

Tilburg University

Cospectral Mates for Generalized Johnson and Grassmann Graphs

Abiad, Aida; D'haeseleer, Jozefien; Haemers, Willem H.; Simoens, Robin

Publication date:
2023

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):

Abiad, A., D'haeseleer, J., Haemers, W. H., & Simoens, R. (2023). *Cospectral Mates for Generalized Johnson and Grassmann Graphs*.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Cospectral mates for generalized Johnson and Grassmann graphs

Aida Abiad* Jozefien D’haeseleer† Willem H. Haemers‡ Robin Simoens§

Abstract

We provide three infinite families of graphs in the Johnson and Grassmann schemes that are not uniquely determined by their spectrum. We do so by constructing graphs that are cospectral but non-isomorphic to these graphs.

MSC Subject: 05C50, 15A18

Keywords: Graph, Eigenvalues, Determined by spectrum, Switching

1 Introduction

A major problem in the field of spectral graph theory is to decide whether a given graph is determined by its spectrum (the eigenvalues of its adjacency matrix). This problem has been solved for many families of graphs; sometimes by proving that the spectrum determines the graph, and sometimes by constructing cospectral mates (non-isomorphic graphs with the same spectrum). Although a conjecture by the third author states that almost all graphs are determined by their spectrum, many well-known graphs seem to have cospectral mates. This is, in particular, the case for several graphs in the Johnson and Grassmann association schemes, see for instance [2, 3, 4, 6, 9]. We contribute to this line of research by constructing cospectral mates for a number of open cases in the Johnson and Grassmann schemes. For this, we use switching techniques due to Godsil and McKay [5] and Wang, Qiu and Hu [11].

This paper is structured as follows. In Section 2, we introduce the necessary definitions. Section 3 contains an overview of previous results on the cospectrality of generalized Johnson and Grassmann graphs. In Section 4, we prove that the generalized Johnson graph $J_{\{2\}}(n, 4)$, $n \geq 8$, is not determined by its spectrum, thereby solving the second open problem of [2]. In Section 5, we show that the generalized Johnson graph $J_{\{1, 2, \dots, \frac{k-1}{2}\}}(2k, k)$ with $k \geq 5$, k odd, is not determined by its spectrum. In Section 6, we

*a.abiad.monge@tue.nl, Department of Mathematics and Computer Science, Eindhoven University of Technology, The Netherlands

Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Belgium
Department of Mathematics and Data Science, Vrije Universiteit Brussel, Belgium

†Jozefien.Dhaeseleer@UGent.be, Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Belgium

‡haemers@uvt.nl, Department of Econometrics and Operations Research, Tilburg University, The Netherlands

§Robin.Simoens@UGent.be, Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Belgium

prove that the q -Kneser graph $K_q(n, k)$ is not determined by its spectrum if $q = 2$. In Section 7 we provide three specific generalized Johnson graphs that are not determined by their spectrum and pose some open questions. Finally, the Appendix contains an overview of all cospectrality results for small graphs in the aforementioned schemes.

2 Preliminaries

Let $\Gamma = (V, E)$ denote a simple loopless graph. The *spectrum* of Γ is the multiset of eigenvalues of the adjacency matrix of Γ . Graphs are *cospectral* if they have the same spectrum. Two graphs that are cospectral but not isomorphic, are called *cospectral mates*. A graph is *determined by its spectrum* (DS) if it has no cospectral mate. Otherwise, it is not determined by its spectrum (NDS).

One of the most well-known tools for constructing cospectral graphs was introduced by Godsil and McKay [5].

Theorem 1 (GM-switching [5]). *Let Γ be a graph and $\{C_1, C_2, \dots, C_k, D\}$ a partition of $V(\Gamma)$ such that the following hold for all $i, j \in \{1, 2, \dots, k\}$:*

- (i) *Any two vertices of C_i have the same number of neighbours in C_j .*
- (ii) *Every vertex in D has exactly $0, \frac{1}{2}|C_i|$ or $|C_i|$ neighbours in C_i .*

For all $i \in \{1, 2, \dots, k\}$, every $u \in C_i$ and every $v \in D$ that has exactly $\frac{1}{2}|C_i|$ neighbours in C_i , reverse the adjacency between u and v . The resulting graph is cospectral with Γ . We say that it is obtained from Γ by GM-switching with respect to C_1, C_2, \dots, C_k .

Another switching technique to construct cospectral graphs was recently introduced by Wang, Qui and Hu [11].

Theorem 2 (WQH-switching [11]). *Let Γ be a graph and C_1, C_2 disjoint subsets of $V(\Gamma)$ such that the following hold:*

- (i) $|C_1| = |C_2|$.
- (ii) *There exists a constant c such that for all $i, j \in \{1, 2\}$, $i \neq j$ and every vertex of C_i , the number of neighbours in C_i minus the number of neighbours in C_j is equal to c .*
- (iii) *Every vertex outside $C_1 \cup C_2$ has either:*
 - (a) $|C_1|$ neighbours in C_1 and 0 neighbours in C_2 ,
 - (b) 0 neighbours in C_1 and $|C_2|$ neighbours in C_2 ,
 - (c) *the same number of neighbours in C_1 and C_2 .*

For every $u \in C_1 \cup C_2$ and every $v \notin C_1 \cup C_2$ for which (a) or (b) holds, reverse the adjacency between u and v . The resulting graph is cospectral with Γ . We say that it is obtained from Γ by WQH-switching with respect to C_1 and C_2 .

