality in mutual claims problems and shows its implications for ϕ-transfer schemes. It also contains the leading example that is used throughout the article. Section 4 introduces ϕ-based transfer rules and their dual rules. Section 5 introduces ϕ-based allocation rules and their dual rules. Finally, Section 6 studies self-duality.

2 Duality in claims problems

A *claims problem* is a pair $(e, c) \in \mathbb{R} \times \mathbb{R}_+^M$ in which M is a finite set of *claimants*, e is a, possibly negative, *estate*, and $c = (c_i)_{i \in M}$ is a vector of rightful non-negative claims on the estate. The class of all claims problems on M is denoted by $\mathcal{C}^M$. Claims problems on M in which the estate is non-negative and the sum of claims exceeds the value of the non-negative estate, i.e., $\sum_{i \in M} c_i > e$, are called *bankruptcy problems*.

Bankruptcy law prescribes that some assets may not be included in the bankruptcy for the purpose of meeting basic domestic needs or sustaining business operations. A bankrupt entity therefore pays claimants only after its basic domestic needs are met or after its business operations are sustained. A claims problem $(e, c) \in \mathcal{C}^M$ with a negative estate, $e < 0$, can thus be thought of as a problem in which claimants get paid only once $-e$ has been honored to the bankrupt entity. As will be seen later, this interpretation allows for a natural approach to duality in claims problems.

Accommodating for both negative estates and for estates exceeding the sum of the claims, we define *claims rules* as a generalization of bankruptcy rules in the following way.

Definition 2.1. A *claims rule* $\varphi\colon \mathcal{C}^M \to \mathbb{R}^M$ prescribes, for all $(e, c) \in \mathcal{C}^M$, an allocation vector $\varphi(e, c)$ that satisfies

(i) $0 \leq \varphi_i(e, c) \leq c_i$ for all $i \in M$,

(ii) $\displaystyle\sum_{i \in M} \varphi_i(e, c) = \min\{\max\{0, e\}, \sum_{i \in M} c_i\}$.

Condition (ii) boils down to $\sum_{i \in M} \varphi_i(e, c) = e$ in a bankruptcy problem (e, c). If the estate is negative, conditions (i) and (ii) imply $\varphi(e, c) = 0$. On the other hand, if the estate can cover all claims, conditions (i) and (ii) imply $\varphi(e, c) = c$.

From the outset, we assume that claims rules satisfy estate monotonicity. A claims rule φ satisfies *estate monotonicity* if, for all $(e, c) \in \mathcal{C}^M$ and $(e', c) \in \mathcal{C}^M$ with $e \leq e'$, it holds that $\varphi(e, c) \leq \varphi(e', c)$.[3] It is well known that a bankruptcy rule is continuous in the estate if it satisfies estate monotonicity, see, e.g., Thomson (2019). The extension of this implication to the context of claims rules is straightforward.

In the examples in this article, we consider one claims rule in particular, namely the *proportional rule*. The proportional rule prescribes a proportional division of the estate with respect to the proportion of a claimant's claim to the total claims. The proportional rule satisfies estate monotonicity.

Definition 2.2. The *proportional rule PROP* is, for all $(e, c) \in \mathcal{C}^M$, and all $i \in M$, defined by

$$\mathrm{PROP}_i(e, c) = \begin{cases} 0 & \text{if } e < 0, \\ \dfrac{c_i}{\sum_{j \in M} c_j} e & \text{if } 0 \leq e < \displaystyle\sum_{j \in M} c_j, \\ c_i & \text{if } e \geq \displaystyle\sum_{j \in M} c_j. \end{cases}$$

To each claims problem corresponds a *dual claims problem* in which the amount to be divided corresponds to an excess amount, or loss, i.e., the amount by which the total amount of claims exceeds the estate. So, in particular, if the estate in a claims problem is negative, the estate in the dual claims problem exceeds the sum of claims. And, if the estate in a claims problem exceeds the sum of claims, the estate in the dual claims problem is negative.

[3]Note that the inequality here is a vector inequality, i.e., for two vectors $x, y \in \mathbb{R}^M$ with M being a finite set, we have $x \leq y$ if and only if $x_i \leq y_i$ for all $i \in M$. We have $x < y$ if and only if $x \leq y$ and $x_i < y_i$ for at least one $i \in M$.

Definition 2.3. Let $(e, c) \in \mathcal{C}^M$. The *dual claims problem* of (e, c) is given by $(\sum_{i \in M} c_i - e, c)$.

Similarly, to each claims rule φ corresponds a *dual claims rule* that first allocates to all claimants their full claim; however, as this may be infeasible, claimants subsequently need to return the excess amount according to the claims rule φ, which now describes how the losses are allocated.

Definition 2.4. Given a claims rule φ, its *dual claims rule* φ^* is, for all $(e, c) \in \mathcal{C}^M$, given by

$$\varphi^*(e, c) = c - \varphi(\sum_{i \in M} c_i - e, c).$$

It follows that $(\varphi^*)^* = \varphi$ and that φ^* is indeed a claims rule.[4] The existing literature on duality in claims problems, which does not allow for a negative estate or for an estate to be strictly larger than the sum of claims, restricts duality of claims rules to the class of bankruptcy problems (cf. Thomson (2019)).

Finally, note that a dual rule prescribes that each claimant receives its full claim if the estate is strictly larger than the sum of claims. If this happens, all debts are paid which leaves a positive leftover estate equal to $e - \sum_{i \in M} c_i$. Hence, in the dual claims problem a value of $e - \sum_{i \in M} c_i$ is the rightful amount to be used for own consumption before the claimants are paid. A dual argument applies in case the estate is negative. In that case, the dual rule prescribes zero to all claimants which leaves a negative leftover estate such that the estate in the dual problem exceeds the sum of claims. Consequently, all claimants get paid their full claim in the dual problem.

3 Duality in mutual claims problems

Mutual claims problems represent financial networks and generalize claims problems by allowing for multiple estates and mutual claims. A mutual claims problem is a pair $(E, C) \in \mathbb{R}^N \times \mathbb{R}_+^{N \times N}$ in which N is a finite set of agents, $E = (e_i)_{i \in N}$ is an *estates vector*, and $C = (c_{ij})_{i,j \in N}$ is a *claims matrix*. Each coordinate e_i of E represents the, possibly negative, estate corresponding to agent $i \in N$. The claims matrix C represents mutual liabilities between agents. Each cell c_{ij} of C represents the rightful non-negative claim of agent $j \in N$ on agent $i \in N$. Row i in C thus captures creditors of agent i, whereas column i of C captures debtors of agent i. We assume that agents have no claim on themselves, i.e., $c_{ii} = 0$ for all $i \in N$. No additional conditions are imposed on the claims matrix, in particular, there is no condition on the relation between claims c_{ij} and c_{ji} for $i \neq j$. The class of all mutual claims problems on N is denoted by $\mathcal{L}^N$.

As we will see, negative estates turn out to be imperative to formulate duality for mutual claims problems. From a duality perspective, a negative estate corresponds to the amount used for own consumption before claimants are paid. Interestingly, an agent that has a

[4] We do not explicitly prove these two statements here because they will be proved later on in the more general setting of mutual claims problems, see Proposition 4.3 and Proposition 4.4, respectively.

negative estate initially, can still make payments to other agents if its incoming payments exceed the value of its negative estate.[5]

To each mutual claims problem corresponds a *dual mutual claims problem*. The *loss* of an agent, being equal to its liabilities to the other agents minus its estate and claims on the other agents, serves as the estate of this agent in a dual mutual claims problem. Agents for which the loss is positive are essentially bankrupt as they are never able to pay off all debts, even if they receive their full claims on the other agents.

Definition 3.1. Let $(E, C) \in \mathcal{L}^N$. The vector $\ell(E, C) \in \mathbb{R}^N$ of *losses* with respect to (E, C) is, for all $i \in N$, given by

$$\ell_i(E, C) = \sum_{j \in N} c_{ij} - e_i - \sum_{j \in N} c_{ji}.$$

The *dual mutual claims problem* of (E, C) is then given by $(\ell(E, C), C)$.

The definition of the loss of an agent is in accordance with the claims problem setting where the excess amount, i.e., the loss, is equal to the sum of claims minus the estate. It is more general here because we further subtract all the claims on other agents.

Negative losses, and thus negative estates in a dual mutual claims problem, are naturally occurring. If the value of the assets in a financial network is positive, then the value of the losses is negative, which is why at least one agent has a negative loss. Formally, if $(E, C) \in \mathcal{L}^N$ is such that $\sum_{i \in N} e_i > 0$, then $\sum_{i \in N} \ell_i(E, C) < 0$, which implies that $\ell_i(E, C) < 0$ for at least one agent $i \in N$ in the dual mutual claims problem $(\ell(E, C), C)$.

The following example illustrates duality in mutual claims problems. In fact, the mutual claims problem in the following example is the leading mutual claims problem throughout this article.

Example 3.1. Consider the mutual claims problem $(E, C) \in \mathcal{L}^N$ given by $N = \{1, 2, 3, 4\}$,

$$E = \begin{pmatrix} 1 \\ -3 \\ 3 \\ 0 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}.$$

The losses of the agents are given by

$$\ell(E, C) = (4, 2, 2, 3) - (1, -3, 3, 0) - (5, 4, 0, 2) = (-2, 1, -1, 1).$$

Hence, the corresponding dual mutual claims problem $(\ell(E, C), C)$ is given by

$$\ell(E, C) = \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}.$$

△

[5] As negative estates are intrinsic to the agents, we refrain from making them artificially non-negative by introducing an outside sector and replacing a negative estate by a liability towards the outside sector. In fact, as we illustrate in Appendix A, even if one were to make estates non-negative in a way that is common in the literature, one runs into problems that make duality unnecessarily complicated.

The following proposition states that, for each agent, the 'loss of its loss' equals its initial estate. Hence, for each mutual claims problem $(E, C) \in \mathcal{L}^N$, the dual of the dual mutual claims problem, $(\ell(\ell(E, C), C), C)$, is equal to (E, C).

Proposition 3.2. *Let $(E, C) \in \mathcal{L}^N$. Then, $\ell(\ell(E, C), C) = E$.*

Proof. Let $i \in N$. Then,

$$\begin{aligned}
\ell_i(\ell(E, C), C) &= \sum_{j \in N} c_{ij} - \ell_i(E, C) - \sum_{j \in N} c_{ji} \\
&= \sum_{j \in N} c_{ij} - \sum_{j \in N} c_{ij} + e_i + \sum_{j \in N} c_{ji} - \sum_{j \in N} c_{ji} \\
&= e_i.
\end{aligned}$$

□

Mutual claims problems and claims problems are related in the following way. For instance, if $(e, c) \in \mathcal{C}^M$ is a claims problem with $M = \{1, 2, \ldots, m\}$, then (e, c) can be associated with any mutual claims problem $(E, C) \in \mathcal{L}^{\{0\} \cup M}$ given by

$$E = \begin{pmatrix} e \\ e_1 \\ \vdots \\ e_m \end{pmatrix}, \ C = \begin{bmatrix} 0 & c_1 & \ldots & c_m \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \tag{3.1}$$

where, for each $i \in M$, e_i is a real number that corresponds to the estate of agent $i \in M$. Moreover, the dual mutual claims problem $(\ell(E, C), C)$ of the mutual claims problem (E, C) given in (3.1), is given by

$$\ell(E, C) = \begin{pmatrix} \sum_{i \in M} c_i - e \\ c_1 - e_1 \\ \vdots \\ c_m - e_m \end{pmatrix}, \ C = \begin{bmatrix} 0 & c_1 & \ldots & c_m \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix},$$

which is a mutual claims problem associated with the dual claims problem $(\sum_{i \in M} c_i - e, c)$ of (e, c).

Let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules in which $\varphi^i \colon \mathcal{C}^{N \setminus \{i\}} \to \mathbb{R}^{N \setminus \{i\}}$ is the claims rule associated with agent $i \in N$ that prescribes how each creditor of agent i is to be paid.[6] We denote the vector of claims on agent $i \in N$ by $\bar{c}_i = (c_{ij})_{j \in N \setminus \{i\}} \in \mathbb{R}_+^{N \setminus \{i\}}$. A ϕ-transfer scheme is a payment matrix $P = (p_{ij})_{i,j \in N}$, where, for $i \neq j$, element p_{ij} denotes the payment from agent i to agent j, in which transfers between agents are dictated by claims rules in ϕ with an additional consistency requirement. Moreover, as agents have no claim on themselves, they pay nothing to themselves in a ϕ-transfer scheme.

[6]The existing literature on financial networks that allows for agent-specific claims rules defines each claims rule on $\mathcal{C}^N$. Since agent $i \in N$ has no claim on itself, we define the claims rule of agent i on $\mathcal{C}^{N \setminus \{i\}}$ instead. Moreover, for self-duality in Section 6, it will be necessary that, for each $i \in N$, a claims rule is defined on $\mathcal{C}^{N \setminus \{i\}}$.

Definition 3.3. Let $(E, C) \in \mathcal{L}^N$, and let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules. The payment matrix $P = (p_{ij})_{i,j \in N}$ is a *ϕ-transfer scheme* for (E, C) if, for all $i \in N$, $p_{ii} = 0$ and, for all $j \in N \setminus \{i\}$,

$$p_{ij} = \varphi^i_j(e_i + \sum_{k \in N} p_{ki}, \bar{c}_i). \tag{3.2}$$

The set of all possible ϕ-transfer schemes for (E, C) is denoted by $\mathcal{P}^\phi(E, C)$.

The consistency requirement on the transfers between agents implies that, for each agent, its payment to any other agent follows from allocating its initial estate plus incoming payments in accordance with its claims rule.

The following example provides a PROP-transfer scheme for the leading mutual claims problem.

Example 3.2. Reconsider the mutual claims problem $(E, C) \in \mathcal{L}^N$ of Example 3.1 given by $N = \{1, 2, 3, 4\}$,

$$E = \begin{pmatrix} 1 \\ -3 \\ 3 \\ 0 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}.$$

Let $\phi = (\text{PROP}, \text{PROP}, \text{PROP}, \text{PROP}) \equiv \text{PROP}$. The payment matrix

$$P = \begin{bmatrix} 0 & 1\frac{1}{2} & 0 & 1\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

is a PROP-transfer scheme for (E, C) because, for all $i \in N$, $p_{ii} = 0$, and

$$\begin{aligned} (1\tfrac{1}{2}, 0, 1\tfrac{1}{2}) &= \text{PROP}(1 + 1 + 1, (2, 0, 2)), && \text{(Agent 1)} \\ (0, 0, 0) &= \text{PROP}(-3 + 1\tfrac{1}{2} + 1 + \tfrac{1}{2}, (2, 0, 0)), && \text{(Agent 2)} \\ (1, 1, 0) &= \text{PROP}(3 + 0, (1, 1, 0)), && \text{(Agent 3)} \\ \text{and } (1, \tfrac{1}{2}, 0) &= \text{PROP}(0 + 1\tfrac{1}{2}, (2, 1, 0)). && \text{(Agent 4)} \end{aligned}$$

$\triangle$

In general, we will see that existence of ϕ-transfer schemes for $(E, C) \in \mathcal{L}^N$ follows from Tarski's fixed-point theorem (Tarski, 1955) applied to the mapping $f^\phi(\cdot\,; E, C)\colon [0^{N \times N}, C] \to [0^{N \times N}, C]$ which is defined by setting, for all $i \in N$, $f^\phi_{ii}(P; E, C) = 0$ and, for all $j \in N \setminus \{i\}$,

$$f^\phi_{ij}(P; E, C) = \varphi^i_j(e_i + \sum_{k \in N} p_{ki}, \bar{c}_i) \tag{3.3}$$

for all $P \in [0^{N\times N}, C]$.[7,8] Indeed, $P = f^\phi(P; E, C)$ implies that $P \in \mathcal{P}^\phi(E, C)$, and vice versa. The mapping f^ϕ is defined on $[0^{N\times N}, C]$, which is a *complete lattice* with respect to $\leq$.[9] A *lattice* is a partially ordered set in which every pair of elements has a greatest lower bound (bottom) and a least upper bound (top) within the lattice, whereas a complete lattice is a lattice in which also every non-empty subset has a bottom and a top within the lattice. The following proposition on the existence of ϕ-transfer schemes generalizes the existing literature as estates may be negative.

Proposition 3.4. *Let $(E, C) \in \mathcal{L}^N$, and let $\phi = (\varphi^i)_{i\in N}$ be a vector of claims rules. Then, the set of ϕ-transfer schemes $\mathcal{P}^\phi(E, C)$ is non-empty. Moreover, there exists a bottom ϕ-transfer scheme $\underline{P}^\phi(E, C)$ such that, for all $P \in \mathcal{P}^\phi(E, C)$, $\underline{P}^\phi(E, C) \leq P$, and a top ϕ-transfer scheme $\overline{P}^\phi(E, C)$ such that, for all $P \in \mathcal{P}^\phi(E, C)$, $\overline{P}^\phi(E, C) \geq P$.*

Proof. To apply Tarski's fixed-point theorem, we will show that the mapping f^ϕ is monotone. Let $P, P' \in [0^{N\times N}, C]$ with $P \leq P'$. Let $i \in N$. Then, $f^\phi_{ii}(P; E, C) = 0 = f^\phi_{ii}(P'; E, C)$, and, for all $j \in N \setminus \{i\}$,

$$f^\phi_{ij}(P; E, C) = \varphi^i_j(e_i + \sum_{k\in N} p_{ki}, \bar{c}_i) \leq \varphi^i_j(e_i + \sum_{k\in N} p'_{ki}, \bar{c}_i) = f^\phi_{ij}(P'; E, C).$$

The inequality follows from estate monotonicity of φ^i for all $i \in N$. Tarski's fixed-point theorem (Tarski, 1955) consequently implies that the set of fixed points of f^ϕ, given by $\mathcal{P}^\phi(E, C)$, is non-empty and a complete lattice with respect to $\leq$. □

Proposition 3.4 extends existing results in the literature. Eisenberg and Noe (2001) shows Proposition 3.4 restricted to proportional claims rules and non-negative estates, whereas Elsinger (2009) shows the result restricted to proportional claims rules when estates may also be negative. Csóka and Herings (2023) shows Proposition 3.4 for arbitrary agent-specific claims rules and non-negative estates. All these articles also use Tarski's fixed-point theorem for the proof. Alternatively, Groote Schaarsberg, Reijnierse, and Borm (2018) uses a constructive proof to show existence of ϕ-transfer schemes when all agents use the same arbitrary claims rule and estates are non-negative. Ketelaars and Borm (2021) extends this constructive proof to arbitrary agent-specific claims rules and explicitly characterize the bottom ϕ-transfer scheme. Moreover, Csóka and Herings (2018) shows Proposition 3.4 for a discrete setup, but not allowing for negative estates.

There exists a one-to-one correspondence between ϕ-transfer schemes in a mutual claims problem and in its corresponding dual mutual claims problem. If the matrix P is a ϕ-transfer scheme for $(E, C) \in \mathcal{L}^N$, with $\phi = (\varphi^i)_{i\in N}$ a vector of claims rules, then the matrix $C - P$ is a ϕ^*-transfer scheme for the dual problem $(\ell(E, C), C)$, and vice versa. Here, $\phi^* = ((\varphi^i)^*)_{i\in N}$ is the vector of dual claims rules in which $(\varphi^i)^*$ is the dual claims rule of φ^i as defined in Definition 2.4.

[7] Here, $[0^{N\times N}, C] = \{P \in \mathbb{R}^{N\times N} \mid \text{for all } i, j \in N,\ 0 \leq p_{ij} \leq c_{ij}\}$.

[8] In the notation, the mapping f^ϕ explicitly incorporates (E, C) as we will use f^ϕ with respect to $(\ell(E, C), C)$ as well.

[9] The partial order $\leq$ is reflexive, antisymmetric, and transitive.

Theorem 3.5. *Let $(E, C) \in \mathcal{L}^N$, and let $\phi = (\varphi^i)_{i\in N}$ be a vector of claims rules with associated dual claims rule vector $\phi^* = ((\varphi^i)^*)_{i\in N}$. Then, $P \in \mathcal{P}^\phi(E, C)$ if and only if $(C - P) \in \mathcal{P}^{\phi^*}(\ell(E, C), C)$.*

Proof. Using Proposition 3.2, it readily follows that it suffices to show that $(C - P) \in \mathcal{P}^{\phi^*}(\ell(E, C), C)$ if $P \in \mathcal{P}^\phi(E, C)$.