We should mention that a more general version of WQH-switching was introduced in [10].

Let n, k be positive integers with $k \leq n$. We now define the graphs of our interest, namely those in the Johnson and Grassmann schemes.

Definition 3. Let $S \subseteq \{0, 1, \dots, k-1\}$. The *generalized Johnson graph* $J_S(n, k)$ has as vertices the k -subsets of $\{1, \dots, n\}$, where two vertices are adjacent if their intersection size is in S .

In particular, $J_{\{0\}}(n, k)$ is the *Kneser graph* $K(n, k)$ and $J_{\{k-1\}}(n, k)$ is the *Johnson graph* $J(n, k)$.

If we replace sets by subspaces and sizes by dimensions in Definition 3, we obtain the definition of a generalized Grassmann graph. A vector space of dimension k is called a *k-space* for short. Furthermore, let \mathbb{F}_q denote the finite field of order q (q is a prime power).

Definition 4. Let $S \subseteq \{0, 1, \dots, k-1\}$. The *generalized Grassmann graph* $J_{q,S}(n, k)$ has as vertices the k -subspaces of \mathbb{F}_q^n , where two vertices are adjacent if their intersection dimension is in S .

In particular, $J_{q,\{0\}}(n, k)$ is the *q-Kneser graph* $K_q(n, k)$ and $J_{q,\{k-1\}}(n, k)$ is the *Grassmann graph* $J_q(n, k)$.

From now on, for all the aforementioned graphs, we assume that $k \leq n/2$, since $J_S(n, k)$ is isomorphic to $J_{\{s+n-2k|s \in S\}}(n, n-k)$. We also assume that $|S| \leq k/2$ because the complement of $J_S(n, k)$ is $J_{\{1, \dots, n\} \setminus S}(n, k)$, and a regular graph is DS if and only if its complement is DS. Note that these observations also hold for Grassmann graphs. Also note that if $S = \emptyset$, the graph is complete or edgeless and therefore trivially DS. Therefore, we also assume that $|S| \geq 1$, and in particular, $k \geq 2$.

3 Previous results

The following results about the cospectrality of generalized Johnson and Grassmann graphs have been obtained by various authors. We present them in a chronological order.

Let q be a prime power and $1 \leq k \leq n/2$.

Theorem 5 ([1, 7]). $K(n, 2)$ is DS if and only if $n \neq 8$.

Theorem 6 ([8]). $K(2k+1, k)$ is DS.

Theorem 7 ([4]). $J_q(2k+1, k)$ is NDS.

Theorem 8 ([3]). $J(n, k)$ and $J_q(n, k)$ are NDS if $k \geq 3$.

Theorem 9 ([6]).

(i) $K(n, k)$ is NDS if there exists an $m < k-1$ for which $\binom{n-k+m}{m} = 2\binom{n-2k+m}{m}$.

(ii) $K_{\{0,2,4,\dots\}}(n, k)$ is NDS if $k \geq 3$.

Theorem 10 ([2]).

(i) $J_{\{0,1,\dots,m\}}(3k-2m-1, k)$ is NDS if $k \geq \max(m+2, 3)$.

(ii) $J_{\{0,1,\dots,m\}}(n, 2m+1)$ is NDS if $m \geq 2$ and $n \geq 4m+2$.

Theorem 11 ([9]). $J_q(n, 2)$ is NDS.

4 $J_{\{2\}}(n, 4)$ is NDS if $n \geq 8$

The authors of [2] found, computationally, that the graph $J_{\{2\}}(8, 4)$ has a cospectral mate by GM-switching with respect to switching sets of size 8, which implies that it is NDS. In this section, we place this graph in an infinite family of graphs that are NDS by using the recent WQH-switching. This solves the second open problem of [2].

We assume that $n \geq 8$, in accordance with the usual assumption that $k \leq n/2$. Note that if $n = 7$, $J_{\{2\}}(n, 4) \cong J_{\{1\}}(7, 3)$ is NDS by Theorem 7. If $n = 6$, then $J_{\{2\}}(n, 4) \cong K(6, 2)$ is DS by Theorem 5 and if $n \leq 5$, then $J_{\{2\}}(n, 4)$ is trivially DS.

In the main result of this section, we use that $J_{\{2\}}(n, 4)$ is edge-regular (adjacent vertices have a constant number of common neighbours), while the cospectral mate is not. Actually, every ‘‘elementary’’ generalized Johnson graph $J_{\{i\}}(n, k)$ is edge-regular, because it is the distance- $(k - i)$ graph of the Johnson graph $J(n, k)$, which is distance-regular.

Lemma 12. *Adjacent vertices of $J_{\{2\}}(n, 4)$ have $\frac{1}{2}n(n + 3) - 26$ common neighbours.*

Proof. Choose any two adjacent vertices v and w . By relabelling the elements of $\{1, 2, \dots, n\}$, we may assume that $v = \{1, 2, 3, 4\}$ and $w = \{1, 2, 5, 6\}$. There are three types of common neighbours.