Let $P = (p_{ij})_{i,j\in N} \in \mathcal{P}^\phi(E, C)$. Let $i \in N$. Then, $c_{ii} - p_{ii} = 0$, and, for all $j \in N \setminus \{i\}$,

$$\begin{aligned} c_{ij} - p_{ij} &= c_{ij} - \varphi^i_j(e_i + \sum_{k\in N} p_{ki}, \bar{c}_i) \\ &= (\varphi^i)^*_j(\sum_{k\in N} c_{ik} - e_i - \sum_{k\in N} p_{ki}, \bar{c}_i) \\ &= (\varphi^i)^*_j(\sum_{k\in N} c_{ik} - \sum_{k\in N} c_{ki} - e_i + \sum_{k\in N} c_{ki} - \sum_{k\in N} p_{ki}, \bar{c}_i) \\ &= (\varphi^i)^*_j(\ell(E, C) + \sum_{k\in N}(c_{ki} - p_{ki}), \bar{c}_i), \end{aligned}$$

where the first equality follows from $P \in \mathcal{P}^\phi(E, C)$, and the second equality follows from duality of φ^i. □

In other words, Theorem 3.5 states that, if we have a ϕ-transfer scheme with respect to what is available, then we also have a corresponding ϕ^*-transfer scheme with respect to what is missing, and vice versa.

If all agents have a strictly negative estate, then the set of ϕ-transfer schemes comprises only of the zero matrix; conversely, if all agents have a strictly negative loss, then the set of ϕ-transfer schemes comprises only of the claims matrix. In the latter, a strictly negative loss means that the estate plus all claims on other agents is strictly larger than the liabilities. These observations are formally stated in the following proposition.

Proposition 3.6. *Let $(E, C) \in \mathcal{L}^N$, and let $\phi = (\varphi^i)_{i\in N}$ be a vector of claims rules. Then,*

(i) if, for all $i \in N$, $e_i < 0$, then $\mathcal{P}^\phi(E, C) = \{0^{N\times N}\}$;

(ii) if, for all $i \in N$, $\ell_i(E, C) < 0$, then $\mathcal{P}^\phi(E, C) = \{C\}$.

Proof. Let $P = (p_{ij})_{i,j\in N}$ be a ϕ-transfer scheme for (E, C). First, we explicitly show (i). Subsequently, we will use a duality argument to show (ii).

(i). Let $e_i < 0$ for all $i \in N$. Let

$$S = \{i \in N \,|\, \text{there is a } k \in N \text{ such that } p_{ik} > 0\}.$$

We first prove that, for all $i \in S$, it holds that

$$e_i \geq \sum_{j\in N} p_{ij} - \sum_{j\in N} p_{ji}. \tag{3.4}$$

To this end, let $i \in S$ and $k \in N$ be such that $p_{ik} > 0$. Then, since

$$\sum_{j \in N} p_{ij} \geq p_{ik} > 0,$$

and, by condition (ii) of claims rule φ^i,

$$\sum_{j \in N} p_{ij} = \min\{\max\{0, e_i + \sum_{j \in N} p_{ji}\}, \sum_{j \in N} c_{ij}\} \leq \max\{0, e_i + \sum_{j \in N} p_{ji}\},$$

it follows that

$$\sum_{j \in N} p_{ij} \leq e_i + \sum_{j \in N} p_{ji}.$$

Next, we show that either $S = \emptyset$ or $S = N$. Let $S \neq \emptyset$ and suppose that $S \neq N$. Then, since

$$\sum_{i \in S} e_i < 0,$$

and, by (3.4),

$$\sum_{i \in S} e_i \geq \sum_{i \in S} \sum_{j \in N} p_{ij} - \sum_{i \in S} \sum_{j \in N} p_{ji} = \sum_{i \in S} \sum_{j \in N \setminus S} p_{ij} - \sum_{i \in S} \sum_{j \in N \setminus S} p_{ji},$$

it would follow that

$$0 = \sum_{i \in S} \sum_{j \in N \setminus S} p_{ji} > \sum_{i \in S} \sum_{j \in N \setminus S} p_{ij} \geq 0.$$

However, if $S = N$, then we arrive at a contradiction as well because (3.4) would imply that

$$0 > \sum_{i \in N} e_i \geq \sum_{i \in N} \sum_{j \in N} p_{ij} - \sum_{i \in N} \sum_{j \in N} p_{ji} = 0,$$

Hence, we can conclude that $S = \emptyset$, so $p_{ij} = 0$ for all $i, j \in N$.

(ii). Let $\ell_i(E, C) < 0$ for all $i \in N$, and let ϕ^* be the dual claims rule vector associated with ϕ. Then, as we have shown in part (i), $\mathcal{P}^{\phi^*}(\ell(E, C), C) = \{0^{N \times N}\}$, so Theorem 3.5 implies that $\mathcal{P}^{\phi}(E, C) = \{C\}$. □

Note, however, that Proposition 3.6 cannot be extended to situations where estates are zero or losses are zero.[10]

[10]For example, consider the three-agent mutual claims problem (E, C) given by

$$E = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, for all $i \in N$, $e_i = 0$ and $\ell_i(E, C) = 0$. In this case, for any ϕ, $\mathcal{P}^{\phi}(E, C) = \{\lambda C \mid \lambda \in [0, 1]\}$.

The following theorem shows that there is a natural relation between the bottom (resp. top) ϕ-transfer scheme of a mutual claims problem and the top (resp. bottom) ϕ^*-transfer scheme of the corresponding dual mutual claims problem. That is, the bottom (resp. top) ϕ-transfer scheme based on the estates is equal to the claims matrix minus the top (resp. bottom) ϕ^*-transfer scheme based on the losses.

Theorem 3.7. *Let $(E, C) \in \mathcal{L}^N$, and let ϕ be a vector of claims rules with associated dual claims rules vector ϕ^*. Then,*

(i) $\underline{P}^{\phi}(E, C) = C - \overline{P}^{\phi^*}(\ell(E, C), C)$;

(ii) $\overline{P}^{\phi}(E, C) = C - \underline{P}^{\phi^*}(\ell(E, C), C)$.

Proof. First, we show that $C - \overline{P}^{\phi^*}(\ell(E, C), C)$ is the bottom ϕ-transfer scheme for (E, C). For all $P \in \mathcal{P}^{\phi^*}(\ell(E, C), C)$, it holds that $P \leq \overline{P}^{\phi^*}(\ell(E, C), C)$, and consequently that $C - \overline{P}^{\phi^*}(\ell(E, C), C) \leq C - P$. From Theorem 3.5 it follows that $(C - P) \in \mathcal{P}^{\phi}(E, C)$ if and only if $P \in \mathcal{P}^{\phi^*}(\ell(E, C), C)$. Therefore, it follows that $\underline{P}^{\phi}(E, C) = C - \overline{P}^{\phi^*}(\ell(E, C), C)$.

Second, by applying the same arguments to $(\ell(E, C), C)$ with respect to ϕ^*, we obtain that $\underline{P}^{\phi^*}(\ell(E, C), C) = C - \overline{P}^{(\phi^*)^*}(\ell(\ell(E, C), C), C) = C - \overline{P}^{\phi}(E, C)$. □

It is well known that ϕ-transfer schemes for a mutual claims problem need not necessarily be unique, so there may exist agents that pay differently under the bottom ϕ-transfer scheme and the top ϕ-transfer scheme.

Definition 3.8. Let $(E, C) \in \mathcal{L}^N$, and let ϕ be a vector of claims rules. Let $\underline{P}^{\phi}(E, C) = (\underline{p}^{\phi}_{ij})_{i,j \in N}$ and $\overline{P}^{\phi}(E, C) = (\overline{p}^{\phi}_{ij})_{i,j \in N}$. Then, the set $\mathcal{I}^{\phi}(E, C) \subseteq N$ is defined by

$$\mathcal{I}^{\phi}(E, C) = \{i \in N \,|\, \underline{p}^{\phi}_{ij} < \overline{p}^{\phi}_{ij} \text{ for some } j \in N\}.$$

From Theorem 3.7 it directly follows that ϕ-transfer schemes for a mutual claims problem are unique if and only if ϕ^*-transfer schemes for the corresponding dual mutual claims problem are unique. In fact, the set of agents for which payments are not uniquely determined in a mutual claims problem with respect to ϕ coincides with the set of agents for which payments are not uniquely determined in the corresponding dual mutual claims problem with respect to ϕ^*.

Corollary 3.9. *Let $(E, C) \in \mathcal{L}^N$, and let ϕ be a vector of claims rules with associated dual claims rules vector ϕ^*. Then, $\mathcal{I}^{\phi}(E, C) = \mathcal{I}^{\phi^*}(\ell(E, C), C)$.*

Eisenberg and Noe (2001) introduces the *fictitious default algorithm* that computes the top PROP-transfer scheme for $(E, C) \in \mathcal{L}^N$ efficiently in at most $|N|$ iterations. However, the algorithm may fall apart in the case that some agents have a negative estate, see, e.g., Example 2 in Elsinger (2009). In the discrete setup, also for the case with non-negative estates, Csóka and Herings (2018) introduces a decentralized clearing process that converges to the bottom ϕ-transfer scheme in a finite number of iterations.

In general, the bottom (resp. top) ϕ-transfer scheme for $(E, C) \in \mathcal{L}^N$ with respect to ϕ can be characterized by a, possibly infinite, iterative procedure with respect to the mapping

f^ϕ as defined in (3.3). More specifically, let $(E, C) \in \mathcal{L}^N$, let ϕ be a vector of claims rules, and, for all $k \in \mathbb{N}$, define $P^{k+1} = f^\phi(P^k; E, C)$. Ketelaars and Borm (2021) shows that, in the case all estates are non-negative, the bottom ϕ-transfer scheme is the limit of the sequence $(P^k)_{k\in\mathbb{N}}$ with starting point $P^1 = 0^{N\times N}$. In the following theorem, we generalize this result by allowing for negative estates. Moreover, as the following theorem states, a characterization of the top ϕ-transfer scheme follows when the starting point P^1 of the iterative procedure is the claims matrix C. Our proof for the characterization of the top ϕ-transfer scheme is based on a duality argument.

Theorem 3.10. *Let $(E, C) \in \mathcal{L}^N$, and let $\phi = (\varphi^i)_{i\in N}$ be a vector of claims rules. Then,*

(i) $\underline{P}^\phi(E, C) = \lim_{k\to\infty} P^k$, where, for all $k \in \mathbb{N}$, $P^{k+1} = f^\phi(P^k; E, C)$ with $P^1 = 0^{N\times N}$;

(ii) $\overline{P}^\phi(E, C) = \lim_{k\to\infty} P^k$, where, for all $k \in \mathbb{N}$, $P^{k+1} = f^\phi(P^k; E, C)$ with $P^1 = C$.