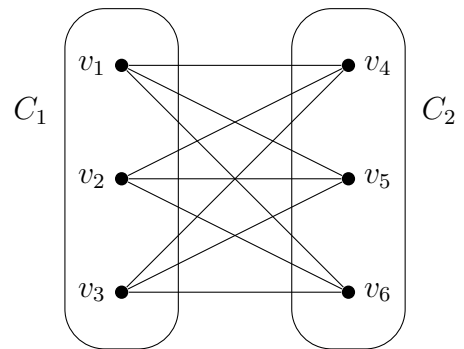
- (i) Vertices that include $\{1, 2\}$. The other two elements can be chosen from $\{7, 8, \dots, n\}$. So there are $\binom{n-6}{2}$ such vertices.
- (ii) Vertices that intersect $\{1, 2\}$ in one element. They must contain one element of $\{3, 4\}$ and one element of $\{5, 6\}$, while the remaining element should be in $\{7, 8, \dots, n\}$. There are $2^3(n - 6)$ such vertices.
- (iii) If a vertex is adjacent to v and w but does not contain 1 or 2, it must be equal to $\{3, 4, 5, 6\}$.

We can now conclude that v and w have $\binom{n-6}{2} + 2^3(n - 6) + 1 = \frac{1}{2}n(n + 3) - 26$ common neighbours. \square

Theorem 13. *$J_{\{2\}}(n, 4)$ is not determined by its spectrum if $n \geq 8$.*

Proof. Define the sets $C_1 := \{v_1, v_2, v_3\}$ and $C_2 := \{v_4, v_5, v_6\}$, where:

- $v_1 := \{1, 2, 3, 4\}$
- $v_2 := \{1, 2, 3, 5\}$
- $v_3 := \{1, 2, 3, 6\}$
- $v_4 := \{1, 4, 5, 6\}$
- $v_5 := \{2, 4, 5, 6\}$
- $v_6 := \{3, 4, 5, 6\}$



Notice the symmetry with respect to the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$. With this in mind, we can prove that C_1 and C_2 form a WQH-switching set of $J_{\{2\}}(n, 4)$, by checking all conditions of Theorem 2. The first two conditions are fulfilled by definition. In order to check

the third condition, select an arbitrary 4-set $v \notin C_1 \cup C_2$. We distinguish seven cases, according to the intersection size of v with $\{1, 2, 3\}$ and $\{4, 5, 6\}$, see Table 1. Because of the symmetry of C with respect to these sets, we assume that $|v \cap \{1, 2, 3\}| \leq |v \cap \{4, 5, 6\}|$. Also note that $|v \cap \{1, 2, 3\}| + |v \cap \{4, 5, 6\}| \leq 4$, and that these cardinalities cannot be 1 and 3 simultaneously, since otherwise the vertex is one of the vertices in $C_1 \cup C_2$. In each case, the requirements of Theorem 2(iii) are met.

$ v \cap \{1, 2, 3\} $	0	0	0	0	1	1	2
$ v \cap \{4, 5, 6\} $	0	1	2	3	1	2	2
# neighbours in C_1	0	0	0	0	1	2	1
# neighbours in C_2	0	0	3	0	1	2	1

Table 1: Given $v \notin C_1 \cup C_2$, the number of neighbours of v in C_1 and C_2 only depend on its intersection size with $\{1, 2, 3\}$ and $\{4, 5, 6\}$.

We are left to prove that the graph obtained by WQH-switching with respect to C_1 and C_2 , is not isomorphic to the original one.

Consider the vertices $v = \{1, 2, 3, 4\} \in C_1$ and $w = \{1, 4, 5, 7\} \notin C_1 \cup C_2$. The neighbours of w are preserved by switching, since $|w \cap \{1, 2, 3\}| = 1$ and $|w \cap \{4, 5, 6\}| = 2$ (see Table 1). In particular, v and w are adjacent in both Γ and Γ' . We show that they have more common neighbours in Γ' than in Γ . For this, we only need to consider neighbours of w for which the adjacency with v is changed during the switching process. We observe from Table 1 that the switching only affects those vertices which have 2 elements in one of the sets $\{1, 2, 3\}, \{4, 5, 6\}$ and none in the other. Neighbours of w that contain 2 elements of $\{1, 2, 3\}$ and none of $\{4, 5, 6\}$, are of the form $\{1, 2, 7, a\}$ or $\{1, 3, 7, a\}$ with $a \geq 8$. So there are a total of $2(n-7)$ vertices that are adjacent to v and w in Γ but not adjacent to v in Γ' . On the other hand, neighbours of w that contain no element of $\{1, 2, 3\}$ and 2 elements of $\{4, 5, 6\}$, are of the form $\{4, 5, a, b\}$, $\{4, 6, 7, a\}$ or $\{5, 6, 7, a\}$ with $a, b \geq 8$. They are not adjacent to v in Γ , but become adjacent to v after switching. So $\binom{n-7}{2} + 2(n-7)$ new common neighbours are created.