Proof. (i). Let the sequence $(P^k)_{k\in\mathbb{N}}$ be such that, for all $k \in \mathbb{N}$, $P^{k+1} = f^\phi(P^k; E, C)$ with $P^1 = 0^{N\times N}$. We will show that the sequence $(P^k)_{k\in\mathbb{N}}$ is monotonically increasing, i.e., $P^1 \leq P^2 \leq P^3 \leq \ldots$, and converges to $\underline{P}^\phi(E, C)$.

Clearly, $0^{N\times N} = P^1 \leq f^\phi(P^1; E, C) = P^2$. Let $k \in \mathbb{N}$. Assume that $P^\ell \leq P^{\ell+1}$ for all $\ell \in \{1, \ldots, k\}$. Then, as the mapping f^ϕ is monotone, which was shown in the proof of Proposition 3.4, $P^{k+1} = f^\phi(P^k; E, C) \leq f^\phi(P^{k+1}; E, C) = P^{k+2}$. Therefore, by induction, the sequence $(P^k)_{k\in\mathbb{N}}$ is monotonically increasing. As the sequence is bounded from above by C, the monotone convergence theorem for sequences implies that it has a limit. Let $P' = \lim_{k\to\infty} P^k$. Then, $P' = \lim_{k\to\infty} P^k = \lim_{k\to\infty} f^\phi(P^k; E, C) = f^\phi(\lim_{k\to\infty} P^k; E, C) = f^\phi(P'; E, C)$, in which the third equality follows from the fact that, for all $i \in N$, φ^i is continuous in the estate, so that f^ϕ is continuous. Hence, it holds that $P' \in \mathcal{P}^\phi(E, C)$.

We will now show that $P' = \underline{P}^\phi(E, C)$. Because $0^{N\times N} = P^1 \leq \underline{P}^\phi(E, C)$, it holds that $P^2 = f^\phi(P^1; E, C) \leq f^\phi(\underline{P}^\phi(E, C); E, C) = \underline{P}^\phi(E, C)$, in which the last equality follows from the fact that $\underline{P}^\phi(E, C) \in \mathcal{P}^\phi(E, C)$. Let $k \in \mathbb{N}$. Assume that $P^{\ell+1} \leq \underline{P}^\phi(E, C)$ for all $\ell \in \{1, \ldots, k\}$. Therefore, $P^{k+2} = f^\phi(P^{k+1}; E, C) \leq f^\phi(\underline{P}^\phi(E, C); E, C) = \underline{P}^\phi(E, C)$. Hence, by induction, it holds that $P^k \leq \underline{P}^\phi(E, C)$ for all $k \in \mathbb{N}$, which implies that $P' \leq \underline{P}^\phi(E, C)$. The payment matrix $\underline{P}^\phi(E, C)$ is the bottom ϕ-transfer scheme for (E, C), i.e., $P' \geq \underline{P}^\phi(E, C)$, so it must hold that $P' = \underline{P}^\phi(E, C)$.

(ii). Applying (i) to $(\ell(E, C), C)$ with respect to ϕ^*, where $\phi^* = ((\varphi^i)^*)_{i\in N}$ is the dual claims rule vector associated with ϕ, we obtain that

$$\underline{P}^{\phi^*}(\ell(E, C), C) = \lim_{k\to\infty} Q^k,$$

where, for all $k \in \mathbb{N}$, $Q^k = (q^k_{ij})_{i,j\in N}$ with $Q^{k+1} = f^{\phi^*}(Q^k; \ell(E, C), C)$ and $Q^1 = 0^{N\times N}$. From Theorem 3.7 it follows that $\underline{P}^{\phi^*}(\ell(E, C), C) = C - \overline{P}^\phi(E, C)$, which implies that

$$\overline{P}^\phi(E, C) = \lim_{k\to\infty} (C - Q^k).$$

Let the sequence $(P^k)_{k\in\mathbb{N}}$ be such that, for all $k \in \mathbb{N}$, $P^{k+1} = f^\phi(P^k; E, C)$ with $P^1 = C$.

To show (ii), it suffices to show that, for all $k \in \mathbb{N}$, $P^k = C - Q^k$. Clearly, $P^1 = C = C - 0^{N\times N} = C - Q^1$. Let $k \in \mathbb{N}$. Assume that $P^\ell = C - Q^\ell$ for all $\ell \in \{1, \ldots, k\}$. Then,

$P^{k+1} = f^{\phi}(P^k; E, C) = f^{\phi}(C - Q^k; E, C) = C - Q^{k+1}$, where the last equality follows from the fact that, for all $i \in N$, $c_{ii} - q_{ii}^{k+1} = 0 = f_{ii}^{\phi}(C - Q^k; E, C)$, and, for all $j \in N \setminus \{i\}$,

$$\begin{aligned}
c_{ij} - q_{ij}^{k+1} &= c_{ij} - f_{ij}^{\phi^*}(Q^k; \ell(E, C), C) \\
&= c_{ij} - (\varphi^i)_j^*(\ell_i(E, C) + \sum_{h \in N} q_{hi}^k, \bar{c}_i) \\
&= \varphi_j^i(\sum_{h \in N} c_{ih} - \ell_i(E, C) - \sum_{h \in N} q_{hi}^k, \bar{c}_i) \\
&= \varphi_j^i(\sum_{h \in N} c_{ih} + e_i + \sum_{h \in N} c_{hi} - \sum_{h \in N} c_{ih} - \sum_{h \in N} q_{hi}^k, \bar{c}_i) \\
&= \varphi_j^i(e_i + \sum_{h \in N}(c_{hi} - q_{hi}^k), \bar{c}_i) \\
&= f_{ij}^{\phi}(C - Q^k; E, C),
\end{aligned}$$

in which the third equality follows from the definition of $(\varphi^i)^*$ as the dual of φ^i. Hence, by induction, it holds that $P^k = C - Q^k$ for all $k \in \mathbb{N}$. □

Theorem 3.10 and the dual relationship as given in Theorem 3.7 highlight that one obtains the top (resp. bottom) ϕ-transfer scheme for the primal problem by computing the bottom (resp. top) ϕ^*-transfer scheme for the dual problem.

As the following example shows, even if agents pay their claimants in accordance with the proportional rule, ϕ-transfer schemes for a mutual claims problem need not necessarily be unique. Under the assumption that all estates are non-negative, multiplicity of PROP-transfer schemes can only occur when at least one agent has an estate of zero (cf. Eisenberg and Noe (2001), and Csóka and Herings (2023)). However, as the dual mutual claims problem in the following example demonstrates, when allowing for possibly negative estates, multiplicity may also occur when some agents have a negative estate, and the remaining agents have a positive estate.

Example 3.3. Reconsider the mutual claims problem $(E, C) \in \mathcal{L}^N$ of Example 3.1 given by $N = \{1, 2, 3, 4\}$,

$$E = \begin{pmatrix} 1 \\ -3 \\ 3 \\ 0 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}.$$

Let $\phi = (\text{PROP}, \text{PROP}, \text{PROP}, \text{PROP}) \equiv \text{PROP}$. Using Theorem 3.10, we can determine the bottom PROP-transfer scheme and the top PROP-transfer scheme for (E, C), which are given by

$$\underline{P}^{\text{PROP}}(E, C) = \begin{bmatrix} 0 & 1\frac{1}{2} & 0 & 1\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 \end{bmatrix} \text{ and } \overline{P}^{\text{PROP}}(E, C) = \begin{bmatrix} 0 & 2 & 0 & 2 \\ \frac{2}{3} & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1\frac{1}{3} & \frac{2}{3} & 0 & 0 \end{bmatrix},$$

respectively. The set of agents that pay differently under both PROP-transfer schemes is given by $\mathcal{I}^{\text{PROP}}(E,C) = \{1,2,4\}$. Note that agent 2 pays a positive amount under the top PROP-transfer scheme, even though its initial estate is negative.

Now, consider the corresponding dual problem $(\ell(E,C),C)$, given by

$$\ell(E,C) = \begin{pmatrix} -2 \\ 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}.$$

Since PROP = PROP*,

$$\underline{P}^{\text{PROP}}(\ell(E,C),C) = C - \overline{P}^{\text{PROP}}(E,C) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix},$$

$$\overline{P}^{\text{PROP}}(\ell(E,C),C) = C - \underline{P}^{\text{PROP}}(E,C) = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 \end{bmatrix},$$

and $\mathcal{I}^{\text{PROP}}(\ell(E,C),C) = \mathcal{I}^{\text{PROP}}(E,C) = \{1,2,4\}$. Finally, note that agent 3 pays off all its claims in every PROP-transfer scheme for (E,C), whereas agent 3 pays nothing in every PROP-transfer scheme for $(\ell(E,C),C)$. △

4 Duality of transfer rules

In financial networks, a clearing mechanism prescribes how payments between agents should take place to settle their mutual liabilities. We formalize a clearing mechanism by means of a *ϕ-based transfer rule*, which assigns to each mutual claims problem exactly one ϕ-transfer scheme.

Definition 4.1. Let ϕ be a vector of claims rules. A *ϕ-based transfer rule* τ^ϕ on $\mathcal{L}^N$ assigns to each $(E,C) \in \mathcal{L}^N$ exactly one transfer scheme $P \in \mathcal{P}^\phi(E,C)$.

We associate to each ϕ-based transfer rule τ^ϕ a *dual ϕ-based transfer rule*. A dual ϕ-based transfer rule first lets all agents settle all their mutual claims, but as there may exist agents for which the estate plus outstanding claims are insufficient to pay off all debts, settling all claims may be infeasible, so an excess amount, i.e., a loss, has to be returned in accordance with τ^ϕ.

Definition 4.2. Given a ϕ-based transfer rule τ^ϕ, its *dual transfer rule* $(\tau^\phi)^*$ is, for all $(E,C) \in \mathcal{L}^N$, given by

$$(\tau^\phi)^*(E,C) = C - \tau^\phi(\ell(E,C),C).$$

Agents with a positive loss are essentially bankrupt because the amount they have at their disposal is insufficient to pay off their debts, even if they are paid in full by their debtors. Such agents have a positive estate in the dual problem and therefore always make repayments. Nevertheless, agents with a negative loss, and thus a negative estate in the dual problem, may not always make repayments because repaying some of the excess they received is of lower priority; this is also what Example 3.3 demonstrates with respect to agents 1 and 3.

As the following proposition states, the dual of a dual ϕ-based transfer rule $(\tau^{\phi})^*$ coincides with τ^{ϕ}.

Proposition 4.3. *Let $(E, C) \in \mathcal{L}^N$, and let ϕ be a vector of claims rules. Then, $((\tau^{\phi})^*)^*(E, C) = \tau^{\phi}(E, C)$.*

Proof. It holds that

$$\begin{aligned}((\tau^{\phi})^*)^*(E, C) &= C - (\tau^{\phi})^*(\ell(E, C), C) \\ &= C - (C - \tau^{\phi}(\ell(\ell(E, C), C), C)) \\ &= \tau^{\phi}(E, C),\end{aligned}$$

where the third equality follows Proposition 3.2. □

Furthermore, a dual ϕ-based transfer rule $(\tau^{\phi})^*$ is a ϕ^*-based transfer rule because it prescribes a ϕ^*-transfer scheme.

Proposition 4.4. *Let $(E, C) \in \mathcal{L}^N$, and let ϕ be a vector of claims rules with associated dual claims rules vector ϕ^*. Then, $(\tau^{\phi})^*(E, C) \in \mathcal{P}^{\phi^*}(E, C)$.*

Proof. As $\tau^{\phi}(\ell(E, C), C) \in \mathcal{P}^{\phi}(\ell(E, C), C)$, Theorem 3.5 implies that $(\tau^{\phi})^*(E, C) = (C - \tau^{\phi}(\ell(E, C), C)) \in \mathcal{P}^{\phi^*}(E, C)$. □

Note that, even though τ^{ϕ^*} and $(\tau^{\phi})^*$ both prescribe a ϕ^*-transfer scheme, they need not necessarily prescribe the same ϕ^*-transfer scheme. The following example illustrates this and additionally illustrates an approach to construct all ϕ-transfer schemes when ϕ is given by PROP.

Example 4.1. Reconsider the mutual claims problem $(E, C) \in \mathcal{L}^N$ of Example 3.1 given by $N = \{1, 2, 3, 4\}$,

$$E = \begin{pmatrix} 1 \\ -3 \\ 3 \\ 0 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix},$$

in which all agents use the proportional rule as their underlying payment mechanism, i.e., $\phi \equiv$ PROP.

Any PROP-transfer scheme for (E, C) can be written as an element-wise convex combination of the bottom PROP-transfer scheme and the top PROP-transfer scheme. More specifically, if $\underline{P}^{\text{PROP}}(E, C) = (\underline{p}_{ij})_{i,j \in N}$ and $\overline{P}^{\text{PROP}}(E, C) = (\overline{p}_{ij})_{i,j \in N}$, consider $P^\lambda = (p^\lambda_{ij})_{i,j \in N}$, where, for all $i, j \in N$,

$$p^\lambda_{ij} = (1 - \lambda_{ij})\underline{p}_{ij} + \lambda_{ij}\overline{p}_{ij}$$

with $\lambda_{ij} \in [0, 1]$. By doing so, we obtain

$$P^\lambda = \begin{bmatrix} 0 & 1\frac{1}{2} + \frac{1}{2}\lambda_{12} & 0 & 1\frac{1}{2} + \frac{1}{2}\lambda_{14} \\ \frac{2}{3}\lambda_{21} & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 + \frac{1}{3}\lambda_{41} & \frac{1}{2} + \frac{1}{6}\lambda_{42} & 0 & 0 \end{bmatrix}.$$

Condition (3.2) of a ϕ-transfer scheme requires that

$$(1\tfrac{1}{2} + \tfrac{1}{2}\lambda_{12}, 0, 1\tfrac{1}{2} + \tfrac{1}{2}\lambda_{14}) = \text{PROP}(1 + 2 + \tfrac{2}{3}\lambda_{21} + \tfrac{1}{3}\lambda_{41}, (2, 0, 2)), \qquad \text{(Agent 1)}$$

$$(\tfrac{2}{3}\lambda_{21}, 0, 0) = \text{PROP}(-3 + 3 + \tfrac{1}{2}\lambda_{12} + \tfrac{1}{6}\lambda_{42}, (2, 0, 0)), \qquad \text{(Agent 2)}$$

$$(1, 1, 0) = \text{PROP}(3 + 0, (1, 1, 0)), \qquad \text{(Agent 3)}$$

$$\text{and } (1 + \tfrac{1}{3}\lambda_{41}, \tfrac{1}{2} + \tfrac{1}{6}\lambda_{42}, 0) = \text{PROP}(0 + 1\tfrac{1}{2} + \tfrac{1}{2}\lambda_{14}, (2, 1, 0)). \qquad \text{(Agent 4)}$$

It follows that $\lambda_{12} = \lambda_{14} = \lambda_{21} = \lambda_{41} = \lambda_{42}$. Therefore, the set of PROP-transfer schemes for (E, C) comprises of

$$P^\lambda = \begin{bmatrix} 0 & 1\frac{1}{2} + \frac{1}{2}\lambda & 0 & 1\frac{1}{2} + \frac{1}{2}\lambda \\ \frac{2}{3}\lambda & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 + \frac{1}{3}\lambda & \frac{1}{2} + \frac{1}{6}\lambda & 0 & 0 \end{bmatrix} \tag{4.1}$$

with $\lambda \in [0, 1]$. Note that $P^\lambda = \underline{P}^{\text{PROP}}(E, C)$ if $\lambda = 0$ and $P^\lambda = \overline{P}^{\text{PROP}}(E, C)$ if $\lambda = 1$.

Since PROP = PROP*, both $\tau^{\text{PROP}^*}(E, C)$ and $(\tau^{\text{PROP}})^*(E, C)$ select exactly one PROP-transfer scheme of the form (4.1). However, note that they need not necessarily prescribe the same one, e.g., $\tau^{\text{PROP}^*}(E, C) = P^{\frac{1}{3}}$, whereas $(\tau^{\text{PROP}})^*(E, C) = P^{\frac{2}{3}}$. △

5 Duality of allocation rules

In the previous section, we focused on ϕ-based transfer rules that prescribe a payment matrix to settle the mutual liabilities between agents in a mutual claims problem. In this section, we focus on the allocations that are the result of transfers between agents in accordance with a ϕ-based transfer rule τ^ϕ. As ϕ-based transfer schemes form the basis for a reallocation of the total estate, we call these reallocations *ϕ-based transfer allocations.*

Definition 5.1. Let $(E, C) \in \mathcal{L}^N$, let ϕ be a vector of claims rules, and let $P = (p_{ij})_{i,j \in N} \in \mathcal{P}^\phi(E, C)$. The vector $a^\phi(P) \in \mathbb{R}^N$ is the *ϕ-based transfer allocation* corresponding to P if, for all $i \in N$,

$$a^\phi_i(P) = e_i + \sum_{j \in N} p_{ji} - \sum_{j \in N} p_{ij}.$$

Note that, indeed, a ϕ-based transfer allocation is a reallocation of the total estate, because, for all $P \in \mathcal{P}^{\phi}(E,C)$,

$$\sum_{i \in N} a_i^{\phi}(P) = \sum_{i \in N} e_i.$$

The following lemma states that an agent has paid off all its debts if it has a strictly positive ϕ-based transfer allocation, whereas an agent has paid nothing to all agents if it has a strictly negative ϕ-based transfer allocation.

Lemma 5.2. *Let $(E,C) \in \mathcal{L}^N$, let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules, let $P = (p_{ij})_{i,j \in N} \in \mathcal{P}^{\phi}(E,C)$, and let $i \in N$. Then,*

(i) if $a_i^{\phi}(P) > 0$, then $p_{ij} = c_{ij}$ for all $j \in N$;

(ii) if $a_i^{\phi}(P) < 0$, then $p_{ij} = 0$ for all $j \in N$.

Proof. We will first show (i) and subsequently use a duality argument to show (ii).

Let $a_i^{\phi}(P) > 0$. Then,

$$0 \leq \sum_{j \in N} p_{ij} < e_i + \sum_{j \in N} p_{ji}. \tag{5.1}$$

Condition (3.2) of a ϕ-transfer scheme and condition (ii) of claims rule φ^i imply that

$$\sum_{j \in N} p_{ij} = \sum_{j \in N} \varphi_j^i(e_i + \sum_{k \in N} p_{ki}, \bar{c}_i) = \min\{\max\{0, e_i + \sum_{j \in N} p_{ji}\}, \sum_{j \in N} c_{ij}\}. \tag{5.2}$$

From (5.1) and (5.2) it follows that $\sum_{j \in N} p_{ij} = \sum_{j \in N} c_{ij}$. Condition (i) of claims rule φ^i consequently implies that $p_{ij} = c_{ij}$ for all $j \in N$.

Second, let $a_i^{\phi}(P) < 0$. Then,

$$-e_i - \sum_{j \in N} p_{ji} + \sum_{j \in N} p_{ij} > 0,$$

which is equivalent to

$$\ell_i(E,C) + \sum_{j \in N}(c_{ji} - p_{ji}) - \sum_{j \in N}(c_{ij} - p_{ij}) > 0. \tag{5.3}$$

Theorem 3.5 implies that $(C - P) \in \mathcal{P}^{\phi^*}(\ell(E,C), C)$, where ϕ^* is the dual claims rules vector associated with ϕ. Therefore, the left-hand side of (5.3) is a ϕ^*-based transfer allocation for $(\ell(E,C), C)$ with respect to $(C - P)$, and by result (i) we must have that $c_{ij} - p_{ij} = c_{ij}$ for all $j \in N$, which means that $p_{ij} = 0$ for all $j \in N$. □

Although a mutual claims problem may lead to different ϕ-transfer schemes, their resulting ϕ-based transfer allocations are identical. Groote Schaarsberg et al. (2018) shows this result for mutual claims problems $(E,C) \in \mathcal{L}^N$ with $E \geq 0$ and $\phi = (\varphi^i)_{i \in N}$ with $\varphi^i = \varphi$ for all $i \in N$. Csóka and Herings (2023) shows this result for arbitrary φ. The proof of Proposition 5.3 extends existing proofs to deal with the case of negative estates.

Proposition 5.3. *Let $(E, C) \in \mathcal{L}^N$, let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules, and let $P, P' \in \mathcal{P}^\phi(E, C)$. Then, $a^\phi(P) = a^\phi(P')$.*

Proof. Let $P = (p_{ij})_{i,j \in N}$ and $P' = (p'_{ij})_{i,j \in N}$. For notational convenience, set $a = a^\phi(P)$ and $a' = a^\phi(P')$.