We conclude that v and w have $\binom{n-7}{2}$ more common neighbours in Γ' than any two adjacent vertices in Γ , which is a strictly positive difference, except when $n = 8$. In order to solve this one specific case, we take a look at an other pair of vertices.

Assume $n = 8$ and consider the vertices $v = \{1, 2, 3, 4\} \in C_1$ and $u = \{5, 6, 7, 8\} \notin C_1 \cup C_2$. They are not adjacent in Γ , but become adjacent after switching, since $|u \cap \{1, 2, 3\}| = 0$ and $|u \cap \{4, 5, 6\}| = 2$ (see Table 1). We give a lower bound on the number of common neighbours of v and u in Γ' . In the original graph, v and u have $\binom{4}{2}^2 = 36$ common neighbours. Three of these are the vertices of C_2 , and are therefore lost after switching. All other common neighbours of v and u in Γ that are no longer common neighbours in Γ' , are those that contain 2 elements of $\{1, 2, 3\}$ and none of $\{4, 5, 6\}$, i.e. the three vertices $\{1, 2, 7, 8\}$, $\{1, 3, 7, 8\}$ and $\{2, 3, 7, 8\}$. We are left with at least $36 - 3 - 3 = 30$ common neighbours, which is strictly more than $\frac{1}{2}8(8+3) - 26 = 18$, the number of common neighbours of any two vertices in Γ (Lemma 12).

We proved that there are always two neighbours in Γ' which have strictly more common neighbours than any two neighbours in Γ . So Γ and Γ' cannot be isomorphic. \square

5 $J_{\{1,2,\dots,\frac{k-1}{2}\}}(2k, k)$ is NDS if $k \geq 5$, k odd

The next result uses GM-switching with respect to two switching sets. Our argument is inspired by the proof of Theorem 10(i), as $J_{\{1,2,\dots,\frac{k-1}{2}\}}(2k, k)$ can be seen as two copies of $J_{\{0,1,\dots,\frac{k-3}{2}\}}(2k-1, k-1)$, connected in a certain way. While our first switching set corresponds to the one used to show Theorem 10(i), the second one is given by its complement.

Theorem 14. $J_{\{1,2,\dots,\frac{k-1}{2}\}}(2k, k)$ is NDS if $k \geq 5$, k odd.

Proof. Denote $\Gamma := J_{\{1,2,\dots,\frac{k-1}{2}\}}(2k, k)$. Let C_1 be the set of k -subsets of $[2k]$ that contain $[k-1]$ and let C_2 be the set of k -subsets of $[2k]$ that are disjoint with $[k-1]$. There is a bijection from C_1 to C_2 , given by the complement.

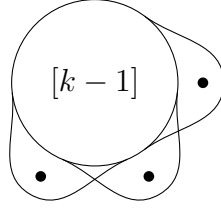


Figure 1: Elements of C_2 include $[k-1]$ and one other element.

We first check the conditions of Theorem 1 in order to show that C_1 and C_2 form a GM-switching set of Γ . It is immediate that $|C_1| = |C_2| = k+1$. Moreover, every two k -sets of C_1 (resp. C_2) intersect in $k-1$ elements, which means that C_1 (resp. C_2) is a coclique. Any vertex in C_1 has k neighbours in C_2 and vice versa. Choose an arbitrary vertex $v \notin C_1 \cup C_2$ and let l be the size of its intersection with $[k-1]$. Then v intersects $l+1$ vertices of C_1 in l elements and the other $k-l$ vertices in $l+1$ elements. Similarly, v intersects $l+1$ vertices of C_2 in $k-l$ elements and $k-l$ vertices in $k-l-1$ elements. So if $l < \frac{k-1}{2}$, then v is adjacent to all vertices of C_1 and none of C_2 . If $l = \frac{k-1}{2}$, then it is adjacent to $\frac{k+1}{2} = \frac{1}{2}|C_1|$ elements of each switching set, and if $l > \frac{k-1}{2}$, then it is adjacent with no vertices of C_1 and all vertices of C_2 . We conclude that C_1 and C_2 fulfil the GM-switching conditions.

Next, we show that the graph Γ' obtained from Γ by GM-switching with respect to C_1 and C_2 , is not isomorphic to Γ . Define $v := [k] \in C_1$ and $w := \{2, 3, \dots, k+1\} \notin C_1 \cup C_2$.

We claim that v and w have strictly less common neighbours after switching. For this, it suffices to count the number of common neighbours that are lost or added during the GM-switching process. Therefore, we only consider those neighbours of w that have $\frac{1}{2}|C_1| = \frac{k+1}{2}$ neighbours in C_1 , i.e. those that contain exactly $\frac{k-1}{2}$ elements of $[k-1]$. First consider the common neighbours that are added. They should not be adjacent to v in the original graph, which means that they must contain the element k . Adjacency with w implies that they must also contain the element 1 and not the element $k+1$. Thus, there are

$$\underbrace{\binom{k-2}{\frac{k-3}{2}}}_{\frac{k-1}{2} \text{ elements of } [k-1], \text{ one of which is } 1} \cdot \underbrace{\binom{k-1}{\frac{k-1}{2}}}_{\frac{k+1}{2} \text{ elements not in } [k-1], \text{ one of which is } k}$$

added common neighbours of v and w . Now consider the common neighbours that are lost. Since these vertices must originally be adjacent to v , they may not contain k . If

they contain the element 1, then they will automatically be adjacent to w . So there are at least

$$\underbrace{\binom{k-2}{\frac{k-3}{2}}}_{\frac{k-1}{2} \text{ elements of } [k-1], \text{ one of which is } 1} \cdot \underbrace{\binom{k}{\frac{k+1}{2}}}_{\frac{k+1}{2} \text{ elements not in } [k]}$$

common neighbours of v and w that are lost during switching, which is already strictly more than the number of added common neighbours.