The proof follows by contradiction. We will assume that $a_k < a'_k$ for some $k \in N$ and show that it implies that $a_i \leq a'_i$ for all $i \in N \setminus \{k\}$, which leads to a contradiction because then

$$\sum_{i \in N} e_i = \sum_{i \in N} a_i < \sum_{i \in N} a'_i = \sum_{i \in N} e_i.$$

So, let $k \in N$ be such that $a_k < a'_k$. If $a'_k > 0$, then, by Lemma 5.2, $p'_{kj} = c_{kj}$ for all $j \in N$ and consequently $p_{kj} \leq p'_{kj}$ for all $j \in N$. On the other hand, if $a'_k \leq 0$, then $a_k < 0$, and Lemma 5.2 implies that $p_{kj} = 0$ for all $j \in N$; again, $p_{kj} \leq p'_{kj}$ for all $j \in N$.

Let

$$S = \{i \in N \,|\, p_{ij} \leq p'_{ij} \text{ for all } j \in N \text{ and } a_i \leq a'_i\}.$$

Clearly, $k \in S$. It suffices to prove that $S = N$.

Suppose that $S \neq N$. Since $\sum_{i \in S} a_i < \sum_{i \in S} a'_i$, we have

$$\sum_{i \in S} \sum_{j \in N} p_{ji} - \sum_{i \in S} \sum_{j \in N} p_{ij} < \sum_{i \in S} \sum_{j \in N} p'_{ji} - \sum_{i \in S} \sum_{j \in N} p'_{ij}.$$

This implies that

$$0 \leq \sum_{i \in S} \sum_{j \in N \setminus S} (p'_{ij} - p_{ij}) < \sum_{i \in S} \sum_{j \in N \setminus S} (p'_{ji} - p_{ji}),$$

and so we have

$$\sum_{i \in S} \sum_{j \in N \setminus S} p_{ji} < \sum_{i \in S} \sum_{j \in N \setminus S} p'_{ji}.$$

Therefore, there must exist an $i \in S$ and $h \in N \setminus S$ such that $p_{hi} < p'_{hi}$.

From $p_{hi} < p'_{hi} \leq c_{hi}$ and (i) of Lemma 5.2 it follows that $a_h \leq 0$; from $p'_{hi} > p_{hi} \geq 0$ and (ii) of Lemma 5.2 it follows that $a'_h \geq 0$.

Hence, as $h \in N \setminus S$, there exists an $\ell \in N$, $\ell \neq h$, such that $p_{h\ell} > p'_{h\ell}$.

Moreover, since $p_{hi} < p'_{hi}$,

$$p_{hi} = \varphi^h_i(e_h + \sum_{j \in N} p_{jh}, \bar{c}_h) < \varphi^h_i(e_h + \sum_{j \in N} p'_{jh}, \bar{c}_h) = p'_{hi}.$$

So, estate monotonicity of φ^h implies that

$$e_h + \sum_{j \in N} p_{jh} < e_h + \sum_{j \in N} p'_{jh}.$$

However, as a consequence this would imply that

$$p_{h\ell} = \varphi^h_\ell(e_h + \sum_{j \in N} p_{jh}, \bar{c}_h) \leq \varphi^h_\ell(e_h + \sum_{j \in N} p'_{jh}, \bar{c}_h) = p'_{h\ell},$$

which contradicts $p_{h\ell} > p'_{h\ell}$. □

From Proposition 5.3 it follows that the reallocation of the total estate according to a ϕ-transfer scheme depends only on ϕ and not on the exact underlying ϕ-transfer scheme.

Definition 5.4. Let ϕ be a vector of claims rules. The *ϕ-based allocation rule* μ^ϕ on $\mathcal{L}^N$ assigns to each $(E, C) \in \mathcal{L}^N$ the ϕ-based transfer allocation $a^\phi(P)$, in which $P \in \mathcal{P}^\phi(E, C)$.[11]

A ϕ-based allocation rule is thus defined as the rule that maps, for all mutual claims problems, the corresponding ϕ-based transfer rules into a single allocation. More specifically, for all $(E, C) \in \mathcal{L}^N$ and $\tau^\phi(E, C)$, $\mu^\phi(E, C) = a^\phi(\tau^\phi(E, C))$.

Given a mutual claims problem $(E, C) \in \mathcal{L}^N$, a ϕ-based allocation rule μ^ϕ prescribes a reallocation of the total estate $\sum_{i\in N} e_i$ on the basis of a ϕ-transfer scheme $P \in \mathcal{P}^\phi(E, C)$. We define the corresponding dual allocation rule $(\mu^\phi)^*$ as the rule that prescribes a reallocation of the total estate $\sum_{i\in N} e_i$ as well, however from the perspective of the corresponding dual problem $(\ell(E, C), C)$ by allocating the losses with respect to μ^ϕ.

Definition 5.5. Given a ϕ-based allocation rule μ^ϕ, its *dual allocation rule* $(\mu^\phi)^*$ is, for all $(E, C) \in \mathcal{L}^N$, given by

$$(\mu^\phi)^*(E, C) = -\mu^\phi(\ell(E, C), C).$$

In both a mutual claims problem $(E, C) \in \mathcal{L}^N$ and its corresponding dual problem $(\ell(E, C), C)$, the total estate $\sum_{i\in N} e_i$ is reallocated among the agents because

$$-\sum_{i\in N} \ell_i(E, C) = \sum_{i\in N}(e_i + \sum_{j\in N} c_{ji} - \sum_{j\in N} c_{ij}) = \sum_{i\in N} e_i.$$

The following theorem states that $(\mu^\phi)^*$ is a ϕ^*-based allocation rule because the allocation that it prescribes is associated with a ϕ^*-transfer scheme prescribed by a ϕ^*-based transfer rule $(\tau^\phi)^*$. In this way, the theorem also shows that the definition of a dual allocation rule $(\mu^\phi)^*$ is natural and consistent in the sense that $(\mu^\phi)^*$ corresponds to $(\tau^\phi)^*$ in the same way as μ^ϕ corresponds to τ^ϕ as discussed earlier.

Theorem 5.6. *Let $(E, C) \in \mathcal{L}^N$, let ϕ be a vector of claims rules with associated dual claims rules vector ϕ^*, and let τ^ϕ be a ϕ-based transfer rule. Then, $(\mu^\phi)^*(E, C) = a^{\phi^*}((\tau^\phi)^*(E, C))$.*

Proof. Let $i \in N$. Then,

$$\begin{aligned}
(\mu^\phi)^*_i(E, C) &= -\mu^\phi_i(\ell(E, C), C) \\
&= -a^\phi_i(\tau^\phi(\ell(E, C), C)) \\
&= -(\ell_i(E, C) + \sum_{j\in N} \tau^\phi_{ji}(\ell(E, C), C) - \sum_{j\in N} \tau^\phi_{ij}(\ell(E, C), C)) \\
&= e_i + \sum_{j\in N} c_{ji} - \sum_{j\in N} c_{ij} - \sum_{j\in N} \tau^\phi_{ji}(\ell(E, C), C) + \sum_{j\in N} \tau^\phi_{ij}(\ell(E, C), C) \\
&= e_i + \sum_{j\in N}(c_{ji} - \tau^\phi_{ji}(\ell(E, C), C)) - \sum_{j\in N}(c_{ij} - \tau^\phi_{ij}(\ell(E, C), C))
\end{aligned}$$

[11] In the literature on mutual claims problems, a ϕ-based allocation rule is also called a *ϕ-based mutual claims rule* or a *ϕ-based mutual liability rule*.

$$= e_i + \sum_{j \in N} (\tau^\phi)^*_{ji}(E, C) - \sum_{j \in N} (\tau^\phi)^*_{ij}(E, C)$$
$$= a_i^{\phi^*}((\tau^\phi)^*(E, C)).$$

The first equality follows from the definition of $(\mu^\phi)^*$ as the dual of μ^ϕ; the sixth equality follows from the definition of $(\tau^\phi)^*$ as the dual of τ^ϕ. □

One can take the dual of μ^ϕ as in Definition 5.5, which gives $(\mu^\phi)^*$. However, one can also take the dual of ϕ as in Definition 2.4, which gives μ^{ϕ^*}. In fact, from Proposition 5.3 and Theorem 5.6 it follows that the dual ϕ-based allocation rule $(\mu^\phi)^*$ coincides with the ϕ^*-based allocation rule μ^{ϕ^*}. Recall from Example 4.1 that a similar result does not necessarily hold for ϕ-based transfer rules.

Corollary 5.7. *Let $(E, C) \in \mathcal{L}^N$, and let ϕ be a vector of claims rules with associated dual claims rules vector ϕ^*. Then, $(\mu^\phi)^*(E, C) = \mu^{\phi^*}(E, C)$.*

Finally, the following proposition states that the dual of a dual ϕ-based allocation rule $(\mu^\phi)^*$ coincides with μ^ϕ.

Proposition 5.8. *Let $(E, C) \in \mathcal{L}^N$, and let ϕ be a vector of claims rules. Then, $((\mu^\phi)^*)^*(E, C) = \mu^\phi(E, C)$.*

Proof. It holds that

$$\begin{aligned} ((\mu^\phi)^*)^*(E, C) &= -(\mu^\phi)^*(\ell(E, C), C) \\ &= -(-\mu^\phi(\ell(\ell(E, C), C), C)) \\ &= \mu^\phi(E, C), \end{aligned}$$

where the third equality follows from Proposition 3.2. □

6 Self-duality of transfer and allocation rules

Claims rules that coincide with their corresponding dual rule, are called *self-dual*. Three well-known self-dual claims rules in the literature are the proportional rule, the *Talmud rule* (Aumann & Maschler, 1985), and the *random-arrival rule* (O'Neill, 1982).

Definition 6.1. A claims rule φ on $\mathcal{C}^M$ is *self-dual* if, for all $(e, c) \in \mathcal{C}^M$, it holds that $\varphi(e, c) = \varphi^*(e, c)$.

In the special case that the estate e in a claims problem $(e, c) \in \mathcal{C}^M$ equals the total loss $\sum_{i \in M} c_i - e$, i.e., if $e = \frac{1}{2}\sum_{i \in M} c_i$, then, for any self-dual claims rule φ,

$$\varphi(e, c) = \varphi^*(e, c) = c - \varphi(\sum_{i \in M} c_i - e, c) = c - \varphi(e, c),$$

which implies that

$$\varphi(e, c) = \frac{1}{2}c.$$

We want to extend the notion of self-duality to the mutual claims problem setting. If $(E, C) \in \mathcal{L}^N$ is such that the estate of each agent equals its loss, i.e., if $E = \ell(E, C)$, then half the claims matrix is a ϕ-transfer scheme for (E, C). However, note that this may not be the only ϕ-transfer scheme for (E, C).

Proposition 6.2. *Let $(E, C) \in \mathcal{L}^N$, and let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules in which, for all $i \in N$, φ^i is self-dual. If $E = \ell(E, C)$, then $\frac{1}{2}C \in \mathcal{P}^\phi(E, C)$.*

Proof. Let $E = \ell(E, C)$. Then, for all $i \in N$,

$$e_i = \frac{1}{2}(\sum_{j \in N} c_{ij} - \sum_{j \in N} c_{ji}). \tag{6.1}$$

Clearly, for all $i \in N$, $\frac{1}{2}c_{ii} = 0$. Therefore, we need to show that, for all $i, j \in N$ with $i \neq j$,

$$\frac{1}{2}c_{ij} = \varphi^i_j(e_i + \sum_{k \in N} \frac{1}{2}c_{ki}, \bar{c}_i).$$

Let $i, j \in N$ with $i \neq j$. Then,

$$\begin{aligned}
\frac{1}{2}c_{ij} &= \frac{1}{2}\varphi^i_j(e_i + \sum_{k \in N} \frac{1}{2}c_{ki}, \bar{c}_i) + \frac{1}{2}(\varphi^i)^*_j(\sum_{k \in N} c_{ik} - e_i - \sum_{k \in N} \frac{1}{2}c_{ki}, \bar{c}_i) \\
&= \frac{1}{2}\varphi^i_j(e_i + \sum_{k \in N} \frac{1}{2}c_{ki}, \bar{c}_i) + \frac{1}{2}\varphi^i_j(\sum_{k \in N} c_{ik} - e_i - \sum_{k \in N} \frac{1}{2}c_{ki}, \bar{c}_i) \\
&= \frac{1}{2}\varphi^i_j(e_i + \sum_{k \in N} \frac{1}{2}c_{ki}, \bar{c}_i) + \frac{1}{2}\varphi^i_j(e_i + \sum_{k \in N} \frac{1}{2}c_{ki}, \bar{c}_i) \\
&= \varphi^i_j(e_i + \sum_{k \in N} \frac{1}{2}c_{ki}, \bar{c}_i).
\end{aligned}$$

The first equality follows from the definition of $(\varphi^i)^*$ as the dual of φ^i; the second equality follows from self-duality of φ^i; the third equality follows from (6.1). □

The notion of self-duality can be extended to ϕ-based allocation rules and ϕ-based transfer rules.

Definition 6.3. A ϕ-based allocation rule μ^ϕ on $\mathcal{L}^N$ is *self-dual* if, for all $(E, C) \in \mathcal{L}^N$, it holds that $\mu^\phi(E, C) = (\mu^\phi)^*(E, C)$.

Thus, self-duality of ϕ-based allocation rules entails that, for all mutual claims problems, the allocation prescribed by μ^ϕ is equal to the allocation prescribed by its corresponding dual rule $(\mu^\phi)^*$.

In the special case that the estate of each agent coincides with its loss in a mutual claims problem $(E, C) \in \mathcal{L}^N$, i.e., if $E = \ell(E, C)$, then a self-dual ϕ-based allocation rule will always allocate zero to all agents.

Proposition 6.4. *Let $(E, C) \in \mathcal{L}^N$, and let μ^ϕ be a self-dual ϕ-based allocation rule. If $E = \ell(E, C)$, then $\mu^\phi(E, C) = (\mu^\phi)^*(E, C) = 0$.*

Proof. Let $E = \ell(E, C)$. Then,

$$\mu^{\phi}(E, C) = (\mu^{\phi})^{*}(E, C) = -\mu^{\phi}(\ell(E, C), C) = -\mu^{\phi}(E, C), \tag{6.2}$$

where the first equality follows from self-duality of μ^{ϕ}, the second equality follows from the definition of $(\mu^{\phi})^{*}$ as the dual of μ^{ϕ}, and the last equality follows from $E = \ell(E, C)$. From (6.2) it follows that $\mu^{\phi}(E, C) = 0$. □

In particular, $E = \ell(E, C)$ implies that $\sum_{i \in N} e_i = 0$. In that case, there may exist an $i \in N$ such that $e_i \neq 0$. Nevertheless, a reallocation of the total estate with respect to a self-dual ϕ-based allocation rule always yields an allocation of zero to all agents.

The following theorem states that self-duality of the claims rules in ϕ carries over to self-duality of a ϕ-based allocation rule μ^{ϕ}, and vice versa.

Theorem 6.5. *Let $\phi = (\varphi^{i})_{i \in N}$ be a vector of claims rules. The ϕ-based allocation rule μ^{ϕ} is self-dual if and only if, for all $i \in N$, φ^{i} is self-dual.*

Proof. First, let $(E, C) \in \mathcal{L}^{N}$ and assume that, for all $i \in N$, φ^{i} is self-dual, so $\phi = \phi^{*}$. Then, it holds that

$$\mu^{\phi}(E, C) = \mu^{\phi^{*}}(E, C) = (\mu^{\phi})^{*}(E, C).$$

Here, the second equality follows from Corollary 5.7. Therefore, μ^{ϕ} is self-dual.

Second, assume that μ^{ϕ} is self-dual. Let $i \in N$, and let $(e, c) \in \mathcal{C}^{N \setminus \{i\}}$. Define $(E, C) \in \mathcal{L}^{N}$, where $e_i = e$, $e_j = 0$ for all $j \in N \setminus \{i\}$, $c_{ii} = 0$, $c_{ij} = c_j$ for all $j \in N \setminus \{i\}$, and $c_{jk} = 0$ for all $j \in N \setminus \{i\}$ and $k \in N$. Then, $P^{\phi}(E, C) = \{P\}$, in which $P = (p_{jk})_{j,k \in N}$ with $p_{jk} = 0$ for all $j \in N \setminus \{i\}$ and $k \in N$, $p_{ii} = 0$, and $p_{ij} = \varphi^{i}_{j}(e, \bar{c}_i) = \varphi^{i}_{j}(e, c)$ for all $j \in N \setminus \{i\}$. Therefore, for all $j \in N \setminus \{i\}$,

$$\mu^{\phi}_{j}(E, C) = e_j + \sum_{k \in N} p_{kj} - \sum_{k \in N} p_{jk} = p_{ij} = \varphi^{i}_{j}(e, c). \tag{6.3}$$

Moreover, let $\phi^{*} = ((\varphi^{j})^{*})_{j \in N}$ be the vector of dual claims rules associated with ϕ. Then, $P^{\phi^{*}}(E, C) = \{P^{*}\}$, in which $P^{*} = (p^{*}_{jk})_{j,k \in N}$ with $p^{*}_{jk} = 0$ for all $j \in N \setminus \{i\}$ and $k \in N$, $p^{*}_{ii} = 0$, and $p^{*}_{ij} = (\varphi^{i})^{*}_{j}(e, \bar{c}_i) = (\varphi^{i})^{*}_{j}(e, c)$ for all $j \in N \setminus \{i\}$. Therefore, for all $j \in N \setminus \{i\}$,

$$\mu^{\phi^{*}}_{j}(E, C) = e_j + \sum_{k \in N} p^{*}_{kj} - \sum_{k \in N} p^{*}_{jk} = p^{*}_{ij} = (\varphi^{i})^{*}_{j}(e, c). \tag{6.4}$$

Hence, for all $j \in N \setminus \{i\}$,

$$\varphi^{i}_{j}(e, c) = \mu^{\phi}_{j}(E, C) = (\mu^{\phi})^{*}_{j}(E, C) = \mu^{\phi^{*}}_{j}(E, C) = (\varphi^{i})^{*}_{j}(e, c).$$

The first equality follows from (6.3); the second equality follows from self-duality of μ^{ϕ}; the third equality follows from Corollary 5.7; the fourth equality follows from (6.4). Thus, φ^{i} is self-dual. □

Hence, a ϕ-based allocation rule μ^ϕ is not self-dual if the claims rule of at least one agent is not self-dual.[12]

We now turn our attention to self-duality of ϕ-based transfer rules. We will see that self-duality of ϕ-based transfer rules is a stronger requirement than self-duality of ϕ-based allocation rules.

A ϕ-based transfer rule τ^ϕ is self-dual if it coincides with its corresponding dual rule $(\tau^\phi)^*$.

Definition 6.6. A ϕ-based transfer rule τ^ϕ on $\mathcal{L}^N$ is *self-dual* if, for all $(E, C) \in \mathcal{L}^N$, it holds that $\tau^\phi(E, C) = (\tau^\phi)^*(E, C)$.

A self-dual ϕ-based transfer rule τ^ϕ thus prescribes, for each mutual claims problem, the same ϕ-transfer scheme as its corresponding dual ϕ-based transfer rule $(\tau^\phi)^*$.

Indeed, as the following theorem states, self-duality of ϕ-based transfer rules is a stronger requirement than self-duality of ϕ-based allocation rules. In the proof, we use the correspondence between μ^ϕ and τ^ϕ as was discussed in Section 5.

Theorem 6.7. *Let $\phi = (\varphi^i)_{i\in N}$ be a vector of claims rules. If a ϕ-based transfer rule τ^ϕ is self-dual, then the ϕ-based allocation rule μ^ϕ is self-dual.*

Proof. Let τ^ϕ be self-dual, let $(E, C) \in \mathcal{L}^N$, and let $i \in N$. Then,

$$\begin{aligned}
\mu_i^\phi(E, C) &= a_i^\phi(\tau^\phi(E, C)) \\
&= a_i^\phi((\tau^\phi)^*(E, C)) \\
&= e_i + \sum_{j\in N}(\tau^\phi)^*_{ji}(E, C) - \sum_{j\in N}(\tau^\phi)^*_{ij}(E, C) \\
&= e_i + \sum_{j\in N}(c_{ji} - \tau^\phi_{ji}(\ell(E, C), C)) - \sum_{j\in N}(c_{ij} - \tau^\phi_{ij}(\ell(E, C), C)) \\
&= -\ell_i(E, C) - \sum_{j\in N}\tau^\phi_{ji}(\ell(E, C), C) + \sum_{j\in N}\tau^\phi_{ij}(\ell(E, C), C) \\
&= -a_i^\phi(\tau^\phi(\ell(E, C), C)) \\
&= -\mu_i^\phi(\ell(E, C), C) \\
&= (\mu^\phi)^*_i(E, C).
\end{aligned}$$

The second equality follows from self-duality of τ^ϕ; the fourth equality follows from the definition of $(\tau^\phi)^*$ as the dual of τ^ϕ; the last equality follows from the definition of $(\mu^\phi)^*$ as the dual of μ^ϕ. □

[12] Furthermore, in Theorem 6.5 it is necessary that, for each $i \in N$, the claims rule φ^i is defined on $\mathcal{C}^{N\setminus\{i\}}$, which is unlike the existing literature that defines φ^i on $\mathcal{C}^N$. To see this, consider $i \in N$, and a claims problem $(e, c) \in \mathcal{C}^N$ with $c_i > 0$. The claim c_i of agent i in (e, c) can not be incorporated as a claim on itself in a mutual claims problem in $\mathcal{L}^N$, so the claims matrix of a mutual claims problem in $\mathcal{L}^N$ associated with (e, c) does not contain information about c_i. As a consequence, a self-dual ϕ-based allocation rule imposes restrictions only on how φ^i should divide e among $N \setminus \{i\}$ and not on how it should divide e among N.