Now, assume by contradiction that $\Gamma \cong \Gamma'$. Let $\lambda(u, w)$ denote the number of common neighbours of u and w , for any vertex u and w defined as before. Since Γ is vertex-transitive, the multiset

$$\Lambda(w) = \{\lambda(u, w) \mid u \not\sim w\}$$

remains the same after switching. The value of $\lambda(u, w)$ stays the same if $u \notin C_1 \cup C_2$, so we can restrict ourselves to vertices in $C_1 \cup C_2$, i.e. the multiset

$$\{\lambda(u, w) \mid u \in C_1 \cup C_2, u \not\sim w\}$$

remains the same. Since w is adjacent to all vertices of C_2 and none of C_1 , this is equivalent to saying that

$$\{\lambda(u, w) \mid u \in C_1\}$$

is invariant. We now apply the same reasoning as in [2]. Define $v' := [k-1] \cup \{k+1\}$. The automorphism induced by the permutation $(k+2 \ k+3 \ \dots \ 2k)$ fixes v, v' and w , but shifts the other vertices of C_1 cyclically. So $\lambda(u, w)$ is constant among all $u \in C_1 \setminus \{v, v'\}$. Together with the observation that $\lambda(v, w) = \lambda(v', w)$, we see that $\lambda(v, w)$ is the same for Γ and Γ' , a contradiction with the above.

We conclude that Γ and Γ' are cospectral mates. Thus, Γ is NDS. \square

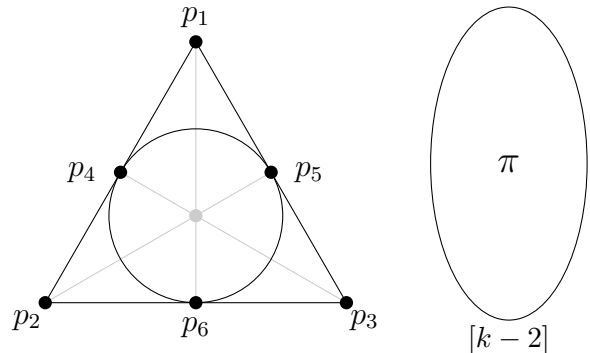
6 q -Kneser graphs are NDS if $q = 2$

In this section, we prove that the q -Kneser graph $K_2(n, k) = J_{2, \{0\}}(n, k)$ is NDS (under the assumption that $k \geq 2$, otherwise the graph is trivially DS) using GM-switching with respect to a switching set of size 4. This result can be seen as a q -analog of Theorem 9(i) in the case that $q = 2$ (and without the extra condition).

Theorem 15. $K_2(n, k)$ is not determined by its spectrum.

Proof. Let $\Gamma := K_2(n, k)$. Define the following 1-spaces in \mathbb{F}_2^n (drawn projectively, as points in $PG(n-1, 2)$).

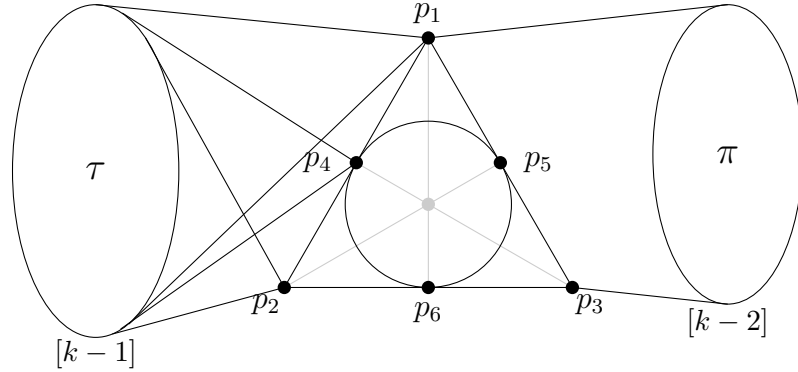
- $p_1 := \langle (1, 0, 0, 0, 0, \dots, 0) \rangle$
- $p_2 := \langle (0, 1, 0, 0, 0, \dots, 0) \rangle$
- $p_3 := \langle (0, 0, 1, 0, 0, \dots, 0) \rangle$
- $p_4 := \langle (1, 1, 0, 0, 0, \dots, 0) \rangle$
- $p_5 := \langle (1, 0, 1, 0, 0, \dots, 0) \rangle$
- $p_6 := \langle (0, 1, 1, 0, 0, \dots, 0) \rangle$



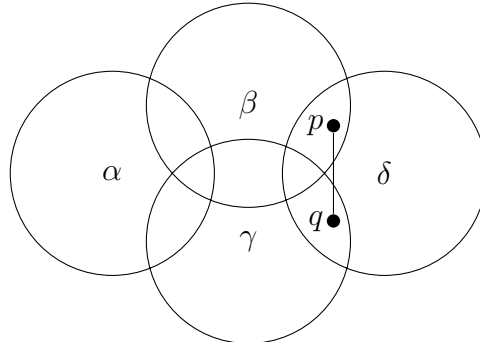
Let π be a $(k-2)$ -space that has a trivial intersection with the 3-space (projective plane) spanned by these points. Define $C := \{p_1p_2\pi, p_1p_3\pi, p_2p_3\pi, p_4p_5\pi\}$.

We first prove that C is a GM-switching set of Γ by checking the conditions of Theorem 1 (where $k = 1$). Every element of C meets every other element of C in a $k-1$ -space, so C is a coclique and therefore regular. Choose an arbitrary k -space not in C and let $R := \{p_1\pi, p_2\pi, \dots, p_6\pi\}$. If the chosen k -space intersects all elements of R trivially, then it also intersects all spaces in C trivially, which means that it is adjacent to all the corresponding vertices of the q -Kneser graph. If it intersects just one of the elements of R non-trivially, then it is adjacent to exactly 2 vertices of C . If the k -space intersects at least two elements of R non-trivially, then it either intersects π nontrivially, or it contains a 2-space in the $k+1$ -space $p_1p_2p_3\pi$. Either case, it intersects every element of C non-trivially, therefore being adjacent to no element of C .

Let Γ' be the graph obtained from Γ by GM-switching with respect to C . We prove that Γ and Γ' are non-isomorphic. Let τ be a $(k-1)$ -space that intersects $p_1p_2p_3\pi$ trivially and consider the pairwise non-adjacent k -spaces $p_1\tau, p_2\tau$ and $p_4\tau$ in Γ' . Then there is exactly one k -space that is adjacent to $p_1\tau$ but not adjacent to both $p_2\tau$ and $p_4\tau$ in Γ' . This is the k -space $p_1p_3\pi$. Indeed, a k -space outside C that intersects both $p_2\tau$ and $p_4\tau$, must intersect $p_1\tau$ as well. So the only spaces that can meet this property are in C . Since the switching reverses adjacency for $p_1\tau, p_2\tau$ and $p_4\tau$, we get that $p_1p_3\pi$ is adjacent to $p_1\tau$ and not adjacent to $p_2\tau$ and $p_4\tau$, while the other elements of C are not.



Now consider three arbitrarily chosen non-adjacent k -spaces in Γ . Call them α, β and γ . Non-adjacency here means that they intersect one another. We prove that the number of k -spaces that intersects α trivially but intersects both β and γ non-trivially, is never equal to one. Let δ be such a space (if it does not exist, we are done). Choose two 1-spaces (projective points) $p \in \beta \cap \delta$ and $q \in \gamma \cap \delta$. The 2-space (projective line) pq does not intersect α , as it lies in δ . There are $2^{k(k-2)} \binom{n-k-1}{k-2}_2 > 1$ different k -spaces through pq that intersect α trivially, and they all meet the above property.



Since the number of k -spaces with this property is different for Γ and Γ' , while it should be invariant under isomorphism, we conclude that Γ and Γ' are not isomorphic. \square

7 Open problems

Besides the three infinite families of graphs that are NDS, we also found three “sporadic” generalized Johnson graphs with a cospectral mate. For each of these three graphs, we provide a pair of switching sets that produces a cospectral mate, see the table below. The fact that they are non-isomorphic was checked by computer.

graph	method	switching sets
$J_{\{1\}}(11, 4)$	WQH	$C_1 = \{\{1, 2, 3, 10\}, \{4, 5, 6, 10\}, \{7, 8, 9, 10\}\}$ $C_2 = \{\{1, 2, 3, 11\}, \{4, 5, 6, 11\}, \{7, 8, 9, 11\}\}$
$J_{\{2,4\}}(10, 5)$	GM	$C_1 = \{\{1, 2, 3, 4, 5\}, \{1, 2, 3, 6, 7\}, \{1, 2, 4, 6, 8\}, \{1, 2, 5, 7, 8\}\}$ $C_2 = \{\{6, 7, 8, 9, 10\}, \{4, 5, 8, 9, 10\}, \{3, 5, 7, 9, 10\}, \{3, 4, 6, 9, 10\}\}$
$J_{\{2,4\}}(12, 6)$	GM	$C_1 = \{\{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 7, 8\}, \{1, 2, 3, 5, 7, 9\}, \{1, 2, 3, 6, 8, 9\}\}$ $C_2 = \{\{7, 8, 9, 10, 11, 12\}, \{5, 6, 9, 10, 11, 12\}, \{4, 6, 8, 10, 11, 12\}, \{4, 5, 7, 10, 11, 12\}\}$

It would be nice to see if any of these switching sets can be extended to an infinite family of generalized Johnson graphs that are NDS. We believe that $J_{\{2,4,\dots\}}(2k, k)$ is NDS if $k \geq 4$, following a similar argument as in [6], but with two switching sets instead of one.