Since self-duality of τ^ϕ implies self-duality of μ^ϕ which, in turn, implies self-duality of the claims rules in the corresponding claims rules vector, we obtain the following corollary.

Corollary 6.8. *Let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules. If a ϕ-based transfer rule τ^ϕ is self-dual, then, for all $i \in N$, φ^i is self-dual.*

At the same time, even if all claims rules in the claims rules vector ϕ are self-dual, a ϕ-based transfer rule need not be self-dual, as the following example demonstrates.

Example 6.1. Reconsider the mutual claims problem $(E, C) \in \mathcal{L}^N$ of Example 3.1 given by $N = \{1, 2, 3, 4\}$,

$$E = \begin{pmatrix} 1 \\ -3 \\ 3 \\ 0 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix},$$

in which all agents use the proportional rule as their underlying payment mechanism, i.e., $\phi \equiv \text{PROP}$.

To guarantee self-duality of τ^{PROP} with respect to (E, C), it is required that $\tau^{\text{PROP}}(E, C) = (\tau^{\text{PROP}})^*(E, C)$. Proposition 4.4 implies that the dual PROP-transfer rule $(\tau^{\text{PROP}})^*$ prescribes a PROP-transfer scheme because the proportional rule is self-dual.

Let $\tau^{\text{PROP}}(E, C) = P^\lambda$ for some $\lambda \in [0, 1]$, in which P^λ is given by (4.1). Similarly, let $(\tau^{\text{PROP}})^*(E, C) = P^{\lambda^*}$ for some $\lambda^* \in [0, 1]$. For example, if $\lambda = \frac{1}{3}$ and $\lambda^* = \frac{2}{3}$, then $\tau^{\text{PROP}}(E, C) = P^{\frac{1}{3}} \neq P^{\frac{2}{3}} = (\tau^{\text{PROP}})^*(E, C)$. Therefore, to guarantee self-duality with respect to (E, C), it is required that $\lambda = \lambda^*$. Furthermore, as

$$(\tau^{\text{PROP}})^*(E, C) = C - \tau^{\text{PROP}}(\ell(E, C), C),$$

it must hold that $\tau^{\text{PROP}}(\ell(E, C), C) = C - P^{\lambda^*} = C - P^\lambda$. △

The following proposition states that any self-dual ϕ-based transfer rule prescribes half the claims matrix if the estate of each agent equals its loss. Note that such a prescription is valid by Proposition 6.2.

Proposition 6.9. *Let $(E, C) \in \mathcal{L}^N$, and let τ^ϕ be a self-dual ϕ-based transfer rule. If $E = \ell(E, C)$, then $\tau^\phi(E, C) = (\tau^\phi)^*(E, C) = \frac{1}{2}C$.*

Proof. Let $E = \ell(E, C)$. Then,

$$\tau^\phi(E, C) = (\tau^\phi)^*(E, C) = C - \tau^\phi(\ell(E, C), C) = C - \tau^\phi(E, C), \tag{6.5}$$

where the first equality follows from self-duality of τ^ϕ, the second equality follows from the definition of $(\tau^\phi)^*$ as the dual of τ^ϕ, and the last equality follows from $E = \ell(E, C)$. From (6.5) it follows that $\tau^\phi(E, C) = \frac{1}{2}C$. □

Finally, provided that the claims rules in ϕ are self-dual, a self-dual ϕ-based transfer rule s^ϕ can be constructed as follows. For each pair $\{(E, C), (\ell(E, C), C)\}$ of mutual claims problem $(E, C) \in \mathcal{L}^N$ and its corresponding dual $(\ell(E,C), C) \in \mathcal{L}^N$, choose $P \in \mathcal{P}^\phi(E,C)$ and define

$$s^\phi(E,C) = \begin{cases} \frac{1}{2}C & \text{if } E = \ell(E,C), \\ P & \text{if } E \neq \ell(E,C), \end{cases}$$

and

$$s^\phi(\ell(E,C),C) = \begin{cases} \frac{1}{2}C & \text{if } E = \ell(E,C), \\ C - P & \text{if } E \neq \ell(E,C). \end{cases}$$

Note that the pair $\{(E,C), (\ell(E,C),C)\}$ consists of exactly one element if $E = \ell(E,C)$, in which case it holds that $s^\phi(E,C) = s^\phi(\ell(E,C),C)$. Let $\mathcal{S}^\phi$ denote the set of all ϕ-based transfer rules of the form s^ϕ as described above. Please note that the collection containing all pairs $\{(E,C), (\ell(E,C),C)\}$ as elements forms a partition of the class of all mutual claims problems $\mathcal{L}^N$. In the special case that $E = \ell(E,C)$, a ϕ-based transfer rule of the form s^ϕ prescribes $\frac{1}{2}C$ in both (E,C) and $(\ell(E,C),C)$, as is required by Proposition 6.9. If $E \neq \ell(E,C)$, a ϕ-based transfer rule of the form s^ϕ selects any ϕ-transfer scheme $P \in \mathcal{P}^\phi(E,C)$ for (E,C), whereas it prescribes $C - P$ for $(\ell(E,C),C)$. Moreover, the payment matrix $C - P$ is ϕ-transfer scheme for $(\ell(E,C),C)$ because Theorem 3.5 implies that $(C-P) \in \mathcal{P}^{\phi^*}(\ell(E,C),C)$, and, because the claims rules in ϕ are self-dual, it holds that $\mathcal{P}^{\phi^*}(\ell(E,C),C) = \mathcal{P}^\phi(\ell(E,C),C)$.

By construction, ϕ-based transfer rules of the form s^ϕ are self-dual, because, for all $(E,C) \in \mathcal{L}^N$,

$$s^\phi(E,C) = C - s^\phi(\ell(E,C),C) = (s^\phi)^*(E,C),$$

which proves the following theorem.

Theorem 6.10. *Let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules. If, for all $i \in N$, φ^i is self-dual, then a ϕ-based transfer rule τ^ϕ is self-dual if and only if $\tau^\phi \in \mathcal{S}^\phi$.*

Hence, provided that agents use self-dual claims rules, only ϕ-based transfer rules of the form s^ϕ are the self-dual, whereas the ϕ-based allocation rule μ^ϕ is always self-dual as per Theorem 6.5. The fact that ϕ-transfer schemes need not necessarily be unique but nonetheless lead to the same ϕ-based transfer allocation is the reason for this difference. Indeed, as there is a choice to be made, self-duality of ϕ-transfer rules is a stronger requirement than self-duality of ϕ-based allocation rules, which is confirmed by Theorem 6.7 and illustrated in Example 6.1.

A Ruling out negative estates is not harmless

In the existing literature, negative estates are often modeled by introducing an artificial agent with an initial estate of zero, no liabilities to other agents, but claims on agents with a negative estate equal to the absolute value of their negative estate (cf. Eisenberg and Noe (2001)). Formally, this approach of dealing with possibly negative estates is as follows. Let $(E, C) \in \mathcal{L}^N$ be a mutual claims problem. The *augmented* mutual claims problem (E', C') on $N' = \{0\} \cup N$ is defined by $E' = (e'_i)_{i \in N'}$ with $e'_i = \max\{e_i, 0\}$ for all $i \in N$, $e'_0 = 0$, $C' = (c'_{ij})_{i,j \in N'}$ with $c'_{ij} = c_{ij}$ for all $i, j \in N$, $c'_{0j} = 0$ for all $j \in N'$, and $c'_{i0} = \max\{-e_i, 0\}$ for all $i \in N$. In (E, C), an agent with a negative estate can only pay agents if its incoming payments exceed the value of its negative estate. Hence, to preserve this priority structure in the augmented mutual claims problem, agents in N are paid only once the liabilities to agent 0 are paid in full.

The following example shows that such an approach is less suitable in the context of duality because, by definition, each time an artificial agent is introduced to obtain the augmented mutual claims problem, this artificial agent has a negative estate in the corresponding dual mutual claims problem, which makes it necessary to define a new artificial agent and a new augmented mutual claims problem, and so on. In this way, the set of agents expands throughout. In particular, to assert that the dual of the dual coincides with the original mutual claims problem, one has to know which agents are artificial and at which point they were introduced.

Example A.1. Consider the mutual claims problem $(E, C) \in \mathcal{L}^N$ of Example 3.3 given by $N = \{1, 2, 3, 4\}$,

$$E = \begin{pmatrix} 1 \\ -3 \\ 3 \\ 0 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}.$$

Then, as the estate of agent 2 is negative, we should augment the mutual claims problem by introducing an artificial agent, agent zero, such that we obtain a mutual claims problem (E', C') on $N' = \{0, 1, 2, 3, 4\}$, which is given by

$$E' = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \\ 0 \end{pmatrix} \text{ and } C' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \end{bmatrix},$$

with corresponding dual problem $(\ell(E', C'), C')$, given by

$$\ell(E', C') = \begin{pmatrix} -3 \\ -2 \\ 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } C' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \end{bmatrix}.$$

Note that, indeed, the dual problem $(\ell(E', C'), C')$ on N' restricted to N is the dual problem as given in Example 3.3; however, as agents 0, 1, and 4 have a negative estate, we should augment the problem $(\ell(E', C'), C')$ to obtain the mutual claims problem (E'', C'') on $N'' = \{0', 0, 1, 2, 3, 4\}$, which is given by

$$E'' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } C'' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \end{bmatrix}.$$

The corresponding dual problem $(\ell(E'', C''), C'') = (E''', C''')$ is given by

$$E''' = \begin{pmatrix} -6 \\ 0 \\ 1 \\ 0 \\ 3 \\ 0 \end{pmatrix} \text{ and } C''' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \end{bmatrix},$$

in which the artificial agent $0'$ has a negative estate. Hence, the dual of the augmented dual problem (E'', C'') is equal to (E''', C'''). Without knowing that agent 0 is an artificial agent introduced in the first step, it is not straightforward to conclude that the appropriate restriction of (E''', C''') indeed corresponds to (E, C). △

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