More generally, it remains an open problem to determine the cospectrality of other graphs in the Johnson and Grassmann schemes (entries “?” in the tables in the Appendix). The smallest open case is $K(9, 3)$, see also [2, Section 5].

Acknowledgements

Aida Abiad is partially supported by the Dutch Research Council through the grant VI.Vidi.213.085 and by the Research Foundation Flanders through the grant 1285921N. Jozefien D’haeseleer is supported by the Research Foundation Flanders through the grant 1218522N.

References

- [1] L. Chang, The uniqueness and nonuniqueness of triangular association schemes, *Science Record* **3** (1959), 604–613.
- [2] S.M. Cioabă, W.H. Haemers, T. Johnston and M. McGinnis, Cospectral mates for the union of some classes in the Johnson association scheme, *Linear Algebra and its Applications* **539** (2018), 219–228.
- [3] E.R. van Dam, W.H. Haemers, J.H. Koolen and E. Spence, Characterizing distance-regularity of graphs by the spectrum, *Journal of Combinatorial Theory, Series A* **113** (2006), 1805–1820.
- [4] E.R. van Dam and J.H. Koolen, A new family of distance-regular graphs with unbounded diameter, *Inventiones Mathematicae* **162** (2005), 189–193.

- [5] C.D. Godsil and B.D. McKay, Constructing cospectral graphs, *Aequationes Mathematicae* **25** (1982), 257–268.
- [6] W.H. Haemers and F. Ramezani, Graphs cospectral with Kneser graphs, *Graphs and Combinatorics* **531** (2010), 159–164.
- [7] A.J. Hoffman, On the uniqueness of the triangular association scheme, *The Annals of Mathematical Statistics* **31** (1960), 492–497.
- [8] T. Huang and C. Liu, Spectral characterization of some generalized Odd graphs, *Graphs and Combinatorics* **15** (1999), 195–209.
- [9] F. Ihringer and A. Munemasa, New strongly regular graphs from finite geometries via switching, *Linear Algebra and its Applications* **580** (2019), 464–474.
- [10] L. Qiu, Y. Ji, W. Wang, On a theorem of Godsil and McKay concerning the construction of cospectral graphs, *Linear Algebra and its Applications* **603** (2020), 265–274.
- [11] W. Wang, L. Qiu and Y. Hu, Cospectral graphs, GM-switching and regular rational orthogonal matrices of level p , *Linear Algebra and its Applications* **563** (2019), 154–177.

Appendix: overview of results

This appendix provides a structured overview of the currently known cospectrality results for some small generalized Johnson and Grassmann graphs. New results are indicated with an asterisk (* or **).

Legend:

⁰ Trivial	¹ Hoffman/Chang (1959)
² Huang, Liu (1999)	³ Van Dam et al. (2006)
⁴ Haemers, Ramezani (2010)	⁵ Cioabă et al. (2018)
*Theorem 13	**Sporadic (Section 7)

$J_S(n, 2)$		S	$ V $
		$\{0\}$	
n	4	DS ⁰	6
	5	DS ¹	10
	6	DS ¹	15
	7	DS ¹	21
	8	NDS ¹	28
	9	DS ¹	36

Table 2: Cospectrality of small generalized Johnson graphs with $k = 2$.

$J_S(n, 3)$		S			$ V $
		$\{0\}$	$\{1\}$	$\{2\}$	
n	6	DS ⁰	NDS ⁴	NDS ³	20
	7	DS ²	NDS ⁴	NDS ³	35
	8	NDS ⁴	NDS ⁴	NDS ³	56
	9	?	NDS ⁴	NDS ³	84
	10	?	NDS ⁴	NDS ³	120
	11	?	NDS ⁴	NDS ³	165

Table 3: Cospectrality of small generalized Johnson graphs with $k = 3$.

$J_S(n, 4)$		S							$ V $
		$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0, 1\}$	$\{0, 2\}$	$\{0, 3\}$	
n	8	DS ⁰	?	NDS*	NDS ³	?	NDS ⁴	?	70
	9	DS ²	?	NDS*	NDS ³	NDS ⁵	NDS ⁴	?	126
	10	?	?	NDS*	NDS ³	?	NDS ⁴	?	210
	11	NDS ⁴	NDS**	NDS*	NDS ³	?	NDS ⁴	?	330
	12	?	?	NDS*	NDS ³	?	NDS ⁴	?	495
	13	?	?	NDS*	NDS ³	?	NDS ⁴	?	715

Table 4: Cospectrality of small generalized Johnson graphs with $k = 4$.

Legend:

⁰Trivial

²Huang, Liu (1999)

²Van Dam, Koolen (2005)

³Van Dam et al. (2006)

⁴Haemers, Ramezani (2010)

⁵Cioabă et al. (2018)

⁶Ihringer, Munemasa (2019)

*Theorem 14

**Sporadic (Section 7)

$J_S(n, 5)$		S														$ V $	
		{0}	{1}	{2}	{3}	{4}	{0, 1}	{0, 2}	{0, 3}	{0, 4}	{1, 2}	{1, 3}	{1, 4}	{2, 3}	{2, 4}		{3, 4}
n	10	DS ⁰	?	?	?	NDS ³	?	?	?	?	NDS*	NDS ⁴	?	?	NDS**	NDS ⁵	252
	11	DS ¹	?	?	?	NDS ³	?	?	?	?	?	NDS ⁴	?	?	?	NDS ⁵	462
	12	?	?	?	?	NDS ³	NDS ⁵	?	?	?	?	NDS ⁴	?	?	?	NDS ⁵	792
	13	?	?	?	?	NDS ³	?	?	?	?	?	NDS ⁴	?	?	?	NDS ⁵	1287
	14	NDS ⁴	?	?	?	NDS ³	?	?	?	?	?	NDS ⁴	?	?	?	NDS ⁵	2002
	15	?	?	?	?	NDS ³	?	?	?	?	?	NDS ⁴	?	?	?	NDS ⁵	3003

Table 5: Cospectrality of small generalized Johnson graphs with $k = 5$.

12

$J_{q,S}(n, 2)$		$q = 2$		$q = 3$		$q = 4$	
		S	$ V $	S	$ V $	S	$ V $
		{0}		{0}		{0}	
n	4	NDS ⁶	35	NDS ⁶	130	NDS ⁶	357
	5	NDS ⁶	155	NDS ²	1210	NDS ²	5797
	6	NDS ⁶	651	NDS ⁶	11 011	NDS ⁶	93 093
	7	NDS ⁶	2667	NDS ⁶	99 463	NDS ⁶	$\approx 10^6$
	8	NDS ⁶	10 795	NDS ⁶	$\approx 10^6$	NDS ⁶	$\approx 10^7$
	9	NDS ⁶	43 435	NDS ⁶	$\approx 10^7$	NDS ⁶	$\approx 10^8$

Table 6: Cospectrality of small generalized Grassmann graphs with $k = 2$ and $q = 2, 3, 4$.

Legend: ¹Van Dam et al. (2006) *Theorem 15

$J_{q,S}(n, 3)$		$q = 2$				$q = 3$				$q = 4$			
		S			$ V $	S			$ V $	S			$ V $
		{0}	{1}	{2}		{0}	{1}	{2}		{0}	{1}	{2}	
n	6	NDS*	?	NDS ¹	1395	?	?	NDS ¹	33 880	?	?	NDS ¹	376 805
	7	NDS*	?	NDS ¹	11 811	?	?	NDS ¹	$\approx 10^6$?	?	NDS ¹	$\approx 10^7$
	8	NDS*	?	NDS ¹	97 155	?	?	NDS ¹	$\approx 10^7$?	?	NDS ¹	$\approx 10^9$
	9	NDS*	?	NDS ¹	$\approx 10^6$?	?	NDS ¹	$\approx 10^9$?	?	NDS ¹	$\approx 10^{11}$
	10	NDS*	?	NDS ¹	$\approx 10^7$?	?	NDS ¹	$\approx 10^{10}$?	?	NDS ¹	$\approx 10^{13}$
	11	NDS*	?	NDS ¹	$\approx 10^8$?	?	NDS ¹	$\approx 10^{11}$?	?	NDS ¹	$\approx 10^{14}$

Table 7: Cospectrality of small generalized Grassmann graphs with $k = 3$ and $q = 2, 3, 4$.

$J_{q,S}(n, 4)$		$q = 2$							$q = 3$							$q = 4$									
		S						$ V $	S						$ V $	S						$ V $			
		{0}	{1}	{2}	{3}	{0, 1}	{0, 2}		{0, 3}	{0}	{1}	{2}	{3}	{0, 1}		{0, 2}	{0, 3}	{0}	{1}	{2}	{3}		{0, 1}	{0, 2}	{0, 3}
n	8	NDS*	?	?	NDS ¹	?	?	?	200 787	?	?	?	NDS ¹	?	?	?	$\approx 10^8$?	?	?	NDS ¹	?	?	?	$\approx 10^{10}$
	9	NDS*	?	?	NDS ¹	?	?	?	$\approx 10^6$?	?	?	NDS ¹	?	?	?	$\approx 10^{10}$?	?	?	NDS ¹	?	?	?	$\approx 10^{12}$
	10	NDS*	?	?	NDS ¹	?	?	?	$\approx 10^8$?	?	?	NDS ¹	?	?	?	$\approx 10^{12}$?	?	?	NDS ¹	?	?	?	$\approx 10^{14}$
	11	NDS*	?	?	NDS ¹	?	?	?	$\approx 10^9$?	?	?	NDS ¹	?	?	?	$\approx 10^{13}$?	?	?	NDS ¹	?	?	?	$\approx 10^{17}$
	12	NDS*	?	?	NDS ¹	?	?	?	$\approx 10^{10}$?	?	?	NDS ¹	?	?	?	$\approx 10^{15}$?	?	?	NDS ¹	?	?	?	$\approx 10^{19}$
	13	NDS*	?	?	NDS ¹	?	?	?	$\approx 10^{11}$?	?	?	NDS ¹	?	?	?	$\approx 10^{17}$?	?	?	NDS ¹	?	?	?	$\approx 10^{22}$

Table 8: Cospectrality of small generalized Grassmann graphs with $k = 4$ and $q = 2, 3, 4$.